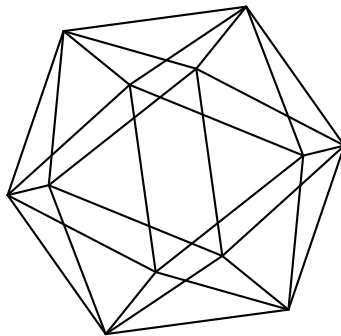# Max-Planck-Institut für Mathematik Bonn

The Ismagilov conjecture over a finite field $\mathbb{F}_p$

by

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Abstract

We construct the so-called quasiregular representations of the group of infinite upper triangular matrices with coefficients in a finite field and give the criteria of theirs irreducibility in terms of the initial measure. These representations are particular case of the Koopman representation hence, we find new conditions of its irreducibility. Since the field $\mathbb{F}_p$ is compact some new operators in the commutant emerges. Therefore, the Ismagilov conjecture in the case of the finite field should be corrected.

Keywords: infinite-dimensional groups, finite field, unitary representation, irreducible representation, quasiregular representation, Koopman's representation, Ismagilov's conjecture, quasi-invariant, ergodic measure
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Contents

1. Introduction

Let $\hat{G}$ be a *dual space of a group* G, i.e., the set of all equivalence classes of unitary irreducible representations of the group G. Our *far reching goal* is to describe $\hat{G}$ for $G = \varinjlim_n G_n$ where $G_n = B(n, \mathbf{k})$ is the group of all upper triangular matrices with units on the principal diagonal with natural inclusion $G_n \subset G_{n+1}$, where $\mathbf{k} = \mathbb{R}$ or $\mathbf{k} = \mathbb{F}_p$ is a finite field $\mathbb{Z}/p\mathbb{Z}$, p is prime.

We mention here only some results concerning representations of algebraic groups over a finite field. The book by G, Lusztig [24] presents a classification of all (complex) irreducible representations of a reductive group with connected centre, over a finite field. To achieve this, the author uses etale intersection cohomology, and detailed information on representations of Weyl groups.

From the article by P. Deligne and G. Lusztig [6]: "Let us consider a connected reductive algebraic group G, defined over a finite field $\mathbb{F}_q$, with *Frobenius map F*. We shall be concerned with the representation theory of the finite group G^F, over field of characteristic 0. In 1968, Macdonalds conjectured, on the basis of the character table known at that time for $(\mathrm{GL}_4, \mathrm{Sp}_4)$ that should be a well defined correspondence which, to any F-stable maximal torus T of G and a character θ of T^F in general position, associate an irreducible representation of G^F; moreover, if T modulo the centre of G is *anisotropic* over $\mathbb{F}_q$, the corresponding representation of G^F should be *caspidal*. In this paper we prove Macdonald's conjecture. More precisely, for T as above and θ an arbitrary character of T^F we construct virtual representations R_T^θ which have all the required properties."

The group $G_n = B(n, \mathbb{F}_p)$ is finite, hence the set $\hat{G}_n$, in principal, is known (it is numerated by the set of all conjugacy classes) and all irreducible representations are contained in the regular representation. "Nevertheless the complete classification of the complex irreducible representations of this group has long been known to be a difficult task" [27] . Recently in 2006, Ning Yan, have introduced in [27] "a natural variant of the orbit method, in which the central role is played by certain *clusters of coadjoint orbits*. This method of clusters leads to the construction of a subring in the representation ring of $B(n, \mathbb{F}_p)$ that is "rich in structure but pleasantly comprehensible".

The article by V. Gorin, A. Vershik and S. Kerov [8] is devoted to the representation theory of locally compact infinite-dimensional group GLB of *almost upper-triangular infinite matrices* over the finite field with q elements. From [8]:" The group GLB consist of all almost triangular matrices of infinite order. An infinite matrix $g = (g_{ij})$, $i, j = 1, 2, \ldots$, is said to be almost triangular if the number of its subdiagonal elements $g_{ij} \neq 0$, $i > j$, is finite. This group was defined by S. Kerov, A. Vershik, and A. Zelevinsky in 1982 as an adequate for $n = \infty$ analogue of general linear groups $GL(n, q)$. It serves as an alternative to $GL(\infty, q)$, whose representation theory is poor. Our most important results are the description of semi-finite unipotent traces (characters) of the group GLB via certain probability measures on the Borel

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subgroup B and the construction of the corresponding von Neumann factor representations of type II_∞."

Coming back to our group $B_0^{\mathbb{N}}(\mathbb{F}_p) = \varinjlim_n G_n$ where $G_n = B(n, \mathbb{F}_p)$, we mention that with the natural homomorphism $p_{n+1}^n : G_{n+1} \to G_n$ (see (1.1)) we can associate an inclusion $\hat{G}_n \to \hat{G}_{n+1}$ therefore, $\hat{G} \supset \bigcup_n \hat{G}_n$. In the case $\mathbf{k} = \mathbb{R}$ one may use Kirillov's orbit method to describe $\hat{G}_n$. We define p_{n+1}^n as follows:

$$B(n+1, \mathbf{k}) \ni x = x^{n+1} x_n \mapsto p_{n+1}^n(x) = x_n \in B(n, \mathbf{k}), \qquad (1.1)$$

where for

$$x = I + \sum_{1 \le k < m \le n+1} x_{km} E_{km} \in B(n+1, \mathbf{k}), \quad \text{we set}$$

$$x^{n+1} = I + \sum_{k=1}^n x_{kn+1} E_{kn+1}, \quad x_n = I + \sum_{1 \le k < m \le n} x_{km} E_{km}. \qquad (1.2)$$

Obviously, $x = x^{n+1} x_n$ and $B(n+1, \mathbf{k})$ is a semi-direct product

$$B(n+1, \mathbf{k}) = \mathbf{k}^n \ltimes B(n, \mathbf{k}). \qquad (1.3)$$

Remark 1.1. The group $B^{\mathbb{N}}(\mathbb{F}_p)$ is compact, the corresponding *Haar measure* on this group is infinite tensor product of the normalised invariant measures on $\mathbb{F}_p$, where μ_{inv}^{kn} is defined by (2.4)

$$h = \mu_{inv} = \otimes_{1 \le k < n} \mu_{inv}^{kn}. \qquad (1.4)$$

Therefore, all irreducible representations of the group $B^{\mathbb{N}}(\mathbb{F}_p)$ are *finite-dimensional* and are contained in the decomposition of the the regular representation of $B^{\mathbb{N}}(\mathbb{F}_p)$. Moreover we have

$$\widehat{B^{\mathbb{N}}(\mathbb{F}_p)} = \bigcup_{n \ge 1} \widehat{B(n, \mathbb{F}_p)}. \qquad (1.5)$$

The group $B_0^{\mathbb{N}}(\mathbb{F}_p) = \varinjlim_n B(n, \mathbb{F}_p)$ is subgroup of $B^{\mathbb{N}}(\mathbb{F}_p)$ therefore, $\widehat{B_0^{\mathbb{N}}(\mathbb{F}_p)} \supset \widehat{B^{\mathbb{N}}(\mathbb{F}_p)}$. We construct the *infinite-dimensional* irreducible representations of the group $B_0^{\mathbb{N}}(\mathbb{F}_p)$ as *quasiregular representations*. The most important observation is that the measure on the homogeneous space X^m (see (2.1) below), we use for this, has the property that its projection and the projection of the Haar measure on subspace $X^{(k)}$ (see (2.6) below) are *orthogonal*.

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Remark 1.2. We show that $\hat{G} \setminus \left(\bigcup_n \hat{G}_n \right) \neq \emptyset$. Namely, $\hat{G} \setminus \left(\bigcup_n \hat{G}_n \right)$ contains "regular" and "quasiregular" representations of the group G.

The criteria of the irreducibilty of quasiregular representations in the case of the field $\mathbb{F}_p$ is established. Some new conditions of the irreducibility are found, if we compare with the Ismagilov conjecture 2.1 in the case $k = \mathbb{R}$.

To construct some representations of a group G we shall use the well-known concept of G−space. Let we have the *measurable action* $\alpha : G \to$ $\mathrm{Aut}(X)$ of the group G on a measurable space (X, μ) with G-*quasi-invariant measure* μ having the following property: $\mu^{\alpha_t} \sim \mu \quad \forall t \in G$. With these dates one can associate a *representation* $\pi^{\alpha,\mu,X} : G \to U(L^2(X, d\mu))$ defined as follows:

$$(\pi_t^{\alpha,\mu,X} f)(x) = (d\mu(\alpha_{t^{-1}}(x))/d\mu(x))^{1/2} f(\alpha_{t^{-1}}(x)), \quad f \in L^2(X, \mu). \quad (1.6)$$

In the case of invariant measure this representations is called *Koopman's representation*, see [11]. Consider the *centralizer* $Z_{\mathrm{Aut}(X)}(\alpha(G))$ of the subgroup $\alpha(G) = \{\alpha_t \in \mathrm{Aut}(X) \mid t \in G\}$ in the group $\mathrm{Aut}(X)$

$$Z_{\mathrm{Aut}(X)}(\alpha(G)) = \{g \in \mathrm{Aut}(X) \mid \{g, \alpha_t\} = g\alpha_t g^{-1}\alpha_t^{-1} = e \; \forall t \in G\}.$$

The following conjecture was proved for some infinite-dimensional groups.

Conjecture 1.1 (Kosyak, [17, 18]). *The representation* $\pi^{\alpha,\mu,X} : G \to$ $U(L^2(X, \mu))$ *is irreducible if and only if*
1) $\mu^g \perp \mu \; \forall g \in Z_{\mathrm{Aut}(X)}(\alpha(G)) \setminus \{e\}$, *(where* $\perp$ *stands for singular),*
2) the measure μ *is* G-*ergodic.*

We recall that a measure μ is G-ergodic if $f(\alpha_t(x)) = f(x) \; \forall t \in G$ implies $f(x) = const \; \mu$ a.e. (almost everywhere) for all functions $f \in L^1(X, \mu)$.

2. Regular and guasiregular representations of the group $B_0^{\mathbb{N}}(\mathbb{F}_p)$

2.1. Regular and quasiregular representations of infinite-dimensional groups, the case of $\mathbf{k} = \mathbb{R}$

Let the group G *be a locally compact,* $X = G$ and h be the Haar measure. If α is right or left action of the group G on itself then $\rho = \pi^{R,h,G}$ and $\lambda = \pi^{L,h,G}$ are well known *right* and *left regular representations. Quasiregular representation* is a particular case of the representation $\pi^{\alpha,\mu,X}$ defined by

(1.6), where $X = H\backslash G$, H is some closed subgroup of G and μ is some G-quasi-invariant measure on X.

Recall the notions of the regular and quasiregular representations for infinite-dimensional groups and the Ismagilov conjecture 2.1. To define a "*regular representation*" for *infinite-dimensional group G* the initial group G as a candidate for X is not suitable since on G there is no Haar (invariant) measure (Weil, [28]) no a G-quasi-invariant measure (Xia Dao-Xing, [26]). It is natural to consider some bigger topological group $\tilde{G}$ containing the initial group G as the dense subgroup and a G-quasi-invariant measure μ on $\tilde{G}$.

Definition 2.1. Representations $T^{\alpha,\mu} = \pi^{\alpha,\mu,\tilde{G}}$ where $\alpha = R$ (resp. $\alpha = L$) we call the *right (resp. the left) regular representation* of the group G.

Conjecture 2.1 (Ismagilov, 1985). *The right regular representation $T^{R,\mu}$: $G \to U(L^2(\tilde{G}, \mu))$ is irreducible if and only if*
 1) $\mu^{L_t} \perp \mu$ for all $t \in G\backslash\{e\}$,
 2) the measure μ is G-ergodic.

Similarly, we can generalize the notion of *quasiregular representation* of a group G associated with some subgroup H using a suitable completion $\tilde{X} = \widetilde{H\backslash G}$ of the homogeneous space $X = H\backslash G$ and constructing some G-right quasi-invariant measure μ on $\tilde{X}$.

Consider the group $G = B_0^{\mathbb{N}}(\mathbb{R}) = \varinjlim_n B(n, \mathbb{R})$. Let us fix the space X and the measure μ on X as follows, where E_{kn} are matrix units of infinite order:
$$X = \tilde{G} = B^{\mathbb{N}} = \{I + x \mid x = \sum_{1 \le k < n} x_{kn} E_{kn}, \ x_{kn} \in \mathbb{R}\},$$

$$d\mu_b(x) = \otimes_{1 \le k < n}(b_{kn}/\pi)^{1/2} \exp(-b_{kn} x_{kn}^2) dx_{kn}, \ b = (b_{kn})_{k<n}.$$

Theorem 2.2 ([12, 13]). *Ismagilov's conjecture holds, i.e., $T^{R,\mu} \in \hat{G}$ if and only if $\mu^{L_t} \perp \mu$ for all $t \in G\backslash\{e\}$, and the measure μ is G-ergodic. Moreover*
$$T^{R,\mu_1} \sim T^{R,\mu_1} \quad \text{if and only if} \quad \mu_1 \sim \mu_2.$$

Quasiregular representations for the group $G = B_0^{\mathbb{N}}$. Let us consider two subgroups of the group $B^{\mathbb{N}}(\mathbb{R})$:
$$B_m = \{I + x \in B^{\mathbb{N}} \mid x = \sum_{m<k<n} x_{kn} E_{kn}, \ x_{kn} \in \mathbb{R}\},$$

$$B^m = \{I + x \in B^{\mathbb{N}} \mid x = \sum_{1 \le k \le m, k < n} x_{kn} E_{kn}\}.$$

The group $B^{\mathbb{N}}(\mathbb{R})$ is a semi-direct product $B^{\mathbb{N}} = B_m \rtimes B^m$. Fix the corresponding decomposition $x = x_m \cdot x^m$. Define $X^m = B_m \backslash B^{\mathbb{N}} \simeq B^m$. Right action R of G on X^m is well defined $R_t(x) := (xt^{-1})^m$, $x \in X^m$, $t \in G$. The measure on X^m is defined by

$$d\mu^m_{(b,a)}(x) = \otimes_{1 \leq k \leq m,\, k < n}(b_{kn}/\pi)^{1/2}\exp(-b_{kn}(x_{kn} - a_{kn})^2)dx_{kn}.$$

Quasiregular representation is defined by $T^{R,\mu_{(b,a)},m} = \pi^{R,\mu_{(b,a)},X^m}$.

Theorem 2.3 ([17]). *Conjecture 1.1 holds, i.e., quasi-regular representation $T^{R,\mu_{(b,a)},m}$ is irreducible if and only if conditions 1) and 2) of Conjecture 1.1 holds. Moreover,*

$$T^{R,\mu_1,m} \sim T^{R,\mu_2,n} \Leftrightarrow m = n \;\; and \;\; \mu_1 \sim \mu_2.$$

Conjecture 1.1 for quasiregular representations of the group $B_0^{\mathbb{N}}(\mathbb{R})$ is proved by A. Kosyak and S. Albeverio in [2] for a tensor product of arbitrary one-dimensional measures and for more general Gaussian measures by A. Kosyak and S. Albeverio in [3].

2.2. Regular and quasiregular representations and criteria of irreducibility, the case $\mathbf{k} = \mathbb{F}_p$

We show that in the case $\mathbf{k} = \mathbb{F}_p$ *Conjecture 1.1 does not hold but may be corrected easily.* More precisely, two conditions of the irreducibility 1) and 2) of the Conjecture 1.1 are not sufficient, since the commutant of the right quasiregular representation may be generated not only by the operators of the left representations, as in the case of $\mathbf{k} = \mathbb{R}$, but also by some operators acting in L^2 on infinite rows and existing only in the case when the corresponding measures are equivalent with the infinite tensor product of the invariant measures (see conditions 3) of Conjecture 2.5.

Define a *quasiregular representations* for the group $B_0^{\mathbb{N}}(\mathbb{F}_p) = \varinjlim_n B(n, \mathbb{F}_p)$. Let us consider two subgroups of the group of all upper triangular matrices $B^{\mathbb{N}}(\mathbb{F}_p)$:

$$B_m(\mathbb{F}_p) = \{I + x \in B^{\mathbb{N}}(\mathbb{F}_p) \mid x = \sum_{m < k < n} x_{kn}E_{kn},\; x_{kn} \in \mathbb{F}_p\},$$

$$B^m(\mathbb{F}_p) = \{I + x \in B^{\mathbb{N}}(\mathbb{F}_p) \mid x = \sum_{1 \leq k \leq m, k < n} x_{kn}E_{kn},\; x_{kn} \in \mathbb{F}_p\}.$$

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The group $B^{\mathbb{N}}(\mathbb{F}_p)$ is semi-direct product $B^{\mathbb{N}}(\mathbb{F}_p) = B_m(\mathbb{F}_p) \ltimes B^m(\mathbb{F}_p)$. Fix the corresponding decomposition $x = x_m \cdot x^m$. Define the homogeneous space

$$X^m := B_m(\mathbb{F}_p) \setminus B^{\mathbb{N}}(\mathbb{F}_p) \simeq B^m(\mathbb{F}_p). \tag{2.1}$$

The measure $\mu_\alpha = \mu_\alpha^{(m)}$ on the space X^m is defined as infinite tensor product:

$$\mu_\alpha^{(m)} := \otimes_{1 \leq k \leq m, k < n} \mu_{\alpha_{kn}} = \otimes_{k=1}^m \mu_\alpha^k, \quad \text{where} \quad \mu_\alpha^k := \otimes_{n=k+1}^\infty \mu_{\alpha_{kn}}, \tag{2.2}$$

of the probability measures $\mu_{\alpha_{kn}}$ on $\mathbb{F}_p$ defined as follows:

$$\mathbb{F}_p \ni r \mapsto \mu_{\alpha_{kn}}(r) = \alpha_{kn}(r) > 0 \quad \text{and} \quad \sum_{r \in \mathbb{F}_p} \alpha_{kn}(r) = 1.$$

The right action $R : B_0^{\mathbb{N}}(\mathbb{F}_p) \to \mathrm{Aut}(X^m)$ of the group $B_0^{\mathbb{N}}(\mathbb{F}_p)$ on the factor space $X^m = B_m(\mathbb{F}_p) \setminus B^{\mathbb{N}}(\mathbb{F}_p)$ is well defined by $R_t(x) = (xt^{-1})^m$.

Lemma 2.4. *The right action of the group $B_0^{\mathbb{N}}(\mathbb{F}_p)$ on the space X^m is admissible, i.e., $(\mu_\alpha)^{R_t} \sim \mu_\alpha \ \forall t \in B_0^{\mathbb{N}}(\mathbb{F}_p)$.*

Quasiregular representation $T^{R, \mu_\alpha, m}$ is defined in the space $L^2(X^m, \mu_\alpha)$ by (3.2). For $1 \leq k \leq m$ define the measures μ_α^k and μ_{inv}^k as follows:

$$\mu_\alpha^k := \otimes_{n=k+1}^\infty \mu_{\alpha_{kn}}, \quad \mu_{inv}^k = \otimes_{n=k+1}^\infty \mu_{inv}^{kn}, \quad \mu_{inv} = \otimes_{k=1}^m \mu_{inv}^k, \tag{2.3}$$

where μ_{inv}^{kn} is the normalized invariant measure on $\mathbb{F}_p$, i.e.,

$$\mu_{inv}^{kn}(r) = p^{-1}, \quad r \in \mathbb{F}_p. \tag{2.4}$$

Conjecture 2.5. *Let $m \in \mathbb{N}$. The quasiregular representation $T^{R, \mu_\alpha, m} : B_0^{\mathbb{N}}(\mathbb{F}_p) \to U(L^2(X^m, \mu_\alpha))$ of the group $B_0^{\mathbb{N}}(\mathbb{F}_p)$ is irreducible if and only if*
  1) $\mu_\alpha^{L_t} \perp \mu_\alpha \ \forall t \in B(m, \mathbb{F}_p) \setminus \{e\}$,
  2) the measure μ_α on the space X^m is $B_0^{\mathbb{N}}(\mathbb{F}_p)$-right-ergodic,
  3) for the measure $\mu_\alpha = \otimes_{k=1}^m \mu_\alpha^k$ holds $\mu_\alpha^k \perp \mu_{inv}^k$ for all $1 \leq k \leq m$ (it is sufficient to verify this condition only for $k = m$),
  4) two irreducible representations $T^{R, \mu_\alpha, m}$ and $T^{R, \mu_\beta, n}$ are equivalent $T^{R, \mu_\alpha, m} \sim T^{R, \mu_\beta, n}$ if and only if $m = n$ and $\mu_\alpha \sim \mu_\beta$.

We would like to mention here a nice *problem to solve*. Define a *regular representation of the group* $B_0^{\mathbb{N}}(\mathbb{F}_p)$ as before. On the group $B^{\mathbb{N}}(\mathbb{F}_p)$ of all upper triangular matrices define the measure μ_α as follows:

$$\mu_\alpha := \otimes_{k<n} \mu_{\alpha_{kn}} = \otimes_{k=1}^\infty \mu_\alpha^k. \tag{2.5}$$

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Lemma 2.6. *The right action of the group $B_0^{\mathbb{N}}(\mathbb{F}_p)$ on the group $B^{\mathbb{N}}(\mathbb{F}_p)$ is admissible, i.e., $(\mu_\alpha)^{R_t} \sim \mu_\alpha \ \forall t \in B_0^{\mathbb{N}}(\mathbb{F}_p)$.*

Conjecture 2.7. *The regular representation $T^{R,\mu_\alpha} : B_0^{\mathbb{N}}(\mathbb{F}_p) \to U(L^2(B^{\mathbb{N}}(\mathbb{F}_p), \mu_\alpha))$ of the group $B_0^{\mathbb{N}}(\mathbb{F}_p)$ is irreducible if and only if*
  1) $\mu_\alpha^{L_t} \perp \mu_\alpha \ \forall t \in B_0^{\mathbb{N}}(\mathbb{F}_p)\backslash\{e\}$,
  2) the measure μ_α on the group $B^{\mathbb{N}}(\mathbb{F}_p)$ is $B_0^{\mathbb{N}}(\mathbb{F}_p)$-right-ergodic,
  3) two irreducible representations T^{R,μ_α} and T^{R,μ_β} are equivalent $T^{R,\mu_\alpha} \sim T^{R,\mu_\beta}$ if and only if $\mu_\alpha \sim \mu_\beta$.

2.3. Idea of the proof of the irreducibility of the regular and quasiregular representations

Below we show that conditions 1)–3) of Conjecture 2.5 and 1)–2) of Conjecture 2.7) are necessary for the irreducibility of the representation $T^{R,\mu_\alpha,m}$ (resp. of T^{R,μ_α}). *The remaining part of the chapter is devoted to the proof of the fact that these conditions are sufficient for the irreducibility of the representations $T^{R,\mu_\alpha,m}$.*

The conditions $1) - 3)$ of the conjecture are necessary for the irreducibility of $T^{R,\mu_\alpha,m}$ and T^{R,μ_α}. Indeed, let conditions 1) does not hold, then $\mu_\alpha^{L_s} \sim \mu_\alpha$ for some $s \in B(m,\mathbb{F}_p)\backslash\{e\}$ therefore, the operator $T_s^{L,\mu_\alpha,m} = \pi_s^{L,\mu_\alpha,X^m}$ is well defined and commutes with the representation $T^{R,\mu_\alpha,m}$. Similarly, the operator T_s^{L,μ_α} commutes with T^{R,μ_α}.

The necessity of the condition 2) is evident. Indeed, if the measure μ_α is not G-ergodic on the space X^m (resp. $X \simeq B^{\mathbb{N}}(\mathbb{F}_p)$) then $X^m = X_1 \cup X_2$ (resp. $X = X_1 \cup X_2$) where X_k are $G-$invariant and $\mu_\alpha(X_k) > 0, \ k = 1, 2$. In this case $L^2(X^m, \mu_\alpha) = H_1 \oplus H_2$ (resp. $L^2(X, \mu_\alpha) = H_1 \oplus H_2$) is a direct sum of two nontrivial G-invariant subspaces.

To explain the condition 3) we define the *elementary representations* $T^{R,\mu_\alpha^k,(k)}$ of the group G as follows. Consider the subspace

$$X^{(k)} = \left\{ I + \sum_{n=k+1}^{\infty} x_{kn} E_{kn} \right\} \tag{2.6}$$

of the space X^m and the projection μ_α^k of the measure μ_α on the subspace $X^{(k)}$, then

$$X^m = X^{(m)} X^{(m-1)} \dots X^{(1)}, \quad \mu_\alpha = \otimes_{k=1}^m \mu_\alpha^k, \quad \text{where} \quad \mu_\alpha^k := \otimes_{n=k+1}^{\infty} \mu_{\alpha_{kn}}.$$

In this case the following decomposition of the representation $T^{R,\mu_\alpha,m}$ holds:

$$T_t^{R,\mu_\alpha,m} = \otimes_{k=1}^m T_t^{R,\mu_\alpha^k,(k)} \quad \text{in} \quad L^2(X^m,\mu_\alpha) = \otimes_{k=1}^m L^2(X^{(k)},\mu_\alpha^k).$$

We shall use the following *notations*

$$T_{kn} := T_{I-E_{kn}}^{R,\mu_\alpha,m}, \quad T_{kn}(r) := T_{I-E_{kn}}^{R,\mu_\alpha^r,(r)}. \tag{2.7}$$

The following decomposition holds for the quasiregular representation $T^{R,\mu_\alpha,m}$:

$$T_{kn} = \otimes_{r=1}^k T_{kn}(r), \; 1 \le k \le m, \quad T_{kn} = \otimes_{r=1}^m T_{kn}(r), \; k > m. \tag{2.8}$$

For the regular representations T^{R,μ_α} in $L^2(B^{\mathbb{N}}(\mathbb{F}_p),\mu_\alpha) = \otimes_{k=1}^\infty L^2(X^{(k)},\mu_\alpha^k)$ we have:

$$T_t^{R,\mu_\alpha,m} = \otimes_{k=1}^\infty T_t^{R,\mu_\alpha^k,(k)}, \quad T_{kn} = \otimes_{r=1}^k T_{kn}(r), \; k < n. \tag{2.9}$$

In Section 4.2 we describe the commutant $(\mathfrak{A}^m)'$ of the von Neumann algebra

$$\mathfrak{A}^m := \left(T_t^{R,\mu_\alpha,m} \mid t \in G\right)''. \tag{2.10}$$

To be more precise, define the *Laplace operators* $\Delta^{(m)}$ and Δ_k where

$$\Delta^{(m)} = \prod_{k=1}^m \Delta_k, \quad \Delta_k := \prod_{n=k+1}^\infty p^{-1} C(T_{kn}(k)) \quad \text{and} \quad C(T) := \sum_{r \in \mathbb{F}_p} T^r. \tag{2.11}$$

By Lemma 4.6 we conclude that *the operator Δ_k is well defined and belongs to the commutant $(\mathfrak{A}^m)'$ of the corresponding von Neumann algebra $\mathfrak{A}^m$ if $\mu_\alpha^k \sim \mu_{inv}^k$ for some $1 \le k \le m$.* This shows that condition 3) of Conjecture 2.5 are necessary conditions of the irreducibility of the representation $T^{R,\mu_\alpha,m}$.

Remark 2.1. We were able to prove Conjecture 2.5 only in the case $m = 1$, p is arbitrary and $m = 2$, $p = 2$. The general case of m and p is open. We shall try to study these cases later.

Remark 2.2. Idea to prove the irreducibility. Roughly speaking, to prove that conditions 1) – 3) are sufficient for the irreducibility, it is sufficient to show that in this case operators $T_{\alpha_{kn}}$ defined by (3.10) and

$$x_{kn} := \text{diag}(0,1,\ldots,p-1) = \sum_{r \in \mathbb{F}_p} r E_{rr}, \quad 1 \le k \le m, \quad k < n, \tag{2.12}$$

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acting on the Hilbert spaces H_{kn}

$$H_{kn} := H_{\alpha_{kn}} := L^2(\mathbb{F}_p, \mu_{\alpha_{kn}}), \qquad (2.13)$$

belong to the von Neumann algebra $\mathfrak{A}^m$ generated by the representation $T^{R,\mu_\alpha,m}$. To be more precise, consider two infinite families of operators X_k and T_r defined as follows: $X_k = (x_{kn} \mid k < n)$ and $T_r = (T_{\alpha_{rn}} \mid r < n)$ for $1 \leq k, r \leq m$. For $m = 1$ we prove that $X_1 \subset \mathfrak{A}^1$ therefore, $(\mathfrak{A}^1)' \subset (L^\infty(X_1))' = L^\infty(X_1)$ since the von Neumann algebra $L^\infty(X_1)$ is maximal abelian (see Definition 3.1). For $m = 2$ we prove that, depending on the measure, one of the families (X_1, X_2), (X_1, T_2), (T_1, X_2), (T_1, T_2) belong to $\mathfrak{A}^2$. For an arbitrary m it is sufficient to prove that one of the following families $(F_1, F_2, \ldots, F_m)$ belongs to the von Neumann algebra $\mathfrak{A}^m$ where F_k is X_k or T_k for $1 \leq k \leq m$. To prove the irreducibility it is sufficient to prove that the von Neumann algebra $L^\infty(F_1, F_2, \ldots, F_m)$ is maximal abelian therefore, $(\mathfrak{A}^m)' \subset L^\infty(F_1, F_2, \ldots, F_m)$ and use the ergodicity of the measure μ_α.

Remark 2.3. For shortness we shall use the same notations A_k for the operator A_k acting on the Hilbert space H_k and the operator $\mathcal{A}_k = I \otimes \cdots \otimes I \otimes A_k \otimes I \otimes \ldots$ acting on the finite $\mathcal{H}_r = \otimes_{n=1}^r H_n$ or infinite tensor product $\mathcal{H} = \otimes_{n=1}^\infty H_n$.

3. The space X and the measure

Let us consider the finite field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ of p elements $\mathbb{F}_p = \{0, 1, ..., p-1\}$. The group $B_0^{\mathbb{N}}(\mathbb{F}_p)$ is defined as the inductive limit (with natural inclusion) $B_0^{\mathbb{N}}(\mathbb{F}_p) = \varinjlim_n B(n, \mathbb{F}_p)$, where $B(n, \mathbb{F}_p)$ is the group of n-by-n upper-triangular matrices with unities on the principal diagonal with entries from $\mathbb{F}_p$. For the group $B_0^{\mathbb{N}}(\mathbb{F}_p)$ we have the following description

$$B_0^{\mathbb{N}}(\mathbb{F}_p) = \{I + \sum_{1 \leq k < n} x_{kn} E_{kn} \mid x_{kn} \in \mathbb{F}_p, \quad x_{kn} = 0 \text{ for large } n\}.$$

Let $B^{\mathbb{N}}(\mathbb{F}_p)$ be the group of all upper-triangular matrices: $B^{\mathbb{N}}(\mathbb{F}_p) = \{I + \sum_{1 \leq k < n} x_{kn} E_{kn} \mid x_{kn} \in \mathbb{F}_p\}$. We have the following semi-direct product $B^{\mathbb{N}}(\mathbb{F}_p) = B_m(\mathbb{F}_p) \ltimes B^m(\mathbb{F}_p)$, where $B^m(\mathbb{F}_p)$ is normal subgroup in $B^{\mathbb{N}}(\mathbb{F}_p)$ and

$$B_m(\mathbb{F}_p) = \{I + x \in B^{\mathbb{N}}(\mathbb{F}_p) \mid x = \sum_{m < k < n} x_{kn} E_{kn}\},$$

$$B^m(\mathbb{F}_p) = \{I + x \in B^{\mathbb{N}}(\mathbb{F}_p) \mid x = \sum_{1 \leq k \leq m, k < n} x_{kn} E_{kn}\},$$

and we shall write $B^{\mathbb{N}}(\mathbb{F}_p) \ni x = x_m \cdot x^m \in B_m(\mathbb{F}_p) \cdot B^m(\mathbb{F}_p) \simeq B^m(\mathbb{F}_p)$. We define the space X^m as the factor-space $X^m = B_m(\mathbb{F}_p) \backslash B^{\mathbb{N}}(\mathbb{F}_p) \simeq B^m(\mathbb{F}_p)$. The right action R_t of the group $B^{\mathbb{N}}(\mathbb{F}_p)$ is correctly defined on the factor-space X^m by the formula $R_t(x) = (xt^{-1})^m$, $t \in B^{\mathbb{N}}(\mathbb{F}_p)$, $x \in B^m(\mathbb{F}_p)$. We have

$$R_t(x) = xt^{-1}, \text{ if } t \in B^m(\mathbb{F}_p), \text{ and } R_t(x) = t_m x t^{-1}, \text{ if } t = t_m t^m \notin B^m(\mathbb{F}_p). \tag{3.1}$$

To prove (3.1) we get

$$B_m(\mathbb{F}_p) \cdot B^m(\mathbb{F}_p) \ni x_m \cdot x^m \overset{x \mapsto xt}{\mapsto} x_m \cdot x^m t_m \cdot t^m = x_m t_m (t_m^{-1} x^m t_m) t^m$$

hence, $(x^m t_m t^m)^m = t_m^{-1} x^m t$ and $(xt^{-1})^m = t_m x t^{-1}$ for $x \in B^m(\mathbb{F}_p)$. We use relation $(t^{-1})_m = (t_m)^{-1}$. The measure μ_α on the space X^m is defined as infinite tensor product

$$\mu_\alpha = \otimes_{1 \leq k \leq m, k < n} \mu_{\alpha_{kn}} = \otimes_{k=1}^m \mu_\alpha^k, \text{ where } \mu_\alpha^k := \otimes_{n=k+1}^\infty \mu_{\alpha_{kn}}$$

of the probability measures $\mu_{\alpha_{kn}}$ on $\mathbb{F}_p$ defined as follows:
$\mathbb{F}_p \ni r \mapsto \mu_{\alpha_{kn}}(r) = \alpha_{kn}(r) > 0$ and $\sum_{r \in \mathbb{F}_p} \alpha_{kn}(r) = 1$.

Lemma 3.1. *We have $\mu_\alpha^{R_t} \sim \mu_\alpha$ for all $t \in B_0^{\mathbb{N}}(\mathbb{F}_p)$.*

Define the unitary representation $T^{R,\mu_\alpha,m} : B_0^{\mathbb{N}}(\mathbb{F}_p) \mapsto U(L^2(X^m, \mu_\alpha))$ in a natural way, i.e., for $f \in L^2(X^m, \mu_\alpha)$ set

$$(T_t^{R,\mu_\alpha,m} f)(x) = (d\mu_\alpha(R_t^{-1}(x))/d\mu_\alpha(x))^{1/2} f(R_t^{-1}(x)), \ t \in B_0^{\mathbb{N}}(\mathbb{F}_p). \tag{3.2}$$

Conjecture 3.2. *The quasiregular representation $T^{R,\mu_\alpha,m}$ of the group $B_0^{\mathbb{N}}(\mathbb{F}_p)$ is irreducible if and only if conditions 1)–3) holds:*
 1) $\mu_\alpha^{L_t} \perp \mu_\alpha \ \forall t \in B(m, \mathbb{F}_p) \backslash \{e\}$,
 2) the measure μ_α is G-ergodic,
 3) for the measure $\mu_\alpha = \otimes_{k=1}^m \mu_\alpha^k$ holds $\mu_\alpha^m \perp \mu_{inv}^m$.
 4) Moreover, $T^{R,\mu_\alpha,m} \sim T^{R,\mu_\beta,n}$ if and only if $m = n$ and $\mu_\alpha \sim \mu_\beta$.

Remark 3.1. In the case of the field $k = \mathbb{R}$ and the measure being a Gaussian product-measure, the irreducibilty holds if and only if the condition 1) and 2) are valid (see [17, 18]) hence, the case $k = \mathbb{F}_p$ is richer.

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The right action R of the group $B_0^{\mathbb{N}}(\mathbb{F}_p)$ on the space X^m is given by the formula (3.1). The left action L of the group $B(m, \mathbb{F}_p)$ on the space X^m is as follows: $L_t(x) = tx$, $t \in B(m, \mathbb{F}_p)$, $x \in X^m$. Let us consider the case $p = 2$ and $m = 2$, i.e., the space X^2. Set $E_{kn}(d) = I + dE_{kn} \in G$, $d \in \mathbb{F}_p$. We have

$$E_{12}(d)x = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x_{12} & x_{13} & \cdots & x_{1n} & \cdots \\ 0 & 1 & x_{23} & \cdots & x_{2n} & \cdots \end{pmatrix} = \begin{pmatrix} 1 & x_{12}+d & x_{13}+dx_{23} & \cdots & x_{1n}+dx_{2n} & \cdots \\ 0 & 1 & x_{23} & \cdots & x_{2n} & \cdots \end{pmatrix}.$$

For $t = I + E_{kn}$, $1 \leq k \leq n$ the right action is $R_t(x) = xt^{-1}$ (see (3.1))

$$\begin{pmatrix} 1 & x_{12} & x_{13} & \cdots & x_{1n} & \cdots \\ 0 & 1 & x_{23} & \cdots & x_{2n} & \cdots \end{pmatrix} \overset{R_{I+E_{1n}}^{-1}}{\longmapsto} \begin{pmatrix} 1 & x_{12} & x_{13} & \cdots & x_{1n}+1 & \cdots \\ 0 & 1 & x_{23} & \cdots & x_{2n} & \cdots \end{pmatrix},$$

$$\begin{pmatrix} 1 & x_{12} & x_{13} & \cdots & x_{1n} & \cdots \\ 0 & 1 & x_{23} & \cdots & x_{2n} & \cdots \end{pmatrix} \overset{R_{I+E_{2n}}^{-1}}{\longmapsto} \begin{pmatrix} 1 & x_{12} & x_{13} & \cdots & x_{1n}+x_{12} & \cdots \\ 0 & 1 & x_{23} & \cdots & x_{2n}+1 & \cdots \end{pmatrix},$$

$$\begin{pmatrix} 1 & x_{12} & x_{13} & \cdots & x_{1k} & \cdots & x_{1n} & \cdots \\ 0 & 1 & x_{23} & \cdots & x_{2k} & \cdots & x_{2n} & \cdots \end{pmatrix} \overset{R_{I+E_{kn}}^{-1}}{\longmapsto} \begin{pmatrix} 1 & x_{12} & x_{13} & \cdots & x_{1k} & \cdots & x_{1n}+x_{1k} & \cdots \\ 0 & 1 & x_{23} & \cdots & x_{2k} & \cdots & x_{2n}+x_{2k} & \cdots \end{pmatrix}.$$

Therefore, we have four actions to study:

$$R_{I+E_{1n}}^{-1} : x_{1n} \mapsto x_{1n} + 1, \quad (x_{1k}, x_{1n}) \mapsto (x_{1k}, x_{1n} + x_{1k}), \qquad (3.3)$$

$$\begin{pmatrix} x_{12} & x_{1n} \\ 1 & x_{2n} \end{pmatrix} \overset{R_{I+E_{2n}}^{-1}}{\longmapsto} \begin{pmatrix} x_{12} & x_{1n}+x_{12} \\ 1 & x_{2n}+1 \end{pmatrix}, \quad \begin{pmatrix} x_{1k} & x_{1n} \\ x_{2k} & x_{2n} \end{pmatrix} \overset{R_{I+E_{kn}}^{-1}}{\longmapsto} \begin{pmatrix} x_{1k} & x_{1n}+x_{1k} \\ x_{2k} & x_{2n}+x_{2k} \end{pmatrix},$$
$$(3.4)$$

and

$$L_{I+dE_{12}} : \begin{pmatrix} x_{1n} \\ x_{2n} \end{pmatrix} \mapsto \begin{pmatrix} x_{1n} + dx_{2n} \\ x_{2n} \end{pmatrix}, \quad d \in \mathbb{F}_p. \qquad (3.5)$$

Set

$$H_{inv} = L^2(\mathbb{F}_p, \mu_{inv}) \quad \text{and} \quad H_\alpha = L^2(\mathbb{F}_p, \mu_\alpha) \qquad (3.6)$$

where the normalized Haar measure μ_{inv} on the additive group $\mathbb{F}_p$ is defined by

$$\mu_{inv}(r) = p^{-1}, \ r \in \mathbb{F}_p, \quad \text{and} \quad \mu_\alpha(r) = \alpha(r), \quad \text{with} \quad \sum_{r \in \mathbb{F}_p} \alpha(r) = 1. \quad (3.7)$$

The operator T_{inv} on the Hilbert space H_{inv} associated with the action $x \mapsto x - 1$ on $\mathbb{F}_p$ is defined by the following formula

$$(T_{inv}f)(x) = \left(\frac{d\mu_{inv}(x-1)}{d\mu_{inv}(x)} \right)^{1/2} f(x-1) = f(x-1), \ f(x) = (f_0, f_1, \ldots, f_{p-1}) \in \mathbb{C}^p.$$

Take the orthonormal basis (o.n.b.) in the space H_α as follows:

$$(e_k^\alpha)_{k\in\mathbb{F}_p}, \quad \text{where } e_k^\alpha = (e_k^\alpha(r))_{r\in\mathbb{F}_p}, \ e_k^\alpha(r) = (\alpha(r))^{-1/2}\delta_{k,r}, \quad k,r\in\mathbb{F}_p. \quad (3.8)$$

For $e_k(r) = (p)^{-1/2}\delta_{kr}$, $k,r\in\mathbb{F}_p$ we get $(Te_k)(r) = e_k(r-1) = e_{k+1}(r)$, so

$$T(\sum_k f_k e_k) = \sum_{k\in\mathbb{F}_p} f_k e_{k+1} = \sum_{k\in\mathbb{F}_p} f_{k-1} e_k \quad \text{hence,} \quad T = \sum_{r\in\mathbb{F}_p} E_{r+1,r}.$$

To define the corresponding operator T_α on the Hilbert space H_α we use the following commutative diagram:

$$
\begin{array}{ccc}
H_\alpha & \overset{T_\alpha}{\to} & H_\alpha \\
U_\alpha \downarrow & & \downarrow \ U_\alpha \\
H_{inv} & \overset{T_{inv}}{\to} & H_{inv}
\end{array}
$$

where $U_\alpha : H_\alpha \to H_{inv}$ is the isomorphisms defined by

$$U_\alpha = (d\mu_\alpha(x)/d\mu_{inv}(x))^{1/2} = \mathrm{diag}((p\alpha(0))^{1/2}, (p\alpha(1))^{1/2}, \ldots, (p\alpha(p-1))^{1/2}).$$

Finally, the operator T_α is equal to $T_\alpha = U_\alpha^{-1}T_{inv}U_\alpha$ hence, we have for $p=2$

$$T_\alpha = \begin{pmatrix} \frac{1}{\sqrt{2\alpha(0)}} & 0 \\ 0 & \frac{1}{\sqrt{2\alpha(1)}} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{2\alpha(0)} & 0 \\ 0 & \sqrt{2\alpha(1)} \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{\frac{\alpha(1)}{\alpha(0)}} \\ \sqrt{\frac{\alpha(0)}{\alpha(1)}} & 0 \end{pmatrix}. \quad (3.9)$$

For general p we have in the basis $(e_k^\alpha)_{k\in\mathbb{F}_p}$

$$T_\alpha = \begin{pmatrix} 0 & 0 & 0 \ldots & 0 & \sqrt{\frac{\alpha(p-1)}{\alpha(0)}} \\ \sqrt{\frac{\alpha(0)}{\alpha(1)}} & 0 & 0 \ldots & 0 & 0 \\ 0 & \sqrt{\frac{\alpha(1)}{\alpha(2)}} & 0 \ldots & 0 & 0 \\ & & \ldots & & \\ 0 & 0 & 0 \ldots & \sqrt{\frac{\alpha(p-2)}{\alpha(p-1)}} & 0 \end{pmatrix}, \quad T_{inv} = \begin{pmatrix} 0 & 0 & 0 \ldots & 0 & 1 \\ 1 & 0 & 0 \ldots & 0 & 0 \\ 0 & 1 & 0 \ldots & 0 & 0 \\ & & \ldots & & \\ 0 & 0 & 0 \ldots & 1 & 0 \end{pmatrix}. \quad (3.10)$$

3.1. The Kakutani criterion

We find the condition of orthogonality $\mu_\alpha^{L_I+dE_{12}} \perp \mu_\alpha$, $d\in\mathbb{F}_p\setminus\{0\}$, using the *Kakutani criterion* [9]. The *Hellinger integral* $H(\mu,\nu)$ for two measures μ and ν on the space X is defined [23] as follows:

$$H(\mu,\nu) = \int_X \sqrt{\frac{d\mu(x)}{d\rho(x)}\frac{d\nu(x)}{d\rho(x)}}d\rho(x),$$

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where ρ is some measure on X such that both measures μ and ν are *absolutely continuous* with respect to the measure ρ. For example, one can take $\rho = \frac{1}{2}(\mu + \nu)$.

Let we have two probability measures μ_α and μ_β on the group $\mathbb{F}_p$ defined as follows: $\mu_\alpha(r) = \alpha(r)$, $\sum_{r \in \mathbb{F}_p} \alpha(r) = 1$ and $\mu_\beta(r) = \beta(r)$, $\sum_{r \in \mathbb{F}_p} \beta(r) = 1$. The Hellinger integral $H(\mu_\alpha, \mu_\beta)$ for two measures μ_α and μ_β is given in this case by

$$H(\mu_\alpha, \mu_\beta) = \int_{\mathbb{F}_p} \sqrt{\frac{d\mu_\alpha(x)}{d\mu_{inv}(x)} \frac{d\mu_\beta(x)}{d\mu_{inv}(x)}} d\mu_{inv}(x) = \sum_{r \in \mathbb{F}_p} \sqrt{\alpha(r)\beta(r)}.$$

Let us consider two probability measures $\mu_\alpha = \otimes_{n \in \mathbb{N}} \mu_{\alpha_n}$ and $\mu_\beta = \otimes_{n \in \mathbb{N}} \mu_{\beta_n}$ defined on the space $(\mathbb{F}_p)^\infty = \mathbb{F}_p \times \mathbb{F}_p \times ...$ as the infinite tensor product, where μ_{α_n} and μ_{β_n}, $n \in \mathbb{N}$ are probability measures defined on the space $\mathbb{F}_p$, as before. The Hellinger integral $H(\mu_\alpha, \mu_\beta)$ for two measures μ_α and μ_β is given in this case by

$$H(\mu_\alpha, \mu_\beta) = \prod_{n \in \mathbb{N}} H(\mu_{\alpha_n}, \mu_{\beta_n}) = \prod_{n \in \mathbb{N}} \sum_{r \in \mathbb{F}_p} \sqrt{\alpha_n(r)\beta_n(r)}.$$

We use the *notation* $\mu^f(\Delta) = \mu(f^{-1}(\Delta))$ for a measure μ on the space X and a measurable bijection $f : X \to X$. For two measures $\mu_\alpha \otimes \mu_\beta$ and $(\mu_\alpha \otimes \mu_\beta)^{L_{I+dE_{12}}}$ on $\mathbb{F}_p \times \mathbb{F}_p$ where $L_{I+dE_{12}} : (x, y) \mapsto (x + dy, y)$ (see (3.5)) we have

$$(\mu_\alpha \otimes \mu_\beta)^{L_{I+dE_{12}}^{-1}}(k, r) = (\alpha(k)\beta(r))^{L_{I+dE_{12}}^{-1}} = \alpha(k + dr)\beta(r). \qquad (3.11)$$

Hence, we have for the Hellinger integral the following expression:

$$H_{12}^d := H\left((\mu_\alpha \otimes \mu_\beta)^{L_{I+dE_{12}}^{-1}}, \mu_\alpha \otimes \mu_\beta\right) = \sum_{r,k \in \mathbb{F}_p} \sqrt{(\alpha(k)\beta(r))^{L_{I+dE_{12}}^{-1}} \alpha(k)\beta(r)}$$

$$= \sum_{r,k \in \mathbb{F}_p} \sqrt{\alpha(k + dr)\beta(r)\alpha(k)\beta(r)} = \sum_{r \in \mathbb{F}_p} \beta(r) \sum_{k \in \mathbb{F}_p} \sqrt{\alpha(k + dr)\alpha(k)}.$$

Lemma 3.3. *For the measure $\mu_\alpha = \otimes_{1 \leq k \leq m, k < n} \mu_{\alpha_{kn}}$ on the space X^m five following conditions are equivalent:*

1) $\mu_\alpha^{L_t} \perp \mu_\alpha, \forall t \in B(m, \mathbb{F}_p) \setminus \{e\}$,

2) $(\mu_\alpha)^{L_{I+dE_{ls}}} \perp \mu_\alpha, \ \forall d \in \mathbb{F}_p \setminus \{0\}, \ 1 \leq l < s \leq m$,

3) $(\mu_\alpha)^{L_{I+E_{ls}}} \perp \mu_\alpha, \quad 1 \leq l < s \leq m$,

4) $\Pi_{ls}^{L,d}(\mu_\alpha) = \prod_{n=s+1}^{\infty} H_{n,ls} = \prod_{n=s+1}^{\infty} \sum_{r \in \mathbb{F}_p} \alpha_{sn}(r) \sum_{k \in \mathbb{F}_p} \sqrt{\alpha_{ln}(k + dr)\alpha_{ln}(k)} = 0,$

5) $S_{ls}^{L,d}(\mu_\alpha) = \sum_{n=s+1}^{\infty} \sum_{r \in \mathbb{F}_p} \alpha_{sn}(r)\Big(1 - \sum_{k \in \mathbb{F}_p \setminus \{0\}} \sqrt{\alpha_{ln}(k + dr)\alpha_{ln}(k)}\Big) = \infty.$

PROOF. Obviously 1) $\Rightarrow$ 2) $\Rightarrow$ 3) $\Rightarrow$ 4). We show also that 4) $\Leftrightarrow$ 5). The implication 5) $\Rightarrow$ 1) will follow from the irreducibility that we prove later.

We show that 2) $\Leftrightarrow$ 3). Indeed, since $(\mu_\alpha)^{L_{I+dE_{ls}}}$ and μ_α are product measures, by Kakutani criterion we conclude that $(\mu_\alpha)^{L_{I+dE_{ls}}}$ and μ_α are orthogonal or equivalent. It is sufficient to show that 3) implies 2) for all $d \in \mathbb{F}_p^* = \mathbb{F}_p \setminus \{0\}$. Let us suppose the opposite, i.e., that for some $d \in \mathbb{F}_p^* :=$ $\mathbb{F}_p \setminus \{0\}$ holds $(\mu_\alpha)^{L_{I+dE_{ls}}} \sim \mu_\alpha$. Since $\mathbb{F}_p^*$ is a multiplicative group there exists an inverse $a = d^{-1} \in \mathbb{F}_p^*$. For this element a we then get

$$\mu_\alpha \sim (\mu_\alpha)^{L_{I+dE_{ls}}^a} = (\mu_\alpha)^{L_{(I+dE_{ls})^a}} = (\mu_\alpha)^{L_{I+adE_{ls}}} = (\mu_\alpha)^{L_{I+E_{ls}}}.$$

This contradicts with 3). We have 4) $\Leftrightarrow$ 5) since

$$\sum_{r \in \mathbb{F}_p} \alpha_{sn}(r) \sum_{k \in \mathbb{F}_p} \sqrt{\alpha_{ln}(k + r)\alpha_{ln}(k)} = \alpha_{sn}(0)$$

$$+ \sum_{r \in \mathbb{F}_p \setminus \{0\}} \alpha_{sn}(r) \sum_{k \in \mathbb{F}_p} \sqrt{\alpha_{ln}(k + r)\alpha_{ln}(k)} = 1 - \sum_{r \in \mathbb{F}_p \setminus \{0\}} \alpha_{sn}(r) + \sum_{r \in \mathbb{F}_p \setminus \{0\}} \alpha_{sn}(r)$$

$$\times \sum_{k \in \mathbb{F}_p} \sqrt{\alpha_{ln}(k + r)\alpha_{ln}(k)} = 1 - \sum_{r \in \mathbb{F}_p \setminus \{0\}} \alpha_{sn}(r)\Big(1 - \sum_{k \in \mathbb{F}_p} \sqrt{\alpha_{ln}(k + r)\alpha_{ln}(k)}\Big).$$

$\square$

Remark 3.2. If $\mu_\alpha^l \sim \mu_{inv}^l$ for some $l, \ 1 \leq l < m$ then by Lemma 3.3, 4) we get $(\mu_\alpha)^{L_{I+dE_{ls}}} \sim \mu_\alpha$ for $l < s \leq m$ hence, the representation is reducible. Therefore, only one condition from the list of conditions 3) in Conjecture 3.2 is independent, namely: $\mu_\alpha^m \perp \mu_{inv}^m$!

PROOF. If we replace the factor μ_α^l in the expression for the measure $\mu_\alpha = \otimes_{k=1}^m \mu_\alpha^k$ by μ_{inv}^l we get the equivalent measure $\mu_{\alpha'}$ and the representation $T^{R,\mu_{\alpha'},m}$ equivalent with the initial one $T^{R,\mu_\alpha,m}$. For this measure we have $(\mu_{\alpha'})^{L_{I+dE_{ls}}} \sim \mu_{\alpha'}$ for $s : l < s \le m$. Indeed, in this case we have $\sum_{k\in\mathbb{F}_p} \sqrt{\alpha_{tn}(k+dr)\alpha_{tn}(k)} = 1$ hence, $\Pi_{ls}^{L,d}(\mu) = 1$. The representation $T^{R,\mu_{\alpha'},m}$ is reducible in this case, since the operator $T_{I+dE_{ts}}^{L,\mu_{\alpha'},m}$ generated by the transformation $L_{I+dE_{ls}}$ is well defined and commutes with the representation $T^{R,\mu_{\alpha'},m}$.

$\square$

Examples. 1) In the particular case $p = 2$ we have

$$H_{n,12} = \beta(0) + 2\beta(1)\sqrt{\alpha(0)\alpha(1)} = \alpha_{2n}(0) + 2\alpha_{2n}(1)\sqrt{\alpha_{1n}(0)\alpha_{1n}(1)}$$

$$= 1 - \alpha_{2n}(1)(1 - 2\sqrt{\alpha_{1n}(0)\alpha_{1n}(1)}).$$

Hence, for $X = X^2$ and $\mathbb{F}_2$ we have

$$\Pi_{12}^{L,1}(\mu_\alpha) = \prod_{n=3}^\infty H_{n,12} = \prod_{n=3}^\infty \left(1 - \alpha_{2n}(1)(1 - 2\sqrt{\alpha_{1n}(0)\alpha_{1n}(1)})\right). \qquad (3.12)$$

We see that $\Pi_{12}^{L,1}(\mu_\alpha) = 0$ if and only if $S_{12}^{L,1}(\mu_\alpha) = \infty$ where

$$S_{12}^L(\mu_\alpha) := S_{12}^{L,1}(\mu_\alpha) = \sum_{n=3}^\infty \alpha_{2n}(1)\left(1 - 2\sqrt{\alpha_{1n}(0)\alpha_{1n}(1)}\right).$$

2) For $X = X^3$ and $\mathbb{F}_2$ we have

$$\Pi_{12}^{L,1}(\mu_\alpha) = \prod_{n=3}^\infty H_{n,12} = \prod_{n=3}^\infty \left(\alpha_{2n}(0) + 2\alpha_{2n}(1)\sqrt{\alpha_{1n}(0)\alpha_{1n}(1)}\right),$$

$$\Pi_{13}^{L,1}(\mu_\alpha) = \prod_{n=4}^\infty H_{n,13} = \prod_{n=4}^\infty \left(\alpha_{3n}(0) + 2\alpha_{3n}(1)\sqrt{\alpha_{1n}(0)\alpha_{1n}(1)}\right),$$

$$\Pi_{23}^{L,1}(\mu_\alpha) = \prod_{n=4}^\infty H_{n,23} = \prod_{n=4}^\infty \left(\alpha_{3n}(0) + 2\alpha_{3n}(1)\sqrt{\alpha_{2n}(0)\alpha_{2n}(1)}\right).$$

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3) For $\mathbb{F}_3$ and X^2 we have

$$
\begin{aligned}
H_{12}^1 =& \beta(0)\left(\alpha(0)+\alpha(1)+\alpha(2)\right) \\
&+\beta(1)\left(\sqrt{\alpha(1)\alpha(0)}+\sqrt{\alpha(2)\alpha(1)}+\sqrt{\alpha(0)\alpha(2)}\right) \\
&+\beta(2)\left(\sqrt{\alpha(2)\alpha(0)}+\sqrt{\alpha(0)\alpha(1)}+\sqrt{\alpha(1)\alpha(2)}\right) \\
=& \beta(0)+(\beta(1)+\beta(2))\left(\sqrt{\alpha(2)\alpha(0)}+\sqrt{\alpha(0)\alpha(1)}+\sqrt{\alpha(1)\alpha(2)}\right),
\end{aligned}
$$

hence, for $\mathbb{F}_3$ and X^2 we have

$$
\Pi_{12}^{L,1}(\mu_\alpha) = \prod_{n=2}^{\infty} H_{n,12} = \prod_{n\in\mathbb{N},n>2}\left(\alpha_{2n}(0)+(\alpha_{2n}(1)+\alpha_{2n}(2))\right) \qquad (3.13)
$$

$$
\times\left(\sqrt{\alpha_{1n}(2)\alpha_{1n}(0)}+\sqrt{\alpha_{1n}(0)\alpha_{1n}(1)}+\sqrt{\alpha_{1n}(1)\alpha_{1n}(2)}\right).
$$

We study first the condition 1) of Lemma 3.3.

Lemma 3.4. *The following three conditions are equivalent:*

1) $\mu_\alpha^l \perp \mu_{inv}^l, \quad 1 \le l \le m,$

2) $\Pi_{ll}(\mu_\alpha) = \displaystyle\prod_{n=l+1}^{\infty} \frac{1}{p}\left(1+\sum_{r\in\mathbb{F}_p}\sum_{k\in\mathbb{F}_p\backslash\{0\}}\sqrt{\alpha_{ln}(k)\alpha_{ln}(k+r)}\right) = 0,$

3) $S_{ll}^L(\mu_\alpha) = \displaystyle\sum_{n=l+1}^{\infty}\sum_{r\in\mathbb{F}_p\backslash\{0\}}\left(1-\sum_{k\in\mathbb{F}_p}\sqrt{\alpha_{ln}(k+r)\alpha_{ln}(k)}\right) = \infty.$

Particular cases. 1) $p = 2$ and $m = 1$. We have only one condition:

$$
S_{11}^L(\mu_\alpha) = \sum_{n=2}^{\infty}\left(1-2\sqrt{\alpha_{1n}(0)\alpha_{1n}(1)}\right) = \infty.
$$

2) The case $p = 2$ and $m \in \mathbb{N}$. We have the following conditions for $1 \le k \le n \le m$:

$$
S_{kn}^L(\mu_\alpha) = \infty, \quad \text{where} \quad S_{kk}^L(\mu_\alpha) = \sum_{r=k+1}^{\infty}\left(1-2\sqrt{\alpha_{kr}(0)\alpha_{kr}(1)}\right), \qquad (3.14)
$$

$$
S_{kn}^L(\mu_\alpha) = \sum_{r=n+1}^{\infty}\alpha_{nr}(1)\left(1-2\sqrt{\alpha_{kr}(0)\alpha_{kr}(1)}\right), \quad k < n. \qquad (3.15)
$$

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Remark 3.3. The conditions 3) of the Conjecture 2.5 mean the following. The space $X^{(k)} = \prod_{n=k+1}^{\infty} (\mathbb{F}_p)_n$ is isomorphic to the set $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : \|x\|_p \leq 1\}$ of entire p-adic numbers of the field $\mathbb{Q}_p$ of all p-adic numbers since $\mathbb{Z}_p$ has the following description: $\mathbb{Z}_p = \{\sum_{n=0}^{\infty} a_n p^n \mid a_n \in \mathbb{F}_p\}$. The measure μ_{inv}^k on $X^{(k)}$ is the Haar measure on $\mathbb{Z}_p$ under this identification.

Remark 3.4. The lemma analogous to Lemma 3.3 holds in the case when we replace the field $k = \mathbb{F}_p$ by the ring $k = \mathbb{Z}$. The measure μ_α on $\mathbb{Z}$ is defined by $\mu_\alpha(r) = \alpha(r) > 0$, $r \in \mathbb{Z}$ such that $\sum_{r \in \mathbb{Z}} \alpha(r) = 1$. The corresponding conditions are the following:

1) $\mu_\alpha^{L_t} \perp \mu_\alpha, \forall t \in B(m, \mathbb{Z}) \setminus \{e\}$,

2) $(\mu_\alpha)^{L_I + dE_{ls}} \perp \mu_\alpha, \ \forall d \in \mathbb{Z} \setminus \{0\}, \ 1 \leq l < s \leq m$,

3) $\Pi_{ls}^{L,d}(\mu_\alpha) = \prod_{n=s+1}^{\infty} H_{n,ls} = \prod_{n=s+1}^{\infty} \sum_{r \in \mathbb{Z}} \alpha_{sn}(r) \sum_{k \in \mathbb{Z}} \sqrt{\alpha_{ln}(k + dr) \alpha_{ln}(k)} = 0$.

3.2. Fourier transform.

Let us consider an additive group of the field $\mathbb{F}_p$. The Haar measure μ_{inv} on $\mathbb{F}_p$ is defined by $\mu_{inv}(r) = 1/p$, $r \in \mathbb{F}_p$. The set of unitary characters $\chi_R(r)$, $R \in \mathbb{F}_p$, are defined as follows:

$$\mathbb{F}_p \ni r \mapsto \chi_R(r) = \exp \frac{2\pi i R r}{p} \in S^1. \qquad (3.16)$$

The Fourier transform F is defined on the space $H_{inv} = L^2(\mathbb{F}_p, \mu_{inv})$ by the formula

$$(Ff)(R) := \tilde{f}(R) := \sqrt{p} \int_{\mathbb{F}_p} f(x) \overline{\chi}_R(x) d\mu_{inv}(x) = \frac{1}{\sqrt{p}} \sum_{r \in \mathbb{F}_p} f(r) \exp\left(-\frac{2\pi i R r}{p}\right).$$

$$(3.17)$$

The operator F is a unitary operator on the space $L^2(\mathbb{F}_p, \mu_{inv})$.

Lemma 3.5. *The image $\tilde{T}_{inv} = FT_{inv}F^{-1}$ of the operator T_{inv} with respect to the Fourier transform is defined by*

$$(\tilde{T}_{inv}\tilde{f})(R) = \exp\left(-\frac{2\pi i R}{p}\right)\tilde{f}(R), \qquad (3.18)$$

i.e., $\tilde{T}_{inv} = \mathrm{diag}(\bar{\chi}_1(0), \bar{\chi}_1(1), \dots, \bar{\chi}_1(p-1)) = \mathrm{diag}(1, \lambda, \dots, \lambda^{p-1})$, $\lambda = \bar{\chi}_1(1)$.

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PROOF. Indeed, by (3.10) we have $T_{inv} : f(x) \mapsto f(x-1)$ hence,

$$\tilde{f}(R) = \frac{1}{\sqrt{p}} \sum_{r \in \mathbb{F}_p} f(r) \exp\left(-\frac{2\pi i R r}{p}\right) \overset{T_{inv}}{\mapsto} \frac{1}{\sqrt{p}} \sum_{r \in \mathbb{F}_p} f(r-1) \exp\left(-\frac{2\pi i R r}{p}\right) =$$

$$\frac{1}{\sqrt{p}} \sum_{s \in \mathbb{F}_p} f(s) \exp\left(-\frac{2\pi i R(s+1)}{p}\right) = \exp\left(-\frac{2\pi i R}{p}\right) \tilde{f}(R).$$

$\square$

To define the Fourier transform $F_\alpha : H_\alpha \to H_{\tilde{\alpha}}$, where the measure $\tilde{\alpha}$ on $\mathbb{F}_p$ is defined by (3.22), we use the following commutative diagram:

$$\begin{array}{ccc} & L^2(\mathbb{F}_p, \mu_\alpha) & \overset{F_\alpha}{\to} & L^2(\mathbb{F}_p, \mu_{\tilde{\alpha}}) \\ U_\alpha & \downarrow & & \downarrow & U_{\tilde{\alpha}} \\ & L^2(\mathbb{F}_p, \mu_{inv}) & \overset{F}{\to} & L^2(\mathbb{F}_p, \mu_{inv}) \end{array}$$

where $(U_\alpha f)(r) = (p\alpha(r))^{1/2} f(r)$. We have $F_\alpha = U_{\tilde{\alpha}}^{-1} F U_\alpha$ hence (compare with the case of the Fourier transform (3.20) in $L^2(\mathbb{R}, \mu)$ defined below)

$$(F_\alpha f)(R) = \frac{1}{\sqrt{\tilde{\alpha}(R)p}} \sum_{r \in \mathbb{F}_p} \sqrt{\alpha(r)} f(r) \exp\left(-\frac{2\pi i R r}{p}\right). \tag{3.19}$$

Remark 3.5. Let us denote by F_{kn}^μ the one-dimensional Fourier transform corresponding to the measure μ_{kn} (see [15] formula (6) and (7))

$$\begin{array}{ccc} & L^2(\mathbb{R}, \mu_{kn}) & \overset{F_{kn}^\mu}{\to} & L^2(\mathbb{R}, \tilde{\mu}_{kn}) \\ U_{kn}^\mu = \left(\frac{d\mu_{kn}(x)}{dx}\right)^{1/2} & \downarrow & & \downarrow & U_{kn}^{\tilde{\mu}} = \left(\frac{\tilde{\mu}_{kn}(y)}{dy}\right)^{1/2} \\ & L^2(\mathbb{R}, dx) & \overset{F}{\to} & L^2(\mathbb{R}, dy). \end{array}$$

By definition, $F_{kn}^\mu = (U_{kn}^{\tilde{\mu}})^{-1} F U_{kn}^\mu$, where

$$(Ff)(y) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) \exp(iyx) dx,$$

so we have

$$(F_{kn}^\mu f)(y) = \left(\frac{d\tilde{\mu}_{kn}(y)}{dy}\right)^{-1/2} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) \exp(iyx) \left(\frac{d\mu_{kn}(x)}{dx}\right)^{1/2} dx. \tag{3.20}$$

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In the case where the Fourier transform $F\mu_{kn}^{1/2}$ of the function $\mu_{kn}^{1/2}$, is positive, we define the density

$$\tilde{\mu}_{kn}(y) := |\,(F\mu_{kn}^{1/2})(y)\,|^2 \tag{3.21}$$

of the corresponding measure $d\tilde{\mu}_{kn}(y) := \tilde{\mu}_{kn}(y)dy$.

Remark 3.6. We compare the conditions $\mu_\alpha^{L_t} \perp \mu_\alpha, \forall\, t \in B(m,\mathbf{k}) \setminus \{e\}$ for $\mathbf{k} = \mathbb{R}$ and $\mathbf{k} = \mathbb{F}_p$ when $m = 2$.

(a) In the case $\mathbf{k} = \mathbb{R}$ we have

$$X^2 = \begin{pmatrix} 1 & x_{12} & x_{13} & \dots & x_{1n} & \dots \\ 0 & 1 & x_{23} & \dots & x_{2n} & \dots \end{pmatrix},$$

$$d\mu_b^m(x) = \otimes_{1 \le k \le 2,\, k<n} \sqrt{\frac{b_{kn}}{\pi}}\, \exp(-b_{kn}x_{kn}^2)dx_{kn}.$$

For the operator $U_{12}^L(t) := T_{I+tE_{12}}^{L,\mu_b,2} = \exp(itA_{12}^L),\ t \in \mathbb{R}$ acting on $L^2(X^2, \mu_b)$, where $A_{12}^L = D_{12} + \sum_{k=3}^\infty x_{2k}D_{1k}$ we have

$$U_{12}^L(t) = \exp\Big(it(D_{12}+\sum_{k=3}^\infty x_{2k}D_{1k})\Big) = \exp(itD_{12})\prod_{k=3}^\infty \exp(itx_{2k}D_{1k}) =: \prod_{k=2}^\infty U_k^L(t),$$

$$S_{12}^L(\mu) = \|A_{12}^L\mathbf{1}\|^2 = \Big\|\Big(D_{12}+\sum_{k=3}^\infty x_{2k}D_{1k}\Big)\mathbf{1}\Big\|^2 = \|D_{12}\mathbf{1}\|^2+\sum_{k=3}^\infty \|x_{2k}\mathbf{1}\|^2\|D_{1k}\mathbf{1}\|_{\tilde{H}_{1k}}^2$$

$$= \|(F_{12}D_{12})\mathbf{1}\|^2+\sum_{k=3}^\infty \|x_{2k}\mathbf{1}\|^2\|(F_{1k}D_{1k})\mathbf{1}\|_{\tilde{H}_{1k}}^2 = \|iy_{12}\mathbf{1}\|^2+\sum_{k=3}^\infty \|x_{2k}\mathbf{1}\|^2\|iy_{1k}\mathbf{1}\|_{\tilde{H}_{1k}}^2$$

$$= \frac{b_{12}}{2} + \sum_{k=3}^\infty \frac{1}{2b_{2k}}\frac{b_{1k}}{2}.$$

(b) In the case $\mathbf{k} = \mathbb{F}_2$ we have X^2 and $\mu_\alpha = \otimes_{1 \le k \le 2, k<n}\mu_{\alpha_{kn}}$. Using (3.5) we get $\big(\begin{smallmatrix} x_{12} \\ 1 \end{smallmatrix}\big) \mapsto \big(\begin{smallmatrix} x_{1n}+1 \\ 1 \end{smallmatrix}\big)$ and $\big(\begin{smallmatrix} x_{1n} \\ x_{2n} \end{smallmatrix}\big) \mapsto \big(\begin{smallmatrix} x_{1n}+x_{2n} \\ x_{2n} \end{smallmatrix}\big)$. Hence, the corresponding operator $U_{12}^L(t) := T_{I+tE_{12}}^{L,\mu_b,2},\ t \in \mathbb{F}_2$ acting on $L^2(X^2, \mu_\alpha)$, has the following form for $t = 1$:

$$U_{12}^L(1) = \otimes_{k=3}^\infty U_k(1),\ \text{where } U_2-1 = (T_{\alpha_{12}}-1),\ U_k-1 = P_{1k}^{(1)}\otimes(T_{\alpha_{1k}}-1),\ k \ge 3.$$

To get the two latter expressions we use (3.10) and (5.5). Therefore, we get

$$\sum_{k=3}^\infty \|(U_k - 1)\mathbf{1}\|^2 = \|(T_{\alpha_{12}} - 1)\mathbf{1}\|_{H_{12}}^2 + \sum_{k=3}^\infty \|P_{2k}^{(1)}\mathbf{1}\|^2\|(T_{\alpha_{1k}} - 1)\mathbf{1}\|_{H_{1k}}^2$$

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$$= \|(\tilde{T}_{\alpha_{12}} - 1)\mathbf{1}\|^2_{\tilde{H}_{12}} + \sum_{k=2}^{\infty} \|P^{(1)}_{2k}\mathbf{1}\|^2 \|(\tilde{T}_{\alpha_{1k}} - 1)\mathbf{1}\|^2_{\tilde{H}_{1k}}$$

$$= \left(1 - 2\sqrt{\alpha_{12}(0)\alpha_{12}(1)}\right) + \sum_{n=3}^{\infty} \alpha_{2n}(1)\left(1 - 2\sqrt{\alpha_{1n}(0)\alpha_{1n}(1)}\right).$$

Remark 3.7. Suppose that the square of the Fourir transform of the square root of the measure μ_α on $\mathbb{F}_p$ is again a measure on $\mathbb{F}_p$. Compare with the case of the field $\mathbb{R}$, Remark 3.5. The latter condition is equivalent with the following one: $F_\alpha \mathbf{1} = \mathbf{1}$ that means by (3.19 the following:

$$(F_\alpha \mathbf{1})(R) = \frac{1}{\sqrt{\tilde{\alpha}(R)p}} \sum_{r \in \mathbb{F}_p} \sqrt{\alpha(r)} \exp\left(-\frac{2\pi i R r}{p}\right) = 1, \quad R \in \mathbb{F}_p. \quad (3.22)$$

For $p = 2$ we get $(F_\alpha \mathbf{1})(0) = (F_\alpha \mathbf{1})(1) = 1$ or

$$(\sqrt{\alpha(0)} + \sqrt{\alpha(1)})/\sqrt{2\tilde{\alpha}(0)} = 1, \quad (\sqrt{\alpha(0)} - \sqrt{\alpha(1)})/\sqrt{2\tilde{\alpha}(1)} = 1,$$

hence, we get if $\alpha(0) > \alpha(1)$

$$\tilde{\alpha}(0) = (1 + 2\sqrt{\alpha(0)\alpha(1)})/2, \quad \tilde{\alpha}(1) = (1 - 2\sqrt{\alpha(0)\alpha(1)})/2.$$

For $p = 3$ we get $(F_\alpha \mathbf{1})(0) = (F_\alpha \mathbf{1})(1) = (F_\alpha \mathbf{1})(2) = 1$ if $\sqrt{\alpha(0)} > (\sqrt{\alpha(1)} + \sqrt{\alpha(2)})/2$ and $\alpha(1) = \alpha(2)$. Indeed, we get

$$(F_\alpha \mathbf{1})(0) = (\sqrt{\alpha(0)} + \sqrt{\alpha(1)} + \sqrt{\alpha(2)})/\sqrt{3\tilde{\alpha}(0)} = 1,$$
$$(F_\alpha \mathbf{1})(1) = (\sqrt{\alpha(0)} + \sqrt{\alpha(1)}\exp(-2\pi i/3) + \sqrt{\alpha(2)}\exp(-4\pi i/3))\sqrt{3\tilde{\alpha}(1)} = 1,$$
$$(F_\alpha \mathbf{1})(2) = (\sqrt{\alpha(0)} + \sqrt{\alpha(1)}\exp(-4\pi i/3) + \sqrt{\alpha(2)}\exp(-8\pi i/3))/\sqrt{3\tilde{\alpha}(2)} = 1$$

hence,

$$\sqrt{\alpha(0)} + \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\sqrt{\alpha(1)} + \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\sqrt{\alpha(2)} =$$

$$\sqrt{\alpha(0)} - \frac{1}{2}(\sqrt{\alpha(1)} + \sqrt{\alpha(2)}) + i\frac{\sqrt{3}}{2}(\sqrt{\alpha(1)} - \sqrt{\alpha(2)}) > 0.$$

For general prime p we get $\sum_{R \in \mathbb{F}_p} \sqrt{\alpha(r)}\exp(-2\pi R r/p) = 1$ for $R \in \mathbb{F}_p$.

3.3. Maximal abelian subalgebra and a simple spectrum

Definition 3.1. An abelian subalgebra of a von Neumann algebra $\mathfrak{A}$ is called maximal if it is not properly included in any other such subalgebra of $\mathfrak{A}$.

Consider a finite-dimensional Hilbert space $H = \mathbb{C}^n$ with the standard scalar product $(x, y) = \sum_{k=1}^{n} x_k \bar{y}_k$.

Definition 3.2. A spectrum $Sp(A)$ of an operator A in an n-dimensional Hilbert space H we call simple if $Sp(A)$ consists of n distinct eigenvalues.

Lemma 3.6. *A von Neumann algebara $L^\infty(A)$ generated by a diagonal operator $A = \mathrm{diag}(\lambda_k)_{k=1}^n$ in $H = \mathbb{C}^n$ is maximal abelian if and only if the spectrum of A is simple. In addition, $L^\infty(A) = \{P(A) \mid \mathrm{ord}\, P \leq n - 1\}$ where $P(x) = \sum_{k=0}^{n-1} a_k x^k$, $a_k \in \mathbb{C}$.*

PROOF. We know that for a von Neumann algebra $\mathfrak{A}$ holds $(\mathfrak{A})' = \mathfrak{A}$ therefore, $L^\infty(A) = (A)''$. We show that

$$(L^\infty(A))' = (A)' = \{\mathrm{diag}(b_k)_{k=1}^n \mid b = (b_k)_{k=1}^n \in \mathbb{C}^n\}.$$

Indeed, let $[A, B] = 0$ where $B = (b_{km})_{k,m=1}^n$ then

$$\lambda_k b_{km} = b_{km} \lambda_m, \quad \text{for all} \quad k \neq m.$$

Therefore, $b_{km} = 0$ for $k \neq m$ since $\lambda_k \neq \lambda_m$. By the same arguments we show that

$$L^\infty(A) = (A)'' = \{\mathrm{diag}(a_k)_{k=1}^n \mid a = (a_k)_{k=1}^n \in \mathbb{C}^n\}.$$

When $A = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_2)$ with $\lambda_1 \neq \lambda_2$ then $L^\infty(A) \neq (L^\infty(A))'$ since

$$L^\infty(A) = \left\{ \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} \mid a_{kn} \in \mathbb{C} \right\}, \quad (L^\infty(A))' = \left\{ \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix} \mid a_{kn} \in \mathbb{C} \right\}.$$

$\square$

Denote by $L^\infty(T_\alpha)$ the von Neumann algebra of operators acting on $H_\alpha \simeq \mathbb{C}^p$ generated by operator T_α, defined by (3.10), i.e., $L^\infty(T_\alpha) = (T_\alpha)''$.

Lemma 3.7. *The von Neumann algebra $L^\infty(T_\alpha)$ is a maximal abelian subalgebra in $B(H_\alpha)$, i.e., $(L^\infty(T_\alpha))' = L^\infty(T_\alpha)$. In addition, $L^\infty(A) = \{P(A) \mid \mathrm{ord}\, P \leq p - 1\}$.*

PROOF. By Lemma 3.18 $T_{inv} \sim \tilde{T}_{inv}$ and $\tilde{T}_{inv}$ is the diagonal operator with different eigenvalues: $Sp(\tilde{T}_{inv}) = \{\exp(-ik\pi/p) \mid k \in \mathbb{F}_p\}$. $\qquad\square$

Lemma 3.8. *The von Neumann algebra $L^\infty(x)$ generated by the operator $x = \mathrm{diag}(k)_{k\in\mathbb{F}_p}$ is a maximal abelian subalgebra in $B(H_\alpha)$, i.e., $(L^\infty(x))' = L^\infty(x)$.*

PROOF. Since the spectrum $Sp(x) = \{k \mid k \in \mathbb{F}_p\}$ of the operator x is simple the proof follows from Lemma 3.6. $\qquad\square$

4. The Laplace operator and the commutant description

4.1. The Laplace operator and the irreducibility

For approximationof an operator x_{kn} defined by (2.12) we shall use the well known result (see for example [4], Chap. I, §52)

$$\min_{x\in\mathbb{R}^n} \Big(\sum_{k=1}^n a_k x_k^2 \mid \sum_{k=1}^n x_k = 1 \Big) = \Big(\sum_{k=1}^n \frac{1}{a_k} \Big)^{-1}, \quad a_k > 0, \ k = 1, 2, \ldots, n.$$

We use the same result in a slightly different form with $b_k \neq 0, \ k = 1, 2, \ldots, n$

$$\min_{x\in\mathbb{R}^n} \Big(\sum_{k=1}^n a_k x_k^2 \mid \sum_{k=1}^n x_k b_k = 1 \Big) = \Big(\sum_{k=1}^n \frac{b_k^2}{a_k} \Big)^{-1}. \tag{4.1}$$

The minimum is realized for $x_k = \frac{b_k}{a_k} \Big(\sum_{k=1}^n \frac{b_k^2}{a_k} \Big)^{-1}$.

Let $p = 2$ and $X = X^1$. Set

$$A_{\alpha_{1n}} := T_{\alpha_{1n}} - 1 = \begin{pmatrix} -1 & \sqrt{\frac{\alpha_{1n}(1)}{\alpha_{1n}(0)}} \\ \sqrt{\frac{\alpha_{1n}(0)}{\alpha_{1n}(1)}} & -1 \end{pmatrix}. \tag{4.2}$$

Remark 4.1. In order to approximate the operator $x_{1k} := \mathrm{diag}(0, 1)$ acting on H_{1k} (see (2.13)) by linear combinations of $T_{kn} - I = x_{1k} \otimes (T_{\alpha_{1n}} - I)$ (see (5.5)) it is sufficient to approximate the identity operator $Id = I$ by linear combinations $\sum_{n=2}^N t_n A_{\alpha_{1n}} = \sum_{n=2}^N t_n(T_{\alpha_{1n}} - 1)$ as $N \to \infty$ (see Lemma 4.1 below).

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Notations. Set $\mathbf{1} = \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes \cdots \in L^2(X^m, \mu_\alpha) = \otimes_{1 \le k \le m, k < n} L^2(\mathbb{F}_p, \mu_{\alpha_{kn}})$, where $\mathbf{1} = (1, 1, \ldots, 1) \in L^2(\mathbb{F}_p, \mu_{\alpha_{kn}})$ and let $c_{1n} = 2\sqrt{\alpha_{1n}(0)\alpha_{1n}(1)}$. As before, for $f \in L^1(X, \mu)$ we use the following notation $Mf = \int_X f(x) d\mu(x)$ and let $\langle f_n \mid n \in \mathbb{N} \rangle$ be a closed subspace generated by the set of vectors $(f_n)_{n \in \mathbb{N}}$ in a space H.

Remark 4.2. Obviously, two series with positive a_n, b_n are *equivalent*:

$$\sum_n \frac{a_n}{a_n + b_n} \sim \sum_n \frac{a_n}{b_n}, \tag{4.3}$$

i.e., they are simultaneously convergent or divergent.

Lemma 4.1. *Three following conditions are equivalent if $p = 2$:*

(i) $\mathbf{1} \in \langle A_{\alpha_{1n}} \mathbf{1} \mid n \ge 2 \rangle$,

(ii) $S_{11}^L(\mu_\alpha) = \sum_{n=2}^\infty \left(1 - 2\sqrt{\alpha_{1n}(0)\alpha_{1n}(1)} \right) =: \sum_{n=2}^\infty (1 - c_{1n}) = \infty$,

(iii) $\mu_\alpha \perp \mu_{inv}$.

PROOF. To prove $(i) \Leftrightarrow (ii)$ set $\xi_n = A_{\alpha_{1n}} \mathbf{1}$. We have

$$(T_{\alpha_{1n}} \mathbf{1}, \mathbf{1}) = \sqrt{\frac{\alpha_{1n}(1)}{\alpha_{1n}(0)}} \alpha_{1n}(0) + \sqrt{\frac{\alpha_{1n}(0)}{\alpha_{1n}(1)}} \alpha_{1n}(1) = 2\sqrt{\alpha_{1n}(0)\alpha_{1n}(1)} = c_{1n},$$

$$M\xi_n = MA_{\alpha_{1n}} \mathbf{1} = (A_{\alpha_{1n}} \mathbf{1}, \mathbf{1}) = ((T_{\alpha_{1n}} - 1)\mathbf{1}, \mathbf{1}) = c_{1n} - 1,$$

$$\|\xi_n\|^2 = \|A_{\alpha_{1n}} \mathbf{1}\|^2 = (A_{\alpha_{1n}} \mathbf{1}, A_{\alpha_{1n}} \mathbf{1}) = ((T_{\alpha_{1n}} - 1)\mathbf{1}, (T_{\alpha_{1n}} - 1)\mathbf{1})$$

$$= 2 - 2(T_{\alpha_{1n}} \mathbf{1}, \mathbf{1}) = 2(1 - c_{1n}).$$

Finally, we have

$$M\xi_n = -\left(1 - 2\sqrt{\alpha_{1n}(0)\alpha_{1n}(1)} \right), \quad \|\xi_n\|^2 = 2\left(1 - 2\sqrt{\alpha_{1n}(0)\alpha_{1n}(1)} \right).$$

If we take $(t_n)_n$ such that $\sum_{n=2}^{N+2} t_n M\xi_n = 1$ we obtain (since $\xi_n - M\xi_n \perp \xi_m - M\xi_m$ for $n \ne m$)

$$\left\| \left(\sum_{n=2}^{N+2} t_n A_{\alpha_{1n}} - 1 \right) \mathbf{1} \right\|^2 = \left\| \sum_{n=2}^{N+2} t_n \left(A_{\alpha_{1n}} - M\xi_n \right) \mathbf{1} \right\|^2 =$$

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$$\sum_{n=2}^{N+2} t_n^2 \| \left(A_{\alpha_{1n}} - M\xi_n \right) \mathbf{1} \|^2 = \| \sum_{n=2}^{N+2} t_n (\xi_n - M\xi_n) \|^2$$

$$= \sum_{n=2}^{N+2} t_n^2 \left(\|\xi_n\|^2 - | M\xi_n |^2 \right).$$

Using (4.1) for $b_n = M\xi_n$ and $a_n = \|\xi_n\|^2 - | M\xi_n |^2$ we conclude that

$$\min_{t \in \mathbb{R}^N} \left(\| \left(\sum_{n=2}^{N+2} t_n A_{\alpha_{1n}} - 1 \right) \mathbf{1} \|^2 \mid \sum_{n=2}^{N+2} t_n M\xi_n = 1 \right) = (S_{11,N}^L(\mu_\alpha))^{-1}$$

where

$$S_{11,N}^L(\mu_\alpha) = \sum_{n=2}^{N+2} \frac{b_n^2}{a_n} = \sum_{n=2}^{N+2} \frac{| M\xi_n |^2}{\|\xi_n\|^2 - | M\xi_n |^2} \overset{(4.3)}{\sim} \sum_{n=2}^{N+2} \frac{| M\xi_n |^2}{\|\xi_n\|^2} =$$

$$\sum_{n=2}^{N+2} \frac{\left(1 - 2\sqrt{\alpha_{1n}(0)\alpha_{1n}(1)} \right)^2}{2 \left(1 - 2\sqrt{\alpha_{1n}(0)\alpha_{1n}(1)} \right)} = \frac{1}{2} \sum_{n=2}^{N+2} \left(1 - 2\sqrt{\alpha_{1n}(0)\alpha_{1n}(1)} \right).$$

To prove $(ii) \Leftrightarrow (iii)$ we have

$$H(\mu_{\alpha_{1n}}, \mu_{inv}) = \int_{\mathbb{F}_2} \sqrt{\frac{d\mu_{\alpha_{1n}}(x)}{d\mu_{inv}(x)}} d\mu_{inv}(x) = \frac{\sqrt{\alpha_{1n}(0)} + \sqrt{\alpha_{1n}(1)}}{\sqrt{2}}.$$

Hence, we get

$$H(\mu_\alpha, \mu_{inv}) = \prod_{n=2}^{\infty} H(\mu_{\alpha_{1n}}, \mu_{inv}) = \prod_{n=2}^{\infty} \frac{\sqrt{\alpha_{1n}(0)} + \sqrt{\alpha_{1n}(1)}}{\sqrt{2}}.$$

By the Kakutani criterion we conclude that $\mu_\alpha \perp \mu_{inv}$ if and only if

$$H(\mu_\alpha, \mu_{inv}) = \prod_{n=2}^{\infty} \frac{\sqrt{\alpha_{1n}(0)} + \sqrt{\alpha_{1n}(1)}}{\sqrt{2}} = 0 \Leftrightarrow$$

$$\prod_{n=2}^{\infty} \left(\frac{\sqrt{\alpha_{1n}(0)} + \sqrt{\alpha_{1n}(1)}}{\sqrt{2}} \right)^2 = 0 \Leftrightarrow \prod_{n=2}^{\infty} \frac{1}{2} \left(1 + 2\sqrt{\alpha_{1n}(0)\alpha_{1n}(1)} \right) = 0$$

$$\Leftrightarrow S_{11}^L(\mu_\alpha) = \sum_{n=2}^{\infty} \left(1 - 2\sqrt{\alpha_{1n}(0)\alpha_{1n}(1)} \right) = \infty.$$

$\square$

Lemma 4.2. *Denote by* $T_n = T_{\alpha_{1n}}$, $n \geq 2$. *The following strong limit of operators* $\Delta_1 = \mathrm{s.}\lim_{k\to\infty} \prod_{n=2}^{k} \frac{I+T_n}{2}$ *is correctly defined if and only if the following equivalent conditions hold:*

$$\mu_\alpha \sim \mu_{inv} \Leftrightarrow \prod_{n=2}^{\infty} \frac{\sqrt{\alpha_{1n}(0)} + \sqrt{\alpha_{1n}(1)}}{\sqrt{2}} > 0 \Leftrightarrow S_{11}^L(\mu_\alpha) < \infty.$$

PROOF. We have

$$\|1/2(I + T_n)\mathbf{1}\|^2 = 1/4\|(I + T_n)\mathbf{1}\|^2 = 1/4[(\mathbf{1}, \mathbf{1}) + 2(T_n\mathbf{1}, \mathbf{1}) + (T_n\mathbf{1}, T_n\mathbf{1})]$$

$$= 1/2(1 + (T_n\mathbf{1}, \mathbf{1})) \overset{(3.10)}{=} 1/2(1 + 2\sqrt{\alpha_{1n}(0)\alpha_{1n}(1)}) = \left(\frac{\sqrt{\alpha_{1n}(0)} + \sqrt{\alpha_{1n}(1)}}{\sqrt{2}}\right)^2.$$

Hence,

$$\lim_{k\to\infty} \|\prod_{n=3}^{k} \frac{I + T_n}{2}\mathbf{1}\|^2 = \left(\prod_{n=3}^{\infty} \frac{\sqrt{\alpha_{1n}(0)} + \sqrt{\alpha_{1n}(1)}}{\sqrt{2}}\right)^2 = H^2(\mu_\alpha, \mu_{inv}). \quad (4.4)$$

$\square$

Consider the space X^1, i.e., the case $m = 1$, and fix a prime $p > 2$. For the operator T_α we have (see (3.10))

$$T_\alpha = \begin{pmatrix} 0 & 0 & 0 \dots & 0 & \sqrt{\frac{\alpha(p-1)}{\alpha(0)}} \\ \sqrt{\frac{\alpha(0)}{\alpha(1)}} & 0 & 0 \dots & 0 & 0 \\ 0 & \sqrt{\frac{\alpha(1)}{\alpha(2)}} & 0 \dots & 0 & 0 \\ & & \dots & & \\ 0 & 0 & 0 \dots & \sqrt{\frac{\alpha(p-2)}{\alpha(p-1)}} & 0 \end{pmatrix} = \sum_{k\in\mathbb{F}_p} \sqrt{\frac{\alpha(k)}{\alpha(k+1)}} E_{k+1,k}.$$

Remark 4.3. To guess the expression for the right version of the operator $A_{\alpha_{1n}}$ defined by (4.2), if $p > 2$, we observe that $A_{\alpha_{1n}} = T_{\alpha_{1n}} - 1 = \sum_{r\in\mathbb{F}_2} T_{\alpha_{1n}}^r - 2$ for $p = 2$. Hence, it is natural to replace the operator $A_{\alpha_{1n}}$ in the case $p = 2$ by the expression $C(T_{\alpha_{1n}}) - p = \sum_{r\in\mathbb{F}_p} T_{\alpha_{1n}}^r - p$ in the case of an arbitrary p.

Lemma 4.3. *Three following conditions are equivalent for an arbitrary p:*
(i) $\mathbf{1} \in \langle \left(\sum_{r\in\mathbb{F}_p} T_{\alpha_{1n}}^{p-r} - p\right)\mathbf{1} \mid n \geq 2 \rangle$,

(ii) $S_{11}^L(\mu_\alpha) = \infty$, *where*

$$S_{11}^L(\mu_\alpha) := \sum_{n=2}^\infty \sum_{r \in \mathbb{F}_p \setminus \{0\}} \left(1 - \sum_{k \in \mathbb{F}_p} \sqrt{\alpha_{1n}(k+r)\alpha_{1n}(k)}\right) =$$

$$\sum_{n=2}^\infty \left(p - \sum_{k,s \in \mathbb{F}_p} \sqrt{\alpha_{1n}(s)\alpha_{1n}(k)}\right) = p \sum_{n=2}^\infty \left[1 - \left(\sum_{k \in \mathbb{F}_p} \sqrt{\frac{\alpha_{1n}(k)}{p}}\right)^2\right],$$

(iii) $\mu_\alpha \perp \mu_{inv}$.

PROOF. To prove $(i) \Leftrightarrow (ii)$ we have

$$T_\alpha = \sum_{k \in \mathbb{F}_p} \sqrt{\frac{\alpha(k)}{\alpha(k+1)}} E_{k+1,k}, \quad T_\alpha^r = \sum_{k \in \mathbb{F}_p} \sqrt{\frac{\alpha(k)}{\alpha(k+r)}} E_{k+r,k}. \qquad (4.5)$$

Using the change of variables $k + p - r = s$, $k = s - p + r = s + r \bmod p$ we get:

$$T_\alpha^{p-r} = \sum_{k \in \mathbb{F}_p} \sqrt{\frac{\alpha(k)}{\alpha(k+p-r)}} E_{k+p-r,k} = \sum_{s \in \mathbb{F}_p} \sqrt{\frac{\alpha(s+r)}{\alpha(s)}} E_{s,s+r}.$$

Finally, we have

$$C(T_\alpha) = \sum_{r \in \mathbb{F}_p} T_\alpha^{p-r} = \sum_{r \in \mathbb{F}_p} \sum_{s \in \mathbb{F}_p} \sqrt{\frac{\alpha(s+r)}{\alpha(s)}} E_{ss+r} = \sum_{k \in \mathbb{F}_p} \sum_{r \in \mathbb{F}_p} \sqrt{\frac{\alpha(r)}{\alpha(k)}} E_{kr}. \quad (4.6)$$

Set $\xi_n = \xi_{\alpha_{1n}} = (C(T_{\alpha_{1n}}) - p)\mathbf{1} = \left(\sum_{r \in \mathbb{F}_p} T_{\alpha_{1n}}^{p-r} - p\right)\mathbf{1}$. We get

$$M\xi_\alpha = \left((C(T) - p)\mathbf{1}, \mathbf{1}\right) = (C(T)\mathbf{1}, \mathbf{1}) - p =$$

$$\stackrel{(4.6)}{=} \sum_{k \in \mathbb{F}_p} \left(\sum_{r \in \mathbb{F}_p} \sqrt{\alpha(r)/\alpha(k)}\right)\alpha(k) - p = \sum_{r,k \in \mathbb{F}_p} \sqrt{\alpha(r)\alpha(k)} - p.$$

To calculate $\|\xi_\alpha\|^2$ we get using (4.6)

$$\|\xi_\alpha\|^2 = \sum_{k \in \mathbb{F}_p} \left|\sum_{r \in \mathbb{F}_p} \sqrt{\frac{\alpha(r)}{\alpha(k)}} - p\right|^2 \alpha(k) =$$

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$$= \sum_{k\in\mathbb{F}_p} \Big(\sum_{r\in\mathbb{F}_p} \alpha(r) + p^2\alpha(k) + \sum_{r,s\in\mathbb{F}_p,\, r\neq s} \sqrt{\alpha(r)\alpha(s)} - 2p \sum_{r\in\mathbb{F}_p} \sqrt{\alpha(r)\alpha(k)} \Big)$$

$$= \sum_{k\in\mathbb{F}_p} \Big(p^2\alpha(k) + \sum_{r,s\in\mathbb{F}_p} \sqrt{\alpha(r)\alpha(s)} - 2p \sum_{r\in\mathbb{F}_p} \sqrt{\alpha(r)\alpha(k)} \Big)$$

$$= p^2 + p \sum_{r,s\in\mathbb{F}_p} \sqrt{\alpha(r)\alpha(s)} - 2p \sum_{r,k\in\mathbb{F}_p} \sqrt{\alpha(r)\alpha(k)}$$

$$= p^2 - p \sum_{r,k\in\mathbb{F}_p} \sqrt{\alpha(r)\alpha(k)} = p\Big(p - \sum_{r,k\in\mathbb{F}_p} \sqrt{\alpha(r)\alpha(k)} \Big).$$

Finally, we get

$$M\xi_\alpha = -\Big(p - \sum_{r,k\in\mathbb{F}_p} \sqrt{\alpha(r)\alpha(k)} \Big), \quad \|\xi_\alpha\|^2 = p\Big(p - \sum_{r,k\in\mathbb{F}_p} \sqrt{\alpha(r)\alpha(k)} \Big).$$

If we take $\sum_{n=2}^{N+2} t_n M\xi_n = 1$ we obtain (since $\xi_n - M\xi_n \perp \xi_m - M\xi_m$ for $n \neq m$)

$$\Big\| \Big(\sum_{n=2}^{N+2} t_n \big(C(T_{\alpha_{1n}}) - p \big) - 1 \Big) \mathbf{1} \Big\|^2 = \Big\| \sum_{n=2}^{N+2} t_n \Big(\big(C(T_{\alpha_{1n}}) - p \big) - M\xi_n \Big) \mathbf{1} \Big\|^2$$

$$= \Big\| \sum_{n=2}^{N+2} t_n (\xi_n - M\xi_n) \Big\|^2 = \sum_{n=2}^{N+2} t_n^2 \Big(\|\xi_n\|^2 - | M\xi_n |^2 \Big).$$

Using (4.1) for $b_n = M\xi_n$ and $a_n = \|\xi_n\|^2 - | M\xi_n |^2$ we conclude that

$$\min_{t\in\mathbb{R}^N} \Big(\Big\| \Big[\sum_{n=2}^{N+2} t_n \big(C(T_{\alpha_{1n}}) - p \big) - I \Big] \mathbf{1} \Big\|^2 \,\Big|\, \sum_{n=2}^{N+2} t_n M\xi_n = 1 \Big) = (S_{11,N}^L(\mu))^{-1},$$

where

$$S_{11,N}^L(\mu) = \sum_{n=2}^{N+2} \frac{| M\xi_n |^2}{\|\xi_n\|^2 - | M\xi_n |^2} \overset{(4.3)}{\sim} \sum_{n=2}^{N+2} \frac{| M\xi_n |^2}{\|\xi_n\|^2} =$$

$$\sum_{n=2}^{N+2} \frac{\Big(p - \sum_{r,k\in\mathbb{F}_p} \sqrt{\alpha_{1n}(r)\alpha_{1n}(k)} \Big)^2}{p\Big(p - \sum_{r,k\in\mathbb{F}_p} \sqrt{\alpha_{1n}(r)\alpha_{1n}(k)} \Big)} = \frac{1}{p} \sum_{n=2}^{N+2} \Big(p - \sum_{r,k\in\mathbb{F}_p} \sqrt{\alpha_{1n}(r)\alpha_{1n}(k)} \Big).$$

To prove $(ii) \Leftrightarrow (iii)$ we have $H(\mu_\alpha, \mu_{inv}) =$

$$\prod_{n=2}^{\infty} H(\mu_{\alpha_{1n}}, \mu_{inv}) = \prod_{n=2}^{\infty} \int_{\mathbb{F}_p} \sqrt{\frac{d\mu_{\alpha_{1n}}(x)}{d\mu_{inv}(x)}} d\mu_{inv}(x) = \prod_{n=2}^{\infty} \sum_{k\in\mathbb{F}_p} \sqrt{\frac{\alpha_{1n}(k)}{p}}. \quad (4.7)$$

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So, by Kakutani's criterion, we conclude that $\mu_\alpha \perp \mu_{inv}$ if and only if

$$H(\mu_\alpha, \mu_{inv}) = \prod_{n=2}^\infty \sum_{k\in\mathbb{F}_p} \sqrt{\frac{\alpha_{1n}(k)}{p}} = 0 \Leftrightarrow \prod_{n=2}^\infty \left(\sum_{k\in\mathbb{F}_p} \sqrt{\frac{\alpha_{1n}(k)}{p}} \right)^2 = 0.$$

We note that

$$\left(\sum_{k\in\mathbb{F}_p} \sqrt{\alpha_{1n}(k)} \right)^2 = \sum_{k\in\mathbb{F}_p} \sum_{r\in\mathbb{F}_p} \sqrt{\alpha_{1n}(k)\alpha_{1n}(r)} = \sum_{r\in\mathbb{F}_p} \sum_{k\in\mathbb{F}_p} \sqrt{\alpha_{1n}(k)\alpha_{1n}(k+r)} =$$

$$1 + \sum_{r\in\mathbb{F}_p\setminus\{0\}} \sum_{k\in\mathbb{F}_p} \sqrt{\alpha_{1n}(k)\alpha_{1n}(k+r)} =: 1 + \sum_{r\in\mathbb{F}_p\setminus\{0\}} c_{1n}(r),$$

where $c_{1n}(r) := \sum_{k\in\mathbb{F}_p} \sqrt{\alpha_{1n}(k)\alpha_{1n}(k+r)}$. Finally, $\mu_\alpha \perp \mu_{inv}$ if and only if

$$\prod_{n=2}^\infty \frac{1}{p}\left(1 + \sum_{r\in\mathbb{F}_p\setminus\{0\}} \sum_{k\in\mathbb{F}_p} \sqrt{\alpha_{1n}(k)\alpha_{1n}(k+r)} \right) = 0 \Leftrightarrow$$

$$S_{11}^L(\mu_\alpha) = \sum_{n\in\mathbb{N},n>2} \sum_{r\in\mathbb{F}_p\setminus\{0\}} \left(1 - \sum_{k\in\mathbb{F}_p} \sqrt{\alpha_{1n}(k+r)\alpha_{1n}(k)} \right) = \infty.$$

$\square$

Lemma 4.4. *Set* $C(T) = \sum_{r\in\mathbb{F}_p} T^r$ *and* $T_n = T_{\alpha_{1n}}$. *The following strong limit* $\Delta_1 = \mathrm{s.}\lim_{k\to\infty} \prod_{n=2}^k p^{-1}C(T_n)$, *is correctly defined if and only if three equivalent conditions hold:*

$$(i) \quad \mu_\alpha \sim \mu_{inv},$$

$$(ii) \quad H(\mu_\alpha, \mu_{inv}) = \prod_{n=2}^\infty \sum_{r\in\mathbb{F}_p} \sqrt{\frac{\alpha_{1n}(r)}{p}} > 0,$$

$$(iii) \quad S_{11}^L(\mu_\alpha) < \infty.$$

PROOF. Using (4.6) we have $\|C(T_n)\mathbf{1}\|^2 =$

$$\sum_{k\in\mathbb{F}_p} \left(\sum_{r\in\mathbb{F}_p} \sqrt{\frac{\alpha_{1n}(r)}{\alpha_{1n}(k)}} \right)^2 \alpha_{1n}(k) = p\left(\sum_{r\in\mathbb{F}_p} \sqrt{\alpha_{1n}(r)} \right)^2 = p^2\left(H(\mu_{\alpha_{1n}}, \mu_{inv}) \right)^2,$$

$$(4.8)$$

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hence, by (4.7) we get

$$\lim_{k\to\infty} \|\prod_{n=2}^{k} p^{-1}C(T_n)\mathbf{1}\|^2 = \prod_{n=2}^{\infty} \Big(\sum_{r\in\mathbb{F}_p} \sqrt{\frac{\alpha_{1n}(r)}{p}}\Big)^2 = \Big(H(\mu_\alpha, \mu_{inv})\Big)^2. \quad (4.9)$$

$\square$

Using Lemmas 4.3 and 4.4 we conclude that

Lemma 4.5. *The following four conditions are equivalent for the measure* μ_α *on the space* X^1:

(i) $\mu_\alpha \sim \mu_{inv}$,

(ii) $S_{11}^L(\mu_\alpha) < \infty$,

(iii) $\mathbf{1} \notin \langle (C(T_{1n}) - p)\,\mathbf{1} \mid n \geq 2 \rangle$,

(iv) *there exist a non trivial limit* $\Delta_1 := \lim_{n\to\infty} \prod_{k=2}^{n} p^{-1}C(T_{1k})$.

Remark 4.4. Using Lemma 4.5 we conclude that condition 3) of Conjecture 2.5 are necessary for the irreducibility of the representation $T^{R,\mu_\alpha,m}$.

Consider the measure $\mu_\alpha = \otimes_{k=1}^{m}\mu_\alpha^k$ on the space X^m and the representation $T^{R,\mu_\alpha,m}$.

Lemma 4.6. *If* $\mu_\alpha^k \sim \mu_{inv}^k$ *for some* $1 \leq k \leq m$ *then the Laplace operator*

$$\Delta_k = s.\lim_{r\to\infty} \prod_{n=k+1}^{r} p^{-1}C(T_{kn}(k))$$

is well defined and commutes with the representation $T^{R,\mu_\alpha,m}$. *In particular, if* $\mu_\alpha \sim \mu_{inv} = \otimes_{k=1}^{m}\mu_{inv}^k$ *then the Laplace operator* $\Delta^{(m)} := \Delta_1\Delta_2\ldots\Delta_m$ *is well defined and commutes with the representation.*

PROOF. The operator Δ_l is well defined by analogue of Lemma 4.4. To prove that Δ_l commutes with the representation, i.e., $[\Delta_l, T_t^{R,\mu_\alpha,m}] = 0$ for all $t \in B_0^{\mathbb{N}}(\mathbb{F}_p)$ it is sufficient to prove commutation $[\Delta_l, T_{kk+1}] = 0$ for all $k \in \mathbb{N}$ since the subgroups $E_{kk+1}(t) = I + tE_{kk+1}$, $t \in \mathbb{F}_p$, $k \in \mathbb{N}$ generate all the group $B_0^{\mathbb{N}}(\mathbb{F}_p)$.

In the case $m = 1$ we prove the commutation relations $[\Delta_1, T_{kk+1}] = 0$ for all $k \in \mathbb{N}$. The latter relation follows from $[C(T_{12}), T_{12}] = 0$ that is evident, since $TC(T) = C(T)T = C(T)$ for T such that $T^p = I$, and the relation $[C(T_{1k})C(T_{1k+1}), T_{kk+1}] = 0$ $k > 2$. We prove more general relations:

$$[C(T_{1k})C(T_{1m}), T_{km}] = 0 \quad \text{for} \quad 1 < k < m. \tag{4.10}$$

We have

$$C(T_{1k})C(T_{1m})T_{km} = C(T_{1k})T_{km}C(T_{1m}) = \sum_{r \in \mathbb{F}_p} T_{1k}^r T_{km} C(T_{1m}) \overset{(4.14)}{=}$$

$$T_{km} \sum_{r \in \mathbb{F}_p} T_{1k}^r T_{1m}^r C(T_{1m}) = T_{km} \sum_{r \in \mathbb{F}_p} T_{1k}^r C(T_{1m}) = T_{km} C(T_{1k}) C(T_{1m}).$$

For the general m we show that $[\Delta_l, T_{kk+1}] = 0$ for $1 \leq l \leq m$ and $k \in \mathbb{N}$. First, using (2.8) we conclude that $[\Delta_l, T_{kk+1}] = 0$ for $1 \leq k < l$. Further, we conclude that $[\Delta_l, T_{kk+1}] = 0$ for $k \geq l$ by analogy with the relation $[\Delta_1, T_{kk+1}] = 0$ for $k \geq 1$. $\qquad\square$

4.2. Commutant of the von Neumann algebra $\mathfrak{A}^m$, case $m = 1$

In this subsection we explain how the Laplace operator Δ_k (see (2.11)) in the commutant $(\mathfrak{A}^m)'$ was found. Let

$$\mathfrak{A}^m = \left(T_t^{R,\mu_\alpha,m} \mid t \in G\right)'' = \left(T_{rk} \mid 1 \leq r < k\right)''$$

be the von-Neumann algebra generated by the representation $T^{R,\mu_\alpha,m}$ acting in the space $H^m = L^2(X^m, \mu_\alpha)$ and let $\mathfrak{A}^{m,n}$ be its von-Neumann subalgebra,

$$\mathfrak{A}^{m,n} = \left(T_{rk} \mid 1 \leq r \leq m, \; r < k \leq n\right)'', \tag{4.11}$$

where T_{kn} are defined by (2.7). We have $\mathfrak{A}^m = \left(\bigcup_{n>m} \mathfrak{A}^{m,n}\right)''$. We would like to describe the *commutant* $(\mathfrak{A}^m)' = \bigcap_{n>m}(\mathfrak{A}^{m,n})'$ of the von Neumann algebra $\mathfrak{A}^m$ First, we shall do this for $m = 1$.

To describe $(\mathfrak{A}^{m,n})'$ it is sufficient to consider the invariant measures μ_{inv} since in the finite-dimensional space $X^{m,n}$ (see below (4.32)) all considered measures are equivalent. Set $H_{kn} = L^2(\mathbb{F}_p, \mu_{inv}^{kn})$, where $\mu_{inv}(r) = \mu_{inv}^{kn}(r) = p^{-1}$. For $m = 1$ and $n = 3$, we denote

$$X = \{(x_{12}, x_{13}) \mid x_{kn} \in \mathbb{F}_p\}, \quad H = H_{12} \otimes H_{13} = L^2(\mathbb{F}_p, \mu_{inv}) \otimes L^2(\mathbb{F}_p, \mu_{inv}).$$

We fix the basis $(e_k)_{k\in\mathbb{F}_p}$ in $L^2(\mathbb{F}_p,\mu_{inv})$, where $e_k(x)=p^{1/2}\delta_{k,x}$, $k,x\in\mathbb{F}_p$ (see 3.8), i.e.,

$$e_0=(p^{1/2},0,\ldots,0),\quad e_2=(0,p^{1/2},0,\ldots,0),\quad\ldots,\quad e_{p-1}=(0,\ldots 0,p^{1/2}).$$

Fix $p=2$. We have in the spaces H_{12} and H_{13}

$$T_{12}=T_{13}=T:=T_{inv}=\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right).$$

In the space $H_{12}\otimes H_{13}$ we get $T_{12}=T_{inv}\otimes I$, $T_{13}=I\otimes T_{inv}$ (see Remark 2.3), i.e.,

$$T_{12}=\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\otimes\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right),\quad T_{13}=\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\otimes\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right).$$

The basis in the space $L^2(\mathbb{F}_p,\mu_{inv})\otimes L^2(\mathbb{F}_p,\mu_{inv})$ is $(e_{kr}:=e_k\otimes e_r)_{k,r\in\mathbb{F}_p}$. Let us fix the *lexicographic order* on the set $(k,r)_{k,r\in\mathbb{F}_p}$. For $p=2$ the basis $(e_{kr})_{k,r}$ in $H_{12}\otimes H_{13}$ is ordered as follows:

$$e_{00}=e_0\otimes e_0,\quad e_{01}=e_0\otimes e_1,\quad e_{10}=e_1\otimes e_0,\quad e_{11}=e_1\otimes e_1.$$

In this basis the operators T_{12} and T_{13} on the space $H_{12}\otimes H_{13}$ have the following form:

$$T_{12}=\left(\begin{smallmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{smallmatrix}\right)=\left(\begin{smallmatrix}0&I\\I&0\end{smallmatrix}\right),\quad T_{13}=\left(\begin{smallmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{smallmatrix}\right)=\left(\begin{smallmatrix}T&0\\0&T\end{smallmatrix}\right),$$

where

$$I=\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right),\quad T=\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right).$$

Consider the general case of p and μ. In the space $H=H_{\alpha_{12}}\otimes H_{\alpha_{13}}\otimes\cdots\otimes H_{\alpha_{1n}}$ the basis $e_{i_2i_3\ldots i_n}$, $i_2,i_3,\ldots,i_n\in\mathbb{F}_p$ is defined by $e_{i_2i_3\ldots i_n}=e_{i_2}\otimes e_{i_3}\otimes\cdots\otimes e_{i_n}$, and the scalar product for two elements f and g in H

$$f=\sum_{i_2,i_3,\ldots,i_n\in\mathbb{F}_p}f_{i_2i_3\ldots i_n}e_{i_2i_3\ldots i_n},\quad g=\sum_{i_2,i_3,\ldots,i_n\in\mathbb{F}_p}g_{i_2i_3\ldots i_n}e_{i_2i_3\ldots i_n}$$

is defined by the formula:

$$(f,g)_{H_{12}\otimes H_{13}\otimes\cdots\otimes H_{1n}}=\sum_{i_2,i_3,\ldots,i_n\in\mathbb{F}_p}f_{i_2i_3\ldots i_n}\overline{g}_{i_2i_3\ldots i_n}.\alpha_{12}(i_2)\alpha_{13}(i_3)\ldots\alpha_{1n}(i_n).$$

$$(4.12)$$

To describe $(\mathfrak{A}^{1,2})'$ when $p = 2$ we take any operator A on the space $H_{12} \otimes H_{13}$

$$A = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \\ c_{11} & c_{12} & d_{11} & d_{12} \\ c_{21} & c_{22} & d_{21} & d_{22} \end{pmatrix}.$$

Since $T_{12} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, $T_{13} = \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}$ the relations $[A, T_{13}] = 0$ and $[A, T_{12}] = 0$ gives us

$$[\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}] = 0 \quad \text{and} \quad [\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}] = 0,$$

or

$$\begin{pmatrix} AT & BT \\ CT & DT \end{pmatrix} = \begin{pmatrix} TA & TB \\ TC & TD \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} B & A \\ D & C \end{pmatrix} = \begin{pmatrix} C & D \\ A & B \end{pmatrix}.$$

The second relation gives us $A = D$ and $B = C$, the first relation gives $TA = AT$ and $TB = BT$ hence, we get

$$\begin{pmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{pmatrix} = \begin{pmatrix} a_{12} & a_{11} \\ a_{22} & a_{21} \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} b_{21} & b_{22} \\ b_{11} & b_{12} \end{pmatrix} = \begin{pmatrix} b_{12} & b_{11} \\ b_{22} & b_{21} \end{pmatrix}.$$

Finally, we get $A = D$, $B = C$, $a_{11} = a_{22}$, $a_{12} = a_{21}$, $b_{11} = b_{22}$, $b_{12} = b_{21}$ where $a_{11}, a_{12}, b_{11}, b_{12} \in \mathbb{C}$ hence,

$$A = \begin{pmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{12} & a_{11} & b_{12} & b_{11} \\ b_{11} & b_{12} & a_{11} & a_{12} \\ b_{12} & b_{11} & a_{12} & a_{11} \end{pmatrix} = a_{11} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + a_{12} \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} + b_{11} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} + b_{12} \begin{pmatrix} 0 & T \\ T & 0 \end{pmatrix}$$

$$= a_{11}(I \otimes I) + a_{12}(I \otimes T) + b_{11}(T \otimes I) + b_{12}(T \otimes T).$$

Finally, in the case of $p = 2$ the following statement is proved

Lemma 4.7. *The von Neumann algebra $L^\infty(T_{12}, T_{13})$ is maximal abelian, i.e., $(L^\infty(T_{12}, T_{13}))' = L^\infty(T_{12}, T_{13})$.*

In the case of an arbitrary $\mathbb{F}_p$ and the space $X^{1,n}$ denote by $L^\infty(T_{12}, \ldots, T_{1n})$ the von Neumann algebra generated by operetors T_{1k}, $2 \le k \le n$.

Lemma 4.8. *The von Neumann algebra $L^\infty(T_{12}, \ldots, T_{1n})$ is maximal abelian, i.e., $(L^\infty(T_{12}, \ldots, T_{1n}))' = L^\infty(T_{12}, \ldots, T_{1n})$. In other words, any operator $A \in (L^\infty(T_{12}, \ldots, T_{1n}))'$ has the following form:*

$$A = \sum_{i_2, i_3, \ldots, i_n \in \mathbb{F}_p} a_{i_2 i_3 \ldots i_n} T_{12}^{i_2} T_{13}^{i_3} \ldots T_{1n}^{i_n}. \tag{4.13}$$

PROOF. The proof follows from the fact that the von Neumann algebra $L^\infty(T_{1k})$ is maximal abelian, i.e., $(L^\infty(T_{1k}))' = L^\infty(T_{1k})$, by Theorem 3.7. Indeed, let us consider the operator $T = T_{inv} = \sum_{k\in\mathbb{F}_p} E_{k+1k}$ (see (3.10)) acting in the space $L^2(\mathbb{F}_p, \mu_{inv})$, then $T_{1k} = \underbrace{I \otimes I \otimes \cdots \otimes I \otimes T}_{k} \otimes I \otimes \cdots \otimes I$.

Finally,

$$(L^\infty(T_{12}, \ldots, T_{1n}))' = (T_{1k} \mid 2 \le k \le n)' = \bigcap_{k=2}^{n}(T_{1k})' = \otimes_{k=2}^{n}(T_{1k})'$$
$$= \otimes_{k=2}^{n}(L^\infty(T_{1k}))' = \otimes_{k=2}^{n}L^\infty(T_{1k}) = L^\infty(T_{12}, \ldots, T_{1n}).$$

$\square$

We calculate explicitly the commutant $(\mathfrak{A}^{1,n})'$ for $p = 2$ and small $n = 3, 4$ to guess the general rule.

Lemma 4.9. *In the case $p = 2$ the commutant $(\mathfrak{A}^{1,3})'$ of the von Neumann algebra $\mathfrak{A}^{1,3} = (T_{12}, T_{13}, T_{23})''$ is generated by operators $T_{12}(I + T_{13})$ and T_{13} or by $T_{12}C(T_{13})$ and T_{13}:*

$$(\mathfrak{A}^{1,3})' = (T_{12}C(T_{13}), T_{13})''.$$

PROOF. Let $A \in (\mathfrak{A}^{1,3})'$, since $(\mathfrak{A}^{1,3})' = (T_{12}, T_{13})' \bigcap (T_{23})'$ so, by Lemma 4.8,

$$(\mathfrak{A}^{1,3})' = (A = aI + bT_{13} + cT_{12} + dT_{12}T_{13} \mid [A, T_{23}] = 0).$$

The operator T_{23} has the following form (see (5.4))

$$T_{23} = \begin{pmatrix} I & 0 \\ 0 & T \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

We prove that for $k < r < s$ holds

$$T_{kr}T_{rs} = T_{rs}T_{kr}T_{ks}, \quad T_{kr}^v T_{rs} = T_{rs}T_{kr}^v T_{ks}^v, \; v \in \mathbb{F}_p, \tag{4.14}$$

in particular, for $(k, r, s) = (1, 2, 3)$ we have $T_{12}T_{23} = T_{23}T_{12}T_{13}$. Let us denote $E_{kr}(t) = I + tE_{kr}, \; t \in \mathbb{F}_p, \; k < r$, then we have by (2.7) $T_{kn}^{-1} := T_{E_{kn}(1)}^{R,\mu_\alpha,m}$. We calculate

$$E_{12}(t)E_{23}(s) = \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & t & ts \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & ts \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

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$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & ts \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = E_{23}(s)E_{12}(t)E_{13}(st),$$

hence, $E_{12}(t)E_{23}(s) = E_{23}(s)E_{12}(t)E_{13}(st)$, or if we take $s = t = 1$ we get (4.14) for $k = 1$, $r = 2$, $s = 3$, $v = 1$. The proof for an arbitrary $v \in \mathbb{F}_p$ is similar.

Using (4.14) we get for $A = a_{00}I + a_{01}T_{13} + a_{10}T_{12} + a_{11}T_{12}T_{13}$:

$$AT_{23} = a_{00}T_{23} + a_{01}T_{13}T_{23} + T_{23}T_{12}T_{13}(a_{10} + a_{11}T_{13}) =$$

$$a_{00}T_{23} + a_{01}T_{13}T_{23} + T_{23}T_{12}(a_{10}T_{13} + a_{11}T_{13}^2) =$$

$$a_{00}T_{23} + a_{001}T_{13}T_{23} + T_{23}T_{12}(a_{10}T_{13} + a_{11}).$$

Similarly, we have

$$T_{23}A = a_{00}T_{23} + a_{01}T_{13}T_{23} + T_{23}T_{12}(a_{10} + a_{11}T_{13}).$$

The condition $AT_{23} = T_{23}A$ gives us $(a_{10}T_{13} + a_{11}) = (a_{10} + a_{11}T_{13})$ or $a_{10} = a_{11}$. Hence, $A = a_{00}I + a_{01}T_{13} + a_{10}T_{12}(I + T_{13})$ and lemma is proved. $\square$

Lemma 4.10. *The commutant $(\mathfrak{A}^{1,4})'$ of the von Neumann algebra $\mathfrak{A}^{1,4}$ is generated by operators $T_{12}(I + T_{13})(I + T_{14})$, $T_{13}(I + T_{14})$ and T_{14} or by operators $T_{12}C(T_{13})C(T_{14})$, $T_{13}C(T_{14})$ and T_{14}:*

$$(\mathfrak{A}^{1,4})' = (T_{12}C(T_{13})C(T_{14}), \ T_{13}C(T_{14}), \ T_{14})''.$$

PROOF. Let $A \in (\mathfrak{A}^{1,4})'$, since $\mathfrak{A}^{1,4} = (T_{12}, T_{13}, T_{14}, T_{23}, T_{24}, T_{34})''$ and $T_{23}T_{34} = T_{34}T_{23}T_{24}$ or $\{T_{23}, T_{34}\} = T_{24}$, where $\{a, b\} := aba^{-1}b^{-1}$ we conclude that $(\mathfrak{A}^{1,4})' = (T_{12}, T_{13}, T_{14})' \bigcap (T_{23}, T_{34})'$ so,

$$(\mathfrak{A}^{1,4})' = (A \in (T_{12}, T_{13}, T_{14})' \mid [A, T_{23}] = [A, T_{34}] = 0).$$

Using Lemma 4.8 we have for $A \in (T_{12}, T_{13}, T_{14})'$, $\quad A =$

$a_{000} + a_{100}T_{12} + a_{010}T_{13} + a_{001}T_{14} + a_{110}T_{12}T_{13} + a_{101}T_{12}T_{14} + a_{011}T_{13}T_{14} + a_{111}T_{12}T_{13}T_{14}$

$= a_{000} + a_{010}T_{13} + a_{001}T_{14} + a_{011}T_{13}T_{14} + T_{12}(a_{100} + a_{110}T_{13} + a_{101}T_{14} + a_{111}T_{13}T_{14})$

$= a_{000} + a_{100}T_{12} + a_{001}T_{14} + a_{101}T_{12}T_{14} + T_{13}(a_{010} + a_{110}T_{12} + a_{011}T_{14} + a_{111}T_{12}T_{14}).$

The condition $[A, T_{23}] = 0$ gives us

$$0 = AT_{23} - T_{23}A = T_{12}T_{23}(a_{100} + a_{110}T_{13} + a_{101}T_{14} + a_{111}T_{13}T_{14})$$

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$$-T_{23}T_{12}\left(a_{100} + a_{110}T_{13} + a_{101}T_{14} + a_{111}T_{13}T_{14}\right).$$

Since $T_{12}T_{23} = T_{23}T_{12}T_{13}$, we have

$$T_{12}T_{23}\left(a_{100} + a_{110}T_{13} + a_{101}T_{14} + a_{111}T_{13}T_{14}\right)$$

$$= T_{23}T_{12}T_{13}\left(a_{100} + a_{110}T_{13} + a_{101}T_{14} + a_{111}T_{13}T_{14}\right)$$

$$= T_{23}T_{12}\left(a_{100}T_{13} + a_{110} + a_{101}T_{13}T_{14}. + a_{111}T_{14}\right).$$

Therefore,

$$a_{100}T_{13} + a_{110} + a_{101}T_{13}T_{14} + a_{111}T_{14} = a_{100} + a_{110}T_{13} + a_{101}T_{14} + a_{111}T_{13}T_{14}$$

hence,

$$a_{100} = a_{110}, \ a_{101} = a_{111}. \tag{4.15}$$

Similarly, we get using condition $[A, T_{34}] = 0$,

$$0 = AT_{34} - T_{34}A = T_{13}T_{34}\left(a_{010} + a_{110}T_{12} + a_{011}T_{14} + a_{111}T_{12}T_{14}\right) -$$

$$T_{34}T_{13}\left(a_{010} + a_{110}T_{12} + a_{011}T_{14} + a_{111}T_{12}T_{14}\right).$$

Since $T_{23}T_{34} = T_{34}T_{23}T_{24}$ we have

$$T_{13}T_{34}\left(a_{010} + a_{110}T_{12} + a_{011}T_{14} + a_{111}T_{12}T_{14}\right)$$

$$= T_{34}T_{13}T_{14}\left(a_{010} + a_{110}T_{12} + a_{011}T_{14} + a_{111}T_{12}T_{14}\right)$$

$$= T_{34}T_{13}\left(a_{010}T_{14} + a_{110}T_{12}T_{14} + a_{011} + a_{111}T_{12}\right)$$

hence,

$$a_{010}T_{14} + a_{110}T_{12}T_{14} + a_{011} + a_{111}T_{12} = a_{010} + a_{110}T_{12} + a_{011}T_{14} + a_{111}T_{12}T_{14},$$

and finally, we get

$$a_{010} = a_{011}, \ a_{110} = a_{111}. \tag{4.16}$$

Using (4.15) and (4.16) we conclude $a_{100} = a_{110} = a_{101} = a_{111}$, $a_{010} = a_{011}$ hence,

$$A = a_{000}I + a_{100}T_{12}(I + T_{13} + T_{14} + T_{13}T_{14}) + a_{010}T_{13}(I + T_{14}) + a_{001}T_{14}$$

$$= a_{000}I + a_{100}T_{12}(I + T_{13})(I + T_{14}) + a_{010}T_{13}(I + T_{14}) + a_{001}T_{14}.$$

$$\square$$

The previous lemmas were proved for $\mathbb{F}_2$ and von Neumann algebras $\mathfrak{A}^{1,n}$ with $n = 3$, $n = 4$. For the general case $\mathfrak{A}^{1,n}$ and arbitrary $\mathbb{F}_p$ we can guess

Lemma 4.11. *The commutant $(\mathfrak{A}^{1,n})'$ of the von Neumann algebra $\mathfrak{A}^{1,n}$ as the linear space is generated by the following operators (we set $\Delta_{n,n}^{1,r} := T_{1n}^r$):*

$$(\mathfrak{A}^{1,n})' = \left(\Delta_{s,n}^{1,r} := T_{1s}^r \prod_{k=s+1}^{n} p^{-1} C(T_{1k}) \mid 2 \leq s \leq n,\, r \in \mathbb{F}_p \setminus \{0\} \right)''. \quad (4.17)$$

The dimension of the von Neumann algebra $(\mathfrak{A}^{1,n})'$ equals to $(n-1)(p-1)+1$.

PROOF. We prove the statement first for $p = 3$ and $n = 3$. Any operator $A \in L^\infty(T_{12}, T_{13}, T_{14})$ can be expressed as follows:

$$A = \sum_{i_1,i_2,i_3 \in \mathbb{F}_3} a_{i_1,i_2,i_3} T_{12}^{i_1} T_{13}^{i_2} T_{14}^{i_3}.$$

Rewrite the operator A in the following form:

$$A = \sum_{i_2,i_3 \in \mathbb{F}_3} a_{0,i_2,i_3} T_{13}^{i_2} T_{14}^{i_3} + T_{12} \sum_{i_2,i_3 \in \mathbb{F}_3} a_{1,i_2,i_3} T_{13}^{i_2} T_{14}^{i_3} + T_{12}^2 \sum_{i_2,i_3 \in \mathbb{F}_3} a_{2,i_2,i_3} T_{13}^{i_2} T_{14}^{i_3},$$

$$A = \sum_{i_1,i_3 \in \mathbb{F}_3} a_{i_1,0,i_3} T_{12}^{i_2} T_{14}^{i_3} + T_{13} \sum_{i_1,i_3 \in \mathbb{F}_3} a_{i_1,1,i_3} T_{12}^{i_2} T_{14}^{i_3} + T_{13}^2 \sum_{i_1,i_3 \in \mathbb{F}_3} a_{i_1,2,i_3} T_{12}^{i_2} T_{14}^{i_3}.$$

Using (4.14) we get

$$T_{12}^r T_{23} = T_{23} T_{12}^r T_{13}^r, \quad T_{13}^r T_{34} = T_{34} T_{13}^r T_{14}^r.$$

Using the relation $A T_{23} = T_{23} A$, we conclude that

$$T_{13} \sum_{i_2,i_3 \in \mathbb{F}_3} a_{1,i_2,i_3} T_{13}^{i_2} T_{14}^{i_3} = \sum_{i_2,i_3 \in \mathbb{F}_3} a_{1,i_2,i_3} T_{13}^{i_2} T_{14}^{i_3},$$

$$T_{13}^2 \sum_{i_2,i_3 \in \mathbb{F}_3} a_{2,i_2,i_3} T_{13}^{i_2} T_{14}^{i_3} = \sum_{i_2,i_3 \in \mathbb{F}_3} a_{2,i_2,i_3} T_{13}^{i_2} T_{14}^{i_3},$$

therefore, we get

$$a_{1,i_2,i_3} = a_{1,i_2+1,i_3}, \quad a_{2,i_2,i_3} = a_{2,i_2+2,i_3}, \quad \forall i_2 \in \mathbb{F}_3. \quad (4.18)$$

Using the relation $AT_{34} = T_{34}A$, we conclude that

$$T_{14} \sum_{i_1,i_3 \in \mathbb{F}_3} a_{i_1,1,i_3} T_{12}^{i_1} T_{14}^{i_3} = \sum_{i_1,i_3 \in \mathbb{F}_3} a_{i_1,1,i_3} T_{12}^{i_1} T_{14}^{i_3},$$

$$T_{14}^2 \sum_{i_1,i_3 \in \mathbb{F}_3} a_{i_1,2,i_3} T_{12}^{i_1} T_{14}^{i_3} = \sum_{i_1,i_3 \in \mathbb{F}_3} a_{i_1,2,i_3} T_{12}^{i_1} T_{14}^{i_3},$$

$$\text{so,} \quad a_{i_1,1,i_3} = a_{i_1,1,i_3+1}, \ a_{i_1,2,i_3} = a_{i_1,2,i_3+2}, \ \forall i_3 \in \mathbb{F}_3. \qquad (4.19)$$

Using (4.18) and (4.19) we conclude that

$$a_{2i_2i_3} = a_{200}, \ a_{1i_2i_3} = a_{100}, \ \forall i_2, i_3, \in \mathbb{F}_3, \ a_{01i_3} = a_{010}, \ a_{02i_3} = a_{020}, \ \forall i_2 \in \mathbb{F}_3.$$

This implies that A has the following form:

$$A = a_{000}I + a_{001}T_{14} + a_{002}T_{14}^2 + a_{010}T_{13}C(T_{14}) + a_{020}T_{13}^2 C(T_{14})+$$

$$a_{100}T_{12}C(T_{13})C(T_{14}) + a_{200}T_{12}^2 C(T_{13})C(T_{14}).$$

To prove the statement for general p and n, set $T_k = T_{1k+1}, \ k = 1, 2, \ldots, n$.
Let an operator A has the following form:

$$A = \sum_{i_1,i_2,\ldots,i_n \in \mathbb{F}_p} a_{i_1,i_2,\ldots,i_n} T_1^{i_1} T_2^{i_2} \ldots T_n^{i_n}.$$

Rewrite the operator A in the following form for all r:

$$A = \sum_{i_r \in \mathbb{F}_p} T_r^{i_r} \sum_{i_1,\ldots,\hat{i}_r,\ldots,i_n \in \mathbb{F}_p} a_{i_1,\ldots,\hat{i}_r,\ldots,i_n} T_1^{i_1} \ldots \hat{T}_r \ldots T_n^{i_n}, \quad 1 \le r \le n$$

where $\hat{T}_r$, (resp. $\hat{i}_r$) means that factor T_r (resp, index $\hat{i}_r$) is absent in the expression. The commutation relations $AT_{kk+1} = T_{kk+1}A$ for $1 \le k \le n$ imply, as before, the following relations

$$T_2^r \sum_{i_2,\ldots,i_n \in \mathbb{F}_p} a_{r,i_2,\ldots,i_n} T_2^{i_2} \ldots T_n^{i_n} = \sum_{i_2,\ldots,i_n \in \mathbb{F}_p} a_{r,i_2,\ldots,i_n} T_2^{i_2} \ldots T_n^{i_n},$$

$$T_3^r \sum_{i_1,\hat{i}_2,i_3\ldots,i_n \in \mathbb{F}_p} a_{i_1,r,i_3,\ldots,i_n} T_1^{i_1} \hat{T}_2 T_3^{i_3} \ldots T_n^{i_n} = \sum_{i_1,\hat{i}_2,i_3\ldots,i_n \in \mathbb{F}_p} a_{i_1,r,i_3,\ldots,i_n} T_1^{i_1} \hat{T}_2 T_3^{i_3} \ldots T_n^{i_n}.$$

In the general case of r, $1 \le r \le n$ we get

$$T_r^s \sum_{i_1,\ldots,\hat{i_r},\ldots,i_n \in \mathbb{F}_p} a_{i_1,\ldots,r,i_{r+1},\ldots,i_n} T_1^{i_1}\ldots\hat{T}_r\ldots T_n^{i_n} = \sum_{i_1,\ldots,\hat{i_r},\ldots,i_n \in \mathbb{F}_p} a_{i_1,\ldots,r,i_{r+1},\ldots,i_n} T_1^{i_1}\ldots\hat{T}_r\ldots T_n^{i_n}.$$

The previous relations implies the analog of the relations (4.18) and (4.19):

$$a_{r,i_2,\ldots,i_n} = a_{r,i_2+r,\ldots,i_n} \ \forall r, \ i_2 \in \mathbb{F}_p, \quad a_{i_1,r,i_3,\ldots,i_n} = a_{i_1,r,i_3+r,\ldots,i_n} \ \forall r, \ i_3 \in \mathbb{F}_p,$$
$$\text{(4.20)}$$
$$a_{i_1,\ldots,r,i_{r+1},\ldots,i_n} = a_{i_1,\ldots,r,i_{r+1}+r,\ldots,i_n} \ \forall r, \ i_{r+1} \in \mathbb{F}_p. \qquad \text{(4.21)}$$

Using (4.20) and (4.21) we conclude that

$$a_{r,i_2,\ldots,i_n} = a_{r,0,\ldots,0}, \quad \forall i_2,\ldots,i_n \in \mathbb{F}_p, \quad a_{0,r,i_3,\ldots,i_n} = a_{0,r,0,\ldots,0}, \ \forall i_3,\ldots,i_n \in \mathbb{F}_p,$$

$$a_{0,\ldots,0,r,i_{r+1},\ldots,i_n} = a_{0,\ldots,0,r,0,\ldots,0} \ \forall i_{r+1},\ldots,i_n \in \mathbb{F}_p.$$

This implies that A has the following form:

$$A = \sum_{r \in \mathbb{F}_p} a_{0,\ldots,0,r} T_n^r + \sum_{r \in \mathbb{F}_p\backslash\{0\}} a_{0,\ldots,0,r,0} T_{n-1}^r C(T_n) + \cdots + \sum_{r \in \mathbb{F}_p\backslash\{0\}} a_{r,0,\ldots,0} T_1^r C(T_2)\ldots C(T_n).$$

$$\square$$

Remark 4.5. We have proved in the previous lemma that the von Neumann algebra $(\mathfrak{A}^{1,n})'$ as the linear space is generated by the operators $\delta_{s,n}^{1,r}$:

$$(\mathfrak{A}^{1,n})' = \left(\delta_{s,n}^{1,r} := T_{1s}^r \prod_{k=s+1}^{n} C(T_{1k}) \mid 2 \le s \le n, \ r \in \mathbb{F}_p \backslash \{0\}\right)''.$$

But the uniform limit $\lim_n \delta_{s,n}^{1,r}$ is divergent, since

$$\| \prod_{k=s+1}^{n} C(T_{1k})\| = \prod_{k=s+1}^{n} \|C(T_{1k})\| = p^{n-s} \to \infty, \quad \text{when} \quad n \to \infty.$$

We use the fact that $\|C(T)\| = p$. Instead of the basis $\delta_{s,n}^{1,r}$, we choose the basis $\Delta_{s,n}^{1,r}$ of the algebra $(\mathfrak{A}^{1,n})'$ in Lemma 4.11 to be shur that the limit $\lim_n \Delta_{s,n}^{1,r}$ is correctly defined. Consider again the expression $C(T) = \sum_{k \in \mathbb{F}_p} T^k$. Since $C(T)T = TC(T) = C(T)$ we get $C(T)^2 = pC(T)$ so, $C(T)$ is "almost projector", i.e., $A^2 = \lambda A$. The operator $C(T)$ has two eigenvalues,

$\lambda_1 = 0$ and $\lambda_2 = p$. Indeed, if $C(T)f = \lambda f$, then $\lambda^2 f = p\lambda f$ so, $\lambda(\lambda - p) = 0$. Therefore, $\|C(T)\| = \max\{0, p\} = p$.

$$\text{Set} \quad c(T) = p^{-1}C(T). \quad \text{We have} \quad c^2(T) = c(T) \tag{4.22}$$

therefore, the eigenvalues of $c(T)$ are 0 and 1 hence, $\|c(T)\| = 1$ so, the operator $\Delta_{s,\infty}^{1,r} = \lim_n \Delta_{s,n}^{1,r}$, at least formally, is correctly defined since

$$\|\Delta_{s,\infty}^{1,r}\| = \lim_n \|\Delta_{s,n}^{1,r}\| = \lim_n \|T_{1s}^r \prod_{k=s+1}^{n} c(T_{1k})\| = \|T_{1s}^r\| \prod_{k=s+1}^{n} \|c(T_{1s})\| = 1.$$

In Lemma 4.14 below we prove that the operator $\lim_n \Delta_{s,n}^{1,r}$ is correctly defined when $\mu_\alpha \sim \mu_{inv}$.

Remark 4.6. The von Neuman algebra $(\mathfrak{A}^{1,n})'$ as an algebra is generated by the following expressions:

$$(\mathfrak{A}^{1,n})' = \left(\Delta_{s,n}^{1,1} \mid 2 \le s \le n\right)''.$$

PROOF. It is sufficient to use Lemma 4.11 and the following relations:

$$\Delta_{s,n}^{1,r}\Delta_{s,n}^{1,t} = \Delta_{s,n}^{1,r+t}, \quad \Delta_{s_1,n}^{1,r_1}\Delta_{s_2,n}^{1,r_2} = \Delta_{s_1,n}^{1,r_1} \quad \text{for} \quad 3 \le s_1 < s_2 \le n.$$

Using the relation $c^2(T) = c(T)$ and $Tc(T) = c(T)$ we get $\Delta_{sn}^{1,r}\Delta_{sn}^{1,l} = \Delta_{sn}^{1,r+l}$. Indeed,

$$\Delta_{sn}^{1,r}\Delta_{sn}^{1,l} = T_{1s}^r \prod_{k=s+1}^{n} c(T_{1k})T_{1s}^l \prod_{t=s+1}^{n} c(T_{1t}) = T_{1s}^{r+l} \prod_{k=s+1}^{n} c^2(T_{1k}) = \Delta_{sn}^{1,r+l}.$$

Similarly, we prove the second relation $\Delta_{s_1,n}^{1,r_1}\Delta_{s_2,n}^{1,r_2} = \Delta_{s_1,n}^{1,r_1}$. $\qquad\square$

Another description of the commutant $(\mathfrak{A}^{1,n})'$. Any operator $A \in L^\infty(T_{12}, \ldots, T_{1n})$ has the following form by (4.13): $A = f(T_{12}, \ldots, T_{1n})$.

Lemma 4.12. *An operator* $A = f(T_{12}, \ldots, T_{1n}) \in L^\infty(T_{12}, \ldots, T_{1n})$ *commutes with* T_{kk+1} *for all* $2 \le k \le n-1$ *if and only if for all* $2 \le k \le n-1$ *holds:*

$$f(T_{12}, \ldots, T_{1k}, T_{1k+1}, \ldots, T_{1n}) = f(T_{12}, \ldots, T_{1k}T_{1k+1}, T_{1k+1}, \ldots, T_{1n}). \tag{4.23}$$

PROOF. Consider first the space $H_{1k} \otimes H_{1k+1}$ and the von Neumann subalgebra $L^\infty(T_{1k}, T_{1k+1})$ in the algebra $B(H_{1k} \otimes H_{1k+1}) = B(H_{1k}) \otimes B(H_{1k+1})$. Take the function $f(T_{1k}, T_{1k+1}) \in L^\infty(T_{1k}, T_{1k+1})$ of the following form $f(T_{1k}, T_{1k+1}) = T_{1k}^r T_{1k+1}^s$. We show that commutation relation $[f, T_{kk+1}] = 0$ implies the relation (4.23), i.e., $f(T_{1k}, T_{1k+1}) = f(T_{1k}T_{1k+1}, T_{1k+1})$. Indeed, using the relations (4.14) $T_{1k}^r T_{km} = T_{km} T_{1k}^r T_{1m}^r$ for $r \in \mathbb{F}_p$ and $2 \leq k < m$ and commutation relation $f(T_{1k}, T_{1k+1}) T_{kk+1} = T_{kk+1} f(T_{1k}, T_{1k+1})$ we get:

$$T_{kk+1} f(T_{1k}, T_{1k+1}) = f(T_{1k}, T_{1k+1}) T_{kk+1} = T_{1k}^r T_{1k+1}^s T_{kk+1} =$$
$$T_{kk+1} T_{1k}^r T_{1k+1}^r T_{1k+1}^s = T_{kk+1} f(T_{1k} T_{1k+1}, T_{1k+1}).$$

Finally, we prove $f(T_{1k}, T_{1k+1}) = f(T_{1k}T_{1k+1}, T_{1k+1})$ for the particalar case of $f(T_{1k}, T_{1k+1}) = T_{1k}^r T_{1k+1}^s$. For the general function $f(T_{1k}, T_{1k+1}) = \sum_{r,s \in \mathbb{F}_p} a_{r,s} \times T_{1k}^r T_{1k+1}^s$ the proof is the same. Similarly, we prove (4.23) for any function $\in L^\infty(T_{12}, \ldots, T_{1n})$:

$$f(T_{12}, \ldots, T_{1k}) = \sum_{i_2,\ldots,i_n \in \mathbb{F}_p} a_{i_2,\ldots,i_n} T_{12}^{i_2} \ldots T_{1n}^{i_n}.$$

$\square$

Lemma 4.13. *If for the function $f \in L^\infty(T_{12}, \ldots, T_{1n})$ holds relation (4.23) then*

$$f = f_s(T_{1s}) \prod_{k=s+1}^{n} c(T_{1k}) \quad \text{for some} \quad s, \ 2 \leq s \leq n, \qquad (4.24)$$

where $f_s(T_{1s}) \in L^\infty(T_{1s})$.

PROOF. If for the function $f(T_{1k}, T_{1k+1}) = \sum_{r,s \in \mathbb{F}_p} a_{r,s} T_{1k}^r T_{1k+1}^s$ holds $f(T_{1k}, T_{1k+1}) = f(T_{1k}T_{1k+1}, T_{1k+1})$ then we get

$$f(T_{1k}T_{1k+1}, T_{1k+1}) = \sum_{r,s \in \mathbb{F}_p} a_{r,s} (T_{1k}T_{1k+1})^r T_{1k+1}^s = \sum_{r,s \in \mathbb{F}_p} a_{r,s} T_{1k}^r T_{1k+1}^{r+s} =$$
$$\sum_{r,t \in \mathbb{F}_p} a_{r,t-r} T_{1k}^r T_{1k+1}^t = \sum_{r,t \in \mathbb{F}_p} a_{r,t} T_{1k}^r T_{1k+1}^t.$$

Therefore, $a_{r,t-r} = a_{r,t}$ for all $r, t \in \mathbb{F}_p$ hence, $a_{r,t} = a_{r,t-kr}$ for all $r, t, k \in \mathbb{F}_p$. Since $\mathbb{F}_p$ is a field, we conclude that

$$a_{r,t} = a_{r,0} \quad \text{for all} \quad r, t \in \mathbb{F}_p. \qquad (4.25)$$

Finally, if we set $f_k(T_{1k}) = \sum_{r \in \mathbb{F}_p} a_{r,0} T_{1k}^r$ we get $f =$

$$\sum_{r,s \in \mathbb{F}_p} a_{r,s} T_{1k}^r T_{1k+1}^s = \sum_{r,s \in \mathbb{F}_p} a_{r,0} T_{1k}^r T_{1k+1}^s = \sum_{r \in \mathbb{F}_p} a_{r,0} T_{1k}^r \sum_{s \in \mathbb{F}_p} T_{1k+1}^s = p^{-1} f_k(T_{1k}) c(T_{1k+1}).$$

If for the function $f(T_{1k}, T_{1k+1}, T_{1k+2}) = \sum_{r,s,t \in \mathbb{F}_p} a_{r,s,t} T_{1k}^r T_{1k+1}^s T_{1k+2}^t$ holds

$$f(T_{1k}, T_{1k+1}, T_{1k+2}) = f(T_{1k} T_{1k+1}, T_{1k+1}, T_{1k+2}) = f(T_{1k}, T_{1k+1} T_{1k+2}, T_{1k+2})$$

we conclude similarly, that $a_{r,s-r,t} = a_{r,s,t} = a_{r,s,t-s}$ for all $r, s, t \in \mathbb{F}_p$ hence,

$$a_{r,s,t} = a_{r,0,0} \quad \text{for all} \quad r, s, t \in \mathbb{F}_p. \tag{4.26}$$

Finally, we get

$$f = \sum_{r,s,t \in \mathbb{F}_p} a_{r,s,t} T_{1k}^r T_{1k+1}^s T_{1k+2}^t = \sum_{r,s,t \in \mathbb{F}_p} a_{r,0,0} T_{1k}^r T_{1k+1}^s T_{1k+2}^t = \sum_{r \in \mathbb{F}_p} a_{r,0,0} T_{1k}^r \times$$

$$\sum_{s \in \mathbb{F}_p} T_{1k+1}^s \sum_{t \in \mathbb{F}_p} T_{1k+2}^t = f_k(T_{1k}) C(T_{1k+1}) C(T_{1k+2}) = p^{-2} f_k(T_{1k}) c(T_{1k+1}) c(T_{1k+2}),$$

where $f_k(T_{1k}) = \sum_{r \in \mathbb{F}_p} a_{r,0,0} T_{1k}^r$. $\qquad\square$

Lemma 4.14. *When $\mu_\alpha \sim \mu_{inv}$ the commutant $(\mathfrak{A}^1)'$ of the von Neumann algebra $\mathfrak{A}^1$ is generated as a linear space by the following expressions:*

$$(\mathfrak{A}^1)' = \left(\Delta_{s,\infty}^{1,r} = \lim_n \Delta_{s,n}^{1,r} := T_{1s}^r \prod_{k=s+1}^{\infty} p^{-1} C(T_{1k}) \mid 2 \leq s, \ r \in \mathbb{F}_p \setminus \{0\} \right)''.$$

$$\tag{4.27}$$

When $\mu_\alpha \perp \mu_{inv} \Leftrightarrow S_{11}^L(\mu) = \infty$ the commutant $(\mathfrak{A}^1)'$ is trivial, i.e., $(\mathfrak{A}^1)' = (\lambda I \mid \lambda \in \mathbb{C})$.

PROOF. Denote by $L^\infty(T_1) = L^\infty(T_{1k} \mid 2 \leq k) = (T_{1k} \mid 2 \leq k)''$ the von Neumann algebra generated by the commuting family of operators $T_1 := (T_{1k} \mid 2 \leq k)$. By definition, we have

$$(\mathfrak{A}^1)' = \left(f \in L^\infty(T_1) \mid [f, T_{kk+1}] = 0, \quad 2 \leq k \right). \tag{4.28}$$

Using the spectral theorem for the family $T_1 = (T_{1k} \mid 2 \leq k)$ of commuting unitary operators T_{1k} we conclude that any element $f \in L^\infty(T_1)$ has the following form: $f(T_1) = \int_{X^1} f(\lambda) dE(\lambda)$ or

$$f(T_1) = f(T_{1k} \mid 2 \leq k) = \int_{X^1} f(\lambda) dE(\lambda) \tag{4.29}$$

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where $X^1 = \prod_{k=2}^{\infty} (\mathbb{F}_p)_k$, f is essentially bounded function on X^1 and E is a common resolution of the identity of the family of operators T_1 defined on cylidrycal sets $\Delta_2 \times \cdots \times \Delta_k$ as follows:

$$E(\Delta_2 \times \cdots \times \Delta_k) := E_2(\Delta_2) \cdots E_k(\Delta_k)$$

where E_k is resolution of the identity of the operators T_{1k}. See details in [5]. Similarly to the proof of Lemma 4.12 we get

Lemma 4.15. *An operator $A = f(T_{12}, \ldots, T_{1n}, \ldots) \in L^{\infty}(T_{1k} \mid 2 \leq k)$ defined by (4.29) commutes with T_{kk+1} for all $2 \leq k$ if and only if for all $2 \leq k$ holds:*

$$f(T_{12}, \ldots, T_{1k}, T_{1k+1}, \ldots, T_{1n}, \ldots) = f(T_{12}, \ldots, T_{1k}T_{1k+1}, T_{1k+1}, \ldots, T_{1n}, \ldots).$$
$$(4.30)$$

Lemma 4.16. *If for the function $f = f(T_1) \in L^{\infty}(T_{12}, \ldots, T_{1n}, \ldots)$ holds relation (4.30) then*

$$f = f_s(T_{1s}) \prod_{k=s+1}^{\infty} c(T_{1k}) \quad \text{for some} \quad s \geq 2, \qquad (4.31)$$

where $f_s(T_{1s}) \in L^{\infty}(T_{1s})$.

$\square$

4.3. The commutant of the von Neumann algebra $\mathfrak{A}^m$, case $m > 1$

Let us consider the restriction $T^{R,m,n}$ of the representation $T^{R,\mu_\alpha,m}$: $B_0^{\mathbb{N}}(\mathbb{F}_p) \mapsto U(L^2(X^m, \mu_\alpha))$ to the subgroup $B(n, \mathbb{F}_p)$, $m \leq n$, of the group $B_0^{\mathbb{N}}(\mathbb{F}_p)$ acting in the space $H^{m,n} = \otimes_{1 \leq r \leq m, r < k \leq n} L^2(\mathbb{F}_p, \mu_{\alpha_{rk}}) = L^2(X^{m,n}, \mu_{\alpha^{m,n}})$ where $\mu_{\alpha^{m,n}} = \otimes_{1 \leq r \leq m, r < k \leq n} \mu_{\alpha_{rk}}$ and

$$X^{m,n} = \begin{pmatrix} 1 & x_{12} & x_{13} & \ldots & x_{1m} & \ldots & x_{1n} \\ 0 & 1 & x_{23} & \ldots & x_{2m} & \ldots & x_{2n} \\ 0 & 0 & 1 & \ldots & x_{3m} & \ldots & x_{3n} \\ & & & \ldots & & \ldots & \\ 0 & 0 & & \ldots & x_{m-1m} & \ldots & x_{m-1n} \\ 0 & 0 & & \ldots & 1 & \ldots & x_{mn} \end{pmatrix}. \qquad (4.32)$$

Let us denote as before by $\mathfrak{A}^{m,n}$ and $\mathfrak{A}^m$ the von-Neumann algebras generated by the representation $T^{R,m,n}$ (respectively by $T^{R,\mu_\alpha,m}$)

$$\mathfrak{A}^{m,n} \left(T_t^{R,m,n} \mid t \in B(n, \mathbb{F}_p) \right)'', \quad \mathfrak{A}^m = \left(T_t^{R,\mu_\alpha,m} \mid t \in B_0^{\mathbb{N}}(\mathbb{F}_p) \right)'' = \left(\bigcup_{n \geq m} \mathfrak{A}^{m,n} \right)''.$$

Obviously, the commutant $(\mathfrak{A}^{m,n})'$ contains the following operators:

$$(\mathfrak{A}^{m,n})' \supset \left(\Delta_{s,n}^{k,s} := T_s^r \prod_{k=s+1}^{n} p^{-1}C(T_{kr}(k)) \mid 1 \leq k \leq m, \ k+1 \leq s \leq n\right)''.$$

Since $\mathfrak{A}^m = (\bigcup_{n \geq m+1} \mathfrak{A}^{m,n})''$ so, $(\mathfrak{A}^m)' = \bigcap_{n \geq m+1} (\mathfrak{A}^{m,n})'$. Hence, the commutant $(\mathfrak{A}^m)'$ is not trivial if there exist a non trivial limit Δ_k, $1 \leq k \leq m$

$$\Delta_k := \lim_{n \to \infty} \prod_{r=k+1}^{n} p^{-1}C(T_{kr}(k)) = \lim_{n \to \infty} \Delta_{k,n}^{k,0}.$$

The latter limit exists, if $\mu_\alpha^k \sim \mu_{inv}^k \Leftrightarrow S_{kk}^L(\mu_\alpha) < \infty$ (see Lemma 4.6).

4.4. Commutant of the von Neumann algebra $(\mathfrak{A}^m)'$, case $m = 2$

Set

$$\Delta_{s,n}^{2,r} := T_{\alpha_{2s}}^r \prod_{k=s+1}^{n} p^{-1}C(T_{\alpha_{2k}}), \quad 3 \leq s < n, \quad \Delta_{n,n}^{2,r} := T_{\alpha_{2n}}^r, \quad r \in \mathbb{F}_p.$$

Lemma 4.17. *Let $x_{12} \in \mathfrak{A}^{2,n}$, then the commutant $(\mathfrak{A}^{2,n})'$ of the von Neumann algebra $\mathfrak{A}^{2,n}$ as the linear space is generated by the following operators:*

$$(\mathfrak{A}^{2,n})' = \left(\Delta_{s,n}^{l,r} := T_{1s}^r \prod_{r=s+1}^{n} p^{-1}C(T_{1r}) \mid 1 \leq l \leq 2, \ 3 \leq s \leq n, \ r \in \mathbb{F}_p \backslash \{0\}\right)''.$$

(4.33)

The dimension of the von Neumann algebra $(\mathfrak{A}^{2,n})'$ equals to $[(n{-}1)(p{-}1){+}1]^2$.

PROOF. Since x_{12}, T_{1k}, $T_{2k} \in \mathfrak{A}^{2,n}$ we conclude that $T_{2k}(2) = T_{\alpha_{2k}} \in \mathfrak{A}^{2,n}$ (see Remark 6.5). Since the commutative family of operators with common simple spectrum lies in $\mathfrak{A}^{2,n}$, i.e., x_{12}, T_{1k}, $T_{\alpha_{2k}} \in \mathfrak{A}^{2,n}$ for $3 \leq k \leq n$ we conclude that

$$(\mathfrak{A}^{2,n})' \subset L^\infty \begin{pmatrix} x_{12} \ T_{\alpha_{13}} \ \cdots \ T_{\alpha_{1n}} \\ T_{\alpha_{23}} \ \cdots \ T_{\alpha_{2n}} \end{pmatrix}. \qquad (4.34)$$

Commutation relation $[f, T_{12}] = 0$ for $f \in L^\infty \begin{pmatrix} x_{12} \ T_{\alpha_{13}} \ \cdots \ T_{\alpha_{1n}} \\ T_{\alpha_{23}} \ \cdots \ T_{\alpha_{2n}} \end{pmatrix}$ means that f does not depend on x_{12}. Fix $n = 4$ and $p = 2$, let $f\begin{pmatrix} T_{\alpha_{13}} \ T_{\alpha_{14}} \\ T_{\alpha_{23}} \ T_{\alpha_{24}} \end{pmatrix} \in L^\infty \begin{pmatrix} T_{\alpha_{13}} \ T_{\alpha_{14}} \\ T_{\alpha_{23}} \ T_{\alpha_{24}} \end{pmatrix}$. We use the following relations (see (2.8))

$$T_{34} = T_{34}(1) \otimes T_{34}(1), \ T_{\alpha_{13}} T_{34}(1) = T_{34}(1) T_{\alpha_{13}} T_{\alpha_{14}}, \ T_{\alpha_{23}} T_{34}(2) = T_{34}(2) T_{\alpha_{23}} T_{\alpha_{24}}$$

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or $\quad [T_{\alpha_{13}}, T_{34}(1)] = T_{34}(1)T_{\alpha_{13}}(T_{\alpha_{14}} - I), \quad [T_{\alpha_{23}}, T_{34}(2)] = T_{34}(2)T_{\alpha_{23}}(T_{\alpha_{24}} - I).$
$$\tag{4.35}$$

Any operator $A \in L^\infty \begin{pmatrix} T_{\alpha_{13}} & T_{\alpha_{14}} \\ T_{\alpha_{23}} & T_{\alpha_{24}} \end{pmatrix}$ has the form

$$A = \sum_{i_1, i_2, j_1, j_2 \in \mathbb{F}_2} a^{i_1, i_2}_{j_1, j_2} T^{i_1}_{\alpha_{13}} T^{i_2}_{\alpha_{14}} T^{j_1}_{\alpha_{23}} T^{j_2}_{\alpha_{24}}.$$

We rewrite A as follows:

$$\begin{aligned}
A &= [a^{00}_{00}I + a^{01}_{00}T_{\alpha_{14}} + a^{00}_{01}T_{\alpha_{24}} + a^{01}_{01}T_{\alpha_{14}}T_{\alpha_{24}}] \\
&\quad + T_{\alpha_{23}}[(a^{00}_{10} + a^{00}_{11}T_{\alpha_{24}}) + T_{\alpha_{14}}(a^{01}_{10} + a^{01}_{11}T_{\alpha_{24}})] \\
&\quad + T_{\alpha_{13}}[(a^{10}_{00} + a^{11}_{00}T_{\alpha_{14}}) + T_{\alpha_{24}}(a^{10}_{01} + a^{11}_{01}T_{\alpha_{14}})] \\
&\quad + T_{\alpha_{13}}T_{\alpha_{23}}[(a^{10}_{10} + a^{10}_{11}T_{\alpha_{24}} + a^{11}_{10}T_{\alpha_{14}} + a^{11}_{11}T_{\alpha_{14}}T_{\alpha_{24}})] \\
&= A_1 + A_2 + A_3 + A_4 \\
&= A_1 + T_{\alpha_{23}}a_2 + T_{\alpha_{13}}a_3 + T_{\alpha_{13}}T_{\alpha_{23}}a_4
\end{aligned}$$

where

$$\begin{aligned}
A_1 &= [a^{00}_{00}I + a^{01}_{00}T_{\alpha_{14}} + a^{00}_{01}T_{\alpha_{24}} + a^{01}_{01}T_{\alpha_{14}}T_{\alpha_{24}}], \\
a_2 &= [(a^{00}_{10} + a^{00}_{11}T_{\alpha_{24}}) + T_{\alpha_{14}}(a^{01}_{10} + a^{01}_{11}T_{\alpha_{24}})], \\
a_3 &= [(a^{10}_{00} + a^{11}_{00}T_{\alpha_{14}}) + T_{\alpha_{24}}(a^{10}_{01} + a^{11}_{01}T_{\alpha_{14}})], \\
a_4 &= [(a^{10}_{10} + a^{10}_{11}T_{\alpha_{24}} + a^{11}_{10}T_{\alpha_{14}} + a^{11}_{11}T_{\alpha_{14}}T_{\alpha_{24}})].
\end{aligned}$$

Since $T_{34}(1) - I = x_{13} \otimes (T_{\alpha_{14}} - I), \ T_{34}(2) - I = x_{23} \otimes (T_{\alpha_{24}} - I)$ (see (5.5) and (6.8)) we get

$$[A_1, T_{34}(1)] = [A_2, T_{34}(1)] = [T_{34}(2), A_1] = [T_{34}(2), A_3] = 0$$

hence,

$$\begin{aligned}
&[A, T_{34}(1) \otimes T_{34}(2)] = [A, T_{34}(1)] \otimes T_{34}(2) + T_{34}(1) \otimes [A, T_{34}(2)] \\
&= \big([A_3, T_{34}(1)] + [A_4, T_{34}(1)]\big) \otimes T_{34}(2) \\
&\quad + T_{34}(1) \otimes \big([A_2, T_{34}(2)] + [A_4, T_{34}(2)]\big) \\
&= \big([T_{\alpha_{13}}, T_{34}(1)]a_3 + [T_{\alpha_{13}}, T_{34}(1)]T_{\alpha_{23}}a_4\big) \otimes T_{34}(2) \\
&\quad + T_{34}(1) \otimes \big([T_{\alpha_{23}}, T_{34}(2)]a_2 + T_{\alpha_{13}}[T_{\alpha_{23}}, T_{34}(2)]a_4\big) \\
&= \big(T_{34}(1)T_{\alpha_{13}}(T_{\alpha_{14}} - I)a_3 + T_{34}(1)T_{\alpha_{13}}T_{\alpha_{23}}(T_{\alpha_{14}} - I)a_4\big) \otimes T_{34}(2) \\
&\quad + T_{34}(1) \otimes \big(T_{34}(2)T_{\alpha_{23}}(T_{\alpha_{24}} - I)a_2 + T_{\alpha_{13}}T_{34}(1)T_{\alpha_{23}}(T_{\alpha_{24}} - I)a_4\big) = 0.
\end{aligned}$$

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Therefore, we have

$$(T_{\alpha_{14}} - I)a_3 = (T_{\alpha_{14}} - I)a_4 = (T_{\alpha_{24}} - I)a_2 = (T_{\alpha_{24}} - I)a_4 = 0$$

or

$$(T_{\alpha_{14}} - I)[(a_{00}^{10} + a_{00}^{11}T_{\alpha_{14}}) + T_{\alpha_{24}}(a_{01}^{10} + a_{01}^{11}T_{\alpha_{14}})] = 0,$$
$$(T_{\alpha_{14}} - I)[(a_{10}^{10} + a_{11}^{10}T_{\alpha_{24}} + a_{10}^{11}T_{\alpha_{14}} + a_{11}^{11}T_{\alpha_{14}}T_{\alpha_{24}})] = 0,$$
$$(T_{\alpha_{24}} - I)[(a_{10}^{00} + a_{11}^{00}T_{\alpha_{24}}) + T_{\alpha_{14}}(a_{10}^{01} + a_{11}^{01}T_{\alpha_{24}})] = 0,$$
$$(T_{\alpha_{24}} - I)[(a_{10}^{10} + a_{11}^{10}T_{\alpha_{24}} + a_{10}^{11}T_{\alpha_{14}} + a_{11}^{11}T_{\alpha_{14}}T_{\alpha_{24}})] = 0.$$

Hence, we get respectively

$$a_{00}^{10} = a_{00}^{11}, \quad a_{01}^{10} = a_{01}^{11}, \quad a_{10}^{10} = a_{10}^{11}, \quad a_{11}^{10} = a_{11}^{11},$$
$$a_{10}^{00} = a_{11}^{00}, \quad a_{10}^{01} = a_{11}^{01}, \quad a_{10}^{10} = a_{11}^{10}, \quad a_{10}^{11} = a_{11}^{11}.$$

At last, using the latter equalities we get

$$A = [a_{00}^{00}I + a_{00}^{01}T_{\alpha_{14}} + a_{01}^{00}T_{\alpha_{24}} + a_{01}^{01}T_{\alpha_{14}}T_{\alpha_{24}}]$$
$$+ T_{\alpha_{23}}[a_{10}^{00}(I + T_{\alpha_{24}}) + a_{10}^{01}T_{\alpha_{14}}(I + T_{\alpha_{24}})]$$
$$+ T_{\alpha_{13}}[a_{00}^{10}(I + T_{\alpha_{14}}) + a_{01}^{10}T_{\alpha_{24}}(I + T_{\alpha_{14}})]$$
$$+ T_{\alpha_{13}}T_{\alpha_{23}}a_{10}^{10}[I + T_{\alpha_{24}} + T_{\alpha_{14}} + T_{\alpha_{14}}T_{\alpha_{24}}].$$

Finally, the basis in the algebra $(\mathfrak{A}^{2,4})'$ can be obtained as a tensor product of two von Neumann algebras

$$(\mathfrak{A}^{1,3,4})' = L^\infty(T_{13}, T_{14}) \bigcap (T_{34})' \quad \text{and} \quad (\mathfrak{A}^{2,3,4})' = L^\infty(T_{\alpha_{23}}, T_{\alpha_{24}}) \bigcap (T_{34})'$$

due to the following relations:

$$(\mathfrak{A}^{2,4})' = (\mathfrak{A}^{1,3,4})' \otimes (\mathfrak{A}^{2,3,4})' = \left(T_{\alpha_{13}}(I + T_{\alpha_{14}}), I, T_{\alpha_{14}}\right)'' \otimes \left(T_{\alpha_{23}}(I + T_{\alpha_{24}}), I, T_{\alpha_{24}}\right)''.$$

$\square$

Set for $s \geq 3$

$$\Delta_{s,\infty}^{1,r} := T_{\alpha_{1s}}^r \prod_{k=s+1}^\infty p^{-1}C(T_{\alpha_{1k}}), \quad \Delta_{s,\infty}^{2,r} := T_{\alpha_{2s}}^r \prod_{k=s+1}^\infty p^{-1}C(T_{\alpha_{2k}}), \; r \in \mathbb{F}_p \setminus \{0\}.$$

Lemma 4.18. *Let $x_{12} \in \mathfrak{A}^2$, and $\mu_\alpha^2 \sim \mu_{inv}^2$ or $S_{11}^L(\mu) < \infty$, $S_{22}^L(\mu) < \infty$, then the commutant $(\mathfrak{A}^2)'$ of the von Neumann algebra $\mathfrak{A}^2$ as the linear space is generated by the following operators:*

$$(\mathfrak{A}^2)' = \left(\Delta_{s,\infty}^{l,r} \mid 1 \le l \le 2, \ 3 \le s, \ r \in \mathbb{F}_p \setminus \{0\} \right)''. \tag{4.36}$$

When $\mu_\alpha^2 \perp \mu_{inv}^2$ or $S_{11}^L(\mu) = S_{22}^L(\mu) = \infty$ and $x_{12} \in \mathfrak{A}^2$, the commutant $(\mathfrak{A}^2)'$ is trivial.

PROOF. Since x_{12}, T_{1k}, $T_{2k} \in \mathfrak{A}^2$ for $k \ge 3$ we conclude that $T_{2k}(2) = T_{\alpha_{2k}} \in \mathfrak{A}^2$ (see Remark 6.5). Since the commutative family of operators with common simple spectrum lies in $\mathfrak{A}^2$, i.e., x_{12}, T_{1k}, $T_{\alpha_{2k}} \in \mathfrak{A}^2$ for $3 \le k$ we conclude that

$$(\mathfrak{A}^2)' \subset L^\infty \left(\begin{smallmatrix} T_{\alpha_{13}} & \cdots & T_{\alpha_{1n}} & \cdots \\ T_{\alpha_{23}} & \cdots & T_{\alpha_{2n}} & \cdots \end{smallmatrix} \right). \tag{4.37}$$

Set $(T_1, T_2) = (T_{\alpha_{1k}}, T_{\alpha_{2k}} \mid 3 \le k)$. Denote by $L^\infty(T_1, T_2) = L^\infty(T_{\alpha_{1k}}, T_{\alpha_{2k}} \mid 3 \le k) = (T_{\alpha_{1k}}, T_{\alpha_{2k}} \mid 3 \le k)''$ the von Neumann algebra generated by the commuting family of operators (T_1, T_2). By definition, we have

$$(\mathfrak{A}^2)' = \left(f \in L^\infty(T_1, T_2) \mid [f, T_{kk+1}] = 0, \ 3 \le k \right). \tag{4.38}$$

Using the spectral theorem for the family (T_1, T_2) of commuting unitary operators $(T_{\alpha_{1k}}, T_{\alpha_{2k}} \mid 3 \le k)$ we conclude that any element $f \in L^\infty(T_1, T_2)$ has the following form:

$$f(T_1, T_2) = f(T_{\alpha_{1k}}, T_{\alpha_{2k}} \mid 3 \le k) = \int_{X_0^2} f(\lambda_1, \lambda_2) dE(\lambda_1, \lambda_2) \tag{4.39}$$

where $X_0^2 = \prod_{k=3}^\infty (\mathbb{F}_p \times \mathbb{F}_p)_k$, f is essentially bounded function on X_0^2 and E is common resolution of the identity of the family of operators (T_1, T_2) defined on cylindrycal sets $\Delta_{13} \times \Delta_{23} \times \cdots \times \Delta_{1n} \times \Delta_{2n}$ as follows:

$$E(\Delta_{13} \times \Delta_{23} \times \cdots \times \Delta_{1n} \times \Delta_{2n}) := \prod_{k=3}^n E_{1k}(\Delta_{1k}) E_{2k}(\Delta_{2k})$$

where E_{rk} is resolution of the identity of the operators $T_{\alpha_{rk}}$ for $1 \le r \le 2$, $3 \le k$ i.e., $T_{rk} = \int_{Sp(T_{rk})} f(\lambda_{rk}) dE_{rk}(\lambda_{rk})$. Similarly to the proof of Lemma 4.12 we get

Lemma 4.19. *An operator $f(T_1, T_2) \in L^\infty(T_1, T_2)$ defined by (4.39) commute with T_{kk+1} for all $2 \leq k$ if and only if for all $2 \leq k$ holds:*

$$f\left(\begin{smallmatrix} T_{\alpha_{13}}, \dots, T_{\alpha_{1k}}, T_{\alpha_{1k+1}}, \dots, T_{\alpha_{1n}}, \dots \\ T_{\alpha_{23}}, \dots, T_{\alpha_{2k}}, T_{\alpha_{2k+1}}, \dots, T_{\alpha_{2n}}, \dots \end{smallmatrix}\right) = f\left(\begin{smallmatrix} T_{\alpha_{13}}, \dots, T_{\alpha_{1k}} T_{\alpha_{1k+1}}, T_{\alpha_{1k+1}}, \dots, T_{\alpha_{1n}}, \dots \\ T_{\alpha_{23}}, \dots, T_{\alpha_{2k}} T_{\alpha_{2k+1}}, T_{\alpha_{2k+1}}, \dots, T_{\alpha_{2n}}, \dots \end{smallmatrix}\right). \quad (4.40)$$

Lemma 4.20. *If for the function $f(T_1, T_2) \in L^\infty(T_1, T_2)$ holds relation (4.40) then for some (s_1, s_2), $3 \leq s_1$, $3 \leq s_2$ holds*

$$f = f_{s_1, s_2}(T_{\alpha_{1s_1}}, T_{\alpha_{2s_2}}) \prod_{k=s_1+1}^{\infty} c(T_{\alpha_{1k}}) \prod_{k=s_2+1}^{\infty} c(T_{\alpha_{2k}}), \quad (4.41)$$

where $f_{s_1, s_2}(T_{\alpha_{1s_1}}, T_{\alpha_{2s_2}}) \in L^\infty(T_{\alpha_{1s_1}}, T_{\alpha_{2s_2}})$.

$\square$

5. The proof of the irreducibilty, case $m = 1$

5.1. The irreducibilty, case $m = 1$, $p = 2$

Let us consider two operators $T_{1n} := T_{\alpha_{1n}}$ and T_{kn} in the space $H = H_{\alpha_{1k}} \otimes H_{\alpha_{1n}} = L^2(X, \mu)$ where $\mu = \mu_{\alpha_{1k}} \otimes \mu_{\alpha_{1n}}$ and $X = \mathbb{F}_p \times \mathbb{F}_p$, $2 \leq k < n$,

$$X = \begin{pmatrix} 1 & x_{1k} & x_{1n} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The basis in the space $H_{1t} := H_{\alpha_{1t}} := L^2(\mathbb{F}_p, \mu_{\alpha_{1t}})$ is $(e_s^\alpha)_{s \in \mathbb{F}_p}$, where $e_s^\alpha(r) = (\alpha_{1t}(r))^{-1/2} \delta_{sr}$, $s, r \in \mathbb{F}_p$ hence, the basis in the space

$$H_{\alpha_{1k}} \otimes H_{\alpha_{1n}} \quad \text{is} \quad (e_{sr} := e_s^\alpha \otimes e_r^\alpha)_{s, r \in \mathbb{F}_p}.$$

We fix the lexicographic order on the set $(s, r)_{s, r \in \mathbb{F}_p}$. So, we have chosen the following basis

$$e_{00}, \quad e_{01}, \quad e_{10}, \quad e_{11}.$$

In this basis the operators T_{1n} and T_{kn} act as follows if the measures $\mu_{\alpha_{1t}}$ are invariant, recall that $(T_{1n}f)(x_{1n}) = f(x_{1n} - 1)$ and $(T_{kn}f)(x_{1k}, x_{1n}) = f(x_{1k}, x_{1n} - x_{1k})$

$$T_{1n} : e_{ij} \to e_{ij+1}, \quad T_{kn} : e_{ij} \to e_{ij+i}. \quad (5.1)$$

For an arbitrary measure μ_α operators T_{1n} and T_{kn} act as follows:

$$T_{1n} : e_{ij} \to \sqrt{\frac{\alpha_{1n}(j)}{\alpha_{1n}(j+1)}} e_{ij+1}, \quad T_{kn} : e_{ij} \to \sqrt{\frac{\alpha_{1n}(j)}{\alpha_{1n}(j+i)}} e_{ij+i}. \qquad (5.2)$$

Using (5.1) we have the following transformation of indices of the basis e_{ij} under the action of T_{1n} and T_{kn}:

$$\begin{vmatrix} & 0 & 1 & 2 & 3 \\ ij & 00 & 01 & 10 & 11 \\ T_{1n} & 01 & 00 & 11 & 10 \\ T_{kn} & 00 & 01 & 11 & 10 \end{vmatrix}.$$

For the invariant measure and in the general case we have respectively

$$T_{1n} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{smallmatrix}\right), \quad T_{kn} = \left(\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{smallmatrix}\right), \qquad (5.3)$$

$$T_{1n} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 0 & a_n^{-1} \\ a_n & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & a_n^{-1} & 0 & 0 \\ a_n & 0 & 0 & 0 \\ 0 & 0 & 0 & a_n^{-1} \\ 0 & 0 & a_n & 0 \end{smallmatrix}\right), \quad T_{kn} = \left(\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & a_n^{-1} \\ 0 & 0 & a_n & 0 \end{smallmatrix}\right), \qquad (5.4)$$

where $a_n = \sqrt{\frac{\alpha_{1n}(0)}{\alpha_{1n}(1)}}$.

Recall that $x_{1k} = \mathrm{diag}(0,1) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ (see (2.12) for notations x_{kn}). We would like to approximate the operator $x_{1k} \cong \cdots \otimes x_{1k} \otimes \ldots$ (see Remark 2.3) on the space $H^1 = \otimes_{n=2}^{\infty} H_{\alpha_{1n}}$ by linear combinations of operators T_{kn}.

Lemma 5.1. *We have*

$$x_{1k}\mathbf{1} = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)\mathbf{1} \in \langle (T_{kn} - I)\mathbf{1} \mid n > k \rangle \Leftrightarrow S_{11}^L(\mu_\alpha) = \infty.$$

PROOF. In the space $H_{\alpha_{1k}} \otimes H_{\alpha_{1n}}$ we have

$$T_{kn} - I = \left(\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & a_n^{-1} \\ 0 & 0 & a_n & -1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} -1 & a_n^{-1} \\ a_n & -1 \end{smallmatrix}\right) = x_{1k} \otimes (T_{\alpha_{1k}} - I)$$

so,

$$T_{kn} - I = x_{1k} \otimes (T_{\alpha_{1n}} - I). \qquad (5.5)$$

Hence, we get $\sum_{n=k+1}^{N+k+1} t_n(T_{kn} - I) \to x_{1k}$. Indeed, in the space $\otimes_{n=k+1}^{\infty} H_{1n}$ we get

$$\sum_{n=k+1}^{N+k+1} t_n(T_{kn} - I) = x_{1k} \otimes I \otimes \ldots \otimes I \otimes \sum_{n=k+1}^{N+k+1} t_n \left(\begin{smallmatrix} -1 & a_n^{-1} \\ a_n & -1 \end{smallmatrix}\right) \otimes I \ldots$$

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$$= x_{1k} \otimes I \otimes ... \otimes I \otimes \Big[\sum_{n=k+1}^{N+k+1} t_n \begin{pmatrix} -1 & a_n^{-1} \\ a_n & -1 \end{pmatrix} \Big] \otimes I... \to x_{1k} \otimes I \otimes \cdots \otimes I ... \cong x_{1k},$$

by Lemma 4.1, where $a_n = \sqrt{\frac{\alpha_{1n}(0)}{\alpha_{1n}(1)}}$. $\qquad\qquad\square$

Since $x_{1k} = \mathrm{diag}(0,1) \in \mathfrak{A}^1$ the proof of the irreducibility for $m = 1$ and $p = 2$ follows from Remark 2.2.

5.2. *The irreducibilty, case $m = 1$, p is arbitrary*

Notation. For an arbitrary p let us denote by $P_{kn}^{(r)}$ the operators $E_{rr} = \mathrm{diag}(\underbrace{0,...,0,1}_{r},0,...,0)$, acting on the spaces $H_{\alpha_{kn}}$, $r \in \mathbb{F}_p$, $1 \leq k < n$.

Let us suppose that we are able to approximate $P_{1k}^{(r)}$ by the operators of the representation, i.e., that $P_{1k}^{(r)} \in \mathfrak{A}^1$, $r \in \mathbb{F}_p$ hence, an operator x_{1k} acting in H_{1k} (see (2.12) and (2.13)) belongs to $\mathfrak{A}^1$:

$$x_{1k} = \mathrm{diag}(0,1,\ldots,p-1) = \sum_{r \in \mathbb{F}_p} r E_{rr} = \sum_{r \in \mathbb{F}_p} r P_{1k}^{(r)} \in \mathfrak{A}^1.$$

In this case the proof follows from Remark 2.2.

In order to find an appropriate combinations to approximate the operators $P_{1k}^{(r)}$, $r \in \mathbb{F}_p$ *we study first the case $p = 3$.* Let us denote (see (3.10))

$$T_\alpha = \begin{pmatrix} 0 & 0 & t_{02} \\ t_{10} & 0 & 0 \\ 0 & t_{21} & 0 \end{pmatrix}, \quad \text{then } T_\alpha^2 = \begin{pmatrix} 0 & t_{01} & 0 \\ 0 & 0 & t_{12} \\ t_{20} & 0 & 0 \end{pmatrix}, \quad \text{where } t_{ij} = \sqrt{\frac{\alpha_{1n}(j)}{\alpha_{1n}(i)}}, \ i,j \in \mathbb{F}_p.$$

$$(5.6)$$

Let $e_{kr} := (e_k^\alpha \otimes e_r^\alpha)_{k,r \in \mathbb{F}_p}$ be the basis in the space $H_{\alpha_{12}} \otimes H_{\alpha_{1n}}$ (see (3.8)). Using (5.1) we have the following transformation of indices of the basis e_{ij} under the action of T_{1n} and T_{2n}:

$$\begin{vmatrix} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ ij & 00 & 01 & 02 & 10 & 11 & 12 & 20 & 21 & 22 \\ T_{1n} & 01 & 02 & 00 & 11 & 12 & 10 & 21 & 22 & 20 \\ T_{2n} & 00 & 01 & 02 & 11 & 12 & 10 & 22 & 20 & 21 \end{vmatrix}.$$

So, the operators T_{1n} and T_{2n} have the following forms in $H_{12} \otimes H_{1n}$:

$$T_{1n} = \begin{pmatrix} 0 & 0 & t_{02} & 0 & 0 & 0 & 0 & 0 & 0 \\ t_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t_{21} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t_{02} & 0 & 0 & 0 \\ 0 & 0 & 0 & t_{10} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t_{21} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_{02} \\ 0 & 0 & 0 & 0 & 0 & 0 & t_{10} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_{21} & 0 \end{pmatrix}, \quad T_{2n} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t_{02} & 0 & 0 & 0 \\ 0 & 0 & 0 & t_{10} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t_{21} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_{01} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_{12} \\ 0 & 0 & 0 & 0 & 0 & 0 & t_{20} & 0 & 0 \end{pmatrix},$$

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$$T_{1n} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & t_{02} \\ t_{10} & 0 & 0 \\ 0 & t_{21} & 0 \end{pmatrix} = \begin{pmatrix} T_\alpha & 0 & 0 \\ 0 & T_\alpha & 0 \\ 0 & 0 & T_\alpha \end{pmatrix}, \quad T_{2n} = \begin{pmatrix} T_\alpha^0 & 0 & 0 \\ 0 & T_\alpha^1 & 0 \\ 0 & 0 & T_\alpha^2 \end{pmatrix}, \qquad (5.7)$$

where $\alpha = \alpha_{1n}$. Note that

$$T_{1n} = \mathrm{diag}(T_\alpha, T_\alpha, T_\alpha), \quad T_{2n} = \mathrm{diag}(I, T_\alpha, T_\alpha^2). \qquad (5.8)$$

Since

$$T_{2n} = \mathrm{diag}(I, T_\alpha, T_\alpha^2), \;\; T_{2n}^2 = \mathrm{diag}(I, T_\alpha^2, T_\alpha) \;\; \mathrm{so}, \;\; C(T_{2n}) = \mathrm{diag}(3, C(T_\alpha), C(T_\alpha)).$$

Similarly, we get

$$T_{1n}T_{2n} = \mathrm{diag}(T_\alpha, T_\alpha^2, I), \quad (T_{1n}T_{2n})^2 = \mathrm{diag}(T_\alpha^2, T_\alpha, I),$$

$$C(T_{1n}T_{2n}) = \mathrm{diag}(C(T_\alpha), C(T_\alpha), 3),$$

$$T_{1n}^2 T_{2n} = \mathrm{diag}(T_\alpha^2, I, T_\alpha), \quad (T_{1n}^2 T_{2n})^2 = \mathrm{diag}(T_\alpha, I, T_\alpha^2),$$

$$C(T_{1n}^2 T_{2n}) = \mathrm{diag}(C(T_\alpha), 3, C(T_\alpha)).$$

So, we can try to approximate

$$\mathrm{diag}(0, I, I) = (P_{12}^{(1)} + P_{12}^{(2)}) \otimes I = (I - P_{12}^{(0)}) \otimes I \quad \text{by combinations of } \;\; C(T_{2n}) - 3,$$

$$\mathrm{diag}(I, I, 0) = (P_{12}^{(0)} + P_{12}^{(1)}) \otimes I = (I - P_{12}^{(2)}) \otimes I \quad \text{by combinations of } \;\; C(T_{1n}T_{2n}) - 3,$$

$$\mathrm{diag}(I, 0, I) = (P_{12}^{(0)} + P_{12}^{(2)}) \otimes I = (I - P_{12}^{(1)}) \otimes I \quad \text{by combinations of } \;\; C(T_{1n}^2 T_{2n}) - 3.$$

In the general case, we can try to approximate

$$(I - P_{1k}^{(p-r)}) \otimes I \quad \text{by combinations of } \;\; \sum_{s \in \mathbb{F}_p} (T_{1n}^r T_{kn})^s - p = C(T_{1n}^r T_{kn}) - p.$$

Lemma 5.2. *We have for $r \in \mathbb{F}_p$ and $k > 1$*

$$(I - P_{1k}^{(p-r)})\mathbf{1} \in \left\langle \left[C(T_{1n}^r T_{kn}) - p \right] \mathbf{1} \mid n > k \right\rangle$$

if and only if $S_{11}^L(\mu_\alpha) = \infty \Leftrightarrow \mu^\alpha \perp \mu_{inv}$.

PROOF. Since $T_{1n} = \mathrm{diag}(T_{\alpha_{1n}}, \dots, T_{\alpha_{1n}})$, $T_{2n} = \mathrm{diag}(I, T_{\alpha_{1n}}, T^2_{\alpha_{1n}}, \dots, T^{p-1}_{\alpha_{1n}})$, $T^s_{2n} = \mathrm{diag}(I, T^s_{\alpha_{1n}}, T^{2s}_{\alpha_{1n}}, \dots, T^{s(p-1)}_{\alpha_{1n}})$, $s \in \mathbb{F}_p$, we get

$$(T^r_{1n} T_{2n})^s = \left[\mathrm{diag}(T_{1n}, T_{1n}, T_{1n}, \dots, T_{1n})\mathrm{diag}(I, T_{1n}, T^2_{1n}, \dots, T^{p-1}_{1n})\right]^s =$$

$$\left[\mathrm{diag}(T^r_{1n}, T^{r+1}_{1n}, T^{r+2}_{1n}, \dots, T^{r+p-1}_{1n})\right]^s = \left[\mathrm{diag}(T^{rs}_{1n}, T^{(r+1)s}_{1n}, T^{r+2}_{1n}, \dots, T^{(r+p-1)s}_{1n})\right].$$

Therefore,

$$\sum_{s \in \mathbb{F}_p} (T^r_{1n} T_{2n})^s = \left(\sum_{s \in \mathbb{F}_p} T^{rs}_{1n}, \sum_{s \in \mathbb{F}_p} T^{(r+1)s}_{1n}, \dots, \sum_{s \in \mathbb{F}_p} T^{(r+p-1)s}_{1n}\right)$$

$$= \left(\underbrace{C(T_{1n}), C(T_{1n}), \dots, p}_{p-r}, \dots, C(T_{1n}).\right)$$

At last, we get $\sum_{s \in \mathbb{F}_p}(T^r_{1n}T_{2n})^s - p =$

$$\left(C(T_{1n}) - p\right)(\underbrace{I, I, \dots, I, 0}_{p-r}, I, \dots, I) = (I - P^{(p-r)}_{12}) \otimes \left(C(T_{1n}) - p\right).$$

Finally, we get when $N \to \infty$

$$\sum_{n=k+1}^{k+N} t_n \left[C(T^r_{1n}T_{2n}) - p\right] - (I - P^{(p-r)}_{12}) \otimes I =$$

$$(I - P^{(p-r)}_{12}) \otimes \left[\sum_{n=k+1}^{k+N} t_n\left(C(T_{1n}) - p\right) - I\right] \to 0.$$

The proof of the latter statement is similar to the proof of Lemma 5.1. $\quad\square$

Finally, we can approximate $P^{(r)}_{1k}$ therefore, $x_{1k} = \sum_{r \in \mathbb{F}_p} r P^{(r)}_{1k} = \sum_{r \in \mathbb{F}_p} r E_{rr} \in \mathfrak{A}^1$. Using the Remark 2.2 we conclude that the representation is irreducible.

6. Irreducibility, case $m = 2$

Let us consider three operators T_{1n}, T_{2n} and T_{kn} on the space $H = L^2(X, \mu) = H_{\alpha_{12}} \otimes H_{\alpha_{1n}} \otimes H_{\alpha_{2n}}$ where $\mu = \mu_{\alpha_{12}} \otimes \mu_{\alpha_{1n}} \otimes \mu_{\alpha_{2n}}$ and

$$X = \begin{pmatrix} 1 & x_{12} & x_{1n} \\ 0 & 1 & x_{2n} \\ 0 & 0 & 1 \end{pmatrix}.$$

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The basis in the space $H_{st} = H_{\alpha_{st}} = L^2(\mathbb{F}_p, \mu_{\alpha_{st}})$ is $(e_k^\alpha)_{k \in \mathbb{F}_p}$ (see (3.8)) hence, the basis in the space $H_{\alpha_{12}} \otimes H_{\alpha_{1n}} \otimes H_{\alpha_{2n}}$ is $(e_{krs} := e_t^\alpha \otimes e_r^\alpha \otimes e_s^\alpha)_{t,r,s \in \mathbb{F}_p}$. We fix the lexicographic order on the set $(t, r, s)_{k,r,s \in \mathbb{F}_p}$. So, we have chosen the following basis

$$e_{000}, \ e_{001}, \ e_{010}, \ e_{011}, \ e_{100}, \ e_{101}, \ e_{110}, \ e_{111}.$$

In this basis the operators T_{1n} and T_{2n} act as follows if the measures $\mu_{\alpha_{st}}$ are invariant (T_{kn} acts on the space $H_{1k} \otimes H_{1n} \otimes H_{2k} \otimes H_{2n}$):

$$T_{1n} : e_{ijl} \to e_{i,j+1,l}, \ T_{2n} : e_{ijl} \to e_{i,j+i,l+1}, \ T_{kn} : e_{ijlr} \to e_{i,j+i,l,r+l} \qquad (6.1)$$

and as follows if the measure is not invariant:

$$T_{1n} : e_{ijl} \to \sqrt{\frac{\alpha_{1n}(j)}{\alpha_{1n}(j+1)}} e_{i,j+1,l}, \ T_{2n} : e_{ijl} \to \sqrt{\frac{\alpha_{1n}(j)\alpha_{2n}(l)}{\alpha_{1n}(j+i)\alpha_{2n}(l+1)}} e_{i,j+i,l+1},$$

$$(6.2)$$

$$T_{kn} : e_{ijlr} \to \sqrt{\frac{\alpha_{1n}(j)\alpha_{2n}(r)}{\alpha_{1n}(j+i)\alpha_{2n}(r+l)}} e_{i,j+i,l,r+l}. \qquad (6.3)$$

Using (6.1) we have the following transformation of indices of the basis e_{ijl} under the action of T_{1n} and T_{2n}:

$$\begin{vmatrix} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ ijl & 000 & 001 & 010 & 011 & 100 & 101 & 110 & 111 \\ T_{1n} & 010 & 011 & 000 & 001 & 110 & 111 & 100 & 101 \\ T_{2n} & 001 & 000 & 011 & 010 & 111 & 110 & 101 & 100 \end{vmatrix}, \quad \text{i.e.,} \quad \begin{vmatrix} ijl & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ T_{1n} & 2 & 3 & 0 & 1 & 6 & 7 & 4 & 5 \\ T_{2n} & 1 & 0 & 3 & 2 & 7 & 6 & 5 & 4 \end{vmatrix}.$$

So, the operators T_{1n} and T_{2n} have the following forms:

$$T_{1n} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & a_n^{-1} \\ a_n & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a^{-1} & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a^{-1} \\ 0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a & 0 & 0 \end{pmatrix},$$

$$T_{2n} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & a_n^{-1} \\ 0 & 0 & a_n & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & b_n^{-1} \\ b_n & 0 \end{pmatrix} = \begin{pmatrix} 0 & b^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a^{-1}b^{-1} \\ 0 & 0 & 0 & 0 & 0 & 0 & a^{-1}b & 0 \\ 0 & 0 & 0 & 0 & 0 & ab^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & ab & 0 & 0 & 0 \end{pmatrix},$$

where

$$a = a_n = \sqrt{\frac{\alpha_{1n}(0)}{\alpha_{1n}(1)}}, \quad b = b_n = \sqrt{\frac{\alpha_{2n}(0)}{\alpha_{2n}(1)}}. \qquad (6.4)$$

To calculate T_{1n} and T_{2n} we use the following formulas

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) = \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix}, \quad \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix}. \tag{6.5}$$

Remark 6.1. The latter formulas are particular case of the formulas below. For $A = (a_{ij})_{ij} \in \mathrm{Mat}(n, \mathbf{k})$ and $B = (b_{rs})_{rs} \in \mathrm{Mat}(m, \mathbf{k})$ we have in the basis $e_i \otimes f_s$

$$A \otimes B = (a_{ij} b_{rs})_{((i,j),(r,s))} \in \mathrm{End}(\mathbf{k}^n \otimes \mathbf{k}^m).$$

Indeed, we get

$$Ae_j = \sum_{i=1}^{n} a_{ij} e_i, \quad Bf_s = \sum_{r=1}^{m} b_{rs} f_r,$$

therefore, we have

$$(A \otimes B) e_i \otimes f_s = Ae_i \otimes Bf_s = \sum_{i=1}^{n} \sum_{r=1}^{m} a_{ij} b_{rs} e_i \otimes f_r.$$

6.1. Irreducibility $m = 2$, $p = 2$

We write $a_n \sim b_n$ if $C_1 a_n \le b_n \le C_2 a_n$. Recall (see Notations before Lemma 4.1) that $c_{kn} = 2\sqrt{\alpha_{kn}(0)\alpha_{kn}(1)}$ and $S_{kn}^L(\mu_\alpha)$ are defined as follows (see (3.14) and (3.15))

$$S_{11}^L(\mu_\alpha) = \sum_{n=2}^{\infty} (1 - c_{1n}), \quad S_{12}^L(\mu_\alpha) = \sum_{n=3}^{\infty} \alpha_{2n}(1)(1 - c_{1n}), \quad S_{22}^L(\mu_\alpha) = \sum_{n=3}^{\infty} (1 - c_{2n}).$$

Theorem 6.1. *Representation $T^{R,\mu,2}$ is irreducible if and only if*
1) $(\mu_\alpha)^{L_I + E_{12}} \perp \mu_\alpha \Leftrightarrow S_{12}^L(\mu_\alpha) = \infty$,
2) $\mu_\alpha^2 \perp \mu_{inv}^2 \Leftrightarrow S_{22}^L(\mu_\alpha) = \infty$.

Let $p = 2$. To approximate x_{1k} and x_{2k} we use the following expressions (see (2.9))

$$T_{kn} - t_{kn}(2) = T_{kn}(1) \otimes (T_{kn}(2) - t_{kn}(2)) + (T_{kn}(1) - I) \otimes t_{kn}(2) \tag{6.6}$$
$$= T_{kn}(1) \otimes T_{kn}^c(2) + \hat{T}_{kn}(r) \otimes t_{kn}(2),$$
$$T_{kn} - t_{kn}(1) = (T_{kn}(1) - t_{kn}(1)) \otimes T_{kn}(2) + t_{kn}(1) \otimes (T_{kn}(2) - I) \tag{6.7}$$
$$= T_{kn}^c(r) \otimes T_{kn}(2) + t_{kn}(1) \otimes \hat{T}_{kn}(r),$$
$$T_{kn}(1) - I = x_{1k} \otimes (T_{\alpha_{1n}} - I), \quad T_{kn}(2) - I = x_{2k} \otimes (T_{\alpha_{2n}} - I), \tag{6.8}$$

where $T_{kn} = T_{kn}(1) \otimes T_{kn}(2)$ (see (6.16)) and we set

$$t_{kn}(r) = (T_{kn}(r)\mathbf{1}, \mathbf{1}), \quad T^c_{kn}(r) = T_{kn}(r) - t_{kn}(r), \quad \hat{T}_{kn}(r) = T_{kn}(r) - 1.$$
(6.9)

Notation. Recall that we denote by x_{kn} the operator $x_{kn} = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) = E_{11}$ on the space $H_{kn} = L^2(\mathbb{F}_2, \mu_{\alpha_{kn}})$, see Remark 2.2.

We find the conditions of the approximation of the operators x_{12} (respectively x_{1k} and x_{2k}, $k > 2$) by the following linear combinations

$$\sum_n t_n(T_{2n} - t_{2n}(2)) \ \Big(\text{respectively by } \sum_n t_n(T_{kn} - t_{kn}(2)) \text{ and } \sum_n t_n(T_{kn} - t_{kn}(1))\Big).$$

We need the following lemma.

Lemma 6.2. *For fixed λ, c, $\mu \in \mathbb{R}^m$ and the matrix $D_m(\lambda, c)$ defined as follows:*

$$D_m(\lambda, c) = \begin{pmatrix} 1 + \lambda_1 + c_1^2 & 1 + c_1 c_2 & \dots & 1 + c_1 c_m \\ 1 + c_2 c_1 & 1 + \lambda_2 + c_2^2 & \dots & 1 + c_2 c_m \\ & & \dots & \\ 1 + c_m c_1 & 1 + c_m c_2 & \dots & 1 + \lambda_m + c_m^2 \end{pmatrix}$$
(6.10)

we have

$$(D_m^{-1}(\lambda, c)\mu, \mu) = \frac{\Gamma(f_m) + \Gamma(f_m, g_m) + \Gamma(f_m, h_m)}{1 + \Gamma(g_m) + \Gamma(h_m) + \Gamma(g_m, h_m)}$$
(6.11)

$$= \frac{\sum_{k=1}^m \frac{\mu_k^2}{\lambda_k} + \sum_{1 \leq k < n \leq m} \frac{(\mu_k - \mu_n)^2}{\lambda_k \lambda_n} + \sum_{1 \leq k < n \leq m} \frac{(c_n \mu_k - c_k \mu_n)^2}{\lambda_k \lambda_n}}{1 + \sum_{k=1}^m \frac{1}{\lambda_k} + \sum_{k=1}^m \frac{c_k^2}{\lambda_k} + \sum_{1 \leq k < n \leq m} \frac{(c_k - c_n)^2}{\lambda_k \lambda_n}},$$
(6.12)

where

$$f_m = \left(\frac{\mu_k}{\sqrt{\lambda_k}}\right)_{k=1}^m, \quad g_m = \left(\frac{1}{\sqrt{\lambda_k}}\right)_{k=1}^m, \quad h_m = \left(\frac{c_k}{\sqrt{\lambda_k}}\right)_{k=1}^m.$$
(6.13)

PROOF. We have for $m = 2$

$$D_2(\lambda, c) = \begin{pmatrix} 1 + \lambda_1 + c_1^2 & 1 + c_1 c_2 \\ 1 + c_2 c_1 & 1 + \lambda_2 + c_2^2 \end{pmatrix},$$

$$F_2(\lambda, c) := \det D_2(\lambda, c) = 1 + \lambda_1 + \lambda_2 + c_1^2 + c_2^2 + (\lambda_1 + c_1^2)(\lambda_2 + c_2^2) - (1 + 2c_1 c_2 + c_1^2 c_2^2)$$

$$= \lambda_1 \lambda_2 \left(1 + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{c_1^2}{\lambda_1} + \frac{c_2^2}{\lambda_2} + \frac{(c_1 - c_2)^2}{\lambda_1 \lambda_2}\right) = \lambda_1 \lambda_2 (1 + \Gamma(g_2) + \Gamma(h_2) + \Gamma(g_2, h_2)),$$

where
$$g_2 = \Big(\frac{1}{\sqrt{\lambda_1}}, \frac{1}{\sqrt{\lambda_2}}\Big), \quad h_2 = \Big(\frac{c_1}{\sqrt{\lambda_1}}, \frac{c_2}{\sqrt{\lambda_2}}\Big).$$

In the general case, we show that

$$F_m(\lambda, c) := \det D_m(\lambda, c) = \prod_{k=1}^{m} \lambda_k \Big(1 + \sum_{k=1}^{m} \frac{1}{\lambda_k} + \sum_{k=1}^{m} \frac{c_k^2}{\lambda_k} + \sum_{1 \le k < n \le m} \frac{(c_k - c_n)^2}{\lambda_k \lambda_n}\Big).$$

$$(6.14)$$

We prove (6.14) first for $m = 3$:

$$F_3(\lambda, c) = \begin{vmatrix} 1 + \lambda_1 + c_1^2 & 1 + c_1 c_2 & 1 + c_1 c_3 \\ 1 + c_2 c_1 & 1 + \lambda_2 + c_2^2 & 1 + c_2 c_3 \\ 1 + c_3 c_1 & 1 + c_3 c_2 & 1 + \lambda_3 + c_3^2 \end{vmatrix} =$$

$$\prod_{k=1}^{3} \lambda_k \Big(1 + \sum_{k=1}^{3} \frac{1 + c_k^2}{\lambda_k} + \sum_{1 \le k < n \le 3} \frac{(c_k - c_n)^2}{\lambda_k \lambda_n}\Big) = \lambda_1 \lambda_2 \lambda_3 + \lambda_2 \lambda_3 (1 + c_1^2) +$$

$$\lambda_1 \lambda_3 (1 + c_2^2) + \lambda_1 \lambda_2 (1 + c_3^2) + \lambda_3 (c_1 - c_2)^2 + \lambda_2 (c_1 - c_3)^2 + \lambda_1 (c_2 - c_3)^2.$$

To prove the latter equality we show that the appropriate derivatives for the both sides of the relation coincide. Indeed, they are equal respectively:

$$\frac{\partial F_3(\lambda, c)}{\partial \lambda_i} \Big|_{\lambda=0} = (c_j - c_k)^2, \quad \frac{\partial^2 F_3(\lambda, c)}{\partial \lambda_i \lambda_j} \Big|_{\lambda=0} = 1 + c_k^2, \quad \frac{\partial^3 F_3(\lambda, c)}{\partial \lambda_1 \lambda_2 \lambda_3} \Big|_{\lambda=0} = 1,$$

where i, j, k is the cyclic permutations of the indices $1, 2, 3$. In the general case, we have for both sides of the equation (6.14)

$$\frac{\partial^{m-2} F_m(\lambda, c)}{\partial \lambda_1 \dots \lambda_{m-2}} \Big|_{\lambda=0} = (c_{m-1} - c_m)^2, \quad \frac{\partial^{m-1} F_m(\lambda, c)}{\partial \lambda_1 \dots \lambda_{m-1}} \Big|_{\lambda=0} = 1 + c_m^2, \quad \frac{\partial^m F_m(\lambda, c)}{\partial \lambda_1 \dots \lambda_m} \Big|_{\lambda=0} = 1$$

and the corresponding cyclic permutations of the indices. This proves (6.14).

Further, for $m = 2$ we have

$$D_2^{-1}(\lambda, c) = \frac{1}{F_2(\lambda, c)} \begin{pmatrix} 1 + \lambda_2 + c_2^2 & -(1 + c_1 c_2) \\ -(1 + c_2 c_1) & 1 + \lambda_1 + c_1^2 \end{pmatrix} \quad \text{and} \quad (D_2^{-1}(\lambda, c)\mu, \mu) =$$

$$(F_2(\lambda, c))^{-1} \Big[(1 + \lambda_2 + c_2^2)\mu_1^2 - 2(1 + c_1 c_2)\mu_1 \mu_2 + (1 + \lambda_1 + c_1^2)\mu_2^2\Big]$$

$$= (F_2(\lambda, c))^{-1} \Big[(\mu_1 - \mu_2)^2 + (c_2 \mu_1 - c_1 \mu_2)^2 + \lambda_2 \mu_1^2 + \lambda_1 \mu_2^2\Big] =$$

$$\frac{\lambda_1 \lambda_2 \left(\frac{\mu_1^2}{\lambda_1} + \frac{\mu_2^2}{\lambda_2} + \frac{(\mu_1 - \mu_2)^2}{\lambda_1 \lambda_2} + \frac{(c_2 \mu_1 - c_1 \mu_2)^2}{\lambda_1 \lambda_2} \right)}{\lambda_1 \lambda_2 \left(1 + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{c_1^2}{\lambda_1} + \frac{c_2^2}{\lambda_2} + \frac{(c_1 - c_2)^2}{\lambda_1 \lambda_2} \right)} = \frac{\Gamma(f_2) + \Gamma(f_2, g_2) + \Gamma(f_2, h_2)}{1 + \Gamma(g_2) + \Gamma(h_2) + \Gamma(g_2, h_2)}.$$

By analogy, we get (6.12) for the genaral m. $\qquad\square$

Lemma 6.3. *We have* $x_{1k} \mathbf{1} \in \langle [T_{kn} - t_{kn}(2)] \mathbf{1} \mid n > k \rangle \Leftrightarrow \Sigma_{1m} \to \infty$ *where*

$$\Sigma_{1m} = (D_m^{-1}(\lambda, c)\mu, \mu) = \frac{\Gamma(f_m) + \Gamma(f_m, g_m) + \Gamma(f_m, h_m)}{1 + \Gamma(g_m) + \Gamma(h_m) + \Gamma(g_m, h_m)}, \qquad (6.15)$$

the vectors f_m, g_m, h_m *are defined by (6.13) and*

$$\lambda_n = \frac{2(1 - c_{1n}^2 + 1 - c_{2n}^2 + 1 - c_{1n}^2 c_{2n}^2)}{(1 - c_{2n})^2}, \quad c_n = c_{1n}, \quad \mu_n = -\frac{\sqrt{2}(1 + c_{2n})(1 - c_{1n})}{(1 - c_{2n})}.$$

PROOF. We have $T_{kn} = T_{kn}(1) \otimes T_{kn}(2)$ where an operator $T_{kn}(1)$ acts on $H_{1k} \otimes H_{1n}$, $T_{kn}(2)$ acts on $H_{2k} \otimes H_{2n}$ and they are defined by

$$T_{kn}(1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & a_n^{-1} \\ 0 & 0 & a_n & 0 \end{pmatrix}, \quad T_{kn}(2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & b_n^{-1} \\ 0 & 0 & b_n & 0 \end{pmatrix}, \quad a_n = \sqrt{\frac{\alpha_{1n}(0)}{\alpha_{1n}(1)}}, \quad b_n = \sqrt{\frac{\alpha_{2n}(0)}{\alpha_{2n}(1)}}. \qquad (6.16)$$

Using (6.6) we get

$$T_{kn} - t_{kn}(2) = T_{kn}(1) \otimes (T_{kn}(2) - t_{kn}(2)) + (T_{kn}(1) - I) \otimes t_{kn}(2).$$

Set

$$a_n^{(r)} := (T_{\alpha_{rn}} - I)\mathbf{1}, \quad r = 1, 2 \quad \text{and} \quad b = (b_n), \quad b_n = t_{kn}(2) M a_n^{(1)}.$$

Take $t = (t_n)_n$ such that $(t, b) = \sum_n t_n t_{kn}(2) M a_n^{(1)} = 1$. Set

$$f_n = \left[T_{kn}(1) \otimes (T_{kn}(2) - t_{kn}(2)) \right] \mathbf{1},$$

$$g_n = \left[(T_{kn}(1) - I) \otimes t_{kn}(2) \right] \mathbf{1} = \left[x_{1k} \otimes (T_{\alpha_{1n}} - I) \otimes t_{kn}(2) \right] \mathbf{1},$$

$$g_n^c = \left[g_n - x_{1k} \otimes M a_n^{(1)} \otimes t_{kn}(2) \right] \mathbf{1} = \left[x_{1k} \otimes (T_{\alpha_{1n}} - c_{1n}) \otimes t_{kn}(2) \right] \mathbf{1}.$$

We use the relation

$$T_{\alpha_{1n}} - 1 - M a_n^{(1)} = T_{\alpha_{1n}} - 1 - (c_{1n} - 1) = T_{\alpha_{1n}} - c_{1n}. \qquad (6.17)$$

Using (6.6–6.8) we have $(T_{kn} - t_{kn}(2))\mathbf{1} = f_n + g_n$ therefore, we get

$$\big\| \big[\sum_n t_n(T_{kn}-t_{kn}(2))-x_{1k}\big]\mathbf{1}\big\|^2 = \big\| \sum_n t_n(f_n+g_n)-x_{1k}\otimes\sum_n t_n t_{kn}(2)M a_n^{(1)}\mathbf{1}\big\|^2$$

$$= \big\| \sum_n t_n T_{kn}(1) \otimes (T_{kn}(2) - t_{kn}(2))\mathbf{1} + x_{1k} \otimes \sum_n t_n t_{kn}(2)\big(T_{\alpha_{1n}} - c_{1n}\big)\mathbf{1}\big\|^2$$

$$= \big\| \sum_n t_n(f_n + g_n^c)\big\|^2 = \sum_{n,m} t_n t_m(h_n, h_m) = \sum_{n,m} t_n t_m a_{nm} = (At, t),$$

where
$$h_n = f_n + g_n^c, \quad A = (a_{nm})_{n,m}, \quad a_{nm} = (h_n, h_m).$$

We use the following estimation for a positively definite operator A acting on a space H and a vector $b \in H$ (see [19]):

$$\min_{t\in H} \Big((At, t) \mid (t, b) = 1 \Big) = (A^{-1}b, b)^{-1}.$$

The minimum is reached for $x = A^{-1}b\,((A^{-1}b, b))^{-1}$.

We calculate $a_{nm} = (h_n, h_m)$ and we show that $b_n = -\frac{1+c_{2n}}{2}(1 - c_{1n})$ and

$$a_{nn}=1-\big((1+c_{2n})/2\big)^2\big(1+c_{1n}^2\big)/2, \tag{6.18}$$

$$a_{nm}=(1+c_{1n}c_{1m})(1-c_{2n})(1-c_{2m})/8. \tag{6.19}$$

Since $(f_n, g_n^c) = 0$ we get $\|f_n + g_n^c\|^2 = \|f_n\|^2 + \|g_n^c\|^2$. Indeed,

$$(f_n, g_n^c) = \big([T_{kn}(1) \otimes (T_{kn}(2) - t_{kn}(2))]\mathbf{1}, x_{1k} \otimes \big(T_{\alpha_{1n}} - c_{1n}\big) \otimes t_{kn}(2)\mathbf{1}\big)$$

$$= t_{kn}(2)(T_{kn}(1)x_{1k} \otimes \big(T_{\alpha_{1n}} - c_{1n}\big)\mathbf{1}, \mathbf{1})((T_{kn}(2) - t_{kn}(2))\mathbf{1}, \mathbf{1}) = 0.$$

We have

$$a_{nn}=\|f_n\|^2 + \|g_n^c\|^2 = \|(T_{kn}(2) - t_{kn}(2))\mathbf{1}\|^2 + t_{kn}^2(2)\|x_{1k}\mathbf{1}\|^2\|(T_{\alpha_{1n}} - c_{1n})\mathbf{1}\|^2. \tag{6.20}$$

For general $f, g \in L^1(X, \mu) \cap L^2(X, \mu)$ we use the following relation:

$$(f - Mf, g - Mg) = (f, g) - Mf\overline{Mg} \quad \text{where} \quad Mf = \int_X f(x)d\mu(x).$$

In what follows we use the fact that $c_{kn} = 2\sqrt{\alpha_{kn}(0)\alpha_{kn}(1)} \in (0, 1]$. We assume that $\alpha_{rk}(s) = 1/2$, $r = 1, 2$, $s \in \mathbb{F}_2$. Obviously,

$$Ma_n^{(r)} = c_{rn} - 1, \quad t_{kn}(r) = \alpha_{1k}(0) + c_{rn}\alpha_{1k}(1) = 2^{-1}(1 + c_{rn}), \quad r = 1, 2, \ k > 2. \tag{6.21}$$

Indeed, we get for $r = 1$

$$t_{kn}(1) = (T_{kn}(1)\mathbf{1}, \mathbf{1}) = \left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & a_n^{-1} \\ 0 & 0 & a_n & 0 \end{pmatrix}\mathbf{1}, \mathbf{1}\right) \overset{(6.16)}{=} ((1, 1, a_n^{-1}, a_n), (1, 1, 1, 1)) \overset{(4.12)}{=}$$

$$\alpha_{1k}(0)\alpha_{1n}(0) + \alpha_{1k}(0)\alpha_{1n}(1) + \sqrt{\frac{\alpha_{1n}(1)}{\alpha_{1n}(0)}}\alpha_{1k}(1)\alpha_{1n}(0) + \sqrt{\frac{\alpha_{1n}(0)}{\alpha_{1n}(1)}}\alpha_{1k}(1)\alpha_{1n}(1)$$

$$= \alpha_{1k}(0) + c_{1n}\alpha_{1k}(1).$$

Further, we have

$$\|(T_{kn}(2) - t_{kn}(2))\mathbf{1}\|^2 = 1 - t_{kn}^2(2) = 1 - \left(\frac{1 + c_{2n}}{2}\right)^2,$$

$$\|(T_{2n}(2) - t_{2n}(2))\mathbf{1}\|^2 = 1 - c_{2n}^2,$$

$$\|x_{1k}\mathbf{1}\|^2 = \left\|\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\|^2 = \|(0, 1)\|^2 = \alpha_{1k}(1) = 1/2, \tag{6.22}$$

$$\|(T_{\alpha_{1n}} - c_{1n})\mathbf{1}\|^2 = 1 - c_{1n}^2. \tag{6.23}$$

Using (6.20) we get (6.18):

$$a_{nn} = 1 - t_{kn}^2(2) + t_{kn}^2(2)\alpha_{1k}(1)(1 - c_{1n}^2) = 1 - t_{kn}^2(2)\frac{1 + c_{1n}^2}{2} = \tag{6.24}$$

$$1 - \left((1 + c_{2n})/2\right)^2\left(1 + c_{1n}^2\right)/2.$$

Since $(f_n, g_m^c) = (g_n^c, f_m) = (g_n^c, g_m^c) = 0$, for $n \neq m$, we get

$$a_{nm} = (f_n + g_n^c, f_m + g_m^c) = (f_n, f_m) + (f_n, g_m^c) + (g_n^c, f_m) + (g_n^c, g_m^c) = (f_n, f_m).$$

Indeed,

$$(f_n, g_m^c) = (T_{kn}(1) \otimes (T_{kn}(2) - t_{kn}(2))\mathbf{1}, x_{1k} \otimes (T_{\alpha_{1m}} - c_{1m}) \otimes t_{km}(2)\mathbf{1}),$$

$$= t_{km}(2)(T_{kn}(1)\mathbf{1}, x_{1k} \otimes (T_{\alpha_{1m}} - c_{1m})\mathbf{1})((T_{kn}(2) - t_{kn}(2))\mathbf{1}, \mathbf{1})$$

$$= C_1((T_{kn}(2) - t_{kn}(2))\mathbf{1}, \mathbf{1}) = 0, \quad (g_n^c, f_m) = C_2((T_{km}(2) - t_{km}(2))\mathbf{1}, \mathbf{1}) = 0,$$

$$(g_n^c, g_m^c) = t_{kn}(2)t_{km}(2)\|x_{1k}\mathbf{1}\|^2((T_{\alpha_{1n}} - c_{1n})\mathbf{1}, (T_{\alpha_{1m}} - c_{1m})\mathbf{1}) = 0,$$

where C_k are some constants. Finally, we get

$$a_{nm} = (f_n, f_m) = (T_{kn}(1)\mathbf{1}, T_{km}(1)\mathbf{1})((T_{kn}(2) - t_{kn}(2))\mathbf{1}, (T_{km}(2) - t_{km}(2))\mathbf{1})$$
$$\tag{6.25}$$
$$= (T_{kn}(1)\mathbf{1}, T_{km}(1)\mathbf{1})\big[(T_{kn}(2)\mathbf{1}, T_{km}(2)\mathbf{1}) - t_{kn}(2)t_{km}(2)\big].$$

Using (6.8) we get

$$(T_{kn}(1)\mathbf{1}, T_{km}(1)\mathbf{1}) = \big([I + x_{1k} \otimes (T_{\alpha_{1n}} - I)]\mathbf{1}, [I + x_{1k} \otimes (T_{\alpha_{1m}} - I)]\mathbf{1}\big) = 1+$$

$$(\mathbf{1}, x_{1k}\otimes(T_{\alpha_{1m}}-I)\mathbf{1}) + (x_{1k}\otimes(T_{\alpha_{1n}}-I)\mathbf{1}, \mathbf{1}) + (x_{1k}\otimes(T_{\alpha_{1n}}-I)\mathbf{1}, x_{1k}\otimes(T_{\alpha_{1m}}-I)\mathbf{1})$$
$$= 1 + \alpha_{1k}(1)[(c_{1m}-1) + (c_{1n}-1) + (c_{1m}-1)(c_{1n}-1)] = 1 + \alpha_{1k}(1)(c_{1n}c_{1m}-1)$$
$$= \alpha_{1k}(0) + \alpha_{1k}(1)c_{1n}c_{1m} = (1 + c_{1n}c_{1m})/2$$

for $\alpha_{1k}(0) = \alpha_{1k}(1) = 1/2$. Finally, we get

$$a_{nm} = \frac{1 + c_{1n}c_{1m}}{2}\left(\frac{1 + c_{2n}c_{2m}}{2} - \frac{1 + c_{2n}}{2}\frac{1 + c_{2m}}{2}\right) = \frac{1 + c_{1n}c_{1m}}{8}(1 - c_{2n})(1 - c_{2m}).$$

This proves (6.19). In addition we have

$$b_n = t_{kn}(2)Ma_n^{(1)} = -(1 + c_{2n})(1 - c_{1n})/2, \quad t_{kn}(2) = (1 + c_{2n})/2.$$

We shall estimate $(A^{-1}b, b)$ for an operator $A = (a_{nm})_{n,m}$ defined as follows:

$$a_{nn} = 1 - \left(\frac{1 + c_{2n}}{2}\right)^2\frac{1 + c_{1n}^2}{2}, \quad a_{nm} = (1 + c_{1n}c_{1m})(1 - c_{2n})(1 - c_{2m})/8.$$

We have $A = DD_m(\lambda, c)D$ where

$$D = \mathrm{diag}(d_n)_{n=1}^m, \quad d_n = (1 - c_{2n})(2\sqrt{2})^{-1}, \quad c = (c_n)_n, \quad c_n = c_{1n}.$$

Finally, we get

$$(A^{-1}b, b) = (D^{-1}D_m^{-1}(\lambda, c)D^{-1}b, b) = (D_m^{-1}(\lambda, c)D^{-1}b, D^{-1}b) = (D_m^{-1}(\lambda, c)\mu, \mu)$$

where $\mu = D^{-1}b$. Lemma 6.2 finish the proof. Since $1 + \lambda_n + c_n^2 = \frac{a_{nn}}{d_n^2}$ we get

$$\lambda_n = \frac{a_{nn} - (1 + c_n^2)d_n^2}{d_n^2} = \frac{1 - \left(\frac{1+c_{2n}}{2}\right)^2\frac{1+c_{1n}^2}{2} - \frac{(1+c_{1n}^2)(1-c_{2n})^2}{8}}{\frac{(1-c_{2n})^2}{8}}$$

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$$= \frac{8 - (1 + c_{1n}^2)[(1 + c_{2n})^2 + (1 - c_{2n})^2]}{(1 - c_{2n})^2}$$

$$= \frac{2[4 - (1 + c_{1n}^2)(1 + c_{2n}^2)]}{(1 - c_{2n})^2} = \frac{2(1 - c_{1n}^2 + 1 - c_{2n}^2 + 1 - c_{1n}^2 c_{2n}^2)}{(1 - c_{2n})^2}.$$

We have

$$\Gamma(g) = \sum_{n=2}^{\infty} \frac{1}{\lambda_n} = \sum_n \frac{(1 - c_{2n})^2}{2(1 - c_{1n}^2 + 1 - c_{2n}^2 + 1 - c_{1n}^2 c_{2n}^2)}.$$

Since

$$b_n = -\frac{(1 + c_{2n})(1 - c_{1n})}{2}, \quad \text{we get} \quad \mu_n = \frac{b_n}{d_n} = -\frac{\sqrt{2}(1 + c_{2n})(1 - c_{1n})}{(1 - c_{2n})}$$

therefore,

$$\Gamma(f) = \sum_{n=2}^{\infty} \frac{\mu_n^2}{\lambda_n} = \sum_n \frac{(1 + c_{2n})^2(1 - c_{1n})^2}{2(1 - c_{1n}^2 + 1 - c_{2n}^2 + 1 - c_{1n}^2 c_{2n}^2)}.$$

$\square$

Lemma 6.4. *We have $x_{12}\mathbf{1} \in \langle [T_{2n} - c_{2n}]\,\mathbf{1} \mid n > k \rangle$ if $\Sigma_{12} = \infty$ where*

$$\Sigma_{12} = \sum_{n=3}^{\infty} \frac{(1 - c_{1n})^2 c_{2n}^2}{(1 - c_{1n})c_{2n}^2 + 1 - c_{2n}}. \tag{6.26}$$

PROOF. Using (6.8) we get $T_{2n} = T_{2n}(1) \otimes T_{2n}(2)$ so,

$$T_{2n} - c_{2n} = (I + x_{12}(T_{1n} - I)) \otimes T_{\alpha 2n} - c_{2n} = x_{12}(T_{1n} - I) \otimes T_{\alpha 2n} + T_{\alpha 2n} - c_{2n}.$$

If we chose $t = (t_n)_{n=3}^m$ such that $\sum_{n=3}^m t_n M\xi_n = 1$ we get

$$\left\| \left[\sum_{n=3}^m t_n(T_{2n} - c_{2n}) - x_{12} \right]\mathbf{1} \right\| = \left\| x_{12}\left(\sum_{n=3}^m t_n(T_{1n} - I) \otimes T_{\alpha 2n} - I \right)\mathbf{1} + \sum_{n=3}^m t_n(T_{\alpha 2n} - c_{2n})\mathbf{1} \right\|^2$$

$$\left\| \sum_{n=3}^m t_n\left[x_{12}(\xi_n - M\xi_n) + \eta_n \right] \right\|^2 = \sum_{n,k=3}^m t_n t_k(h_n, h_k) = \sum_{n=3}^m t_n^2(h_n, h_n),$$

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since $(h_n, h_k) = 0$ for $n \neq k$, where $h_n = x_{12}(\xi_n - M\xi_n) + \eta_n$. Indeed, we show that

$$(h_n, h_k) = 0, \text{ for } n \neq k, \quad (h_n, h_n) \sim 2(1 - c_{1n})c_{2n}^2 - (1 - c_{1n})^2 c_{2n}^2 + 1 - c_{2n}^2. \quad (6.27)$$

We have

$$(h_n, h_n) = (x_{12}(\xi_n - M\xi_n) + \eta_n, x_{12}(\xi_n - M\xi_n) + \eta_n) =$$

$$(x_{12}\mathbf{1}, \mathbf{1})\|\xi_n - M\xi_n\|^2 + \|\eta_n\|^2 + 2(x_{12}\mathbf{1}, \mathbf{1})(\xi_n - M\xi_n, \eta_n) =$$

$$\alpha_{12}(1)\big[2(1 - c_{1n}) - (1 - c_{1n})^2 c_{2n}^2\big] + 1 - c_{2n}^2 + 2\alpha_{12}(1)(c_{1n} - 1)(1 - c_{2n}^2) =$$

$$2\alpha_{12}(1)(1 - c_{1n})c_{2n}^2 + 1 - c_{2n}^2 - \alpha_{12}(1)(1 - c_{1n})^2 c_{2n}^2$$

since

$$(\xi_n - M\xi_n, \eta_n) = \big(\big[(T_{\alpha_{1n}} - 1)T_{\alpha_{2n}} - (c_{1n} - 1)c_{2n}\big]\mathbf{1}, (T_{\alpha_{2n}} - c_{2n})\mathbf{1}\big) = ((T_{\alpha_{1n}} - 1)\mathbf{1}, \mathbf{1})$$

$$\times (T_{\alpha_{2n}}\mathbf{1}, (T_{\alpha_{2n}} - c_{2n})\mathbf{1}) - (c_{1n} - 1)c_{2n}((T_{\alpha_{2n}} - c_{2n})\mathbf{1}, \mathbf{1}) = (c_{1n} - 1)(1 - c_{2n}^2).$$

Finally, we get

$$\min_{t \in \mathbb{R}^{m-2}} \bigg(\sum_{n=3}^{m} t_n^2(h_n, h_n) \mid \sum_{n=3}^{m} t_n M\xi_n = 1\bigg) \to 0 \Leftrightarrow \sum_{n=3}^{\infty} \frac{|M\xi_n|^2}{(h_n, h_n)} \sim \Sigma_{12} = \infty.$$

We use the following equivalence

$$\sum_{n=3}^{\infty} \frac{|M\xi_n|^2}{(h_n, h_n)} = \sum_{n=3}^{\infty} \frac{(1 - c_{1n})^2 c_{2n}^2}{2\alpha_{12}(1)(1 - c_{1n})c_{2n}^2 + 1 - c_{2n}^2 - \alpha_{12}(1)(1 - c_{1n})^2 c_{2n}^2}$$

$$\sim \sum_{n=3}^{\infty} \frac{(1 - c_{1n})^2 c_{2n}^2}{(1 - c_{1n})c_{2n}^2 + 1 - c_{2n}} = \Sigma_{12}.$$

$\square$

To find the conditions of the approximation of x_{2k} it is sufficient to interchange c_{1n} and c_{2n} in Lemma 6.3. Namely, by analogy we have the following lemma.

Lemma 6.5. *We have $x_{2k}\mathbf{1} \in \langle [T_{kn} - t_{kn}(1)]\,\mathbf{1} \mid n > k \rangle \Leftrightarrow \Sigma_{2m} \to \infty$ where*

$$\Sigma_{2m} := (D_m^{-1}(\lambda, c)\mu, \mu) = \frac{\Gamma(f_m^{(2)}) + \Gamma(f_m^{(2)}, g_m^{(2)}) + \Gamma(f_m^{(2)}, h_m^{(2)})}{1 + \Gamma(g_m^{(2)}) + \Gamma(h_m^{(2)}) + \Gamma(g_m^{(2)}, h_m^{(2)})}, \qquad (6.28)$$

the vectors $f_m^{(2)}$, $g_m^{(2)}$, $h_m^{(2)}$ are defined by (6.13) and

$$\lambda_n = \frac{2(1 - c_{1n}^2 + 1 - c_{2n}^2 + 1 - c_{1n}^2 c_{2n}^2)}{(1 - c_{1n})^2}, \quad c_n = c_{2n}, \quad \mu_n = -\frac{\sqrt{2}(1 + c_{1n})(1 - c_{2n})}{(1 - c_{1n})}.$$

Finally, to approximate x_{1k} or x_{2k} it is sufficient to have respectively $\Sigma_{1m} \to \infty$ or $\Sigma_{2m} \to \infty$ where

$$\Sigma_{1m} = \frac{\Gamma(f_m) + \Gamma(f_m, g_m) + \Gamma(f_m, h_m)}{1 + \Gamma(g_m) + \Gamma(h_m) + \Gamma(g_m, h_m)} \sim$$

$$\frac{1 + \sum_{k=1}^{3}\Gamma(x_k) + \sum_{1 \le k < r \le 3}\Gamma(x_k, x_r) + \Gamma(x_1, x_2, x_3)}{1 + \Gamma(x_2) + \Gamma(x_3) + \Gamma(x_2, x_3)}$$

(where $x_1 = f_m, \quad x_2 = g_m, \quad x_3 = h_m$)

$$= \frac{\det[I + \gamma(f_m, g_m, h_m)]}{\det[I + \gamma(g_m, h_m)]} \sim \frac{\det[I + \gamma(F^m, G^m, H_1^m)]}{\det[I + \gamma(G^m, H_1^m)]},$$

$$\Gamma(g) = \sum_n \frac{1}{\lambda_n} = \sum_n \frac{(1 - c_{2n})^2}{2(1 - c_{1n}^2 + 1 - c_{2n}^2 + 1 - c_{1n}^2 c_{2n}^2)}, \quad \Gamma(h) = \sum_n \frac{c_{1n}^2}{\lambda_n} \le \Gamma(g),$$

$$\Gamma(f) = \sum_n \frac{\mu_n^2}{\lambda_n} = \sum_n \frac{(1 + c_{2n})^2(1 - c_{1n})^2}{2(1 - c_{1n}^2 + 1 - c_{2n}^2 + 1 - c_{1n}^2 c_{2n}^2)},$$

and

$$\Sigma_{2m} = \frac{\Gamma(f_m^{(2)}) + \Gamma(f_m^{(2)}, g_m^{(2)}) + \Gamma(f_m^{(2)}, h_m^{(2)})}{1 + \Gamma(g_m^{(2)}) + \Gamma(h_m^{(2)}) + \Gamma(g_m^{(2)}, h_m^{(2)})} \sim$$

$$\frac{1 + \sum_{k=1}^{3}\Gamma(x_k) + \sum_{1 \le k < r \le 3}\Gamma(x_k, x_r) + \Gamma(x_1, x_2, x_3)}{1 + \Gamma(x_2) + \Gamma(x_3) + \Gamma(x_2, x_3)}$$

(where $x_1 = f_m^{(2)}, \quad x_2 = g_m^{(2)}, \quad x_3 = h_m^{(2)}$)

$$= \frac{\det[I + \gamma(f_m^{(2)}, g_m^{(2)}, h_m^{(2)})]}{\det[I + \gamma(g_m^{(2)}, h_m^{(2)})]} \sim \frac{\det[I + \gamma(G^m, F^m, H_2^m)]}{\det[I + \gamma(F^m, H_2^m)]},$$

$$\Gamma(g^{(2)}) = \sum_n \frac{1}{\lambda_n} = \sum_n \frac{(1 - c_{1n})^2}{2(1 - c_{1n}^2 + 1 - c_{2n}^2 + 1 - c_{1n}^2 c_{2n}^2)}, \quad \Gamma(h^{(2)}) = \sum_n \frac{c_{2n}^2}{\lambda_n} \leq \Gamma(g^{(2)}),$$

$$\Gamma(f^{(2)}) = \sum_n \frac{\mu_n^2}{\lambda_n} = \sum_n \frac{(1 + c_{1n})^2(1 - c_{2n})^2}{2(1 - c_{1n}^2 + 1 - c_{2n}^2 + 1 - c_{1n}^2 c_{2n}^2)}.$$

Since $1 < 1 + c_{rn} \leq 2$ finally, we get

$$\Sigma_{1m} \simeq \frac{\Gamma(F^m) + \Gamma(F^m, G^m) + \Gamma(F^m, H_1^m)}{1 + \Gamma(G^m) + \Gamma(H_1^m) + \Gamma(G^m, H_1^m)}, \tag{6.29}$$

$$\Sigma_{2m} \simeq \frac{\Gamma(G^m) + \Gamma(G^m, F^m) + \Gamma(G^m, H_2^m)}{1 + \Gamma(F^m) + \Gamma(H_2^m) + \Gamma(F^m, H_2^m)}, \tag{6.30}$$

where we denote

$$F_n = \frac{1 - c_{1n}}{(1 - c_{1n} + 1 - c_{2n} + 1 - c_{1n}c_{2n})^{1/2}}, \quad G_n = \frac{1 - c_{2n}}{(1 - c_{1n} + 1 - c_{2n} + 1 - c_{1n}c_{2n})^{1/2}}, \tag{6.31}$$

$$F = (F_n)_n, \ G = (G_n)_n, \ H_1 = (H_n^1)_n, \quad H_2 = (H_n^2)_n, \ H_n^1 = G_n c_{1n}, \ H_n^2 = F_n c_{2n},$$

$$F^m = (F_n)_{n=2}^m, \quad G^m = (G_n)_{n=2}^m, \quad H_1^m = (H_n^1)_{n=2}^m, \quad H_2^m = (H_n^2)_{n=2}^m. \tag{6.32}$$

Lemma 6.6. *If* $S_{11}^L(\mu_\alpha) + S_{22}^L(\mu_\alpha) = \infty$ *then*

$$\Gamma(f) + \Gamma(g) = \infty, \quad \Gamma(f^{(2)}) + \Gamma(g^{(2)}) = \infty \quad and \quad \Gamma(F) + \Gamma(G) = \infty. \tag{6.33}$$

PROOF. Since $c_{kn} = 2\sqrt{\alpha_{kn}(0)\alpha_{kn}(1)} \in (0, 1]$ we conclude that

$$\Gamma(f) \sim \Gamma(g^{(2)}) \sim \Gamma(F) = \sum_{n=3}^{\infty} \frac{(1 - c_{1n})^2}{(1 - c_{1n} + 1 - c_{2n} + 1 - c_{1n}c_{2n})} \tag{6.34}$$

$$\sim \sum_{n=3}^{\infty} \frac{(1 - c_{1n})^2(1 + c_{2n})^2}{(1 - c_{1n} + 1 - c_{2n} + 1 - c_{1n}c_{2n})},$$

$$\Gamma(g) \sim \Gamma(f^{(2)}) \sim \Gamma(G) = \sum_{n=3}^{\infty} \frac{(1 - c_{2n})^2}{(1 - c_{1n} + 1 - c_{2n} + 1 - c_{1n}c_{2n})} \tag{6.35}$$

$$\sim \sum_{n=3}^{\infty} \frac{(1 + c_{1n})^2(1 - c_{2n})^2}{(1 - c_{1n} + 1 - c_{2n} + 1 - c_{1n}c_{2n})}.$$

If $\Gamma(f) + \Gamma(g) < \infty$ or $\Gamma(f^{(2)}) + \Gamma(g^{(2)}) < \infty$ we get $\Gamma(F) + \Gamma(G) < \infty$ and

$$\sum_{n=3}^{\infty} \frac{(1 + c_{1n})^2(1 - c_{2n})^2}{(1 - c_{1n} + 1 - c_{2n} + 1 - c_{1n}c_{2n})} + \sum_{n=3}^{\infty} \frac{(1 - c_{1n})^2(1 + c_{2n})^2}{(1 - c_{1n} + 1 - c_{2n} + 1 - c_{1n}c_{2n})} < \infty$$

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therefore, $\Sigma < \infty$ where

$$\Sigma := \sum_{n=3}^{\infty} \frac{[(1+c_{1n})(1-c_{2n}) + (1-c_{1n})(1+c_{2n})]^2}{4(1-c_{1n}+1-c_{2n}+1-c_{1n}c_{2n})} =$$

$$\sum_{n=3}^{\infty} \frac{(1-c_{1n}c_{2n})^2}{(1-c_{1n}+1-c_{2n}+1-c_{1n}c_{2n})}.$$

Finally, $\Gamma(F) + \Gamma(G) + \Sigma < \infty$ hence,

$$\infty > \sum_{n} \frac{(1-c_{1n}+1-c_{2n}+1-c_{1n}c_{2n})^2}{(1-c_{1n}+1-c_{2n}+1-c_{1n}c_{2n})} = \sum_{n}(1-c_{1n}+1-c_{2n}+1-c_{1n}c_{2n})$$

$$> S_{11}^L(\mu_\alpha) + S_{22}^L(\mu_\alpha) = \infty.$$

This contradiction proves (6.33). □

Remark 6.2. We have proved the convergence $\sum_{n=N_1}^{N_2} t_n[T_{kn}-t_{kn}(r)] \to x_{rk}$ for $r = 1, 2$ only on the vector $f = \mathbf{1}$. The same argument holds for the total set of vectors of the form $f = \otimes_{k=1}^{n} f_k \otimes \mathbf{1} \otimes \mathbf{1} \cdots$ in the space $L^2(X^2, \mu^2) = H_{12} \otimes_{k=3}^{\infty} (H_{1k} \otimes H_{2k})$. Hence, $x_{rk} \in \mathfrak{A}^2$. In what follows we will use the same arguments.

It is useful to use the analogy with the case of the field $\mathbf{k} = \mathbb{R}$.

Remark 6.3. In the case of the field $\mathbf{k} = \mathbb{R}$ the generators A_{kn} and A_{2n} of the corresponding one-parameter groups has the following form [19]:

$$A_{kn} = x_{1k}D_{1n} + x_{2k}D_{2n}, \quad A_{2n} = x_{12}D_{1n} + D_{2n}.$$

If we are able to approximate the variable x_{2n}, but not the x_{1n}. It is reasonable to use the following expressions

$$A_{kn} - x_{2k}A_{2n} = (x_{1k} - x_{12}x_{2k})D_{1n} \tag{6.36}$$

in order to approximate first the expression $x_{1k} - x_{12}x_{2k}$ by linear combinations $\sum_n t_n(x_{1k} - x_{12}x_{2k})D_{1n}^2$. Further, we can approximate the variable x_{12} by $\sum_n t_k(x_{1k} - x_{12}x_{2k})$. After we can approximate the variable x_{1k} (see also the details in [17]).

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Let $x_{2k} = \text{diag}(0,1) \in \mathfrak{A}^2$. We *try to guess* an analogue of the expression $A_{kn} - x_{2k}A_{2n}$. The analogue of A_{kn} is $T_{kn} - I = C(T_{kn}) - p$, by Remark 4.3. So, the analogue of $A_{kn} - x_{2k}A_{2n}$ is $T_{kn} - I - x_{2k} \otimes (T_{2n} - I)$. We shall use the following combinations:

$$T_{kn} - I - x_{2k} \otimes (T_{2n} - I). \tag{6.37}$$

For $k < n$ set

$$\tau_{kn} = (T_{\alpha_{kn}} - I). \tag{6.38}$$

Using (5.5) we get

$$T_{kn} = (I + x_{1k}\tau_{1n})(I + x_{2k}\tau_{2n}) = I + x_{1k}\tau_{1n} + x_{2k}\tau_{2n} + x_{1k}x_{2k}\tau_{1n}\tau_{2n},$$

$$T_{2n} = (I + x_{12}\tau_{1n})(I + \tau_{2n}) = I + x_{12}\tau_{1n} + \tau_{2n} + x_{12}\tau_{1n}\tau_{2n},$$

therefore, we get

$$T_{kn} - I - x_{2k} \otimes (T_{2n} - I) = (x_{1k} - x_{12}x_{2k})\tau_{1n} + (x_{1k}x_{2k} - x_{12}x_{2k})\tau_{1n}\tau_{2n}. \tag{6.39}$$

Lemma 6.7. *We have*

$$(x_{1k} - x_{12}x_{2k})\mathbf{1} \in \langle [T_{kn} - I - x_{2k}(T_{2n} - I)]\,\mathbf{1} \mid n > k \rangle$$

if and only if $\Delta(f'_m, g'_m) \to \infty$ where

$$\Delta(f'_m, g'_m) = \frac{\Gamma(f'_m) + \Gamma(f'_m, g'_m)}{1 + \Gamma(g'_m)}, \tag{6.40}$$

and f'_m, $g'_m \in \mathbb{R}^{m-2}$ are defined as follows:

$$f'_m = (\sqrt{1 - c_{1n}})_{n=3}^m, \quad g'_m = (\sqrt{1 - c_{1n}}(1 - c_{2n}))_{n=3}^m. \tag{6.41}$$

PROOF. Set $b_n = M(T_{\alpha_{1n}} - I)\mathbf{1} = M\tau_{1n}\mathbf{1} = c_{1n} - 1$,

$$f_n = [(x_{1k} - x_{12}x_{2k})\tau_{1n} + (x_{1k}x_{2k} - x_{12}x_{2k})\tau_{1n}\tau_{2n}]\mathbf{1},$$

$$f_n^c = [(x_{1k} - x_{12}x_{2k})(\tau_{1n} - M\tau_{1n}\mathbf{1}) + (x_{1k}x_{2k} - x_{12}x_{2k})\tau_{1n}\tau_{2n}]\mathbf{1}$$
$$= [(x_{1k} - x_{12}x_{2k})(T_{\alpha_{1n}} - c_{1n}) - (x_{1k}x_{2k} - x_{12}x_{2k}) \otimes (T_{\alpha_{1n}} - I) \otimes (T_{\alpha_{2n}} - I)]\mathbf{1}.$$

Take $t = (t_n)_{n=3}^N$ such that $\sum_{n=3}^N t_n b_n = 1$, then we get

$$\left\| \sum_{n=3}^N t_n f_n - (x_{1k} - x_{12}x_{2k})\mathbf{1} \right\|^2 = \left\| \sum_{n=3}^N t_n f_n^c \right\|^2 = \sum_{n=3}^N t_n t_m (f_n^c, f_m^c) = (At, t),$$

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where
$$A = (a_{nm})_{n,m=3}^{N}, \quad a_{nm} = (f_n^c, f_m^c)_{n,m=3}^{N}.$$

As before, we use the following estimation for a positively definite operator A acting in a space H and a vector $b \in H$:

$$\min_{t \in H} \left((At, t) \mid (t, b) = 1 \right) = (A^{-1}b, b)^{-1}.$$

We show that

$$a_{nn} = (2 - c_{2n})(1 - c_{1n}^2)/2 + (1 - c_{1n})(1 - c_{2n}), \tag{6.42}$$

$$a_{nm} = d_n d_m, \; n \neq m, \quad \text{where} \quad d_n = (1 - c_{1n})(1 - c_{2n})/2. \tag{6.43}$$

Set

$$h = (x_{1k} - x_{12}x_{2k}), \quad g = (x_{1k}x_{2k} - x_{12}x_{2k}),$$

$$\xi_n = (T_{\alpha_{1n}} - c_{1n})\mathbf{1}, \quad \eta_n = (T_{\alpha_{1n}} - I) \otimes (T_{\alpha_{2n}} - I)\mathbf{1}.$$

We can suppose that $\mu_{\alpha_{1k}} = \mu_{\alpha_{2k}}$ are invariant measures for fixed k. This does not change the equivalence class of the measure μ hence, the equivalence class of the representation. Then using (6.22) we get

$$a_{nn} = (f_n^c, f_n^c) = \|h\|^2 \|\xi_n\|^2 + \|g\|^2 \|\eta_n\|^2 + 2(g, h)(\xi_n, \eta_n),$$

$$\|h\|^2 = \|(x_{1k} - x_{12}x_{2k})\mathbf{1}\|^2 = ((x_{1k}^2 - 2x_{12}x_{1k}x_{2k} + x_{12}^2 x_{2k}^2)\mathbf{1}, \mathbf{1}) = 1/2,$$

$$\|g\|^2 = \|(x_{1k}x_{2k} - x_{12}x_{2k})\mathbf{1}\|^2 = (x_{1k}^2 x_{2k}^2 - 2x_{12}x_{1k}x_{2k}^2 + x_{12}^2 x_{2k}^2)\mathbf{1}, \mathbf{1}) = 1/4,$$

$$(g, h) = ((x_{1k} - x_{12}x_{2k})\mathbf{1}, (x_{1k}x_{2k} - x_{12}x_{2k})\mathbf{1})$$

$$= ((x_{1k}^2 x_{2k} - x_{12}x_{1k}x_{2k} - x_{12}x_{1k}x_{2k}^2 + x_{12}^2 x_{2k}^2)\mathbf{1}, \mathbf{1}) = 1/4,$$

$$\|\xi_n\|^2 = 1 - c_{1n}^2, \quad \|\eta_n\|^2 = 4(1 - c_{1n})(1 - c_{2n}), \quad (\xi_n, \eta_n) = (1 - c_{1n}^2)(1 - c_{2n}).$$

Finally, we get (6.42). Indeed, we have

$$a_{nn} = (1 - c_{1n}^2)/2 + (1 - c_{1n})(1 - c_{2n}) + (1 - c_{1n}^2)(1 - c_{2n})/2$$

$$= (2 - c_{2n})(1 - c_{1n}^2)/2 + (1 - c_{1n})(1 - c_{2n}).$$

We show that $(\xi_n, \eta_n) = -(1 - c_{1n}^2)(1 - c_{2n})$. Indeed, we get

$$(\xi_n, \eta_n) = ((T_{\alpha_{1n}} - c_{1n})\mathbf{1}, (T_{\alpha_{1n}} - I) \otimes (T_{\alpha_{2n}} - I)]\mathbf{1}) = ((T_{\alpha_{1n}} - c_{1n})\mathbf{1}, (T_{\alpha_{1n}} - I)\mathbf{1}) \times$$

$$((T_{\alpha_{2n}} - I)\mathbf{1}, \mathbf{1}) = \|(T_{\alpha_{1n}} - c_{1n})\mathbf{1}\|^2((T_{\alpha_{2n}} - I)\mathbf{1}, \mathbf{1}) = -(1 - c_{1n}^2)(1 - c_{2n}).$$

Further, since $(\xi_n, \xi_m) = (\xi_n, \eta_m) = 0$ for $n \neq m$ we get

$$a_{nm} = (f_n^c, f_m^c) = (h\xi_n + g\eta_n, h\xi_m + g\eta_m) = (g, g)(\eta_n, \eta_m) = d_n d_m/4,$$

where $d_n = \frac{(1-c_{1n})(1-c_{2n})}{2}$. This proves (6.43). We use the fact that $(g, g) = 1/4$ and

$$(\eta_n, \eta_m) = ((T_{\alpha_{1n}} - I) \otimes (T_{\alpha_{2n}} - I)\mathbf{1}, (T_{\alpha_{1m}} - I) \otimes (T_{\alpha_{2m}} - I)\mathbf{1})$$

$$= ((T_{\alpha_{1n}} - I) \otimes (T_{\alpha_{2n}} - I)\mathbf{1}, \mathbf{1})(T_{\alpha_{1m}} - I)(T_{\alpha_{2m}} - I)\mathbf{1}, \mathbf{1})$$

$$= (1 - c_{1n})(1 - c_{2n})(1 - c_{1m})(1 - c_{2m}).$$

Since a_{nm} is a product $a_{nm} = d_n d_m/4$ we can use the particular case of Lemma 6.2 for $c = 0$. to calculate $(A^{-1}b, b)$. We have $A = DD_m(\lambda)D$ where $D = \mathrm{diag}(D_n)_{n=1}^m$ and $D_m(\lambda)$ is defined by (6.10). Finally, we get if we set $D^{-1}b = \mu$

$$(A^{-1}b, b) = (D_m^{-1}(\lambda, c)\mu, \mu) \overset{(6.11)}{=} \frac{\Gamma(f_m) + \Gamma(f_m, g_m)}{1 + \Gamma(g_m)} = \Delta(f_m, g_m),$$

where $f_m = (\mu_k/\sqrt{\lambda_k})_{k=1}^m$, $g_m = \left(1/\sqrt{\lambda_k}\right)_{k=1}^m$ (see (6.13)). To calculate $\Delta(f_m, g_m)$ we have $\lambda_n = \frac{a_{nn}}{d_n^2} - 1 = \frac{a_{nn} - d_n^2}{d_n^2}$ and $\mu_n = \frac{b_n}{d_n} = -\frac{2}{(1-c_{2n})}$ therefore,

$$\lambda_n = \frac{(2 - c_{2n})(1 - c_{1n}^2)/2 + (1 - c_{1n})(1 - c_{2n}) - (1 - c_{1n})^2(1 - c_{2n})^2/4}{(1 - c_{1n})^2(1 - c_{2n})^2/4}$$

$$= \frac{2(2 - c_{2n})(1 + c_{1n}) + 4(1 - c_{2n}) - (1 - c_{1n})(1 - c_{2n})^2}{(1 - c_{1n})(1 - c_{2n})^2}$$

$$= \frac{2(1 + c_{1n})(2 - c_{2n}) + (1 - c_{2n})[4 - (1 - c_{1n})(1 - c_{2n})]}{(1 - c_{1n})(1 - c_{2n})^2}.$$

Finally, we get $\Gamma(g) = \sum_{n=3}^\infty \frac{1}{\lambda_n}$ and $\Gamma(f) = \sum_{n=3}^\infty \frac{\mu_n^2}{\lambda_n}$ or

$$\Gamma(g) = \sum_{n=3}^\infty \frac{(1 - c_{1n})(1 - c_{2n})^2}{f(x_n, y_n)}, \quad \Gamma(f) = 4\sum_{n=3}^\infty \frac{(1 - c_{1n})}{f(x_n, y_n)},$$

where $f(x_n, y_n) = 2(2 + x_n)(1 + y_n) + y_n(4 - x_n y_n)$ and $x_n = 1 - c_{1n}$, $y_n = 1 - c_{2n}$.

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Since $4 \leq f(x_n, y_n) \leq 16$ we conclude that $\Gamma(f) \sim \Gamma(f')$ and $\Gamma(g) \sim \Gamma(g')$ where f' and g' are defined as follows (see (6.41))

$$f' = (\sqrt{1 - c_{1n}})_{n=3}^{\infty}, \quad g' = (\sqrt{1 - c_{1n}}(1 - c_{2n}))_{n=3}^{\infty}.$$

Hence, $\Gamma(f') = \sum_{n=3}^{\infty}(1 - c_{1n}) = S_{11}^{L}(\mu) = \infty, \quad \Gamma(g') = \sum_{n=3}^{\infty}(1 - c_{1n})(1 - c_{2n})^2.$

Finally, we get $\quad \lim_{m} \Delta(f_m, g_m) \sim \lim_{m} \Delta(f'_m, g'_m) = \dfrac{\Gamma(f') + \Gamma(f', g')}{1 + \Gamma(g')}.$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The schema of the proof of irreducibility for $m = 2$. Recall some notations (see (6.29)–(6.32))

$$\Sigma_{12} = \sum_{n=2}^{\infty} \frac{(1 - c_{1n})^2 c_{2n}^2}{(1 - c_{1n}) c_{2n}^2 + 1 - c_{2n}}, \quad d_n = 1 - c_{1n} + 1 - c_{2n} + 1 - c_{1n} c_{2n},$$

$$\Gamma(F^m) = \|F^m\|^2 = \sum_{n=3}^{m} \frac{(1 - c_{1n})^2}{d_n}, \quad \Gamma(G^m) = \|G^m\|^2 = \sum_{n=3}^{m} \frac{(1 - c_{2n})^2}{d_n},$$

$$\Gamma(H_1^m) = \|H_1^m\|^2 = \sum_{n=3}^{m} \frac{(1 - c_{2n})^2 c_{1n}^2}{d_n}, \quad \Gamma(H_2^m) = \|H_2^m\|^2 = \sum_{n=3}^{m} \frac{(1 - c_{1n})^2 c_{2n}^2}{d_n},$$

$$\Sigma_{1m} \simeq \frac{\Gamma(F^m) + \Gamma(F^m, G^m) + \Gamma(F^m, H_1^m)}{1 + \Gamma(G^m) + \Gamma(H_1^m) + \Gamma(G^m, H_1^m)},$$

$$\Sigma_{2m} \simeq \frac{\Gamma(G^m) + \Gamma(G^m, F^m) + \Gamma(G^m, H_2^m)}{1 + \Gamma(F^m) + \Gamma(H_2^m) + \Gamma(F^m, H_2^m)},$$

$$\Delta(f, g) = \frac{\Gamma(f) + \Gamma(f, g)}{1 + \Gamma(g)}, \quad f = (f_n)_{n \in \mathbb{N}}, \; g = (g_n)_{n \in \mathbb{N}},$$

$$\Gamma(f') = \sum_{n=3}^{\infty}(1 - c_{1n}), \quad \Gamma(g') = \sum_{n=3}^{\infty}(1 - c_{1n})(1 - c_{2n})^2,$$

$$S_{11}^{L}(\mu) = \sum_{n=2}^{\infty}(1 - c_{1n}), \; S_{22}^{L}(\mu) = \sum_{n=3}^{\infty}(1 - c_{2n}), \; S_{12}^{L}(\mu) = \sum_{n=3}^{\infty}\alpha_{2n}(1)(1 - c_{1n}).$$

To prove the irreducibility, consider different cases.

Case 1. Let $\Sigma_{12} = \infty$ then by Lemma 6.4 we conclude that $x_{12} \in \mathfrak{A}^2$. By (2.9)) and (5.5) we get

$$T_{2n} = T_{2n}(1) \otimes T_{2n}(2), \quad T_{2n}(1) = I + x_{12} \otimes (T_{\alpha_{1n}} - I), \quad T_{1n} = T_{\alpha_{1n}}, \quad T_{2n}(2) = T_{\alpha_{2n}},$$
$$(6.44)$$
$$T_{kn} = T_{kn}(1) \otimes T_{kn}(2), \quad T_{kn}(1) = I + x_{1k} \otimes (T_{\alpha_{1n}} - I), \quad T_{kn}(2) = I + x_{2k} \otimes (T_{\alpha_{2n}} - I).$$
$$(6.45)$$

Remark 6.4. We note that
   (a) if $x_{12} \in \mathfrak{A}^2$ then $T_{\alpha_{2n}} \in \mathfrak{A}^2$ for $n \geq 3$;
   (b) if $x_{1k} \in \mathfrak{A}^2$ then $x_{2k} \in \mathfrak{A}^2$ for $k \geq 3$;
   (c) if $x_{12}, x_{2k} \in \mathfrak{A}^2$ for $n \geq 3$ then $x_{1k} \in \mathfrak{A}^2$ for $k \geq 3$.
   To prove (a) it is sufficient to use (6.44) and $T_{1n} = T_{\alpha_{1n}} \in \mathfrak{A}^2$. Since $x_{12}, T_{1n} \in \mathfrak{A}^2$ then $T_{2n}^{-1}(1) = T_{2n}(1) \in \mathfrak{A}^2$ therefore, $T_{\alpha_{2n}} = T_{2n}(2) = T_{2n}^{-1}(1)T_{2n} \in \mathfrak{A}^2$.
   (b) Since $T_{1k}, T_{kn} \in \mathfrak{A}^2, T_{1n} = T_{\alpha_{1n}}$ using (6.3) we conclude that $T_{kn}(1)^{-1} = T_{kn}(1) \in \mathfrak{A}^2$ therefore, $T_{kn}(1)^{-1}T_{kn} = T_{kn}(2) \in \mathfrak{A}^2$ and finally, $x_{2k} \in \mathfrak{A}^2$ by Lemma 6.8 that is an analogue of Lemma 5.1. For an arbitrary p we can use the relation $T_{kn}(1)^{-1} = T_{kn}(1)^{p-1} \in \mathfrak{A}^2$.
   (c) By (a) we get $T_{\alpha_{2n}} \in \mathfrak{A}^2$. Since $x_{2k} \in \mathfrak{A}^2$ we conclute that $T_{kn}(2) \in \mathfrak{A}^2$ therefore, $T_{kn}(1) \in \mathfrak{A}^2$ hence, $x_{1k} \in \mathfrak{A}^2$ by Lemma 5.1.

Lemma 6.8. *We have $x_{2k}\mathbf{1} \in \langle (T_{kn}(2) - I)\mathbf{1} \mid n > k \rangle$ if and only if $S_{22}^L(\mu) = \infty$.*

Remark 6.5. Since $x_{12}, T_{1n}, T_{2n+1} \in \mathfrak{A}^2$, $n \geq 2$ we conclude by Remark 6.4 (a) that $T_{\alpha_{2n}} \in \mathfrak{A}^2, n > 2$. Finally, we get $x_{12}, T_{\alpha_{1n}}, T_{\alpha_{2n}} \in \mathfrak{A}^2$ for all $n \geq 3$. This family of operators is commuting and has common simple spectrum [5] therefore, the von Neumann algebra

$$L^\infty(x_{12}, T_1, T_2) := L^\infty \begin{pmatrix} x_{12} & T_{\alpha_{13}} & \cdots & T_{\alpha_{1n}} & \cdots \\ & T_{\alpha_{23}} & \cdots & T_{\alpha_{2n}} & \cdots \end{pmatrix} \ni f \begin{pmatrix} x_{12} & T_{\alpha_{13}} & \cdots & T_{\alpha_{1n}} & \cdots \\ & T_{\alpha_{23}} & \cdots & T_{\alpha_{2n}} & \cdots \end{pmatrix},$$

generated by this family is maximal abelian subalgebra in $\mathfrak{A}^2$ and consists of all L^∞ functions, i.e., bounded operator-valued functions depending on the variables $(x_{12}, T_{\alpha_{1n}}, T_{\alpha_{2n}}, \ n \geq 3)$. Therefore, $(\mathfrak{A}^2)' \subset L^\infty(x_{12}, T_1, T_2)' = L^\infty(x_{12}, T_1, T_2)$ hence, any operator $f = f(x_{12}, T_{\alpha_{1n}}, T_{\alpha_{2n}}, \ n \geq 3)$ from $(\mathfrak{A}^2)'$ belongs to $L^\infty(x_{12}, T_1, T_2)$. Since $T_{12} \in \mathfrak{A}^2$ the relation $[f, T_{12}] = 0$ implies that the operator f does not depend on x_{12}. Finally, any operator from $(\mathfrak{A}^2)'$ is a function f in commuting family $(T_{\alpha_{1n}}, T_{\alpha_{2n}}, n \geq 3)$.

$$(\mathfrak{A}^2)' \subset L^\infty \begin{pmatrix} T_{\alpha_{13}} & \cdots & T_{\alpha_{1n}} & \cdots \\ T_{\alpha_{23}} & \cdots & T_{\alpha_{2n}} & \cdots \end{pmatrix}. \tag{6.46}$$

The commutation $[f, T_{kk+1}] = 0$ for all $k \geq 1$ implies by Lemma 4.14 and its analogue that f depends only on the following expressions:

$$\Delta_{s,\infty}^{1,r} = T_{\alpha_{1s}}^r \prod_{k=s+1}^{\infty} p^{-1} C(T_{\alpha_{1k}}), \ \ \Delta_{s,\infty}^{2,r} = T_{\alpha_{2s}}^r \prod_{k=s+1}^{\infty} p^{-1} C(T_{\alpha_{2k}}), \ r \in \mathbb{F}_p \backslash \{0\}, \ s \geq 3.$$

But the latter expressions are well defined if and only if $S_{11}^L(\mu) < \infty$ and $S_{22}^L(\mu) < \infty$ by Lemma 4.5 and its analogue.

Case 2. Let $\Sigma_{12} < \infty$ and $\Gamma(G) < \infty$ then $\Gamma(F) = \infty$. This conditions are incompatible. Indeed, since $d_n = 1 - c_{1n} + 1 - c_{2n} + 1 - c_{1n}c_{2n} < 3$ we get

$$\infty > \Gamma(G) = \sum_{n=3}^{\infty} \frac{(1-c_{2n})^2}{d_n} > \frac{1}{3} \sum_{n=3}^{\infty} (1-c_{2n})^2. \tag{6.47}$$

Therefore, $\lim_n c_{2n} = 1$ hence, $\infty > \Sigma_{12} \sim \|F\|^2 = \infty$, contradiction. Indeed, for an arbitrary $\varepsilon > 0$ we have for sufficiently big N

$$\infty > \Sigma_{12} = \sum_{n=3}^{\infty} \frac{(1-c_{1n})^2 c_{2n}^2}{(1-c_{1n})c_{2n}^2 + 1 - c_{2n}} \geq (1-\varepsilon)^2 \sum_{n=N}^{\infty} \frac{(1-c_{1n})^2}{(1-c_{1n})c_{2n}^2 + 1 - c_{2n}} >$$

$$(1-\varepsilon)^2 \sum_{n=N}^{\infty} \frac{(1-c_{1n})^2}{1 - c_{1n} + 1 - c_{2n}} \sim \|F\|^2 = \infty.$$

Case 3. Let $\Sigma_{12} < \infty$ and $\Gamma(F) < \infty$. Then $\Gamma(H_2) < \infty$ and $\Gamma(G) = \infty$ by Lemma 6.6. Therefore, $\Sigma_{2m} \to \infty$ hence, $x_{2k} \in \mathfrak{A}^2$ for $k \geq 3$. Further, using notations $\tau_{1n} = T_{\alpha_{1n}} - I$ and $\tau_{2n} = T_{\alpha_{2n}} - I$ (see (6.38)) we get

$$T_{kn} - I - x_{2k}(T_{2n} - I) = (x_{1k} - x_{12}x_{2k})\tau_{1n} + (x_{1k}x_{2k} - x_{12}x_{2k})\tau_{1n}\tau_{2n} \tag{6.48}$$

(see (6.39)). By Lemma 6.7 we conclude that $x_{1k} - x_{12}x_{2k} \in \mathfrak{A}^2$ if $\Delta(f_m', g_m') \to \infty$ where

$$\Delta(f_m', g_m') = \frac{\Gamma(f_m') + \Gamma(f_m', g_m')}{1 + \Gamma(g_m')}, \tag{6.49}$$

$$\Gamma(f_m') = \sum_{n=3}^{m} (1-c_{1n}), \quad \Gamma(g_m') = \sum_{n=3}^{m} (1-c_{1n})(1-c_{2n})^2. \tag{6.50}$$

The case (3) splits into two cases (30), when $\Delta(f_m', g_m') \to \infty$ and case (31), when $\|f' - tg'\| < \infty$ for some $t \in \mathbb{R} \backslash \{0\}$.

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In the case **30**, by Lemma 6.7, we conclude that

$$x_{1k} - x_{12}x_{2k} \in \mathfrak{A}^2 \quad \text{if} \quad \Delta(f'_m, g'_m) \to \infty.$$

We use Lemmas 6.9 and 6.10 proved in [22]:

Lemma 6.9. *Let $f = (f_k)_{k \in \mathbb{N}}$ and $g = (g_k)_{k \in \mathbb{N}}$ be two real vectors such that $\|f\|^2 = \infty$ where $\|f\|^2 = \sum_k f_k^2$. Denote by $f_{(n)}$, $g_{(n)} \in \mathbb{R}^n$ their projections to the subspace $\mathbb{R}^n$, i.e., $f_{(n)} = (f_k)_{k=1}^n$, $g_{(n)} = (g_k)_{k=1}^n$ and set*

$$\Delta(f_{(n)}, g_{(n)}) = \frac{\Gamma(f_{(n)}) + \Gamma(f_{(n)}, g_{(n)})}{\Gamma(g_{(n)}) + 1} \quad \text{then} \quad \lim_{n \to \infty} \Delta(f_{(n)}, g_{(n)}) = \infty \quad (6.51)$$

in the following cases:

$$(a) \qquad\qquad\qquad \|g\|^2 < \infty,$$

$$(b) \qquad \|g\|^2 = \infty, \quad \text{and} \quad \lim_{n \to \infty} \frac{\|f_{(n)}\|}{\|g_{(n)}\|} = \infty,$$

$$(c) \quad \|f\|^2 = \|g\|^2 = \|f + sg\|^2 = \infty, \quad \text{for all} \quad s \in \mathbb{R} \setminus \{0\}.$$

PROOF. Obviously $\lim_{n \to \infty} \Delta(f_{(n)}, g_{(n)}) = \infty$ if conditions (a) or (b) hold. The implication $(c) \Rightarrow (6.51)$ is based on the following lemma. $\qquad\square$

Lemma 6.10. *Let $f = (f_k)_{k \in \mathbb{N}}$ and $g = (g_k)_{k \in \mathbb{N}}$ be two real vectors such that*

$$\|f\|^2 = \|g\|^2 = \|C_1 f + C_2 g\|^2 = \infty, \quad \text{for all} \quad (C_1, C_2) \in \mathbb{R}^2 \setminus \{0\}, \quad (6.52)$$

$$\text{then} \quad \lim_{n \to \infty} \frac{\Gamma(f_{(n)}, g_{(n)})}{\Gamma(g_{(n)})} = \infty \quad \text{and} \quad \lim_{n \to \infty} \frac{\Gamma(f_{(n)}, g_{(n)})}{\Gamma(f_{(n)})} = \infty \quad (6.53)$$

Obviously, $\Gamma(f') = \|f'\|^2 = S_{11}^L(\mu) = \infty$. Consider the following cases:
    case (a), when $\Gamma(g') < \infty$ then $\Delta(f'_m, g'_m) \to \infty$,
    case (b), when $\Gamma(g') = \infty$ and $\Gamma(f'_m)/\Gamma(g'_m) \to \infty$ then $\Delta(f'_m, g'_m) \to \infty$,
    case (c), when $\Gamma(f'_m)/\Gamma(g'_m) \leq C$ and $\|C_1 f' + C_2 g'\|^2 = \infty$ for all $(C_1, C_2) \in \mathbb{R}^2 \setminus \{0\}$ then $\Delta(f'_m, g'_m) \to \infty$ by Lemma 6.10.

Remark 6.6. We can approximate x_{12} by linear combination of $x_{1k} - x_{12}x_{2k}$ due to Lemma 6.17 if $\sigma_{12}^{(0)}(\mu) = \infty$ (see (6.65)). The divergence $\sigma_{12}^{(0)}(\mu) = \infty$ follows from the inequality $\alpha_{kn}(0)\alpha_{kn}(1) \leq 1/4$ based on the relation $(1-x)x \leq 1/4$ for $x \in [0,1]$ and the divergence $\sigma_{12}(\mu) = \sum_{n=3}^{\infty} \alpha_{2n}^2(1) = \infty$ which follows from Lemma 6.11. The convergence $\sum_n (1 - c_{1n})^2 < \infty$ follows from the fact that $\Gamma(F) < \infty$ (see (6.47)).

Lemma 6.11. *Let* $S_{12}^L(\mu)=\infty$ *and* $\sum_n(1-c_{1n})^2<\infty$ *then* $\sigma_{12}(\mu):=\sum_n\alpha_{2n}^2(1)=\infty$.

PROOF. Using Cauchy-Schwarz inequality $\mid(x,y)\mid\leq\|x\|\cdot\|y\|$ for $x,y\in\mathbb{R}^m$ we get

$$\Big(\sum_{n=1}^m\alpha_{2n}(1)(1-c_{1n})\Big)^2\leq\Big(\sum_{n=1}^m\alpha_{2n}^2(1)\Big)\Big(\sum_{n=1}^m(1-c_{1n})^2\Big),\quad\text{for all}\quad m\in\mathbb{N}.$$

Therefore,

$$\Big(\sum_{n=1}^\infty\alpha_{2n}^2(1)\Big)\geq\Big(\sum_{n=1}^\infty(1-c_{1n})^2\Big)^{-1}\Big(\sum_{n=1}^\infty\alpha_{2n}(1)(1-c_{1n})\Big)^2=\infty.$$

$\square$

Finally, in the cases (a), (b) or (c) we can approximate $x_{1k}-x_{12}x_{2k}$. Then, by Remark 6.6 we can approximate x_{12}. Therefore, we can approximate all the variables $x_{1n},x_{2n+1},\ n\geq2$ and the proof is completed.

31. In the opposite case $(\bar{a})\cap(\bar{b})\cap(\bar{c})$, i.e., in the case $\Gamma(f_m')/\Gamma(g_m')\leq C$ and

$$\|f'-tg'\|^2=\sum_n(1-c_{1n})[1-t(1-c_{1n})]^2<\infty\tag{6.54}$$

for some $t\in\mathbb{R}\setminus\{0\}$ consider again the expressions $T_{kn}-I-x_{2k}(T_{2n}-I)$ see (6.39)

$$T_{kn}-I-x_{2k}(T_{2n}-I)=(x_{1k}-x_{12}x_{2k})\tau_{1n}+(x_{1k}x_{2k}-x_{12}x_{2k})\tau_{1n}\tau_{2n}.\tag{6.55}$$

Lemma 6.12. *We can approximate by* $\sum_{n=3}^m t_n[T_{kn}-I-x_{2k}(T_{2n}-I)]$ *the following expression:*

$$(x_{1k}-x_{12}x_{2k})+\beta(x_{1k}x_{2k}-x_{12}x_{2k})=x_{1k}-(1-s)x_{12}x_{2k}-sx_{1k}x_{2k}\tag{6.56}$$

where $s:=-\beta\in[0,1]$ *and* β *is defined as follows:*

$$\beta=\lim_m\beta_m^{(3)},\quad\beta_m^{(3)}=-\Big(\sum_{n=3}^m\frac{1-c_{1n}}{1+c_{1n}}\Big)^{-1}\sum_{n=3}^m\frac{(1-c_{1n})(1-c_{2n})}{1+c_{1n}}.\tag{6.57}$$

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PROOF. First, we show that $\sum_{n=3}^{m} t_n \tau_{1n} \to I$ for an appropriate $t = (t_n)_{n=3}^{m}$. Second, we show that $\lim_{m\to\infty} \sum_{n=3}^{m} t_n \tau_{1n} \tau_{2n} = \beta$. Indeed, set $b_n := M\tau_{1n}\mathbf{1}$ and $b_n^{(3)} := M\tau_{1n}\tau_{2n}\mathbf{1}$ then we get

$$b_n = M(T_{1n} - I)\mathbf{1} = (c_{1n} - 1), \quad b_n^{(3)} = M(T_{1n} - I)(T_{\alpha_{2n}} - I)\mathbf{1} = (1 - c_{1n})(1 - c_{2n}). \tag{6.58}$$

By Lemma 5.1 we conclude that $\sum_{n=3}^{m} t_n \tau_{1n} \to I$ if and only if $S_{11}^{L}(\mu) = \sum_{n=2}^{\infty}(1 - c_{1n}) = \infty$. Indeed, if we set $t = (t_n)_{n=3}^{m}$, $b = (b_n)_{n=3}^{m}$ we get

$$\sum_{n=3}^{m} t_n \tau_{1n} \to I \Leftrightarrow \min_{t\in\mathbb{R}^{m-2}} \left(\sum_{n=3}^{m} t_n^2 (1 - c_{1n})^2 \mid (t,b) = 1 \right) =$$

$$\left(\sum_{n=3}^{m} \frac{b_n^2}{a_n} = \sum_{n=3}^{m} \frac{(1 - c_{1n})^2}{1 - c_{1n}^2} = \sum_{n=3}^{m} \frac{1 - c_{1n}}{1 + c_{1n}} \right)^{-1} \to 0 \Leftrightarrow \sum_{n=2}^{\infty}(1 - c_{1n}) = \infty,$$

where

$$a_n = \|(T_{1n} - c_{1n})\mathbf{1}\|^2 = 1 - c_{1n}^2, \quad t_n = \frac{b_n}{a_n}\left(\sum_{n=3}^{m} \frac{b_n^2}{a_n} \right)^{-1} = -\frac{1}{1 + c_{1n}}\left(\sum_{n=3}^{m} \frac{1 - c_{1n}}{1 + c_{1n}} \right)^{-1}. \tag{6.59}$$

Further, we get

$$\left\| \left[\sum_{n=3}^{m} t_n \tau_{1n}\tau_{2n} - \beta_m^{(3)} \right]\mathbf{1} \right\|^2 = \left\| \left[\sum_{n=3}^{m} t_n \tau_{1n}\tau_{2n} - \sum_{n=3}^{m} t_n(1 - c_{1n})(1 - c_{2n}) \right]\mathbf{1} \right\|^2 =$$

$$\left\| \sum_{n=3}^{m} t_n [\tau_{1n}\tau_{2n} - (1 - c_{1n})(1 - c_{2n})]\mathbf{1} \right\|^2 = \sum_{n=3}^{m} t_n^2 \|[\tau_{1n}\tau_{2n} - (1 - c_{1n})(1 - c_{2n})]\mathbf{1}\|^2$$

$$= \sum_{n=3}^{m} t_n^2 [2(1 - c_{1n})2(1 - c_{2n}) - (1 - c_{1n})^2(1 - c_{2n})^2] \sim \sum_{n=1}^{m} t_n^2(1 - c_{1n})(1 - c_{2n}),$$

since we have $3 < 4 - (1 - x)(1 - y) \le 4$ for $x, y \in (0, 1]$ and

$$\|[\tau_{1n}\tau_{2n} - (1 - c_{1n})(1 - c_{2n})]\mathbf{1}\|^2 = \|(T_{1n} - I)(T_{\alpha_{2n}} - I)\mathbf{1}\|^2 - (1 - c_{1n})^2(1 - c_{2n})^2 =$$

$$2(1 - c_{1n})2(1 - c_{2n}) - (1 - c_{1n})^2(1 - c_{2n})^2 = (1 - c_{1n})(1 - c_{2n})[4 - (1 - c_{1n})(1 - c_{2n})].$$

We show that $\sum_{n=1}^{m} t_n^2(1 - c_{1n})(1 - c_{2n}) \to 0$. Indeed, we have

$$\sum_{n=3}^{m} t_n^2(1 - c_{1n})(1 - c_{2n}) = \left(\sum_{n=3}^{m} \frac{1 - c_{1n}}{1 + c_{1n}} \right)^{-2} \sum_{n=3}^{m} \frac{(1 - c_{1n})(1 - c_{2n})}{1 + c_{1n}} \le$$

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$$4\Big(\sum_{n=3}^{m}(1-c_{1n})\Big)^{-2}\sum_{n=3}^{m}(1-c_{1n}) = 4\Big(\sum_{n=3}^{m}(1-c_{1n})\Big)^{-1} \to 0.$$

Obviously, the sequence $\beta_m^{(3)}$ defined by (6.57) is bounded $\beta_m^{(3)} \in [-1,\, 0]$ for all $m \in \mathbb{N}$ therefore, there exists a subsequence having the limit $\beta \in [-1,\, 0]$. $\qquad\square$

We have to study two cases: (310), when $s \neq 1$ and (311), when $s = 1$.

310. When $s \neq 1$ the proof of the irreducibility is finished. Indeed, we conclude by Lemma 6.14 (we shall prove this lemma below) that $x_{12} \in \mathfrak{A}^2$ if $\sigma_{12}^{(s)}(\mu) = \infty$. The divergence $\sigma_{12}^{(s)}(\mu) = \infty$ follows from $\Sigma_k \alpha_{2k}^2(1) = \infty$ and the estimation

$$\alpha_{rk}(0)\alpha_{rk}(1) \leq 1/4, \quad \alpha_{2k}(0) + (1-s)^2\alpha_{2k}(1) \leq 1.$$

The divergence $\Sigma_k \alpha_{2k}^2(1) = \infty$ follows from $\Gamma(F) = \Sigma_n(1-c_{1n})^2 < \infty$ and Lemma 6.11.

Further, $x_{2k} \in \mathfrak{A}^2$ hence, we get using (6.56) $x_{1k} - sx_{1k}x_{2k} = x_{1k}(1 - sx_{2k}) = x_{1k} \otimes \big(\begin{smallmatrix} 1 & 0 \\ 0 & 1-s \end{smallmatrix}\big) \in \mathfrak{A}^2$. Finally, we conclude that $x_{1k} \in \mathfrak{A}^2$ since

$$(1 - sx_{2k})^{-1} = \big(\begin{smallmatrix} 1 & 0 \\ 0 & 1-s \end{smallmatrix}\big)^{-1} = \big(\begin{smallmatrix} 1 & 0 \\ 0 & (1-s)^{-1} \end{smallmatrix}\big) = (I - x_{2k}) + (1-s)^{-1}x_{2k} \in \mathfrak{A}^2.$$

Now we get $x_{1k},\ x_{2k+1} \in \mathfrak{A}^2$ for $k \geq 2$, this finish the proof in the case $s \neq 1$.

We prove Lemma 6.14 to finish the case $s \neq 1$ before passing to the case $s = 1$. To approximate x_{12}, we correct a little bit the expression $x_{1k} - sx_{1k}x_{2k}$.

Lemma 6.13. *For $s \in \mathbb{R}$ we have*

$$\min_{(t_1,t_2)\in\mathbb{R}^2} \|(x_{1k} - sx_{1k}x_{2k} + t_1 + t_2 x_{2k})\mathbf{1}\|^2 = \|(x_{1k} - Mx_{1k}\mathbf{1})(1 - sx_{2k})\mathbf{1}\|^2$$

$$= \alpha_{1k}(0)\alpha_{1k}(1)[\alpha_{2k}(0) + (1-s)^2\alpha_{2k}(1)].$$

We see that $(x_{1k} - Mx_{1k}\mathbf{1})(1 - sx_{2k}) = x_{1k} - sx_{1k}x_{2k} - Mx_{1k}\mathbf{1} + s(Mx_{1k}\mathbf{1})x_{2k}$ hence, minimum we have for $t_1 = -Mx_{1k}\mathbf{1},\ t_2 = s(Mx_{1k}\mathbf{1})$.

PROOF. We note that the distance $d(f_{n+1}; \langle f_1, ..., f_n \rangle)$ of the vector f_{n+1} in a Hilbert space H from the hyperplane $\langle f_1, ..., f_n \rangle$ generated by vectors

$f_1, ..., f_n$ may be calculated in terms of the *Gram determinants* $\Gamma(f_1, f_2, ..., f_k)$ corresponding to the set of vectors $f_1, f_2, ..., f_k$ (see [7]):

$$d^2(f_{n+1}; \langle f_1, ..., f_n \rangle) = \min_{t=(t_k) \in \mathbb{R}^n} \left\| f_{n+1} + \sum_{k=1}^{n} t_k f_k \right\|^2 = \frac{\Gamma(f_1, f_2, ..., f_{n+1})}{\Gamma(f_1, f_2, ..., f_n)},$$

$$(6.60)$$

where the Gram determinant is defined by $\Gamma(f_1, f_2, ..., f_n) = \det \gamma(f_1, f_2, ..., f_n)$ and $\gamma(f_1, f_2, ..., f_n)$ is the *Gram matrix*

$$\gamma(f_1, f_2, ..., f_n) = \begin{pmatrix} (f_1, f_1) & (f_1, f_2) & \cdots & (f_1, f_n) \\ (f_2, f_1) & (f_2, f_2) & \cdots & (f_2, f_n) \\ & & \cdots & \\ (f_n, f_1) & (f_n, f_2) & \cdots & (f_n, f_n) \end{pmatrix}.$$

Let us denote $f_0 = x_{1k}(1 - sx_{2k})\mathbf{1}, \ f_1 = \mathbf{1}, \ f_2 = x_{2k}\mathbf{1}$. We have

$$\min_{(t_1, t_2) \in \mathbb{R}^2} \| f_0 + t_1 f_1 + t_2 f_2 \|^2 = \frac{\Gamma(f_0, f_1, f_2)}{\Gamma(f_1, f_2)}.$$

Since we have for operators x_{1k} and x_{2k} (acting on the spaces H_{1k} and H_{2k} respectively) the same expressions: $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix} \right)$ and $1 - sx_{2k} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) - \left(\begin{smallmatrix} 0 & 0 \\ 0 & s \end{smallmatrix} \right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1-s \end{smallmatrix} \right)$ (to be more precise we write)

$$x_{1k} = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix} \right) \otimes \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right), \quad 1 - sx_{2k} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) \otimes \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1-s \end{smallmatrix} \right),$$

we get

$$(f_0, f_0) = \| x_{1k}\mathbf{1} \|^2 \| (1 - sx_{2k})\mathbf{1} \|^2 = \alpha_{1k}(1)(\alpha_{2k}(0) + (1 - s)^2 \alpha_{2k}(1)),$$

$$(f_0, f_1) = (x_{1k}(1-sx_{2k})\mathbf{1}, \mathbf{1}) = (x_{1k}\mathbf{1}, \mathbf{1})((1-sx_{2k})\mathbf{1}, \mathbf{1}) = \alpha_{1k}(1)(\alpha_{2k}(0) + (1-s)\alpha_{1k}(1)),$$

$$(f_0, f_2) = (x_{1k}(1-sx_{2k})\mathbf{1}, x_{2k}\mathbf{1}) = (x_{1k}\mathbf{1}, \mathbf{1})((1-sx_{2k})\mathbf{1}, x_{2k}\mathbf{1}) = \alpha_{1k}(1)(1-s)\alpha_{2k}(1),$$

$$(f_1, f_1) = 1, \quad (f_1, f_2) = (\mathbf{1}, x_{2k}\mathbf{1}) = \alpha_{2k}(1), \quad (f_2, f_2) = (x_{2k}\mathbf{1}, x_{2k}\mathbf{1}) = \alpha_{2k}(1).$$

Finally, we conclude that

$$\Gamma(f_0, f_1, f_2) = \begin{vmatrix} (f_0, f_0) & (f_0, f_1) & (f_0, f_2) \\ (f_1, f_0) & (f_1, f_1) & (f_1, f_2) \\ (f_2, f_0) & (f_2, f_1) & (f_2, f_2) \end{vmatrix} =$$

$$\begin{vmatrix} \alpha_{1k}(1)(\alpha_{2k}(0)+(1-s)^2\alpha_{2k}(1)) & \alpha_{1k}(1)(\alpha_{2k}(0)+(1-s)\alpha_{2k}(1)) & \alpha_{1k}(1)(1-s)\alpha_{2k}(1) \\ \alpha_{1k}(1)(\alpha_{2k}(0) + (1-s)\alpha_{2k}(1)) & 1 & \alpha_{2k}(1) \\ \alpha_{1k}(1)(1-s)\alpha_{2k}(1) & \alpha_{2k}(1) & \alpha_{2k}(1) \end{vmatrix}$$

$$= \alpha_{1k}(1)\alpha_{2k}(1) \begin{vmatrix} \alpha_{2k}(0)+(1-s)^2\alpha_{2k}(1) & \alpha_{2k}(0)+(1-s)\alpha_{2k}(1) & (1-s) \\ \alpha_{1k}(1)(\alpha_{2k}(0)+(1-s)\alpha_{2k}(1)) & 1 & 1 \\ \alpha_{1k}(1)(1-s)\alpha_{2k}(1) & \alpha_{2k}(1) & 1 \end{vmatrix}$$

$$= \alpha_{1k}(1)\alpha_{2k}(1) \begin{vmatrix} \alpha_{2k}(0)+(1-s)^2\alpha_{1k}(0)\alpha_{2k}(1) & \alpha_{2k}(0) & 0 \\ \alpha_{1k}(1)\alpha_{2k}(0) & \alpha_{2k}(0) & 0 \\ \alpha_{1k}(1)(1-s)\alpha_{2k}(0) & \alpha_{2k}(1) & 1 \end{vmatrix}$$

$$= \alpha_{1k}(1)\alpha_{2k}(1) \begin{vmatrix} \alpha_{2k}(0)+(1-s)^2\alpha_{1k}(0)\alpha_{2k}(1) & \alpha_{2k}(0) \\ \alpha_{1k}(1)\alpha_{2k}(0) & \alpha_{2k}(0) \end{vmatrix}$$

$$= \alpha_{1k}(1)\alpha_{2k}(1)\alpha_{2k}(0)\left(\alpha_{2k}(0)+(1-s)^2\alpha_{1k}(0)\alpha_{2k}(1)-\alpha_{1k}(1)\alpha_{2k}(0)\right)$$

$$= \alpha_{1k}(0)\alpha_{1k}(1)\alpha_{2k}(0)\alpha_{2k}(1)\left(\alpha_{2k}(0)+(1-s)^2\alpha_{2k}(1)\right).$$

For $\Gamma(f_1, f_2)$ we have

$$\Gamma(f_1, f_2) = \begin{vmatrix} (f_1, f_1) & (f_1, f_2) \\ (f_2, f_1) & (f_2, f_2) \end{vmatrix} = \begin{vmatrix} 1 & \alpha_{2k}(1) \\ \alpha_{2k}(1) & \alpha_{2k}(1) \end{vmatrix} = \alpha_{2k}(0)\alpha_{2k}(1),$$

hence, $\Gamma(f_0, f_1, f_2)(\Gamma(f_1, f_2))^{-1} = \alpha_{1k}(0)\alpha_{1k}(1)\left(\alpha_{2k}(0)+(1-s)^2\alpha_{2k}(1)\right)$ and

$$\|(x_{1k} - Mx_{1k}\mathbf{1})(1 - sx_{2k})\mathbf{1}\|^2 = \|(x_{1k} - Mx_{1k}\mathbf{1})\|^2\|(1 - sx_{2k})\mathbf{1}\|^2$$

$$= \alpha_{1k}(0)\alpha_{1k}(1)\left(\alpha_{2k}(0)+(1-s)^2\alpha_{2k}(1)\right). \qquad \square$$

By Lemma 6.13 we have for optimal t_1 and t_2

$$x_{1k}-sx_{1k}x_{2k}-(1-s)x_{12}x_{2k}+t_1+t_2x_{2k} = (x_{1k}-Mx_{1k}\mathbf{1})(1-sx_{2k})-(1-s)x_{12}x_{2k}.$$

Lemma 6.14. *For $s \neq 1$ we have*

$$-(1-s)x_{12}\mathbf{1}\in\langle[(x_{1k}-Mx_{1k}\mathbf{1})(1-sx_{2k})-(1-s)x_{12}x_{2k}]\,\mathbf{1}\mid k\geq 3\rangle \Leftrightarrow \sigma_{12}^{(s)}(\mu)=\infty,$$

where $\sigma_{12}^{(s)}(\mu) := \sum_k \dfrac{\alpha_{2k}^2(1)}{\alpha_{2k}(0)\alpha_{2k}(1)+\alpha_{1k}(0)\alpha_{1k}(1)(\alpha_{2k}(0)+(1-s)^2\alpha_{2k}(1))}.$

PROOF. We can procced as before. Let us denote

$$\xi_k = x_{2k}\mathbf{1}, \quad \eta_k^s = (x_{1k} - Mx_{1k}\mathbf{1})(1 - sx_{2k})\mathbf{1}, \quad \text{then} \quad M\xi_k = \alpha_{2k}(1),$$

$$\|\xi_k-M\xi_k\|^2 = \alpha_{2k}(0)\alpha_{2k}(1), \quad \|\eta_k^s\|^2 = \alpha_{1k}(0)\alpha_{1k}(1)\left(\alpha_{2k}(0)+(1-s)^2\alpha_{2k}(1)\right).$$

If we take $(t_k)_k$ such that $\sum_{k=3}^{N+3} t_k M \xi_k = 1$ we obtain

$$\left\|\left(\sum_{k=3}^{N+3} t_k \left[(x_{1k} - M x_{1k}\mathbf{1})(1 - x_{2k}) - (1 - s)x_{12}x_{2k}\right] + (1 - s)x_{12}\right)\mathbf{1}\right\|^2 =$$

$$\left\|\sum_{k=3}^{N+3} t_k [\eta_k^s - (1 - s)x_{12}(\xi_k - M\xi_k)]\right\|^2 = \sum_{k=2}^{N+2} t_k^2 \|\eta_k^s - (1 - s)x_{12}(\xi_k - M\xi_k)\|^2$$

$$= \sum_{k=3}^{N+3} t_k^2 \left(\|\eta_k^s\|^2 + \|(1 - s)x_{12}\mathbf{1}\|^2 \|(\xi_k - M\xi_k)\|^2\right).$$

Hence,

$$\min_{t \in \mathbb{R}^N} \left(\sum_{k=3}^{N+3} t_k^2 \left(\|\eta_k^s\|^2 + \|(1 - s)x_{12}\mathbf{1}\|^2 \|(\xi_k - M\xi_k)\|^2\right) \mid \sum_{k=3}^{N+3} t_k M\xi_k = 1\right)$$

$$= \left(\sum_{k=3}^{N+3} \frac{|M\xi_k|^2}{(1 - s)^2 \|x_{12}\mathbf{1}\|^2 \|\xi_k - M\xi_k\|^2 + \|\eta_k^s\|^2}\right)^{-1}$$

$$= \left(\sum_{k=3}^{N} \frac{\alpha_{2k}^2(1)}{(1 - s)^2 \alpha_{12}(1)\alpha_{2k}(0)\alpha_{2k}(1) + \alpha_{1k}(0)\alpha_{1k}(1)(\alpha_{2k}(0) + (1 - s)^2\alpha_{2k}(1))}\right)^{-1} \sim$$

$$\left(\sum_{k=3}^{N} \frac{\alpha_{2k}^2(1)}{\alpha_{2k}(0)\alpha_{2k}(1) + \alpha_{1k}(0)\alpha_{1k}(1)(\alpha_{2k}(0) + (1 - s)^2\alpha_{2k}(1))}\right)^{-1}, \quad \text{if } s \neq 1.$$

$\square$

311. When $s = 1$ we get from (6.56) $x_{1k} - x_{1k}x_{2k}$. The condition (6.54)

$$\|f' - tg'\|^2 = \sum_n (1 - c_{1n})[1 - t(1 - c_{2n})]^2 < \infty, \quad \text{for some} \quad t \in \mathbb{R}\backslash\{0\}, \quad (6.61)$$

splits into two cases (3110), when $t \neq 1$ and (3111), when $t = 1$. We show that in the first and the second case we get respectively:

$$\sum_n (1 - c_{1n})c_{2n}^2 = \infty \quad \text{and} \quad \sum_n (1 - c_{1n})\alpha_{2n}^2(1) = \infty. \quad (6.62)$$

To approximate x_{12}, under the above conditions we use the following expression (see (6.39)) in the first case

$$T_{kn} - I - x_{2k}(T_{2n} - I) - (x_{1k} - x_{1k}x_{2k})(T_{1n} - I) = (x_{1k} - x_{12}x_{2k})\tau_{1n} +$$

$$(x_{1k}x_{2k} - x_{12}x_{2k})\tau_{1n}\tau_{2n} - (x_{1k} - x_{1k}x_{2k})\tau_{1n} = (x_{1k}x_{2k} - x_{12}x_{2k})\tau_{1n}(I + \tau_{2n}).$$
(6.63)

In the second case, if we multiply the latter expression by $T_{2n} = (I + x_{12}\tau_{1n})(I + \tau_{2n})$ we get (see (6.66))

$$(x_{1k}x_{2k} - x_{12}x_{2k})\tau_{1n}(I + \tau_{2n})(I + x_{12}\tau_{1n})(I + \tau_{2n}) = (x_{1k}x_{2k} - x_{12}x_{2k}(2x_{1k} - I))\tau_{1n}.$$
(6.64)

Consider the case **3110**, when $t \neq 1$.

Lemma 6.15. *Let $f, g \notin l_2$ where $f = (f_n)_{n \in \mathbb{N}}$, $g = (g_n)_{n \in \mathbb{N}}$. If for some $t \in \mathbb{R}$ holds $tf + (1 - t)g \in l_2$ then such a t is unique.*

PROOF. Set $H(t) = tf + (1 - t)g$, $t \in \mathbb{R}$. Suppose that $H(t_1)$, $H(t_2) \in l_2$ for two different t_1 and t_2. Then we get the contradiction, since for some $s \in \mathbb{R}$ holds $f = sH(t_1) + (1 - s)H(t_2)$ and by assumption we get $l_2 \not\ni f = sH(t_1) + (1 - s)H(t_2) \in l_2$. We note that $s = (1 - t_2)(t_1 - t_2)^{-1}$. $\square$

Remark 6.7. The condition (6.61) for $t \neq 1$ implies the first condition of (6.62). Indeed, by Lemma 6.15 we get $\|f' - g'\|^2 = \sum_n (1 - c_{1n})c_{2n}^2 = \infty$ for $t = 1$.

Lemma 6.16. *We have*

$$(x_{1k}x_{2k} - x_{12}x_{2k})\mathbf{1} \in \langle (x_{1k}x_{2k} - x_{12}x_{2k})\tau_{1n}(I + \tau_{2n})\mathbf{1} \mid n > k \rangle$$

if and only if $\Sigma_{12}^{(1)} := \sum_n (1 - c_{1n})c_{2n}^2 = \infty$.

PROOF. It is sufficient to show that $\sum_n t_n\big[(T_{\alpha_{1n}} - I) \otimes T_{\alpha_{2n}}\big] \to 1$ if and only if $\Sigma_{12}^{(1)} = \infty$. Set $\xi_n = \big[(T_{\alpha_{1n}} - I) \otimes T_{\alpha_{2n}}\big]\mathbf{1}$ and $\xi_n^c = \xi_n - M\xi_n$, then

$$M\xi_n = (c_{1n} - 1)c_{2n}, \quad \|\xi_n\|^2 = 2(1 - c_{1n}), \quad \|\xi_n^c\|^2 = \|\xi_n\|^2 - |M\xi_n|^2.$$

Indeed, we have
$$\|\xi_n\|^2 = \|\big[(T_{\alpha_{1n}} - I) \otimes T_{\alpha_{2n}}\big]\mathbf{1}\|^2 =$$
$$\|(T_{\alpha_{1n}} - I)\mathbf{1}\|^2 = \|T_{\alpha_{1n}}\mathbf{1}\|^2 - 2(T_{\alpha_{1n}}\mathbf{1}, \mathbf{1}) + 1 = 2(1 - c_{1n}).$$

Take $(t_n)_n$ such that $\sum_{n=2}^{N} t_n M\xi_n = 1$ then

$$\Big\| \Big(\sum_{n=2}^{N+2} t_n \big[(T_{\alpha_{1n}} - I) \otimes T_{\alpha_{2n}} \big] - I \Big) \mathbf{1} \Big\|^2 = \Big\| \Big(\sum_{n=2}^{N+2} t_n \big[(T_{\alpha_{1n}} - I) \otimes T_{\alpha_{2n}} \big] - \sum_{n=2}^{N+2} t_n M\xi_n \Big) \mathbf{1} \Big\|^2$$

$$= \sum_{n=2}^{N+2} t_n^2 \big\| \big[(T_{\alpha_{1n}} - I) \otimes T_{\alpha_{2n}} - M\xi_n \big] \mathbf{1} \big\|^2 = \sum_{n=2}^{N+2} t_n^2 \| \xi_n^c \|^2.$$

Finally, we get

$$\min_{t \in \mathbb{R}^N} \Big(\sum_{n=2}^{N+2} t_n^2 \| \xi_n^c \|^2 \ \Big| \ \sum_{n=2}^{N+2} t_n M\xi_n = 1 \Big) = (\Sigma_{12,N}^{(1)})^{-1} \quad \text{where}$$

$$\Sigma_{12,N}^{(1)} := \sum_{n=2}^{N+2} \frac{|M\xi_n|^2}{\|\xi_n^c\|^2} = \sum_{n=2}^{N+2} \frac{|M\xi_n|^2}{\|\xi_n\|^2 - |M\xi_n|^2} \sim$$

$$\sum_{n=2}^{N+2} \frac{|M\xi_n|^2}{\|\xi_n\|^2} = \sum_{n=2}^{N+2} \frac{(1-c_{1n})^2 c_{2n}^2}{2(1-c_{1n})} = \frac{1}{2} \sum_{n=2}^{N+2} (1-c_{1n}) c_{2n}^2.$$

$\square$

Now we get $x_{1k}x_{2k} - x_{12}x_{2k}, \ x_{1k} - x_{1k}x_{2k} \in \mathfrak{A}^2$ hence, $x_{1k} - x_{12}x_{2k} \in \mathfrak{A}^2$. Using Lemma 6.14 for $s = 0$ we get

Lemma 6.17. *We have $x_{12}\mathbf{1} \in \langle (x_{1k} - x_{12}x_{2k})\mathbf{1} \mid n > k \rangle$ if and only if $\sigma_{12}^{(0)}(\mu) = \infty$ where*

$$\sigma_{12}^{(0)}(\mu) = \sum_k \frac{\alpha_{2k}^2(1)}{\alpha_{2k}(0)\alpha_{2k}(1) + \alpha_{1k}(0)\alpha_{1k}(1)}. \tag{6.65}$$

We use the following obvious implications:

$$\Gamma(F) < \infty \overset{(6.47)}{\Rightarrow} \sum_n (1-c_{1n})^2 < \infty \overset{\text{Lemma 6.11}}{\Rightarrow} \sum_n \alpha_{2n}^2(1) = \infty \Rightarrow \sigma_{12}^{(0)}(\mu) = \infty.$$

By Lemma 6.17 we conclude that $x_{12} \in \mathfrak{A}^2$ hence, $x_{1k}, \ x_{2k+1} \in \mathfrak{A}^2$ for all $k \geq 2$. This finish the proof in this case.

Consider the case **3111**, when $t = 1$. Since $p = 2$ we get $(I + \tau_{2n})^2 = T_{2n}^2(2) = T_{\alpha_{2n}}^2 = I$ and $\tau_{1n}^2 = -2\tau_{1n}$. Indeed, we get

$$\tau_{1n}^2 = (T_{\alpha_{1n}} - I)^2 = T_{\alpha_{1n}}^2 - 2T_{\alpha_{1n}} + I = -2(T_{\alpha_{1n}} - I) = -2\tau_{1n}.$$

Hence, we have

$$(x_{1k}x_{2k}-x_{12}x_{2k})\tau_{1n}(I+\tau_{2n})(I+x_{12}\tau_{1n})(I+\tau_{2n})=(x_{1k}x_{2k}-x_{12}x_{2k})\tau_{1n}(I+x_{12}\tau_{1n})$$

$$=[x_{1k}x_{2k}-x_{12}x_{2k}-2x_{12}(x_{1k}x_{2k}-x_{12}x_{2k})]\tau_{1n}=[x_{1k}x_{2k}-x_{12}x_{2k}(2x_{1k}-I)]\tau_{1n}.$$
$$(6.66)$$

The condition $S_{11}^L(\mu)=\infty$ implies $\sum_{n=1}^m t_n\tau_{1n}\to I$ therefore,

$$x_{1k}x_{2k}-x_{12}x_{2k}(2x_{1k}-I)\in\mathfrak{A}^2.$$

Since $x_{1k}-x_{1k}x_{2k}\in\mathfrak{A}^2$ we conclude finally, that $x_{1k}-x_{12}x_{2k}(2x_{1k}-I)\in\mathfrak{A}^2$.

Remark 6.8. The condition (6.61) for $t=1$ means that $\sum_n(1-c_{1n})c_{2n}^2<\infty$. This implies that $\sum_n(1-c_{1n})\alpha_{2n}^2(1)=\infty$. Indeed, otherwise, if we suppose that
$\sum_n(1-c_{1n})\alpha_{2n}^2(1)<\infty$ we obtain the contradiction with the condition $S_{12}^L(\mu)=\sum_n(1-c_{1n})\alpha_{2n}(1)=\infty$. In fact, since $c_{2n}^2=4\alpha_{2n}(0)\alpha_{2n}(1)$, we get

$$\infty>\sum_n(1-c_{1n})c_{2n}^2=4\sum_n(1-c_{1n})(\alpha_{2n}(1)-\alpha_{2n}^2(1)).$$

By Lemma 6.18 below we conclude that $x_{12}\in\mathfrak{A}^2$. Since $x_{2k}\in\mathfrak{A}^2$ we conclude, by Remark 6.4 (c), that $x_{1k}\in\mathfrak{A}^2$ hence, $x_{1k},x_{2k+1}\in\mathfrak{A}^2$ for $k\geq 2$ and the proof is finished.

Lemma 6.18. We have $x_{12}\mathbf{1}\in\langle[x_{1k}-x_{12}x_{2k}(2x_{1k}-I)]\mathbf{1}\mid k>2\rangle$ if and only if $\sum_n(1-c_{1n})\alpha_{2n}^2(1)=\infty$.

PROOF. Set $\eta_k=x_{1k}\mathbf{1}$ and $\xi_k=x_{2k}(2x_{1k}-I)\mathbf{1}$ then

$$M\eta_k=\alpha_{1k}(1),\quad M\xi_k=\alpha_{2k}(1)(-\alpha_{1k}(0)+\alpha_{1k}(1)),$$

$$\|\eta_k\|^2=\alpha_{1k}(1),\quad\|\xi_k\|^2=\alpha_{2k}(1)(\alpha_{1k}(0)+\alpha_{1k}(1))=\alpha_{2k}(1),$$

since

$$2x_{1k}-I=\begin{pmatrix}0&0\\0&2\end{pmatrix}-\begin{pmatrix}1&0\\0&1\end{pmatrix}=\begin{pmatrix}-1&0\\0&1\end{pmatrix}.$$

Set $h_n=\eta_k-M\eta_k-x_{12}(\xi_k-M\xi_k)$ then

$$a_n=\|h_n\|^2=\|\eta_k-M\eta_k-x_{12}(\xi_k-M\xi_k)\|^2.$$

We have
$$\| \sum_{n=1}^{m} t_n[(x_{1k} - Mx_{1k}) - x_{12}\xi_k]\mathbf{1} - x_{12}\mathbf{1}\|^2 =$$
$$\| \sum_{n=1}^{m} t_n[(x_{1k} - Mx_{1k}) - x_{12}(\xi_k - M\xi_k)]\mathbf{1}\|^2 = \sum_{n=1}^{m} t_n^2 a_n.$$

To calculate a_n we get
$$a_n = \|h_n\|^2 = \|[(\eta_k - M\eta_k) - x_{12}(\xi_k - M\xi_k)]\mathbf{1}\|^2 = \|\eta_k - M\eta_k\|^2 +$$
$$\|x_{12}(\xi_k - M\xi_k)\mathbf{1}\|^2 - (x_{12}\mathbf{1}, \mathbf{1})(\eta_k - M\eta_k, \xi_k - M\xi_k) =$$
$$\|x_{1k}\mathbf{1}\|^2 - | Mx_{1k}\mathbf{1} |^2 + \frac{1}{2}[\|\xi_k\|^2 - | M\xi_k |^2] - (\eta_k - M\eta_k, \xi_k - M\xi_k) =$$
$$\alpha_{1k}(1) - \alpha_{1k}^2(1) + \frac{1}{2}[\alpha_{2k}(1) - \alpha_{2k}^2(1)(-\alpha_{1k}(0) + \alpha_{1k}(1))^2] - (\eta_k - M\eta_k, \xi_k - M\xi_k).$$

Since
$$(\eta_k - M\eta_k, \xi_k - M\xi_k) = (\eta_k, \xi_k) - M\eta_k M\xi_k = (x_{1k}\mathbf{1}, x_{2k}(2x_{1k} - I)\mathbf{1}) - M\eta_k M\xi_k$$
$$= (x_{2k}\mathbf{1}, x_{1k}\mathbf{1}) - M\eta_k M\xi_k = \alpha_{1k}(1)\alpha_{2k}(1) - \alpha_{1k}(1)\alpha_{2k}(1)(-\alpha_{1k}(0) + \alpha_{1k}(1))$$
$$= \alpha_{1k}(1)\alpha_{2k}(1)(1 + \alpha_{1k}(0) - \alpha_{1k}(1)) = 2\alpha_{1k}(0)\alpha_{1k}(1)\alpha_{2k}(1),$$

we conclude that
$$a_k = \alpha_{1k}(1) - \alpha_{1k}^2(1) + \frac{1}{2}[\alpha_{2k}(1) - \alpha_{2k}^2(1)(\alpha_{1k}(0) - \alpha_{1k}(1))^2] - 2\alpha_{1k}(0)\alpha_{1k}(1)\alpha_{2k}(1)$$
$$\sim a_k' := \alpha_{1k}(0)\alpha_{1k}(1) + \frac{1}{2}\alpha_{2k}(1)[1 - 4\alpha_{1k}(0)\alpha_{1k}(1)] = \frac{c_{1k}^2}{4} + \frac{1}{2}\alpha_{2k}(1)[1 - c_{1k}^2].$$

Finally, we get
$$\min_{t \in \mathbb{R}^m} \Big(\sum_{n=1}^{m} t_n^2 a_n \mid \sum_{n=1}^{m} t_n b_n = 1 \Big) = \Big(\sum_{n=1}^{m} \frac{b_n^2}{a_n} \Big)^{-1} \quad \text{where} \quad b_n = \frac{1}{2}M\xi_k \quad \text{and}$$

$$\sum_{n=1}^{\infty} \frac{b_n^2}{a_n} \sim \sum_{n=1}^{\infty} \frac{b_n^2}{a_n'} = \sum_{k=1}^{\infty} \frac{\frac{1}{4}\alpha_{2k}^2(1)(-\alpha_{1k}(0) + \alpha_{1k}(1))^2}{\alpha_{1k}(0)\alpha_{1k}(1) + \frac{1}{2}\alpha_{2k}(1)[1 - 4\alpha_{1k}(0)\alpha_{1k}(1)]} =$$

$$\sum_{k=1}^{\infty} \frac{\frac{1}{4}\alpha_{2k}^2(1))(1 - c_{1k}^2)}{\frac{c_{1k}^2}{4} + \frac{1}{2}\alpha_{2k}(1)(1 - c_{1k}^2)} \geq \frac{1}{3}\sum_{k=1}^{\infty} \alpha_{2k}^2(1))(1 - c_{1k}^2) \sim \sum_{k=1}^{\infty} \alpha_{2k}^2(1)(1 - c_{1k}),$$

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since $1 \leq 1 + c_{1k} < 2$, $c_{1k}^2 \leq 1$ and $\alpha_{2k}(1)[1 - c_{1k}^2] < 1$. We use the following relation for $x = \alpha_{1k}(1)$:

$$(-\alpha_{1k}(0) + \alpha_{1k}(1))^2 = (1 - 2x)^2 = 1 - 4x(1 - x) = 1 - c_{1k}^2.$$

$\square$

Case 4. Let $\Sigma_{12} < \infty$ and $\Gamma(F) = \Gamma(G) = \infty$. Condition $\Sigma_{12} < \infty$ implies $\Gamma(H_2) < \infty$ hence, $\Sigma_{2m} \sim \Delta(G_m, F_m)$. We have two cases:

    (4I),   when $\Delta(G_m, F_m) \to \infty$;

    (4II), when $\Gamma(G_m)/\Gamma(F_m) \leq C$ and $\|G - tF\|^2 < \infty$ for some $t \in \mathbb{R} \backslash \{0\}$.

In the first case (4I), we can approximate x_{2k} and we are reduced to the case (3) but some particular cases should be considered in addition.

In the second case (4II), we show that by linear combinations of the expressions
$$T_{kn} - I = x_{1k}\tau_{1n} + x_{2k}\tau_{2n} + x_{1k}x_{2k}\tau_{1n}\tau_{2n}$$
we can approximate $x_{1k} + \beta_1^{(2)}x_{2k} - \beta_1^{(3)}x_{1k}x_{2k}$ or $\beta_2^{(1)}x_{1k} + x_{2k} - \beta_2^{(3)}x_{1k}x_{2k}$, see Lemma 6.76.

Case **4I.** We follow step by step the case (3) with the same notations, just replacing 3 by 4. We know that $\Delta(G_m, F_m) \to \infty$ in two cases, due to Lemmas 6.9 and 6.10:

    case (b), when $\Gamma(G_m)/\Gamma(F_m) \to \infty$,

    case (c), when $\Gamma(G_m)/\Gamma(F_m) \leq C$ for all $m > 3$ and $\|C_1F + C_2G\|^2 = \infty$ for all $(C_1, C_2) \in \mathbb{R}^2 \setminus \{0\}$.

Consider the following expression (see (6.55))

$$T_{kn} - I - x_{2k}(T_{2n} - I) = (x_{1k} - x_{12}x_{2k})\tau_{1n} + (x_{1k}x_{2k} - x_{12}x_{2k})\tau_{1n}\tau_{2n}.$$

The case (4I) splits into two cases (40), when $\Delta(f'_m, g'_m) \to \infty$ and (41), when

$$\Gamma(f'_m)/\Gamma(g'_m) \leq C \quad \text{and} \quad \|f' - tg'\|^2 = \sum_n (1 - c_{1n})[1 - t(1 - c_{1n})]^2 < \infty$$

for some $t \in \mathbb{R} \setminus \{0\}$ (see (6.54)), as in the case (31).

In the case **40**, by Lemma 6.7, we conclude that $x_{1k} - x_{12}x_{2k} \in \mathfrak{A}^2$ since $\Delta(f'_m, g'_m) \to \infty$. The case (40) splits into two cases: (400), when

$\sigma_{12}(\mu) = \sum_n \alpha_{2n}^2(1) = \infty$ and the case (401), when $\sigma_{12}(\mu) = \sum_n \alpha_{2n}^2(1) < \infty$. In the case (400) we can approximate x_{12} and the proof is finished.

In the case **401**, the condition $\sigma_{12}(\mu) = \sum_n \alpha_{2n}^2(1) < \infty$ implies $\lim_n c_{2n} = 0$ indeed,

$$\lim_n c_{2n}^2 = \lim_n 4\alpha_{2n}(1)(1 - \alpha_{2n}(1)) = 0.$$

Since $\sum_n (1 - c_{1n})\alpha_{2n}^2(1) < \sum_n \alpha_{2n}^2(1) < \infty$ we conclude that $\sum_n (1 - c_{1n})c_{2n}^2 = \infty$. Indeed, use the fact that $S_{12}^L(\mu) = \sum_n (1 - c_{1n})\alpha_{2n}(1) = \infty$ and consider the equality

$$\sum_n (1 - c_{1n})c_{2n}^2 = 4\sum_n (1 - c_{1n})(\alpha_{2n}(1) - \alpha_{2n}^2(1)).$$

Example 6.1. Let $1 - c_{1n} = \frac{1}{n^\beta}$, $c_{2n} = \frac{1}{n^\alpha}$ where $\alpha, \beta > 0$. We show that the conditiosn of the divergence of the following series, which gives us the case (401),

$$\sum_n (1-c_{1n}) = \sum_n (1-c_{2n}) = \sum_n (1-c_{1n})\alpha_{2n}(2) = \sum_n (1-c_{1n})c_{2n}^2 = \|F\|^2 = \|G\|^2 = \infty$$

are as follows:

$$D = \{(\alpha, \beta) \in \mathbb{R}^2 \mid 2\alpha + \beta \leq 1, \ 2\alpha + 2\beta > 1, \ 4\alpha > 1\}. \tag{6.67}$$

Indeed, we have

$$S_{11}^L(\mu) = \sum_n (1 - c_{1n}) = \sum_n \frac{1}{n^\beta} = \infty \quad \text{for} \quad \beta \in (0, 1],$$

$$S_{22}^L(\mu) = \sum_n (1 - c_{2n}) = \sum_n (1 - \frac{1}{n^\alpha}) = \infty \quad \text{for} \quad \alpha > 0.$$

To find $x = \alpha_{2n}(1)$ we use the identity $c_{2n}^2 = 4\alpha_{2n}(0)\alpha_{2n}(1) = 4(1 - x)x$ (see notation c_{1n} before Lemma 4.1). We have $x^2 - x + \frac{c_{2n}^2}{4}$. The roots are as follows:

$$x_1 = \left(1 - \sqrt{1 - c_{2n}^2}\right)/2 = c_{2n}^2 \left(2\left(1 + \sqrt{1 - c_{2n}^2}\right)\right)^{-1}, \quad x_2 = \left(1 + \sqrt{1 - c_{2n}^2}\right)/2. \tag{6.68}$$

Only the first root is suitable since $\alpha_{2n}(1) \to 0$. We have $\alpha_{2n}(1) \sim c_{2n}^2/4$. Therefore,

$$S_{12}^L(\mu) = \sum_n (1-c_{1n})\alpha_{2n}(1) \sim \sum_n (1-c_{1n})c_{2n}^2 = \sum_n \frac{1}{n^{2\alpha+\beta}} = \infty \text{ for } 2\alpha+\beta \leq 1,$$

$$\Sigma_{12} = \sum_{n=2}^{\infty} \frac{(1 - c_{1n})^2 c_{2n}^2}{(1 - c_{1n}) c_{2n}^2 + 1 - c_{2n}} \sim \sum_n (1 - c_{1n})^2 c_{2n}^2 = \sum_n \frac{1}{n^{2\alpha + 2\beta}} < \infty \quad \text{for} \quad 2\alpha + 2\beta > 1,$$

$$\sigma_{12}(\mu) = \sum_n \alpha_{2n}^2(1) \sim \sum_n c_{2n}^4 = \sum_n \frac{1}{n^{4\alpha}} < \infty \quad \text{for} \quad 4\alpha > 1,$$

that proves (6.67). Further, we get

$$\|F_m\|^2 \sim \sum_{n=1}^{m} \frac{(1 - c_{1n})^2}{1 - c_{1n} + 1 - c_{2n}} \sim \sum_{n=1}^{m} (1 - c_{1n})^2 = \sum_{n=1}^{m} \frac{1}{n^{2\beta}} \sim \frac{m^{1-2\beta}}{1 - 2\beta} \to \infty,$$

$$\|G_m\|^2 \sim \sum_{n=1}^{m} \frac{(1 - c_{2n})^2}{1 - c_{1n} + 1 - c_{2n}} \sim \sum_{n=1}^{m} (1 - c_{2n})^2 = \sum_{n=1}^{m} (1 - \frac{1}{n^\alpha})^2 \sim m \to \infty.$$

Therefore,

$$\Delta(G_m, F_m) \to \infty \quad \text{since} \quad \|G_m\|^2 / \|F_m\|^2 \sim (1 - 2\beta) m^{2\beta} \to \infty.$$

In addition we get for $f_m' = (\sqrt{1 - c_{1n}})_{n=1}^m$ and $g_m' = (\sqrt{1 - c_{1n}}(1 - c_{1n}))_{n=1}^m$

$$\|f_m'\|^2 / \|g_m'\|^2 = \sum_{n=1}^{m} (1 - c_{1n}) / \sum_{n=1}^{m} (1 - c_{1n})(1 - c_{2n})^2 \to \lim_n (1 - c_{2n})^{-2} = 1.$$

For all $t \in \mathbb{R}$ we have $\|g_m' - t f_m'\|^2 = \sum_{n=1}^m (1 - c_{1n})(1 - c_{2n} - t)^2 \to \infty$. Indeed, for $t = 1$ we get $\|g_m' - f_m'\|^2 = \sum_{n=1}^m (1 - c_{1n}) c_{2n}^2 \to \infty$ hence, $\|g_m' - t f_m'\|^2 \to \infty$ for all $t \in \mathbb{R}$. Therefore, $\Delta(f_m', g_m') \to \infty$ by Lemma 6.10 and we are in the case (401).

Lemma 6.19. *We have*

$$x_{12} \mathbf{1} \in \langle [T_{2n}, x_{1n} - x_{12} x_{2n}] \, x_{2n} \mathbf{1} \mid n \geq 3 \rangle \tag{6.69}$$

if and only if $\Sigma_{12}^{(2)} = \sum_n \alpha_{1n}(1) \alpha_{2n}(0) = \infty$.

PROOF. Recall (see (6.4)), that

$$a_n = \sqrt{\alpha_{1n}(0)/\alpha_{1n}(1)} \quad b_n = \sqrt{\alpha_{2n}(0)/\alpha_{2n}(1)}.$$

We show that

$$[T_{2n}, x_{1n} - x_{12} x_{2n}] = 2 x_{12} \left(\left(\begin{smallmatrix} 0 & a_n^{-1} \\ 0 & 0 \end{smallmatrix} \right) \otimes \left(\begin{smallmatrix} 0 & 0 \\ b_n & 0 \end{smallmatrix} \right) - \left(\begin{smallmatrix} 0 & 0 \\ a_n & 0 \end{smallmatrix} \right) \otimes \left(\begin{smallmatrix} 0 & b_n^{-1} \\ 0 & 0 \end{smallmatrix} \right) \right), \tag{6.70}$$

therefore,
$$[T_{2n}, x_{1n} - x_{12}x_{2n}]\, x_{2n} = -2x_{12} \left(\begin{smallmatrix} 0 & 0 \\ a_n & 0 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 0 & b_n^{-1} \\ 0 & 0 \end{smallmatrix}\right). \qquad (6.71)$$

Indeed, since
$$x_{1n} = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right), \quad T_{\alpha_{1n}} = \left(\begin{smallmatrix} 0 & a_n^{-1} \\ a_n & 0 \end{smallmatrix}\right), \quad T_{\alpha_{2n}} = \left(\begin{smallmatrix} 0 & b_n^{-1} \\ b_n & 0 \end{smallmatrix}\right),$$

we get
$$[T_{\alpha_{1n}}, x_{1n}] = \left(\begin{smallmatrix} 0 & a_n^{-1} \\ -a_n & 0 \end{smallmatrix}\right), \quad [T_{\alpha_{2n}}, x_{2n}] = \left(\begin{smallmatrix} 0 & b_n^{-1} \\ -b_n & 0 \end{smallmatrix}\right).$$

Using the (2.8) and (5.5) we get
$$T_{2n} = T_{2n}(1) \otimes T_{2n}(2) = \big(x_{12}(T_{\alpha_{1n}} - I) + I\big) \otimes T_{\alpha_{2n}}$$

that implies (6.70). Indeed, we have
$$[T_{2n}, x_{1n} - x_{12}x_{2n}] = \big[\big(x_{12}(T_{\alpha_{1n}} - I) + I\big) \otimes T_{\alpha_{2n}}, x_{1n} - x_{12}x_{2n}\big] =$$

$$x_{12}\Big([T_{\alpha_{1n}}, x_{1n}]T_{\alpha_{2n}} - T_{\alpha_{1n}}[T_{\alpha_{2n}}, x_{2n}]\Big) =$$

$$x_{12}\left(\left(\begin{smallmatrix} 0 & a_n^{-1} \\ -a_n & 0 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 0 & b_n^{-1} \\ b_n & 0 \end{smallmatrix}\right) - \left(\begin{smallmatrix} 0 & a_n^{-1} \\ a_n & 0 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 0 & b_n^{-1} \\ -b_n & 0 \end{smallmatrix}\right)\right),$$

$$x_{12}\left(\left(\begin{smallmatrix} 0 & 0 & 0 & a_n^{-1}b_n^{-1} \\ 0 & 0 & a_n^{-1}b_n & 0 \\ 0 & -a_n b_n^{-1} & 0 & 0 \\ -a_n b_n & 0 & 0 & 0 \end{smallmatrix}\right) - \left(\begin{smallmatrix} 0 & 0 & 0 & a_n^{-1}b_n^{-1} \\ 0 & 0 & -a_n^{-1}b_n & 0 \\ 0 & a_n b_n^{-1} & 0 & 0 \\ -a_n b_n & 0 & 0 & 0 \end{smallmatrix}\right)\right) =$$

$$2x_{12}\left(\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & a_n^{-1}b_n & 0 \\ 0 & -a_n b_n^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right) = 2x_{12}\left(\left(\begin{smallmatrix} 0 & a_n^{-1} \\ 0 & 0 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 0 & 0 \\ b_n & 0 \end{smallmatrix}\right) - \left(\begin{smallmatrix} 0 & 0 \\ a_n & 0 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 0 & b_n^{-1} \\ 0 & 0 \end{smallmatrix}\right)\right).$$

Now we show that $\sum_{n=3}^m t_n \left(\begin{smallmatrix} 0 & 0 \\ a_n & 0 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 0 & b_n^{-1} \\ 0 & 0 \end{smallmatrix}\right) \to I$ if and only if $\sum_n \alpha_{1n}(1)\alpha_{2n}(0) = \infty$.

Indeed, set
$$\Xi_n = \left(\begin{smallmatrix} 0 & 0 \\ a_n & 0 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 0 & b_n^{-1} \\ 0 & 0 \end{smallmatrix}\right) \quad \text{and} \quad \xi_n = \left(\begin{smallmatrix} 0 & 0 \\ a_n & 0 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 0 & b_n^{-1} \\ 0 & 0 \end{smallmatrix}\right) \mathbf{1}.$$

We get
$$\Big\|\Big(\sum_{n=3}^m t_n \Xi_n - I\Big)\mathbf{1}\Big\|^2 = \Big\|\sum_{n=3}^m t_n(\xi_n - M\xi_n)\Big\|^2 = \sum_{n=3}^m t_n^2 \|\xi_n - M\xi_n\|^2 \to 0$$

under the condition $\sum_{n=3}^{m} t_n M\xi_n = 1$ if and only if $\sum_n \frac{b_n^2}{a_n} \sim \Sigma_{12}^{(2)} = \infty$. Indeed, we have

$$\sum_n \frac{b_n^2}{a_n} = \sum_n \frac{c_{1n}^2 c_{2n}^2/16}{\alpha_{1n}(0)\alpha_{2n}(1) - c_{1n}^2 c_{2n}^2/16} \sim \sum_n \frac{\alpha_{1n}(0)\alpha_{1n}(1)\alpha_{2n}(0)\alpha_{2n}(1)}{\alpha_{1n}(0)\alpha_{2n}(1)} = \Sigma_{12}^{(2)} = \infty,$$

where

$$b_n = M\xi_n, \quad a_b = \|\xi_n - M\xi_n\|^2 = \|\xi_n\|^2 - |M\xi_n|^2,$$

$$b_n = M\xi_n = \left(\left(\begin{smallmatrix} 0 & 0 \\ a_n & 0 \end{smallmatrix} \right) \mathbf{1}, \mathbf{1} \right) \left(\left(\begin{smallmatrix} 0 & b_n^{-1} \\ 0 & 0 \end{smallmatrix} \right) \mathbf{1}, \mathbf{1} \right) = c_{1n}c_{2n}/4,$$

$$\|\xi_n\|^2 = \| \left(\begin{smallmatrix} 0 & 0 \\ a_n & 0 \end{smallmatrix} \right) \mathbf{1}\|^2 \| \left(\begin{smallmatrix} 0 & b_n^{-1} \\ 0 & 0 \end{smallmatrix} \right) \mathbf{1}\|^2 = \alpha_{1n}(0)\alpha_{2n}(1).$$

□

The condition $\Sigma_{12}^{(2)} = \sum_n \alpha_{1n}(1)\alpha_{2n}(0) = \infty$ follows from two facts:
(a) $\lim_n \alpha_{2n}(0) = 1$ since $\lim_n c_{2n} = 0$ and
(b) $\lim_k \alpha_{1n_k}(1) = 1/2$ since $\lim_k c_{1n_k} = 1$ (see (6.68)), that follows from

$$\Sigma_{12} \sim \sum_n (1 - c_{1n})^2 c_{2n}^2 < \infty \quad \text{and} \quad S_{12}^L \sim \sum_n (1 - c_{1n}) c_{2n}^2 = \infty. \quad (6.72)$$

The first equivalence follows from $\lim_n[(1 - c_{1n})c_{2n}^2 + 1 - c_{2n}] = 1$. Indeed, the condition $1 - c_{1n} \geq \varepsilon > 0$ contradicts (6.72) therefore, for some subsequence $(n_k)_k$ we have $\lim_k c_{1n_k} = 1$ hence, $\Sigma_{12}^{(2)} \sim \sum_n \alpha_{1n}(1) > \sum_k \alpha_{1n_k}(1) = \infty$.

In the case **41**, when $\|f' - tg'\|^2 < \infty$, by Lemma 6.12, we can approximate the following expression:

$$x_{1k} - (1 - s)x_{12}x_{2k} - sx_{1k}x_{2k}.$$

We have two cases: (410), when $s \neq 1$ and (411), when $s = 1$. The case (410) splits into two cases (4100), when $\sum_k \alpha_{2k}^2(1) = \infty$ and (4101), when $\sum_k \alpha_{2k}^2(1) < \infty$.

In the case **4100** we can approximate x_{12} and the proof is finished.

In the case **4101** the condition $\sum_k \alpha_{2k}^2(1) < \infty$ implies that $\lim_n c_{2n} = 0$ and we are reduced to the case (401).

In the case **411**, when $s = 1$, we get $x_{1k} - x_{1k}x_{2k} \in \mathfrak{A}^2$ and we can consider the expression $(x_{1k}x_{2k} - x_{12}x_{2k})\tau_{1n}(I + \tau_{2n})$, see (6.63). The case (411) splits into two cases: (4110), corresponding to the cases $t \neq 1$ and $t = 1$ in (6.61), when $\sum_n (1 - c_{1n})c_{2n}^2 = \infty$ and (4111), when $\sum_n (1 - c_{1n})c_{2n}^2 < \infty$ hence, $\sum_n (1 - c_{1n})\alpha_{2n}^2(1) = \infty$ (see cases (3110), (3111) and (6.62)).

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In the case **4110** we can approximate $x_{1k}x_{2k} - x_{12}x_{2k}$, by Lemma 6.16. Since $x_{1k} - x_{1k}x_{2k} \in \mathfrak{A}^2$ we get $x_{1k} - x_{12}x_{2k} \in \mathfrak{A}^2$ hence, we can approximate x_{12}, by Lemma 6.17, when $\sum_k \alpha_{2k}^2(1) = \infty$. This finish the proof.

When $\sum_k \alpha_{2k}^2(1) < \infty$ we conclude that $\lim_n c_{2n} = 0$ and we are in the case (401).

In the case **4111**, as in the case (3111), we can use the expression $x_{1k} - x_{12}x_{2k}(2x_{1k}-I)$ (see (6.66)). By Lemma 6.18 we can approximate x_{12} since in this case $\sum_n(1-c_{1n})\alpha_{2n}^2(1) = \infty$ (see Remark 6.8)). Since $x_{12}, \; x_{2n} \in \mathfrak{A}^2$, by Remark 6.4 (c), we conclude that $x_{1k} \in \mathfrak{A}^2$. Finally, we have $x_{1k}, \; x_{2k+1} \in \mathfrak{A}^2$ for all $k \geq 2$.

Case 4II. Let for some $t \in \mathbb{R}\backslash\{0\}$ holds $\|G-tF\|^2 < \infty$ and $\Gamma(G_m)/\Gamma(F_m) \leq C$. This means that

$$\|G - tF\|^2 = \sum_{n=3}^\infty \frac{|\,(1-c_{2n}) - t(1-c_{1n})\,|^2}{d_n}, \quad \frac{\sum_{n=3}^m (1-c_{2n})^2/d_n}{\sum_{n=3}^m (1-c_{1n})^2/d_n} \leq C$$

where $d_n = 1 - c_{1n} + 1 - c_{2n} + 1 - c_{1n}c_{2n}$. Set $x_n = 1 - c_{1n}, \; y_n = 1 - c_{2n}$, and

$$d_1(x,y) = 2x + 2y - xy, \quad d_2(x,y) = x + y, \quad x,y \in [0,\,1].$$

Lemma 6.20. *We have for* $x, y \in [0,\,1]$

$$d_2(x,y) \leq d_1(x,y) \leq 2d_2(x,y). \tag{6.73}$$

PROOF. Indeed, since $x + y - xy = 1 - (1-x)(1-y) \in [0,1]$ we get (6.73) $x + y \leq 2x + 2y - xy \leq 2(x+y)$. $\qquad\square$

Using the relations

$$d_n = 1-c_{1n}+1-c_{2n}+1-c_{1n}c_{2n} = d_1(x_n,y_n), \quad 1-c_{2n}+1-c_{1n}c_{2n} = d_2(x_n,y_n)$$

and Lemma 6.20 we conclude that the following equivalences hold:

$$\Sigma_{12} = \sum_n \frac{x_n^2(1-y_n)^2}{x_n(1-y_n)^2 + y_n}, \quad \|F\|^2 \sim \sum_n \frac{x_n^2}{x_n + y_n}, \tag{6.74}$$

$$\|G\|^2 \sim \sum_n \frac{y_n^2}{x_n + y_n}, \quad \|G - tF\|^2 \sim \sum_n \frac{(y_n - tx_n)^2}{x_n + y_n}. \tag{6.75}$$

We have to consider only the following three possibilities:

(a) the case when $1 > x_n \geq \varepsilon > 0$ for all $n \in \mathbb{N}$, the set of all limit points is $[\varepsilon, 1]$;

(b) the case when $\lim_n x_n = 0$, the set of limit points is one point 0;

(c) the intermediate case, when the set of all limit points is the segment $[0, 1]$, in this case we have $\mathbb{N}_0$ and $\mathbb{N}_1$ two infinite subsets of $\mathbb{N}$ such that

$$x_n \geq C > 0 \quad \forall n \in \mathbb{N}_0, \quad \text{and} \quad \lim_{n \in \mathbb{N}_1} x_n = 0.$$

Consider the expression $T_{kn} - I = x_{1k}\tau_{1n} + x_{2k}\tau_{2n} + x_{1k}x_{2k}\tau_{1n}\tau_{2n}$.

Lemma 6.21. *We can approximate by linear combinations $\sum_{n=3}^{m} t_n(T_{kn} - I)$ the following expressions:*

$$x_{1k} + \beta_1^{(2)}x_{2k} - \beta_1^{(3)}x_{1k}x_{2k}, \quad or \quad \beta_2^{(1)}x_{1k} + x_{2k} - \beta_2^{(3)}x_{1k}x_{2k} \qquad (6.76)$$

where

$$\beta_1^{(2)} = \lim_m \frac{\sum_{n=3}^{m} \frac{1-c_{2n}}{1+c_{1n}}}{\sum_{n=3}^{m} \frac{1-c_{1n}}{1+c_{1n}}}, \qquad \beta_1^{(3)} = \lim_m \frac{\sum_{n=3}^{m} \frac{(1-c_{1n})(1-c_{2n})}{1+c_{1n}}}{\sum_{n=3}^{m} \frac{1-c_{1n}}{1+c_{1n}}}, \qquad (6.77)$$

$$\beta_2^{(1)} = \lim_m \frac{\sum_{n=3}^{m} \frac{1-c_{1n}}{1+c_{2n}}}{\sum_{n=3}^{m} \frac{1-c_{2n}}{1+c_{2n}}}, \qquad \beta_2^{(3)} = \lim_m \frac{\sum_{n=3}^{m} \frac{(1-c_{1n})(1-c_{2n})}{1+c_{2n}}}{\sum_{n=3}^{m} \frac{1-c_{2n}}{1+c_{2n}}}. \qquad (6.78)$$

PROOF. Indeed, to obtain the first expression or the second one in (6.76) we use the fact that $\sum_{n=3}^{m} t_n\tau_{1n} \to I$ or $\sum_{n=3}^{m} t_n\tau_{2n} \to I$ (see Remark 4.1 and Lemma 4.1) where t_n are defined respectively by the following formulas (see (6.59)):

$$t_n = -\frac{1}{1+c_{1n}}\left(\sum_{n=3}^{m} \frac{1-c_{1n}}{1+c_{1n}}\right)^{-1}, \quad t_n = -\frac{1}{1+c_{2n}}\left(\sum_{n=3}^{m} \frac{1-c_{2n}}{1+c_{2n}}\right)^{-1}.$$

Further, we should proceed exactly as in the proof of Lemma 6.12. $\qquad \square$

Example 6.2. Let $x_n = C \in (0, 1)$ for all $n \in \mathbb{N}$, then $\Sigma_{12} < \infty$ if and only if $\sum_n (1 - y_n)^2 = \sum_n c_{2n}^2 < \infty$. Indeed, we have

$$\infty > \Sigma_{12} = \sum_n \frac{x_n^2(1-y_n)^2}{x_n(1-y_n)^2 + y_n} > \sum_n \frac{x_n^2(1-y_n)^2}{x_n + y_n} = C^2 \sum_n \frac{(1-y_n)^2}{C + y_n} \sim \sum_n (1-y_n)^2.$$

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We show that $\|F\|^2 = \|G\|^2 = \infty$ and $\|G - tF\|^2 < \infty$ for some $t \neq 0$. Indeed, we have

$$\|F\|^2 \sim \sum_n \frac{x_n^2}{x_n + y_n} \sim \sum_n \frac{C^2}{C+1} = \infty, \quad \|G\|^2 \sim \sum_n \frac{y_n^2}{x_n + y_n} \sim \sum_n \frac{1}{C+1} = \infty,$$

$$\|G - tF\|^2 \sim \sum_n \frac{|y_n - tx_n|^2}{x_n + y_n} \sim \sum_n \frac{|y_n - tC|^2}{C+1} < \infty \quad \text{for} \quad tC = 1.$$

Since $1 - y_n = c_{2n}$ we conclude that $\sum_n c_{2n}^2 < \infty$ therefore, $\lim_n c_{2n} = 0$ and finally, we conclude by Lemma 6.21 (see (6.77)) Toeplitz theorem 6.22 and Lemma 6.23 that $\beta_1^{(2)} = \frac{1}{C} > 1$ and $\beta_1^{(3)} = 1$. Set $\beta := \frac{1}{C} > 1$ then we get

$$x_{1k} + \beta x_{2k} - x_{1k}x_{2k} = x_{1k}(1 - x_{2k}) + \beta x_{2k} \in \mathfrak{A}^2.$$

A regular matrix summability method is a matrix transformation of a convergent sequence which preserves the limit.

Theorem 6.22 (Otto Toeplitz [25]). *An infinite matrix $(a_{i,j})_{i,j \in \mathbb{N}}$ with complex-valued entries defines a regular summability method, i.e.,*

$$\lim_{n \to \infty} t_n = \lim_{n \to \infty} s_n \quad \text{where} \quad t_n := \sum_{n=1}^{n_k} a_{kn} s_n \tag{6.79}$$

if and only if it satisfies all of the following properties:

$(I) \quad \lim_{i \to \infty} a_{i,j} = 0 \quad j \in \mathbb{N}$ *(every column sequence converges to 0),*

$(II) \quad \lim_{i \to \infty} \sum_{j=0}^{\infty} a_{i,j} = 1$ *(the row sums converge to 1),*

$(III) \quad \sup_i \sum_{j=0}^{\infty} |a_{i,j}| < \infty$ *(the absolute row sums are bounded).*

Lemma 6.23 (A particular case of the Toeplitz theorem). *Let us have three sequences of real numbers $(a_n), (b_n)$ and (α_n) with $(a_n) > 0$, $\sum_{n \in \mathbb{N}} a_n = \infty$, $\sum_{k=1}^{m} | b_k | (\sum_{k=1}^{m} a_k)^{-1} \leq C$, $m \in \mathbb{N}$, for some $C > 0$ and $\lim_n \alpha_n = \alpha \neq 0$. Set*

$$\beta_m = \sum_{k=1}^{m} b_k \Big(\sum_{k=1}^{m} a_k \Big)^{-1}, \quad \beta_m(\alpha] = \sum_{k=1}^{m} \alpha_k b_k \Big(\sum_{k=1}^{m} a_k \Big)^{-1}, \tag{6.80}$$

$$\beta_m[\alpha) = \sum_{k=1}^{m} b_k \Big(\sum_{k=1}^{m} \alpha_k a_k \Big)^{-1}, \quad \beta_m(\alpha) = \sum_{k=1}^{m} \alpha_k b_k \Big(\sum_{k=1}^{m} \alpha_k a_k \Big)^{-1}.$$

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If the limit exists $\lim_m \beta_m = \beta \in \mathbb{R}$, *then the following limits also exist and we have*

$$\lim_m \beta_m(\alpha] = \alpha\beta, \quad \lim_m \beta_m[\alpha) = \alpha^{-1}\beta, \quad \lim_m \beta_m(\alpha) = \beta. \tag{6.81}$$

To prove that $I - x_{2k} \in \mathfrak{A}^2$ we calculate $[T_{1k}, x_{1k}]$. The operators x_{1k} and $T_{\alpha_{1k}}$ have the following form in $H_{1k} = L^2(\mathbb{F}_2, \mu_{\alpha_{1k}})$ (see (3.9), (2.12) and (6.4)):

$$x_{1k} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad T_{1k} = \begin{pmatrix} 0 & a_n^{-1} \\ a_n & 0 \end{pmatrix} \quad \text{where} \quad a_n = \sqrt{\alpha_{1k}(0)/\alpha_{1k}(1)}.$$

We show that

$$[T_{1k}, x_{1k}]^2 = -I, \quad [T_{1k}, x_{1k}(I - x_{2k})]^2 = -(I - x_{2k}). \tag{6.82}$$

Indeed, we get

$$[T_{1k}, x_{1k}] = T_{1k}x_{1k} - x_{1k}T_{1k} = \begin{pmatrix} 0 & a_n^{-1} \\ -a_n & 0 \end{pmatrix}.$$

This implies (6.82) since $[T_{1k}, x_{1k}]^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, and $x_{2k}^2 = x_{2k}$.

Finally, we get $x_{2k} \in \mathfrak{A}^2$ therefore, $x_{1k} - x_{1k}x_{2k} \in \mathfrak{A}^2$ and we can use the following expression (see (6.66))

$$[T_{kn} - I - x_{2k}(T_{2n} - I) - (x_{1k} - x_{1k}x_{2k})(T_{1n} - 1)]T_{2n} = [x_{1k}x_{2k} - x_{12}x_{2k}(2x_{1k} - I)]\tau_{1n}.$$

Since $\sum_n (1 - c_{1n})c_{2n}^2 < \sum_n c_{2n}^2 < \infty$ we conclude by Remark 6.8 that $\sum_n (1 - c_{1n})\alpha_{2n}^2(1) = \infty$. By Lemma 6.18 we get $x_{12} \in \mathfrak{A}^2$. Since $x_{12}, x_{2n} \in \mathfrak{A}^2$, by Remark 6.4 (c), we conclude that $x_{1k} \in \mathfrak{A}^2$. Finally, we have $x_{1k}, x_{2k+1} \in \mathfrak{A}^2$ for all $k \geq 2$ and the proof of the irreducibility of the example is finished.

Consider now the general case **(a)**. Since $0 < \varepsilon \leq x_n = 1 - c_{1n} < 1$ for all $n \in \mathbb{N}$ we conclude that for some subsequence we get $\lim_k (1 - c_{1n_k}) = C_1 \in [\varepsilon, 1]$. As in Example 6.2 we conclude that $\Sigma_{12} \sim \sum_n (1 - y_n)^2 < \infty$. We can repeat then step by step the proof of the irreducibility as it was done in the Example 6.2.

The case **(c)** is similar to the case (a). In this case we conclude that

$$\infty > \Sigma_{12} = \sum_{n \in \mathbb{N}_0} \frac{x_n^2(1 - y_n)^2}{x_n(1 - y_n)^2 + y_n} > \sum_{n \in \mathbb{N}_0} \frac{x_n^2(1 - y_n)^2}{x_n + y_n} \geq C^2 \sum_{n \in \mathbb{N}_0} \frac{(1 - y_n)^2}{1 + y_n} \sim \sum_{n \in \mathbb{N}_0} (1 - y_n)^2.$$

Therefore, $\sum_{n\in\mathbb{N}_0}(1-y_n)^2 < \infty$. Set $\mathbb{N}_0^m = \mathbb{N}_0 \bigcap [1, m]$ and define $\beta_0^{(2)}$ and $\beta_0^{(3)}$ as follows:

$$\beta_0^{(2)} = \lim_m \frac{\sum_{n\in\mathbb{N}_0^m} \frac{1-c_{2n}}{1+c_{1n}}}{\sum_{n\in\mathbb{N}_0^m} \frac{1-c_{1n}}{1+c_{1n}}}, \qquad \beta_0^{(3)} = \lim_m \frac{\sum_{n\in\mathbb{N}_0^m} \frac{(1-c_{1n})(1-c_{2n})}{1+c_{1n}}}{\sum_{n\in\mathbb{N}_0^m} \frac{1-c_{1n}}{1+c_{1n}}}. \qquad (6.83)$$

Since $\lim_{n\in\mathbb{N}_0} c_{2n} = 0$ we conclude that

$$\beta_0^{(2)} = (\lim_{n\in\mathbb{N}_0} x_n)^{-1} = C_1^{-1} \in [1, \, C^{-1}) \quad \text{and} \quad \beta_0^{(3)} = 1.$$

We repeat step by step the proof done in the Example 6.2 to the case **(a)**.

We show that the case **(b)** can not be realized. Indeed, let $\lim_n x_n = 0$. Since for some $t \in \mathbb{R} \setminus \{0\}$ holds

$$\|G - tF\|^2 \sim \sum_{n=3}^{\infty} \frac{|\, y_n - tx_n\,|^2}{x_n + y_n} < \infty, \quad \text{so}$$

$$0 = \lim_n \frac{|\, y_n - tx_n\,|^2}{x_n + y_n} \geq \frac{1}{2} \lim_n |\, y_n - tx_n\,|^2 = \frac{1}{2}(\lim_n y_n - t\lim_n x_n)^2 = \frac{1}{2}(\lim_n y_n)^2.$$

Therefore, $\lim_n y_n = 0$. This contradicts with two conditions:

$$\Sigma_{12} < \infty \quad \text{and} \quad \|F\|^2 \sim \sum_n \frac{x_n^2}{x_n + y_n} = \infty.$$

Indeed, fix some $\varepsilon > 0$. For sufficiently big $N \in \mathbb{N}$ we get

$$\infty > \Sigma_{12} > \sum_{n>N} \frac{x_n^2(1-y_n)^2}{x_n + y_n} \geq (1-\varepsilon)^2 \sum_{n>N} \frac{x_n^2}{x_n + y_n} \sim \|F\|^2 = \infty.$$

We give another proof of the irreducibility in the **case 1**.

The case $\Sigma_{12} = \infty$, in fact, is included in the cases (2), (3) and (4), we shall denote them respectively by (2*), (3*) and (4*). Since $\Sigma_{12} = \infty$ we have $x_{12} \in \mathfrak{A}^2$.

Case **(2*)**. Let $\Gamma(G) < \infty$. Then $\Gamma(H_1) < \infty$ therefore, $\Sigma_{1m} \to \infty$ hence, $x_{1k} \in \mathfrak{A}^2$ for $k \geq 3$. In addition $x_{12} \in \mathfrak{A}^2$. Since $x_{1k}, \, T_{\alpha_{1n}} \in \mathfrak{A}^2$ we conclude, by Remark 6.4(b), that $x_{2k} \in \mathfrak{A}^2$, $k > 3$. Finally, $x_{1k}, \, x_{2k+1} \in \mathfrak{A}^2$ for $k \geq 2$.

Case **(3*)**. Let $\Gamma(F) < \infty$. Then $\Gamma(H_2) < \infty$ therefore, $\Sigma_{2m} \to \infty$ hence, $x_{2k} \in \mathfrak{A}^2$ for $k \geq 3$. As in the case **(30)** (a), (b) or (c) we get $x_{1k} - x_{12}x_{2k} \in$

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$\mathfrak{A}^2$. Since $x_{12}, x_{2k} \in \mathfrak{A}^2$ for $k \geq 3$ we conclude that $x_{1k}, x_{2k+1} \in \mathfrak{A}^2$ for $k \geq 2$. The proof is finished.

In the opposite case, i.e., $(\bar{a}) \cap (\bar{b}) \cap (\bar{c})$, by Lemma 6.12, we can approximate the following expression: $x_{1k} - (1 - s)x_{12}x_{2k} - sx_{1k}x_{2k}$. In the case $s \neq 1$ since $x_{12}, x_{2k} \in \mathfrak{A}^2$ we conclude that $x_{1k} - sx_{1k}x_{2k} = x_{1k}(1 - sx_{2k}) \in \mathfrak{A}^2$ hence, $x_{1k} \in \mathfrak{A}^2$ (see the case **(310)**). The proof is finished.

If $s = 1$ we get $x_{1k} - x_{1k}x_{2k} \in \mathfrak{A}^2$. Since $x_{12} \in \mathfrak{A}^2$ we conclude that $T_{\alpha_{2k}} \in \mathfrak{A}^2$ for $k \geq 3$, by Remark 6.4 (a) therefore, (see (6.82))

$$[T_{\alpha_{2k}}, x_{1k} - x_{1k}x_{2k}]^2 = (-x_{1k}[T_{\alpha_{2k}}, x_{2k}])^2 = -x_{1k}. \tag{6.84}$$

At last, we have $x_{1k}, x_{2k+1} \in \mathfrak{A}^2$ for $k \geq 2$ and the proof is finished.

Case **(4*)**. Let $\Sigma_{12} = \Gamma(F) = \Gamma(G) = \infty$, then $x_{12} \in \mathfrak{A}^2$ and $T_{\alpha_{2n}} \in \mathfrak{A}^2$ for $n \geq 3$. Using Lemma 6.21, we can approximate by linear combinations $\sum_{n=3}^{m} t_n(T_{kn} - I)$ the following expressions:

$$x_{1k} + \beta_1^{(2)}x_{2k} - \beta_1^{(3)}x_{1k}x_{2k}, \quad \text{or} \quad \beta_2^{(1)}x_{1k} + x_{2k} - \beta_2^{(3)}x_{1k}x_{2k}$$

since one of two sequence

$$\sum_{n=3}^{m}(1 - c_{1n})\left(\sum_{n=3}^{m}(1 - c_{2n})\right)^{-1} \quad \text{or} \quad \sum_{n=3}^{m}(1 - c_{2n})\left(\sum_{n=3}^{m}(1 - c_{1n})\right)^{-1}$$

should be bounded.

Because of the symmetry between the first and the second rows, i.e., between variables $(x_{1k})_k$ and $(x_{2k})_k$, it is sufficient to consider the case when $x_{1k} + \beta_1^{(2)}x_{2k} - \beta_1^{(3)}x_{1k}x_{2k} \in \mathfrak{A}^2$ where $0 < \beta_1^{(3)} \leq \beta_1^{(2)} < \infty$. By (6.82) we get

$$[T_{\alpha_{1k}}, x_{1k}(I - \beta_1^{(3)}x_{2k}) + \beta_1^{(2)}x_{2k}]^2 = -(I - \beta_1^{(3)}x_{2k}),$$

therefore, $x_{2k} \in \mathfrak{A}^2$ for $k \geq 3$ when $\beta_1^{(3)} > 0$. By Remark 6.4 (c), we get that $x_{1k} \in \mathfrak{A}^2$ for $k \geq 3$ and the proof is finished.

Let $\beta_1^{(2)} > \beta_1^{(3)} = 0$, then $x_{1k} + \beta_1^{(2)}x_{2k} \in \mathfrak{A}^2$. We prove the following

Lemma 6.24. *The von Neumann algebra C_n generated by operators $T_{\alpha_{1n}}$, $T_{\alpha_{2n}}$ and $x_{1n} + \beta x_{2n}$ is irreducible in the space $H_n := L^2(\mathbb{F}_2, \mu_{\alpha_{1n}}) \otimes L^2(\mathbb{F}_2, \mu_{\alpha_{2n}})$ for $\beta \in (0, 1]$.*

PROOF. Using (6.5), (5.4), (2.12) and Remark 2.3 we get

$$
T_{\alpha_{1n}} = \begin{pmatrix} 0 & 0 & a_n^{-1} & 0 \\ 0 & 0 & 0 & a_n^{-1} \\ a_n & 0 & 0 & 0 \\ 0 & a_n & 0 & 0 \end{pmatrix}, \quad T_{\alpha_{2n}} = \begin{pmatrix} 0 & b_n^{-1} & 0 & 0 \\ b_n & 0 & 0 & 0 \\ 0 & 0 & 0 & b_n^{-1} \\ 0 & 0 & b_n & 0 \end{pmatrix}, \quad x_{1n} + \beta x_{2n} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1+\beta \end{pmatrix}
$$

$$(6.85)$$

where a_n, b_n are defined by (6.4). Indeed, we have

$$
T_{\alpha_{1n}} \otimes I = \begin{pmatrix} 0 & a_n^{-1} \\ a_n & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a_n^{-1} & 0 \\ 0 & 0 & 0 & a_n^{-1} \\ a_n & 0 & 0 & 0 \\ 0 & a_n & 0 & 0 \end{pmatrix},
$$

$$
I \otimes T_{\alpha_{2n}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & b_n^{-1} \\ b_n & 0 \end{pmatrix} = \begin{pmatrix} 0 & b_n^{-1} & 0 & 0 \\ b_n & 0 & 0 & 0 \\ 0 & 0 & 0 & b_n^{-1} \\ 0 & 0 & b_n & 0 \end{pmatrix},
$$

$$
x_{1n} \otimes I + \beta I \otimes x_{2n} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \beta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} =
$$

$$
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \beta \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1+\beta \end{pmatrix}.
$$

In the case $\beta \in (0, 1)$ the commutant $(x_{1n} + \beta x_{2n})'$ consists of all diagonal operators $D(\lambda) = \mathrm{diag}(\lambda_1, \ldots, \lambda_4)$ since eigenvalues of $x_{1n} + \beta x_{2n}$ are distinct. The commutation relation $[D, T_{\alpha_{1n}} \otimes I] = 0$ implies $\lambda_1 = \lambda_3$, $\lambda_2 = \lambda_4$. The commutation relation $[D, I \otimes T_{\alpha_{2n}}, I \otimes T_{\alpha_{2n}}] = 0$ implies $\lambda_1 = \lambda_2$, $\lambda_3 = \lambda_4$. Finally, we get $D(\lambda) = \lambda I$. In the case $\beta = 1$ the commutant $(x_{1n} + \beta x_{2n})'$ consists of all operators of the form

$$
D = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & b & 0 \\ 0 & c & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}.
$$

The commutation relation $[D, T_{\alpha_{1n}} \otimes I] = 0$ implies $b = c = 0$, $\lambda_1 = \lambda_3$, $\lambda_2 = \lambda_4$. The commutation relation $[D, I \otimes T_{\alpha_{2n}}, I \otimes T_{\alpha_{2n}}] = 0$ implies $\lambda_1 = \lambda_2$, $\lambda_3 = \lambda_4$. Hence, in this case we get $D = \lambda I$. □

The irreducibility of the representation in the case $\beta_1^{(2)} > \beta_1^{(3)} = 0$ follows from the fact that von Neumann algebra $\mathfrak{A} = (T_{12}, x_{12})'' \otimes_{n=3}^\infty C_n$ is irreducible since the commutant $\mathfrak{A}'$ is trivial by Lemma 6.24. Indeed, we have $\mathfrak{A}' = (T_{12}, x_{12})' \otimes_{n=3}^\infty C_n'$.

When $\beta_1^{(2)} = \beta_1^{(3)} = 0$ we get $x_{1n} \in \mathfrak{A}^2$. By Remark 6.4 (b), we conclude that $x_{2k} \in \mathfrak{A}^2$ for $k \geq 3$ and the proof is finished.

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