Noncommutative stacks.

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Introduction

One of the purposes of this work is to introduce a noncommutative analogue of Artin's and Deligne-Mumford algebraic stacks in the most natural and sufficiently general way. We start with quasi-coherent modules on fibered categories, then define stacks and prestacks. We define formally smooth, formally unramified, and formally étale cartesian functors. This provides us with enough tools to extend to stacks the glueing formalism we developed in [KR3] for presheaves and sheaves of sets.

Quasi-coherent presheaves and sheaves on a fibered category.

Quasi-coherent sheaves on geometric (i.e. locally ringed topological) spaces were introduced in fifties. The notion of quasi-coherent modules was extended in an obvious way to ringed sites and toposes at the moment the latter appeared (in SGA), but it was not used much in this generality. Recently, the subject was revisited by D. Orlov in his work on quasi-coherent sheaves in commutative an noncommutative geometry [Or] and by G. Laumon an L. Moret-Bailly in their book on algebraic stacks [LM-B].

Slightly generalizing [R4], we associate with any functor F (regarded as a category over a category) the category of 'quasi-coherent presheaves' on F (otherwise called 'quasi-coherent presheaves of modules' or simply 'quasi-coherent modules') and study some basic properties of this correspondence in the case when the functor defines a fibered category. Imitating [Gir], we define the quasi-topology of 1-descent (or simply 'descent') and the quasi-topology of 2-descent (or 'effective descent') on the base of a fibered category (i.e. on the target of the functor F).

If the base is endowed with a quasi-topology, τ, we introduce the notion of a 'sheaf of modules' on (F,τ). We define the category $Qcoh(F,\tau)$ of quasi-coherent sheaves on (F,τ) as the intersection of the category $Qcoh(F)$ of quasi-coherent presheaves on F and the category of sheaves of modules on (F,τ).

If the quasi-topology τ is coarser than the quasi-topology of 1-descent, than every quasi-coherent module on F is a sheaf of modules on (F,τ), i.e. $Qcoh(F,\tau) = Qcoh(F)$. In this case, we show, under certain natural conditions on a presheaf of sets on the base X, the existence of a 'coherator' which is, by definition, a right adjoint to the embedding of the category $Qcoh(F/X)$ of quasi-coherent modules on X into the category of sheaves of modules on X (that is on F/X). This fact is important, because the existence of the coherator on X garantees the existence of the direct image functor (between quasi-coherent modules) for any morphism from a presheaf of sets to X.

The relation of this formalism with the classical notions and those used in [Or] is as follows. With any ringed category $(\mathcal{A},\mathcal{O})$, one can naturally associate a fibered category F: its fiber over an object T of $\mathcal{A}$ is the category opposite to the category of $\mathcal{O}(T)$-modules. The category $Qcoh(F)$ of quasi-coherent modules on the fibered category F is equivalent to the category of quasi-coherent $\mathcal{O}$-modules in the sense of [Or]. If τ is a topology on

$\mathcal{A}$, then the category $Qcoh(F, \tau)$ is equivalent to the category of quasi-coherent sheaves of $\mathcal{O}$-modules in the classical (i.e. [SGA]) sense. In particular, if F is the fibered category of modules over (commutative) affine schemes and the presheaf X is represented by a scheme (or an algebraic space) X, then the category $Qcoh(F/X)$ is naturally equivalent to the category of quasi-coherent sheaves on the scheme (resp. on the algebraic space) X.

A standard noncommutative example is the ringed category $(\mathbf{Aff}, \mathcal{O})$, where $\mathbf{Aff}$ is the category opposite to the category of associative rings and the presheaf of rings $\mathcal{O}$ assigns to any object $\mathbf{Spec}R$ of $\mathbf{Aff}$ (corresponding to the ring R) the ring R. Thus, to any presheaf of sets X on $\mathbf{Aff}$, we assign the category $Qcoh(X)$ of quasi-coherent modules on X. If the presheaf X is representable by $\mathbf{Spec}R$, then $Qcoh(X)$ is equivalent to the category $R - mod$ of left R-modules. If X is a locally affine space, the category $Qcoh(X)$ is described via affine covers and relations.

We show, among other facts, that the canonical topology on $\mathbf{Aff}$ (i.e. the strongest topology such that every representable presheaf of sets on $\mathbf{Aff}$ is a sheaf) is precisely the topology of 1-descent. In the commutative case, this fact was established by D. Orlov [Or].

Noncommutative stacks.

A stack is a sheaf of categories on a (pre)topology. One of the classical examples is the category of (commutative) affine schemes with Zariski topology with the category of quasi-coherent sheaves assigned to each affine scheme. The sheaf property means an effective descent property for quasi-coherent sheaves.

In noncommutative algebraic geometry, (pre)topologies are replaced by Q-categories (where Q stands for 'quotient'), and stacks are sheaves of categories (in an appropriate sense) on a Q-category. We introduce monopresheaves of categories, otherwise called 'prestacks', and epipresheaves of categories the meaning of which is explained below.

A standard example of a noncommutative stack is the Q-category of associative unital rings with faithfully flat morphisms (the category of rings is a quotient of the category of flat covers) with the category of left R-modules assigned to each ring R. The Q-category in this example gives rise to a quasi-topology which is coarser than the quasi-topology of effective descent. The latter is equivalent to the fact that this is, indeed, an example of a noncommutative stack.

In general, every quasi-topology on a category $\mathcal{A}$ induces a structure of a Q-category on the opposite category. A fibered category over the category $\mathcal{A}$ is a stack (resp. prestack) if the quasi-topology is coarser that the quasi-topology of 2-descent (resp. of 1-descent).

There is another interpretaion of a Q-category illustrated by the following example: the Q-category $\mathbf{CRings}$ of commutative unital rings with ring epimorphisms having a nilpotent kernel. This example suggests that Q-categories might be regarded also as "categories of thickennings".

In this case, sheaves (resp. monopresheaves) of categories over the category $\mathbf{CAff}$ of commutative affine schemes (i.e. the category opposite to the category $\mathbf{CRings}$) can be interpreted as 'formally etale' (resp. formally unramified) fibered categories; and epipresheaves of categories are formally smooth fibered categories over $\mathbf{CAff}$. An appropriate (not quite obvious) choice of a Q-category produces a noncommutative version of formally étale, formally unramified, and formally smooth fibered categories over the category $\mathbf{Aff}$ of noncommutative affine schemes.

We consider the relative situation (i.e. morphisms of fibered categories) and define the notions of formally étale, formally unramified, and formally smooth cartesian functors. Note that even in the case of stacks over the site of commutative schemes, our notions are applicable to a much larger class of cartesian functors (morphisms of stacks) than the conventional ones, since they do not require any representability assumptions.

We define smooth, unramified, and étale morphisms as formally smooth (resp. formally unramified, resp. formally étale) locally finitely presentable morphisms.

Given a *quasi-pretopology* (i.e a class of coverings) τ on the category of stacks (or fibered categories) over a Q-category, we define 'τ-locally affine stacks' as stacks which are locally representable. Using the notions of smooth and étale cartesian functors, we introduce smooth, étale, and Zariski quasi-topologies on the category of stacks over a given Q-category. In particular, we define smooth, étale and Zariski quasi-topologies on the category of stacks over the category **Aff** of noncommutative affine schemes endowed with the flat quasi-topology.

Locally affine stacks on **Aff** with respect to smooth (resp. étale) quasi-topology are noncommutative versions of algebraic Artin's (resp. Deligne-Mumford's) stacks.

We define 'local constructions' on fibered categories (in particular, on stacks) which is a device to transfer certain functorial constructions of (noncommutative) 'varieties' defined over an affine base to constructions of 'varieties' over stacks, in particular, over arbitrary locally affine spaces. Among them are affine and projective vector-fibers corresponding to a quasi-coherent module on a locally ringed category, and Grassmannians corresponding to a pair of locally projective quasi-coherent modules on a locally ringed category.

The paper is organized as follows.

In Section 1, we introduce and study quasi-coherent presheaves and sheaves of modules on an arbitrary fibered category. This includes quasi-topologies of 1- and 2-descent, and the existence (under certain conditions) of a coherator.

In Section 2, we apply the facts and constructions of Section 1 to the fibered categories associated with ringed categories (in particular, to ringed sites and toposes).

The second part of the work is dedicated to *noncommutative stacks*.

Section 3 contains preliminaries on representability of fibered categories and cartesian functors.

Noncommutative stacks and prestacks are defined in Section 4.

In Section 5, we give some standard examples of stacks.

In Section 6, we obtain (as a natural extension of the results and definitions of the first part of the work) the notions of (formally) smooth and (formally) étale morphisms of fibered categories.

In Section 7, we introduce locally affine stacks. In particular, we define noncommutative versions of resp. Artin's stacks and Deligne-Mumford's stacks.

In Section 8, we define 'local constructions' on fibered categories (in particular, on stacks) and illustrate them with several applications.

Our introduction to stacks is restricted to several basic notions illustrated by a couple of examples which we have chosen to sketch here because they are a natural extension of the sheaf theory and the facts on formally smooth and formally étale morphisms presented in [KR3], and also the next step after introducing quasi-coherent presheaves on fibered categories and descent quasi-topologies discussed in Section 1.

A more detailed exposition of stacks will appear in a subsequent paper.

The paper has two appendices. The first appendix is a short summary on Q-categories. Appendix 2 contains preliminaries on fibered categories.

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I. Quasi-coherent sheaves on fibered categories.

1. Quasi-coherent modules on a fibered category.

1.1. Modules and quasi-coherent modules on a category over a category. Let $\mathcal{E}$ be a category (which belongs to some universum $\mathfrak{U}$) and $\mathfrak{F} = (\mathcal{F}, \mathcal{F} \xrightarrow{\pi} \mathcal{E})$ a category over $\mathcal{E}$. Denote by $\mathcal{M}od(\mathfrak{F})$ the category opposite to the category of all sections of $\mathfrak{F}$. We shall call objects of $\mathcal{M}od(\mathfrak{F})$ *modules* on $\mathfrak{F}$.

We denote by $Qcoh(\mathfrak{F})$ the category opposite to the category $Cart_{\mathcal{E}}(\mathcal{E}, \mathfrak{F})$ of cartesian sections of $\mathfrak{F}$. In other words, $Qcoh(\mathfrak{F}) = (Lim\mathfrak{F})^{op}$ (cf. A2.5.5). Objects of $Qcoh(\mathfrak{F})$ will be called *quasi-coherent modules* on $\mathfrak{F}$.

Any morphism $\mathfrak{F} \longrightarrow \mathfrak{G}$ of $\mathcal{E}$-categories induces a functor $\mathcal{M}od(\mathfrak{F}) \longrightarrow \mathcal{M}od(\mathfrak{G})$. Thus we have a functor $\mathcal{M}od : Cat/\mathcal{E} \longrightarrow Cat$ from the category of $\mathcal{E}$-categories to the category of categories.

Similarly, the map $\mathfrak{F} \longmapsto Qcoh(\mathfrak{F})$ extends to a functor $Qcoh : Cart_{\mathcal{E}} \longrightarrow Cat$ from the category of cartesian functors over $\mathcal{E}$ to Cat.

1.1.1. Proposition. *The functor $Qcoh : Cart_{\mathcal{E}} \longrightarrow Cat$ preserves small products.*

Proof. In fact, by A2.6.6 and A2.6.6.2, the functor $Lim : Cart_{\mathcal{E}} \longrightarrow Cat$ preserves small products. The functor $Qcoh$ is, by definition, the composition of Lim and the canonical automorphism $Cat \longrightarrow Cat,\ C \longmapsto C^{op}$. ■

1.1.2. Modules and quasi-coherent modules on a fibered category. Let $\mathfrak{F}$ be a fibered category corresponding to a pseudo-functor $\mathcal{E}^{op} \longrightarrow Cat$,

$$Ob\mathcal{E} \ni X \longmapsto \mathcal{F}_X,\ Hom\mathcal{E} \ni f \longmapsto f^*,\ Hom\mathcal{E} \times_{Ob\mathcal{E}} Hom\mathcal{E} \ni (f, g) \longmapsto c_{f,g} \qquad (1)$$

(cf. A2.7, A2.7.1). Then the category $\mathcal{M}od(\mathfrak{F})$ can be described as follows. An object of $\mathcal{M}od(\mathfrak{F})$ is a function which assigns to each $T \in Ob\mathcal{E}$ an object $M(T)$ of the fiber $\mathcal{F}_T$ and to each morphism $T \xrightarrow{f} T'$ a morphism $f^*(M(T')) \xrightarrow{\xi_f} M(T)$ such that $\xi_{gf} \circ c_{f,g} = \xi_f \circ f^*(\xi_g)$. Morphisms are defined in a natural way.

An object (M, ξ) of $\mathcal{M}od(\mathfrak{F})$ belongs to the subcategory $Qcoh(\mathfrak{F})$ iff ξ_f is an isomorphism for all $f \in Hom\mathcal{E}$.

1.1.3. Proposition. *Let $\mathfrak{F}$ be a fibered category over $\mathcal{E}$. Suppose the category $\mathcal{E}$ has a final object, $T_\bullet$. Then*

(a) The category $Qcoh(\mathfrak{F})$ is equivalent to the category $\mathcal{F}_{T_\bullet}^{op}$ dual to the fiber of $\mathfrak{F}$ at the final object $T_\bullet$.

(b) The inclusion functor $Qcoh(\mathfrak{F}) \longrightarrow \mathcal{M}od(\mathfrak{F})$ has a right adjoint.

Proof. (a) The equivalence is given by the functor $Qcoh(\mathfrak{F}) \longrightarrow \mathcal{F}_{T_\bullet}$ which assigns to every quasi-coherent module M on $\mathfrak{F}$ the object $M(T_\bullet)$ of $\mathcal{F}_{T_\bullet}$. The quasi-inverse functor maps any object L of $\mathcal{F}_{T_\bullet}$ to the quasi-coherent module $L^\sim$ which assigns to each object S of $\mathcal{E}$ the object $f^*(L)$. Here f is the unique morphism $S \longrightarrow T_\bullet$.

(b) The composition of the functor $\mathcal{M}od(\mathfrak{F}) \longrightarrow \mathcal{F}_{T_\bullet}^{op}$, $M \longmapsto M(T_\bullet)$, with the equivalence $\mathcal{F}_{T_\bullet}^{op} \longrightarrow Qcoh(\mathfrak{F})$ constructed in (a) is a right adjoint to the inclusion functor $Qcoh(\mathfrak{F}) \longrightarrow \mathcal{M}od(\mathfrak{F})$. ■

1.1.4. Base change and quasi-coherent modules.

1.1.4.1. Proposition. *Let $\mathfrak{F} = (\mathcal{F}, \mathcal{F} \xrightarrow{\pi} \mathcal{E})$ be a category over $\mathcal{E}$ and $\mathcal{E}' \longrightarrow \mathcal{E}$ a functor. Let $\mathfrak{F} \times_{\mathcal{E}} \mathcal{E}'$ denote the induced category $(\mathcal{F} \times_{\mathcal{E}} \mathcal{E}', \mathcal{F} \times_{\mathcal{E}} \mathcal{E}' \longrightarrow \mathcal{E}')$.*

Then $Qcoh(\mathfrak{F} \times_{\mathcal{E}} \mathcal{E}')$ is isomorphic to the full subcategory of $Hom_{\mathcal{E}}(\mathcal{E}', \mathfrak{F})^{op}$ objects of which are $\mathcal{E}$-functors that transform any morphism into a cartesian morphism.

If $\mathfrak{F}$ is a fibered category over $\mathcal{E}$ and $\mathcal{F}_c$ is the subcategory of $\mathcal{F}$ formed by all cartesian morphisms of $\mathcal{F}$, then $ObQcoh(\mathfrak{F} \times_{\mathcal{E}} \mathcal{E}') \simeq ObHom_{\mathcal{E}}(\mathcal{E}', \mathcal{F}_c)$.

Proof. The assertion follows from A2.6.7.2. ■

Let $\mathfrak{U}$, $\mathfrak{V}$ be two universums such that $\mathfrak{U} \in \mathfrak{V}$. Denote by $Cart_{\mathfrak{U},\mathfrak{V}}$ the 2-category objects of which are categories over categories $\mathfrak{F} = (\mathcal{F} \xrightarrow{\pi} \mathcal{E})$ such that the base $\mathcal{E}$ belongs to $\mathfrak{V}$ and each fiber belongs to $\mathfrak{U}$. Denote by $\mathfrak{M}Cart_{\mathfrak{U},\mathfrak{V}}$ all cartesian functors (1-morphisms of $Cart_{\mathfrak{U},\mathfrak{V}}$)

$$\begin{tikzcd} \mathcal{F}' \arrow[r, "u"] \arrow[d, "\pi'"'] & \mathcal{F} \arrow[d, "\pi"] \\ \mathcal{E}' \arrow[r, "v"] & \mathcal{E} \end{tikzcd} \qquad (1)$$

such that the functors induced on fibers are category equivalences. It follows that $\mathfrak{M}Cart_{\mathfrak{U},\mathfrak{V}}$ is a 2-subcategory of the 2-category $Cart_{\mathfrak{U},\mathfrak{V}}$.

1.1.4.2. Proposition. *The map $\mathfrak{F} \longmapsto Qcoh(\mathfrak{F})$ extends to a pseudo-functor*

$$Qcoh : \mathfrak{M}Cart_{\mathfrak{U},\mathfrak{V}}^{op} \longrightarrow Cat_{\mathfrak{V}}.$$

Proof. Let (1) be an arbitrary cartesian morphism. It can be decomposed in two cartesian morphisms

$$\begin{tikzcd} \mathcal{F}' \arrow[r, "u'"] \arrow[d, "\pi'"'] & \mathcal{F} \times_{\mathcal{E}} \mathcal{E}' \arrow[r, "v^\sim"] \arrow[d] & \mathcal{F} \arrow[d, "\pi"] \\ \mathcal{E}' \arrow[r, "Id_{\mathcal{E}'}"] & \mathcal{E}' \arrow[r, "v"] & \mathcal{E} \end{tikzcd}, \qquad (2)$$

where the right square is the canonical pull-back. By 1.1.4.1, the right square of (2) induces a functor

$$Qcoh(\mathfrak{F}) \longrightarrow Qcoh(\mathfrak{F} \times_{\mathcal{E}} \mathcal{E}') \tag{3}$$

and by 1.1.1, the left square of (2) induces a functor

$$Qcoh(\mathfrak{F}') \longrightarrow Qcoh(\mathfrak{F} \times_{\mathcal{E}} \mathcal{E}'). \tag{4}$$

The morphism (1) belongs to $\mathfrak{M}Cart$ iff u' in (2) is a category equivalence, which implies that (4) is a category equivalence. Taking the composition of (3) with a quasi-inverse to (4), we assign to the morphism (1) a functor $Qcoh(\mathfrak{F}) \longrightarrow Qcoh(\mathfrak{F}')$. This correspondence defines a pseudo-functor $\mathfrak{M}Cart^{op} \longrightarrow Cat$. ■

1.1.5. Quasi-coherent modules on presheaves of sets. Let X be a presheaf of sets on the base $\mathcal{E}$, i.e. a functor $\mathcal{E}^{op} \longrightarrow$ **Sets**. Then we have a functor $\mathcal{E}/X \longrightarrow \mathcal{E}$ and the category $\mathfrak{F}/X := \mathfrak{F} \times_{\mathcal{E}} \mathcal{E}/X$ over $\mathcal{E}/X$ obtained via a base change (as usual, we identify $\mathcal{E}$ with a full subcategory of the category $\mathcal{E}^\wedge$ of presheaves of sets on $\mathcal{E}$ formed by representable presheaves). Notice that any morphism of the category $\mathcal{E}/X$ over $\mathcal{E}$ is cartesian. Therefore, by 1.1.4.1, the category $Qcoh(\mathfrak{F}/X)$ is equivalent to the category $Cart_{\mathcal{E}}(\mathcal{E}/X, \mathfrak{F})^{op}$ opposite to the category of cartesian functors $\mathcal{E}/X \longrightarrow \mathfrak{F}$.

1.1.5.1. The canonical extension of a fibered category. Following [Gir], we denote the category $Cart_{\mathcal{E}}(\mathcal{E}/X, \mathfrak{F}) = Qcoh(\mathfrak{F}/X)^{op}$ by $\mathfrak{F}^+(X)$. The correspondence $X \longmapsto \mathfrak{F}^+(X)$ extends to a pseudo-functor, hence defines a fibered category over $\mathcal{E}^\wedge$ which is called (in [Gir]) the *canonical extension* of $\mathfrak{F}$ onto $\mathcal{E}^\wedge$.

1.1.5.2. Proposition. *Let $\mathfrak{F}$ be a fibered category over $\mathcal{E}$ and X an object of the category $\mathcal{E}^\wedge$ of presheaves of sets on $\mathcal{E}$.*

(a) If the functor X is a representable by an object, x, of the category $\mathcal{E}$, then the category $Qcoh(\mathfrak{F}/X)$ is equivalent to the category $\mathfrak{F}_x^{op}$ opposite to the fiber $\mathfrak{F}_x$ over x.

(b) Suppose $X = colim(X_i)$ for some diagram $I \longrightarrow \mathcal{E}^\wedge$, $i \longmapsto X_i$. Then the natural functor $Qcoh(\mathfrak{F}/X) \longrightarrow limQcoh(\mathfrak{F}/X_i)$ is an isomorphism.

(c) $X \longmapsto Qcoh(\mathfrak{F}/X)$ is a sheaf of categories on $\mathcal{E}^\wedge$ for the canonical topology.

Proof. (a) This fact is a consequence of 1.1.3.

(b) The assertion follows from the isomorphism $Cart(\mathcal{E}/X, \mathfrak{F}) \xrightarrow{\sim} lim(Cart(\mathcal{E}/X_i, \mathfrak{F})$ proven in [Gir] 3.2.4.

The assertion (c) follows from (b). ■

1.2. The quasi-topology and topology of $\mathfrak{F} - i$-descent. Recall that a functor is called *0-faithful* (resp. *1-faithful*, resp. *2-faithful*) if it is faithful (resp. fully faithful, resp. an equivalence ([Gir], 0.5.1.1).

1.2.1. Definition. Let $\mathfrak{F}$ be a fibered category over $\mathcal{E}$, h_X the presheaf represented by an object X of $\mathcal{E}$. A subpresheaf T of h_X is called a subpresheaf of $\mathfrak{F} - i$*-descent*, $i = 0,\ 1,\ 2$, if the corresponding functor $Qcoh(\mathfrak{F}/X) \longrightarrow Qcoh(\mathfrak{F}/T)$ (or, equivalently, $\mathfrak{F}_X^{op} \longrightarrow Qcoh(\mathfrak{F}/T)$) is i-faithful.

A family of morphisms $\mathfrak{X} = \{X_i \to X |\ i \in I\}$ is said to be of $\mathfrak{F} - i$-descent if the subpresheaf of h_X associated with $\mathfrak{X}$ is of $\mathfrak{F} - i$-descent.

The difinition 1.2.1 is equivalent to the usual definition of a sieve of $\mathfrak{F} - i$-descent (cf. [Gir], II.1.1.1 and II.1.1.1.1). Another terminology: $\mathfrak{F} - 1$-descent is called simply $\mathfrak{F}$-*descent* and $\mathfrak{F} - 2$-descent is called also *effective descent*.

1.2.2. The quasi-topology of $\mathfrak{F} - i$-descent. For any $X \in Ob\mathcal{E}$, denote by $\mathfrak{T}_{\mathfrak{F},i}(X)$ the set of all subpresheaves of h_X which are of $\mathfrak{F} - i$-descent. This defines a quasi-topology, $\mathfrak{T}_{\mathfrak{F},1}$, which we call the *quasi-topology of* $\mathfrak{F} - i$-*descent*.

1.2.2.1. Proposition. *The quasi-topology of* $\mathfrak{F} - 1$-*descent is the finest quasi-topology such that for any* $X \in Ob\mathcal{E}$ *and any* $x, y \in Ob\mathfrak{F}_X$, *the presheaf*

$$Hom_X(x,y) : \mathcal{E}/X \longrightarrow \mathbf{Sets}, \quad (Y \xrightarrow{f} X) \longmapsto Hom_Y(f^*(x), f^*(y)), \tag{1}$$

is a sheaf on $\mathcal{E}/X$ *for the induced quasi-topology.*

Proof. The presheaf $Hom_X(x,y)$ being a sheaf for all $x, y \in Ob\mathfrak{F}_X$ is equivalent to the full faithfulness of the functor $\mathfrak{F}_X^{op} \longrightarrow Qcoh(\mathfrak{F}/T)$ for any $T \in \mathfrak{T}(X)$. The assertion follows now from the definition of the quasi-topology of $\mathfrak{F} - 1$-descent (see also 1.2.1). ■

1.2.3. Definition. Let $\mathfrak{F}$ be a fibered category over $\mathcal{E}$, h_X the presheaf represented by an object X of $\mathcal{E}$. A subpresheaf T in h_X is called a subpresheaf of *universal* $\mathfrak{F} - i$-*descent*, $i = 0,\ 1,\ 2$, if for any morphism $Y \longrightarrow X$ in $\mathcal{E}$, the subpresheaf $T \times_{h_X} h_Y$ of h_Y is of $\mathfrak{F} - i$-descent.

1.2.4. The topology of $\mathfrak{F} - i$-descent. For any $X \in Ob\mathcal{E}$, denote by $\mathfrak{T}^u_{\mathfrak{F},i}(X)$ the set of all subpresheaves of h_X which are of universal $\mathfrak{F} - i$-descent. This defines a topology which is called the *topology of* $\mathfrak{F} - i$-*descent*.

1.2.4.1. Proposition. *The topology of* $\mathfrak{F} - 1$-*descent is the finest topology such that for any* $X \in Ob\mathcal{E}$ *and any* $x, y \in Ob\mathfrak{F}_X$, *the presheaf*

$$Hom_X(x,y) : \mathcal{E}/X \longrightarrow \mathbf{Sets}, \quad (Y \xrightarrow{f} X) \longmapsto Hom_Y(f^*(x), f^*(y)), \tag{1}$$

is a sheaf on $\mathcal{E}/X$ *for the induced topology.*

Proof. The assertion follows from 1.2.2.1. ■

1.2.5. Coinduced topologies and $\mathfrak{F} - i$-descent. Let $\mathfrak{T}$ be a topology on the category $\mathcal{A}$, and let $\mathfrak{T}^\wedge$ denote the coinduced topology on the category $\mathcal{A}^\wedge$ of prespaces. Recall that the topology $\mathfrak{T}^\wedge$ is defined as follows: for any prespace X, a sieve $V \hookrightarrow h_X = Hom(-, X)$ is a refinement of X for $\mathfrak{T}^\wedge$ iff for any $S \in Ob\mathcal{A}$ and any morphism $S \longrightarrow X$, the sieve $V \times_{h_X} h_S \hookrightarrow h_S$ is a refinement of S for $\mathfrak{T}$.

1.2.5.1. Example. If $\mathfrak{T}$ is the discrete topology on $\mathcal{E}$, then the coincduced topology on $\mathcal{E}^\wedge$ coincides with the canonical topology on $\mathcal{E}^\wedge$.

A morphism $f : X \longrightarrow Y$ is called *covering* (resp. *bicovering*) if the induced morphism of associated sheaves, $f^a : X^a \longrightarrow Y^a$ is an epimorphism (resp. an isomorphism).

1.2.6. Proposition. *Let* $\mathfrak{F}$ *be a fibered category over a category* $\mathcal{E}$ *and* i *an integer* $0 \leq i \leq 2$. *Let* $\mathfrak{F}^+$ *denote the canonical extension of* $\mathfrak{F}$ *onto* $\mathcal{E}^\wedge$ *(cf. 1.1.5.1).*

(a) The topology of $\mathfrak{F}^+ - i$-descent is the topology coinduced by the topology of $\mathfrak{F} - i$-descent.

(b) A morphism $X \xrightarrow{f} Y$ in $\mathcal{E}^\wedge$ is bicovering for the topology of $\mathfrak{F}^+ - i$-descent iff for any morphism $Y' \longrightarrow Y$, the corresponding functor

$$Qcoh(\mathfrak{F}/Y') \longrightarrow Qcoh(\mathfrak{F}/X \times_Y Y') \tag{1}$$

is i-faithful. The converse is true if $i = 2$ (i.e. (1) is a category equivalence), or if f is a monomorphism.

Proof. The assertions (a) and (b) are equivalent to the assertions resp. (iii) and (iv) of II.11.3 in [Gir]. ■

1.2.7. Proposition. *Let $\mathfrak{X} = \{X_i \to X \mid i \in I\}$ be a family of arrows in $\mathcal{E}^\wedge$, and let $X' \xrightarrow{\tau} X$ be the image of $\mathfrak{X}$. Then $\mathfrak{X}$ is of $\mathfrak{F} - i$-descent iff the corresponding inverse image functor $\tau^* = Qcoh(\tau) : Qcoh(\mathfrak{F}/X) \longrightarrow Qcoh(\mathfrak{F}/X')$ is i-faithful.*

Proof. The assertion is equivalent to the assertion II.1.1.3.1 in [Gir] (which is a part of the argument of II.1.1.3). ■

1.2.8. Canonical topology on presheaves of sets and the effective descent. If $\mathfrak{X} = \{X_i \to X \mid i \in I\}$ is a cover for the canonical topology on $\mathcal{E}^\wedge$, then the image X' of $\mathfrak{X}$ coincides with X. By 1.2.7, the family $\mathfrak{X}$ is a cover for the effective $\mathfrak{F}$-descent (or $\mathfrak{F} - 2$-descent) topology. In particular, the topology of the effective $\mathfrak{F}^+$-descent is finer than the canonical topology (hence any subcanonical topology) on $\mathcal{E}^\wedge$.

1.2.8.1. Remark. One can deduce the latter fact directly from the part (a) of 1.2.6 as follows. Let $\mathfrak{F}$ be a fibered category over $\mathcal{E}$. If the topology of $\mathfrak{F} - i$-descent is finer than a topology $\mathfrak{T}$ than, evidently, the coinduced topology $\mathfrak{T}^\wedge$ on $\mathcal{E}^\wedge$ is coarser than the topology of $\mathfrak{F}^+ - i$-descent, because by 1.2.6 the topology of $\mathfrak{F}^+ - i$-descent is coinduced by the topology of $\mathfrak{F} - i$-descent). In particular, the topology $\mathfrak{T}_d^\wedge$ coinduced by the discrete topology is coarser than the topology of the effective $\mathfrak{F}^+$-descent. But, as it has been already observed (in 1.2.5.1), $\mathfrak{T}_d^\wedge$ coincides with the canonical topology on $\mathcal{E}^\wedge$.

1.3. Sheaves of modules.

1.3.1. Sheaves of modules on a cofibered category. Let $\mathfrak{F}$ be a cofibered category over $\mathcal{E}$ corresponding to a pseudo-functor $\mathcal{E} \longrightarrow Cat$,

$$Ob\mathcal{E} \ni X \longmapsto \mathcal{F}_X, \ Hom\mathcal{E} \ni f \longmapsto f_\bullet, \ Hom\mathcal{E} \times_{Ob\mathcal{E}} Hom\mathcal{E} \ni (f,g) \longmapsto ((fg)_\bullet \xrightarrow{c_{f,g}} f_\bullet g_\bullet) \tag{1}$$

(cf. A2.7, A2.7.1). Then the category $\mathcal{M}od(\mathfrak{F})$ can be described as follows. An object of $\mathcal{M}od(\mathfrak{F})$ is a function which assigns to each $T \in Ob\mathcal{E}$ an object $M(T)$ of the fiber $\mathcal{F}_T$ and to each morphism $T \xrightarrow{f} T'$ a morphism $M(T') \xrightarrow{\xi_f} f_\bullet(M(T))$ such that $c_{f,g} \circ \xi_{gf} = f_\bullet(\xi_g) \circ \xi_f$. Morphisms are defined in a natural way.

Let M be an object of $\mathcal{M}od(\mathfrak{F})$. For any object X of the category $\mathcal{E}$ and any sub-presheaf $R \hookrightarrow h_X$, we have a cone

$$\{M(X) \xrightarrow{\xi_f} f_\bullet(M(Y)) \mid Y \xrightarrow{f} X \text{ factors by } R \hookrightarrow h_X\}. \tag{2}$$

Denote by $\mathfrak{T}_M(X)$ the set of all subpresheaves $R \hookrightarrow h_X$ such that the cone (2) is terminal. The correspondence $\mathfrak{T}_M : X \longmapsto \mathfrak{T}_M(X)$ is a quasi-topology on $\mathcal{E}$.

Let $\mathfrak{T}$ be a quasi-topology on $\mathcal{E}$. We say that $M \in Ob\mathcal{M}od(\mathfrak{F})$ is a sheaf, or a *sheaf of modules* on $(\mathfrak{F}, \mathfrak{T})$, if the quasi-topology $\mathfrak{T}_M$ is finer than $\mathfrak{T}$; i.e. for any $X \in Ob\mathcal{E}$ and any $R \in \mathfrak{T}(X)$, the cone (2) is terminal. We denote by $\mathcal{M}od(\mathfrak{F}, \mathfrak{T})$ the full subcategory of $\mathcal{M}od(\mathfrak{F})$ formed by sheaves of modules.

1.3.2. Sheaves of modules on a fibered category. Notice that the cone 1.3.1(2) above is terminal iff for any $z \in Ob\mathfrak{F}_X$, the cone

$$\big\{Hom_{\mathfrak{F}_X}(z, M(X)) \xrightarrow{\mathfrak{F}_X(z,\xi_f)} Hom_{\mathfrak{F}_X}(z, f_\bullet(M(Y)))|\ Y \xrightarrow{f} X \text{ factors by } R \hookrightarrow h_X\big\} \qquad (3)$$

is terminal. But the cone (3) is naturally isomorphic to the cone

$$\big\{Hom_{\mathfrak{F}_X}(z, M(X)) \longrightarrow Hom_{\mathfrak{F}_Y}(f^*(z), M(Y))|\ Y \xrightarrow{f} X \text{ factors by } R \hookrightarrow h_X\big\} \qquad (4)$$

Here the map $Hom_{\mathfrak{F}_X}(z, M(X)) \longrightarrow Hom_{\mathfrak{F}_X}(f^*(z), M(Y))$ sends a $z \xrightarrow{\alpha} M(X)$ to the composition of $f^*(\alpha) : f^*(z) \longrightarrow f^*(M(X))$ and the morphism $f^*(M(X)) \longrightarrow M(Y)$.

This observation gives rise to the following

1.3.2.1. Definition. Let $\mathfrak{F}$ be a fibered category over $\mathcal{E}$, and let $\mathfrak{T}$ be a quasi-topology on $\mathcal{E}$. We call a presheaf of modules M on $\mathfrak{F}$ a *sheaf* if for any $X \in Ob\mathcal{E}$ and any $R \in \mathfrak{T}(X)$, the cone (4) is terminal for all $z \in Ob\mathfrak{F}_X$.

If $\mathfrak{F}$ is a bifibered category, then this definition is equivalent to that of 1.3.1. We denote by $\mathcal{M}od(\mathfrak{F}, \mathfrak{T})$ the category of sheaves of modules on $(\mathfrak{F}, \mathfrak{T})$

1.3.2.2. Quasi-coherent sheaves. We denote by $Qcoh(\mathfrak{F}, \mathfrak{T})$ the intersection of the categories $Qcoh(\mathfrak{F})$ and $\mathcal{M}od(\mathfrak{F}, \mathfrak{T})$ and call objects of this category *quasi-coherent sheaves on* $(\mathfrak{F}, \mathfrak{T})$.

1.3.3. Proposition. *Let $\mathfrak{F}$ be a fibered category over $\mathcal{E}$, and let $\mathfrak{T}$ be a quasi-topology on $\mathcal{E}$. The following conditions are equivalent:*

(a) The quasi-topology of $\mathfrak{F} - 1$-descent is finer than $\mathfrak{T}$.

(b) For any $X \in Ob\mathcal{E}$, the category $Qcoh(\mathfrak{F}/X)$ is a subcategory of the category $\mathcal{M}od(\mathfrak{F}/X, \mathfrak{T}/X)$ of sheaves of modules on $\mathcal{E}/X$ (with the induced quasi-topology $\mathfrak{T}/X$).

Proof. By definition, a subpresheaf R of h_X is of $\mathfrak{F}$-descent if the inverse image functor $Qcoh(\mathfrak{F}/h_X) \longrightarrow Qcoh(\mathfrak{F}/R)$ of the embedding $R \hookrightarrow h_X$ is a fully faithful functor. By 1.1.5.2(a), the category $Qcoh(\mathfrak{F}/h_X)$ is equivalent to $\mathfrak{F}_X^{op}$. Let M be a quasi-coherent module on $\mathfrak{F}$. And let z be any object of $\mathfrak{F}_X$. Since M is quasi-coherent, for any morphism $f : Y \longrightarrow X$ in $\mathcal{E}$, the object $M(Y)$ is isomorphic to $f^*(M(X)$. Thus the cone 1.3.2(4) is isomorphic to the cone

$$\{Hom_{\mathfrak{F}_X}(z, M(X)) \longrightarrow Hom_{\mathfrak{F}_X}(f^*(z), f^*(M(X)))|\ Y \xrightarrow{f} X \text{ factors by } R \hookrightarrow h_X\}. \qquad (5)$$

The cone (5) is terminal, because if $\mathfrak{T}$ is of $\mathfrak{F}$-descent, then for any $X \in Ob\mathcal{E}$ and any $z, x \in Ob\mathfrak{F}_X$, the presheaf of sets

$$Hom_{\mathfrak{F}_X}(z, y) : (Y, Y \xrightarrow{f} X) \longmapsto Hom_{\mathfrak{F}_Y}(f^*(z), f^*(x))$$

is a sheaf on $\mathcal{E}/X$ for the induced quasi-topology (see 1.2.2.1). This imply also the assertion (b). ■

1.3.3.1. Thus the quasi-topology (resp. topology) of $\mathfrak{F}-1$-descent is the finest among quasi-topologies (resp. topologies) $\mathfrak{T}$ on $\mathcal{E}$ such that for any $X \in Ob\mathcal{E}$, quasi-coherent modules on $\mathcal{E}/X$ are sheaves on $(\mathcal{E}/X, \mathfrak{T}/X)$. In particular, if $\mathfrak{T}$ is coarser than the quasi-topology of $\mathfrak{F}-1$-descent, then all quasi-coherent modules on $\mathfrak{F}$ are sheaves on the quasi-site $(\mathcal{E}, \mathfrak{T})$.

1.3.4. Fibered category of sheaves of modules over presheaves of sets. Let $\mathfrak{F}$ be a fibered category over $\mathcal{E}$ and $\mathfrak{T}$ a quasi-topolgy on $\mathcal{E}$. To any $X \in Ob\mathcal{E}^\wedge$, we assign the category $\mathcal{M}od(\mathfrak{F}/X, \mathfrak{T}/X)$ of sheaves of modules on $\mathcal{E}/X$ (with the induced quasi-topology $\mathfrak{T}/X$). This correspondence extends to a functor $(\mathcal{E}^\wedge)^{op} \longrightarrow Cat$, hence defines a fibered category, $\mathfrak{Mod}^+(\mathfrak{F}, \mathfrak{T})$.

1.3.4.1. Note. Let $\mathfrak{Mod}(\mathfrak{F}, \mathfrak{T})$ be the restriction of the fibered category $\mathfrak{Mod}^+(\mathfrak{F}, \mathfrak{T})$ to $\mathcal{E}$. Recall that a canonical extension, $\mathfrak{Mod}(\mathfrak{F}, \mathfrak{T})^+$, of the fibered category $\mathfrak{Mod}(\mathfrak{F}, \mathfrak{T})$ onto $\mathcal{E}^\wedge$ is defined by $\mathfrak{Mod}(\mathfrak{F}, \mathfrak{T})^+_X = Qcoh(\mathfrak{Mod}(\mathfrak{F}, \mathfrak{T})/X)$. It follows from definitions that this extension coincides with the fibered category $\mathfrak{Mod}^+(\mathfrak{F}, \mathfrak{T})$.

1.3.4.2. Lemma. *The quasi-topology $\mathfrak{T}$ is of effective $\mathfrak{Mod}(\mathfrak{F}, \mathfrak{T})$-descent.*

Proof. The argument is left to the reader. ■

1.3.5. Proposition. *Let $\mathfrak{F}$ be a fibered category over $\mathcal{E}$ and $\mathfrak{T}$ a quasi-topolgy on $\mathcal{E}$ which is coarser than the $\mathfrak{F}-1$ descent quasi-topology. Let*

$$\mathfrak{R} \overset{p_1}{\underset{p_2}{\rightrightarrows}} \mathfrak{U} \xrightarrow{f} X \qquad (1)$$

be a diagram in $\mathcal{E}^\wedge$ such that $\mathfrak{R}$ and $\mathfrak{U}$ are representable, $f \circ p_1 = f \circ p_2$, and the morphism $Cok(p_1, p_2) \longrightarrow X$ corresponding to f is bicovering. Then the inclusion functor

$$Qcoh(\mathfrak{F}/X) \xrightarrow{\mathfrak{q}_X^*} \mathcal{M}od(\mathfrak{F}/X, \mathfrak{T}/X)$$

has a right adjoint.

Proof. The condition that the canonical morphism $Cok(p_1, p_2) \longrightarrow X$ is bicovering means that the inverse image functor $Qcoh(\mathfrak{F}/X) \longrightarrow Qcoh(\mathfrak{F}/Cok(p_1, p_2))$ is a category equivalence. Thus, we can (and will) assume that the diagram of presheaves of sets (1) is exact.

Consider the quasi-commutative diagram

$$\begin{array}{ccccc} Qcoh(\mathfrak{F}/X) & \xrightarrow{f^*} & Qcoh(\mathfrak{F}/\mathfrak{U}) & \overset{p_1^*}{\underset{p_2^*}{\rightrightarrows}} & Qcoh(\mathfrak{F}/\mathfrak{R}) \\ \mathfrak{q}_X^* \downarrow & & \downarrow \mathfrak{q}_{\mathfrak{U}}^* & & \downarrow \mathfrak{q}_{\mathfrak{R}}^* \\ \mathcal{M}od(\mathfrak{F}/X, \mathfrak{T}/X) & \xrightarrow{f^\bullet} & \mathcal{M}od(\mathfrak{F}/\mathfrak{U}, \mathfrak{T}/\mathfrak{U}) & \overset{p_1^\bullet}{\underset{p_2^\bullet}{\rightrightarrows}} & \mathcal{M}od(\mathfrak{F}/\mathfrak{R}, \mathfrak{T}/\mathfrak{R}) \end{array} \qquad (2)$$

corresponding to the diagram (1). Since the functors $\mathfrak{U}$ and $\mathfrak{R}$ are representable, the functors $\mathfrak{q}_{\mathfrak{U}}^*$ and $\mathfrak{q}_{\mathfrak{R}}^*$ have right adjoints, resp. $\mathfrak{q}_{\mathfrak{U}*}$ and $\mathfrak{q}_{\mathfrak{R}*}$. Recall that the functor $\mathfrak{q}_{\mathfrak{U}*}$ (resp. $\mathfrak{q}_{\mathfrak{R}*}$ assigns to every sheaf M its value at $\mathfrak{U}$ (resp. at $\mathfrak{R}$) (cf. 1.1.3). This implies that

$$\mathfrak{q}_{\mathfrak{R}*} \circ p_i^{\bullet} = p_i^* \circ \mathfrak{q}_{\mathfrak{U}*}. \tag{3}$$

By 1.1.5.2, the subdiagram

$$Qcoh(\mathfrak{F}/X) \xrightarrow{f^*} Qcoh(\mathfrak{F}/\mathfrak{U}) \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} Qcoh(\mathfrak{F}/\mathfrak{R})$$

of (2) is exact. This means that the category $Qcoh(\mathfrak{F}/X)$ can be identified with a category whose objects are pairs (M, ϕ), where $M \in Ob\mathfrak{F}_{\mathfrak{U}}$ and ϕ an isomorphism $p_1^*(M) \xrightarrow{\sim} p_2^*(M)$. Morphisms $(M, \phi) \longrightarrow (M', \phi')$ are given by morphisms $M \xrightarrow{g} M'$ such that $p_2^*(g) \circ \phi = \phi' \circ p_1^*(g)$. The functor f^* maps (M, ϕ) to M and a morphism $(M, \phi) \xrightarrow{g} (M', \phi')$ to $M \xrightarrow{g} M'$.

Similarly, it follows from 1.3.4.2 and 1.1.5.2 that the subdiagram

$$\mathcal{M}od(\mathfrak{F}/X, \mathfrak{T}/X) \xrightarrow{f^{\bullet}} \mathcal{M}od(\mathfrak{F}/\mathfrak{U}, \mathfrak{T}/\mathfrak{U}) \underset{p_2^{\bullet}}{\overset{p_1^{\bullet}}{\rightrightarrows}} \mathcal{M}od(\mathfrak{F}/\mathfrak{R}, \mathfrak{T}/\mathfrak{R})$$

is exact, hence the category $\mathcal{M}od(\mathfrak{F}/X, \mathfrak{T}/X)$ of sheaves on X admits an analogous description: its objects are pairs (L, ψ), where $L \in Ob\mathcal{M}od(\mathfrak{F}/\mathfrak{U}, \mathfrak{T}/\mathfrak{U})$ and ψ an isomorphism $p_1^{\bullet}(L) \xrightarrow{\sim} p_2^{\bullet}(L)$, etc.. The functor $\mathfrak{q}_X^*$ maps an object (M, ϕ) of the category $Qcoh(\mathfrak{F}/X)$ to the object $(\mathfrak{q}_{\mathfrak{U}}^*(M), \mathfrak{q}_{\mathfrak{R}}^*(\phi))$. A right adjoint to $\mathfrak{q}_X^*$ is induced by the right adjoint $\mathfrak{q}_{\mathfrak{U}*}$ to the inclusion functor $\mathfrak{q}_{\mathfrak{U}}^*$.

In fact, let $(L, p_1^{\bullet}(L) \xrightarrow{\psi} p_2^{\bullet})$ be an object of $\mathcal{M}od(\mathfrak{F}/\mathfrak{U}, \mathfrak{T}/\mathfrak{U})$. Thanks to (3), we have isomorphisms:

$$p_1^*\mathfrak{q}_{\mathfrak{U}*}(L) \xrightarrow{\sim} \mathfrak{q}_{\mathfrak{R}*}p_1^{\bullet}(L) \xrightarrow{\mathfrak{q}_{\mathfrak{R}*}(\psi)} \mathfrak{q}_{\mathfrak{R}*}p_2^{\bullet}(L) \xrightarrow{\sim} p_2^*\mathfrak{q}_{\mathfrak{U}*}(L)$$

the composition of which, ψ', defines an object $(\mathfrak{q}_{\mathfrak{U}*}(L), \psi')$ of the category $Qcoh(\mathfrak{F}/X)$. The map $(L, \psi) \longmapsto (\mathfrak{q}_{\mathfrak{U}*}(L), \psi')$ extends to functor

$$\mathcal{M}od(\mathfrak{F}/X, \mathfrak{T}/X) \xrightarrow{\mathfrak{q}_{X*}} Qcoh(\mathfrak{F}/X).$$

It is left to the reader to check that the functor $\mathfrak{q}_{X*}$ is a right adjoint to $\mathfrak{q}_X^*$. ■

1.3.5.1. Remarks. (i) If the quasi-topology in 1.3.5 is of effective descent, it suffices to require that the canonical morphism $Cok(p_1, p_2) \xrightarrow{\hat{f}} X$ is a cover.

(ii) The argument of 1.3.5 is valid in a more general setting. Namely, one can replace representability of $\mathfrak{U}$ and $\mathfrak{R}$ by the existence of right adjoints to the inclusion functors

$Qcoh(\mathfrak{F}/\mathfrak{U}) \xrightarrow{\mathfrak{q}^*_{\mathfrak{U}}} \mathcal{M}od(\mathfrak{F}/\mathfrak{U}, \mathfrak{T}/\mathfrak{U})$ and $Qcoh(\mathfrak{F}/\mathfrak{R}) \xrightarrow{\mathfrak{q}^*_{\mathfrak{R}}} \mathcal{M}od(\mathfrak{F}/\mathfrak{R}, \mathfrak{T}/\mathfrak{R})$ which satisfy the condition (3). Notice that in this case a right adjoint, $\mathfrak{q}_{X*}$, to $\mathfrak{q}^*_X$ satisfies this condition too: $f^* \circ \mathfrak{q}_{X*} \simeq \mathfrak{q}_{\mathfrak{U}*} \circ f^{\bullet}$ (cf. the argument of 1.3.5).

1.3.5.2. Corollary. *Let $\mathfrak{F}$ be a fibered category over $\mathcal{E}$, and let*

$$\mathfrak{R} \underset{p_2}{\overset{p_1}{\rightrightarrows}} \mathfrak{U} \xrightarrow{f} X \qquad (1)$$

be an exact diagram in $\mathcal{E}^\wedge$ such that $\mathfrak{R}$ and $\mathfrak{U}$ are representable. Then the inclusion functor $\mathfrak{q}^*_X : Qcoh(\mathfrak{F}/X) \hookrightarrow \mathcal{M}od(\mathfrak{F}/X)$ has a right adjoint.

Proof. Let C be a category with the discrete topology, $\mathfrak{T}_d$. The corresponding coinduced topology $\mathfrak{T}^\wedge_d$ on the category $C^\wedge$ of presheaves of sets on C can be described in terms of covers as follows. A set of morphisms $\{U_i \longrightarrow X \mid i \in J\}$ is a cover iff the corresponding presheaf morphism $\coprod_{i\in J} U_i \longrightarrow X$ is surjective; in particular, any surjective presheaf morphism $U \longrightarrow X$ is a cover in the topology $\mathfrak{T}^\wedge_d$. This shows that the topology $\mathfrak{T}^\wedge_d$ is the canonical topology on $C^\wedge$.

Taking $C = \mathcal{E}$ endowed with the discrete topology, notice that for any $S \in Ob\mathcal{E}$ every quasi-coherent module on S is a sheaf; i.e. the condition of 1.3.5 holds for the discrete topology on $\mathcal{E}$. Moreover, by 1.2.8, the canonical topology on $\mathcal{E}^\wedge$ is of effective descent. Thus the assertion follows from 1.3.5 and 1.3.5.1(i). ∎

2. Modules and quasi-coherent modules over a ringed category.

A ringed category is a pair $(\mathcal{A}, \mathcal{O})$, where $\mathcal{A}$ is a category and $\mathcal{O}$ is a presheaf of rings on $\mathcal{A}$. For any arrow $T \xrightarrow{f} T'$, let $\mathcal{O}(T)-mod \xrightarrow{f_*} \mathcal{O}(T')-mod$ be the pull-back functor corresponding to the ring morphism $\mathcal{O}(T') \xrightarrow{\mathcal{O}(f)} \mathcal{O}(T)$. The map which assigs to each object T of $\mathcal{A}$ the category $\mathcal{O}(T) - mod^{op}$ opposite to the category of left $\mathcal{O}(T)$-modules and to each morphism $T \xrightarrow{f} T'$ the functor $(\mathcal{O}(T) - mod)^{op} \xrightarrow{f^{op}_*} (\mathcal{O}(T') - mod)^{op}$ is a functor $\mathcal{A} \longrightarrow Cat$ which defines a cofibered category, $\mathfrak{M}(\mathcal{A}, \mathcal{O})$, over $\mathcal{A}$ with the fiber $(\mathcal{O}(T)-mod)^{op}$ over $T \in Ob\mathcal{A}$. The category $\mathfrak{M}(\mathcal{A}, \mathcal{O})$ is fibered (hence bifibered), because for any $T \xrightarrow{f} T'$, the functor f_* has a left adjoint,

$$\mathcal{O}(T') - mod \xrightarrow{f^*} \mathcal{O}(T) - mod, \quad M \longmapsto \mathcal{O}(T) \otimes_{\mathcal{O}(T')} M,$$

or, equivalently, f^{op}_* has a right adjoint, $(f^*)^{op}$.

2.1. Modules over noncommutative affine schemes. Our standard example of a ringed category is the category $Aff_k = Alg^{op}_k$ of affine k-schemes endowed with the presheaf $\mathcal{O}$ which assigns to $\mathbf{Spec}R$ the k-algebra R. The corresponding bifibered category $\mathfrak{M}(\mathcal{A}, \mathcal{O})$ will be called the bifibered *category of modules over noncommutative affine k-schemes.*

2.2. Presheaves of modules. Let $\mathcal{O} - mod$ denote the category $\mathcal{M}od(\mathfrak{M}(\mathcal{A}, \mathcal{O}))$ of modules on the fibered category $\mathfrak{M}(\mathcal{A}, \mathcal{O})$. An object of $\mathcal{O}-mod$ is a function which assigns to each $T \in Ob\mathcal{A}$ an $\mathcal{O}$-module $M(T)$ and to each morphism $T \xrightarrow{f} T'$ an $\mathcal{O}(T')$-module morphism $M(T') \xrightarrow{\gamma_f} f_*(M(T))$ such that $\gamma_{gf} = g_*(\gamma_f) \circ \gamma_g$. Objects of the category $\mathcal{O} - mod$ are called *presheaves of $\mathcal{O}$-modules* on $\mathcal{A}$.

2.3. Quasi-coherent modules. We define the category $Qcoh(\mathcal{A}, \mathcal{O})$ of *quasi-coherent modules on* $(\mathcal{A}, \mathcal{O})$ as the category $Qcoh(\mathfrak{M}(\mathcal{A}, \mathcal{O}))$ of quasi-coherent modules on the fibered category $\mathfrak{M}(\mathcal{O})$. It follows from definitions that an object of the category $Qcoh(\mathcal{A}, \mathcal{O})$ is a presheaf M of $\mathcal{O}$-modules such that for any morphism $T \xrightarrow{f} T'$, the dual to γ_f morphism

$$f^*(M(T')) = \mathcal{O}(T) \otimes_{\mathcal{O}(T')} M(T', \xi') \xrightarrow{\gamma_f^\vee} M(T, \xi)$$

is an isomorphism.

2.4. Proposition. *Suppose the category $\mathcal{A}$ has a final object, $T_\bullet$. Then the category $Qcoh(\mathcal{A}, \mathcal{O})$ is equivalent to the category $\mathcal{O}(T_\bullet) - mod$ of left $\mathcal{O}(T_\bullet)$-modules.*

Proof. The assertion is a special case of 1.1.3. ∎

2.5. Example. Let $(\mathcal{A}, \mathcal{O})$ be as in 2.1; i.e. $\mathcal{A} = Aff_k$ and $\mathcal{O}(\mathbf{Spec}R) = R$ for any k-algebra R. Notice that $\mathbf{Spec}k$ is a final object of the category Aff_k. It follows from 2.4 (or 1.1.3) that the category $Qcoh(\mathcal{A}, \mathcal{O})$ is equivalent to the category $k - mod$ of left k-modules.

2.6. Modules on prespaces. Fix a ringed category $(\mathcal{A}, \mathcal{O})$. Consider the category $\mathcal{A}^\wedge$ of presheaves of sets on $\mathcal{A}$. We shall call presheaves of sets on $\mathcal{A}$ *prespaces*, or $\mathcal{A}$-*prespaces*, and regard representable presheaves as *affine prespaces*. For any $X \in Ob\mathcal{A}^\wedge$, we have the category $\mathcal{A}/X$ of objects over X and the canonical functor $\mathcal{A}/X \xrightarrow{p_X} \mathcal{A}$. The presheaf of rings $\mathcal{O}$ induces a presheaf of rings $\mathcal{O}_X$ on $\mathcal{A}/X$ so that p_X becomes a morphism of ringed categories. The bifibered category $\mathfrak{M}(\mathcal{A}/X, \mathcal{O}_X)$ is naturally isomorphic to the category obtained from $\mathfrak{M}(\mathcal{A}, \mathcal{O})$ via the base change by the functor p_X.

Thus we have the category $\mathcal{O}_X - mod$ of presheaves of $\mathcal{O}_X$-modules which we denote by $\mathcal{M}od_X$ and its full subcategory of quasi-coherent $\mathcal{O}_X$-modules which we denote by $Qcoh_X$ (instead of $Qcoh(\mathcal{A}/X, \mathcal{O}_X)$).

If $X \in Ob\mathcal{A}^\wedge$ is representable by an object T_X of $\mathcal{A}$, then the category $\mathcal{A}/X$ has a final object, and by 2.4, the category $Qcoh_X$ is equivalent to the category $\mathcal{O}(T_X) - mod$ of left $\mathcal{O}(T_X)$-modules.

2.6.1. Fibered category of modules over prespaces. Let $X, Y \in Ob\mathcal{A}^\wedge$. To any morphism $X \xrightarrow{f} Y$, there corresponds a functor

$$\mathcal{A}/X \xrightarrow{f^a} \mathcal{A}/Y, \quad (T, \xi) \longmapsto (T, f \circ \xi), \tag{1}$$

which lifts to a fibered category morphism

$$\mathfrak{M}(\mathcal{O}_X) \longrightarrow \mathfrak{M}(\mathcal{O}_Y), \ (M, (T, \xi)) \longmapsto (M, (T, f \circ \xi)).$$

The latter induces the 'pull-back' functor $\mathcal{M}od_Y \xrightarrow{f^\bullet} \mathcal{M}od_X$ which maps each module M on Y to the module $f^\bullet(M)$ on X defined by $f^\bullet(M)(R.\xi) = M(R, f\circ\xi)$. The map assigning to any presheaf morphism $X \xrightarrow{f} Y$ the functor $(f^\bullet)^{op} : \mathcal{M}od_Y^{op} \longrightarrow \mathcal{M}od_X^{op}$ extends to a pseudo-functor, hence defines a *fibered category of modules over prespaces*. This fibered category is, actually, bifibered because for any prespace morphism f, the functor $f^\bullet$ has a right adjoint, $f_\bullet$.

2.6.2. The fibered category of quasi-coherent modules. For any prespace morphism $X \xrightarrow{f} Y$, the corresponding functor $\mathfrak{M}(\mathcal{A}/Y, \mathcal{O}_Y) \longrightarrow \mathfrak{M}(\mathcal{A}/X, \mathcal{O}_X)$ (cf. 2.6.1) is cartesian, hence the functor $\mathcal{M}od_Y \xrightarrow{f^\bullet} \mathcal{M}od_X$ maps quasi-coherent modules to quasi-coherent modules, i.e. it induces a right exact functor $Qcoh_Y \xrightarrow{f^*} Qcoh_X$ which we call *inverse image functor* of f.

Suppose X, Y are objects of $\mathcal{A}$. Then a morphism f determines a ring morphism $\mathcal{O}(Y) \xrightarrow{\mathcal{O}(f)} \mathcal{O}(X)$, $Qcoh_X \simeq \mathcal{O}(X)-mod$, $Qcoh_Y \simeq \mathcal{O}(Y)-mod$ (cf. 2.4), and the functor f^* is equivalent to $\mathcal{O}(X)\otimes_{\mathcal{O}(Y)} : \mathcal{O}(Y) - mod \longrightarrow \mathcal{O}(X) - mod$.

In general, one might interpret f^* as the functor $M \longmapsto \mathcal{O}_X \otimes_{\mathcal{O}_Y} M$.

The map $X \longmapsto Qcoh_X^{op}$, $f \longmapsto (f^*)^{op}$ extends to a pseudo-functor from the category $\mathcal{A}^\wedge$ of $\mathcal{A}$-prespaces to Cat which defines the fibered category $\mathfrak{Qcoh}(\mathcal{A}, \mathcal{O})$ of quasi-coherent modules on $\mathcal{A}$-prespaces.

2.7. Quasi-topology and topology of descent.

2.7.1. Lemma. *Let $(\mathcal{A}, \mathcal{O})$ be a ringed category, and let $\mathfrak{T}$ be a topology (resp. quasi-topology) on $\mathcal{A}$.*

(a) A presheaf M of $\mathcal{O}$-modules is a sheaf on $(\mathfrak{M}(\mathcal{O}), \mathfrak{T})$ (cf. 1.3) iff M is a sheaf of abelian groups.

(b) The topology (resp. quasi-topology) $\mathfrak{T}$ is coarser than the topology (resp. quasi-topology) of $\mathfrak{M}(\mathcal{O}) - 1$-descent iff for any $X \in Ob\mathcal{A}$ and any $\mathcal{O}(X)$-module L, the presheaf on $\mathcal{A}/X$ which assigns to any object $(S, S \xrightarrow{f} X)$ of $\mathcal{A}/X$ the $\mathcal{O}(X)$-module $f_*f^*(L) = f_*(\mathcal{O}(S) \otimes_{\mathcal{O}(X)} L)$ is a sheaf of $\mathcal{O}(X)$-modules.

In particular, if $\mathfrak{T}$ is coarser than the quasi-topology of $\mathfrak{M}(\mathcal{O}) - 1$-descent, than $\mathcal{O}$ is a sheaf of rings on $(\mathcal{A}, \mathfrak{T})$.

Proof. (a) The assertion follows from definitions.

(b) By 1.3.3, the (quasi-)topology of $\mathfrak{M}(\mathcal{O}) - 1$-descent is the finest (quasi-)topology such that for any $X \in Ob\mathcal{A}$, quasi-coherent modules on $\mathfrak{M}(\mathcal{O}_X)$ are sheaves on $(\mathcal{A}, \mathfrak{T})$. Quasi-coherent modules on $\mathfrak{M}(\mathcal{O}_X)$ map each object $(S, S \xrightarrow{f} X)$ of $\mathcal{A}/X$ the $\mathcal{O}(X)$-module $f_*f^*(L) = f_*(\mathcal{O}(S) \otimes_{\mathcal{O}(X)} L)$ for some $\mathcal{O}(X)$-module L (cf. 2.4), hence the assertion. ■

2.7.2. Canonical topology and the descent topology on the category of commutative affine schemes. Let $\mathcal{A}$ be the category **CAff** of commutative affine schemes, $\mathcal{O}$ a presheaf of rings on $\mathcal{A}$ which assigns to any affine scheme $(X, \mathcal{O}_X)$ the ring $\Gamma\mathcal{O}_X$ of global sections of the structure sheaf $\mathcal{O}_X$.

2.7.2.1. Lemma. *The quasi-topology of $\mathfrak{M}(\mathcal{O}) - 1$-descent is a topology.*

Proof. Let

$$\{\mathbf{Spec}R_i \xrightarrow{\phi_i} \mathbf{Spec}R \mid i \in I\} \qquad (1)$$

be a family of scheme morphisms. It follows from 2.7.1(b) (and from the isomorphism $\mathbf{Spec}(R_i \otimes_R R_j) \simeq X_i \times_X X_j$, where $X_i = \mathbf{Spec}R_i$ and $X = \mathbf{Spec}R$) that (1) is a cover for the quasi-topology of $\mathfrak{M}(\mathcal{O}) - 1$-descent iff for any R-module M the diagram

$$M \longrightarrow \prod_{i\in I} R_i \otimes_R M \rightrightarrows \prod_{i,j\in I} R_i \otimes_R R_j \otimes_R M \qquad (2)$$

is exact. In particular, for any morphism $\mathbf{Spec}S \longrightarrow \mathbf{Spec}R$, the diagram

$$S \otimes_R M \longrightarrow \prod_{i\in I} R_i \otimes_R (S \otimes_R M) \rightrightarrows \prod_{i,j\in I} R_i \otimes_R R_j \otimes_R (S \otimes_R M) \qquad (3)$$

is exact. The latter means that the family of morphisms $\{\mathbf{Spec}S \times_X X_i \longrightarrow \mathbf{Spec}S \mid i \in I\}$ is a cover for the quasi-topology of $\mathfrak{M}(\mathcal{O}) - 1$-descent, hence the quasi-topology of $\mathfrak{M}(\mathcal{O}) - 1$-descent is a topology. ■

2.7.2.2. Proposition. *The canonical topology on* **CAff** *coincides with the topology of* $\mathfrak{M}(\mathcal{O}) - 1$*-descent.*

Proof. The family (1) is a cover for the canonical topology iff for any morphism $\mathbf{Spec}S \longrightarrow \mathbf{Spec}R$ the diagram of R-modules

$$S \longrightarrow \prod_{i\in I} R_i \otimes_R S \rightrightarrows \prod_{i,j\in I} R_i \otimes_R R_j \otimes_R S \qquad (4)$$

is exact. If (1) is a cover in the topology of $\mathfrak{M}(\mathcal{O}) - 1$-descent, then the diagram (4) is exact. This shows that the canonical topology on $\mathcal{A}$ is finer than the topology of $\mathfrak{M}(\mathcal{O}) - 1$-descent.

On the other hand, to any R-module M there corresponds an augmented R-algebra $S_M = R \oplus M$ with zero multiplication on M. One can easily check that the exactness of the diagram (4) for $S = S_M$ is equivalent to the exactness of the diagram (2). ■

2.7.3. Canonical topology and the descent topology on the category of noncommutative affine schemes. Proposition 2.7.2.2 extends to the noncommutative case. Namely, there is the following

2.7.3.1. Proposition. *Let $(\mathcal{A}, \mathcal{O})$ be the ringed category of (noncommutative) affine k-schemes; i.e. $\mathcal{A}$ is the category $\mathbf{Aff}_k = Alg_k^{op}$ of affine k-schemes and the presheaf $\mathcal{O}$ is defined by $\mathcal{O}(\mathbf{Spec}R) = R$ for any associative k-algebra R (cf. 2.1). Then the topology of $\mathfrak{M}(\mathcal{O}) - 1$-descent coincides with the canonical topology on $\mathcal{A} = \mathbf{Aff}_k$.*

Proof. A family

$$\{\mathbf{Spec}R_i \xrightarrow{\phi_i} \mathbf{Spec}R \mid i \in I\} \qquad (1)$$

of morphisms of $\mathbf{Aff}_k$ is a cover for the descent quasi-topology iff for any R-module M the diagram

$$M \longrightarrow \prod_{i\in I} R_i \otimes_R M \rightrightarrows \prod_{i,j\in I} R_i \star_R R_j \otimes_R M \qquad (2)$$

is exact.

The family (1) is a cover for the canonical topology iff for any morphism $\mathbf{Spec}S \longrightarrow \mathbf{Spec}R$, the diagram

$$S \longrightarrow \prod_{i\in I} R_i \star_R S \rightrightarrows \prod_{i,j\in I} R_i \star_R R_j \star_R S \tag{3}$$

is exact.

(a) Suppose M is an R-bimodule. Let S_M be an algebra which is isomorphic to $R \oplus M$ as R-bimodule with zero multiplication on M. Then we have a commutative diagram

$$\begin{array}{ccccc} M & \longrightarrow & \prod_{i\in I} R_i \otimes_R M & \rightrightarrows & \prod_{i,j\in I} R_i \star_R R_j \otimes_R M \\ \downarrow & & \downarrow & & \downarrow \\ S_M & \longrightarrow & \prod_{i\in I} R_i \star_R S_M & \rightrightarrows & \prod_{i,j\in I} R_i \star_R R_j \star_R S_M \end{array} \tag{4}$$

If (1) is a cover for the canonical topology, then lower row in (4) is an exact diagram. Vertical arrows in (4) define a morphism from the diagram (2) to the diagram

$$S_M \longrightarrow \prod_{i\in I} R_i \star_R S_M \rightrightarrows \prod_{i,j\in I} R_i \star_R R_j \star_R S_M \tag{5}$$

which is a retraction (in particular, vertical arrows are split monomorphisms). This implies that the diagram (2) is exact too.

(b) Let M be an arbitrary left R-module. Denote by S_M the algebra $S_{M'}$, where M' is the R-bimodule $M \otimes_k R$. The argument (a) applied to $S_{M'}$ proves that the diagram (2) is exact, i.e. (1) is a cover for the $\mathfrak{M}(\mathcal{O}) - 1$ descent topology.

(c) Let $\{\mathbf{Spec}R_i \xrightarrow{\phi_i} \mathbf{Spec}R \mid i \in I\}$ be a cover for the $\mathfrak{M}(\mathcal{O}) - 1$ descent topology. Then for any morphism $\mathbf{Spec}S \longrightarrow \mathbf{Spec}R$, the family $\{\mathbf{Spec}(R_i \star S) \longrightarrow \mathbf{Spec}S \mid i \in I\}$ is a cover for the $\mathfrak{M}(\mathcal{O}) - 1$ descent topology; i.e. for any S-module M, the diagram

$$M \longrightarrow \prod_{i\in I} R_i \star_R S \otimes_S M \rightrightarrows \prod_{i,j\in I} R_i \star_R R_j \star_R S \otimes_S M$$

is exact. Taking $M = S$, we obtain the exact diagram (3). Thus, $\{\mathbf{Spec}R_i \xrightarrow{\phi_i} \mathbf{Spec}R \mid i \in I\}$ is a cover for the canonical topology. ■

2.7.3.2. Corollary. *A subpresheaf $T \xrightarrow{\iota} X$ of a representable presheaf X on $\mathbf{Aff}_k$. is a refinement in the canonical topology iff the inverse image functor $Qcoh_X \xrightarrow{\iota^*} Qcoh_T$ is fully faithful.*

Proof. This follows from the fact that the canonical topology on $\mathbf{Aff}_k$ is the topology of $\mathfrak{M}(\mathcal{O}) - 1$-descent. ■

2.7.3.3. Corollary. *Every quasi-coherent module on $S \in Ob\mathbf{Aff}_k$ is a sheaf for the canonical topology on $\mathbf{Aff}_k/S$. In particular, for any subcanonical topology $\mathfrak{T}_S$ on $\mathbf{Aff}_k/S$, all quasi-coherent modules on S are sheaves.*

Proof. The fact follows from 2.7.3.1 and 1.3.3. ■

2.7.3.4. Note. The assertion 2.7.3.2 is proven in [Or] for the commutative case. The corollary 2.7.3.3 is also a result by D. Orlov [Or, Proposition 4.9].

2.8. Sheaves of modules. Let $\mathfrak{T}$ be a topology on the category $\mathcal{A}$. The category of sheaves of modules on the fibered category $\mathfrak{M}(\mathcal{A},\mathcal{O})$ coincides with the category $\mathcal{M}od(\mathcal{A},\mathfrak{T};\mathcal{O})$ of sheaves of left $\mathcal{O}$-modules on the site $(\mathcal{A},\mathfrak{T})$ in the conventional sense. The category $\mathcal{M}od(\mathcal{A},\mathfrak{T};\mathcal{O})$ is a Grothendieck category with small products, i.e. an abelian category satisfying the Grothendieck's conditions AB5 and AB3* which has a family of generators (cf. [SGA4], II).

For any prespace X, denote by $\mathcal{M}od_X^{\mathfrak{T}}$ the full subcategory of $\mathcal{M}od_X$ whose objects are *sheaves* of modules with respect to the topology $\mathfrak{T}_X$ induced by $\mathfrak{T}$ on $\mathcal{A}/X$.

Let $X \in Ob\mathcal{A}^\wedge$ and $\mathcal{A}/X \xrightarrow{p_X} \mathcal{A}$ the canonical functor. A presheaf M of $\mathcal{O}_X$-modules is a sheaf on $(X,\mathfrak{T}_X)$ (i.e. on the site $(\mathcal{A}/X,\mathfrak{T}_X)$) iff for any $S \in Ob\mathcal{A}$ and $S \xrightarrow{\xi} X$, the presheaf $\xi^\bullet(M)$ is a sheaf on S.

In particular, a quasi-coherent module M on X is a sheaf on $(X,\mathfrak{T}_X)$ iff for any $S \in Ob\mathcal{A}$ and $S \xrightarrow{\xi} X$, the inverse image $\xi^*(M)$ of M is a sheaf on S.

The functor $\mathcal{A}/X \xrightarrow{p_X} \mathcal{A}$ induces a functor $\mathcal{M}od(\mathcal{A},\mathfrak{T}) \xrightarrow{p_X^\bullet} \mathcal{M}od_X^{\mathfrak{T}}$. The functor $p_X^\bullet$ has a right adjoint, $p_{X\bullet}$, and a left adjoint, $p_{X!}$.

Similarly, for any prespace morphism $X \xrightarrow{f} Y$, the functor $\mathcal{A}/X \xrightarrow{p_f} \mathcal{A}/Y$ induces an 'inverse image' functor $\mathcal{M}od_Y^{\mathfrak{T}} \xrightarrow{p_f^\bullet} \mathcal{M}od_X^{\mathfrak{T}}$ which has a right adjoint, $p_{f\bullet}$, and a left adjoint, $p_{f!}$.

2.8.1. Coherator. Suppose every quasi-coherent module on $(\mathcal{A},\mathcal{O})$ is a sheaf on the site $(\mathcal{A},\mathfrak{T})$, i.e. $Qcoh(\mathcal{A},\mathcal{O})$ is a full subcategory of the category $\mathcal{M}od(\mathcal{A},\mathfrak{T};\mathcal{O})$ of the sheaves of $\mathcal{O}$-modules on $(\mathcal{A},\mathfrak{T})$. A right adjoint (if any) to the inclusion functor $Qcoh(\mathcal{A},\mathcal{O}) \xrightarrow{\psi_X^{\mathfrak{T}}} \mathcal{M}od(\mathcal{A},\mathfrak{T};\mathcal{O})$ is called a *coherator on the ringed site* $(\mathcal{A},\mathfrak{T};\mathcal{O})$.

Since $\mathcal{M}od(\mathcal{A},\mathfrak{T};\mathcal{O})$ is a Grothendieck category with small products, the existence of the coherator implies that the category $Qcoh(\mathcal{A},\mathcal{O})$ of quasi-coherent modules is a Grothendieck category with small products too (see [BD], 5.39).

If every quasi-coherent module on a prespace X is a sheaf on X, i.e. $Qcoh_X$ is a (full) subcategory of $\mathcal{M}od_X^{\mathfrak{T}}$, we have the notion of a *coherator on* $(X,\mathfrak{T}_X)$.

2.8.2. Proposition. *Suppose the topology $\mathfrak{T}$ on $\mathcal{A}$ is such that for any $S \in Ob\mathcal{A}$, quasi-coherent modules on S are sheaves on $(\mathcal{A}/S,\mathfrak{T}_S)$. Let $X \xrightarrow{f} Y$ be a prespace morphism. If there exists a coherator on $(Y,\mathfrak{T}_Y)$. Then the inverse image functor f^* : $Qcoh_Y \longrightarrow Qcoh_X$ has a right adjoint, f_* (a direct image functor of f). In particular, any morphism to an affine space has a direct image.*

Proof. In fact, the functor $\mathcal{M}od_Y \xrightarrow{f^\bullet} \mathcal{M}od_X$ has a right adjoint, $f_\bullet$. The pair of adjoint functors $f^\bullet, f_\bullet$ induces a pair of adjoint functors

$$\mathcal{M}od_Y^{\mathfrak{T}} \xrightarrow{f_{\mathfrak{T}}^\bullet} \mathcal{M}od_X^{\mathfrak{T}}, \quad \mathcal{M}od_X^{\mathfrak{T}} \xrightarrow{f_{\mathfrak{T}\bullet}} \mathcal{M}od_Y^{\mathfrak{T}}.$$

The composition of $f_{\mathfrak{T}}^{\bullet}$ with the inclusion functor $Qcoh_Y \xrightarrow{j_Y} \mathcal{M}od_Y^{\mathfrak{T}}$ equals to the composition of $Qcoh_Y \xrightarrow{f^*} Qcoh_X$ and the inclusion functor $Qcoh_X \xrightarrow{j_X} \mathcal{M}od_X^{\mathfrak{T}}$. Since the functor j_Y has a right adjoint, a coherator $\mathcal{M}od_Y^{\mathfrak{T}} \xrightarrow{\psi_Y} Qcoh_Y$, the functor $f_{\mathfrak{T}}^{\bullet} \circ j_Y = j_X \circ f^*$ is left adjoint to the functor $\psi_Y \circ f_{\mathfrak{T}\bullet}$. Denote by f_* the composition $\psi_Y \circ f_{\bullet} \circ j_X : Qcoh_X \longrightarrow Qcoh_Y$. Thus defined functor f_* is a right adjoint to f^*. In fact, for any $L \in ObQcoh_Y$ and $M \in ObQcoh_X$, we have functorial isomorphisms:

$$Qcoh_Y(L, \psi_Y \circ f_{\mathfrak{T}\bullet} \circ j_X(M)) \simeq \mathcal{M}od_X(f_{\mathfrak{T}}^{\bullet} \circ j_Y(L), j_X(M))$$

$$= \mathcal{M}od_X(j_X \circ f^*(L), j_X(M)) \simeq Qcoh_X(f^*(L), M),$$

hence the assertion. ■

2.8.3. A formula for the coherator. Assume that the inclusion functor j_X has a right adjoint, $\mathcal{M}od_X^{\mathfrak{T}} \xrightarrow{\psi_X} Qcoh_X$. Since ψ_X is left exact, it preserves kernels of pairs of morphisms. In particular, it maps the exact diagram (3) to the exact diagram

$$\psi_X(M) \longrightarrow \psi_X \pi_{\bullet}(M(\mathcal{U}, \pi)) \rightrightarrows \psi_X \nu_{\bullet}(M(\mathcal{R}, \nu)). \qquad (4)$$

The equality $j_U \circ \pi^* = \pi^{\bullet} \circ j_X$ (reflecting the fact that $\pi^{\bullet}$ maps quasi-coherent modules to quasi-coherent modules) implies that $\pi_* \circ \psi_{\mathcal{U}} \simeq \psi_X \circ \pi_{\bullet}$. Similarly, $\nu_* \circ \psi_{\mathcal{R}} \simeq \psi_X \circ \nu_{\bullet}$. Therefore the diagram (4) is isomorphic to the diagram

$$\psi_X(M) \longrightarrow \pi_* \psi_{\mathcal{U}}(M(\mathcal{U}, \pi)) \rightrightarrows \nu_* \psi_{\mathcal{R}}(M(\mathcal{R}, \nu)). \qquad (5)$$

Since the diagram (4) is exact, the diagram (5) is exact too; i.e. $\psi_X(M)$ is isomorphic to the kernel of the pair of arrows $\pi_* \psi_{\mathcal{U}} \pi^{\bullet}(M) \rightrightarrows \nu_* \psi_{\mathcal{R}} \nu^{\bullet}(M)$.

2.8.4. Proposition. *Suppose the topology $\mathfrak{T}$ on $\mathcal{A}$ is such that for any $S \in Ob\mathcal{A}$, quasi-coherent modules on S are sheaves on $(\mathcal{A}/S, \mathfrak{T}_S)$. Let X be a prespace such that there exists a diagram*

$$\mathfrak{R} \underset{p_2}{\overset{p_1}{\rightrightarrows}} \mathfrak{U} \xrightarrow{\pi} X \qquad (1)$$

where $\mathcal{R}$ and $\mathcal{U}$ are representable, $\pi \circ p_1 = \pi \circ p_2$, and the morphism $Cok(p_1, p_2) \longrightarrow X$ induced by π is bicovering. Then the inclusion functor $j_X : Qcoh_X \longrightarrow \mathcal{M}od_X^{\mathfrak{T}}$ has a right adjoint.

Proof. The assertion follows from 1.3.5 and 1.3.5.2. ■

2.8.5. Quasi-coherent modules on prespaces and quasi-coherent modules on the associated spaces. Fix a ringed site $(\mathcal{A}, \mathfrak{T}; \mathcal{O})$.

2.8.5.1. Lemma. *Let $(\mathcal{A}, \mathfrak{T})$ be a site, X a prespace on $\mathcal{A}$ and X^a the associated space (i.e. the associated sheaf of sets). The canonical morphism $X \longrightarrow X^a$ is a cover in the coinduced topology $\mathfrak{T}^{\wedge}$.*

Proof. Let $H_{\mathfrak{T}}$ denote the corresponding Heller's functor $\mathcal{A}^\wedge \longrightarrow \mathcal{A}^\wedge$ defined by $H_{\mathfrak{T}}(X)(T) = colim(\mathcal{A}^\wedge(S,X)|\ S \in \mathfrak{T}(T))$ for all $T \in Ob\mathcal{A}$. It follows from the definition of $H_{\mathfrak{T}}$ that that the canonical morphism $X \xrightarrow{\tau} H_{\mathfrak{T}}(X)$ is a cover in the topology $\mathfrak{T}^\wedge$. The associated sheaf, X^a, is isomorphic to $H^2_{\mathfrak{T}}(X)$ and the canonical morphism $X \longrightarrow X^a$ corresponds to the composition $\tau H_{\mathfrak{T}}(X) \circ \tau(X)$ of two covers, hence it is a cover itself. ■

The following fact is well known (see [SGA4], II, or [Or], 2.4).

2.8.5.2. Proposition. *For any prespace X, the canonical morphism $X \xrightarrow{\mathfrak{j}_X} X^a$ induces an equivalence of categories $\mathcal{M}od^{\mathfrak{T}}_{X^a} \xrightarrow{\mathfrak{j}^\bullet_X} \mathcal{M}od^{\mathfrak{T}}_X$.*

Proof. The assertion follows from the fact that $X \longrightarrow X^a$ is a cover in the coinduced topology $\mathfrak{T}^\wedge$. Details are left to the reader. ■

2.8.5.3. Corollary. *Suppose the topology $\mathfrak{T}$ on $\mathcal{A}$ is of 1-descent, i.e. quasi-coherent modules on S are sheaves on $(\mathcal{A}/S, \mathfrak{T}_S)$. Let X be a prespace and $X \xrightarrow{\mathfrak{j}_X} X^a$ the canonical morphism. Then the inverse image functor $Qcoh_{X^a} \xrightarrow{\mathfrak{j}^*_X} Qcoh_X$ is fully faithful.*

Proof. In the commutative diagram

$$\begin{array}{ccc}
Qcoh_{X^a} & \xrightarrow{\mathfrak{j}^*_X} & Qcoh_X \\
\downarrow & & \downarrow \\
\mathcal{M}od^{\mathfrak{T}}_{X^a} & \xrightarrow{\mathfrak{j}^\bullet_X} & \mathcal{M}od^{\mathfrak{T}}_X
\end{array}$$

the vertical arrows are full embeddings and the lower horizontal arrow, $\mathfrak{j}^\bullet_X$, is a category equivalence, hence $\mathfrak{j}^*_X$ is fully faithful. ■

2.9. A description of the category of quasi-coherent modules. Let $(\mathcal{A}, \mathcal{O})$ be a ringed category. Let X be a presheaf of sets on $\mathcal{A}$ such that there exists a diagram

$$\mathfrak{R} \underset{p_2}{\overset{p_1}{\rightrightarrows}} \mathfrak{U} \xrightarrow{\pi} X \qquad (1)$$

where $\pi \circ p_1 = \pi \circ p_2$, and the morphism $Cok(p_1, p_2) \longrightarrow X$ induced by π is bicovering. Then the category $Qcoh_X$ of quasi-coherent modules on X is equivalent to the category $Ker(p_1^*, p_2^*)$ whose objects are pairs (M, ϕ), where M is an object of $Qcoh_{\mathfrak{U}}$ and ϕ is an isomorphism $p_1^*(M) \xrightarrow{\sim} p_2^*(M)$. Morphisms from (M, ϕ) to (M', ϕ') are given by arrows $M \xrightarrow{g} M'$ which make the diagram

$$\begin{array}{ccc}
p_1^*(M) & \xrightarrow{\phi} & p_2^*(M) \\
{\scriptstyle p_1^*(g)}\downarrow & & \downarrow{\scriptstyle p_2^*(g)} \\
p_1^*(M') & \xrightarrow{\phi'} & p_2^*(M')
\end{array}$$

commute. If $\mathfrak{U}$ and $\mathfrak{R}$ are representable by objects resp. $\mathcal{U}$ and $\mathcal{R}$, then $Qcoh_{\mathfrak{U}} = \mathcal{O}(\mathcal{U}) - mod$, $Qcoh_{\mathfrak{R}} = \mathcal{O}(\mathcal{R}) - mod$, and inverse image functor p_i^*, $i = 1, 2$, is the tensoring $L \longmapsto \mathcal{O}(\mathcal{R}) \otimes_{\mathcal{O}(\mathcal{U})} L$ corresponding to the ring morphism $\mathcal{O}(\mathcal{U}) \xrightarrow{\mathcal{O}(p_i)} \mathcal{O}(\mathcal{R})$.

This follows from the argument of 1.3.5. In particular, we have the following

2.9.1. Proposition. *Let $(\mathcal{A}, \mathcal{O})$ be a ringed category and $\mathfrak{T}$ a quasi-topology on $\mathcal{A}$ which is coarser than the quasi-topology of effective descent. Let $\mathfrak{R} \overset{p_1}{\underset{p_2}{\rightrightarrows}} \mathfrak{U} \xrightarrow{\pi} X$ be an exact diagram of sheaves of sets on $\mathcal{A}$. Then the category $Qcoh_X$ of quasi-coherent modules on X is equivalent to the category $Ker(p_1^*, p_2^*)$.*

If $\mathfrak{R}$, $\mathfrak{U}$ are representable by objects resp. $\mathcal{R}$ and $\mathcal{U}$ of the category $\mathcal{A}$, then the category $Ker(p_1^*, p_2^*)$ is described by a linear algebra data: its objects are pairs (L, ϕ), where L is an $\mathcal{O}(\mathcal{U})$-module and ϕ is an $\mathcal{O}(\mathcal{R})$-module isomorphism $p_1^*(L) \xrightarrow{\sim} p_2^*(L)$.

2.9.2. Remark. The category $Ker(p_1^*, p_2^*)$ in 2.9.1 is equivalent to the category of quasi-coherent modules on the cokernel of the pair $\mathfrak{R} \overset{p_1}{\underset{p_2}{\rightrightarrows}} \mathfrak{U}$ If $Cok(p_1, p_2) \xrightarrow{g} X$ is a presheaf morphism such that the corresponding map of associated sheaves is an isomorphism (and $\mathfrak{T}$ is coarser than the quasi-topology of 2-descent), than g^* is an equivalence of $Qcoh_X$ and $Qcoh_{Cok(p_1,p_2)}$. This allows to find the category of quasi-coherent modules on a space without finding the space itself. We illustrate this observation in the following examples.

2.9.3. Application: quasi-coherent modules on Grassmannians. Let R be a k-algebra, M and V projective left R-modules of finite type. Then the functors $G_{M,V}$ and $\mathfrak{R}_{M,V}$ in the exact diagram

$$\mathfrak{R}_{M,V} \overset{p_1}{\underset{p_2}{\rightrightarrows}} G_{M,V} \xrightarrow{\ \pi\ } Gr_{M,V} \tag{1}$$

defining the functor $Gr_{M,V}$, are corepresentable by k-algebras over R resp. $R \longrightarrow \mathcal{G}_{M,V}$ and $R \longrightarrow \mathcal{R}_{M,V}$ (cf. [KR3, 10.1.3]), which implies that the category $Qcoh_{Gr_{M,V}}$ of quasi-coherent modules on $Gr_{M,V}$ is defined by a linear algebra data: it is equivalent to the category $Ker(p_1^*, p_2^*)$ whose objects are pairs (L, ϕ), where L is a $\mathcal{G}_{M,V}$-module and ϕ is an $\mathcal{R}_{M,V}$-module isomorphism $p_1^*(L) \xrightarrow{\sim} p_2^*(L)$. The inverse image functor p_i^*, $i = 1, 2$, is isomorphic to the tensoring $L \longmapsto \mathcal{R}_{M,V} \otimes_{\mathcal{G}_{M,V}} L$ corresponding to an algebra morphism $\mathcal{G}_{M,V} \xrightarrow{\tilde{p}_i} \mathcal{R}_{M,V}$ representing p_i.

Let $\mathfrak{T}$ be a quasi-topology on the category $\mathbf{Aff}_k/\mathbf{Spec}R$ of affine k-schemes over $\mathbf{Spec}R$. Let $Gr^{\mathfrak{T}}_{M,V}$ be the $\mathfrak{T}$-Grassmannian corresponding to $Gr_{M,V}$, i.e. a sheaf of sets (a 'space') associated to $Gr_{M,V}$. If $\mathfrak{T}$ is coarser than the quasi-topology of effective descent, then the category $Qcoh_{Gr_{M,V}}$ of quasi-coherent modules on $Gr_{M,V}$ is naturally equivalent to the category $Qcoh_{Gr^{\mathfrak{T}}_{M,V}}$ of quasi-coherent modules on $Gr^{\mathfrak{T}}_{M,V}$.

2.9.4. Noncommutative projective space. Let M be the free R-module of the rank n+1, V the free R-module of the rank 1. In this case, we denote the functor $Gr_{M,V}$ by

$\mathcal{P}^n_R$. If a quasi-topology $\mathfrak{T}$ on the category $\mathbf{Aff}_k/\mathbf{Spec}R$ of affine k-schemes over $\mathbf{Spec}R$ is coarser than the quasi-topology of 2-descent, than the category $Qcoh_{\mathcal{P}^n_R}$ of quasi-coherent modules on $\mathcal{P}^n_R$ is equivalent to the category of quasi-coherent modules on the associated projective space $\mathcal{P}^{n^{\mathfrak{T}}}_R$.

2.9.5. The commutative case. Let $R = k$. Denote by $Gr^c_{M,V}$ the restriction of the presheaf $Gr_{M,V}$ to the subcategory $\mathbf{CAff}_k$ of commutative affine k-schemes (i.e. the opposite category to the category $CAlg_k$ of commutative k-algebras). We assume that the rank of the k-module M at each point of $Spec(k)$ is greater than, or equal to the rank of the k-module V at this point; otherwise the functor $Gr^c_{M,V}$ maps every commutative k-algebra to the empty set. The exact diagram (1) induces an exact diagram

$$\mathfrak{R}^c_{M,V} \overset{\mathfrak{p}_1}{\underset{\mathfrak{p}_2}{\rightrightarrows}} G^c_{M,V} \xrightarrow{\ \pi\ } Gr^c_{M,V}, \tag{2}$$

where $\mathfrak{R}^c_{M,V}$ and $G^c_{M,V}$ denote the restrictions of presheaves resp. $\mathfrak{R}_{M,V}$ and $G_{M,V}$ to the subcategory $\mathbf{CAff}_k$. If $\mathfrak{R}_{M,V}$ and $G_{M,V}$ are representable by the algebras resp. $\mathcal{R}_{M,V}$ and $\mathcal{G}_{M,V}$, then the presheaves $\mathfrak{R}^c_{M,V}$ and $G^c_{M,V}$ are representable by *abelianizations* (quotients by the commutant) of these algebras, $\mathcal{R}^c_{M,V}$ and $\mathcal{G}^c_{M,V}$. By 2.9.2, the category of quasi-coherent modules on $Gr^c_{M,V}$ is isomorphic to the kernel $Ker(\mathfrak{p}_1^*, \mathfrak{p}_2^*)$ of the pair of the inverse image functors

$$\mathcal{G}^c_{M,V} - mod \overset{\mathfrak{p}_1^*}{\underset{\mathfrak{p}_2^*}{\rightrightarrows}} \mathcal{R}^c_{M,V} - mod. \tag{3}$$

Now we regard (2) as an exact sequence of presheaves of sets on the ringed site $(\mathbf{CAff}_k, \mathcal{O})$ of commutative k-schemes for a flat (**fpqc**, or **fppf**) topology. The presheaves $\mathfrak{R}^c_{M,V}$ and $G^c_{M,V}$ are sheaves because the flat topology is subcanonical (all representable presheaves are sheaves). The presheaf $Gr^c_{M,V}$ is not a sheaf, but the sheaf associated to $Gr^c_{M,V}$ is isomorphic to the Grassmannian $Grassm_{M,V}$. Since the **fpqc** topology (hence the **fppf** topology) is coarser than the 2-descent topology, the category of quasi-coherent modules on the Grassmannian $Grassm_{M,V}$ is equivalent to the category of quasi-coherent modules on the presheaf $Gr^c_{M,V}$, i.e. to the kernel of the pair of functors (3).

II. Noncommutative stacks.

A stack is a *sheaf of categories on a (pre)topology.* A classical example is the category of (commutative) affine schemes with Zariski topology with the category of quasi-coherent sheaves assigned to each affine scheme. The sheaf property means an effective descent property for quasi-coherent sheaves.

In noncommutative geometry, (pre)topologies are replaced by Q-categories, and stacks are sheaves of categories (in an appropriate sense) on a Q-category. A standard example is the Q-category of rings with faithfully flat covers (see [KR3, 9.3]) with the category of left R-modules assigned to each ring R.

3. Preliminaries: representable fibered categories and representable cartesian functors. Finiteness conditions.

3.1. Categories over $\mathcal{E}$ representable by a presheaf of sets. Fix a category $\mathcal{E}$. For any presheaf $\mathcal{E}^{op} \xrightarrow{S}$ **Sets**, we have a category $\mathcal{E}/S$ over $\mathcal{E}$. For any $X \in Ob\mathcal{E}$, the fiber $(\mathcal{E}/S)_X$ is a discrete category formed by all objects $(X, X \to S)$. In particular, it is empty if $S(X) = \emptyset$. Any morphism $X \xrightarrow{f} Y$ of the category $\mathcal{E}$ induces a functor

$$(\mathcal{E}/S)_Y \xrightarrow{f^*} (\mathcal{E}/S)_X, \quad (Y, Y \xrightarrow{\xi} S) \longmapsto (X, \xi \circ f).$$

The map $f \longmapsto f^*$ is a functor $\mathcal{E}^{op} \longrightarrow Cat$, and $\mathcal{E}/S$ is a fibered category corresponding to this functor. Note that every morphism of the category $\mathcal{E}/S$ is cartesian.

3.1.1. Proposition. *The map*

$$S \longmapsto \mathcal{E}/S, \ S \in Ob\mathcal{E}^{\wedge}, \quad (S \xrightarrow{g} T) \longmapsto \Big(\mathcal{E}/S \longrightarrow \mathcal{E}/T, \ (X, \xi) \longmapsto (X, g \circ \xi), \Big)$$

is a fully faithful functor, $\mathfrak{h}^{\mathcal{E}}$, from $\mathcal{E}^{\wedge}$ to the category of categories over $\mathcal{E}$. The functor $\mathfrak{h}^{\mathcal{E}}$ preserves finite limits.

Proof is left to the reader. ∎

3.2. Definition. Let $\mathcal{E}'$ be a full subcategory of $\mathcal{E}^{\wedge}$. A category $\mathcal{F}$ over $\mathcal{E}$ is called $\mathcal{E}'$*-representable* if it is $\mathcal{E}$-equivalent to the category $\mathcal{E}/S$ for some object S of $\mathcal{E}'$.

In particular, any $\mathcal{E}'$-representable category over $\mathcal{E}$ is fibered.

3.2.1. Standard choices of $\mathcal{E}'$. For an arbitrary category $\mathcal{E}$, the standard choices of $\mathcal{E}'$ are the category $\mathcal{E}$ itself, the category $\mathcal{E}^{\wedge}$ of presheaves on $\mathcal{E}$, and the subcategory of left exact functors $\mathcal{E}^{op} \longrightarrow$ **Sets**.

3.2.2. Standard choices in the case of a Q-category. Let $\mathbb{A} = (\bar{A} \overset{u}{\rightleftarrows} A)$ be a Q-category, and let $\mathcal{E} = A^{op}$. Main examples of $\mathcal{E}'$ in this case are:

(a) The subcategory of $\mathbb{A}$-*spaces*, i.e. sheaves of sets on $\mathbb{A}$.
(b) The subcategory of locally affine $\mathbb{A}$-spaces.
(c) The subcategory of $\mathbb{A}$-monopresheaves on $\mathbb{A}$.
(d) The category A^{op}. We shall call A^{op}-representable fibered categories *affine.*

3.3. Representable cartesian functors. Fix a full subcategory $\mathcal{E}'$ of $\mathcal{E}^{\wedge}$. A cartesian functor $\mathcal{F} \xrightarrow{\Phi} \mathcal{G}$ between categories over $\mathcal{E}$ is called $\mathcal{E}'$*-representable* if for any $T \in Ob\mathcal{E}'$ and any cartesian functor $\mathcal{E}/T \longrightarrow \mathcal{G}$, the fibered product $\mathcal{E}/T \times_{\mathcal{G}} \mathcal{F}$ is $\mathcal{E}'$-representable.

3.3.1. Affine cartesian functors. We call a cartesian functor between categories over $\mathcal{E}$ *affine* if it is $\mathcal{E}$-representable.

3.3.2. Proposition. *(a) Any $\mathcal{E}$-equivalence is $\mathcal{E}'$-representable for any full subcategory $\mathcal{E}'$ of $\mathcal{E}^{\wedge}$. In particular it is affine.*

(b) Suppose $\mathcal{E}'$ has finite products taken in $\mathcal{E}^{\wedge}$. Then for any $\mathcal{E}'$-representable category $\mathcal{F}$ over $\mathcal{E}$, the structure morphism of $\mathcal{F}$ is $\mathcal{E}'$-representable.

(c) Suppose $\mathcal{E}'$ has a final object. Then a structure morphism of a category $\mathcal{F}$ over $\mathcal{E}$ is $\mathcal{E}'$-representable iff $\mathcal{F}$ is $\mathcal{E}'$-representable.

(c') Suppose $\mathcal{E}$ has a final object. Then a structure morphism of a category $\mathcal{F}$ over $\mathcal{E}$ is affine iff $\mathcal{F}$ is affine.

Proof. (a) The assertion follows from definitions.

(b) Let $X \in Ob\mathcal{E}'$. The only $\mathcal{E}$-functor $\mathcal{E}/X \longrightarrow \mathcal{E}$ is the canonical functor,

$$(V,\xi) \longmapsto V,\ \ ((V,\xi) \xrightarrow{f} (V',\xi')) \longmapsto (V \xrightarrow{f} V').$$

If $\mathcal{F} \simeq \mathcal{E}/S$ for some $S \in Ob\mathcal{E}'$, then $\mathcal{F} \times_{\mathcal{E}} \mathcal{E}/X \simeq \mathcal{E}/S \times_{\mathcal{E}} \mathcal{E}/X \simeq \mathcal{E}/(X \times S)$, hence the structure morphism $\mathcal{F} \longrightarrow \mathcal{E}$ is $\mathcal{E}'$-representable.

(c) Suppose the structure morphism $F : \mathcal{F} \longrightarrow \mathcal{E}$ is $\mathcal{E}'$-representable. This means that $\mathcal{F} \times_{\mathcal{E}} \mathcal{E}/X$ is $\mathcal{E}'$-representable for any $X \in Ob\mathcal{E}'$. Taking as X a final object of the category $\mathcal{E}'$, we obtain that $\mathcal{F} \times_{\mathcal{E}} \mathcal{E} \simeq \mathcal{F}$ is representable.

(c') This assertion is a special case of (c). ■

3.3.3. Proposition. *Let $\mathcal{F}$ be a category over $\mathcal{E}$. Suppose $\mathcal{E}'$ is closed under finite limits taken in $\mathcal{E}^\wedge$. Then the following conditions are equivalent:*

(i) Any cartesian morphism $\mathcal{E}/S \longrightarrow \mathcal{F}$, $S \in Ob\mathcal{E}'$, is $\mathcal{E}'$-representable.

(ii) The diagonal morphism $\mathcal{F} \xrightarrow{\Delta_{\mathcal{F}}} \mathcal{F} \times_{\mathcal{E}} \mathcal{F}$ is $\mathcal{E}'$-representable.

Proof. (i) $\Rightarrow$ (ii). Let $X \in Ob\mathcal{E}'$, and let $f, g : \mathcal{E}/X \longrightarrow \mathcal{F}$ be arbitrary cartesian morphisms over $\mathcal{E}$. Consider the following canonical commutative diagram

$$\begin{array}{ccccc}
\mathcal{F} \times_{\mathcal{F}\times_{\mathcal{E}}\mathcal{F}} \mathcal{E}/X & \longrightarrow & \mathcal{E}/X \times_{\mathcal{F}} \mathcal{E}/X & \longrightarrow & \mathcal{F} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{E}/X & \xrightarrow{\Delta_{\mathcal{E}/X}} & \mathcal{E}/X \times_{\mathcal{E}} \mathcal{E}/X & \xrightarrow{(f,g)} & \mathcal{F} \times_{\mathcal{E}} \mathcal{F}
\end{array} \qquad (1)$$

formed by two universal squares. By (i), the fiber product $\mathcal{E}/X \times_{\mathcal{F}} \mathcal{E}/X$ is $\mathcal{E}'$-representable. Since $\mathcal{E}'$ has products, $X \times X$ exists in $\mathcal{E}'$, and $\mathcal{E}/X \times_{\mathcal{E}} \mathcal{E}/X \simeq \mathcal{E}/(X \times X)$, i.e. it is representable by $X \times X$. Since $\mathcal{E}'$ has fibered products and the embedding

$$\mathcal{E} \longrightarrow Cat/\mathcal{E},\ \ X \longmapsto \mathcal{E}/X,$$

preserves limits (cf. 2.2), the category $\mathcal{F} \times_{\mathcal{F}\times_{\mathcal{E}}\mathcal{F}} \mathcal{E}/X$ over $\mathcal{E}$ is $\mathcal{E}'$-representable.

(ii) $\Rightarrow$ (i). For any two morphisms, $\mathcal{E}/X \xrightarrow{f} \mathcal{F} \xleftarrow{g} \mathcal{E}/Y$, of $Cart_{\mathcal{E}}$, the square

$$\begin{array}{ccc}
\mathcal{E}/X \times_{\mathcal{F}} \mathcal{E}/Y & \longrightarrow & \mathcal{E}/X \times_{\mathcal{E}} \mathcal{E}/Y \\
\downarrow & & \downarrow{\scriptstyle f \times g} \\
\mathcal{F} & \xrightarrow{\Delta_{\mathcal{F}}} & \mathcal{F} \times_{\mathcal{E}} \mathcal{F}
\end{array}$$

is cartesian, hence the assertion. ■

3.4. Definition. Let $(\mathcal{P})$ be a property of morphisms of the category $\mathcal{E}'$ stable under base change. We say that an $\mathcal{E}'$-representable morphism $\mathcal{F} \xrightarrow{f} \mathcal{G}$ of categories over $\mathcal{E}$

satisfies $(\mathcal{P})$ if for any $X \in Ob\mathcal{E}'$ and any morphism $\mathcal{E}/X \longrightarrow \mathcal{G}$, the (morphism of $\mathcal{E}'$ representing the) projection $\mathcal{E}/X \times_{\mathcal{G}} \mathcal{F} \longrightarrow \mathcal{E}/X$ satisfies $(\mathcal{P})$.

3.4.1. Proposition. *Let $\mathcal{P}$ be the property of morphisms of the category $\mathcal{E}' \subseteq \mathcal{E}^{\wedge}$ stable under base change.*

(a) The class of $\mathcal{E}'$-representable morphisms of $Cat/\mathcal{E}$ having the property $\mathcal{P}$ is stable under base change.

(b) If the property $\mathcal{P}$ of morphisms of $\mathcal{E}'$ is stable under composition, then same holds for the property $\mathcal{P}$ of $\mathcal{E}'$-representable morphisms of $Cat/\mathcal{E}$.

Proof is left to the reader. ■

3.4.2. Examples. Let $\mathbb{A} = (\bar{A} \overset{u}{\rightleftarrows} A)$ be a Q-category. Take $\mathcal{E} = A^{op}$ and $\mathcal{E}' = \mathcal{E}^{\wedge} = A^{\vee}$. The following classes of morphisms of $A^{\vee}$ are stable under composition and base change (cf. 5.4 and 5.6):

3.4.2.1. The class $\mathcal{P}^{\mathbb{A}}_{fsm}$ of formally $\mathbb{A}$-smooth morphisms.

3.4.2.2. The class $\mathcal{P}^{\mathbb{A}}_{funr}$ of formally $\mathbb{A}$-unramified morphisms.

3.4.2.3. The class $\mathcal{P}^{\mathbb{A}}_{\text{fét}}$ of formally $\mathbb{A}$-étale morphisms.

By 3.4.1, the class of formally $\mathbb{A}$-smooth (resp. formally $\mathbb{A}$-étale, resp. formally $\mathbb{A}$-unramified) $A^{\vee}$-representable morphisms of Cat/A^{op} is stable under composition and base change.

3.5. Finitely presentable and locally finitely presentable cartesian functors. Let $\mathcal{E}$ be a category and $\mathcal{E}'$ a full subcategory of $\mathcal{E}^{\wedge}$. We assume that $\mathcal{E}'$ contains all representable functors and is closed under limits of filtered projective systems.

Let $\mathcal{F}$ and $\mathcal{G}$ be fibered categories over $\mathcal{E}$. A cartesian functor $\mathcal{F} \xrightarrow{\Phi} \mathcal{G}$ over $\mathcal{E}$ will be called *$\mathcal{E}'$-finitely presentable*, if for any filtered projective system $D \xrightarrow{\mathfrak{D}} \mathcal{E}'$, the canonical square

$$\begin{array}{ccc} \operatorname{colim} \mathcal{F}^{+}_{\mathfrak{D}_\mu} & \longrightarrow & \mathcal{F}^{+}_{\lim \mathfrak{D}} \\ {\scriptstyle \Phi'} \downarrow & & \downarrow {\scriptstyle \Phi_{\lim \mathfrak{D}}} \\ \operatorname{colim} \mathcal{G}^{+}_{\mathfrak{D}_\mu} & \longrightarrow & \mathcal{G}^{+}_{\lim \mathfrak{D}} \end{array}$$

is 2-cartesian. Here $\mathcal{F}^{+}$ and $\mathcal{G}^{+}$ are the canonical extensions of the fibered categories resp. $\mathcal{F}$ and $\mathcal{G}$ onto $\mathcal{E}^{\wedge}$ (cf. 1.1.5.1); and $\lim \mathfrak{D}$ is taken in $\mathcal{E}^{\wedge}$.

We call an $\mathcal{E}'$-finitely presentable cartesian functor *locally finitely presentable* if $\mathcal{E}' = \mathcal{E}$ and *finitely presentable* if $\mathcal{E}' = \mathcal{E}^{\wedge}$.

4. Stacks and prestacks over a Q-category.

Fix a Q-category $\mathbb{A} = (\bar{A} \overset{u}{\rightleftarrows} A)$. Let $\mathcal{F}$ be a category over A^{op}. Any object $\bar{y}$ of $\bar{A}$ defines two categories over A, the category $\bar{y}\backslash u^*$ and the category $u_*(\bar{y})\backslash A$, and the A-functor

$$\theta_{\bar{y}} : \bar{y}\backslash u^* \longrightarrow u_*(\bar{y})\backslash A, \quad (z, \bar{y} \xrightarrow{g} u^*(z)) \longmapsto (z, \eta_u^{-1}(z) \circ u_*(g)). \qquad (1)$$

Here η_u denotes an adjunction isomorphism $Id_A \longrightarrow u_*u^*$. Note that all morphisms of the A-categories $u_*(\bar{y})\backslash A$ and $\bar{y}\backslash u^*$ are cartesian and cocartesian. Hence any functor $\bar{y}\backslash u^* \longrightarrow u_*(\bar{y})\backslash A$, in particular $\theta_{\bar{y}}$, is (co)cartesian. Therefore $\theta_{\bar{y}}$ induces a functor

$$Cart_{A^{op}}((u_*(\bar{y})\backslash A)^{op}, \mathcal{F}) \longrightarrow Cart_{A^{op}}((\bar{y}\backslash u^*)^{op}, \mathcal{F}), \quad \Phi \longmapsto \Phi \circ \theta_{\bar{y}}. \tag{2}$$

Note that the A-categories $u_*(\bar{y})\backslash A$ and $\bar{y}\backslash u^*$ are cofibered over A, hence their opposite categories are fibered over A^{op}.

4.1. Definition. We call a fibered category $\mathcal{F}$ over A^{op} an $\mathbb{A}$-*stack* (resp. an $\mathbb{A}$-*prestack*) if for any $\bar{y} \in \bar{A}$, the functor (2) is an equivalence (resp. fully faithful).

4.1.1. The 2-categories of $\mathbb{A}$-stacks and $\mathbb{A}$-prestacks. We denote by $Stacks_{\mathbb{A}}$ (resp. $Prest_{\mathbb{A}}$) the full 2-subcategory of the 2-category $Fib_{/A^{op}}$ (of fibered categories over A^{op}, see A2.6.1) objects of which are $\mathbb{A}$-stacks (resp. $\mathbb{A}$-prestacks).

4.1.2. Note. In the case $\mathbb{A}$ is the Q-category associated with a Grothendieck site, the definition of an $\mathbb{A}$-stack is equivalent to the usual one (cf. [Giraud], Ch. II, 1.2.1).

4.2. A reformulation. Let $\mathbb{A} = (\bar{A} \overset{u}{\rightleftarrows} A)$ be a Q-category and $\mathfrak{F} = (\mathcal{F}, \mathcal{F} \xrightarrow{\pi} A^{op})$ a category over A^{op}. For any $\bar{y} \in Ob\bar{A}$, the category $(u_*(\bar{y})\backslash A)^{op}$ is isomorphic (by Yoneda's lemma) to the category $A^{op}/A(u_*(\bar{y}), -)$. By a similar reason, the category $(\bar{y}\backslash u^*)^{op}$ is isomorphic to $A^{op}/\bar{A}(\bar{y}, u^*(-))$. Therefore,

$$Cart_{A^{op}}((u_*(\bar{y})\backslash A)^{op}, \mathcal{F}) \simeq Cart_{A^{op}}(A^{op}/A(u_*(\bar{y}), -), \mathcal{F}) = \mathcal{F}^+_{A(u_*(\bar{y}),-)} \simeq \mathcal{F}_{u_*(\bar{y})}$$

and

$$Cart_{A^{op}}((\bar{y}\backslash u^*)^{op}, \mathcal{F}) \simeq Cart_{A^{op}}(A^{op}/A(\bar{y}, u^*(-)), \mathcal{F}) = \mathcal{F}^+_{\bar{A}(\bar{y},u^*(-))},$$

where $\mathcal{F}^+$ is the canonical extension of $\mathcal{F}$ onto $A^\vee$ defined by $\mathcal{F}^+_X = Cart_{A^{op}}(A^{op}/X, \mathcal{F}) = Qcoh(\mathcal{F}/X)^{op}$ (cf. 1.1.5.2). Here we identify the category $A^\vee$ of functors $A \longrightarrow$ **Sets** with the category of presheaves of sets on A^{op}. The morphism (2) above is a part of the commutative diagram

$$\begin{tikzcd}
Cart_{A^{op}}((u_*(\bar{y})\backslash A)^{op}, \mathcal{F}) \arrow[r] \arrow[d, "\wr"'] & Cart_{A^{op}}((\bar{y}\backslash u^*)^{op}, \mathcal{F}) \arrow[d, "\wr"] \\
\mathcal{F}^+_{A(u_*(\bar{y}),-)} \arrow[r, "\alpha^{\mathcal{F}}_{\bar{y}}"] & \mathcal{F}^+_{\bar{A}(\bar{y},u^*(-))}
\end{tikzcd}$$

where the lower horizontal arrow is induced by the canonical morphism

$$\bar{A}(\bar{y}, u^*(-)) \longrightarrow A(u_*(\bar{y}), -).$$

Thus, a fibered category $\mathcal{F}$ over A^{op} is an $\mathbb{A}$-prestack (resp. an $\mathbb{A}$-stack) iff the canonical functor

$$\mathcal{F}^+_{A(u_*(\bar{y}),-)} \xrightarrow{\alpha^{\mathcal{F}}_{\bar{y}}} \mathcal{F}^+_{\bar{A}(\bar{y},u^*(-))}$$

is fully faithful (resp. an equivalence of categories) for all $\bar{y} \in Ob\bar{A}$.

4.3. $\mathbb{A}$-stacks and $\mathbb{A}$-spaces. Representable $\mathbb{A}$-stacks and $\mathbb{A}$-prestacks. Let X be a functor $A \longrightarrow$ **Sets**. Consider the category A^{op}/X corresponding to the Yoneda's embedding

$$A^{op} \longrightarrow A^{\vee} = \mathbf{Sets}^{A}, \quad x \longmapsto A(x,-).$$

It is a fibered category over A^{op}.

4.3.1. Proposition. *There are canonical isomorphisms*

$$Cart_{A^{op}}((u_*(\bar{y})\backslash A)^{op}, A^{op}/X) \xrightarrow{\sim} X \circ u_*(\bar{y}) = \hat{u}^*(X)(\bar{y}) \simeq \bar{A}^{\vee}(\bar{y}, X \circ u_*) \qquad (1)$$

and

$$Cart_{A^{op}}((\bar{y}\backslash u^*)^{op}, A^{op}/X) \xrightarrow{\sim} \hat{u}^!(X)(\bar{y}) \simeq A^{\vee}(\bar{A}(\bar{y}, u^*(-)), X) \qquad (2)$$

(cf. 3.6) such that the diagram

$$\begin{array}{ccc} Cart_{A^{op}}((u_*(\bar{y})\backslash A)^{op}, A^{op}/X) & \longrightarrow & Cart_{A^{op}}((\bar{y}\backslash u^*)^{op}, A^{op}/X) \\ \wr\downarrow & & \downarrow\wr \\ A^{\vee}(u_*(\bar{y}), X) \simeq \hat{u}^*(X)(\bar{y}) & \longrightarrow & \hat{u}^!(X)(\bar{y}) = A^{\vee}(\bar{A}(\bar{y}, u*(-)), X) \end{array} \qquad (3)$$

is commutative.

Proof. The category $(z\backslash A)^{op}$ is naturally identified with A^{op}/z; in particular, we can replace $(u_*(\bar{y})\backslash A)^{op}$ by $A^{op}/u_*(\bar{y})$. Any A^{op}-functor $A^{op}/z \longrightarrow A^{op}/X$, $z \in ObA$, maps an object $(V, V \to z)$ to an object $(V, V \to X)$ and is uniquely determined by the image, $(z, z \xrightarrow{\xi} X)$, of the initial object (z, id_z). Since by Yoneda's lemma, $A^{\vee}(z, X) \simeq X(z)$, this defines a canonical isomorphism $Cart_{A^{op}}(A^{op}/z, A^{op}/X) \xrightarrow{\sim} X(z)$. Taking $z = u_*(\bar{y})$, we obtain the isomorphism $Cart_{A^{op}}((u_*(\bar{y})\backslash A)^{op}, A^{op}/X) \xrightarrow{\sim} X \circ u_*(\bar{y})$.

The isomorphism

$$Cart_{A^{op}}((\bar{y}\backslash u^*)^{op}, A^{op}/X) \xrightarrow{\sim} A^{\vee}(\bar{A}(\bar{y}, u^*(-)), X)$$

is defined in an obvious way. ■

4.3.2. Corollary. *The fibered category A^{op}/X is an $\mathbb{A}$-stack (resp. an $\mathbb{A}$-prestack) if and only if X is a sheaf (resp. a monopresheaf) on $\mathbb{A}$.*

4.3.3. Definition. An $\mathbb{A}$-stack (resp. an $\mathbb{A}$-prestack) $\mathcal{F}$ is called *representable* if it is A^{op}-equivalent to the $\mathbb{A}$-stack (resp. $\mathbb{A}$-prestack) A^{op}/X for some $\mathbb{A}$-sheaf (resp. $\mathbb{A}$-monopresheaf) of sets X on $\mathbb{A}$.

4.4. $\mathbb{A}$-prestacks and presheaves $Hom_S(x,y)$. Let $\mathcal{F}$ be a fibered category over $\mathcal{E}$ defined by a pseudo-functor $\mathcal{E}^{op} \longrightarrow Cat$, given by

$$Ob\mathcal{E} \ni X \longmapsto \mathcal{F}_X, \quad Hom\mathcal{E} \ni f \longmapsto f^*, \quad c_{g,f} : f^*g^* \xrightarrow{\sim} (gf)^*.$$

For any $S \in Ob\mathcal{E}$ and any pair of objects, $x,\ y$ of the fiber $\mathcal{F}_S$, we denote by $Hom_S(x,y)$ a presheaf of sets on $\mathcal{E}/S$ defined as follows. $Hom_S(x,y)(T,f) = Hom_T(f^*(x), f^*(y))$ for any object $(T, T \xrightarrow{f} S)$ of $\mathcal{E}/S$. To any morphism $h : (T,f) \longrightarrow (U,g)$, the presheaf $Hom_S(x,y)$ assigns the map

$$Hom_S(x,y)(U,g) := Hom_U(g^*(x), g^*(y)) \longrightarrow Hom_S(x,y)(T,f) := Hom_T(f^*(x), f^*(y))$$

given by

$$(g^*(x) \xrightarrow{\xi} g^*(y)) \longmapsto c_{g,h}(y) \circ h^*(\xi) \circ c_{g,h}^{-1}(x) \in Hom_T(f^*(x), f^*(y)).$$

4.4.1. Proposition. *Let $\mathbb{A} = (\bar{A} \overset{u}{\rightleftarrows} A)$ be a Q-category. A fibered category $\mathcal{F}$ over A^{op} is an $\mathbb{A}$-prestack iff for any $S \in ObA$ and any pair of objects $x,\ y$ of $\mathcal{F}_S$, the presheaf $Hom_S(x,y) : S\backslash A = (A^{op}/S)^{op} \longrightarrow$ **Sets** is a sheaf on the Q-category $S\backslash\mathbb{A}$ (cf. 2.2.1).*

Proof is left to the reader. ■

4.5. Sub(pre)stacks. Let $\mathcal{F}$ be an $\mathbb{A}$-stack, and let $\mathcal{G}$ be a full subcategory of $\mathcal{F}$ such that $\mathcal{G}$ is a fibered category over A^{op} and the inclusion functor $\mathcal{G} \longrightarrow \mathcal{F}$ is cartesian. Then $\mathcal{G}$ is an $\mathbb{A}$-prestack.

4.5.1. Definition. In the setting above, let S be an object of A and x an object of $\mathcal{F}_S$. We say that x *belongs locally to $Ob\mathcal{G}$* if there exists $\bar{S} \in Ob\bar{A}$ such that $u_*(\bar{S}) \simeq S$ and for any $(T,\xi) \in Ob\bar{S}\backslash u^*$ the object $u_{_{S}*}(\xi)^*(x)$ of $\mathcal{F}_T$ is isomorphic to an object of $\mathcal{G}_T$. Here $u_{_{S}*}(\xi) = \eta_u^{-1}(T) \circ u_*(\xi)$.

4.5.2. Lemma. *A subprestack $\mathcal{G}$ of an $\mathbb{A}$-stack $\mathcal{F}$ is an $\mathbb{A}$-stack iff for any $S \in ObA$, any $x \in Ob\mathcal{F}_S$ which locally belongs to $Ob\mathcal{G}$ is isomorphic to an $x' \in Ob\mathcal{G}$.*

Proof is left to the reader. ■

4.6. Local epimorphisms. A canonical decomposition of a cartesian functor. Fix a Q-category $\mathbb{A} = (\bar{A} \overset{u}{\rightleftarrows} A)$. Let $\mathcal{X}$, $\mathcal{Y}$ be fibered categories over A^{op}. We call a cartesian functor $\mathcal{X} \xrightarrow{F} \mathcal{Y}$ a *local epimorphism* if for every $v \in ObA^{op}$ and any $z \in Ob\mathcal{Y}_v$, there exist $\bar{y} \in Ob\bar{A}$ and $z' \in Ob\mathcal{X}^+_{\bar{A}(\bar{y},u^*(-))}$ such that $u_*(\bar{y}) \simeq v$ and $F^+(z')$ is isomorphic to the image of z by the (inverse image of the) composition of the canonical morphism $\bar{A}(\bar{y}, u^*(-)) \longrightarrow A(u_*(\bar{y}), -)$ and the isomorphism $A(u_*(\bar{y}), -) \xrightarrow{\sim} A(v, -)$.

4.6.1. Lemma. *The composition of local epimorphisms is a local epimorphism. If $\mathcal{X} \xrightarrow{F} \mathcal{Y} \xrightarrow{G} \mathcal{Z}$ are cartesian functors such that $G \circ F$ is a local epimorphism, then G is a local epimorphism.*

Proof is left to the reader. ■

Let $\mathcal{X} \xrightarrow{F} \mathcal{Y}$ be an arbitrary cartesian functor. We denote by $\mathcal{Y}^F$ a full subcategory of $\mathcal{Y}$ whose objects are $z \in Ob\mathcal{Y}$ such that there exists $\bar{y} \in Ob\bar{A}$ and $z' \in Ob\mathcal{X}^+_{\bar{A}(\bar{y},u^*(-))}$ such that $u_*(\bar{y}) \simeq \pi(z)$ and $F^+(z')$ is isomorphic to the image of z by the (inverse image of the) composition $\bar{A}(\bar{y}, u^*(-)) \longrightarrow A(u_*(\bar{y}), -) \xrightarrow{\sim} A(\pi(z), -)$. Here π denotes the projection $\mathcal{Y} \longrightarrow A^{op}$. The cartesian functor F factorizes into

$$\mathcal{X} \xrightarrow{F_{\mathfrak{e}}} \mathcal{Y}^F \xrightarrow{F_{\mathfrak{m}}} \mathcal{Y}, \tag{1}$$

where $F_{\mathfrak{e}}$ is a local epimorphism and $F_{\mathfrak{m}}$ is the natural embedding. This decomposition is universal in the sense that if $\mathcal{X} \xrightarrow{F'} \mathcal{Y}' \xrightarrow{F''} \mathcal{Y}$ is an arbitrary factorization of F into a local epimorphism, F', and a fully faithful functor, F'', then there is a commutative diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r, "F'"] \arrow[dr, "F_{\mathfrak{e}}"'] & \mathcal{Y}' \arrow[r, "F''"] \arrow[d, "I"] & \mathcal{Y} \\ & \mathcal{Y}^F \arrow[ur, "F_{\mathfrak{m}}"'] & \end{tikzcd} \tag{2}$$

where $\mathcal{Y}' \xrightarrow{I} \mathcal{Y}^F$ is a fully faithful cartesian functor.

4.6.2. Proposition. *In the above setting, suppose $\mathcal{Y}$ is an $\mathbb{A}$-stack. Then $\mathcal{Y}^F$ is a substack of $\mathcal{Y}$. If $\mathcal{X}$ is a stack too, then the decomposition (1) is unique up to isomorphism; i.e. the functor I in the diagram (2) is an equivalence of categories.*

Proof. (a) If $\mathcal{Y}$ is a stack, then $\mathcal{Y}^F$ is a substack of $\mathcal{Y}$ by 4.5.2.

(b) Suppose now that both $\mathcal{X}$ and $\mathcal{Y}$ are $\mathbb{A}$-stacks. We can and will assume that F is a local epimorphism, i.e. $\mathcal{Y}^F = \mathcal{Y}$. We need to show that for every $v \in ObA$, the functor $\mathcal{Y}'_v \xrightarrow{I_v} \mathcal{Y}_v$ in (2) is a category equivalence. Since the functor I_v is fully faithful, it suffices to show that for every object z of the category $\mathcal{Y}_v$, there exists an object z'' of $\mathcal{X}_v$ such that $F_v(z'') \simeq z$.

Let $z \in Ob\mathcal{Y}_v$. Since F is a local epimorphism, there exist $\bar{y} \in Ob\bar{A}$ and $z' \in Ob\mathcal{X}^+_{\bar{A}(\bar{y},u^*(-))}$ such that $u_*(\bar{y}) \simeq v$ and $F^+(z')$ is isomorphic to the image of z by the (inverse image of the) composition $\bar{A}(\bar{y}, u^*(-)) \longrightarrow A(u_*(\bar{y}), -) \xrightarrow{\sim} A(v, -)$. Since $\mathcal{X}$ is a stack, the canonical functor

$$\mathcal{X}^+_{A(u_*(\bar{y}),-)} \xrightarrow{\alpha^{\mathcal{X}}_{\bar{y}}} \mathcal{X}^+_{\bar{A}(\bar{y},u^*(-))}$$

is an equivalence of categories for all $\bar{y} \in Ob\bar{A}$ (see 4.2). Since $\mathcal{X}^+_{A(u_*(\bar{y}),-)} \simeq \mathcal{X}^+_{A(v,-)} \simeq \mathcal{X}_v$, this implies that there exists an object z'' of $\mathcal{X}_v$ (defined uniquely up to isomorphism) whose image in $\mathcal{X}^+_{A(u_*(\bar{y}),-)}$ is isomorphic to z'. Since $\mathcal{Y}$ is a stack, we have category equivalence

$$\mathcal{Y}_v \simeq \mathcal{Y}^+_{A(u_*(\bar{y}),-)} \xrightarrow{\alpha^{\mathcal{Y}}_{\bar{y}}} \mathcal{Y}^+_{\bar{A}(\bar{y},u^*(-))}$$

Thus, we have a 2-commutative diagram

$$\begin{tikzcd} \mathcal{X}^+_{\bar{A}(\bar{y},u^*(-))} \arrow[d, "F^+"'] & \mathcal{X}_v \arrow[l] \arrow[d, "F_v"] \\ \mathcal{Y}^+_{\bar{A}(\bar{y},u^*(-))} & \mathcal{Y}_v \arrow[l] \end{tikzcd}$$

whose horizontal arrows are category equivalences. In particular, $F_v(z'') \simeq z$, hence the assertion. ∎

4.6.3. Corollary. *Let $\mathcal{X}$ and $\mathcal{Y}$ be $\mathbb{A}$-stacks. A cartesian functor $\mathcal{X} \xrightarrow{F} \mathcal{Y}$ is an equivalence iff it is a fully faithful local epimorphism.*

4.6.4. Proposition. *Let*

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "G'"] \arrow[d, "F'"'] & \mathcal{X} \arrow[d, "F"] \\ \mathcal{Y}' \arrow[r, "G"] & \mathcal{Y} \end{tikzcd} \qquad (3)$$

be a 2-cartesian square of morphisms of $\mathbb{A}$-stacks. If F is a local epimorphism (resp. fully faithful, resp. an equivalence), then F' is the same.

If F', G are local epimorphisms, then F is a local epimorphism.

Proof. (a) We can assume that $\mathcal{X}'$ is the canonical pull-back of the pair of functors $\mathcal{Y}' \xrightarrow{G} \mathcal{Y} \xleftarrow{F} \mathcal{X}$; i.e. objects of $\mathcal{X}'$ are triples $(x, y; \phi)$, where $x \in Ob\mathcal{X}$, $y \in Ob\mathcal{Y}'$, and ϕ is an isomorphism $F(x) \xrightarrow{\sim} G(y)$. For any $v \in ObA$, the fiber $\mathcal{X}'_v$ is the (standard) pull-back of the pair of functors $\mathcal{Y}'_v \xrightarrow{G_v} \mathcal{Y} \xleftarrow{F_v} \mathcal{X}_v$; i.e. spanned by objects $(x, y; \phi)$ such that $\pi(\phi) = id_v$. Here π is the projection $\mathcal{Y} \longrightarrow A^{op}$. The functor $\mathcal{X}'_v \xrightarrow{F'_v} \mathcal{Y}'_v$ maps an object $(x, y; \phi)$ to x and a morphism $(x, y; \phi) \xrightarrow{(f,g)} (x', y'; \phi)$ to $x \xrightarrow{f} x'$. It follows from the construction that if F_v is fully faithful, then F'_v is fully faithful, and if F is a local epimorphism, then F' is a local epimorphism.

(b) If F' and G are local epimorphisms, then, by 4.6.1, $G \circ F'$ is a local epimorphism, hence $F \circ G'$ is a local epimorphism. By 4.6.1, the latter implies that F is a local epimorphism. ∎

4.6.5. Cartesian squares and the canonical decomposition of cartesian functors. Let

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "G'"] \arrow[d, "F'"'] & \mathcal{X} \arrow[d, "F"] \\ \mathcal{Y}' \arrow[r, "G"] & \mathcal{Y} \end{tikzcd} \qquad (3)$$

be a 2-cartesian square of morphisms of $\mathbb{A}$-stacks. Consider the canonical decomposition $\mathcal{X} \xrightarrow{F_{\mathfrak{e}}} \mathcal{Y}^F \xrightarrow{F_{\mathfrak{m}}} \mathcal{Y}$ of the cartesian functor $\mathcal{X} \xrightarrow{F} \mathcal{Y}$. Then we have a diagram of cartesian functors

$$\begin{tikzcd} \mathcal{X}'' \arrow[r, "\widetilde{F}'_{\mathfrak{e}}"] \arrow[d, "G'"'] & \widetilde{\mathcal{Y}}^F \arrow[r, "F'_{\mathfrak{m}}"] \arrow[d, "G''"] & \mathcal{Y}' \arrow[d, "G"] \\ \mathcal{X} \arrow[r, "F_{\mathfrak{e}}"] & \mathcal{Y}^F \arrow[r, "F_{\mathfrak{m}}"] & \mathcal{Y} \end{tikzcd}$$

with both squares 2-cartesian: we construct first the right square, and then take pull-back of $\mathcal{X} \xrightarrow{F_{\mathfrak{e}}} \mathcal{Y}^F \xleftarrow{G''} \widetilde{\mathcal{Y}}^F$. By 4.6.4, $F'_{\mathfrak{m}}$ is fully faithful and $F'_{\mathfrak{e}}$ is a local epimorphism. Since the square

$$\begin{array}{ccc} \mathcal{X}'' & \xrightarrow{F'_{\mathfrak{m}} \circ \widetilde{F}'_{\mathfrak{e}}} & \mathcal{Y}' \\ {\scriptstyle G'} \Big\downarrow & & \Big\downarrow {\scriptstyle G} \\ \mathcal{X} & \xrightarrow{F} & \mathcal{Y} \end{array}$$

is 2-cartesian, as well as the square (3), there is an equivalence $\mathcal{X}' \xrightarrow{\Phi} \mathcal{X}''$. Taking $F'_{\mathfrak{e}} = \widetilde{F}'_{\mathfrak{e}} \circ \Phi$, we obtain a diagram

$$\begin{array}{ccccc} \mathcal{X}' & \xrightarrow{F'_{\mathfrak{e}}} & \widetilde{\mathcal{Y}}^F & \xrightarrow{F'_{\mathfrak{m}}} & \mathcal{Y}' \\ {\scriptstyle G'} \Big\downarrow & & \Big\downarrow {\scriptstyle G''} & & \Big\downarrow {\scriptstyle G} \\ \mathcal{X} & \xrightarrow{F_{\mathfrak{e}}} & \mathcal{Y}^F & \xrightarrow{F_{\mathfrak{m}}} & \mathcal{Y} \end{array}$$

with both squares 2-cartesian, with $F'_{\mathfrak{e}}$ a local epimorphism and $F'_{\mathfrak{m}}$ fully faithful, and such that $F' = F'_{\mathfrak{m}} \circ F'_{\mathfrak{e}}$.

5. Standard examples.

5.1. Stacks and prestacks of categories of quasi-coherent modules. Let $\mathfrak{F} = (\mathcal{F} \to \mathcal{E})$ be a fibered category and $\mathfrak{T}$ a quasi-topology on $\mathcal{E}$. If $\mathfrak{T}$ is coarser than the quasi-topology of effective descent (resp. the 1-descent quasi-topology), then the fibered category $\mathfrak{F}$ is a stack (resp. a prestack) over $(\mathcal{E}, \mathfrak{T})$. This follows from definitions and 1.2.

Let $\mathfrak{T}^\wedge$ be the coinduced topology on $\mathcal{E}^\wedge$, and let $\mathfrak{F}^+$ be a canonical extension of $\mathfrak{F}$ to $\mathcal{E}^\wedge$. Recall that for any presheaf of sets X, the fiber $\mathfrak{F}^+_X$ is the category opposite to the category $Qcoh(\mathfrak{F}/X)$ of quasi-coherent modules on X (cf. 1.1.5.1). If $\mathfrak{T}$ is coarser than the topology of effective descent (resp. the 1-descent topology), then the fibered category $\mathfrak{F}^+$ over $\mathcal{E}^\wedge$ is a stack (resp. a prestack) over $(\mathcal{E}^\wedge, \mathfrak{T}^\wedge)$.

5.2. The fibered category of noncommutative affine schemes. Let $\mathfrak{F}$ be the fibered category $\mathfrak{M}(\mathbf{Aff}_k, \mathcal{O})$ of modules over affine noncommutative k-schemes (associated with the ringed category $(\mathbf{Aff}_k, \mathcal{O})$ of affines k-schemes); i.e. $\mathcal{E} = \mathbf{Aff}_k = Alg_k^{op}$ and the fiber over $\mathbf{Spec}R$ is the category $R - mod^{op}$ opposite to the category of left R-modules (see 2.1).

5.2.1. Subcanonical topologies. Let $\mathfrak{T}$ be a topology on $\mathbf{Aff}_k$. By 2.7.3.1, $\mathfrak{F}$ is a prestack over $(\mathbf{Aff}_k, \mathfrak{T})$ iff the topology $\mathfrak{T}$ is subcanonical (i.e. if every representable functor on $\mathbf{Aff}_k$ is a sheaf).

5.2.2. The fpqc quasi-topology. Let $\mathfrak{T}_{fpqc}$ be **fpqc** quasi-topolgy on $\mathbf{Aff}_k$ defined via covers consisting of finite conservative families of flat morphisms. The fibered category $\mathfrak{F} = \mathfrak{M}(\mathbf{Aff}_k, \mathcal{O})$ is a stack over $(\mathbf{Aff}_k, \mathfrak{T}_{fpqc})$.

6. Formally $\mathbb{A}$-smooth, formally $\mathbb{A}$-unramified, and formally $\mathbb{A}$-étale cartesian functors.

Given a Q-category $\mathbb{A}$, we define the notions of formally $\mathbb{A}$-smooth, formally $\mathbb{A}$-unramified, and formally $\mathbb{A}$-étale cartesian functor without using the representability hypothesis.

6.1. Definition. Let $\mathbb{A} = (\bar{A} \overset{u}{\rightleftarrows} A)$ be a Q-category, and let $\mathcal{X}$, $\mathcal{Y}$ be categories over A^{op}. We call a cartesian functor $\mathcal{X} \xrightarrow{f} \mathcal{Y}$ *formally* $\mathbb{A}$*-smooth* if for any $\bar{y} \in Ob\bar{A}$ and for any pair of morphisms $A^{op}/u_*(\bar{y}) \xrightarrow{g} Y$, $(\bar{y}\backslash u^*)^{op} \xrightarrow{g'} X$ such that the diagram

$$\begin{array}{ccc} (\bar{y}\backslash u^*)^{op} & \xrightarrow{g'} & \mathcal{X} \\ \theta_{\bar{y}} \big\downarrow & & \big\downarrow f \\ A^{op}/u_*(\bar{y}) & \xrightarrow{g} & \mathcal{Y} \end{array} \qquad (1)$$

2-commutes, there exists a morphism $A(u_*(\bar{y}), -) \xrightarrow{\gamma} \mathcal{X}$ such that $\gamma \circ \theta_{\bar{y}} \simeq g'$ and $f \circ \gamma \simeq g$. In other words, the diagram

$$\begin{array}{ccc} (\bar{y}\backslash u^*)^{op} & \xrightarrow{g'} & \mathcal{X} \\ \theta_{\bar{y}} \big\downarrow & \nearrow_{\gamma} & \big\downarrow f \\ A^{op}/u_*(\bar{y}) & \xrightarrow{g} & \mathcal{Y} \end{array} \qquad (2)$$

2-commutes.

6.2. Definition. We call a cartesian functor $\mathcal{X} \xrightarrow{f} \mathcal{Y}$ *formally* $\mathbb{A}$*-unramified* if for any $\bar{y} \in Ob\bar{A}$ and any pair of cartesian functors, $A(u_*(\bar{y}), -) \xrightarrow{g} \mathcal{Y}$, $\bar{A}(\bar{y}, u^*(-)) \xrightarrow{g'} \mathcal{X}$, for which the diagram 6.1(1) 2-commutes, there exists at most one (up to isomorphism) cartesian functor, $A^{op}/u_*(\bar{y}) \xrightarrow{\gamma} \mathcal{X}$, such that the diagram 6.1(2) 2-commutes.

6.2.1. Note. Let a cartesian functor $\mathcal{X} \xrightarrow{f} \mathcal{Y}$ is a monofunctor in the following sense: if $\mathcal{Z} \underset{g_2}{\overset{g_1}{\rightrightarrows}} \mathcal{X}$ is a pair of cartesian functors such that $f \circ g_1 \simeq f \circ g_2$, then $g_1 \simeq g_2$. Then f is formally $\mathbb{A}$-unramified.

6.3. Definition. We call a cartesian functor $\mathcal{X} \xrightarrow{f} \mathcal{Y}$ *formally* $\mathbb{A}$*-étale* if it is both formally $\mathbb{A}$-smooth and formally $\mathbb{A}$-unramified.

6.4. Definition. We call a fibered category $\mathcal{X}$ over A^{op} formally $\mathbb{A}$-smooth (resp. formally $\mathbb{A}$-unramified, resp. formally $\mathbb{A}$-étale) iff the functor, $\mathcal{X} \xrightarrow{p} A^{op}$, is formally $\mathbb{A}$-smooth (resp. formally $\mathbb{A}$-unramified, resp. formally $\mathbb{A}$-étale).

6.4.1. Note. It follows from definitions that a fibered category over A^{op} is $\mathbb{A}$-étale if and only if it is an $\mathbb{A}$-stack. Thus, a formally $\mathbb{A}$-étale morphism can be regarded as a generalization of a notion of an $\mathbb{A}$-stack.

6.5. Proposition. *(a) A composition of formally $\mathbb{A}$-smooth (resp. formally $\mathbb{A}$-unramified, resp. formally $\mathbb{A}$-étale) cartesian functors is formally $\mathbb{A}$-smooth (resp. formally $\mathbb{A}$-unramified, resp. formally $\mathbb{A}$-étale).*

(b) Let $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ be categories over A^{op}, and let $\mathcal{X} \xrightarrow{f} \mathcal{Y}$ and $Y \xrightarrow{h} Z$ be cartesian functors.

(i) Suppose $h \circ f$ is formally $\mathbb{A}$-unramified. Then f is formally $\mathbb{A}$-unramified.

(ii) Suppose h is formally $\mathbb{A}$-unramified. If $\mathcal{X} \xrightarrow{h \circ f} \mathcal{Z}$ is formally $\mathbb{A}$-smooth (resp. formally $\mathbb{A}$-étale, then f is formally $\mathbb{A}$-smooth (resp. formally $\mathbb{A}$-étale).

(c) Let $S \in ObCart_{A^{op}}$, and let $(\mathcal{X}, \xi) \xrightarrow{f} (\mathcal{Y}, \mu)$, $(\mathcal{X}', \xi') \xrightarrow{f'} (\mathcal{Y}', \mu')$ be morphisms of objects over S. The morphisms f, f' are formally $\mathbb{A}$-unramified (resp. formally $\mathbb{A}$-smooth, resp. formally $\mathbb{A}$-étale) iff the morphism $f \times_S f' : \mathcal{X} \times_S \mathcal{X}' \longrightarrow \mathcal{Y} \times_S \mathcal{Y}'$ has the corresponding property.

Proof. The argument is similar to the one of 5.4. ■

6.6. Smooth, unramified, and étale cartesian functors. Let $\mathbb{A} = (\bar{A} \overset{u}{\rightleftarrows} A)$ be a Q-category, and let $\mathcal{X}$, $\mathcal{Y}$ be fibered categories over A^{op}. We call a cartesian functor $\mathcal{X} \xrightarrow{f} \mathcal{Y}$ $\mathbb{A}$-smooth (resp. $\mathbb{A}$-unramified, resp. $\mathbb{A}$-étale) if it is locally finitely presentable (cf. 3.5) and formally $\mathbb{A}$-smooth (resp. formally $\mathbb{A}$-unramified, resp. formally $\mathbb{A}$-étale).

7. Locally affine stacks. $\mathbb{A}$-stacks are viewed as a natural generalization of $\mathbb{A}$-spaces. Morphisms of spaces are replaced by 1-morphisms of stacks, i.e. cartesian functors. Below we extend to stacks the content of Sections 8 and 9 of [KR3].

7.1. Affine τ-covers, locally affine τ-stacks. Zariski τ-stacks. Let $Stacks^1_{\mathbb{A}}$ denote the 1-category of $\mathbb{A}$-stacks: its objects are $\mathbb{A}$-stacks and morphisms cartesian functors. Let τ be a quasi-topology on the category $Stacks^1_{\mathbb{A}}$.

We call a τ-cover $\{\mathcal{X}_u \xrightarrow{u} \mathcal{X} \mid u \in \mathcal{U}\}$ *affine*, if all $\mathcal{X}_u$ are affine (i.e. corepresentable by objects of A).

We call an $\mathbb{A}$-stack $\mathcal{X}$ *τ-locally affine* if it has an affine τ-cover.

We call a τ-cover $\{\mathcal{X}_u \xrightarrow{u} \mathcal{X} \mid u \in \mathcal{U}\}$ *2-affine* if it is affine and for any $u, v \in \mathcal{U}$, the stack $\mathcal{X}_u \times_{\mathcal{X}} \mathcal{X}_v$ has an affine τ-cover.

An affine τ-cover $\{\mathcal{X}_u \xrightarrow{u} \mathcal{X} \mid u \in \mathcal{U}\}$ will be called a *Zariski affine τ-cover* if each u separates non-isomorphic cartesian functors $A^{op}/T \longrightarrow \mathcal{X}_u$ for all $T \in ObA$.

We call an $\mathbb{A}$-stack $\mathcal{X}$ a *Zariski τ-stack* if it has a Zariski affine τ-cover.

A τ-cover $\{X_u \xrightarrow{u} X \mid u \in \mathcal{U}\}$ will be called a *Zariski 2-affine τ-cover* if it is Zariski affine and for any $u, v \in \mathcal{U}$, the space $X_u \times_X X_v$ has a Zariski affine τ-cover. We call an $\mathbb{A}$-stack $\mathcal{X}$ a *Zariski τ-stack* if it has a Zariski affine τ-cover.

7.2. Semiseparated and weakly separated covers and stacks. We call a τ-cover $\{\mathcal{X}_u \xrightarrow{u} \mathcal{X} \mid u \in \mathcal{U}\}$

– *weakly separated* if $\mathcal{X}_u$ and the pull-back $\mathcal{X}_u \times_{\mathcal{X}} \mathcal{X}_v$ are affine (i.e. corepresentable) for all $u, v \in \mathcal{U}$,

– *semiseparated* if the stack $\mathcal{X}_u$ and the morphism $\mathcal{X}_u \xrightarrow{u} \mathcal{X}$ are affine for all u.

Clearly semiseparated covers are weakly separated, and weakly separated covers are 2-covers. In particular, a stack which has a weakly separated cover is locally affine. We call a stack which has a semiseparated (resp. weakly separated) affine cover *semiseparated* (resp. *weakly separated*).

7.2.1. Proposition. *Suppose that $\mathbb{A} = (\bar{A} \overset{u}{\rightleftarrows} A)$ is subcanonical (i.e. all corepresentable functors $A \longrightarrow$ **Sets** are $\mathbb{A}$-stacks), and A has products. Then every separated locally affine $\mathbb{A}$-stack is τ-semiseparated.*

Proof. The argument imitates that of [KR3, 8.4]. Details are left to the reader. ∎

7.3. Natural quasi-topologies on $Stacks_{\mathbb{A}}$. For a cartesian functor $\mathcal{Y} \xrightarrow{\varphi} \mathcal{X}$, let Λ_φ denote the family of all pairs of cartesian functors $\mathcal{Z} \overset{g_1}{\underset{g_2}{\rightrightarrows}} \mathcal{Y}$ such that $\varphi \circ g_1 \simeq \varphi \circ g_2$. We call a cartesian functor $\mathcal{Y} \xrightarrow{\varphi} \mathcal{X}$ *strictly epimorphic* if the following condition holds: for any cartesian functor $\mathcal{Y} \xrightarrow{\psi} \mathcal{Z}$ such that $\Lambda_\varphi \subseteq \Lambda_\psi$, there exists a unique up to isomorphism cartesian functor $\mathcal{X} \xrightarrow{\gamma} \mathcal{Z}$ such that $\psi \simeq \gamma \circ \varphi$.

We call a set of morphisms $\{\mathcal{X}_u \xrightarrow{u} \mathcal{X} \mid u \in \mathcal{U}\}$ *strictly epimorphic* if the corresponding cartesian functor $\coprod_{u \in \mathcal{U}} \mathcal{X}_u \xrightarrow{\mathbf{u}} \mathcal{X}$ is strictly epimorphic.

Let $\mathcal{P}$ be a class of morphisms in $Stacks_{\mathbb{A}}$ (cartesian functors) which contains all identical morphisms and is closed under the composition (that is $\mathcal{P}$ is a subcategory of $Stacks_{\mathbb{A}}$ having same objects as $Stacks_{\mathbb{A}}$). We call a set of morphisms $\{\mathcal{X}_u \xrightarrow{u} \mathcal{X} \mid u \in \mathcal{U}\}$ from a *$\mathcal{P}$-cover* of $\mathcal{X}$ if

(i) it is strictly epimorphic;

(ii) all morphisms of $\mathcal{U}$ belong to $\mathcal{P}$.

This defines a quasi-topology, $\tau^{\mathcal{P}}$, on $Stacks_{\mathbb{A}}$ and its quasi-compact version, $\tau_f^{\mathcal{P}}$. It remains to choose the class $\mathcal{P}$.

7.4. Formally étale, formally smooth, and formally Zariski quasi-topologies. Let $\mathbb{A}_1 = (\bar{A}_1 \rightleftarrows A)$ be another Q-category with the same underlying category A (thought as the Q-category of thickennings). Then we have the following choices:

– the class $\mathcal{P}_{\text{fét}}$ of formally $\mathbb{A}_1$-étale cartesian functors,

– the class $\mathcal{P}_{fsm}$ of formally $\mathbb{A}_1$-smooth cartesian functors,

– the class $\mathcal{P}_{fzar}$ of *formally $\mathbb{A}_1$-open immersions* which we define as formally $\mathbb{A}_1$-étale cartesian functors $\mathcal{X} \xrightarrow{\varphi} \mathcal{Y}$ such that Λ_φ consists of isomorphic pairs of functors.

We denote the corresponding quasi-topologies resp. by $\tau_{\text{fét}}$, τ_{fsm}, and τ_{fzar} and call them resp.*formally étale, formally smooth* and *formally Zariski* quasi-topology.

Note that the category of Zariski τ-stacks (cf. 7.1) is the same for all three quasi-topologies; i.e. it coincides with the category of τ_{fzar}-locally affine $\mathbb{A}$-stacks. We call them *formally Zariski $\mathbb{A}_1$-stacks.*

7.4.1. Smooth, étale, and Zariski quasi-topologies. Our main choices are

– the class $\mathcal{P}_{\text{ét}}$ of $\mathbb{A}_1$-étale cartesian functors,

– the class $\mathcal{P}_{sm}$ of $\mathbb{A}_1$-smooth cartesian functors,

– the class $\mathcal{P}_{zar}$ of $\mathbb{A}_1$-*open immersions.*

We denote the corresponding quasi-topologies resp. by $\tau_{\text{ét}}$, τ_{sm}, and τ_{zar} and call them resp. *étale, smooth* and *Zariski* quasi-topology.

7.5. Noncommutative DM-stacks and Artin's algebraic stacks.

7.5.1. Commutative Zariski and algebraic stacks. Let A be the category $CAlg_k$ of commutative k-algebras, $\bar{A}_1$ the full subcategory of A^2 formed by k-algebra epimorphisms with nilpotent kernels, $\bar{A}$ the full subcategory of A^2 formed by faithfully flat morphisms. If τ is the étale topology, $\tau = \tau_{\text{ét}}$, then locally affine separated stacks are DM-stacks. If τ is the smooth topology, then the locally affine stacks are Artin's stacks.

7.5.2. Noncommutative DM-stacks. Let A be the category $\mathfrak{A}ss_k$ of associative k-algebras with morphisms defined up to conjugation (cf. 4.6). Let $\bar{A}$ be the full subcategory of A^2 formed by equivalence classes of faithfully flat algebra morphisms, and let $\bar{A}_1$ be the full subcategory of A^2 formed by equivalence classes (with respect to the conjugation, see 4.6) of k-algebra epimorphisms with nilpotent kernels. If τ is the étale quasi-topology (cf. 7.4.1), then locally affine $\mathbb{A}$-stacks in this setting seem to be an adequate noncommutative version of DM-algebraic stacks.

7.5.3. Noncommutative Artin's stacks. Let A be the category Alg_k of associative k-algebras, $\bar{A}_1$ the full subcategory of A^2 formed by k-algebra epimorphisms with nilpotent kernels, $\bar{A}$ the full subcategory of A^2 formed by faithfully flat morphisms. If τ is a smooth quasi-pretopology τ_{sm} (cf. 7.3.1), locally affine $\mathbb{A}$-stacks can be regarded as a noncommutative version of Artin's algebraic stacks.

8. Representable cartesian functors and local constructions.

Fix a category $\mathcal{E}$ and a full subcategory $\mathcal{E}'$ of the category $\mathcal{E}^\wedge$ of presheaves of sets on $\mathcal{E}$ which contains (the image of) $\mathcal{E}$ and is closed under fibered products. We denote by $\widetilde{\mathcal{E}}'$ the fibered category over $\mathcal{E}$ defined as follows: for any $V \in Ob\mathcal{E}$, the fiber $\widetilde{\mathcal{E}}'_V$ is $\mathcal{E}'/V$. For any morphism $U \xrightarrow{\phi} V$, its inverse image functor, ϕ^*, assigns to any object $(X, X \to V)$ of the category $\mathcal{E}'/V$ its pull-back $(X \times_{\phi,V} U, X \times_{\phi,V} U \to U)$.

8.1. Proposition. *Let $\mathfrak{X}$ be a fibered category over $\mathcal{E}$. There is a natural equivalence between the category of $\mathcal{E}'$-representable cartesian functors $\mathcal{Y} \longrightarrow \mathfrak{X}$ and the category of cartesian functors $\mathfrak{X} \longrightarrow \widetilde{\mathcal{E}}'$.*

Proof. An argument is left to the reader. ∎

8.1.1. Local constructions. We call any cartesian functor $\mathfrak{X} \longrightarrow \widetilde{\mathcal{E}}'$ a *local $\mathcal{E}'$-construction on* $\mathfrak{X}$. Local $\mathcal{E}$-constructions will be called *affine.*

8.2. Relative local constructions. Fix a functor $\mathcal{A} \xrightarrow{\Phi} \mathcal{E}$. Let $\widetilde{\mathcal{E}}'_\Phi$ denote the fibered category $\widetilde{\mathcal{E}}' \times_{\mathcal{E}} \mathcal{A} = \widetilde{\mathcal{E}}' \times_{\Phi,\mathcal{E}} \mathcal{A}$ over $\mathcal{A}$.

Let $\mathfrak{X}$ be a fibered category over $\mathcal{A}$. We call any cartesian functor $\mathfrak{X} \longrightarrow \widetilde{\mathcal{E}}' \times_{\mathcal{E}} \mathcal{A}$ a *local* $(\mathcal{E}', \Phi)$*-construction on* $\mathfrak{X}$, or simply a *local construction on* $\mathfrak{X}$, if it is clear what are the subcategory $\mathcal{E}'$ and the functor Φ.

Local $(\mathcal{E}, \Phi)$-constructions will be called *affine.*

8.3. Local constructions over a ringed category. Let $(\mathcal{A}, \mathcal{O})$ be a ringed category. Let $\mathcal{E}$ be the category **Aff** of noncommutative affine schemes. Let τ be a pretopology on $\mathcal{E}$ and $\mathcal{E}'$ the category of τ-locally affine spaces. The presheaf of rings $\mathcal{O}$ induces a functor

$$\Phi = \Phi_{\mathcal{O}} : \mathcal{A} \longrightarrow \mathcal{E}, \quad U \longmapsto \mathbf{Spec}(\mathcal{O}(U)).$$

A local $(\mathcal{E}, \Phi)$-construction on a fibered category $\mathfrak{X}$ over $\mathcal{A}$ is a family of functors $F_U : \mathfrak{X}_U \longrightarrow (\widetilde{\mathcal{E}'} \times_{\mathcal{E}} \mathcal{A})_U$, $U \in Ob\mathcal{A}$, such that for any $U \in Ob\mathcal{A}$ and any object x of the fiber $\mathfrak{X}_U$ over U, $F_U(x)$ is a τ-locally affine space over $\mathbf{Spec}(\mathcal{O}(U))$; and for any morphism $U \xrightarrow{\phi} V$ and any $x \in Ob\mathfrak{X}_V$, we are given an isomorphism of spaces

$$F_U(\phi^*(x)) \xrightarrow{\sim} F_V(x) \prod_{\mathbf{Spec}(\mathcal{O}(V))} \mathbf{Spec}(\mathcal{O}(V)).$$

8.3.1. Affine local constructions. A local construction $\mathfrak{X} \xrightarrow{F} \mathcal{E}' \times_{\mathcal{E}} \mathcal{A}$ is affine iff for any $U \in Ob\mathcal{A}$ and any $x \in Ob\mathfrak{X}_U$, the object $F_U(x)$ is an affine space over $\mathbf{Spec}(\mathcal{O}(U))$, i.e. $F_U(x)$ is isomorphic to a pair $(\mathbf{Spec}(\mathcal{R}(U,x)), \mathbf{Spec}(\mathcal{R}(U,x)) \longrightarrow \mathbf{Spec}(\mathcal{O}(U)))$ corresponding to a ring morphism $\mathcal{O}(U) \longrightarrow \mathcal{R}(U,x)$ defined uniquely up to isomorphism.

Thus, an affine local construction on $\mathfrak{X}$ can be described as a function which assigns to every pair (U,x), where $U \in Ob\mathcal{A}$ and $x \in Ob\mathfrak{X}_U$, a ring morphism $\mathcal{O}(V) \longrightarrow \mathcal{R}(U,x)$ and to any morphism $U \xrightarrow{\phi} V$ of $\mathcal{A}$ and any $x \in Ob\mathfrak{X}_V$, a morphism $\mathcal{R}(V,x) \xrightarrow{\xi_\phi} \mathcal{R}(U,\phi^*(x))$ such that the square

$$\begin{array}{ccc}
\mathcal{R}(V,x) & \xrightarrow{\xi_\phi} & \mathcal{R}(U,\phi^*(x)) \\
\uparrow & & \uparrow \\
\mathcal{O}(V) & \xrightarrow{\mathcal{O}(\phi)} & \mathcal{O}(U)
\end{array}$$

is cocartesian. In other words, the induced ring morphism

$$\mathcal{R}(V,x) \star_{\mathcal{O}(V)} \mathcal{O}(U) \xrightarrow{\xi'_\phi} \mathcal{R}(U,\phi^*(x))$$

is an isomorphism. The morphisms ξ_ϕ should satisfy standard compatibility conditions with respect to the composition of morphisms of $\mathcal{A}$.

8.3.2. Vector fibers over a ringed category. Let $\mathfrak{Bmod}(\mathcal{A}, \mathcal{O})$ denote the *fibered category of* $(\mathcal{A}, \mathcal{O})$*-bimodules* determined by the pseudo-functor $\mathcal{A}^{op} \longrightarrow Cat$ which assigns to each object V of $\mathcal{A}$ the category opposite to the category of $\mathcal{O}(V)$-bimodules and to any morphism $U \xrightarrow{\phi} V$ the functor opposite to the functor

$$\widetilde{\phi}^* : \mathcal{M} \longmapsto \mathcal{O}(U) \otimes_{\mathcal{O}(V)} \mathcal{M} \otimes_{\mathcal{O}(V)} \mathcal{O}(U).$$

Let $\mathfrak{X}$ be a fibered category over $\mathcal{A}$. We define $\mathcal{O}$*-bimodules on* $\mathfrak{X}$ as cartesian functors $\mathcal{M} : \mathfrak{X} \longrightarrow \mathfrak{Bmod}(\mathcal{A}, \mathcal{O})$. This means that $\mathcal{M}$ is a function which assigns to each pair

(V,x), where $V \in Ob\mathcal{A}$ and $x \in Ob\mathfrak{X}_V$, an $\mathcal{O}(V)$-bimodule $\mathcal{M}(V,x)$ and to each morphism $U \xrightarrow{\phi} V$ a bimodule isomorphism

$$\mathcal{O}(U) \otimes_{\mathcal{O}(V)} \mathcal{M}(V,x) \otimes_{\mathcal{O}(V)} \mathcal{O}(U) \xrightarrow{\zeta_\phi} \mathcal{M}(U,\phi^*(x)) \tag{1}$$

which satisfies the usual compatibility conditions.

For any $V \in Ob\mathcal{A}$ and $x \in \mathfrak{X}_V$, let $\mathcal{R}(V,x)$ be the tensor algebra, $T_{\mathcal{O}(V)}(\mathcal{M}(V,x))$, of the $\mathcal{O}(V)$-bimodule $\mathcal{M}(V,x)$. Fix a morphism $U \xrightarrow{\phi} V$ of $\mathcal{A}$. The $\mathcal{O}(V)$-bimodule morphism $\mathcal{M}(V,x) \longrightarrow \mathcal{M}(U,\phi^*(x))$ and the ring morphism $\mathcal{O}(V) \xrightarrow{\mathcal{O}(\phi)} \mathcal{O}(U)$ induce ring morphisms $T_{\mathcal{O}(V)}(\mathcal{M}(V,x)) \longrightarrow T_{\mathcal{O}(U)}(\mathcal{M}(U,\phi^*(x))) \longleftarrow \mathcal{O}(U)$ which, in turn, determines a morphism

$$\mathcal{O}(U) \star_{\mathcal{O}(V)} T_{\mathcal{O}(V)}(\mathcal{M}(V,x)) \longrightarrow T_{\mathcal{O}(U)}(\mathcal{M}(U,\phi^*(x))). \tag{2}$$

Since the $\mathcal{O}$-bimodule $\mathcal{M}$ is quasi-coherent, (1) is a bimodule isomorphism, which implies that the ring morphism (2) is an isomorphism.

We denote the affine scheme $\mathbf{Spec}(T_{\mathcal{O}(V)}(\mathcal{M}(V,x)))$ by $\mathbb{V}_{\mathcal{O}(V)}(\mathcal{M}(V,x))$ and the local construction on $\mathfrak{X}$ given by $(V,x) \longmapsto \mathbb{V}_{\mathcal{O}(V)}(\mathcal{M}(V,x))$ by $\mathbb{V}_{\mathcal{O}}(\mathcal{M})$.

8.3.3. Vector fibers associated with pairs of quasi-coherent modules. Let $\mathfrak{M}(\mathcal{A},\mathcal{O})$ be the fibered category of $(\mathcal{A},\mathcal{O})$-modules (cf. 2). Let $\mathfrak{X}$ be a fibered category over $\mathcal{A}$. We define *$\mathcal{O}$-modules on $\mathfrak{X}$* as cartesian functors $\mathcal{M} : \mathfrak{X} \longrightarrow \mathfrak{M}(\mathcal{A},\mathcal{O})$. This means that $\mathcal{M}$ is a function which assigns to each pair (V,x), where $V \in Ob\mathcal{A}$ and $x \in Ob\mathfrak{X}_V$, an $\mathcal{O}(V)$-module $\mathcal{M}(V,x)$ and to each morphism $U \xrightarrow{\phi} V$ an $\mathcal{O}(U)$-module isomorphism

$$\mathcal{O}(U) \otimes_{\mathcal{O}(V)} \mathcal{M}(V,x) \xrightarrow{\zeta_\phi} \mathcal{M}(U,\phi^*(x)) \tag{3}$$

which satisfies the usual compatibility conditions.

Let $\mathcal{M}$ and $\mathcal{P}$ be two $\mathcal{O}$-modules on $\mathfrak{X}$. The pair $(\mathcal{M},\mathcal{P})$ defines a functor

$$Hom_{\mathcal{O},\mathfrak{X}}(\mathcal{M},\mathcal{P}) : \mathfrak{X} \longrightarrow \widetilde{\mathcal{E}}^\wedge \times_{\mathcal{E}} \mathcal{A} \tag{4}$$

which assigns to each pair (U,x), $U \in Ob\mathcal{A}$, $x \in Ob\mathfrak{X}_U$, the functor

$$\mathcal{O}(U)\backslash Rings \longrightarrow Sets, \quad (\mathcal{O}(U) \xrightarrow{\phi} \mathcal{R}) \longmapsto Hom_{\mathcal{R}}(\phi^*(\mathcal{M}(U,x)), \phi^*(\mathcal{P}(U,x))) \tag{5}$$

naturally defined on morphisms.

8.3.3.1. Proposition. *Suppose that for every $U \in Ob\mathcal{A}$ and $x \in Ob\mathfrak{X}_U$, the $\mathcal{O}(U)$-module $\mathcal{P}(U,x)$ is projective of finite type. Then the functor (4) is an affine local construction.*

Proof. Set for convenience $M = \mathcal{M}(U,x)$ and $P = \mathcal{P}(U,x)$. Then

$$Hom_{\mathcal{R}}(\phi^*(M), \phi^*(P)) \simeq Hom_{\mathcal{O}(U)}(M, \phi_*\phi^*(P)) = Hom_{\mathcal{O}(U)}(M, \mathcal{R} \otimes_{\mathcal{O}(U)} P) \simeq$$

$$Hom_{\mathcal{O}(U)}(M, Hom^{\mathcal{O}(U)}(P^\vee, \mathcal{R})) \simeq Hom_{\mathcal{O}(U)^e}(M \otimes P^\vee, \mathcal{R}) \simeq$$
$$\mathcal{O}(U)\backslash\mathbf{Rings}(T_{\mathcal{O}(U)}(M \otimes P^\vee), \mathcal{R})$$

Here $Hom^{\mathcal{O}(U)}(P^\vee, S)$ is the (left) $\mathcal{O}(U)$-module of right $\mathcal{O}(U)$-module morphisms from $P^\vee$ to S, $\mathcal{O}(U)^e := \mathcal{O}(U) \otimes \mathcal{O}(U)^o$, and $T_{\mathcal{O}(U)}(M \otimes P^\vee)$ is the tensor algebra of the $\mathcal{O}(U)$-bimodule $M \otimes P^\vee$. This shows that (4) is isomorphic to the vector fiber $\mathbb{V}_{\mathcal{O}}(\mathcal{M} \otimes \mathcal{P}^\vee)$ of the quasi-coherent $\mathcal{O}$-bimodule $\mathcal{M} \otimes \mathcal{P}^\vee$. ∎

It is useful to have an analogue of 8.3.3.1 for a family of modules. Let $\{\mathcal{M}_i, \mathcal{P}_i \mid i \in J\}$ be a family of $\mathcal{O}$-modules on $\mathfrak{X}$. These family defines a functor

$$\prod_{i \in J} Hom_{\mathcal{O},\mathfrak{X}}(\mathcal{M}_i, \mathcal{P}_i) : \mathfrak{X} \longrightarrow \widetilde{\mathcal{E}}^\wedge \times_{\mathcal{E}} \mathcal{A} \qquad (6)$$

which assigns to each pair (U, x), $U \in Ob\mathcal{A}$, $x \in Ob\mathfrak{X}_U$, the functor

$$\mathcal{O}(U)\backslash Rings \longrightarrow Sets, \quad (\mathcal{O}(U) \xrightarrow{\phi} \mathcal{R}) \longmapsto \prod_{i \in J} Hom_{\mathcal{R}}(\phi^*(\mathcal{M}_i(U,x)), \phi^*(\mathcal{P}_i(U,x)))$$

naturally defined on morphisms.

8.3.3.2. Proposition. *Suppose that for every $U \in Ob\mathcal{A}$ and $x \in Ob\mathfrak{X}_U$ and for all $i \in J$, the $\mathcal{O}(U)$-module $\mathcal{P}(U,x)$ is projective of finite type. Then the functor (6) is an affine local construction on $\mathfrak{X}$ isomorphic to $\mathbb{V}(\oplus_{i \in J}(\mathcal{M}_i \otimes \mathcal{P}_i^\vee))$.*

Proof. The argument is similar to that of 8.3.3.1. Details are left to the reader. ∎

8.3.4. The construction of isomorphisms. Fix a ringed category $(\mathcal{A}, \mathcal{O})$ and a fibered category $\mathfrak{X}$ over $\mathcal{A}$. Let $\mathcal{M}$ and $\mathcal{P}$ be $\mathcal{O}$-modules on $\mathfrak{X}$. We denote by $G^{\mathcal{O},\mathfrak{X}}_{\mathcal{M},\mathcal{V}}$ the functor $\mathfrak{X} \longrightarrow \widetilde{\mathcal{E}}^\wedge \times_{\mathcal{E}} \mathcal{A}$ which assigns to every pair (U, x), $U \in Ob\mathcal{A}$, $x \in Ob\mathfrak{X}_U$, the functor

$$G^{\mathcal{O}(U)}_{\mathcal{M}(U,x),\mathcal{V}(U,x)} : \mathcal{O}(U)\backslash Rings \longrightarrow Sets \qquad (7)$$

defined as follows: to every ring morphism $\mathcal{O}(U) \xrightarrow{\phi} \mathcal{R}$, the functor (7) assigns the set of all pairs of $\mathcal{R}$-module morphisms $\phi^*(\mathcal{P}(U,x)) \longrightarrow \phi^*(\mathcal{M}(U,x)) \longrightarrow \phi^*(\mathcal{P}(U,x))$ the composition of which is the identical morphism.

8.3.4.1. Proposition. *Let $\mathcal{M}(U,x)$ and $\mathcal{P}(U,x)$ be projective $\mathcal{O}(U)$-modules of finite type for all $U \in Ob\mathcal{A}$ and $x \in Ob\mathfrak{X}$. Then the functor $G^{\mathcal{O},\mathfrak{X}}_{\mathcal{M},\mathcal{V}}$ is an affine local construction on $\mathfrak{X}$.*

Proof. (a) For convenience, we set $M = \mathcal{M}(U,x)$ and $P = \mathcal{P}(U,x)$. For any ring morphism $\mathcal{O}(U) \xrightarrow{\phi} \mathcal{R}$, the set $G^{\mathcal{O}(U)}_{M,P}(\mathcal{R}, \phi)$ is the kernel of the pair of morphisms

$$Hom_{\mathcal{R}}(\phi^*(M), \phi^*(P)) \times Hom_{\mathcal{R}}(\phi^*(P), \phi^*(M)) \rightrightarrows Hom_{\mathcal{R}}(\phi^*(P), \phi^*(P)) \qquad (8)$$

where one arrow assigns to each pair (u, v) the composition, $u \circ v$, of morphisms u and v, and the other one maps each pair (u, v) to the identity morphism, $id_{\phi^*(P)}$. Since (8) depends functorially on everything, the functor $G^{\mathcal{O},\mathfrak{X}}_{\mathcal{M},\mathcal{V}}$ is the kernel of a pair of functor morphisms

$$Hom_{\mathcal{O},\mathfrak{X}}(\mathcal{M},\mathcal{P}) \times Hom_{\mathcal{O},\mathfrak{X}}(\mathcal{P},\mathcal{M}) \rightrightarrows Hom_{\mathcal{O},\mathfrak{X}}(\mathcal{P},\mathcal{P}) \tag{9}$$

Since $\mathcal{M}(U,x)$ and $\mathcal{P}(U,x)$ are projective $\mathcal{O}(U)$-modules of finite type for all $U \in Ob\mathcal{A}$ and $x \in Ob\mathfrak{X}$, the functors $Hom_{\mathcal{O},\mathfrak{X}}(\mathcal{M},\mathcal{P})$, $Hom_{\mathcal{O},\mathfrak{X}}(\mathcal{P},\mathcal{M})$, and $Hom_{\mathcal{O},\mathfrak{X}}(\mathcal{P},\mathcal{P})$ are affine local constructions on $\mathfrak{X}$ resp. $\mathbb{V}_{\mathcal{O},\mathfrak{X}}(\mathcal{M}\otimes\mathcal{P}^\vee)$, $\mathbb{V}_{\mathcal{O},\mathfrak{X}}(\mathcal{P}\otimes\mathcal{M}^\vee)$, and $\mathbb{V}_{\mathcal{O},\mathfrak{X}}(\mathcal{P}\otimes\mathcal{P}^\vee)$. Thus the diagram (9) is equivalent to the diagram

$$\mathbb{V}_{\mathcal{O},\mathfrak{X}}(\mathcal{M}\otimes\mathcal{P}^\vee \oplus \mathcal{P}\otimes\mathcal{M}^\vee) \rightrightarrows \mathbb{V}_{\mathcal{O},\mathfrak{X}}(\mathcal{P}\otimes\mathcal{P}^\vee) \tag{10}$$

(see 8.3.3.2). The kernel of a pair of morphisms between two affine local constructions is an affine local construction. ■

Let $\mathcal{M}$ and $\mathcal{P}$ be $\mathcal{O}$-modules on $\mathfrak{X}$. We denote by $Iso^{\mathcal{O},\mathfrak{X}}_{\mathcal{M},\mathcal{P}}$ the functor $\mathfrak{X} \longrightarrow \widetilde{\mathcal{E}}^\wedge \times_{\mathcal{E}} \mathcal{A}$ which assigns to every pair (U,x), $U \in Ob\mathcal{A}$, $x \in Ob\mathfrak{X}_U$, the functor

$$Iso^{\mathcal{O}(U)}_{\mathcal{M}(U,x),\mathcal{P}(U,x)} : \mathcal{O}(U)\backslash\mathbf{Rings} \longrightarrow Sets$$

that assigns every ring morphism $\mathcal{O}(U) \xrightarrow{\phi} \mathcal{R}$ the set of isomorphisms $\phi^*(\mathcal{M}(U,x)) \xrightarrow{\sim} \phi^*(\mathcal{P}(U,x))$.

8.3.4.2. Proposition. *Let $\mathcal{M}(U,x)$ and $\mathcal{P}(U,x)$ be projective $\mathcal{O}(U)$-modules of finite type for all $U \in Ob\mathcal{A}$ and $x \in Ob\mathfrak{X}$. Then the functor $Iso^{\mathcal{O},\mathfrak{X}}_{\mathcal{M},\mathcal{V}}$ is an affine local construction on $\mathfrak{X}$.*

Proof. The functor $Iso^{\mathcal{O},\mathfrak{X}}_{\mathcal{M},\mathcal{V}}$ is naturally identified with the fiber product of the pair of morphisms

$$G^{\mathcal{O},\mathfrak{X}}_{\mathcal{M},\mathcal{P}} \xrightarrow{\varphi} Hom_{\mathcal{O},\mathfrak{X}}(\mathcal{M},\mathcal{P}) \times Hom_{\mathcal{O},\mathfrak{X}}(\mathcal{P},\mathcal{M}) \xleftarrow{\psi} G^{\mathcal{O},\mathfrak{X}}_{\mathcal{P},\mathcal{M}}, \tag{11}$$

where φ is the natural embedding, ψ is the composition of the natural imbedding

$$G^{\mathcal{O},\mathfrak{X}}_{\mathcal{P},\mathcal{M}} \xrightarrow{\varphi} Hom_{\mathcal{O},\mathfrak{X}}(\mathcal{P},\mathcal{M}) \times Hom_{\mathcal{O},\mathfrak{X}}(\mathcal{M},\mathcal{P})$$

and the isomorphism

$$Hom_{\mathcal{O},\mathfrak{X}}(\mathcal{P},\mathcal{M}) \times Hom_{\mathcal{O},\mathfrak{X}}(\mathcal{M},\mathcal{P}) \xrightarrow{\sim} Hom_{\mathcal{O},\mathfrak{X}}(\mathcal{M},\mathcal{P}) \times Hom_{\mathcal{O},\mathfrak{X}}(\mathcal{P},\mathcal{M})$$

defined by $(u,v) \longmapsto (v,u)$. By 8.3.3.2, the diagram (11) is isomorphic to the diagram

$$G^{\mathcal{O},\mathfrak{X}}_{\mathcal{M},\mathcal{P}} \longrightarrow \mathbb{V}_{\mathcal{O},\mathfrak{X}}(\mathcal{M}\otimes\mathcal{P}^\vee \oplus \mathcal{P}\otimes\mathcal{M}^\vee) \longleftarrow G^{\mathcal{O},\mathfrak{X}}_{\mathcal{P},\mathcal{M}}. \tag{12}$$

By 8.3.4.1, the functors $G^{\mathcal{O},\mathfrak{X}}_{\mathcal{M},\mathcal{P}}$ and $G^{\mathcal{O},\mathfrak{X}}_{\mathcal{P},\mathcal{M}}$ are affine local constructions on $\mathfrak{X}$. Therefore the pull-back of (5) is an affine local construction on $\mathfrak{X}$. ■

8.4. Grassmannians. For any associative unital ring S and any pair $M,\ P$ of left S-modules, we define the 'Grassmannian' $Gr^S_{M,P}$ as the functor $S\backslash\mathbf{Rings} \longrightarrow Sets$ which assigns to every ring morphism $S \xrightarrow{\phi} T$ the isomorphism class of coretractions (– splittable epimorphisms) $\phi^*(M) \longrightarrow \phi^*(P)$. We have a canonical exact sequence of functors

$$\mathfrak{R}^S_{M,P} \ \underset{p_2}{\overset{p_1}{\rightrightarrows}}\ G^S_{M,P} \ \xrightarrow{\ \pi\ }\ Gr^S_{M,P}. \tag{1}$$

Here π is the natural epimorphism (sending a split pair of arrows $\phi^*(P) \xrightarrow{v} \phi^*(M) \xrightarrow{u} \phi^*(P)$ to the class of the epimorphism u) and $\mathfrak{R}^S_{M,P}$ is the 'functor of relations', i.e. $\mathfrak{R}^S_{M,P} = G^S_{M,P} \prod_{Gr^S_{M,P}} G^S_{M,P}$.

Fix a ringed category $(\mathcal{A}, \mathcal{O})$ and a fibered category $\mathfrak{X}$ over $\mathcal{A}$. Let $\mathcal{M}$ and $\mathcal{P}$ be $\mathcal{O}$-modules on $\mathfrak{X}$. Since the functors $Gr^S_{M,P}$ and $\mathfrak{R}^S_{M,P}$ (as well as $G^S_{M,P}$) in the diagram (1) depend functorially on $S,\ M$, and P, they determine functors $\mathfrak{X} \longrightarrow \widetilde{\mathcal{E}}^\wedge \times_{\mathcal{E}} \mathcal{A}$ which we denote resp. by $Gr^{\mathcal{O},\mathfrak{X}}_{\mathcal{M},\mathcal{V}}$ and $\mathfrak{R}^{\mathcal{O},\mathfrak{X}}_{\mathcal{M},\mathcal{V}}$. Thus, the diagram (1) induces an exact diagram

$$\mathfrak{R}^{\mathcal{O},\mathfrak{X}}_{\mathcal{M},\mathcal{V}} \ \underset{p_2}{\overset{p_1}{\rightrightarrows}}\ G^{\mathcal{O},\mathfrak{X}}_{\mathcal{M},\mathcal{V}} \ \xrightarrow{\ \pi\ }\ Gr^{\mathcal{O},\mathfrak{X}}_{\mathcal{M},\mathcal{V}} \tag{2}$$

of functors $\mathfrak{X} \longrightarrow \widetilde{\mathcal{E}}^\wedge \times_{\mathcal{E}} \mathcal{A}$.

8.4.1. Proposition. *Let $\mathcal{M}(U,x)$ and $\mathcal{P}(U,x)$ be projective $\mathcal{O}(U)$-modules of finite type for all $U \in Ob\mathcal{A}$ and $x \in Ob\mathfrak{X}$. Then (2) is the exact diagram of local constructions on $\mathfrak{X}$ (in particular, $Gr^{\mathcal{O},\mathfrak{X}}_{\mathcal{M},\mathcal{P}}$ is a local construction), and the local constructions $\mathfrak{R}^{\mathcal{O},\mathfrak{X}}_{\mathcal{M},\mathcal{V}}$ and $G^{\mathcal{O},\mathfrak{X}}_{\mathcal{M},\mathcal{V}}$ are affine.*

Proof. If the S-modules M and V are projective of finite type, then, by [KR3, 10.1.3], the functor $\mathfrak{R}^S_{M,P}$ (and $G^S_{M,P}$) is representable. By [KR3, 10.1.4], all three functors in (1) are compatible with the base change. The latter means that if $S \xrightarrow{\phi} T$ is a ring morphism, and $\mathfrak{G}^S_{M,P}$ denotes any of the functors in the diagram (1), then

$$\mathfrak{G}^T_{\phi^*(M),\phi^*(P)} \simeq \mathfrak{G}^S_{M,P} \prod_{\mathbf{Spec}S} \mathbf{Spec}T.$$

This implies the assertion. ∎

Appendix 1: Q-categories and quasi-topologies.

The following is a glossary extracted from the first 3 sections of [KR3].

A1.1. Q-categories and Q^{o}categories. A Q-category is a pair, $\bar{A} \underset{u^*}{\overset{u_*}{\rightleftarrows}} A$, of functors such that the functor u^* is fully faithful and left adjoint to u_*, which implies that A is a quotient category of $\bar{A}$ and u_* is a localization functor.

A *morphism* from a Q-category $\bar{A} \overset{u}{\rightleftarrows} A$ to a Q-category $\bar{A}' \overset{u'}{\rightleftarrows} A'$ is a triple $(\Phi, \bar{\Phi}, \phi)$, where $\Phi : A \longrightarrow A'$ and $\bar{\Phi} : \bar{A} \longrightarrow \bar{A}'$ are functors and ϕ is a functor isomorphism $\Phi u_* \longrightarrow u'_* \bar{\Phi}$. The composition of two morphisms, is defined by

$$(\Phi', \bar{\Phi}', \phi') \circ (\Phi, \bar{\Phi}, \phi) = (\Phi'\Phi, \bar{\Phi}'\bar{\Phi}, \bar{\Phi}'\phi \circ \phi'\Phi)$$

A Q^o-*category* is a pair of functors $\bar{A} \underset{u^*}{\overset{u_*}{\rightleftarrows}} A$ such that the functor u^* is fully faithful and a right adjoint to u_*. In other words, the pair of functors $\bar{A} \underset{u^*}{\overset{u_*}{\rightleftarrows}} A$ is a Q^o-category iff the pair of dual functors, $\bar{A}^{op} \xrightarrow{u_*^{op}} A^{op} \xrightarrow{u^{*op}} \bar{A}^{op}$, is a Q-category.

A1.2. Examples of Q-categories and Q^o-categories.

A1.2.1. The Q-category of cosieves. Fix a category A. Denote by $\mathbf{S}A$ the *category of cosieves* on A defined as follows. Objects of $\mathbf{S}A$ are pairs (x, R), where $x \in ObA$ and R is a cosieve in $x\backslash A$. Morphisms from (x, R) to (x', R') are given by morphisms $x \xrightarrow{f} x'$ such that $R'_f \subseteq R$. Here R'_f is a cosieve in $x\backslash A$ objects of which are all pairs $(v, \xi \circ f)$ such that $(v, \xi) \in ObR'$. There is a functor $u^* : A \longrightarrow \mathbf{S}A$ which assigns to each object x of A the pair $(x, x\backslash A)$. The functor u^* is fully faithful and has a canonical right adjoint,

$$u_* : \mathbf{S}A \longrightarrow A, \quad (x, R) \longmapsto x.$$

This defines a Q-category of cosieves, $\mathbf{S}A \rightleftarrows A$.

Cosieves in $x\backslash A$ are in a natural one-to-one correspondence with subfunctors of the functor $A(x, -)$. Thus the Q-category of cosieves is isomorphic to a Q-category, $\bar{A} \rightleftarrows A$, defined as follows. Objects of $\bar{A}$ are pairs (x, R), where $x \in ObA$, R is a subfunctor of $A(x, -)$. Morphisms from (x, R) to (y, S) are morphisms $x \xrightarrow{f} y$ such that the morphism $A(f, -) : A(y, -) \longrightarrow A(x, -)$ induces a morphism $S \longrightarrow R$ of the subfunctors. The functor u_* maps a pair (x, R) to x. The functor u^* assigns to any object x of A the pair $(x, A(x, -))$.

A1.2.2. Quasi-(co)sites and (co)sites. Let A be a category, and let $\mathfrak{T}$ be a map which assigns to every object x of A a set $\mathfrak{T}(x)$ of subfunctors of $A(x, -)$ which contains $A(x, -)$ itself. We identify the pair $(A, \mathfrak{T})$ with the full Q-subcategory $\bar{A}_{\mathfrak{T}} \rightleftarrows A$ of the Q-category $\mathbf{S}A \rightleftarrows A$ of cosieves objects of which are all pairs (x, R), where $x \in ObA$ and $R \in \mathfrak{T}(x)$.

We call $\mathfrak{T}$ a *quasi-cotopology*, and the pair $(A, \mathfrak{T})$ a *quasi-cosite*, if the following two conditions hold:

(a) for any pair $R,\ R' \in \mathfrak{T}(x)$, $R \cap R' \in \mathfrak{T}(x)$,

(b) if $R \in \mathfrak{T}(x)$ and R' is a subfunctor of $A(x,-)$ containing R, then $R' \in \mathfrak{T}(x)$.

Quasi-topologies and *quasi-sites* correspond to quasi-cotopologies and quasi-cosites on the dual category A^{op}.

Grothendieck sites are quasi-sites. Recall that a site is a pair $(A, \mathfrak{T})$, where $\mathfrak{T}$ is a *topology*, i.e. a map which assigns to each $x \in ObA$ a set $\mathfrak{T}(x)$ of subfunctors of $A(-,x)$ (called *refinements of* x) satisfying the conditions:

(i) for any $R \in \mathfrak{T}(x)$ and any arrow $f : y \longrightarrow x$, the subfunctor $R_f = R \times_{A(-,x)} A(-,y)$ of $A(-y)$ is a refinement of y (i.e. it belongs to $\mathfrak{T}(y)$.

(ii) If $R \in \mathfrak{T}(x)$ and $\widetilde{R}$ is a subfunctor of $A(-,x)$ such that $\widetilde{R}_f \in \mathfrak{T}(y)$ for any $f \in R(y),\ y \in ObA,$, then $\widetilde{R} \in \mathfrak{T}(x)$.

Q-categories dual to Grothendieck sites are called *cosites*. The Q-category of cosieves, $(\mathbf{S}A \rightleftarrows A)$ and its Q-subcategory $\mathbb{A}_{dis} = (\bar{A}_{dis} \rightleftarrows A)$, where $\bar{A}_{dis}$ is formed by all pairs $(x, x\backslash A),\ x \in ObA$, are two extreme examples of cosites.

Cosites might be regarded as a topological data in terms of “closed sets”. In particular, if $(\bar{A} \overset{u}{\rightleftarrows} A)$ is a cosite, then A might be viewed as the category of closed sets of a would-be space.

A1.2.3. A quasi-cosite associated with a Q-category. Fix a Q-category $(\bar{A} \overset{u}{\rightleftarrows} A)$. To any $\bar{y} \in Ob\bar{A}$, we assign the comma category $\bar{y}\backslash u^*$ of pairs (f,x), where f is a morphism $\bar{y} \longrightarrow u^*(x)$. The functor u_* induces a morphism, $\Phi^\sim = (\Phi, id_{u_*}, Id_A)$, from $\mathbb{A} = (\bar{A} \overset{u}{\rightleftarrows} A)$ to the cosite $\mathbf{S}A \rightleftarrows A$. Here Φ is a functor $\bar{A} \longrightarrow \mathbf{S}A$ which assigns to any object $\bar{y}$ of $\bar{A}$ the pair $(u_*(\bar{y}), R_{\bar{y}})$, where $R_{\bar{y}}$ denotes the cosieve in $u_*(\bar{y})\backslash A$ formed by all $(v, u_*(\bar{y}) \overset{\xi}{\to} v)$ such that $\xi = \eta_u^{-1}(v) \circ \bar{\xi}$ for some $\bar{\xi} : \bar{y} \longrightarrow u^*(v)$. The *quasi-cosite,* $\mathfrak{T}\mathbb{A} = (\mathfrak{T}_{\mathbb{A}} \rightleftarrows A)$*, associated with* $\mathbb{A}$ is the smallest quasi-cosite containing the image of the functor Φ. The triple (Id_A, Φ, id) is a canonical morphism from $\mathbb{A}$ to the Q-category $(\mathbf{S}A \rightleftarrows A)$ of cosieves on A.

A1.2.3.1. Note. If $\mathbb{A}$ is a quasi-cosite, then $\mathfrak{T}\mathbb{A}$ is naturally isomorphic to $\mathbb{A}$.

Dually, with every Q^o-category $\mathbb{A}$, one can associate a quasi-site which is naturally isomorphic to $\mathbb{A}$ if $\mathbb{A}$ is a quasi-site.

Suppose $\mathbb{A}$ has the property:

(*) for any $\bar{y} \in Ob\bar{A}$ and any morphism $x \xrightarrow{f} u_*(\bar{y})$, there exists a morphism $\bar{f}$: $\bar{x} \longrightarrow \bar{y}$ and an isomorphism $\alpha : u_*(\bar{x}) \longrightarrow x$ such that $u_*(\bar{f}) = f \circ \alpha$.

Then the quasi-cosite associated with $\mathbb{A}$ is a cosite.

A1.2.4. Quasi-pretopologies. Let A be a category and τ a function which assigns to each object x of A a family, τ_x, of sets of arrows to x ('covers' of x) which contains $\{x \overset{id_x}{\to} x\}$. This data defines a category, A_τ, whose objects are all pairs $(x, \mathcal{U})$, where $x \in ObA$, $\mathcal{U} \in \tau_x$; we shall call them *covers.* Morphisms from $(x, \mathcal{U})$ to $(y, \mathcal{V})$ are morphisms $x \xrightarrow{f} y$ such that any arrow $x_u \overset{u}{\to} x$ in $\mathcal{U}$ factors through an arrow of $\mathcal{V}$. The functor $A_\tau \longrightarrow A$ which assigns to every pair $(x, \mathcal{U})$ the object x and to every morphism $(x, \mathcal{U}) \xrightarrow{f}$

$(y,\mathcal{V})$ the morphism $x \xrightarrow{f} y$ is right adjoint to the fully faithful functor $A \longrightarrow A_\tau$ which maps every object x of A to $(x, \{id_x\})$. This defines a Q^o-category $\mathbb{A}_\tau = (A_\tau \rightleftarrows A)$.

The function τ is called *quasi-pretopology* if the following conditions hold:

(a) The composition of covers is a cover; i.e. if $\{x_u \to x \mid u \in \mathcal{U}\}$ is a cover (i.e. it belongs to τ_x), and $\{x_u^\nu \to x_u \mid \nu \in \mathcal{U}_u\}$ is a cover for any $u \in \mathcal{U}$, then the set of compositions of arrows, $\{x_u^\nu \to x \mid u \in \mathcal{U},\ \nu \in \mathcal{U}_u\}$, is a cover.

(b) For any two covers, $\mathcal{U}$, $\mathcal{U}'$, of x, there exists a third cover, $\mathcal{U}''$, and morphisms of covers $\mathcal{U} \leftarrow \mathcal{U}'' \to \mathcal{U}'$.

The dual notion carries an awkward name *quasi-precotopology.*

A1.2.4.1. One of the main fpqc-cocovers. Let A be the category Alg_k of associative unital k-algebras. We call a finite set of k-algebra morphisms $\{R \xrightarrow{\phi_i} R_i \mid i \in J\}$ an **fpqc**-*cocover* of R if $\{R_i \otimes_R - \mid i \in J\}$ is a conservative family of exact functors. The class of **fpqc**-cocovers forms a quasi-precotopology on Alg_k, or a quasi-pretopology on the dual category $\mathbf{Aff}_k$ of affine noncommutative k-schemes.

A1.2.4.2. Quasi-topology associated with a quasi-pretopology. Consider the quasi-site $\mathfrak{T}\mathbb{A}_\tau$ associated with $\mathbb{A}_\tau$. The functor $\Phi : A_\tau \longrightarrow \mathfrak{T}A_\tau$ assigns to every cover $(x,\mathcal{U})$ the pair $(x, R_{\mathcal{U}})$, where $R_{\mathcal{U}}$ is the sieve associated with the set of arrows $\mathcal{U}$: it consists of all arrows to x which factor through some of the arrows of $\mathcal{U}$.

If for any morphism $f : y \longrightarrow x$ and any $\mathcal{U} \in \tau_x$, there exists $\mathcal{V} \in \tau_y$ such that f is a morphism $(y,\mathcal{V}) \longrightarrow (x,\mathcal{U})$, then the quasi-site associated with $\mathbb{A}_\tau$ is a site. In particular, if τ is a Grothendieck pretopology, we obtain this way the site associated with τ.

A1.2.4.3. Covers. Let $\mathbb{A} = (\bar{A} \overset{u}{\rightleftarrows} A)$ be a quasi-site. A set of arrows $\mathcal{U} = \{x_i \to x \mid i \in J\}$ in A is called a *cover* (or an $\mathbb{A}$-cover) of x, if the pair $(x, R_{\mathcal{U}})$, where $R_{\mathcal{U}}$ is the sieve associated to $\mathcal{U}$, is an object of $\bar{A}$.

It follows from the definition of a quasi-site that

(i) every set of arrows to x which contains a cover is a cover;

(ii) if $\mathcal{U}$ and $\mathcal{U}'$ are covers of x, then $\mathcal{U} \times_x \mathcal{U}' = \{x_u \times_x x_v \longrightarrow x \mid u \in \mathcal{U},\ v \in \mathcal{U}'\}$ is a cover of x, provided the pull-backs $x_u \times_x x_v$ exist for all $u \in \mathcal{U},\ v \in \mathcal{U}'$.

A1.2.5. The Q-category and the $\mathbf{Q}^o$-category of morphisms of a category. Fix a category A. Consider the category A^2 objects of which are morphisms of the category A, and morphisms from $x \xrightarrow{f} y$ to $x' \xrightarrow{f'} y'$ are commutative squares

$$\begin{tikzcd} x \arrow[r, "g"] \arrow[d, "f"'] & x' \arrow[d, "f'"] \\ y \arrow[r, "h"] & y' \end{tikzcd} \qquad (1)$$

Denote by u^* the functor $A \longrightarrow A^2$ which assigns to any object x of A the object $x \xrightarrow{id_x} x$ and to any morphism f the corresponding commutative square. The functor u^* is fully faithful and has a right adjoint, u_*, which maps any object $x \xrightarrow{f} y$ of A^2 to x

and any morphism (1) to $x \xrightarrow{g} x'$. In fact, $u_* u^* = Id_A$, and there is a natural morphism $\epsilon_u : u^* u_* \to Id_{A^2}$ which assigns to any object $x \xrightarrow{f} y$ of the category A^2 the morphism

$$\begin{array}{ccc} x & \xrightarrow{id_x} & x \\ {\scriptstyle id_x}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ x & \xrightarrow{f} & y \end{array}$$

from $u^* u_*(x \xrightarrow{f} y)$ to $(x \xrightarrow{f} y)$. One can see that $id : Id_A \to u_* u^*$ and ϵ_u are adjunction morphisms.

Dually, the functor u^* has a natural left adjoint, $u_!$, which assigns to any object $x \xrightarrow{f} y$ of A^2 the object y and to any morphism (1) the morphism $y \xrightarrow{h} y'$.

A1.2.5.1. Q-subcategories of $(A^2 \rightleftarrows A)$**.** Let $\bar{A}$ be a full subcategory of the category A^2 which contains all objects $x \xrightarrow{id_x} x$. Then the functor u^* takes values in the subcategory $\bar{A}$, hence it induces a structure of a Q-subcategory, $\mathbb{A} = (\bar{A} \overset{u}{\rightleftarrows} A)$, of the Q-category $A^2 \rightleftarrows A$. The functor $u_! : A^2 \longrightarrow A$ induces a functor $\bar{A} \longrightarrow A$ left adjoint to u^*.

A1.2.6. The Q-category of infinitesimal algebra epimorphisms. Let A be the category Alg_k of associative unital k-algebras, and let $\bar{A}$ be the full subcategory of the category Alg_k^2 of k-algebra morphisms whose objects are epimorphisms with a nilpotent kernel.

A1.2.6.1. The commutative version of A1.2.6 (i.e. A is the category $CAlg_k$ of commutative algebras and $\bar{A}$ is the category of commutative algebra epimorphisms with nilpotent kernel) can be interpreted as the category of infinitesimal extensions of affine schemes over k.

A1.2.6.2. The Q-category of thickennings of a scheme. A non-affine version of the example A1.2.6.1 is the Q-category of thickennings of a scheme. Fix a scheme $\mathbf{X}$. Let A be the category of (Zariski) open subschemes of $\mathbf{X}$, and $\bar{A}$ the category of thickennings: objects of the category $\bar{A}$ are nilpotent scheme closed immersions $U \longrightarrow T$, where U is any open subscheme of $\mathbf{X}$. The fully faithful functor $u^* : A \longrightarrow \bar{A}$, $U \longmapsto (U \xrightarrow{id_U} U)$, is left adjoint to the functor $u_* : \bar{A} \longrightarrow A$ sending an immersion $U \longrightarrow T$ to U.

A1.2.8. Q-categories of functors. Fix a category C. To any Q-category $\mathbb{A} = (\bar{A} \overset{u}{\rightleftarrows} A)$, we assign a Q-category $C^{\mathbb{A}} = (C^{\bar{A}} \overset{C^u}{\rightleftarrows} C^A)$. Here C^A denotes the category of functors $A \longrightarrow C$ and C^u is a morphism with the inverse image functor

$$C^{u_*} : C^A \longrightarrow C^{\bar{A}}, \ F \longmapsto F \circ u_*,$$

and the direct image functor $C^{u^*} : G \longmapsto G \circ u^*$. If C is the category **Sets**, we shall write $\mathbb{A}^\vee = (\bar{A}^\vee \overset{u^\vee}{\rightleftarrows} A^\vee)$ instead of $\mathbf{Sets}^{\mathbb{A}} = (\mathbf{Sets}^{\bar{A}} \rightleftarrows \mathbf{Sets}^A)$.

A1.3. Sheaves, monopresheaves and epipresheaves in a Q-category. Given a Q-category $\mathbb{A} = (\bar{A} \underset{u^*}{\overset{u_*}{\rightleftarrows}} A)$, an object x of the category A is called an $\mathbb{A}$-*sheaf* if the canonical map

$$\bar{A}(\bar{y}, u^*(x)) \longrightarrow A(u_*(\bar{y}), x), \quad g \longmapsto \eta_x^{-1} \circ u_*(g), \tag{1}$$

is an isomorphism for all $\bar{y} \in Ob\bar{A}$. Here η_u is an adjunction isomorphism $Id_A \xrightarrow{\sim} u_* u^*$.

We denote by $\mathfrak{F}\mathbb{A}$ the full subcategory of the category A generated by $\mathbb{A}$-sheaves.

A1.3.1.2. $\mathbb{A}$-monopresheaves. We call an object x of the category A an $\mathbb{A}$-*monopresheaf*, or an $\mathbb{A}$-*separated presheaf*, if the canonical map (1) is injective for any $\bar{y} \in Ob\bar{A}$. We denote by $\mathfrak{M}\mathbb{A}$ the full subcategory of A formed by $\mathbb{A}$-monopresheaves.

A1.3.1.3. The canonical morphism ρ_u. Let x be an object of the category A such that the functor $A(u_*(-), x)$ is representable, i.e. $A(u_*(-), x) \simeq \bar{A}(-, u^!(x))$ for some $u^!(x) \in Ob\bar{A}$. There is a canonical morphism $\rho_u(x) : u^*(x) \longrightarrow u^!(x)$ corresponding to the isomorphism $\eta_u^{-1}(x) : u_* u^*(x) \longrightarrow x$. It follows from the definitions that x is an $\mathbb{A}$-monopresheaf iff the morphism $\rho_u(x)$ is a monomorphism. Note, however, that x can be a monopresheaf without the functor $A(u_*(-), x)$ being representable.

A1.3.1.4. $\mathbb{A}$-epipresheaves. We call an object x of A an $\mathbb{A}$-*epipresheaf* if the functor $A(u_*(-), x)$ is representable and the canonical morphism $\rho_u(x) : u^*(x) \longrightarrow u^!(x)$ (cf. A1.3.1.3) is a strict epimorphism.

We denote by $\mathfrak{E}\mathbb{A}$ the full subcategory of the category A formed by $\mathbb{A}$-epipresheaves.

It follows from A1.3.1.3 that an object x of A is an $\mathbb{A}$-sheaf iff it is an $\mathbb{A}$-monopresheaf and an $\mathbb{A}$-epipresheaf.

A1.4. Examples.

A1.4.1. Sheaves on a quasi-pretopology. Suppose A is a category with fibered products and τ a quasi-pretopology on A identified with the corresponding Q-category $\mathbb{A}_\tau = (\bar{A}_\tau \rightleftarrows A)$ (cf. A1.2.4). A presheaf F of sets on A is a sheaf iff for any τ-cover $\{x_u \longrightarrow x \mid u \in \mathcal{U}\}$ the canonical diagram

$$F(x) \longrightarrow \prod_{u \in \mathcal{U}} F(x_u) \rightrightarrows \prod_{u,v \in \mathcal{U}} F(x_u \times_x x_v)$$

is exact. A presheaf F on A with values in a category C is a sheaf in $C^{\mathbb{A}_\tau^{op}}$ iff the presheaf of sets $C(z, F(-))$ is a sheaf for any $z \in ObC$. Thus, if τ is a pretopology, we recover the usual notion of a sheaf.

A1.4.2. Epipresheaves and sheaves on the Q-category of thickennings. Let A be the category of associative unital k-algebras and $\bar{A}$ the category of k-algebra epimorphisms with a nilpotent kernel (see A1.2.6). Epipresheaves of sets on $\mathbb{A} = (\bar{A} \overset{u}{\rightleftarrows} A)$ are formally smooth functors.

Appendix 2: Fibered categories.

Main references are Exposé VI in [SGA1] and Exposé VI in [SGA4]. The purpose of this appendix is to recall basic notions and fix notations. All categories we consider belong to a fixed universum, $\mathfrak{U}$.

A2.1. Categories over a category. Fix a category $\mathcal{E}$. Let $(A, A \xrightarrow{F} \mathcal{E})$ and $(B, B \xrightarrow{G} \mathcal{E}$ be objects of the category $Cat/\mathcal{E}$. For any two morphisms, Φ, Ψ from (A,F) to (B,G) (called *$\mathcal{E}$-functors*), an *$\mathcal{E}$-morphism* $\Phi \to \Psi$ is defined as any functor morphism $\phi : \Phi \longrightarrow \Psi$ such that $G(\phi(x)) = id_{F(x)}$ for all $x \in ObA$. This defines a subcategory, $Hom_{\mathcal{E}}((A,F),(B,G))$, of the category $Hom(A,B)$ of all functors from A to B. The composition

$$Hom(A,B) \times Hom(B,C) \longrightarrow Hom(A,C)$$

induces a composition

$$Hom_{\mathcal{E}}((A,F),(B,G)) \times Hom_{\mathcal{E}}((B,G),(C,H)) \longrightarrow Hom_{\mathcal{E}}((A,F),(C,H)).$$

The map $((A,F),(B,G)) \longmapsto Hom_{\mathcal{E}}((A,F),(B,G))$ defines a functor

$$(Cat/\mathcal{E})^{op} \times Cat/\mathcal{E} \longrightarrow Cat.$$

A2.2. Inner hom. For any two categories $\mathcal{F}$, $\mathcal{G}$ over $\mathcal{E}$ and any category $\mathcal{H}$, there is an isomorphism

$$Hom(\mathcal{H}, Hom_{\mathcal{E}}(\mathcal{F},\mathcal{G})) \xrightarrow{\sim} Hom_{\mathcal{E}}(\mathcal{F} \times \mathcal{H}, \mathcal{G})$$

functorial in all three arguments).

A2.3. Base change. If $\mathcal{F}$ and $\mathcal{E}'$ are two categories over $\mathcal{E}$, $\mathcal{F} \times_{\mathcal{E}} \mathcal{E}'$ denotes their product in $Cat/\mathcal{E}$. Recall that $Ob(\mathcal{F} \times_{\mathcal{E}} \mathcal{E}') = Ob\mathcal{F} \times_{Ob\mathcal{E}} Ob\mathcal{E}'$ and $Hom(\mathcal{F} \times_{\mathcal{E}} \mathcal{E}') = Hom\mathcal{F} \times_{Hom\mathcal{E}} Hom\mathcal{E}'$. Fixing $\gamma : \mathcal{E}' \longrightarrow \mathcal{E}$, we obtain the *base change functor*

$$Cat/\mathcal{E} \longrightarrow Cat/\mathcal{E}', \quad (\mathcal{F} \longmapsto \mathcal{F} \times_{\mathcal{E}} \mathcal{E}', \pi_{\mathcal{E}'}),$$

where $\pi_{\mathcal{E}'}$ is the canonical projection.

For any two categories, $\mathcal{F}$, $\mathcal{G}$, over $\mathcal{E}$, the projection $\mathcal{G} \times_{\mathcal{E}} \mathcal{E}' \longrightarrow \mathcal{G}$ induces a category isomorphism

$$Hom_{\mathcal{E}'}(\mathcal{F} \times_{\mathcal{E}} \mathcal{E}', \mathcal{G} \times_{\mathcal{E}} \mathcal{E}') \longrightarrow Hom_{\mathcal{E}}(\mathcal{F} \times_{\mathcal{E}} \mathcal{E}', \mathcal{G}).$$

The inverse morphism sends any $\mathcal{E}$-functor Φ to the $\mathcal{E}'$-functor $\Phi \times_{\mathcal{E}} \mathcal{E}'$.

A2.3.1. Proposition. *If an $\mathcal{E}$-functor $\Phi : \mathcal{F} \longrightarrow \mathcal{G}$ is fully faithful, then for any base change $\mathcal{E}' \longrightarrow \mathcal{E}$, the corresponding $\mathcal{E}'$-functor $\Phi \times_{\mathcal{E}} \mathcal{E}' : \mathcal{F} \times_{\mathcal{E}} \mathcal{E}' \longrightarrow \mathcal{G} \times_{\mathcal{E}} \mathcal{E}'$ is fully faithful too.*

A2.3.2. Definition. An $\mathcal{E}$-functor $\Phi : \mathcal{F} \longrightarrow \mathcal{G}$ is called an *$\mathcal{E}$-equivalence* if there exists $\mathcal{E}$-functor $\Psi : \mathcal{G} \longrightarrow \mathcal{F}$ and $\mathcal{E}$-isomorphisms $\Phi \circ \Psi \xrightarrow{\sim} Id_{\mathcal{G}}$, $\Psi \circ \Phi \xrightarrow{\sim} Id_{\mathcal{F}}$.

A2.3.3. Proposition. *The following conditions on an $\mathcal{E}$-functor $\Phi : \mathcal{F} \longrightarrow \mathcal{G}$ are equivalent:*

(i) Φ is an $\mathcal{E}$-equivalence.

(ii) For any category $\mathcal{E}'$ over $\mathcal{E}$, the functor $\Phi \times_{\mathcal{E}} \mathcal{E}' : \mathcal{F} \times_{\mathcal{E}} \mathcal{E}' \longrightarrow \mathcal{G} \times_{\mathcal{E}} \mathcal{E}'$ is an equivalence of categories.

(iii) Φ is an equivalence of categories, and for any $X \in Ob\mathcal{E}$, the functor $\Phi_X : \mathcal{F}_X \longrightarrow \mathcal{G}_X$ induced by Φ is an equivalence of categories.

A2.4. Cartesian morphisms. Inverse image functors. Fix a category $\mathcal{E}$ and an object $\mathcal{A} = (A, A \xrightarrow{F} \mathcal{E})$ of the category $Cat/\mathcal{E}$. For any $X \in Ob\mathcal{E}$, we denote by $\mathcal{A}_X$ the *fiber of F in X* which is the subcategory $F^{-1}(id_X)$ of A. For any $f : X \to Y$ of $\mathcal{E}$ and $x, y \in ObA$ such that $F(x) = X,\ F(y) = Y$, we set $\mathcal{A}_f(x, y) := \{\xi : x \to y |\ F(\xi) = f\}$.

A2.4.1. Cartesian morphisms. A morphism $\xi \in A(x, y)$ is called *cartesian* if for any $x' \in Ob\mathcal{A}_X$ and any $\xi' : x' \to y$ such that $F(\xi') = f := F(\xi)$, there exists a unique *X-morphism* $u : x' \to x$ (that is $Fu = id_X$) such that $\xi' = \xi \circ u$. In other words, for any $y \in \mathcal{A}_{F(x)}$, the map

$$\mathcal{A}_X(x', x) \longrightarrow \mathcal{A}_f(x', y), \quad v \longmapsto \xi \circ v,$$

is bijective. This means also that the pair (x, ξ) represents the functor

$$\mathcal{A}_X^{op} \longrightarrow \mathbf{Sets}, \quad x' \longmapsto \mathcal{A}_f(x', y).$$

If for a morphism $f \in \mathcal{E}(X, Y)$, there exists a cartesian morphism $\xi : x \to y$ such that $F(\xi) = f$, then the object x is defined uniquely up to isomorphism and is called *inverse image of y by f*. The standard notation: $x = f^*(y)$. The morphism $\xi : f^*(y) \to y$ is then denoted by ξ_f, or by $\xi_f(y)$.

A2.4.2. Inverse image functor. Suppose an inverse image exists for all $y \in \mathcal{A}_Y$. Then the map $y \longmapsto (f^*(y), \xi_f(y))$ defines a functor $f^* : \mathcal{A}_Y \longrightarrow \mathcal{A}_X$.

In fact, fix objects $y,\ y'$ of $\mathcal{A}_Y$ and a cartesian morphisms $\xi_f(y) : f^*(y) \longrightarrow y$ and $\xi_f(y') : f^*(y') \longrightarrow y'$. For any morphism $\phi : y \to y'$ of $\mathcal{A}_Y$, there exists a unique morphism, $f^*(\phi) : f^*(y) \longrightarrow f^*(y')$, such that the diagram

$$\begin{array}{ccc} f^*(y) & \xrightarrow{\xi_f(y)} & y \\ {\scriptstyle f^*(\phi)}\downarrow & & \downarrow{\scriptstyle \phi} \\ f^*(y') & \xrightarrow{\xi_f(y')} & y' \end{array} \qquad (1)$$

commutes.

A2.4.3. Note. Let

$$\begin{array}{ccc} x & \xrightarrow{\xi} & y \\ {\scriptstyle \psi}\downarrow & & \downarrow{\scriptstyle \phi} \\ x' & \xrightarrow{\xi'} & y' \end{array}$$

be a commutative diagram in A such that $\psi \in Iso\mathcal{A}_X$ and $\phi \in Iso\mathcal{A}_Y$. Then ξ is cartesian iff ξ' is cartesian.

A2.5. Cartesian functors. Let $\mathcal{A} = (A, F)$, $\mathcal{B} = (B, G)$ be $\mathcal{E}$-categories. An $\mathcal{E}$-functor $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$ is called a *cartesian functor* if it transforms cartesian morphisms to cartesian morphisms. The full subcategory of $Hom_{\mathcal{E}}(\mathcal{A}, \mathcal{B})$ formed by cartesian functors is denoted by $Cart_{\mathcal{E}}(\mathcal{A}, \mathcal{B})$.

A2.5.1. Proposition. *(a) Any $\mathcal{E}$-equivalence is a cartesian functor.*

(a') Given an $\mathcal{E}$-equivalence $\Phi : \mathcal{A} = (A, F) \longrightarrow \mathcal{B}$, a morphism ξ of A is cartesian iff $\Phi(\xi)$ is cartesian.

(b) Any $\mathcal{E}$-functor which is isomorphic to a cartesian functor is cartesian.

(c) Composition of cartesian functors is a cartesian functor.

A2.5.2. Corollary. *Let $\Phi : \mathcal{A} = (A, F) \longrightarrow \mathcal{B}$ be an $\mathcal{E}$-equivalence. Then for any $\mathcal{E}$-category $\mathcal{C}$, the functors $\Psi \longmapsto \Psi \circ \Phi$ and $\Psi \longmapsto \Phi \circ \Psi$ induce equivalence of categories:*

$$Cart_{\mathcal{E}}(\mathcal{B}, \mathcal{C}) \xrightarrow{\sim} Cart_{\mathcal{E}}(\mathcal{A}, \mathcal{C})$$

$$Cart_{\mathcal{E}}(\mathcal{C}, \mathcal{A}) \xrightarrow{\sim} Cart_{\mathcal{E}}(\mathcal{C}, \mathcal{B}).$$

A2.5.3. The category $Cart_{\mathcal{E}}$. We denote by $Cart_{\mathcal{E}}$ the category objects of which are same as objects of $Cat/\mathcal{E}$ and morphisms are cartesian functors.

A2.5.3.1. The category $Cart$. We denote by $Cart$ the subcategory of Cat^2 whose objects are same as objects of Cat^2 and morphisms from $A' \xrightarrow{F'} \mathcal{E}'$ to $A \xrightarrow{F} \mathcal{E}$ are commutative diagrams

$$\begin{array}{ccc} A' & \xrightarrow{G} & A \\ {\scriptstyle F'}\downarrow & & \downarrow{\scriptstyle F} \\ \mathcal{E}' & \xrightarrow{G'} & \mathcal{E} \end{array}$$

such that G is a cartisian functor from $(A', G' \circ F')$ to (A, F).

A2.5.4. Colimits. Let B, C be categories and $\mathcal{S}$ a family of morphisms of B. Denote by $Hom_{\mathcal{S}}(B, C)$ the category of functors $B \to C$ which transform morphisms of $\mathcal{S}$ into isomorphisms.

Let $\mathcal{A} = (A, F)$ be a category over $\mathcal{E}$ and $\mathcal{S}_{\mathcal{A}}$ the family of cartesian morphisms of A. The $\mathcal{E}$-category $\mathcal{A}$ defines two functors $Cat \longrightarrow$ **Sets**:

$$C \longmapsto Hom_{\mathcal{S}_{\mathcal{A}}}(A, C), \tag{1}$$

$$C \longmapsto Cart_{\mathcal{E}}(\mathcal{A}, (C \times \mathcal{E}, P_{\mathcal{E}})). \tag{2}$$

Here $P_{\mathcal{E}}$ is the natural projection $C \times \mathcal{E} \longrightarrow \mathcal{E}$.

A2.5.4.1. Lemma. *The functors (1) and (2) are canonically isomorphic.*

Proof. Cartesian morphisms of $C \times \mathcal{E}$ are all morphisms of the form (m, f), where m is an isomorphism of A. ∎

A2.5.4.2. Corollary. *For any $\mathcal{E}$-category $\mathcal{A} = (A, F)$, the functor*

$$Cat \longrightarrow \mathbf{Sets}, \quad C \longmapsto Cart_{\mathcal{E}}(\mathcal{A}, (C \times \mathcal{E}, P_{\mathcal{E}})), \tag{2}$$

is representable by the category $(\mathcal{S}_{\mathcal{A}})^{-1}A$.

A2.5.4.3. Definition. The functor (2) and the category $(\mathcal{S}_{\mathcal{A}})^{-1}A$ representing it are denoted by $Colim\mathcal{A}/\mathcal{E}$ and are called a *colimit of A over $\mathcal{E}$*.

A2.5.5. Limits.

A2.5.5.1. Proposition. *Let $\mathcal{A} = (A, F)$ be a category over $\mathcal{E}$. The functor*

$$Cat \longrightarrow \mathbf{Sets}, \quad C \longmapsto Cart_{\mathcal{E}}((C \times \mathcal{E}, P_{\mathcal{E}}), \mathcal{A}),$$

is representable by the category $Cart_{\mathcal{E}}(\mathcal{E}, \mathcal{A})$ of $\mathcal{E}$-cartesian functors $\mathcal{E} \to \mathcal{A}$.

Proof. Let A be a category and $G : (C \times \mathcal{E}) \longrightarrow \mathcal{A}$ a cartesian functor. For any $z \in ObC$, the functor $\mathcal{E} \longrightarrow \mathcal{A}$, $X \longmapsto G(z, X)$, is cartesian. This gives a map $Cart_{\mathcal{E}}((C \times \mathcal{E}, P_{\mathcal{E}}) \longrightarrow Cart_{\mathcal{E}}(\mathcal{E}, \mathcal{A})$ functorial in A. This map is a bijection. ∎

A2.5.5.2. Definition. The category $Cart_{\mathcal{E}}(\mathcal{E}, \mathcal{F})$ is called the category of *cartesian sections of $\mathcal{A}$ over $\mathcal{E}$*. It is also called a *limit of $\mathcal{A}$ over $\mathcal{E}$* and is denoted by $Lim\mathcal{A}/\mathcal{E}$.

A2.6. Fibered and prefibered categories.

A2.6.1. Definitions. (a) A category $\mathcal{A} = (A, A \xrightarrow{F} \mathcal{E})$ over $\mathcal{E}$ is called *prefibered* if for any morphism $f : X \to Y$ of $\mathcal{E}$, an inverse image functor exists.

(b) A prefibered category $\mathcal{A}$ over $\mathcal{E}$ is called *fibered* if the composition of cartesian morphisms is a cartesian morphism.

A2.6.1.1. The 2-category of fibered categories over $\mathcal{E}$. We denote by $Fib_{/\mathcal{E}}$ the 2-category of fibered categories over $\mathcal{E}$. Its 1-morphisms are cartesian functors and 2-morphisms natural transformation of (cartesian) functors.

A2.6.2. Fibered and prefibered subcategories. Let $\mathcal{A} = (A, F)$ be a fibered (resp. prefibered) category over $\mathcal{E}$. A subcategory B of A is called a *fibered subcategory of $\mathcal{A}$* (resp. a *prefibered subcategory of $\mathcal{A}$*) if $\mathcal{B} = (B, F|_{\mathcal{B}})$ is a fibered (resp. prefibered) category and the inclusion functor is a cartesian functor $\mathcal{B} \longrightarrow \mathcal{A}$.

A2.6.2.1. Lemma. *Let $\mathcal{A} = (A, F)$ be a fibered (resp. prefibered) category over $\mathcal{E}$. If B is a full subcategory of A, then B is a fibered (resp. prefibered) subcategory of $\mathcal{A}$ iff for any morphism $f : X \to Y$ of $\mathcal{E}$ and for any $y \in Ob\mathcal{B}_Y$, the inverse image, $f^*_{\mathcal{A}}(y)$ is $\mathcal{A}_X$-isomorphic to an object of $\mathcal{B}_X$.*

A2.6.2.2. Example. Let $\mathcal{A} = (A, F)$ be a fibered category over $\mathcal{E}$. And let B be a subcategory of A having same objects; morphisms of B are cartesian morphisms of

$\mathcal{A}$. In particular, for any $X \in Ob\mathcal{E}$, morphisms of $\mathcal{B}_X$ are all isomorphisms of $\mathcal{A}_X$. The subcategory B is a fibered subcategory of $\mathcal{A}$.

A2.6.3. Proposition. *Let $F : A \longrightarrow \mathcal{E}$ be a functor. The following conditions are equivalent:*

(a) All morphisms of A are cartesian.

(b) $\mathcal{A} = (A, F)$ is a fibered category and all fibers are groupoids.

If the equivalent conditions (a), (b) hold, (A, F) is called *fibered category of groupoids.*

If $\mathcal{E}$ is a groupoid, then the conditions (a), (b) are equivalent to the condition

(c) The category A is a groupoid, and the functor $F : A \longrightarrow \mathcal{E}$ is *transportable.* The latter means that for any isomorphism $f : X \to Y$ of $\mathcal{E}$ and any object x of $\mathcal{A}_X$, there exists an isomorphism $\xi : x \to y$ such that $F(\xi) = f$.

A2.6.4. Proposition. *Let $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$ be an $\mathcal{E}$-equivalence. Then $\mathcal{A}$ is a fibered (resp. prefibered) category over $\mathcal{E}$ iff $\mathcal{B}$ is such.*

Proof. The assertion follows from the fact that a morphism ξ of A is cartesian iff $\Phi(\xi)$ is cartesian. ∎

A2.6.5. Proposition. *Let $\mathcal{A}_1$, $\mathcal{A}_2$ be categories over $\mathcal{E}$ and let $\xi = (\xi_1, \xi_2)$ be a morphism of $\mathcal{A} = \mathcal{A}_1 \times_{\mathcal{E}} \mathcal{A}_2$. The morphism ξ is cartesian iff ξ_1, ξ_2 are cartesian.*

A2.6.6. Proposition. *Let $\mathcal{A} = \mathcal{A}_1 \times_{\mathcal{E}} \mathcal{A}_2$, and let $\Psi = (\Psi_1, \Psi_2)$ be an $\mathcal{E}$-functor $\mathcal{B} \to \mathcal{A}$. The functor Ψ is cartesian iff Ψ_1 and Ψ_2 are cartesian. Thus one has a category isomorphism*

$$Cart_{\mathcal{E}}(\mathcal{B}, \mathcal{A}_1 \times_{\mathcal{E}} \mathcal{A}_2) \xrightarrow{\ \sim\ } Cart_{\mathcal{E}}(\mathcal{B}, \mathcal{A}_1) \times Cart_{\mathcal{E}}(\mathcal{B}, \mathcal{A}_2)$$

In particular, there is a natural isomorphism of categories

$$Lim(\mathcal{A}_1 \times_{\mathcal{E}} \mathcal{A}_2/\mathcal{E}) \xrightarrow{\ \sim\ } Lim(\mathcal{A}_1/\mathcal{E}) \times Lim(\mathcal{A}_2/\mathcal{E})$$

Proof. The assertion follows from A2.6.5. ∎

A2.6.6.1. Corollary. *Let $\mathcal{A}_1$, $\mathcal{A}_2$ be fibered (resp. prefibered) categories over $\mathcal{E}$. Then $\mathcal{A}_1 \times_{\mathcal{E}} \mathcal{A}_2$ is a fibered (resp. prefibered) category over $\mathcal{E}$.*

A2.6.6.2. Remark. The results above hold for fibered products of any (small) set of categories over $\mathcal{E}$.

A2.6.7. Proposition. *Let $\mathcal{A} = (A, F)$ be a category over $\mathcal{E}$ and $G : \mathcal{E}' \longrightarrow \mathcal{E}$ a functor. Let $\mathcal{A}' = (A', F')$, where $A' = A \times_{\mathcal{E}} \mathcal{E}'$and F' is the projection $A' \to \mathcal{E}'$. A morphism, ξ', of A' is cartesian iff its image, ξ, in A is cartesian.*

A2.6.7.1. Corollary. *For any cartesian functor $\mathcal{F} : \mathcal{A} \longrightarrow \mathcal{B}$ of categories over $\mathcal{E}$ and any functor $G : \mathcal{E}' \longrightarrow \mathcal{E}$, the functor $\mathcal{F}' = \mathcal{F} \times_{\mathcal{E}} \mathcal{E}' : \mathcal{A}' = \mathcal{A} \times_{\mathcal{E}} \mathcal{E}' \longrightarrow \mathcal{B}' = \mathcal{B} \times_{\mathcal{E}} \mathcal{E}'$ is cartesian.*

Thus the functor $Hom_{\mathcal{E}}(\mathcal{A},\mathcal{B})\longrightarrow Hom_{\mathcal{E}'}(\mathcal{A}',\mathcal{B}')$ induces a functor

$$Cart_{\mathcal{E}}(\mathcal{A},\mathcal{B})\longrightarrow Cart_{\mathcal{E}'}(\mathcal{A}',\mathcal{B}').$$

Taking into consideration the canonical isomorphism

$$Hom_{\mathcal{E}'}(\mathcal{A}',\mathcal{B}') \xrightarrow{\sim} Hom_{\mathcal{E}}(\mathcal{A}\times_{\mathcal{E}}\mathcal{E}',\mathcal{B}),$$

one can see that cartesian $\mathcal{E}'$-functors correspond to $\mathcal{E}$-functors $\mathcal{A}\times_{\mathcal{E}}\mathcal{E}'\longrightarrow\mathcal{B}$ transforming any morphism the first projection of which is cartesian into a cartesian morphism of $\mathcal{B}$.

A2.6.7.2. Corollary. *The category $Lim(\mathcal{A}'/\mathcal{E}')$ is isomorphic to the full subcategory of $Hom_{\mathcal{E}}(\mathcal{E}',\mathcal{A})$ formed by $\mathcal{E}$-functors which transform any morphism into a cartesian morphism.*

In particular, if $\mathcal{A}$ is a fibered category over $\mathcal{E}$ and $\mathcal{A}_c$ is the subcategory of $\mathcal{A}$ morphisms of which are all cartesian morphisms of $\mathcal{A}$, then there is a bijection

$$ObLim(\mathcal{A}'/\mathcal{E}') \xrightarrow{\sim} ObHom_{\mathcal{E}}(\mathcal{E}',\mathcal{A}_c).$$

A2.6.8. Proposition. *Let $\mathcal{A}$ be a fibered (resp. prefibered) category over $\mathcal{E}$. Then for any functor $\mathcal{E}'\longrightarrow\mathcal{E}$, $\mathcal{A}' := \mathcal{A}\times_{\mathcal{E}}\mathcal{E}'$ is a fibered (resp. prefibered) category over $\mathcal{E}'$.*

A2.6.9. Proposition. *Let $\mathcal{A}$ and $\mathcal{B}$ be prefibered categories over $\mathcal{E}$, Φ a cartesian functor $\mathcal{A}\longrightarrow\mathcal{B}$. The functor Φ is faithful (resp. fully faithful, resp. $\mathcal{E}$-equivalence) iff for any $X\in Ob\mathcal{E}$, the induced functor $\Phi_X:\mathcal{A}_X\longrightarrow\mathcal{B}_X$ is faithful (resp. fully faithful, resp. an equivalence).*

Proof. The fact follows from definitions. ■

A2.6.10. Proposition. *Let $\mathcal{A}=(A,F)$ be a prefibered category over $\mathcal{E}$. It is fibered iff the following condition holds:*

(Fib) Let $\xi: x\to y$ be a cartesian morphism over $f: X\to Y$ (i.e. $F(\xi)=f$). For any morphism $g: V\to X$ and any $v\in Ob\mathcal{A}_V$, the map

$$Hom_g(v,x)\longrightarrow Hom_{fg}(v,x),\quad u\longmapsto \xi\circ u,$$

is bijective.

A2.6.10.1. Corollary. *Let $\mathcal{A}=(A,F)$ be a category over $\mathcal{E}$, ξ a morphism of A.*

(a) If ξ an isomorphism, then ξ is cartesian and $F(\xi)$ is an isomorphism.

(b) If $\mathcal{A}$ is fibered, then the inverse is true.

A2.6.10.2. Corollary. *Let $\xi: x\to y$ and $\alpha: v\to x$ be morphisms of a fibered category $\mathcal{A}$ over $\mathcal{E}$. Suppose ξ is cartesian. Then α is cartesian iff $\xi\circ\alpha$ is cartesian.*

A2.7. Fibered categories and pseudo-functors. A pseudo-functor $\mathcal{E}^{op}\longrightarrow Cat$ is given by the following data:

(i) A map $Ob\mathcal{E}\longrightarrow ObCat,\ X\longmapsto\mathcal{A}_X$.

(ii) A map $Hom\mathcal{E} \longrightarrow HomCat$ which associates to any $f : X \to Y$ a functor $f^* : \mathcal{A}_Y \longrightarrow \mathcal{A}_X$.

(iii) A map which associates to any pair of composable morphisms, $S \xrightarrow{f} T \xrightarrow{g} U$ a functor morphism $c_{g,f} : f^*g^* \longrightarrow (gf)^*$.

This data should satisfy the following conditions:

(a) $c_{f,id_S} = id_{f^*} = c_{id_T,f}$ for any morphism $f : S \to T$ of $\mathcal{E}$,

(b) For any composable morphsims, $f : S \to T,\ g : T \to U,\ h : U \to V$ of $\mathcal{E}$, the diagram

$$\begin{array}{ccc}
f^*g^*h^* & \xrightarrow{c_{f,g}h^*} & (gf)^*h^* \\
{\scriptstyle f^*c_{h,g}}\big\downarrow & & \big\downarrow{\scriptstyle c_{h,gf}} \\
f^*(hg)^* & \xrightarrow{c_{hg,f}} & (hgf)^*
\end{array}$$

commutes.

Pseudo-functors $\mathcal{E}^{op} \longrightarrow Cat$ form a category defined in a natural way.

A2.7.1. Prefibered categories and pseudo-functors. Let $\mathcal{A} = (A, F)$ be a prefibered category over $\mathcal{E}$. Then there is a function which assigns to any morphism f of $\mathcal{E}$ its inverse image functor, f^* in such a way that $(id_X)^* = Id_{\mathcal{A}_X}$ for all $X \in Ob\mathcal{E}$.

Let $f : X \to Y$ and $g : V \to X$ be morphisms of $\mathcal{E}$ and y an object of $\mathcal{A}_Y$. There exists a unique V-morphism

$$c_{f,g}(y) : g^*f^*(y) \longrightarrow (fg)^*(y)$$

such that the diagram

$$\begin{array}{ccc}
g^*f^*(y) & \xrightarrow{\xi_g(f^*(y))} & f^*(y) \\
{\scriptstyle c_{f,g}(y)}\big\downarrow & & \big\downarrow{\scriptstyle \xi_f(y)} \\
(fg)^*(y) & \xrightarrow{\xi_{fg}(y)} & y
\end{array}$$

is commutative. These morphisms are functorial in y, i.e. $c_{f,g} = \{c_{f,g}(y) | y \in Ob\mathcal{A}_Y\}$ is a morphism $g^*f^* \longrightarrow (fg)^*$ of functors $\mathcal{A}_Y \longrightarrow \mathcal{A}_V$. One can check that they satisfy the conditions (a), (b) of A2.7.1.

Conversely, let $\mathcal{E}^{op} \longrightarrow Cat,\ X \longmapsto \mathcal{A}_X,\ f \longmapsto f^*$ be a pseudo-functor. Set $Ob\mathcal{A} = \coprod_{X \in Ob\mathcal{E}} \mathcal{A}_X = \{(X, x)|\ X \in Ob\mathcal{E},\ x \in Ob\mathcal{A}_X\}$. A morphism from $\bar{x} = (X, x)$ to $\bar{y} = (Y, y)$ is a pair (f, ξ), where f is a morphism $X \to Y$, ξ a morphism $x \to f^*(y)$. A composition is defined by

$$(f, \xi) \circ (g, \mu) := c_{f,g}(y) \circ g^*(\xi) \circ \mu. \tag{1}$$

Set $h_f(\bar{x}, \bar{y}) := Hom_{\mathcal{A}_X}(x, f^*(y))$ and $\mathcal{A}(\bar{x}, \bar{y}) = \coprod_{f \in \mathcal{E}(X,Y)} h_f(\bar{x}, \bar{y})$. The composition (1) defines the composition on $\mathcal{A}$. The projection functor, $F : \mathcal{A} \longrightarrow \mathcal{E}$, is given by the maps $(X, x) \longmapsto X,\ (f, \xi) \longmapsto f$.

The $\mathcal{E}$-category $F : \mathcal{A} \longrightarrow \mathcal{E}$ is fibered iff all morphisms $c_{f,g}$ are isomorphisms.

A2.8. Limits, Colimits, and pseudo-functors. Let $\mathcal{A} = (A, F)$ be a prefibered category corresponding to a pseudo-functor $\mathcal{E}^{op} \longrightarrow Cat$,

$$Ob\mathcal{E} \ni X \longmapsto \mathcal{A}_X,\ Hom\mathcal{E} \ni f \longmapsto f^*,\ Hom\mathcal{E} \times_{Ob\mathcal{E}} Hom\mathcal{E} \ni (f, g) \longmapsto c_{f,g}. \tag{1}$$

A2.8.1. Colimits of pseudo-functors. The composition of inclusion functors $\mathcal{A}_X \hookrightarrow A$ and the canonical functor $A \longrightarrow Colim\mathcal{A}/\mathcal{E}$ provide for any $X \in Ob\mathcal{E}$ a functor $q_X : \mathcal{A}_X \longrightarrow Colim\mathcal{A}/\mathcal{E}$, and for any morphism $f : X \to Y$ of $\mathcal{E}$ a diagram commutative up to isomorphism:

$$\begin{tikzcd} \mathcal{A}_Y \arrow[rr, "f^*"] \arrow[dr, "q_Y"'] & & \mathcal{A}_X \arrow[dl, "q_X"] \\ & Colim\mathcal{A}/\mathcal{E} & \end{tikzcd}$$

Thus $Colim\mathcal{A}/\mathcal{E}$ is a colimit in the sense of pseudo-functors of the pseudo-functor $\mathcal{E}^{op} \longrightarrow Cat$.

Note that if $X \longmapsto \mathcal{A}_X$, $f \longmapsto f^*$ is a functor, the category $Colim\mathcal{A}/\mathcal{E}$ is not, in general, the colimit of this functor.

A2.8.2. Limits of pseudo-functors. Fix a pseudo-functor

$$\mathcal{E}^{op} \longrightarrow Cat,\ X \longmapsto \mathcal{A}_X,\ f \longmapsto f^*,\ (f,g) \longmapsto c_{f,g}.$$

For any $X \in Ob\mathcal{E}$, denote by p_X the functor $Lim\mathcal{A}/\mathcal{E} \longrightarrow \mathcal{A}_X$ of evaluation at X. For any morphism $f : X \to Y$ of $\mathcal{E}$, there is a diagram

$$\begin{tikzcd} \mathcal{A}_Y \arrow[rr, "f^*"] & & \mathcal{A}_X \\ & Lim\mathcal{A}/\mathcal{E} \arrow[ul, "p_Y"] \arrow[ur, "p_X"'] & \end{tikzcd}$$

commutative up to isomorphism. This means that $Lim\mathcal{A}/\mathcal{E}$ is a limit in the sense of pseudo-functors of the pseudo-functor $\mathcal{E}^{op} \longrightarrow Cat$.

If $X \longmapsto \mathcal{A}_X$, $f \longmapsto f^*$ is a functor, the category $Lim\mathcal{A}/\mathcal{E}$ is not, in general, the limit of this functor.

A2.9. Cofibered and bifibered categories. Fix a category $\mathcal{A} = (A, A \xrightarrow{F} \mathcal{E})$, over $\mathcal{E}$. A morphism $\xi : x \to y$ of A is called *cocartesian* if it is a cartesian morphism of the category $\mathcal{A}^{op} := (A^{op}, F^{op})$ over $\mathcal{E}^{op}$. This means that for any $x' \in Ob\mathcal{A}_Y$, the map $\mathcal{A}_X(y, y') \longrightarrow Hom_f(x, y')$, $u \longmapsto u \circ \xi$, is bijective. In this case, (y, ξ) is called a *direct image of x by f*. If it exists for any $x \in \mathcal{A}_X$, then there exists a *direct image functor* $f_* : \mathcal{A}_X \longrightarrow \mathcal{A}_Y$. It is defined (uniquely up to isomorphism) by an isomorphism of bifunctors

$$\mathcal{A}_Y(f_*(x), y) \xrightarrow{\sim} Hom_f(x, y).$$

A2.9.1. Suppose f_* exists. Then f^* exists iff f_* has a right adjoint.

In fact, the functor f^* is defined (uniquely up to isomorphism) by a functorial isomorphism $\mathcal{A}_X(x, f^*(y)) \xrightarrow{\sim} Hom_f(x, y)$. Therefore we have a functorial isomorphism $\mathcal{A}_X(x, f^*(y)) \simeq \mathcal{A}_Y(f_*(x), y)$.

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