ABOUT NON-PROJECTIVE THREEFOLDS

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Abstract. We show that a smooth complete threefold over $\mathbb{C}$, which is not projective contains a rational curve. This gives a partial answer to a question of J. Kollar.

1. Introduction.

Hironaka in [3] give examples of smooth complete algebraic threefolds, which are not projective. In his construction Hironaka uses blowing up of smooth curves, in particular his examples contain rational curves. In a natural way appears a question - whether a smooth complete algebraic threefold, which is not projective, must contain a rational curve. In [5] (5.17, p. 148) Janos Kollar posed even more general problem:

Conjecture. *Let X be a smooth and proper algebraic space which is not projective. Then X contains a rational curve.*

In this paper we show that this conjecture is true at least for smooth algebraic threefolds over $\mathbb{C}$ (here a threefold means an algebraic variety of dimension three). We also show that if X is a smooth complete threefold, and there is a divisor D which is positive on the cone $NE(X)$ of effective curves of X (i.e., $D.C > 0$ for every integral curve $C \subset X$), then X is projective.

2. Preliminaries.

We start with the following result of Kawamata (see [4], appendix):

Theorem 2.1. *Let X be a normal projective variety with at most canonical singularities and let $f : X \to Y$ be a birational map to a normal projective variety Y. Assume that $-K_X$ is f-ample. Then every fiber of f of positive dimension is uniruled.*

We need also the following result (see [6], 3.54, p. 110):

Theorem 2.2. *Let X_1, X_2 be terminal projective varieties with numerically effective canonical divisors. Then any birational mapping $f : X_1 - \to X_2$ is an isomorphism in codimension 1.*

In the sequel the following result of J. Kollar will be important (it can be deduced from Remark 1.8 and Theorem 1.9 on page 290 in [5]). For the sake of completness we give here a simple proof.

Theorem 2.3. *Let X, Y be normal complete varieties of dimension three. Assume that X is $\mathbb{Q}$-factorial and with terminal singularities. Let $f : X - \to Y$ be a rational mapping. If Y does not contain rational curves then f extends to a morphism $f : X \to Y$.*

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Before we prove Theorem 2.3 we need:

Lemma 2.1. *Let X be a $\mathbb{Q}$-factorial projective threefold with terminal singularities. Let Z be any complete threefold such that there is a birational morphism $p : Z \to X$. Then every fiber of p of positive dimension is uniruled.*

Proof. Since X is $\mathbb{Q}-$factorial, we have that the exceptional set of p is a divisor. First we show that every irreducible component F of E is a uniruled surface. Let Z' be a projective cover of Z. Since it is enough to prove our statement for Z', we can assume that Z is projective. Let $r : X' \to X$ be a resolution of singularities of X. We have a birational mappings $g : X- \to Z$ and $g' : X'- \to Z$ induced by p^{-1}. By a sequence of blowings up's with smooth centers, we can resolve intederminancy of g' to obtain a regular mapping $G : X'' \to Z$. We have the following diagram

$$\begin{tikzcd} X'' \arrow[d, "\pi_m"] \arrow[dddr, "G"] & \\ X_{m-1} \arrow[d, "\pi_{m-1}"] & \\ \vdots \arrow[d, "\pi_1"] & \\ X \arrow[r, dashed, "g"] & Z \end{tikzcd}$$

where $\pi_1 = r$, $X_1 = X'$ and for $j > 1$ the π_j are blowing up's with smooth centers. Let us note that fibers of all π_j with $j > 1$ are projective spaces. By the Minimal Model Program the mapping $\pi_1 = r$ is a composition of divisioral and small contractions (and their inverses). Thus by Theorem 2.1 if π_1 contracts an irreducible divisor D, then D is uniruled. Moreover if the image of D is a curve C then the generic fiber (hence every fiber) of the mapping $r : D \to C$ must be uniruled. Of course the same is true if the image of D is a point. Trivially the same holds for π_j, if $j > 1$.

Now let F be an irreducible component of the exceptional set E of p. Let $F' \subset X''$ be a divisor, such that $G(F') = F$. From the diagram above it is clear that F' is contracted by π. By the preceding remarks it means that there is a divisor F'' birationally equivalent to F which is contracted by some π_j. It is easy to see that a generic fiber of the contraction $\pi_j : F'' \to X_{j-1}$ is uniruled. Hence the same is true for every fiber and we can replace here F'' by F'. This means that fibers of the mapping $p : F \to X$ are dominated by uniruled sets, hence these fibers are uniruled. $\square$

Now we can prove Theorem 2.3.

Proof. Let us consider the graph Z of the mapping f. It is a closed subset of $X \times Y$. Let $p : Z \to X$ and $q : Z \to Y$ be canonical projections. If p is not an isomorphism, then there is an exceptional divisor E, which is uniruled over X. In particular there is a rational curve $\Gamma \subset Z$, such that $p(\Gamma) = point$. This implies that q is an injection on Γ, in particular the variety Y contains a rational curve $q(\Gamma)$. This is a contradiction. Hence the mapping p is an isomorphism and $f = q \circ p^{-1}$ is a morphism. $\square$

At the end of this section we prove one basic lemma:

Lemma 2.2. *Let X be a smooth complete variety and let $U \subset X$ be an open quasi-projective subset of X. Then there exists a smooth projective variety Z and a birational morphism $f : Z \to X$, such that the inverse mapping f^{-1} is defined on an open subset V which contains U. Moreover, codim $X \setminus V \geq 2$.*

Proof. By the Nagata theorem (see [7], Theorem 3.2) there is a projective Z containing U and dominating X. Indeed, take any projective variety X' which contains U, then Z can be obtained as a join of some blowing-ups of X'. We can resolve the singularities of Z, hence we can assume that Z is smooth. Let $p : Z \to X$ be a birational morphism, which is an isomorphism on U. Then p^{-1} defines an open embedding into Z outside the exceptional locus S, which by smoothness and the Zariski Main Theorem, is of codimension ≥ 2 in X. Now it is enough to take $V = X \setminus S$. □

Corollary 2.1. *Let X be a smooth complete non-projective threefold and let $U \subset X$ be a maximal quasi-projective subset of X. Then the set $G := X \setminus U$ has a pure dimension* 1.

Proof. Let $f : Z \to X$ be a birational morphism as above. Assume that G has a point a as an isolated component. Let $V = U \cup \{a\}$ and let $W = f^{-1}(X \setminus \{a\})$. We can glue V and W along $U \cong f^{-1}(U)$ to obtain a new algebraic complete variety Z'. Let you note that we have a birational morphism $f' : Z \to Z'$, which is induced by f and which is an isomorphism outside the set $E = f^{-1}(a)$. Let $p = (f')^{-1}$ and let $D = p^*(O(1))$.

For an integral curve C and a point $P \in C$ we denote by $m_P(C)$ the multiplicity of the point P on a curve C. Let $m(C) = \mathrm{Sup}_{P \in C} m_P(C)$.

Since the linear system given by D has at most one base point a and this system is very ample outside a, it is easy to check that for an integral curve $C \subset Z'$ we have $D.C \geq m(C)$. Hence by the Seshadri theorem ([2], p. 37) we have that D is ample. Consequently Z' is projective and V is quasi-projective, a contradiction with maximality of U. □

3. Main Result.

Now we are ready to prove the following:

Theorem 3.1. *Let X be a smooth complete algebraic threefold which is not projective. Then X contains a rational curve.*

Proof. Assume that X does not contain a rational curve. By the Lemma 2.2 we can cover X by open subsets U_i, $i = 1, ..., r$, such that for each i:

1) $X \setminus U_i$ is a curve, we will denote it by G_i;

2) there is a $\mathbb{Q}$ factorial and terminal projective variety X_i and a birational morphism $f_i : X_i \to X$, which induces an isomorphism $X_i \setminus f_i^{-1}(G_i) \to X \setminus G_i$.

Now we use the Minimal Model Program (see e.g., [6]) to modify varieties X_i in this way, that conditions 1,2 are still fulfilled, but X_i are terminal minimal models. First note that X_i are not uniruled, hence they have not Fano contractions. Now assume that e.g., X_1 is not a minimal model. Hence K_{X_1} is not numerically effective and we can contract some extremal ray. Let $\pi : X_1 \to X_1'$ be a contraction. Now two cases can happen

a) π is a divisorial contraction,

b) π is a small contraction.

In the case a) the variety X_1' is $\mathbb{Q}$ factorial and terminal projective variety. Let $f_1' : X_1' - \to X$ be the induced birational mapping. By Theorem 2.3 the mapping f_1' is in fact a morphism. It is easy to see that this morphism induces an isomorphism $X_1' \setminus (f_1')^{-1}(G_1) \to X \setminus G_1$ and we can set $X_1 = X_1'$ and $f_1 = f_1'$.

Let us consider the case b). Let $\varphi : X_1 - \to X^+$ be a flip and let $f^+ : X^+ - \to X$ be the induced birational mapping. By Theorem 2.3 the mapping f^+ is in fact a morphism. Thus we have a birational morphism $f^+ : X^+ \to X$ and X^+ is $\mathbb{Q}$ factorial and terminal projective variety (see the diagram below).

$$\begin{tikzcd}
X_1 \arrow[rr, dashed, "\varphi"] \arrow[dr, "\pi"] \arrow[ddr, bend right, "f_1"'] & & X^+ \arrow[dl, "\pi^+"'] \arrow[ddl, bend left, "f^+"] \\
& X_1' & \\
& \mathrm{X} &
\end{tikzcd}$$

We show that f^+ is an isomorphism outside the curve $(f^+)^{-1}(G_1)$. Assume that $(f^+)^{-1}$ is not defined at point $a \notin G_1$. By the Zariski Main Theorem (smooth case) the mapping f^+ must contract a divisor H, and $a \in f^+(H)$. Note that by Lemma 2.1 the divisor H is uniruled. Let $H' = \varphi^{-1}(H)$ be a proper transform of H. Of course the divisor H' is also uniruled. Since the flip φ is an ismorphism in codimension 1 and $f_1 = (f^+) \circ \varphi$, we have that $a \in cl(f_1(H')) = f_1(H')$. Consequently $H' \not\subset f_1^{-1}(G_1)$ and since H' is uniruled this is a contradiction. Finally we can set $X_1 = X^+$ and $f_1 = f^+$. By the construction we have:

i) X_1 is a $\mathbb{Q}$ factorial and terminal projective variety and f_1 is a birational morphism $f_1 : X_1 \to X$, which induces an isomorphism $X_1 \setminus f_1^{-1}(G_1) \to X \setminus G_1$.

We can continue our procedure and in a finite number of steps we get that X_1 is a $\mathbb{Q}$ factorial and terminal projective variety with nef canonical divisor. Similarly we can assume that all X_i have the same properties. Take X_1. Since the variety X is not projective, we have that the mapping f_1 is not an isomorphism. In particular it means that it contracts a divisor H. Let $a \in f_1(H)$. There is a number j, such that $a \in U_j$. Consider the mapping $F = f_j^{-1} \circ f_1 : X_1 - \to X_j$. Thus the mapping F is a birational morphism of minimal terminal models, which is not an isomorphism in codimension 1. This is a contradiction (see Theorem 2.2). □

Remark 3.1. In Theorem 3.1 it is enough to assume that X is $\mathbb{Q}-$factorial terminal threefold. The proof is the same.

Remark 3.2. From our proof it follows that Theorem 3.1 holds in dimension n if the MMP holds in this dimension.

At the end of this paper we give another criterion for a smooth threefold X to be projective:

Theorem 3.2. *Let X be a smooth algebraic threefold. Assume that there is a divisor H, such that $H.C > 0$ for every integral curve $C \subset X$. Then X is a projective variety.*

Proof. By the Lemma 2.2 there is an open subset U, such that:

1) $X \setminus U$ is a curve, we will denote it by G;

2) there is a smooth projective variety Z and a birational morphism $f : Z \to X$, which induces an isomorphism $Z \setminus f^{-1}(G) \to X \setminus G$.

Let $g = f^{-1}$ and take $D = g^*(O(1))$. For an integral curve C and a point $P \in C$ we denote by $m_P(C)$ the multiplicity of the point P on a curve C. For a curve C let $m(C) = \mathrm{Sup}_{P \in C} m_P(C)$. It is easy to check that if an integral curve C is not a component of the curve G then $D.C \geq m(C)$. Indeed, for every integral curve $C \not\subset G$ and a point $c \in C$ there is a non-zero section ρ of D (which is a pullback of an appropriate section of $O(1)$ by g) such that $c \in |\rho|$ but $C \not\subset |\rho|$.

Since there is only finite number of curves which are components of the curve G and the divisor H is positive on each integral curve, we have that for $m >> 0$ the line bundle $D_1 = D + mO(H)$ satisfies $D_1.C \geq m(C)$ for every integral curve C. Hence by the Seshadri theorem ([2], p. 37) we have that D_1 is ample. □

Remark 3.3. There is a smooth complete threefold X and an effective divisor D on X such that $D.C > 0$ for every integral curve $C \subset X$ but D is not ample (such an example is constructed e.g. in [2], Example 10.8, p. 57). However Theorem 3.2 shows that anyway the existence of such a divisor (which may be even not-effective) implies that X is projective.

Remark 3.4. Let X be a non-projective smooth threefold. We can cover X by maximal quasi-projective open subsets U_i and consider curves $G_i = X \setminus U_i$. Let $G_i = \bigcup G_{ij}$ be a decomposition of G_i onto irreducible components. We have only finite number of such sets U_i (see [8]). In particular it follows that we have only finite number of "interesting" curves G_{ij}. The proof of Theorem 3.1 suggests that these curves should be rational. It can be proved that they are in some sense isolated in the cone $NE(X)$ of effective curves. I believe that these curves are the key to understanding the geometry of smooth non-projective threefolds.

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