Homotopy and homology of fibred spaces

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Abstract

We study fibred spaces with fibres in a structure category $\mathfrak{F}$ and we show that cellular approximation, Blakers–Massey theorem, Whitehead theorems, obstruction theory, Hurewicz homomorphism, Wall finiteness obstruction, and Whitehead torsion theorem hold for fibred spaces. For this we introduce the cohomology of fibred spaces.

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1 Introduction

A fibred space X is a map

$$p\colon X \to \bar{X}$$

for which fibres $p^{-1}x$, $x \in \bar{X}$, are objects in a structure category $\mathfrak{F}$. For example a G-space X for a topological group G which acts properly on X yields the fibred space

$$p\colon X \to X/G$$

where X/G is the orbit space. Here fibres are objects in the orbit category $\mathfrak{F}_G$ consisting of G-orbits G/H and G-equivariant maps. Moreover fibre bundles [14, 31] and stratified fibre bundles [5] are examples of fibred spaces. In this paper we study the homotopy theory of fibred spaces. In particular we show that fundamental results like Whitehead theorems, cohomology, obstructions to the extension of maps, Wall-finiteness obstruction and Whitehead torsion are available for fibred spaces. These results form the bulk of the book [22] in the case of G-spaces. As it turns out homotopy theory of G-spaces to a large extent is just a specialization of the homotopy theory of fibred spaces.

The theory of $\mathfrak{F}$-fibred spaces yields a new flexibility since the structure category $\mathfrak{F}$ is a parameter of the theory. For example all constructions and obstructions are natural in $\mathfrak{F}$ and such naturality is useful similarly as in the case of naturality of coefficients in cohomology. By comparing the orbit category $\mathfrak{F}_G$ with other structure categories $\mathfrak{F}$ one also obtains applications to the theory of transformation groups. On the other hand our theory can be applied to stratified bundles like stratified vector bundles which appear as tangent bundles of stratified manifolds, see [21, 4, 5]. Obstructions to non trivial sections of stratified vector bundles can be studied by use of the obstruction theory of fibred spaces (such problems were treated in [29]).

A self-contained proof of the results in this paper is highly elaborate and requires a rewriting of the book [22] (more than 400 pages) for fibred spaces. In this paper, however, we prove the results by use of the axiomatic approach of [3]. For this we only need to show that the Blakers–Massey theorem holds in the category of fibred spaces. This implies that we can use the methods in [3] leading, in particular, to a new construction of the cellular chain complex $C_*(X, A)$ which is simpler than the construction of Bredon [6, 7, 33, 22].

Our homotopy theory of fibred spaces with fibres in the (topological enriched) category $\mathfrak{F}$ is closely related to the theory of continuous $\mathfrak{F}^{\mathrm{op}}$-diagrams of spaces (compare with the principal diagram in [5]). In case $\mathfrak{F}$ is discrete such diagrams are treated in chapter A of [3]. It is easy to transform the results in this paper to the theory of continuous diagrams by use of [3].

The methods of [3], in fact, yield further results on fibred spaces not treated in this paper. We leave it to the reader to study the "model lifting property" of the chain complex $C_*(X, A)$, the "obstruction for the realizability" of a chain complex, the "tower of categories" for the classification of homotopy classes of maps between fibred spaces and the associated spectral sequences.

On the other hand we refer to the recent book of Crabb and James [10] for a different perspective on spaces of fibres and for a detailed account of the areas of fibrewise topology and fibrewise homotopy theory.

2 Fibre families

Let $\mathfrak{F}$ be a small category together with a faithful functor $F\colon \mathfrak{F} \to \mathbf{Top}$ to the category $\mathbf{Top}$ of compactly generated Hausdorff spaces. Then $\mathfrak{F}$ is termed a *structure category* and F is the *fibre functor* on $\mathfrak{F}$. In many examples the functor F is actually the inclusion of a subcategory $\mathfrak{F}$ of $\mathbf{Top}$ so that in this case we need not to mention the fibre functor F.

A *fibre family* with fibres in $\mathfrak{F}$ (or a $(\mathfrak{F}, F)$-family) is a topological space

X, termed *total space*, together with a map $p_X \colon X \to \bar{X}$, termed *projection* to the *base space* $\bar{X}$, and for every $b \in \bar{X}$ a selected homeomorphism $\Phi_b \colon p_X^{-1} b \approx F X_b$ where X_b is an object in $\mathfrak{F}$, called *fibre*, depending on $b \in \bar{X}$. The homeomorphism Φ_b is termed *chart* at b. The family $(p_X : X \to \bar{X}, X_b, \Phi_b, b \in \bar{X})$ is denoted simply by X. A fibre family is also termed a *fibred space* with fibres in $\mathfrak{F}$.

Given two $\mathfrak{F}$-families X and Y a $\mathfrak{F}$-*map* from X to Y is a pair of maps $(f, \bar{f})$ such that the following diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p_X"'] & Y \arrow[d, "p_Y"] \\ \bar{X} \arrow[r, "\bar{f}"] & \bar{Y}, \end{tikzcd}$$

commutes, and such that for every $b \in \bar{X}$ the composition given by the dotted arrow of the diagram

$$\begin{tikzcd} p_X^{-1}(b) \arrow[r, "f|p^{-1}(b)"] & p_Y^{-1}(\bar{f}b) \arrow[d, "\Phi_{\bar{f}b}"] \\ FX_b \arrow[u, "\Phi_b^{-1}"] \arrow[r, dotted] & FY_{\bar{f}b} \end{tikzcd}$$

is a morphism in the image of the functor F. That is, there exists a morphism $\phi \colon X_b \to Y_{\bar{f}b}$ in $\mathfrak{F}$ such that the dotted arrow is equal to $F(\phi)$. We will often denote FX_b by X_b and it will be clear from the context whether X_b denotes an object in $\mathfrak{F}$ or a space in **Top** given by the functor F. If a $\mathfrak{F}$-map $f = (f, \bar{f})$ is a $\mathfrak{F}$-isomorphism then f and $\bar{f}$ are homeomorphisms but the converse need not be true.

If X is a $\mathfrak{F}$-family and Z is a topological space, then the product $X \times Z$ in **Top** is a $\mathfrak{F}$-family with projection $p_{X \times Z} = p \times 1_Z \colon X \times Z \to \bar{X} \times Z$. The fibre over a point $(b, z) \in \bar{X} \times Z$ is equal to $p_X^{-1}(b) \times \{z\}$; using the chart $\Phi_b : p_X^{-1}(b) \to X_b$ the chart $\Phi_{(b,z)}$ is defined by $(x, z) \mapsto \Phi_b(x) \in X_b$ where of course we set $(X \times Z)_{(b,z)} = X_b$. In particular, by taking $Z = I$ the unit interval we obtain the cylinder object $IX = X \times I$ and therefore the notion of *homotopy*: two $\mathfrak{F}$-maps $f_0, f_1 \colon X \to Y$ are $\mathfrak{F}$-homotopic (in symbols $f_0 \sim f_1$) if there is a $\mathfrak{F}$-map $F \colon IX \to Y$ such that $f_0 = Fi_0$ and $f_1 = Fi_1$. Here i_0 and i_1 are the inclusions $X \to X \times I$ at the levels 0 and 1 respectively.

Let $\mathfrak{F}$-**Top** be the category consisting of $\mathfrak{F}$-families $p_X : X \to \bar{X}$ in **Top** and $\mathfrak{F}$-maps. Homotopy of $\mathfrak{F}$-maps yields a natural equivalence relation $\sim$ on $\mathfrak{F}$-**Top** so that the homotopy category $(\mathfrak{F}\text{-}\mathbf{Top})/_{\sim}$ is defined.

(2.1) Definition. If V is a fibre in $\mathfrak{F}$ and $\bar{X}$ is a space in **Top**, then the projection onto the first factor $p_1\colon X = \bar{X}\times V \to \bar{X}$ yields the *product family* with fibre V; the charts $\Phi_b\colon \{b\}\times V \to V = X_b$ are given by projection and $X_b = V$ for all $b\in\bar{X}$. If X is a $\mathfrak{F}$-family $\mathfrak{F}$-isomorphic to a product family then X is said to be a *trivial* $\mathfrak{F}$*-bundle*. In general a $\mathfrak{F}$*-bundle* is a locally trivial family of fibres, i.e. a family X over $\bar{X}$ such that every $b\in\bar{X}$ admits a neighborhood U for which $X|U$ is trivial. Here $X|U$ is the *restriction* of the family X defined by $U\subset\bar{X}$.

Given a family Y with projection $p_Y\colon Y\to\bar{Y}$ and a map $\bar{f}\colon\bar{X}\to\bar{Y}$, the pull-back $X=\bar{f}^*Y$ is the total space of a family of fibres given by the vertical dotted arrow of the following pull-back diagram in **Top**.

$$\begin{tikzcd} X = \bar{f}^*Y \arrow[r, dotted] \arrow[d, dotted] & Y \arrow[d, "p"] \\ \bar{X} \arrow[r, "\bar{f}"] & \bar{Y}. \end{tikzcd} \tag{2.2}$$

The charts are defined as follows: For every $b\in\bar{X}$ let $X_b = Y_{\bar{f}b}$, and Φ_b : $p_X^{-1}(b)\to X_b$ the composition $p_X^{-1}(b)\to p_Y^{-1}(\bar{f}b)\approx Y_{\bar{f}b} = X_b$ where the map $p_X^{-1}(b)\to p_Y^{-1}(\bar{f}b)$ is a homeomorphism since X is a pull-back.

A $\mathfrak{F}$-map $i\colon A\to Y$ is termed a *closed inclusion* if $\bar{i}\colon\bar{A}\to\bar{Y}$ is an inclusion, $\bar{i}\bar{A}$ is closed in $\bar{Y}$ and the following diagram is a pull-back:

$$\begin{tikzcd} i^*Y = A \arrow[r, "i"] \arrow[d, "p_A"'] & Y \arrow[d, "p_Y"] \\ \bar{A} \arrow[r, "\bar{i}"] & \bar{Y} \end{tikzcd} \tag{2.3}$$

Hence a closed inclusion $i\colon A\to Y$ induces homeomorphisms on fibres.

The push-out construction can be extended to the category $\mathfrak{F}$-**Top**, provided the push-out is defined via a closed inclusion, see [5] (2.5).

(2.4) Lemma. *Given* $\mathfrak{F}$*-families* A, X, Y *and* $\mathfrak{F}$*-maps* $f\colon A\to X$, $i\colon A\to Y$ *with* i *a closed inclusion the push-out diagram in* $\mathfrak{F}$-**Top**

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

exists and $X\to Z$ *is a closed inclusion.*

Let $A \subset Y$ be a closed inclusion then also $A \times I \subset Y \times I$ is a closed inclusion and the *relative cylinder* $I_A Y$ is defined by the push-out

$$\begin{tikzcd} A \times I \arrow[r, "\mathrm{pr}"] \arrow[d] & A \arrow[d] \\ Y \times I \arrow[r] & I_A Y \end{tikzcd}$$

in $\mathfrak{F}$-**Top**. A map $F\colon I_A Y \to X$ is termed a *homotopy* $f_0 \sim f_1$ rel A with $f_0 = Fi_0$ and $f_1 = Fi_1$. Let $g\colon A \to X$ given. Then

$$[Y, X]^A_{\mathfrak{F}} = [Y, X]^g_{\mathfrak{F}} = \{f\colon Y \to X, f|A = g\} /_{\sim \text{ rel } A} \tag{2.5}$$

denotes the set of homotopy classes relative A or under A. We call a closed inclusion (Y, Y') a *pair* in $\mathfrak{F}$-**Top** and maps between pairs are defined as usual. Moreover for a closed inclusion $A \subset Y'$ and a map $g\colon A \to X'$ we obtain the set $[(Y, Y'), (X, X')]^A_{\mathfrak{F}}$ of homotopy classes of pair maps under A.

(2.6) Remark. All results in this paper remain true if **Top** is the category of all topological spaces and if the fibre functor F satisfies condition $(*)$: For every object V in $\mathfrak{F}$ the space $F(V)$ is locally compact and Hausdorff. In [5] we assumed the fibres to be also second countable, in order to deal with metrizable spaces and avoid many unnecessary technicalities, which do not occur in homotopy theory of $\mathfrak{F}$-families.

3 $\mathfrak{F}$-complexes

For an object V in $\mathfrak{F}$ the family $FV \to *$ with base space a singleton is termed a $\mathfrak{F}$-*point* also denoted by V. A disjoint union of $\mathfrak{F}$-points is called a $\mathfrak{F}$-*set*. This is a $\mathfrak{F}$-family for which the base space has the discrete topology. Let D^n be the unit disc in $\mathbb{R}^n$ and S^{n-1} its boundary with base point $* \in S^{n-1}$. The complement $e^n = D^n \setminus S^{n-1}$ is the *open cell* in D^n. A $\mathfrak{F}$-cell is a product family $V \times e^n \to e^n$ with $V \in \mathfrak{F}$.

We say that a $\mathfrak{F}$-family X is obtained from a $\mathfrak{F}$-family A by *attaching* n-cells if a $\mathfrak{F}$-set Z together with a $\mathfrak{F}$-map f is given, such that the following diagram

$$\begin{tikzcd} Z \times S^{n-1} \arrow[r, "f"] \arrow[d] & A \arrow[d] \\ Z \times D^n \arrow[r, "\Phi"] & X = A \cup_f (Z \times D^n) \end{tikzcd} \tag{3.1}$$

is a push-out in $\mathfrak{F}$-**Top**. The inclusion $Z \times S^{n-1} \to Z \times D^n$ is a closed inclusion, therefore the push-out exists and the induced map $A \to X$ is a closed inclusion and $X \setminus A = Z \times e^n$ is a union of open $\mathfrak{F}$-cells. If Z is a $\mathfrak{F}$-point then we say that X is obtained from A by attaching a $\mathfrak{F}$-cell and Φ is the *characteristic map* of the $\mathfrak{F}$-cell.

(3.2) Definition. A *relative* $\mathfrak{F}$*-complex* (X, A) is a family X and a filtration

$$A = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_n \subset X_{n+1} \subset \cdots \subset X$$

of $\mathfrak{F}$-families X_n, $n \geq -1$, such that for every $n \geq 0$ the $\mathfrak{F}$-family X_n is obtained from X_{n-1} by attaching n-cells and

$$X = \lim_{n \geq 0} X_n.$$

The spaces X_n are termed *n-skeleta* of (X, A). If A is empty we call X a $\mathfrak{F}$*-complex*. Then X is a union of $\mathfrak{F}$-cells, that is, there are the $\mathfrak{F}$-sets Z_n of n-cells in $X \setminus A$ such that $X \setminus A$ is the union of $Z_n \times e^n$ for $n \geq 0$.

(3.3) Example. Let G be a compact Lie group and let $\mathfrak{F}$ be the *category of orbits* of G, that is, $\mathfrak{F}$ is the subcategory of **Top** consisting of spaces G/H, where H is a closed subgroup of G, and G-equivariant maps $G/H \to G/H'$. Then each G-CW-complex (see [33]) is a $\mathfrak{F}$-complex.

(3.4) Example. Each $\mathfrak{F}$-stratified fibre bundle with finitely many strata is $\mathfrak{F}$-homotopy equivalent to a $\mathfrak{F}$-complex [5].

A map $f\colon (K, A) \to (X, A)$ under A between relative $\mathfrak{F}$-complexes is termed *cellular* if $f(K_n) \subset X_n$ for $n \geq 0$.

(3.5) Theorem (Cellular approximation). *Let (K, A) and (X, A) be relative $\mathfrak{F}$-complexes; let L be a subcomplex of K and $g\colon K \to X$ a $\mathfrak{F}$-map under A such that the restriction to $L \subset K$ is cellular. Then there is a cellular map $f\colon K \to X$ extending $g|L$ and a homotopy $f \sim g$* rel *L.*

Proof. The proof is identical to the classical case of CW-complexes, once lemma (5.1) is proved. Alternatively one can use IV.5.8 Baues [3] to obtain the result by use of the Blakers-Massey theorem (5.3) below. *q.e.d.*

(3.6) Remark. In [5] we have seen that $\mathfrak{F}$-complexes are stratified bundles which generalize the classical notion of bundle. The bundle theorem of [5] shows: Let $\mathfrak{F}$ be a structure category which is a groupoid. Then a $\mathfrak{F}$-complex $X \to \bar{X}$ is a $\mathfrak{F}$-bundle over $\bar{X}$. Conversely, each $\mathfrak{F}$-bundle $X \to \bar{X}$ over a CW-complex $\bar{X}$ is $\mathfrak{F}$-isomorphic to a $\mathfrak{F}$-complex.

(3.7) Remark. The definition of $\mathfrak{F}$-complex follows the lines of [2] (page 194). After Whitehead, Bredon defined G-complexes for G finite [6], independently Matumoto [23] and Illmann [16] extended the definition to the case G a compact Lie group. See [33] for a general approach of G-CW complexes and [22] for its consequences in K-theory of G-complexes. A notion of fibrewise CW-complex was presented by James in [18] (see also the decomposition of Bredon $O(n)$-manifolds W_k^{2n-1} there) in the framework of fibrewise topology [17]. While in the case of G-CW-complexes the structure category is evidently the orbit category of G, the structure category of a fibrewise complex [18] is the full subcategory of **Top** containing the fibres and the fibres are assumed to be compact.

4 Homotopy groups

A $\mathfrak{F}$-family X in $\mathfrak{F}$-**Top** is *pointed* if a $\mathfrak{F}$-map $*_V\colon V \to X$ is given, where V denotes the $\mathfrak{F}$-point given by the object V in $\mathfrak{F}$. Such a map is termed a *base point* of X. A *pointed pair* $(X, Y, *_V)$ of $\mathfrak{F}$-families is given by a closed inclusion $Y \subset X$ and a base point $*_V\colon V \to Y$. As usual a base point is chosen in the sphere S^{n-1} so that the pair $(D^n, S^{n-1}, *)$ is pointed in **Top**. For every $n \geq 0$ and $\mathfrak{F}$-point V let S^n_V and D^{n+1}_V denote the pointed $\mathfrak{F}$-family $S^n_V = V \times S^n$ over S^n and $D^{n+1}_V = V \times D^{n+1}$ over D^{n+1} with $S^n_V \subset D^{n+1}_V$.

(4.1) Definition. For every pointed $\mathfrak{F}$-family $(X, *_V)$ the *homotopy group*

$$\pi_n^{\mathfrak{F}}(X; *_V) = [S^n_V, X]^V_{\mathfrak{F}} \tag{4.2}$$

is the set of homotopy classes relative V of $\mathfrak{F}$-maps $f\colon S^n_V \to X$ as in the commutative diagram (4.3) in $\mathfrak{F}$-**Top**.

$$\begin{tikzcd} & V \times * \arrow[r, equal] \arrow[d, tail] & V \arrow[d, "*_V"] \\ S^n_V \arrow[r, equal] & V \times S^n \arrow[r, "f"] & X \end{tikzcd} \tag{4.3}$$

Moreover the *relative homotopy group*

$$\pi_{n+1}^{\mathfrak{F}}(X, Y; *_V) = \big[(D^{n+1}_V, S^n_V), (X, Y)\big]^V_{\mathfrak{F}} \tag{4.4}$$

is the set of homotopy classes relative V of pair maps $f\colon (D^{n+1}_V, S^n_V) \to (X, Y)$.

The homotopy groups above can also be described as ordinary homotopy groups of function spaces. For a fibre family X and V in $\mathfrak{F}$ let X^V be the space of all $\mathfrak{F}$-maps $V \to X$ with the compact-open topology in **Top**. For example the faithful fibre functor F yields the identification

$$W^V = \hom_{\mathfrak{F}}(V, W)$$

so that $\mathfrak{F}$ is a topological enriched category. Function spaces yield the functor

$$X^{\circ} \colon \mathfrak{F}^{\mathrm{op}} \to \mathbf{Top} \tag{4.5}$$

which carries V to X^V and $\alpha \colon V \to W$ to the induced map $\alpha^* \colon X^W \to X^V$ with $\alpha^*(x) = x \circ F(\alpha)$.

We have canonical isomorphisms

$$\pi_n^{\mathfrak{F}}(X; *_V) = \pi_n(X^V; *_V) \tag{4.6}$$

$$\pi_n^{\mathfrak{F}}(X, Y; *_V) = \pi_n(X^V, Y^V; *_V)$$

where the right hand side denotes the homotopy groups in **Top**.

We will prove in section 8 the following Whitehead theorem in $(\mathfrak{F}\text{-}\mathbf{Top})^A$.

(4.7) Theorem. *A map $f \colon (X, A) \to (Y, A)$ under A between relative $\mathfrak{F}$-complexes is a $\mathfrak{F}$-homotopy equivalence under A if and only if f induces isomorphisms*

$$f_* \colon \pi_n^{\mathfrak{F}}(X; *_V) \xrightarrow{\simeq} \pi_n^{\mathfrak{F}}(Y; f *_V)$$

for all $n \geq 0$, basepoints $*_V \colon V \to X$, and objects V in $\mathfrak{F}$.

(4.8) Corollary. *Let f be a $\mathfrak{F}$-map $f \colon (X, A) \to (Y, A)$ under A between relative $\mathfrak{F}$-complexes. Then the following statements are equivalent.*

- (i) *The $\mathfrak{F}$-map f is a $\mathfrak{F}$-homotopy equivalence under A;*
- (ii) *For every V in $\mathfrak{F}$ the induced map $f^V \colon X^V \to Y^V$ is a weak homotopy equivalence in* **Top**.

Proof. It is clear that *(i)* $\implies$ *(ii)*. But if f^V is a weak equivalence in **Top** for every V, then by (4.7) *(i)* holds. *q.e.d.*

(4.9) Remark. It is clear that in (4.8) we can replace *(ii)* with the following: For every V in $\mathfrak{F}$ the induced map $f^V \colon X^V \to Y^V$ is a homotopy equivalence in **Top**. Thus we obtain a generalization to $\mathfrak{F}$-complexes of James–Segal theorem [19] for G-ANR's. See also [13].

5 The Blakers-Massey theorem

We say that a pair (X, Y) is *n-connected* if for all basepoints $*_V : V \to Y$, $V \in \mathfrak{F}$ and $1 \le r \le n$ the homotopy group $\pi_r^{\mathfrak{F}}(X, Y; *_V) = 0$ are trivial and $\pi_0^{\mathfrak{F}}(Y; *_V) \to \pi_0^{\mathfrak{F}}(X; *_V)$ is surjective.

(5.1) Lemma. *For a relative $\mathfrak{F}$-complex (X, A) the pair (X, X_n) is n-connected, $n \ge 0$.*

Proof. Let $V \in \mathfrak{F}$. As in the classical case it suffices to prove that for an attachment $X = A \cup_\alpha (V \times D^n)$ as in (3.1) the pair (X, A) is $(n-1)$-connected. For this we use similar arguments and notations as in the proof of 13.5 of [12]. Let $g \colon v \times (B^r, S^{r-1}) \to (X, A)$ be a map which represents an element in $\pi_r^{\mathfrak{F}}(X, A; *_V)$, with $r < n$. Using the classical map $\chi \colon (I^r, \partial I^r, J^{r-1}) \to (B^r, S^{r-1}, *)$ we obtain the composite $f' = g\chi$ which plays the role of f in the proof of 13.5 of [12]. The map f' induces the commutative diagram

$$\begin{array}{ccc} V \times (I^r, \partial I^r, J^{r-1}) & \xrightarrow{f'} & (A \cup (V \times e^n), A, A_*) \\ \downarrow & & \downarrow \\ (I^r, \partial I^r, J^{r-1}) & \xrightarrow{f} & (\bar{A} \cup e^n, \bar{A}, *) \end{array}$$

Here $* \in \bar{A}$ is determined by $*_V$ and $A_* \in \mathfrak{F}$ is the fibre of A over $*$. As in 13.5 of [12] we obtain for f the subspace $\bar{U} \subset I^r$ with $f(\bar{U}) \subset e^n$ and the homotopy $h_t \colon \bar{U} \to e^n$ $(t \in [0, 1])$ relative $\partial\bar{U}$ with $h_0 = f|\bar{U}$. Hence we obtain by f' the commutative diagram

$$\begin{array}{ccc} V \times \bar{U} & \xrightarrow{f'|(V \times \bar{U})} & V \times e^n \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q} \\ \bar{U} & \xrightarrow{f|\bar{U}} & e^n \end{array}$$

The map $f'|(V \times \bar{U})$ is a $\mathfrak{F}$-map which is determined by $(f|\bar{U})_q$ and the coordinate $f'' \colon V \times \bar{U} \to V$. We define a $\mathfrak{F}$-homotopy

$$h'_t \colon V \times \bar{U} \to V \times e^n$$

by $h'_t = (h_t p, f'')$. We have $h_0 = f'|(V \times \bar{U})$ and h'_t us a $\mathfrak{F}$-homotopy relative $V \times \partial\bar{U}$. Hence we can define a $\mathfrak{F}$-homotopy

$$H'_t \colon V \times I^r \to A \cup (V \times e^n) \quad \text{with } H'_0 = f'$$

which induces $H_t\colon I^r \to \bar{A} \cup e^n$ in the proof of 13.5 of [12]. We obtain H'_t by

$$H'_t(x,u) = \begin{cases} h'_t(x,u) & \text{for } u \in \bar{U},\, x \in V \\ f'(x,u) & \text{for } u \in I^r \setminus U,\, x \in V \end{cases}$$

Now we can choose $p \in e^n(1/2) \subset e^n$ with $p \notin \mathrm{image}(\mathrm{H}_1)$; see 13.5 [12]. Hence the map H'_1 has a factorization

$$H'_1\colon V \times I^r \to A \cup (V \times (e^n \setminus p)) \subset A \cup (V \times e^n).$$

Here the inclusion $A \to A \cup (V \times (e^n \setminus p))$ is an $\mathfrak{F}$-homotopy equivalence. This completes the proof. *q.e.d.*

(5.2) Lemma. *Let (X, A) be a relative $\mathfrak{F}$-complex and let (X, A) be n-connected, with $n \geq 0$. Then there exists a relative $\mathfrak{F}$-complex (Y, A) with $Y_0 = Y_1 = \cdots = Y_n = A$ and a $\mathfrak{F}$-homotopy equivalence $Y \to X$ under A.*

Proof. The map $Y \to X$ is obtained inductively by attaching "ball pairs" and collapsing the non-attached part of the boundary of the ball pair; compare the proof of (6.14) in [32]. *q.e.d.*

(5.3) Theorem. *Let (X, A) a relative $\mathfrak{F}$-complex with subcomplexes X_1,X_2, such that $X = X_1 \cup X_2$. Let A be the intersection $A = X_1 \cap X_2$. If (X_1, A) is n_1-connected and (X_2, A) is n_2-connected, with $n_1, n_2 \geq 0$, then for each basepoint $*_V\colon V \to A$ with $V \in \mathfrak{F}$ the induced map*

$$i_*\colon \pi_r^{\mathfrak{F}}(X_1, A; *_V) \to \pi_r^{\mathfrak{F}}(X, X_2; *_V)$$

is an isomorphism if $r < n_1 + n_2$ and is a surjection if $r = n_1 + n_2$.

(5.4) Remark. Using the spaces (X^V, X_1^V, X_2^V, A^V) and the fact that A^V is a $(\mathfrak{F}, F^V)$-neighborhood deformation retract in X_1^V and X_2^V it is possible to prove theorem (5.3) also by using the adjoints of maps and the homotopy excision theorem of Spanier [30].

Proof. As in the proof of 16.27 of [12] it suffices to consider the case when $A = X_1 \cap X_2$ and X_1 and X_2 are obtained from A by attaching a single cell, namely $X = X_1 \cup_A X_2$ with

$$X_1 = A \cup_\beta (S \times D^m)$$

$$X_2 = A \cup_\alpha (R \times D^n)$$

with $n = n_1 + 1$ and $m = n_2 + 1$ and R, $S \in \mathfrak{F}$. In this case we generalize the proof of 13.6 of [12] as follows.

Let $p \in e^n$ and $q \in e^m$ and let $R_p = R$ and $S_q = S$ be the fibres of X over p and q respectively. Then we obtain the commutative diagram

$$\begin{array}{ccc} \pi_r^{\mathfrak{F}}(X_1, A; *_V) & \xrightarrow{i_*} & \pi_r^{\mathfrak{F}}(X, X_2; *_V) \\ \downarrow \cong & & \downarrow \cong \\ \pi_r^{\mathfrak{F}}(X \smallsetminus R_p, X \smallsetminus R_p \smallsetminus S_q; *_V) & \xrightarrow{j_*} & \pi_r^{\mathfrak{F}}(X, X \smallsetminus S_q; *_V) \end{array} \tag{5.5}$$

in which the vertical arrows are isomorphisms. This is readily seen by applying (2.4). The diagram is the analogue of 13.7 of [12].

Next we consider an $\mathfrak{F}$-map

$$\begin{array}{ccc} V \times I^r & \xrightarrow{h'} & X = A \cup (R \times E^n) \cup (e^m \cup S) \\ \downarrow & & \downarrow \\ I^r & \xrightarrow{h} & \bar{X} = \bar{A} \cup e^n \cup e^m \end{array}$$

which induces the map h in 13.9 of [12]. Using H_t in the proof of 13.9 of [12] we obtain a $\mathfrak{F}$-homotopy

$$\begin{array}{ccc} V \times I^r & \xrightarrow{H'_t} & X \\ \downarrow & & \downarrow \\ I^r & \xrightarrow[H_t]{} & \bar{X} \end{array}$$

by defining for $(x_1, \ldots, x_r) \in I^r$, $v \in V$

$$H'_t(x_1, \ldots, x_3, v) = h'(x_1, \ldots, x_{r-1}, \lambda, v)$$

with

$$\lambda = 1 - (1 - x_r)(1 - t\varphi(x_1, \ldots, x_{r-1}))$$

where φ is the map in the proof of 13.9 of [12]. One readily checks that (H'_t, H_t) is a well-defined $\mathfrak{F}$-homotopy.

We can now apply the arguments in the proof of 13.6 of [12]. We first show that if $r \leq m + n - 2$, i_* is surjective. Let

$$f' \colon V \times (I^r, \partial I^r, J^{r-1}) \to (X, X_2, X_*)$$

which represents an element $\{f'\}$ in $\pi_r^{\mathfrak{F}}(X, X_2; *_V)$ as in the proof of (5.1) above. Hence f' induces a map

$$f\colon (I^r, \partial I^r, J^{r-1}) \to (\bar{X}, \bar{X}_2, *)$$

with $\bar{X} = \bar{A} \cup e^n \cup e^m$. We obtain for f the homotopy f_t as in the proof of 13.6 of [12] and we define the $\mathfrak{F}$-homotopy

$$f_t'\colon V \times (I^r, \partial I^r, J^{r-1}) \to (X, X_2, X_*)$$

which induces f_t by

$$f_t'(v,u) = \begin{cases} h_t'(v,u) & \text{for } u \in \bar{U},\ v \in V \\ k_t'(v,u) & \text{for } u \in \bar{V},\ v \in V \\ f'(v,u) & \text{for } u \in I^r \setminus \bar{U} \setminus \bar{V},\ v \in V. \end{cases}$$

Here h_t' is defined already in the proof of (5.1) and k_t' is defined in the same way as h_t'. Hence $f_t'\colon f \simeq f_t'$ is a $\mathfrak{F}$-homotopy and f_1' represents $\{f'\}$. Moreover since $H_1(I^r) \subset \bar{X} \setminus p$ we see that $\{f'\} = \{H_1'\}$ is in the image of j_* in (5.5). This shows that j_*, and hence i_*, is onto. In a similar way one follows the argument in the proof of 13.6 of [12] to show that i_* is injective for $n < n_1 + n_2$. *q.e.d.*

6 The category $\Pi_{\mathfrak{F}}(X)$ and $\Pi_{\mathfrak{F}}(X)$-modules

Let X be a $\mathfrak{F}$-family and let $x, y\colon V \to X$ be $\mathfrak{F}$-points in X. Then we can consider homotopies $F\colon x \simeq y$ with $F\colon IV \to X$. Such a homotopy is termed a *$\mathfrak{F}$-path* in X. Homotopy classes of such $\mathfrak{F}$-paths relative the boundary $V \coprod V$ of the cylinder IV are termed *$\mathfrak{F}$-tracks* $\{F\}\colon x \Longrightarrow y$. Hence a $\mathfrak{F}$-track is an element

$$\{F\} \in [IV, X]^{V \amalg V} = [IV, X]^{(x,y)}.$$

Addition of homotopies yields the composite of tracks H, G denoted by $H \square G$. Accordingly let $\Pi_{\mathfrak{F}}^V(X)$ be the following groupoid. Objects are $\mathfrak{F}$-maps $V \to X$ and morphisms are $\mathfrak{F}$-tracks. For a closed inclusion $A \subset X$ let

(6.1) $$\Pi_{\mathfrak{F}}^V(X, A) \subset \Pi_{\mathfrak{F}}^V(X)$$

be the full subgroupoid consisting of objects $x\colon V \to A \subset X$.

The following category $\Pi_{\mathfrak{F}}(X)$ plays the role of the fundamental groupoid of a space X in classical homotopy theory.

(6.2) Definition. Let X be a $\mathfrak{F}$-family. First we define the category $\mathbf{P}_{\mathfrak{F}}(X)$ as follows. Objects are pairs (V, x) where V is an object in $\mathfrak{F}$ and

$$x\colon V \to X$$

is a $\mathfrak{F}$-point in X. A morphism $(V, x) \to (W, y)$ is a pair (a, α) where $\alpha\colon V \to W$ is a morphism in $\mathfrak{F}$ and $a\colon x \Longrightarrow y\alpha$ is a track. We associate to the morphism (a, α) the diagram

$$\begin{tikzcd} V \arrow[rr, "\alpha"] \arrow[dr, "x"'] \arrow[rr, phantom, shift right=4ex, "\overset{a}{\Longrightarrow}"] & & W \arrow[dl, "y"] \\ & X & \end{tikzcd}$$

The composition of such morphisms according to the diagram

$$\begin{tikzcd} U \arrow[r, "\beta"] \arrow[d, "z"] \arrow[dr, phantom, "\overset{b}{\Longrightarrow}"] & V \arrow[r, "\alpha"] \arrow[d, "x"] \arrow[dr, phantom, "\overset{a}{\Longrightarrow}"] & W \arrow[d, "y"] \\ X \arrow[r, equal] & X \arrow[r, equal] & X \end{tikzcd}$$

is defined by

$$(a, \alpha)(b, \beta) = ((a\beta)\Box b, \alpha\beta).$$

Two morphisms (a, α) and $(a', \alpha')\colon (V, x) \to (W, y)$ are *equivalent* if there is a track

$$h\colon \alpha \Longrightarrow \alpha' \quad \text{in } W^V$$

such that $(yh)\Box a = a'$. This is a natural equivalence relation on $\mathbf{P}_{\mathfrak{F}}(X)$ so that we obtain the quotient category

(6.3) $$\Pi_{\mathfrak{F}}(X) = \mathbf{P}_{\mathfrak{F}}(X)/_\sim$$

which is termed the (discrete) *fundamental category* of X.

For a closed inclusion $A \subset X$ in $\mathfrak{F}$-**Top** let

(6.4) $$\Pi_{\mathfrak{F}}(X, A) \subset \Pi_{\mathfrak{F}}(X)$$

be the full subcategory consisting of all objects (V, x) with $x\colon V \to A \subset X$.

The cellular approximation theorem shows for a 0-connected relative $\mathfrak{F}$-complex (X, A) that the inclusion $X_2 \subset X$ induces an isomorphism

(6.5) $$\Pi_{\mathfrak{F}}(X_2, A) = \Pi_{\mathfrak{F}}(X, A).$$

(6.6) Remark. For a space X in **Top** let $\Pi(X)$ be the fundamental groupoid of X. Then we get the functor

$$\Pi(X^\circ)\colon \mathfrak{F}^{\mathrm{op}} \to \mathbf{Grd}$$

which carries V to the fundamental groupoid $\Pi(X^V)$ of the function space X^V. We have $\Pi(X^V) = \Pi^V_{\mathfrak{F}}(X)$. Now the category $\mathbf{P}_{\mathfrak{F}}(X)$ above coincides with the "integration category" $\int_{\mathfrak{F}} \Pi(X^\circ)$, see [24] and compare with [22].

A left (resp. right) $\Pi_{\mathfrak{F}}(X)$-*module* is a covariant (resp. contravariant) functor

$$M\colon \Pi_{\mathfrak{F}}(X) \to \mathbf{Ab}$$

where $\mathbf{Ab}$ is the category of abelian groups. Let $\mathbf{Mod}(\Pi_{\mathfrak{F}}(X)^{\mathrm{op}})$ be the category of right modules. Morphisms are natural transformations.

(6.7) Proposition. *For $n \geq 2$ the homotopy group yields a right module*

$$\pi_n^{\mathfrak{F}}(X)\colon \Pi_{\mathfrak{F}}(X)^{\mathrm{op}} \to \mathbf{Ab}.$$

Moreover for $n \geq 3$ relative homotopy groups of a pair (Y, X) yield the right module

$$\pi_n^{\mathfrak{F}}(Y, X)\colon \Pi_{\mathfrak{F}}(X)^{\mathrm{op}} \to \mathbf{Ab}.$$

Proof. An object (V, x) of $\Pi_{\mathfrak{F}}(X)$ yields a basepoint $x\colon V \to X$, hence the abelian group $\pi_n^{\mathfrak{F}}(X; x)$ is defined for $n \geq 2$. If $(a, \alpha)\colon (V, x) \to (W, y)$ is a morphism in $\mathbf{P}_{\mathfrak{F}}(X)$ and $n \geq 2$ then there is a canonical induced homomorphism of abelian groups defined by the composition

(6.8)
$$\begin{tikzcd}
\pi_n(X^W; y) \arrow[d, equal] & \pi_n(X^V; y\alpha) \arrow[d, equal] & \pi_n(X^V; x) \arrow[d, equal] \\
\pi_n^{\mathfrak{F}}(X; y) \arrow[r, "\pi_n(\alpha^*)"] & \pi_n^{\mathfrak{F}}(X; y\alpha) \arrow[r, "a^\#"] & \pi_n^{\mathfrak{F}}(X; x)
\end{tikzcd}$$

where

$$\alpha^*(y) = y\alpha\colon V \xrightarrow{\alpha} W \xrightarrow{y} X$$

and $a^\#$ is the isomorphism of homotopy groups induced by the track a. It is not difficult to check that the structure of $\mathbf{P}_{\mathfrak{F}}(X)$-module induces a structure of $\Pi_{\mathfrak{F}}(X)$-module, since the homomorphism in (6.8) does not depend on the representative (a, α) in $\mathbf{P}_{\mathfrak{F}}(X)$ of a morphism in $\Pi_{\mathfrak{F}}(X)$. If $n \geq 3$ the same argument can be applied to the relative group $\pi_n^{\mathfrak{F}}(Y, X; x)$. *q.e.d.*

Let Z be a $\mathfrak{F}$-set and let $\alpha\colon Z \to X$ be a $\mathfrak{F}$-map. Hence (Z, α) is given by a family of $\mathfrak{F}$-maps $\alpha|V\colon V \to X$ with $V \in Z$. We define the *free right* $\Pi_{\mathfrak{F}}(X)$*-module* in $\mathbf{Mod}(\Pi_{\mathfrak{F}}(X)^{\mathrm{op}})$

$$M[Z,\alpha]\colon \Pi_{\mathfrak{F}}(X)^{\mathrm{op}} \to \mathbf{Ab} \tag{6.9}$$

by the direct sum

$$M[Z,\alpha] = \bigoplus_{V \in Z} \mathbb{Z}\hom(-, (V, \alpha|V)).$$

Here $\mathbb{Z}\hom(-, (V, \alpha|V))$ denotes the module which carries (W, y) to the free abelian group generated by the hom-set $\hom((W, y), (V, \alpha|V))$ of all morphisms $(W, y) \to (V, \alpha|V)$ in $\Pi_{\mathfrak{F}}(X)$. A morphism

$$\varphi\colon M[Z,\alpha] \to M \tag{6.10}$$

in $\mathbf{Mod}(\Pi_{\mathfrak{F}}(X)^{\mathrm{op}})$ is uniquely determined by a family of elements

$$\varphi_V \in M(V, \alpha|V) \quad \text{for } V \in Z$$

such that $\varphi(1_{(V,\alpha|V)}) = \varphi_V$. Here $1_{(V,\alpha|V)}$ denotes the identity of the object $(V, \alpha|V)$ in $\Pi_{\mathfrak{F}}(X)$.

We now consider for (Z, α) above the pushout

$$\begin{tikzcd}
Z \times * \arrow[r, equal] \arrow[d] & Z \arrow[r, "\alpha"] & X \arrow[d] \\
Z \times S^n \arrow[r, equal] & S^n_Z \arrow[r] & X \cup_\alpha S^n_Z
\end{tikzcd}$$

The projection $Z \times S^n \to Z$ induces the retraction

$$r_X\colon X \cup_\alpha S^n_Z \to X$$

which is the identity on X. For $n \geq 1$ let

$$\pi_n^{\mathfrak{F}}(X \cup_\alpha S^n_Z)_X = \ker\left((r_X)_*\colon \pi_n^{\mathfrak{F}}(X \cup_\alpha S^n_Z) \to \pi_n^{\mathfrak{F}}(X)\right).$$

We define the *partial suspension*

$$E\colon \pi_n^{\mathfrak{F}}(X \cup_\alpha S^n_Z)_X \to \pi_{n+1}^{\mathfrak{F}}(X \cup_\alpha S^{n+1}_Z)_X \tag{6.11}$$

by the composition $E = j^{-1}\pi_*\partial^{-1}$

$$\begin{tikzcd}
\pi_{n+1}^{\mathfrak{F}}(X \cup_\alpha D^{n+1}_Z, X \cup_\alpha S^n_Z) \arrow[r, "\partial", "\cong"'] \arrow[d, "\pi_*"'] & \pi_n^{\mathfrak{F}}(X \cup_\alpha S^n_Z)_X \\
\pi_{n+1}^{\mathfrak{F}}(X \cup_\alpha S^{n+1}_Z, X) & \pi_{n+1}^{\mathfrak{F}}(X \cup_\alpha S^{n+1}_Z)_X \arrow[l, "\cong"', "j"]
\end{tikzcd}$$

Here π is induced by $D^{n+1}/S^n = S^{n+1}$ and the isomorphisms ∂ and j are induced by the homotopy exact sequences of pairs. The Blakers-Massey theorem implies:

(6.12) Proposition. *E is an isomorphism for $n \geq 2$ and is surjective for $n = 1$.*

(6.13) Proposition. *There is an isomorphism of modules $(n \geq 2)$*

$$\varphi\colon M[Z,\alpha] \cong \pi_n^{\mathfrak{F}}(X \cup_\alpha S^n_Z)_X$$

which is given by the family of maps

$$\varphi_V\colon S^n_V \subset S^n_Z \to X \cup_\alpha S^n_Z$$

with $V \in Z$. Moreover φ is compatible with E.

Proof. Without loss of generality, by additivity, we can assume that $Z = W$; moreover, by considering the mapping cylinder of $\alpha\colon W \to X$ in $\mathfrak{F}$-**Top** we can assume that $\alpha\colon W \to X$ is a closed inclusion (actually, a $\mathfrak{F}$-cofibration). Let (V,x) be an arbitrary object in $\Pi_{\mathfrak{F}}(X)$. We want to show that φ induces an isomorphism of groups

$$\varphi\colon M[W,\alpha](V,x) = \mathbb{Z}\hom((V,x),(W,\alpha)) \to \pi_n^{\mathfrak{F}}(X \cup_\alpha S^n_W; x)_X. \tag{6.14}$$

Let X^V_x denote the path-component in X^V containing $x \in X^V$ and $W^V_x = X^V_x \cap \alpha^V(W^V)$; let $\alpha_x\colon W^V_x \subset X^V_x$ be the inclusion. Moreover, let $\widetilde{X^V_x} = E_X$ be the universal covering space of X^V_x. We can assume that any space that we are considering in this proof has a universal cover by taking a CW-approximation weakly equivalent to it. The elements of $\widetilde{X^V_x}$ correspond bijectively with tracks $x \Longrightarrow \xi$, where $\xi\colon V \to X$ is any $\mathfrak{F}$-map. The covering projection in this case is the evaluation of the track at 1. We can define E_W by the following pull-back diagram.

$$\begin{tikzcd} E_W \arrow[r, tail, "\widetilde{\alpha_x}"] \arrow[d, two heads] & \widetilde{X^V_x} = E_X \arrow[d, two heads] \\ W^V_x \arrow[r, tail, "\alpha_x"] & X^V_x \end{tikzcd} \tag{6.15}$$

Then $E_X \cup_{\widetilde{\alpha_x}} (E_W \times S^n)$ is the universal covering of $X^V_x \cup_{\alpha_x} (W^V_x \times S^n)$.

Consider the morphisms (b,β) in $\mathbf{P}_{\mathfrak{F}}(X)((V,x),(W,\alpha))$. Then β and b are elements in $\widetilde{X^V_x}$ and W^V_x respectively, compatible with the pull-back diagram

(6.15). Hence there is a map λ as in the following diagram.

$$\begin{tikzcd}
\mathbf{P}_{\mathfrak{F}}(X)((V,x),(W,\alpha)) \arrow[r, "\lambda"] \arrow[d] & E_W \arrow[d] \\
\Pi_{\mathfrak{F}}(X)((V,x),(W,\alpha)) \arrow[r, dotted, "\bar{\lambda}", "\cong"'] & \pi_0(E_W).
\end{tikzcd}$$

Here two morphisms (b,β) and (b',β') in $\mathbf{P}_{\mathfrak{F}}(X)((V,x),(W,\alpha))$ have images under λ in the same path-component if and only if there is $h\colon I \to E_W$ such that $h(0) = b$, $h(1) = b'$. But this happens if and only if (b,β) and (b',β') belong to the same equivalence class in $\Pi_{\mathfrak{F}}(X)$, and thus there is an induced bijection $\bar{\lambda}$ as in the diagram.

Now, using the exact homotopy sequence of the pair, the property of universal covering spaces, the homotopy excision, and the isomorphism induced by $\bar{\lambda}$ we get the equations:

$$\begin{aligned}
\pi_n^{\mathfrak{F}}(X \cup_\alpha S^n_W; x)_X &= \pi_n^{\mathfrak{F}}(X \cup_\alpha S^n_W, X; x) \\
&= \pi_n(X^V \cup_{\alpha^V} S^n_{W^V}, X^V; x) \\
&= \pi_n(X^V_x \cup_{\alpha^V_x} S^n_{W^V_x}, X^V_x; x) \\
&= \pi_n(E_X \cup_{\widetilde{\alpha_x}} (E_W \times S^n), E_X; x) \\
&= \pi_n(E_X \cup_{\widetilde{\alpha_x}} (E_W \times S^n), E_X) \\
&\cong \bigoplus_{C \subset E_W} \pi_n(S^n \times C, C) \\
&= \mathbb{Z}\pi_0(E_W) \\
&= \mathbb{Z}\Pi_{\mathfrak{F}}(X)((V,x),(W,\alpha))
\end{aligned}$$

where the sum $\bigoplus$ ranges over the set of path-components $C \subset E_W$. We can apply the homotopy excision since E_X is simply connected and $W = W \times *$ is a $\mathfrak{F}$-neighborhood deformation retract in S^n_W. Therefore we have the isomorphism

$$M[W,\alpha](V,x) = \mathbb{Z}\hom((V,x),(W,\alpha)) \cong \pi_n^{\mathfrak{F}}(X \cup_\alpha S^n_W; x)_X.$$

It is not difficult, by chasing along the chain of equalities above, to see that this is the same homomorphism induced by φ as in (6.14) and that it commutes with the partial suspension E as claimed. Since φ is a morphism of $\mathbf{Mod}(\Pi_{\mathfrak{F}}(X)^{\mathrm{op}})$-modules, the proof is complete. *q.e.d.*

7 Homology and cohomology

Let (X, A) be a relative $\mathfrak{F}$-complex. We say that (X, A) is *normalized* if the attaching maps

$$f_n \colon S^{n-1}_{Z_n} = Z_n \times S^{n-1} \to X_{n-1} \tag{7.1}$$

of n cells in X $(n \geq 1)$ carry the basepoint $*$ of S^{n-1} to the 0-skeleton, that is $f_n(Z_n \times *) \subset X_0$ or equivalently if the attaching maps are cellular. Moreover (X, A) is *reduced* if $X_0 = A$ so that X is obtained from A by attaching $\mathfrak{F}$-cells of dimension ≥ 1.

(7.2) Lemma. *Let (X, A) be a relative $\mathfrak{F}$-complex which is 0-connected. Then there exists a relative $\mathfrak{F}$-complex (Y, A) which is reduced and normalized together with a $\mathfrak{F}$-homotopy equivalence $Y \to X$ under A.*

Proof. First we obtain a reduced $\mathfrak{F}$-complex (Y', A) by (5.2). Using the cellular approximation theorem (3.5) the attaching maps in Y' are $\mathfrak{F}$-homotopic to normalized attaching maps. This can be used to construct inductively Y. *q.e.d.*

If (X, A) is reduced and normalized we obtain for $n \geq 1$ the commutative diagram

$$\begin{tikzcd} Z_n \times S^{n-1} \arrow[r, equal] & S^{n-1}_{Z_n} \arrow[r, "f_n"] & X_{n-1} \\ Z_n \times * \arrow[u] \arrow[r, equal] & Z_n \arrow[r, "\alpha_n"] & A \arrow[u] \end{tikzcd} \tag{7.3}$$

where α_n is the *basepoint map* associated to the attaching map f_n. We identify the closed cell D^n with the (reduced) cone over the pointed space S^{n-1} so that we obtain the pinch map under S^{n-1} $(n \geq 1)$

$$D^n \xrightarrow{\bar{\mu}} D^n \vee S^n.$$

This map induces the coaction map

$$\begin{tikzcd} X_n \arrow[r, "\mu"] \arrow[d, equal] & X_n \cup_{\alpha_n} S^n_{Z_n} \arrow[d, equal] \\ X_{n-1} \cup_{f_n} (Z_n \times D^n) \arrow[r] & X_{n-1} \cup_{f_n} (Z_n \times (D^n \vee S^n)) \end{tikzcd} \tag{7.4}$$

where the bottom row is $1_{X_{n-1}} \cup (Z_n \times \bar{\mu})$. On the other hand we have for the 1 dimensional disk D^1 with boundary points $\partial D^1 = \{\partial_0, \partial_1\}$ and basepoint $\partial_0 = *$ the map

$$(S^1, \partial_0) \xrightarrow{\bar{\mu}_0} (D^1 \cup_{\partial_1} S^1, \partial_0)$$

which is defined by composing obvious tracks $\partial_0 \to \partial_1$ in D^1, $* \Longrightarrow *$ in S^1 and $\partial_1 \Longrightarrow \partial_0$ in D^1. If X is a normalized $\mathfrak{F}$-complex we have the $\mathfrak{F}$-set $X_0 = Z_0$ and the inclusion $\alpha_0 \colon Z_0 \subset X$. Then $\bar{\mu}_0$ induces the map

(7.5)
$$\begin{tikzcd} S^1_{Z_1} \arrow[r, "\mu_0"] \arrow[d, equal] & X \cup_{\alpha_0} S^1_{Z_0} \\ Z_1 \times S^1 \arrow[r] & Z_1 \times (D^1 \cup_{\partial_1} S^1) \arrow[u] \end{tikzcd}$$

Here the bottom row is $Z_1 \times \bar{\mu}_0$ and the right hand side is $\bar{f}_1 \cup (S^1 \times \alpha_1^1)$ where $\bar{f}_1$ is the characteristic map of 1-cells and $\alpha_1^1 = f_1|(Z_1 \times \partial_1)$. The map μ_0 restricted to $Z_1 \times *$ is given by $\alpha_1 = f_1|(Z_1 \times \partial_0)$.

(7.6) Definition. Let (X, A) be a $\mathfrak{F}$-complex which is reduced and normalized. Then we define the *chain complex* $C_*(X, A)$ in $\mathbf{Mod}(\Pi_{\mathfrak{F}}(X, A)^{\mathrm{op}})$ as follows. For $n \leq 0$ let $C_n(X, A) = 0$ and for $n \geq 1$ let

$$C_n(X, A) = M[Z_n, \alpha_n]$$

be defined as in (6.9) by the $\mathfrak{F}$-set Z_n of n-cells in (X, A) and the basepoint map $\alpha_n \colon Z_n \to A \subset X$. The differential

$$d \colon C_{n+1}(X, A) \to C_n(X, A)$$

is defined for $V \in Z_{n+1}$ by elements d_V obtained as follows. Let ∂_V be the composite ($n \geq 1$)

$$\partial_V \colon S^n_V \subset S^n_{Z_{n+1}} \xrightarrow{f_{n+1}} X_n \xrightarrow{\mu} X_n \cup_{\alpha_n} S^n_{Z_n} \subset X \cup_{\alpha_n} S^n_{Z_n}$$

which for $n \geq 2$ represents the element

$$d_V \in \pi_n^{\mathfrak{F}}(X \cup_{\alpha_n} S^n_{Z_n}; \alpha_{n+1}|V)_X = M[Z_n, \alpha_n](V, \alpha_{n+1}|V).$$

Compare (6.6) and (6.9). For $n = 1$ we have the partial suspension

$$E \colon \pi_1^{\mathfrak{F}}(X \cup_{\alpha_1} S^1_{Z_1}; \alpha_2|V)_X \to \pi_2^{\mathfrak{F}}(X \cup_{\alpha_1} S^2_{Z_1}; \alpha_2|V)_X$$

and we set $d_V = E(\partial_V)$.

If X is a normalized $\mathfrak{F}$-complex we define the chain complex C_*X in $\mathbf{Mod}(\Pi_{\mathfrak{F}}(X,X_0)^{\mathrm{op}})$ by

$$\begin{aligned} C_n(X) &= C_n(X,X_0) \quad \text{for } n \geq 1 \text{ and} \\ C_0(X) &= M[Z_0,\alpha_0] \end{aligned}$$

where $Z_0 = X_0$ is the 0-skeleton of X which is an $\mathfrak{F}$-set and $\alpha_0\colon Z_0 \subset X$ is the inclusion. Moreover the differential

$$d\colon C_{n+1}(X) \to C_n(X)$$

is defined as above for $n > 1$. For $n = 1$ the differential

$$d\colon C_1(X) \to C_0(X)$$

is defined by $d_V = E(\mu_0|S^1_V)$ for $V \in Z_1$ where μ_0 is the map in (7.5) and E is the partial suspension. We may consider $C_*(X,A)$ and $C_*(X)$ also as chain complexes in $\mathbf{Mod}(\Pi_{\mathfrak{F}}(X)^{\mathrm{op}})$.

(7.7) Proposition. *$C_*(X,A)$ and $C_*(X)$ are well defined chain complexes.*

Proof. Since ($\mathfrak{F}$-**Top**, **T**) is homological (B.3), the *chain functor C_** is defined as in V.2.4, page 255 of [3], and it is easy to see that it coincides with the chain functor C_* defined above. *q.e.d.*

(7.8) Definition. Let (X,A) be a 0-connected relative $\mathfrak{F}$-complex. By lemma (7.2) we can assume that (X,A) is reduced and normalized. Given a *right* $\Pi_{\mathfrak{F}}(X)$-module M we define the cohomology of (X,A) with coefficients in M by

$$H^*(X,A;M) = H^*\left(\mathrm{hom}(C_*(X,A),M)\right),$$

where hom is defined in the abelian category of right $\Pi_{\mathfrak{F}}(X)$-modules.

In a similar way, if N is a left $\Pi_{\mathfrak{F}}(X)$-module we can define the *homology* of (X,A) with coefficients in N by

$$H_*(X,A;N) = H_*(C_*(X,A) \otimes N),$$

where $\otimes$ stands for the tensor product of a right and a left $\Pi_{\mathfrak{F}}(X)$-module. Here $H^*(X,A;M)$ and $H_*(X,A;M)$ are abelian groups.

Finally, the *total homology* of (X,A) is defined by

$$H_*(X,A) = H_*(C_*(X,A)).$$

The total homology is the homology of a chain complex of right $\Pi_{\mathfrak{F}}(X)$-modules, so that $H_*(X,A)$ is again a $\Pi_{\mathfrak{F}}(X)$-module.

(7.9) Remark. This is a generalization of the Bredon–Illman cohomology [6] for G-spaces. See Bröcker for the singular Bredon homology [7]. See [26, 25, 24] for singular Bredon cohomology of discrete diagrams.

8 The Whitehead theorem

(8.1) Theorem. *Let A be a $\mathfrak{F}$-family and let $f\colon (X,A) \to (Y,A)$ be a cellular map between normalized reduced relative $\mathfrak{F}$-complexes. Then f is a $\mathfrak{F}$-homotopy equivalence under A if and only if f induces an isomorphism*

$$\varphi = \Pi(f)\colon \Pi^V_{\mathfrak{F}}(X,A) \to \Pi^V_{\mathfrak{F}}(Y,A)$$

of groupoids for all $V \in \mathfrak{F}$ and any one of the following cases is true:

(i) *f induces a homotopy equivalence of $\Pi_{\mathfrak{F}}(X,A)$-chain complexes*

$$f_*\colon C_*(X,A) \to C_*(Y,A),$$

where we use the isomorphism φ to identify $\Pi_{\mathfrak{F}}(X,A)$ and $\Pi_{\mathfrak{F}}(Y,A)$.

(ii) *The induced map between homology*

$$f_*\colon H_*(X,A) \to \varphi^* H_*(Y,A)$$

is an isomorphism.

(iii) *For every $\Pi_{\mathfrak{F}}(Y,A)$-module M the induced map*

$$f^*\colon H^*(Y,A;M) \to H^*(X,A;\varphi^*M)$$

is an isomorphism.

Proof. This is a consequence of the general homological Whitehead theorem V.7.1 of [3]. It is only needed to show that any reduced and normalized $\mathfrak{F}$-complex is a **T**-complex in the sense of IV.2.2 of [3], where **T** the theory of $\mathfrak{F}$-graphs (see definition (B.1)). This follows from lemma (B.2) in the appendix. *q.e.d.*

Proof of theorem (4.7). By (5.1) the **T**-complexes in $\mathfrak{F}$-**Top** are **T**-good in the sense of IV.3.7 [3]. Hence by (7.2) and IV.3.11 of [3] we know that the Whitehead theorem holds for 0-connected relative $\mathfrak{F}$-complexes (X',A) and (Y',A). This can be used to prove the general case in (4.7) as follows. Let $f\colon (X,A) \to (Y,A)$ be a Whitehead equivalence as in (4.7). We may assume that f is cellular. Hence f yields a restriction $\alpha\colon X_0 \to Y_0 \subset Y$ which is surjective in $\pi_0^{\mathfrak{F}}$. Let $M_\alpha = IX_0 \cup_\alpha Y$ be the mapping cylinder (in $\mathfrak{F}$-**Top**) of α, i.e. the push-out of $i_0\colon X_0 \to IX_0$ and α (see (A.6)). Then $i_1\colon X_0 \to IX_0$ yields a pair (M_α, X_0) which is 0-connected. Moreover $Y \to M_\alpha$ is a $\mathfrak{F}$-homotopy equivalence and the composite $X \to Y \to M_\alpha$ is by the homotopy extension property of $X_0 \subset X$ $\mathfrak{F}$-homotopic to a map $g\colon X \to M_\alpha$ which is the identity on X_0. Hence $g\colon (X,X_0) \to (M\alpha, Y_0)$ is a Whitehead equivalence between 0-connected pairs and hence a $\mathfrak{F}$-homotopy equivalence. *q.e.d.*

9 The Whitehead sequence and obstruction theory

Let (X, A) a reduced normalized $\mathfrak{F}$-complex and $\Pi_{\mathfrak{F}}(X, A)$ the (restricted) fundamental category. Then the homotopy groups $\pi_n^{\mathfrak{F}}(X)$ and the homology groups $H_n(X, A)$ are $\Pi_{\mathfrak{F}}(X, A)^{\mathrm{op}}$-modules and it is possible to define a Hurewicz homomorphism

$$h\colon \pi_n^{\mathfrak{F}}(X) \to H_n(X, A)$$

as follows.

Let $x\colon V \to X$ be a basepoint and $\alpha\colon S_V^n \to X$ a representative of the element $[\alpha] \in \pi_n^{\mathfrak{F}}(X; x)$. By (3.5) we can assume that $\alpha S_V^n \subset X_n$. Consider the difference $\mu\alpha - i\alpha$ of the two composites

$$\begin{tikzcd}
S_V^n \arrow[r, "\alpha"] \arrow[d, "\alpha"] & X_n \arrow[d, "\mu"] & \\
X_n \arrow[r, "i"] & X_n \cup_{\alpha_n} S_{Z_n}^n \arrow[r, tail, "j"] & X \cup_{\alpha_n} S_{Z_n}^n
\end{tikzcd}$$

where μ is the coaction map. Then $j(\mu\alpha - i\alpha)$ is a cycle in $C_n(X, A)$ that represents an element

$$h(\alpha) = \{j(\mu\alpha - i\alpha)\} \in H_n(X, A).$$

It is not difficult to see that $h(\alpha)$ does not depend upon the $\mathfrak{F}$-homotopy class of α and that h is a $\Pi_{\mathfrak{F}}(X, A)$-homomorphism, termed the *Hurewicz* homomorphism.

The groups $\Gamma_n(X, A)$, if $n \geq 3$, are defined as follows. If $x\colon V \to A$ is an object of $\Pi_{\mathfrak{F}}(X, A)$ then

$$\Gamma_n(X, A)(x) = \text{Image}\left[\pi_n^{\mathfrak{F}}(X_{n-1}; x) \to \pi_n^{\mathfrak{F}}(X_n; x)\right]$$

For $n = 1, 2$ the definition is more complicate (see section V.5, page 262, of [3]).

(9.1) Theorem. *Let (X, A) be a reduced and normalized $\mathfrak{F}$-complex Then the following sequence of $\Pi_{\mathfrak{F}}(X, A)^{\mathrm{op}}$-modules is exact*

$$\longrightarrow \Gamma_n(X, A) \longrightarrow \pi_n^{\mathfrak{F}}(X) \xrightarrow{h} H_n(X, A) \longrightarrow \Gamma_{n-1}(X, A) \longrightarrow \dots$$
$$\longrightarrow \Gamma_2(X, A) \longrightarrow \pi_2^{\mathfrak{F}}(X) \xrightarrow{h} H_2(X, A) \twoheadrightarrow \Gamma_1(X, A) \longrightarrow 0$$

*Furthermore, the sequence is natural in (X, A) in the category $\mathfrak{F}$-**Top**^{A} of $\mathfrak{F}$-families under A.*

Proof. It is a direct consequence of theorem V.5.4, page 264 [3]. *q.e.d.*

The cohomology defined above is suitable for obstruction theory as follows.

(9.2) Theorem. *Let (X, A) a normalized reduced relative $\mathfrak{F}$-complex and $f\colon A \to Y$ a $\mathfrak{F}$-map that admits an extension $g\colon X_n \to Y$, for $n \geq 2$. Then its restriction $g|X_{n-1}$ has an extension $g'\colon X_{n+1} \to Y$ to the $(n+1)$-skeleton if and only if the obstruction*

$$\mathcal{O}(g|X_{n-1}) \in H^{n+1}(X, A; g^*\pi_n^{\mathfrak{F}} Y)$$

vanishes, where $g_*\colon \Pi_{\mathfrak{F}}(X, A) \to \Pi(Y)$ induces the functor $g^*\colon \mathbf{Mod}(\Pi_{\mathfrak{F}}(Y)) \to \mathbf{Mod}(\Pi_{\mathfrak{F}}(X, A))$.

Proof. The obstruction $\mathcal{O}(g|X_{n-1})$ is defined as in chapter V of [3]. See theorem V.4.4, page 262. *q.e.d.*

10 Wall finiteness obstruction

A $\mathfrak{F}$-complex X is *finite* if it has only finitely many cells or equivalently if $\bar{X}$ is a finite CW-complex. A *domination* of Y in $\mathfrak{F}$-**Top** is given by $\mathfrak{F}$-maps

$$Y \xrightarrow{f} X \xrightarrow{g} Y$$

and a $\mathfrak{F}$-homotopy $H\colon gf \sim 1_Y$. The domination has dimension $\leq n$ if the dimension of X is $\leq n$ and the domination is finite if X is a finite $\mathfrak{F}$-complex. For a ringoid R let $K_0(R)$ denote the reduced projective class group of R. If $\mathbf{C}$ is a category let $\mathbb{Z}\mathbf{C}$ be the ringoid associated to $\mathbf{C}$ for which

$$\hom_{\mathbb{Z}\mathbf{C}}(X, Y) = \mathbb{Z}[\hom_{\mathbf{C}}(X, Y)]$$

is the free abelian group generated by $\hom_{\mathbf{C}}(X, Y)$.

(10.1) Theorem. *Let Y be a $\mathfrak{F}$-complex which admits a finite domination in $\mathfrak{F}$-**Top**. Then the finiteness obstruction*

$$[Y] = [C_*(Y)] \in K_0(\mathbb{Z}\Pi_{\mathfrak{F}}(Y, Y_0))$$

is defined, with the property that the obstruction $[Y] = 0$ is trivial if and only if Y is $\mathfrak{F}$-homotopy equivalent to a finite $\mathfrak{F}$-complex. Moreover, if the domination of Y has dimension $\leq n$ and $[Y] = 0$ then Y is $\mathfrak{F}$-homotopy equivalent to a finite $\mathfrak{F}$-complex of dimension $\leq \mathrm{Max}(3, n)$.

(10.2) Remark. A result like (10.1) was proved by different methods in the case of G-spaces in theorem 14.6 of Lück [22], without the dimension estimate.

(10.3) Lemma. *Let $\tilde{X}$ be a finite domination of Y. Then there exists a finite domination X of Y for which $f\colon Y \to X$ induces an isomorphism $\pi_0^{\mathfrak{F}}(Y; *_V) \cong \pi_0^{\mathfrak{F}}(X; f*_V)$ for every basepoint $*_V$.*

Proof. We may assume that $\tilde{f}\colon Y \to \tilde{X}$ and $\tilde{g}\colon \tilde{X} \to Y$ are cellular maps which yield the composite $\lambda = i\tilde{f}_0\tilde{g}_0\colon \tilde{X}_0 \to \tilde{X}_0 \subset \tilde{X}$. Let X be obtained by attaching 1-cells to $\tilde{X}$ as in the push-out diagram

$$\begin{tikzcd}
\tilde{X}_0 \times \partial I \arrow[r] \arrow[d, "{(i,\lambda)}"] & \tilde{X}_0 \times I \arrow[d] \\
\tilde{X} \arrow[r, "j"] & X
\end{tikzcd}$$

where $i\colon \tilde{X}_0 \to \tilde{X}$ is the inclusion. Since $\tilde{g}i \sim \tilde{g}\lambda$, there exists an extension $g\colon X \to Y$ of $\tilde{g}$ with $gj = \tilde{g}$. Hence for $f = j\tilde{f}$ we obtain the domination

$$Y \xrightarrow{f} X \xrightarrow{g} Y$$

with $gf = \tilde{g}\tilde{f} \sim 1_Y$. This shows that $f_*\colon \pi_0^{\mathfrak{F}}(Y; *_V) \to \pi_0^{\mathfrak{F}}(X; f*_V)$ is surjective. In fact, f is also surjective since an element of $\pi_0^{\mathfrak{F}}(X; f*_V)$ can be represented by a $\mathfrak{F}$-map $\xi\colon V \to X_0 = \tilde{X}_0$ via the cellular approximation theorem (3.5). For the inclusion $j_0\colon X_0 \subset X$ we get the following homotopy

$$j_0 = ji\xi \sim j\lambda\xi = jif_0g_0\xi = j_0f_0(\eta) = f_*(\eta),$$

where $\eta = g_0\xi$. *q.e.d.*

Proof of (10.1). Let X be a finite domination of Y as in (10.3). We can choose tg to be cellular and we obtain by restriction of g the map $\alpha = ig_0\colon X_0 \to Y_0 \subset Y$. We use α to construct the mapping cylinder $Y' = M_\alpha = Y \cup_\alpha IX_0$, so that (M_α, X_0) is a 0-connected relative $\mathfrak{F}$-complex and $M_\alpha \simeq Y$ in $\mathfrak{F}$-**Top**. The domination X of Y now yields a domination X' of M_α by the composites

$$Y' \simeq Y \xrightarrow{f} X \simeq X' \simeq X \xrightarrow{g} Y \simeq Y'$$

where the composites $f'\colon Y' \to X'$ and $g'\colon X' \to Y'$ can be chosen to be maps under X_0 and where the $\mathfrak{F}$-homotopy $g'f' \sim 1_{Y'}$ can be chosen to be a $\mathfrak{F}$-homotopy relative X_0. Moreover we can assume by the approximation lemma

(7.2) that (Y', X_0) and (X', X_0) are replaced by normalized and reduced relative $\mathfrak{F}$-complexes (Y'', X_0) and (X'', X_0). Hence we can apply theorem VII.2.5 of [3] which shows that a finiteness obstruction

$$[Y''] = [C_*(Y'', X_0)] \in K_0(\mathbb{Z}\Pi_{\mathfrak{F}}(X, A))$$

is defined and has the property that $[Y''] = 0$ if and only if there exists a finite relative $\mathfrak{F}$-complex (Z, X_0) which is reduced and normalized with $\dim(Z) \leq \mathrm{Max}(3, n)$, where $n = \dim(X'')$, and such that Z is $\mathfrak{F}$-homotopy equivalent to Y'' relative X_0. Since we have homotopy equivalences of chain complexes $C_*(Y) \simeq C_*(Y') \simeq C_*(Y', X_0) \simeq C_*(Y'', X_0)$, and since by the finiteness of X_0 we have $[C_*(Y')] = [C_*(Y', X_0)]$, we see that $[C_*(Y)] = [C_*(Y'', X_0)]$. This yields the results in (10.1). *q.e.d.*

11 Whitehead torsion

For $n \geq 0$ let $\mathbb{R}^{n+1}_+$ and $\mathbb{R}^{n+1}_-$ be the subspaces of $\mathbb{R}^{n+1}$ defined by the elements $(x_0, \ldots, x_n) \in \mathbb{R}^{n+1}$ with $x_0 \geq 0$ and $x_0 \leq 0$ respectively. Let D^{n+1} denote the unit sphere in $\mathbb{R}^{n+1}$ and S^n its boundary. A *ball pair* is a tuple $(\square^{n+1}, S^n, P^n, Q^n)$ homeomorphic to the Euclidean ball pair

$$(D^{n+1}, S^n, S^n \cap \mathbb{R}^{n+1}_+, S^n \cap \mathbb{R}^{n+1}_-).$$

We can assume that the basepoint of D^{n+1} is in $P^n \cap Q^n = S^{n-1}$. For every object W in $\mathfrak{F}$ and $n \geq 0$ a ball pair in $\mathfrak{F}$-**Top** is defined as

$$(\square^{n+1}_W, S^n_W, P^n_W, Q^n_W) = W \times (\square^{n+1}, S^n, P^n, Q^n),$$

and denoted simply by $(\square^{n+1}_W, Q^n_W)$, where $\square^{n+1}_W = D^{n+1}_W$. If Z is a finite $\mathfrak{F}$-set, then we can define $\square^{n+1}_Z$ as the coproduct in $\mathfrak{F}$-**Top** of the $\square^{n+1}_W$ with $W \in Z$. The pair $(\square^{n+1}_Z, Q^n_Z)$ is also termed *ball pair* in $\mathfrak{F}$-**Top**, and P^n_Z is termed the *complement of Q^n_Z in the boundary.* It is easy to see that $\square^{n+1}_Z$ is a $\mathfrak{F}$-complex and P^n_Z and Q^n_Z are subcomplexes. There are $\mathfrak{F}$-cells only in dimensions 0, $n-1$, n and $n+1$: the 0-skeleton is Z, the $(n-1)$-skeleton is $\partial Q^n_Z = \partial P^n_Z = S^{n-1}_Z$, the n-skeleton is $\partial \square^{n+1}_Z = S^n_Z = P^n_Z \cup_{S^{n-1}_Z} Q^n_Z$, and the $(n+1)$-skeleton is $\square^{n+1}_Z$ itself.

Now consider a ball pair $(\square^{n+1}_Z, Q^n_Z)$ with the complement P^n_Z in the boundary, a $\mathfrak{F}$-complex L and a $\mathfrak{F}$-map $f\colon Q^n_Z \to L$ with the property that $f(Q^n_Z, S^{n-1}_Z) \subset (L_n, L_{n-1})$. Let K be the push-out in $\mathfrak{F}$-**Top**

$$\begin{tikzcd} P^n_Z \arrow[r, "f"] \arrow[d, tail] & L \arrow[d, tail, "i"] \\ \square^{n+1}_Z \arrow[r] & K \end{tikzcd}$$

It is easy to check that K is a $\mathfrak{F}$-complex with subcomplex L. As in [3], VIII.6, page 329, if K' is a complex with L as subcomplex and $\mathfrak{F}$-isomorphic to K under L, then K' is termed an *elementary expansion* of L; moreover, there is an associated canonical $\mathfrak{F}$-retraction $r\colon K' \to L$ termed *elementary collapse.* If S is a subcomplex of L then we say that i (resp. r) is an elementary expansion (resp. collapse) *relative* S.

Now let K and L two finite $\mathfrak{F}$-complexes both containing a subcomplex S. If $j\colon L \to K$ is an *expansion relative* S we write $L \nearrow K$ rel S; is $r\colon K \to L$ is a *collapse relative* S, we write $K \searrow L$ rel S. A *formal deformation relative* S (i.e. a finite composition of expansions and collapses relative S) is denoted by $L \curvearrowright L$. A $\mathfrak{F}$-map $f\colon L \to K$ under S is a *simple homotopy equivalence* rel S if f is $\mathfrak{F}$-homotopic rel S to a formal deformation relative S.

Let L be a finite $\mathfrak{F}$-complex. Consider the set of all pairs (K, L) such that L is a subcomplex of K, K is a finite $\mathfrak{F}$-complex, and the inclusion $L \subset K$ is a $\mathfrak{F}$-homotopy equivalence. Two pairs (K, L) and (K', L) are equivalent if and only if $K \curvearrowright K'$ rel L. Let $[K, L]$ denote the class of the pair (K, L) and let $\mathbf{Wh}(L)$ denote the set of equivalence classes. It is possible to define an addition in $\mathbf{Wh}(L)$ by

$$[K, L] + [K', L] = [K \cup_L K', L]$$

so that $(\mathbf{Wh}(L), +)$ is an abelian group (lemma VIII.8.7, page 334 of [3]). Given a cellular $\mathfrak{F}$-map $f\colon L \to L'$ between finite $\mathfrak{F}$-complexes, there is an induced map

$$f_*\colon \mathbf{Wh}(L) \to \mathbf{Wh}(L')$$

defined by

$$f_*[K, L] = [K \cup_L M_f, L'],$$

where M_f is the mapping cylinder of f in $\mathfrak{F}$-**Top** (see (A.6) below). Actually **Wh** is a functor from the category of finite $\mathfrak{F}$-complexes and $\mathfrak{F}$-homotopy classes of maps to the category of abelian groups.

Now consider a $\mathfrak{F}$-homotopy equivalence $f\colon X \to L$ of finite $\mathfrak{F}$-complexes; the *Whitehead torsion* of f is defined by

$$\tau(f) = f_*[M_f, X] = [M_f \cup_X L, L] \in \mathbf{Wh}(L).$$

(11.1) Theorem. *A $\mathfrak{F}$-homotopy equivalence $f\colon X \to L$ between finite $\mathfrak{F}$-complexes is a simple homotopy equivalence if and only if the torsion vanishes $\tau(f) = 0 \in \mathbf{Wh}(L)$. Moreover:*

(*i*) *For every pair of homotopy equivalences f and g the derivation property holds:*

$$\tau(gf) = \tau(g) + g_*\tau(f).$$

(*ii*) *Consider the following double push-out diagram*

$$\begin{tikzcd}
K'_0 \arrow[rrr] \arrow[ddd] & & & K'_2 \arrow[ddd] \\
& K_0 \arrow[ul, "f_0"', "\simeq"] \arrow[r, "i_2"] \arrow[d, "i_1"] \arrow[dr, "i_0"] & K_2 \arrow[ur, "f_2", "\simeq"'] \arrow[d] & \\
& K_1 \arrow[r] \arrow[dl, "f_1"', "\simeq"] & K \arrow[dr, dotted, "f", "\simeq"'] & \\
K'_1 \arrow[rrr] & & & K'
\end{tikzcd}$$

where f_0, f_1 and f_2 are $\mathfrak{F}$-homotopy equivalences and all the non-diagonal maps are inclusions of subcomplexes. Then the push-out map f is a $\mathfrak{F}$-homotopy equivalence and the addition formula holds:

$$\tau(f) = \tau(f_1 \cup_{f_0} f_2) = i_1\tau(f_1) + i_2\tau(f_2) - i_0\tau(f_0).$$

Proof. Let $\mathcal{D}$ denote the set of finite $\mathfrak{F}$-sets. Since $(\mathfrak{F}\text{-}\mathbf{Top}, \mathcal{D})$ is a cellular I-category (see (B.3)), theorem (11.1) is a consequence of VIII.8.3, VIII.8.4 and VIII.8.5 (page 335) of [3]. *q.e.d.*

Following Ranicki [28], if $\mathbf{A}$ is a small additive category with sum denoted by $\oplus$, then the *isomorphism torsion group* $K_1^{\text{iso}}(\mathbf{A})$ is the abelian group with one generator $\tau(f)$ for each isomorphism $f\colon M \to N$ in $\mathbf{A}$, and relations

$$\tau(gf) = \tau(f) + \tau(g)$$

$$\tau(f \oplus f') = \tau(f) + \tau(f')$$

for all isomorphisms $f\colon M \to N$, $f'\colon M' \to N'$ and $g\colon N \to P$ in $\mathbf{A}$. If R is a ringoid, then it is possible to define $K_1^{\text{iso}}(R)$ as the isomorphism torsion group of the additive category $\mathbf{A}(R)$ consisting of finitely generated free R-modules. A *trivial* unit in $K_1^{\text{iso}}(\mathbb{Z}\Pi_{\mathfrak{F}}(K))$ is represented by an automorphism of the free $\mathbb{Z}\Pi_{\mathfrak{F}}(K)$-module $\mathbb{Z}\Pi_{\mathfrak{F}}(K)(-,(V,x))$ that can be written as $\pm f_*$ for some automorphism of (V,x) in $\Pi_{\mathfrak{F}}(K)$.

(11.2) Theorem. *Let K be a finite $\mathfrak{F}$-complex. Then there is an isomorphism*

$$\tau\colon \mathbf{Wh}(K) \cong K_1^{\mathbf{iso}}(\mathbb{Z}\Pi_{\mathfrak{F}}(K))/T$$

between the Whitehead group of K and the quotient of $K_1^{\text{iso}}(\mathbb{Z}\Pi_{\mathfrak{F}}(K))$ by the subgroup T generated by all trivial units. It is defined by associating to $[X, K] \in \mathbf{Wh}(K)$ the torsion of the contractible $\Pi_{\mathfrak{F}}(K)$-chain complex $C_*(X', K)$, where X' is a normalization of X rel K.

Proof. It is a consequence of theorem VIII.12.7, page 348, of [3]. *q.e.d.*

In case of transformation groups this is theorem 14.16, page 286, of [22] (in the same book the equivariant finiteness obstruction can be found). Compare with the equivariant Whitehead torsion of Hauschild [13], Dovermann and Rothenberg [11], Illman [15] and Anderson [1].

Appendix A. Basic homotopy theory in $\mathfrak{F}$-Top

We proved several results above by refering to [3]. We now describe some results that are needed in order to apply the abstract theory of [3] to $\mathfrak{F}$-**Top**.

(A.1) Definition. A $\mathfrak{F}$-map $i\colon Y \to X$ is a $\mathfrak{F}$-cofibration if i is a closed inclusion and the following homotopy extension property holds: for every commutative diagram in $\mathfrak{F}$-**Top**

$$\begin{tikzcd}
Y \arrow[r, rightarrowtail, "i"] \arrow[d, rightarrowtail, "i_0"] & X \arrow[d, rightarrowtail, "i'_0"] \arrow[ddr, bend left, "f"] & \\
Y \times I \arrow[r, rightarrowtail, "i \times 1_I"] \arrow[drr, bend right, "H"'] & X \times I \arrow[dr, dotted, "G"] & \\
 & & Z
\end{tikzcd}$$

there is a $\mathfrak{F}$-map $G\colon X \times I \to Z$ such that $G \circ i'_0 = f$ and $G \circ (i \times 1_I) = H$; in the diagram i_0 is the map defined by $y \mapsto (y, 0)$ for every $y \in Y$, i'_0 the same for X.

Equivalently a closed inclusion $i\colon Y \to X$ in $\mathfrak{F}$-**Top** is a $\mathfrak{F}$-cofibration if and only if the $\mathfrak{F}$-map $(Y \times I) \cup_Y X \to X \times I$ admits a retraction (that is a left inverse). This implies that $\bar{Y} \to \bar{X}$ and $Y \to X$ are cofibrations in **Top**.

(A.2) Lemma. *For a cofibration $i\colon A \to M$ in $\mathfrak{F}$-**Top** and a $\mathfrak{F}$-map $h\colon A \to Y$, in the push-out diagram (which exists by (2.4))*

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "i"] & Y \arrow[d, "\bar{i}"] \\ M \arrow[r] & X = M \cup_A Y \end{tikzcd}$$

the induced $\mathfrak{F}$-map $\bar{i}$ is also a cofibration. Moreover, I carries this push-out diagram into a push-out diagram (i.e. $(M \cup_A Y) \times I = (M \times I) \cup_{A\times I} (Y \times I)$).

Proof. The proof is formally identical to the proof for compactly generated spaces with no structure group (see (5.1) and (5.4) of [34], pages 22–23); or, directly applying the homotopy extension property (A.1) and the assumptions on the fibres, is the same as the proof in **Top** (see e.g. lemma 2.3.6, page 56 of [27]). *q.e.d.*

The next result shows that basic homotopy theory is available in $\mathfrak{F}$-**Top**. For the axioms and properties of I-categories and cofibration categories we refer the reader to [3]. As in section 1 the cylinder in $\mathfrak{F}$-**Top** is given by $IX = X \times I$.

(A.3) Theorem. *The category $\mathfrak{F}$-**Top** with the cylinder IX is an I-category. Moreover, it is a cofibration category, with cofibrations the $\mathfrak{F}$-cofibrations and weak equivalences the $\mathfrak{F}$-homotopy equivalences. All objects are fibrant and cofibrant in $\mathfrak{F}$-**Top**.*

Proof. To show that the axioms of I-category are fulfilled, since lemma (A.2) holds, it suffices to use the argument of [3], proposition (8.2), with the same homeomorphism $\alpha\colon I^2 \to I^2$, to prove the relative cylinder axion, see page 222 of [3]. Then we can apply Theorem (7.4) of [3], page 223. *q.e.d.*

The theorem implies that the following lemmata are true in $\mathfrak{F}$-**Top**. Corollaries (A.4), (A.5) and (A.6) are immediate consequences of the first three axioms of a cofibration category.

(A.4) Corollary (Composition). *Isomorphisms in $\mathfrak{F}$-**Top** are $\mathfrak{F}$-homotopy equivalences and also cofibrations. For two maps*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

if any two of f, g and gf are $\mathfrak{F}$-homotopy equivalences then so is the third. The composite of $\mathfrak{F}$-cofibrations is a $\mathfrak{F}$-cofibration.

(A.5) Corollary (Pushout). *Let $i\colon A \to M$ be a $\mathfrak{F}$-cofibration and $f\colon A \to Y$ be a $\mathfrak{F}$-map. Consider the pushout in $\mathfrak{F}$-**Top**,*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, tail, "i"] & Y \arrow[d, "\bar{i}"] \\
M \arrow[r, "\bar{f}"'] & Y \cup_f M
\end{tikzcd}$$

where $\bar{i}$ is a $\mathfrak{F}$-cofibration by (A.2). If f is a $\mathfrak{F}$-homotopy equivalence, then so is $\bar{f}$; if i is a $\mathfrak{F}$-homotopy equivalence, then so is $\bar{i}$.

(A.6) Corollary (Factorization). *Every $\mathfrak{F}$-map $f\colon X \to Y$ can be factorized as*

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, tail, "i"'] & & Y \\
& Y' \arrow[ur, "\simeq", "g"'] &
\end{tikzcd}$$

*where g is a $\mathfrak{F}$-homotopy equivalence and i a $\mathfrak{F}$-cofibration. The $\mathfrak{F}$-family Y' is the mapping cylinder (in $\mathfrak{F}$-**Top**) of f, i.e. the push-out of $i_0\colon X \to IX$ and f.*

(A.7) Corollary (Gluing lemma). *Consider the following diagram,*

$$\begin{tikzcd}
A' \arrow[rrr, "h'"] \arrow[ddd, tail] & & & Y' \arrow[ddd] \\
& A \arrow[ul, "\simeq"] \arrow[r, "h"] \arrow[d, tail] & Y \arrow[ur, "\simeq"'] \arrow[d] & \\
& M \arrow[dl, "\simeq"] \arrow[r] & Y \cup_A M \arrow[dr, dotted, "\simeq"] & \\
M' \arrow[rrr] & & & Y' \cup_{A'} M'
\end{tikzcd}$$

*where the two squares are push-outs in $\mathfrak{F}$-**Top**, and $A \to M$ and $A' \to M'$ are $\mathfrak{F}$-cofibrations. The dotted map exists as a consequence of (A.2). If the arrows $A \to A'$, $M \to M'$ and $Y \to Y'$ are $\mathfrak{F}$-homotopy equivalences, then so is the push-out map $Y \cup_A M \to Y' \cup_{A'} M'$.*

Proof. See lemma (1.2) of [2], page 84. *q.e.d.*

(A.8) Corollary (Lifting lemma). *Consider a $\mathfrak{F}$-cofibration $i\colon A \to M$, a $\mathfrak{F}$-homotopy equivalence $p\colon X \to Y$, and two maps $f\colon A \to X$, $g\colon M \to Y$ such that $gi = pf$. Then there is a $\mathfrak{F}$-map $h\colon M \to X$ such that $hi = f$ and $ph \sim g$ rel A. The map h is termed the* lifting *of the following diagram, and is unique up to homotopy* rel A.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, tail, "i"] & X \arrow[d, "p", "\simeq"'] \\ M \arrow[r, "g"] \arrow[ur, dotted, "h"] & X \end{tikzcd}$$

Proof. See lemma (1.11) of [2], page 90. *q.e.d.*

The following is Dold's theorem in $\mathfrak{F}$-**Top**.

(A.9) Corollary. *Consider the commutative diagram.*

$$\begin{tikzcd} X \arrow[rr, "g"] & & Y \\ & A \arrow[ul, tail] \arrow[ur, tail] & \end{tikzcd}$$

If g is a $\mathfrak{F}$-homotopy equivalence, then g is a $\mathfrak{F}$-homotopy equivalence under A, that is, there is a $\mathfrak{F}$-map $f\colon Y \to X$ such that $gf \sim 1_Y$ rel A *and $fg \sim 1_X$* rel A.

Proof. See corollary (2.12) of [2], page 96. *q.e.d.*

(A.10) Remark. Let $\mathfrak{F}$ be a structure category with fibre functor F and $i\colon A \to X$ a $(\mathfrak{F}, F)$-cofibration. Then $\bar{i}\colon \bar{A} \to \bar{X}$ and $i\colon A \to X$ are (closed) cofibrations in **Top**. Moreover, since the fibre functor F is faithful, the induced map i^V on function spaces

$$i^V\colon A^V \to X^V$$

is a $(\mathfrak{F}, F^V)$-cofibration for every $V \in \mathfrak{F}$.

Proof. Since $i\colon A \to X$ is a $(\mathfrak{F}, F)$-cofibration, there is a $(\mathfrak{F}, F)$-retraction

$$r\colon X \times I \to A \times I \cup X \times \{0\}.$$

By definition of $\mathfrak{F}$-map, this implies that i and $\bar{i}$ are cofibrations in **Top**, where clearly $A \subset X$ and $\bar{A} \subset \bar{X}$ are closed. Moreover, assume that F is faithful, so that the function spaces A^V, X^V are defined for every $V \in \mathfrak{F}$. Then there exists the induced map

$$r^V\colon (X \times I)^V \to (A \times I \cup X \times \{0\})^V.$$

Since $(X \times I)^V \cong X^V \times I$ and

$$(A \times I \cup X \times \{0\})^V \cong A^V \times I \cup X^V \times \{0\},$$

and the homeomorphisms are actually $(\mathfrak{F}, F^V)$-isomorphisms, there is a $(\mathfrak{F}, F^V)$-retraction

$$X^V \times I \to A^V \times I \cup X^V \times \{0\}.$$

Since $i^V\colon A^V \to X^V$ is a closed inclusion, it is a $(\mathfrak{F}, F^V)$-cofibration. *q.e.d.*

We recall now the definition of $\mathfrak{F}$-deformation retract, following Whitehead [34], page 24. A $\mathfrak{F}$-family A is a $\mathfrak{F}$-deformation retract of $X \supset A$ if there is a $\mathfrak{F}$-homotopy $F\colon X \times I \to X$ such that $F(x,t) = x$ for each $(x,t) \in A \times I \cup X \times \{0\}$ and $F(X \times \{1\}) \subset A$.

(A.11) Lemma. *Let X be a $(\mathfrak{F}, F)$-family and $A \subset X$ a closed inclusion. Then for every $V \in \mathfrak{F}$ the induced map $A^V \to X^V$ is a closed inclusion of $(\mathfrak{F}, F^V)$-families. Moreover, if A is a $(\mathfrak{F}, F)$-deformation retract of X, then A^V is a $(\mathfrak{F}, F^V)$-deformation retract of X^V. A is a deformation retract of X in* **Top** *and $\bar{A}$ is a deformation retract of $\bar{X}$ in* **Top**.

Proof. The inclusion $A^V \to X^V$ is a closed inclusion since $A^V \subset X^V$ is equal to the pre-image of $\bar{A}$ under the map $X^V \to \bar{X}$. Furthermore, any $(\mathfrak{F}, F)$-deformation retraction $F\colon X \times I \to X$ yields a $(\mathfrak{F}, F^V)$-deformation retraction $F^V\colon X^V \times I \to X^V$. *q.e.d.*

Appendix B. The category of coefficients

We here describe the coefficient functor for $\mathfrak{F}$-**Top** which is a basic ingredient of the theory in [3].

(B.1) Definition. Let A be a $\mathfrak{F}$-space. Consider pairs of $\mathfrak{F}$-maps $x, y\colon Z \to A$ where Z is a $\mathfrak{F}$-set. We associate with x, y the $\mathfrak{F}$-space $X(x,y)$ given by the push-out in $\mathfrak{F}$-**Top**

$$\begin{tikzcd}
Z \times \{0,1\} \arrow[r, "{(x,y)}"] \arrow[d] & A \arrow[d] \\
Z \times I \arrow[r] & X(x,y)
\end{tikzcd}$$

Hence $(X(x,y),A)$ is a reduced 1-dimensional relative $\mathfrak{F}$-complex. We call it a *$\mathfrak{F}$-graph under A*. Let $\mathbf{T}$ denote the category *of $\mathfrak{F}$-graphs*: Objects are $\mathfrak{F}$-graphs $X = (X,A) = (X(x,y),A)$ and morphisms are $\mathfrak{F}$-homotopy classes rel A of $\mathfrak{F}$-maps, so that

$$\mathbf{T}(X,Y) = [X,Y]^A_{\mathfrak{F}}$$

for $\mathfrak{F}$-graphs X, Y under A. The coproduct $X \vee_A Y = X \cup_A Y$ in $\mathbf{T}$ is defined by the push-out of $X \leftarrow A \rightarrow Y$ in $\mathfrak{F}$-$\mathbf{Top}$. Since finite coproducts exists in $\mathbf{T}$, $\mathbf{T}$ is a theory. Let $\mathbf{T}^\sharp$ denote the full subcategory of $\mathbf{T}$ consisting of $\mathfrak{F}$-graphs $X(x,y)$ for which the $\mathfrak{F}$-set Z above is finite. Since $\mathfrak{F}$ is a small category, we see that $\mathbf{T}^\sharp$ is a small category. Let $\mathbf{model}(\mathbf{T}^\sharp)$ denote the *category of models of $\mathfrak{F}$-graphs*, that is the category of functors $(\mathbf{T}^\sharp)^{\mathrm{op}} \to \mathbf{Set}$ which carry coproducts in $\mathbf{T}^\sharp$ to products in $\mathbf{Set}$.

(B.2) Lemma. *A $\mathbf{T}$-complex in $\mathfrak{F}$-$\mathbf{Top}^A$ as defined in [3] is the same as a relative $\mathfrak{F}$-complex (X,A) which is reduced and normalized.*

Proof. According to IV.2.2 of [3] a $\mathbf{T}$-complex $X_{\geq 1}$ in $\mathfrak{F}$-$\mathbf{Top}$ is given by a sequence of $\mathfrak{F}$-cofibrations

$$X_{\geq 1} = (X_1 \subset X_2 \subset \dots)$$

where X_1 is an object in $\mathbf{T}$ and (X_{n+1}, X_n) is a principal cofibration with attaching map $\partial_{n+1}\colon \Sigma^{n-1}C_{n+1} \to X_n$, $n \geq 1$, where C_{n+1} is a cogroup in $\mathbf{T}$ (that is, a 1-dimensional spherical object S^1_α under A). Since $\Sigma^{n-1}S^1_\alpha = S^n_\alpha$, we have the result. *q.e.d.*

(B.3) Remark. The category $\mathfrak{F}$-$\mathbf{Top}$ is a cofibration category under $\mathbf{T}$ that satisfies the *Blakers–Massey property* (see [3], page 245). Hence it follows from proposition 1.2, page 250 of [3] that $\mathfrak{F}$-$\mathbf{Top}$ is a homological cofibration category under $\mathbf{T}$. Moreover $\mathfrak{F}$-$\mathbf{Top}$ is a cellular I-category since the ball pair axiom holds, as a consequence of the ball pair axiom in $\mathbf{Top}$ and the cellular approximation theorem (3.5) (see page 327 of [3]).

We recall that a *track category*, i.e. a *category enriched in groupoids*, is a 2-category all of whose 2-cells are invertible (see [20] for details). In particular the category of groupoids $\mathbf{Grd}$ is a track category. Objects are groupoids, morphisms are functors between groupoids and tracks are natural transformations. A *track functor*, or else a 2-functor, between track categories is a groupoid enriched functor.

Let $\mathfrak{F}$ be a topological enriched category. Then there is a canonical track category $[\![\mathfrak{F}]\!]$ associated to $\mathfrak{F}$ defined as follows. Objects and morphisms are the same as $\mathfrak{F}$ and 2-cells are tracks $b\colon \alpha \Longrightarrow \alpha'$ in $W^V = \hom_{\mathfrak{F}}(V, W)$ with $\alpha, \alpha'\colon V \to W$ and V, W objects in $\mathfrak{F}$

A $[\![\mathfrak{F}^{\mathrm{op}}]\!]$-diagram is a track functor $[\![\mathfrak{F}^{\mathrm{op}}]\!] \to \mathbf{Grd}$. A morphism between $[\![\mathfrak{F}^{\mathrm{op}}]\!]$-diagrams is a natural transformation of track functors. An example of a track functor is given by

$$[\![\Pi_{\mathfrak{F}}(X^\circ, A^\circ)]\!]\colon [\![\mathfrak{F}]\!] \to \mathbf{Grd},$$

with (X, A) a pair of $\mathfrak{F}$-families. The image of an object $V \in [\![\mathfrak{F}]\!]$ is the groupoid $\Pi(X^V, A^V)$. The image of a morphism $\lambda\colon V \to W$ in $[\![\mathfrak{F}]\!]$ is the morphism of groupoids $\lambda^*\colon \Pi(X^W, A^W) \to \Pi(X^V, A^V)$ induced by the map $X^\lambda\colon X^W \to X^V$. The image of a track $b\colon \lambda \Longrightarrow \lambda'$ in $[\![\mathfrak{F}]\!]$ is given by the natural transformation

$$\begin{tikzcd}[column sep=huge] \Pi(X^W, A^W) \arrow[r, bend left, "\Pi(\lambda)"] \arrow[r, bend right, "\Pi(\lambda')"'] \arrow[r, phantom, "\Downarrow b^*"] & \Pi(X^V, A^V) \end{tikzcd}$$

in $\mathbf{Grd}$.

With the aid of the track category $[\![\mathfrak{F}]\!]$ we define the category $(\mathfrak{F}, A)$-$\mathbf{Grd}$ as follows. Objects are track functors $F\colon [\![\mathfrak{F}^{\mathrm{op}}]\!] \to \mathbf{Grd}$ under $[\![\Pi(A^\circ)]\!]$ such that the natural transormation

$$\iota\colon [\![\Pi(A^\circ)]\!] \to F$$

for each $V \in \mathfrak{F}$ yields a functor

$$\iota\colon \Pi(A^V) \to F^V$$

which is the identity on objects. A morphism in $(\mathfrak{F}, A)$-$\mathbf{Grd}$ is a natural transformation of track functors under $[\![\Pi(A^\circ)]\!]$.

(B.4) Theorem. *Let $\mathbf{T}^\sharp$ denote the theory of finite $\mathfrak{F}$-graphs under A. Then there is an equivalence of categories*

$$\mathbf{model}(\mathbf{T}^\sharp) \simeq (\mathfrak{F}, A)\text{-}\mathbf{Grd}.$$

Proof. Consider the following functor

$$\xi\colon \mathbf{model}(\mathbf{T}^\sharp) \to (\mathfrak{F}, A)\text{-}\mathbf{Grd}.$$

Let $M\colon (\mathbf{T}^\sharp)^{\mathrm{op}} \to \mathbf{Set}$ be a model of $\mathbf{T}^\sharp$. Then $\xi(M)$ is the $[\![\mathfrak{F}^{\mathrm{op}}]\!]$-diagram of groupoids under $[\![\Pi(A^\circ)]\!]$ defined as follows. Let V be an object of $\mathfrak{F}$.

Objects of the groupoid $\xi(M)(V)$ are the same objects as $\Pi^V_{\mathfrak{F}}(A) = \Pi(A^V)$, i.e. $\mathfrak{F}$-points of type $x_V : V \to A$. For every pair of objects x_V, y_V in $\xi(M)(V)$ the homset is given by the set $M(X(x_V, y_V))$. Let x_V, y_V and z_V be points in A^V. The natural $\mathfrak{F}$-map

$$X(x_V, z_V) \to X(x_V \cup y_V, y_V \cup z_V) = X(x_V, y_V) \vee X(y_V, z_V)$$

induces under M a function

$$\begin{tikzcd}
M(X(x_V \cup y_V, y_V \cup z_V)) \arrow[d, equal] & \\
M(X(x_V, y_V)) \times M(X(y_V, z_V)) \arrow[r] & M(X(x_V, z_V))
\end{tikzcd}$$

which yields the composition law for morphisms in $\xi(M)$. Since there is a natural $\mathfrak{F}$-isomorphism $X(x_V, y_V) \simeq X(y_V, x_V)$, we obtain that $\xi(M)(V)$ is a groupoid.

Now let $\lambda : V \to W$ be a morphism in $\mathfrak{F}$. The induced $\mathfrak{F}$-map $A^W \to A^V$ is a function on objects of $\xi(M)$. Given x_W and y_W in A^W the induced $\mathfrak{F}$-map

$$X(x_W\lambda, y_W\lambda) \to X(x_W, y_W)$$

has an image under M

$$\begin{tikzcd}
\xi(M)(W)(x_W, y_W) \arrow[d, equal] & \xi(M)(V)(x_W\lambda, y_W\lambda) \arrow[d, equal] \\
M(X(x_W, y_W)) \arrow[r] & M(X(x_W\lambda, y_W\lambda))
\end{tikzcd}$$

Hence $\xi(M)(\lambda)$ is a morphism of groupoids, and $\xi(M)$ is a functor $\mathfrak{F}^{\mathrm{op}} \to$ **Grd**. It is not difficult to show that it is a functor under $\Pi(A^\circ)$.

It is left to define the values of $\xi(M)$ on the tracks of $[\![\mathfrak{F}^{\mathrm{op}}]\!]$. A track $b : \lambda \Longrightarrow \lambda'$ in $[\![\mathfrak{F}^{\mathrm{op}}]\!]$ is represented by a $\mathfrak{F}$-map

$$b : V \times I \to W.$$

Hence for every $x_W : W \to A$ the track b induces a $\mathfrak{F}$-map $x_W b : V \times I \to A$, and hence a $\mathfrak{F}$-map

$$X(x_W\lambda, x_W\lambda') \to A.$$

By taking the values of the model M we obtain a function

$$* = M(A) \to M(X(x_W\lambda, x_W\lambda')),$$

hence we obtain an element in $M(X(x_W\lambda, x_W\lambda'))$ which is therefore a morphism in $\xi(M)(V)(x_W\lambda, x_W\lambda')$. This defines the natural transormation induced by b. By the elementary properties of M it is easy to prove that ξ is a track functor.

Conversely, let $\mathbf{C}$ be an object in $(\mathfrak{F}, A)$-$\mathbf{Grd}$. Then to each $\mathfrak{F}$-graph $X(x_Z, y_Z)$ with $x_Z, y_Z\colon Z \to A$ we can associate the set

$$\xi'(\mathbf{C})(X(x_Z, y_Z) = \mathrm{Mor}\,(\llbracket\Pi(X(x_Z, y_Z)^\circ, A^\circ)\rrbracket, \mathbf{C})$$

of all morphisms in $(\mathfrak{F}, A)$-$\mathbf{Grd}$ from $\llbracket\Pi(X(x_Z, y_Z)^\circ, A^\circ)\rrbracket$ to $\mathbf{C}$. It is clear that $\xi'(\mathbf{C})$ is a functor $\left(\mathbf{T}^\sharp\right)^{\mathrm{op}} \to \mathbf{Set}$. It is also a model by (B.8) below. It is possible to prove that this construction yields a functor

$$\xi'\colon (\mathfrak{F}, A)\text{-}\mathbf{Grd} \to \mathbf{model}(\mathbf{T}^\sharp).$$

The functors ξ and ξ' are equivalences of categories (the details are left to the reader). *q.e.d.*

We define the *coefficient functors* c and $\pi_{\mathfrak{F}}$

(B.5)
$$\begin{tikzcd}
 & \mathbf{model}(\mathbf{T}^\sharp) \arrow[d, "\xi"', "\simeq"] \\
(\mathfrak{F}\text{-}\mathbf{Top})^A/_{\simeq \text{ rel } A} \arrow[ur, "c"] \arrow[r, "\pi_{\mathfrak{F}}"] & (\mathfrak{F}, A)\text{-}\mathbf{Grd}
\end{tikzcd}$$

as follows. For every $\mathfrak{F}$-family (X, A) under A (where $A \to X$ is a $\mathfrak{F}$-cofibration) let $\pi_{\mathfrak{F}}(X, A) = \llbracket\Pi(X^\circ, A^\circ)\rrbracket$. Furthermore, let $c(X, A)$ be the model in $\mathbf{model}(\mathbb{T}^\sharp)$ that associates to the $\mathfrak{F}$-graph $X(x, y)$ with $x, y\colon Z \to A$ the set of $\mathfrak{F}$-homotopy classes rel A of $\mathfrak{F}$-maps extending 1_A

$$X(x, y) \mapsto [X(x, y), X]^A_{\mathfrak{F}}.$$

(B.6) Proposition. *The diagram (B.5) is commutative:*

$$\xi c = \pi_{\mathfrak{F}}.$$

Proof. The object $\xi c(X, A)$ in $(\mathfrak{F}, A)$-$\mathbf{Grd}$ associates to $V \in \llbracket\mathfrak{F}^{\mathrm{op}}\rrbracket$ the groupoid $\xi c(X, A)^V$ with objects the points in A^V; the morphisms are the sets

$$c(X, A)(X(x, y)) = [X(x, y), X]^A_{\mathfrak{F}}$$

where $x, y\colon V \to A$ are $\mathfrak{F}$-points. The morphism set $[X(x, y), X]^A_{\mathfrak{F}}$ is naturally isomorphic to $\Pi(X^V, A^V)(x, y)$. The same argument can be applied to tracks, so that we get $\xi c = \pi_{\mathfrak{F}}$ as claimed. *q.e.d.*

Now we define the category $(\mathfrak{F}, A)$-**Set**. Objects are pairs of $\mathfrak{F}$-maps $x, y\colon Z \to A$ where Z is a $\mathfrak{F}$-set. A morphism $(x, y) \to (x', y')$ is a $\mathfrak{F}$-map $u\colon Z \to Z'$ such that $x'u = x$ and $y'u = y$.

Consider now an object $F\colon [\![\mathfrak{F}^{\mathrm{op}}]\!] \to \mathbf{Grd}$ in $(\mathfrak{F}, A)$-**Grd**. For every object $V \in \mathfrak{F}$ we denote by F^V the groupoid associated to V. For every $V \in \mathfrak{F}$ consider the two maps

$$F_1^V \overset{\partial_0}{\underset{\partial_1}{\rightrightarrows}} A^V$$

from the set of arrows of F^V to A^V given by the source ∂_0 and the target ∂_1 in the groupoid F^V. This yields an object $x_V, y_V\colon Z_V = V \times F_1^V \to A$ in $(\mathfrak{F}, A)$-**Set**. The coproduct of all such objects is an object $\Phi(F) = x, y\colon Z \to A$. This defines the forgetful functor

$$\Phi\colon (\mathfrak{F}, A)\text{-}\mathbf{Grd} \to (\mathfrak{F}, A)\text{-}\mathbf{Set}.$$

The left-adjoint of Φ denoted by $<,>$ sends an object $x, y\colon Z \to A$ in $(\mathfrak{F}, A)$-**Set** to the free object $< x, y >$ in $(\mathfrak{F}, A)$-**Grd**.

(B.7) Proposition. *For every $x, y\colon Z \to A$, we have a natural isomorphism*

$$< x, y >= \pi_{\mathfrak{F}}(X(x, y), A).$$

Moreover a result corresponding to the Seifert–van Kampen theorem holds. The proof is a modification of the Brown's proof [8, 9] of the Seifert–van Kampen theorem for groupoids.

(B.8) Proposition. *If X_1 and X_2 are $\mathfrak{F}$-complexes relative A both containing a copy of a relative $\mathfrak{F}$-complex (X_0, A) as a subcomplex then*

$$\pi_{\mathfrak{F}}(X_1 \cup_{X_0} X_2) = \pi_{\mathfrak{F}}(X_1) \cup_{\pi_{\mathfrak{F}}(X_0)} \pi_{\mathfrak{F}}(X_2).$$

*Here the right hand side denotes the push-out in $(\mathfrak{F}, A)$-**Grd**.*

(B.9) Remark. In [3] the coefficient functor c maps to the category **Coef** which can be identified with

$$(\mathfrak{F}, A)\text{-}\mathbf{Grd} \simeq \mathbf{model}(\mathbf{T}^\sharp) \simeq \mathbf{Coef}$$

such that the coefficient functor c in [3] mapping to **Coef** coincides with $\pi_{\mathfrak{F}}$ and c in (B.5). Moreover the *enveloping functor* U of [3] (pages ≥ 150) can be identified with the composite

$$U\colon \mathbf{Coef} \xrightarrow{\cong} (\mathfrak{F}, A)\text{-}\mathbf{Grd} \xrightarrow{\bar{U}} \mathbf{Ringoids}$$

where $\bar{U}$ carries F to the category $\mathbb{Z}\left[(\int_{\mathfrak{F}} F)/_{\sim}\right]$. Here the equivalence relation $\sim$ for the integration category $\int_{\mathfrak{F}} F$ is defined as in (6.3) above. Hence for a reduced relative $\mathfrak{F}$-complex (X, A) we get

$$Uc(X, A) = \bar{U}(\pi_{\mathfrak{F}}(X, A)) = \mathbb{Z}\Pi_{\mathfrak{F}}(X, A)$$

where we use the discrete fundamental category in (6.3).

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