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COMMUTING RATIONAL FUNCTIONS REVISITED

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Abstract. Let B be a rational function of one complex variable, which is neither a Lattès map nor conjugate to $z^{\pm n}$ or $\pm T_n$, and let C_B be the set of all rational functions commuting with B. We show that the quotient of C_B by a certain equivalence relation is a finite group G_B. We describe generators of G_B in terms of the fundamental group of a special graph associated with B, providing a method for describing C_B, and calculate G_B for several classes of rational functions.

1. Introduction

In this paper we study commuting rational functions, that is rational solutions of the functional equation

$$B \circ X = X \circ B. \tag{1}$$

More precisely, we fix a function $B \in \mathbb{C}(z)$ of degree at least two and study the set C_B consisting of all $X \in \mathbb{C}(z)$ such that (1) holds.

Functional equation (1) was investigated already by Julia [3] and Fatou [4]. In particular, they showed that commuting rational functions X and B of degree at least two have a common repelling periodic point and the same Julia set $J = J(X) = J(B)$. Using Poincaré functions, Julia and Fatou proved that if commuting X and B have no iterate in common and $J \neq \mathbb{CP}^1$, then, up to a conjugacy, X and B are either powers or Chebyshev polynomials. The assumption $J \neq \mathbb{CP}^1$ was removed by Ritt [14], who used a topological-algebraic method. Ritt proved that solutions of (1) having no iterate in common reduce either to powers, or to Chebyshev polynomials, or to Lattès maps. A proof of the Ritt theorem based on modern dynamical methods was given by Eremenko [1].

All the above results assume that X and B have no iterate in common. However, commuting rational functions X and B which *do* have a common iterate, that is satisfy

$$X^{\circ l} = B^{\circ k} \tag{2}$$

for some $l, k \geqslant 1$ also exist. The simplest examples of such functions can be obtained setting

$$X = R^{\circ l_1}, \qquad B = R^{\circ l_2},$$

where R is an arbitrary rational function and $l_1, l_2 \geqslant 0$. More generally, we can set

$$X = \mu_1 \circ R^{\circ l_1}, \qquad B = \mu_2 \circ R^{\circ l_2}, \tag{3}$$

where μ_1 and μ_2 are Möbius transformations commuting with R and between themselves. However, it was shown already by Ritt ([14]) that commuting rational functions satisfying (2) are not exhausted by functions of the form (3).

Although Ritt's method provides some insight on the structure of commuting rational functions X and B satisfying (2), it does not permit to describe this class of functions in an explicit way, and Ritt concluded his paper saying "we think that the example given above makes it conceivable that no great order may reign in this class". Notice also that the Ritt method uses some chains of transformations involving both functions X and B. By this reason, it provides an information rather about *pairs* of commuting functions than about the set C_B for a given function B. In particular, if B is not *special*, that is if B is neither a Lattès map, nor conjugate to $z^{\pm n}$ or $\pm T_n$, then essentially all the information about C_B provided by the Ritt method reduces to the fact that any element of C_B has a common iterate with B.

Functional equation (1) is a particular case of the functional equation

$$A \circ X = X \circ B, \tag{4}$$

where A and B are rational functions of degree at least two. In case if (4) is satisfied for some rational function X of degree at least two, the function B is called *semiconjugate* to the function A. Semiconjugate rational functions were investigated in the recent papers [5], [7], [9], [10], [11]. In particular, it was shown in [5] that solutions of (4) satisfying $\mathbb{C}(X, B) = \mathbb{C}(z)$, called *primitive*, can be described in terms of group actions on $\mathbb{CP}^1$ or $\mathbb{C}$, implying strong restrictions on a possible form of A, B and X.

Any solution of (4) can be reduced to a primitive one by a simple iterative process. Namely, by the Lüroth theorem, the field $\mathbb{C}(X, B)$ is generated by some rational function W. Thus, if a solution A, X, B of (4) is not primitive, then there exists a rational function W of degree greater than one such that $\mathbb{C}(X, B) = \mathbb{C}(W)$ and the equalities

$$X = X' \circ W, \quad B = B' \circ W \tag{5}$$

hold for some rational functions X' and B'. Substituting these expressions in (4) we see that the triple $A, X', W \circ B'$ is another solution of (4). This new solution is not necessary primitive. Nevertheless, $\deg X' < \deg X$. Therefore, after a finite number of similar transformations we will arrive to a primitive solution. The considered transformations

$$X \to X', \qquad B \to W \circ B' \tag{6}$$

are clear analogues of the "transformations of the first type" used by Ritt, which transform commuting functions X and B satisfying (5) to commuting functions $W \circ X'$ and $W \circ B'$. However, transformations (6) have certain advantages since they do not affect the function A, and the corresponding iterative process always stops after a finite number of steps. Moreover, it was shown in [11] that for a fixed non-special rational function B the number of required steps in order to reach a primitive solution can be somehow "controlled", implying some finiteness results for solutions of equations (1) and (4).

Specifically, regarding to equation (1), it was shown in [11] that if B is not special, then there exist *finitely many* rational functions $X_1, X_2, \dots, X_n$ such that X commutes with B if and only if

$$X = X_j \circ B^{\circ k}$$

for some j, $1 \leqslant j \leqslant r$, and $k \geqslant 0$. Moreover, r and the degrees of X_j, $1 \leqslant j \leqslant r$, can be bounded by numbers depending on $\deg B$ only. This result immediately implies the Ritt theorem in its part concerning non-special functions. Indeed, if X commutes with B, then any iterate $X^{\circ l}$, $l \geqslant 1$, does. Thus, by the Dirichlet box principle, there exist distinct l_1, l_2 such that

$$X^{\circ l_1} = X_j \circ B^{\circ k_1}, \qquad X^{\circ l_2} = X_j \circ B^{\circ k_2}$$

for the same j and some k_1, $k_2 \geqslant 0$. Therefore, if, say, $l_2 > l_1$, then

$$X^{\circ l_2} = X^{\circ l_1} \circ B^{\circ k_2 - k_1},$$

implying that (2) holds for $l = l_2 - l_1$ and $k = k_2 - k_1$, since X and B commute.

The main results of this paper are following. First, for any non-special rational function B we introduce the structure of a finite group on the quotient of C_B by a certain equivalence relation. Second, we describe generators of this group in terms of the fundamental group of a special graph associated with B, providing a method for describing C_B. Finally, we calculate G_B for several classes of rational functions. Notice that our method of describing C_B reduces the problem to the following two easier problems: finding all functional decompositions $F = U \circ V$ of a given rational function F into a composition of rational functions U and V of degree at least two, and finding all Möbius transformations commuting with F.

In more details, for a non-special rational function B we define an equivalence relation $\underset{B}{\sim}$ on the set C_B, setting $A_1 \underset{B}{\sim} A_2$ if

$$A_1 \circ B^{\circ l_1} = A_2 \circ B^{\circ l_2}$$

for some $l_1 \geqslant 0$, $l_2 \geqslant 0$, and show that the multiplication of classes induced by the functional composition of their representatives provides $C_B/\underset{B}{\sim}$ with the structure of a finite group G_B. The existence of the group structure on $C_B/\underset{B}{\sim}$ offers a new look at the problem of describing C_B, and permits to describe properties of C_B in group theoretic terms. For example, the statement that the group G_B is trivial is equivalent to the statement that for a given rational function B any element of C_B is an iterate of B.

We describe generators of G_B using a special finite graph Γ_B defined as follows. Let B be a rational function. A rational function $\widehat{B}$ is called an *elementary transformation* of B if there exist rational functions U and V such that $B = V \circ U$ and $\widehat{B} = U \circ V$. We will say that rational functions B and A are *equivalent* and write $A \sim B$ if there exists a chain of elementary transformations between B and A (this equivalence relation should not be confused with the previous one where the subscript B is used). Since for any Möbius transformation μ the equality

$$B = (B \circ \mu^{-1}) \circ \mu$$

holds, the equivalence class $[B]$ of a rational function B is a union of conjugacy classes. Moreover, by the result of [7], the equivalence class $[B]$ consists of finitely many conjugacy classes, unless B is a flexible Lattès map. The graph Γ_B is defined as a multigraph whose vertices are in a one-to-one correspondence with some fixed representatives B_i of conjugacy classes in $[B]$, and whose multiple edges connecting the vertices corresponding to B_i to B_j are in a one-to-one correspondence with solutions of the system

$$B_i = V \circ U, \quad B_j = U \circ V$$

in rational functions. In these terms, the main result of the paper about the group G_B is a construction of a group epimorphism from the fundamental group of the graph Γ_B to the group G_B.

The paper is organized as follows. In the second section we describe the set C_B in terms of elementary transformations. In the third section we define the group G_B. In the fourth and the fifth sections we define the graph Γ_B and construct a group epimorphism from $\pi_1(\Gamma_B)$ to Γ_B. We also show that if $A \sim B$, then the groups G_A and G_B are isomorphic.

In the sixth section we calculate the group G_B for certain classes of rational functions, and consider some examples. Specifically, we show that for a wide class of rational functions which we call *generically decomposable* the group G_B is isomorphic to the group of Möbius transformations commuting with B, implying that $X \in C_B$ if and only if $X = \mu \circ B^{\circ l}$, where μ is such a transformation and $l \geqslant 0$. We also show that for a polynomial B the group G_B is metacyclic. Finally, we discuss in details the example of commuting rational functions B and X satisfying condition (2) from the paper of Ritt [14]. In particular, we calculate the group G_B which turns out to be a cyclic group of order three. We also provide a different example of this kind.

2. The set C_B and elementary transformations

Let B be a rational function of degree at least two. Denote by C_B the set of all rational functions commuting with B.

Lemma 2.1. *The set C_B is closed with respect to the operation of composition, that is $A_1, A_2 \in C_B$ implies $A_1 \circ A_2 \in C_B$. Furthermore, if $A \circ U \in C_B$ and $U \in C_B$, then $A \in C_B$.*

Proof. Indeed, if $A_1, A_2 \in C_B$, then

$$A_1 \circ A_2 \circ B = A_1 \circ B \circ A_2 = B \circ A_1 \circ A_2.$$

On the other hand, if

$$B \circ (A \circ U) = (A \circ U) \circ B$$

and $U \in C_B$, then

$$B \circ A \circ U = A \circ U \circ B = A \circ B \circ U,$$

implying that

$$B \circ A = A \circ B.$$ □

Stress out that we allow to elements of C_B to have degree one, that is to be Möbius transformations. All Möbius transformations commuting with B obviously form a group denoted by $Aut(B)$ and called the *symmetry group* of B. Since any $\mu \in Aut(B)$ maps periodic points of B of order $l \geqslant 1$ to themselves, and any Möbius transformation is defined by its values at any three points, the symmetry group of any rational function is finite. In particular, $Aut(B)$ is one of the five well known finite rotation groups of the sphere: A_4, S_4, A_5, C_n, D_{2n}. Notice that the property of $\mu \in Aut(B)$ to map periodic points of B to periodic points can be used for a practical description of $Aut(B)$.

For any decomposition $B = V \circ U$, where U and V are rational functions, the rational function $\widehat{B} = U \circ V$ is called an *elementary transformation* of B. Rational functions B and A are called *equivalent* if there exists a chain of elementary

transformations between B and A. Since for any Möbius transformation μ the equality
$$B = (B \circ \mu^{-1}) \circ \mu$$
holds, the equivalence class $[B]$ of a rational function B is a union of conjugacy classes. Thus, the relation $\sim$ can be considered as a weaker form of the classical conjugacy relation. An equivalence class $[B]$ contains infinitely many conjugacy classes if and only if B is a flexible Lattès map (see [7]).

The following lemma is obtained by a direct calculation (see [10], Lemma 3.1).

Lemma 2.2. *Let*
$$L: \ B \to B_1 \to B_2 \to \cdots \to B_s \tag{7}$$
be a sequence of elementary transformations, and U_i, V_i, $1 \leqslant i \leqslant s$, *rational functions such that*
$$B = V_1 \circ U_1, \quad B_i = U_i \circ V_i, \qquad 1 \leqslant i \leqslant s,$$
and
$$U_i \circ V_i = V_{i+1} \circ U_{i+1}, \quad 1 \leqslant i \leqslant s-1.$$
Then the functions
$$U = U_s \circ U_{s-1} \circ \cdots \circ U_1, \qquad V = V_1 \circ \cdots \circ V_{s-1} \circ V_s \tag{8}$$
make the diagram
$$\begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{B} & \mathbb{CP}^1 \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U} \\ \mathbb{CP}^1 & \xrightarrow{B_s} & \mathbb{CP}^1 \\ {\scriptstyle V}\downarrow & & \downarrow{\scriptstyle V} \\ \mathbb{CP}^1 & \xrightarrow{B} & \mathbb{CP}^1, \end{array}$$
commutative and satisfy the equalities
$$V \circ U = B^{\circ s}, \qquad U \circ V = B_s^{\circ s}.$$
□

It follows from Lemma 2.2, that any sequence of elementary transformation (7) such that $B_s = B$ gives rise to a rational function U commuting with B, and the main result of this section states that for non-special B any element of C_B can be obtained in this way.

We will need a technical result concerning solutions of the functional equation
$$A \circ C = D \circ B \tag{9}$$
in rational functions. Say that a solution A, C, D, B of (9) is *good* if the algebraic curve
$$A(x) - D(y) = 0$$
is irreducible and the functions B and C satisfy the condition
$$\mathbb{C}(B, C) = \mathbb{C}(z).$$

If $\deg A = \deg B$, then any one of the above conditions implies the other one (see [5], Lemma 2.1). Therefore, a solution A, X, B of (4) is primitive if and only if the corresponding solution A, X, X, B of (9) is good.

The following result (see [11], Theorem 2.17, and [12], Theorem 2.10) states that "gluing together" two commutative diagrams corresponding to good solutions of (9) we obtain again a good solution of (9) (see the diagram below).

$$\begin{array}{ccccc} \mathbb{CP}^1 & \xrightarrow{B} & \mathbb{CP}^1 & \xrightarrow{W} & \mathbb{CP}^1 \\ \downarrow C & & \downarrow D & & \downarrow V \\ \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1 & \xrightarrow{U} & \mathbb{CP}^1 . \end{array}$$

Theorem 2.3. *Assume that A, C, D, B and U, D, V, W are good solutions of* (9)*. Then $U \circ A$, C, V, $W \circ B$ is also a good solution of* (9)*.* □

Theorem 2.3 obviously implies the following corollary.

Corollary 2.4. *Let A, X, B be a primitive solution of* (4)*. Then for any $l \geqslant 1$ the triple $A^{\circ l}, X, B^{\circ l}$ is also a primitive solution of* (4)*.* □

The following theorem provides a description of the set C_B in terms of elementary transformations.

Theorem 2.5. *Let B be a non-special rational function of degree at least two. Then a rational function X belongs to C_B if and only if there exists a sequence of elementary transformation* (7) *such that $B_s = B$ and $X = U_s \circ U_{s-1} \circ \dots \circ U_1$.*

Proof. The sufficiency follows from Lemma 2.2. In the other direction, assume that $X \in C_B$. If X is a Möbius transformation, then the chain

$$B = (B \circ \mu^{-1}) \circ \mu \to \mu \circ (B \circ \mu^{-1}) = B$$

is as required. Assume now that $\deg X \geqslant 2$. Considering the quadruple B, X, X, B as a solution of equation (9), and using the iterative process described in the introduction, we can construct a sequence (7) and a commutative diagram

$$\begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{B} & \mathbb{CP}^1 \\ U \downarrow & & \downarrow U \\ \mathbb{CP}^1 & \xrightarrow{B_s} & \mathbb{CP}^1 \\ X_0 \downarrow & & \downarrow X_0 \\ \mathbb{CP}^1 & \xrightarrow{B} & \mathbb{CP}^1, \end{array}$$

such that U is defined by (8), the equality

$$X = X_0 \circ U$$

holds, and the triple B, X_0, B_s is a primitive solution of (4). In order to prove the theorem we only must show that $\deg X_0 = 1$. Indeed, in this case changing U_s to $X_0 \circ U_s$ and B_s to $X_0^{-1} \circ B_s \circ X_0$, without loss of generality we may assume that $X_0 = z$, so that $B_s = B$ and (7) is the sequence required.

Assume in contrary that $\deg X_0 > 1$. By Corollary 2.4, for any $l \geqslant 1$ the triple $B^{\circ l}, X_0, B_s^{\circ l}$ is a good solution of (4). On the other hand, by the Ritt theorem, there exist k and l such that $X^{\circ k} = B^{\circ l}$. Thus,

$$B^{\circ l} = X^{\circ k} = X_0 \circ (U \circ X^{\circ k-1}),$$

implying that the curve

$$(U \circ X^{\circ k-1})(x) - y = 0$$

is a component of the curve

$$B^{\circ l}(x) - X_0(y) = 0.$$

Moreover, this component is proper because $\deg X_0 > 1$. Since this contradicts to the fact that $B^{\circ l}, X_0, B_s^{\circ l}$ is a good solution of (4), we conclude that $\deg X_0 = 1$. □

3. The group G_B

Define an equivalence relation $\underset{B}{\sim}$ on the set C_B, setting $A_1 \underset{B}{\sim} A_2$ if

$$A_1 \circ B^{\circ l_1} = A_2 \circ B^{\circ l_2} \tag{10}$$

for some $l_1 \geqslant 0$, $l_2 \geqslant 0$ (in order to distinguish this relation with the relation $\sim$ introduced in the previous section we use the subscript B). It is easy to see that $\underset{B}{\sim}$ is really an equivalence relation. Indeed, $\underset{B}{\sim}$ is clearly reflexive and symmetric. Furthermore, if equalities (10) and

$$A_2 \circ B^{\circ n_1} = A_3 \circ B^{\circ n_2}$$

hold, and $n_1 \geqslant l_2$, then

$$A_1 \circ B^{\circ(l_1+n_1-l_2)} = A_2 \circ B^{\circ n_1} = A_3 \circ B^{\circ n_2},$$

implying that $A_1 \underset{B}{\sim} A_3$. Similarly, if $l_2 \geqslant n_1$, then

$$A_3 \circ B^{\circ(n_2+l_2-n_1)} = A_2 \circ B^{\circ l_2} = A_1 \circ B^{\circ n_1}.$$

Lemma 3.1. *Let* $\mathbf{A}$ *be an equivalence class of* $\underset{B}{\sim}$. *For any* $n \geqslant 1$ *the class* $\mathbf{A}$ *contains at most one rational function of degree* n. *Furthermore, if* $A_0 \in \mathbf{A}$ *is a function of minimal possible degree, then any* $A \in \mathbf{A}$ *has the form* $A = A_0 \circ B^{\circ l}$, $l \geqslant 1$. *Alternatively, the function* A_0 *can be described as a unique function in* $\mathbf{A}$ *which is not a rational function in* B.

Proof. If $\deg A_1 = \deg A_2$ in (10), then $l_1 = l_2$, implying that $A_1 = A_2$. Furthermore, if

$$A \circ B^{\circ l_1} = A_0 \circ B^{\circ l_2} \tag{11}$$

and $l_1 > l_2$, then

$$A_0 = A \circ B^{\circ(l_1-l_2)},$$

implying that $\deg A < \deg A_0$ in contradiction with the assumption. Therefore, $l_1 \leqslant l_2$ and

$$A = A_0 \circ B^{\circ(l_2-l_1)}.$$

Finally, A_0 is not a rational function in B, since if $A_0 = A' \circ B$, then A' commutes with B by Lemma 2.1, implying that $A' \underset{B}{\sim} A_0$ with $\deg A' < \deg A_0$. On the other hand, if *two* functions A and A_0 in the class $\mathbf{A}$ are not a rational functions in B, then (11) implies that $l_1 = l_2$ and $A = A_0$. □

For a rational function B denote by G_B the set of equivalence classes of $\underset{B}{\sim}$. Define a binary operation on the set G_B as follows. If $\mathbf{A}_1$ and $\mathbf{A}_2$ are equivalence classes of $\underset{B}{\sim}$, and $A_1 \in \mathbf{A}_1$ and $A_2 \in \mathbf{A}_2$ are their representatives, then $\mathbf{A}_1 \cdot \mathbf{A}_2$ is defined as the equivalence class containing $A_1 \circ A_2$. It is easy to see that this operation is well-defined. Indeed, assume that $A_1 \underset{B}{\sim} A_1'$ and $A_2 \underset{B}{\sim} A_2'$. Then

$$A_1 \circ B^{\circ l_1} = A_1' \circ B^{\circ l_1'}$$

and

$$A_2 \circ B^{\circ l_2} = A_2' \circ B^{\circ l_2'},$$

implying that

$$A_1 \circ B^{\circ l_1} \circ A_2 \circ B^{\circ l_2} = A_1' \circ B^{\circ l_1'} \circ A_2' \circ B^{\circ l_2'}. \tag{12}$$

In turn, since $A_1, A_2 \in C_B$, equality (12) implies that

$$A_1 \circ A_2 \circ B^{\circ (l_1+l_2)} = A_1' \circ A_2' \circ B^{\circ (l_1'+l_2')},$$

and hence

$$A_1 \circ A_2 \underset{B}{\sim} A_1' \circ A_2'.$$

Theorem 3.2. *The set G_B equipped with the operation $\cdot$ is a finite group.*

Proof. By definition, if $A_i \in \mathbf{A}_i$, $1 \leqslant i \leqslant 3$, then $(\mathbf{A}_1 \cdot \mathbf{A}_2) \cdot \mathbf{A}_3$ and $\mathbf{A}_1 \cdot (\mathbf{A}_2 \cdot \mathbf{A}_3)$ are classes containing the functions $(A_1 \circ A_2) \circ A_3$ and $A_1 \circ (A_2 \circ A_3)$ correspondingly. On the other hand,

$$(A_1 \circ A_2) \circ A_3 = A_1 \circ (A_2 \circ A_3),$$

since $\circ$ is an associative operation on the set of rational functions. Therefore, the classes $(\mathbf{A}_1 \cdot \mathbf{A}_2) \cdot \mathbf{A}_3$ and $\mathbf{A}_1 \cdot (\mathbf{A}_2 \cdot \mathbf{A}_3)$ coincide, and hence the operation $\cdot$ satisfies the associative axiom.

Further, the class $\mathbf{E}$ containing the function z and consisting of all iterates of B obviously serves as the unit element. Moreover, for any class $\mathbf{X}$ there exists a class $\mathbf{X}^{-1}$ such that

$$\mathbf{X} \cdot \mathbf{X}^{-1} = \mathbf{X}^{-1} \circ \mathbf{X} = \mathbf{E}. \tag{13}$$

Indeed, by Theorem 2.5, for any $X \in \mathbf{X}$ there exists a sequence of elementary transformation (7) such that

$$X = U_s \circ U_{s-1} \circ \cdots \circ U_1,$$

and, by Lemma 2.2, the functions X and

$$Y = V_s \circ V_{s-1} \circ \cdots \circ V_1$$

satisfy

$$X \circ Y = Y \circ X = B^{\circ s}. \tag{14}$$

Since (14) implies by Lemma 2.1 that $Y \in C_B$, we see that defining $\mathbf{X}^{-1}$ as the class containing the rational function Y we satisfy (13).

Finally, by the result of the paper [11] cited in the introduction, there exist at most finitely many rational functions $A \in C_B$ which are not rational functions in B, implying by Lemma 3.1 that the group G_B is finite. □

Notice that the above proof provides a method for actual finding $\mathbf{X}^{-1}$. On the other hand, merely the existence of the inverse element follows from the Ritt theorem. Indeed, since for any $X \in \mathbf{X}$ there exist $n \geqslant 1$ and $m \geqslant 1$ such that

$$X^{\circ n} = B^{\circ m},$$

for any class $\mathbf{X}$ there exists n such that $\mathbf{X}^n = \mathbf{e}$, implying that (13) holds for $\mathbf{X}^{-1} = \mathbf{X}^{n-1}$. Notice also that the Ritt theorem by itself does not imply that the group G_B is finite, although implies that any its element has finite order.

For $X \in C_B$ we will denote by $\boldsymbol{X}$ the element of G_B corresponding to the equivalence class of $\underset{B}{\sim}$ containing X.

Lemma 3.3. *The map $\mu \to \boldsymbol{\mu}$ is a group monomorphism from the group $Aut(B)$ to the group G_B.*

Proof. Since functions from $Aut(B)$ have degree one, it follows from Lemma 3.1 that $\boldsymbol{\mu_1} = \boldsymbol{\mu_2}$ if and only if $\mu_1 = \mu_2$. Therefore, the map $\tau : \mu \to \boldsymbol{\mu}$ is injective, and it is easy to see that τ is a homomorphism of groups. □

We will denote the image of $Aut(B)$ under the group monomorphism $\mu \to \boldsymbol{\mu}$ by $Aut_G(B)$.

Lemma 3.4. *The following conditions are equivalent.*

1) *Any $X \in C_B$ has the form $X = \mu \circ B^{\circ l}$ for some $\mu \in Aut(B)$ and $l \geqslant 0$.*
2) *Any $X \in C_B$ of degree at least two is a rational function in B.*
3) *The group G_B coincides with $Aut_G(B)$.*

Proof. It is easy to see that 1) and 3) are equivalent, and that 1) implies 2). Assume now that 2) holds, and let $X \in C_B$ be a function of degree at least two. By the assumption, $X = R_1 \circ B$ for some $R \in \mathbb{C}(z)$. Since $R_1 \in C_B$ by Lemma 2.1, using this assumption again we conclude that either $R_1 \in Aut(B)$, or there exists $R_2 \in \mathbb{C}(z)$ such that $R_1 = R_2 \circ B$ and $R_2 \in C_B$. It is clear that continuing this process we will eventually obtain a representation $X = \mu \circ B^l$ for some $\mu \in Aut(B)$ and $l \geqslant 0$. □

4. The graph Γ_B

Let B be a rational function of degree at least two. Define Γ_B as a multigraph whose vertices are in a one-to-one correspondence with some fixed representatives of conjugacy classes in $[B]$, and whose multiple edges connecting vertices corresponding to representatives B_i to B_j are in a one-to-one correspondence with solutions of the system

$$B_i = V \circ U, \quad B_j = U \circ V \tag{15}$$

in rational functions. Stress out that Γ_B may have loops. They correspond to solutions of the equations

$$B_i = U \circ V = V \circ U.$$

Lemma 4.1. *The graph Γ_B does not depend on a choice of representatives of conjugacy classes in $[B]$.*

Proof. Indeed, for any Möbius transformations α and β, to any solution U, V of system (15) corresponds a solution U', V' of the system

$$\alpha \circ B_i \circ \alpha^{-1} = V' \circ U', \quad \beta \circ B_j \circ \beta^{-1} = U' \circ V', \tag{16}$$

defined by the formulas

$$U' = \beta \circ U \circ \alpha^{-1}, \quad V' = \alpha \circ V \circ \beta^{-1}. \tag{17}$$

Furthermore, it is easy to see that formulas (17) provide a one-to-one correspondence between solutions of (15) and (16). □

Theorem 4.2. *Let B a rational function of degree at least two. Then the graph Γ_B is finite, unless B is a flexible Lattès map.*

Proof. By the main result of the paper [7], the class $[B]$ contains infinitely many conjugacy classes if and only if B is a flexible Lattès map. Therefore, if B is not such a map, the graph Γ_B contains only finitely many vertices. Let us show now that the number of edges connecting two vertices is finite.

Recall that two decompositions

$$B = V \circ U, \qquad B = V' \circ U' \tag{18}$$

of a rational function B into compositions of rational functions are called *equivalent* if there exists a Möbius transformation μ such that

$$V' = V \circ \mu^{-1}, \quad U' = \mu \circ U. \tag{19}$$

It is well known that equivalence classes of decompositions of B are in one-to-one correspondence with imprimitivity systems of the monodromy group $Mon(B)$ of B. In particular, there exists at most finitely many such classes. This implies that in order to prove the finiteness of the number of edges adjacent to the vertices corresponding to B_i and B_j it is enough to show that for any fixed solution U, V of (15) there exist only finitely many solutions U', V' of (15) such that decompositions (18) are equivalent. Since equalities (19) combined with the equality

$$U \circ V = U' \circ V'$$

imply the equality

$$U \circ V = \mu \circ U \circ V \circ \mu^{-1},$$

the last statement follows from the finiteness of the group $Aut(U \circ V)$. □

Since in this paper we consider non-special rational functions B, the corresponding graphs Γ_B are always finite by Theorem 4.2. Notice that the results of [11] imply that the number of vertices of Γ_B can be bounded by a number depending on B only (see Remark 5.2 in [11]). Nevertheless, there exists no *absolute* bound for a number of vertices of Γ_B, and it is easy to construct rational functions B of degree n for which the graph Γ_B contains $\approx \log_2 n$ vertices (see [5], p. 1241).

We always will assume that the representative of the conjugacy class of the function B in Γ_B is the function B itself. Abusing notation, below we will call the functions B_j simply "vertices" of Γ_B. Notice that for each vertex B_j of Γ_B there exists at least one loop starting and ending at B which corresponds to the solution

$$B = B \circ z = z \circ B \tag{20}$$

of (15). More generally, the solutions

$$B = (\mu^{-1} \circ B) \circ \mu = \mu \circ (\mu^{-1} \circ B), \qquad \mu \in Aut(B), \tag{21}$$

give rise to $|Aut(B)|$ loops.

Example 1. Assume that B is an *indecomposable* rational function, that is such a function that the equality $B = V \circ U$ implies that at least one of the functions U and V has degree one. In this case the equivalence class $[B]$ obviously consists of a unique conjugacy class. Thus, Γ_B has a unique vertex, and all edges of Γ_B are loops corresponding to solutions

$$B = U \circ V = V \circ U. \tag{22}$$

Moreover, since B is indecomposable, for any solution U, V of (22) one of the functions U, V has degree one. Assuming without loss of generality that $\deg U = 1$, we see that

$$U^{-1} \circ B = V = B \circ U^{-1},$$

implying that $U \in Aut(B)$. Therefore, Γ_B has the form shown on Fig. 1 with the

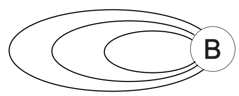

FIGURE 1

number of loops equal $|Aut(B)|$.

Example 2. Assume now that a rational function B has, up to equivalency (19), a unique decomposition $B = V \circ U$ into a composition of rational functions of degree at least two, and that the same is true for the function $B_1 = U \circ V$. In this case graph Γ_B may have two distinct forms. Namely, if B_1 and B are not conjugate, then Γ_B has the form shown on Fig. 2, where all loops correspond to

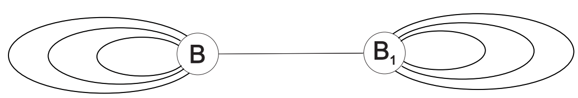

FIGURE 2

some automorphisms. Notice that under considered conditions the groups $Aut(B)$ and $Aut(B_1)$ are isomorphic (see Lemma 6.3 below), implying that the number of loops attached to B and B_1 is the same.

On the other hand, if B_1 is conjugate to B, that is if

$$U \circ V = \alpha^{-1} \circ V \circ U \circ \alpha,$$

then the graph Γ_B has only one vertex, and considering instead of the functions U and V the functions $U \circ \alpha$ and $V = \alpha^{-1} \circ V$, we can assume that

$$B = V \circ U = U \circ V.$$

By the assumption, the decompositions $B = V \circ U$ and $B = U \circ V$ are equivalent, that is

$$U = V \circ \mu, \qquad V = \mu^{-1} \circ U, \tag{23}$$

where μ is a Möbius transformation, implying that

$$V = \mu^{-1} \circ U = \mu^{-1} \circ V \circ \mu. \tag{24}$$

Since (24) implies that $\mu \in Aut(V)$, it follows now from (23) that

$$B = \mu \circ V^{\circ 2}, \qquad \mu \in Aut(V). \tag{25}$$

Notice that (25) implies that $\mu \in Aut(B)$. Thus, Γ_B has $|Aut(B)|$ loops corresponding to decompositions (21) and one additional loop corresponding to the decomposition

$$B = V \circ (V \circ \mu) = (V \circ \mu) \circ V.$$

Example 3. Set

$$B = -\frac{2z^2}{z^4+1} = -\frac{2}{z^2 + \frac{1}{z^2}}.$$

The function B is an invariant for the finite automorphism group of $\mathbb{CP}^1$ generated by the transformations

$$z \to \frac{1}{z}, \qquad z \to -z,$$

and its monodromy group $Mon(B)$ is the Klein four group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ having three proper imprimitivity systems. Corresponding decompositions of B are:

$$B = -\frac{2}{z^2-2} \circ \frac{z^2+1}{z}, \qquad B = -\frac{2}{z^2+2} \circ \frac{z^2-1}{z},$$

and

$$B = \frac{z^2-1}{z^2+1} \circ \frac{z^2-1}{z^2+1}. \tag{26}$$

Using for example the "Maple" system, one can check that the function

$$B_1 = \frac{z^2+1}{z} \circ -\frac{2}{z^2-2} = -\frac{1}{2}\,\frac{z^4-4\,z^2+8}{z^2-2}$$

has three critical values in $\mathbb{CP}^1$, and the corresponding permutations in $Mon(B_1)$ are (12)(34), (1243), and (14). On the other hand, the function

$$B_2 = \frac{z^2-1}{z} \circ -\frac{2}{z^2+2} = \frac{1}{2}\,\frac{z^2\left(z^2+4\right)}{z^2+2}$$

has four critical values, and the corresponding permutations in $Mon(B_2)$ are the permutations (12)(34), (23), (12)(34), (14). Since B_1 and B_2 have a different number of critical values, they are not conjugate. Furthermore, it is easy to see that the both groups $Mon(B_1)$ and $Mon(B_2)$ have a unique proper imprimitivity system $\{1,4\},\{2,3\}$, implying in particular that B is not conjugate to B_1 or B_2. Finally, one can check by a direct calculations, solving the system

$$\frac{az+b}{cz+d} \circ B = B \circ \frac{az+b}{cz+d}$$

in a, b, c, d, that the functions B, B_1, B_2 have no automorphisms. Summing up, we conclude that the graph Γ_B has the form shown on Fig. 3.

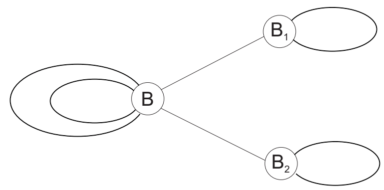

FIGURE 3

5. THE EPIMORPHISM $\pi_1(\Gamma_B) \to G_B$

The graph Γ_B was defined above as a purely combinatorial object. We can however consider it as a one-dimensional CW complex in $\mathbb{R}^3$. In this case we can provide each edge of Γ_B, including loops, with two opposite *orientations*. With each oriented edge e of Γ_B we associate a rational function $\mathcal{F}(e)$ as follows. Assume first that e corresponds to solution (15) with *different* B_i and B_j. Then we set $\mathcal{F}(e) = U$, if the initial point of e is B_i and the final point is B_j, and $\mathcal{F}(e) = V$, if the orientation is opposite. In case if e is a loop, we simply chose and fix, once of all, $\mathcal{F}(e)$ equal U for one of the two corresponding oriented edges, and $\mathcal{F}(e)$ equal V for the opposite oriented edge. For each oriented *path*

$$l = e_n e_{n-1} \dots e_1$$

set

$$\mathcal{F}(l) = \mathcal{F}(e_n) \circ \mathcal{F}(e_{n-1}) \circ \dots \circ \mathcal{F}(e_1).$$

Clearly, this definition implies that if

$$l = l_2 l_1$$

is a path obtained by a concatenation of the paths l_1 and l_2, then

$$\mathcal{F}(l) = \mathcal{F}(l_2) \circ \mathcal{F}(l_1). \tag{27}$$

Us usual, we will denote by l^{-1} the path l traversed in the opposite direction.

Lemma 5.1. *Let l be an oriented path in Γ_B from a vertex B_{i_1} to a vertex B_{i_2} consisting of k oriented edges. Then*

$$B_{i_2} \circ \mathcal{F}(l) = \mathcal{F}(l) \circ B_{i_1}, \tag{28}$$

and

$$\mathcal{F}(l^{-1}) \circ \mathcal{F}(l) = B_{i_1}^{\circ k}, \qquad \mathcal{F}(l) \circ \mathcal{F}(l^{-1}) = B_{i_2}^{\circ k}. \tag{29}$$

Proof. Since any oriented path l corresponds to a sequence of elementary transformations, the lemma follows from Lemma 2.2. □

If l is a closed path in Γ_B starting and ending at B, then (28) implies that the function $\mathcal{F}(l)$ commutes with B, while equalities (29) reduce to the equalities

$$\mathcal{F}(l^{-1}) \circ \mathcal{F}(l) = \mathcal{F}(l) \circ \mathcal{F}(l^{-1}) = B^{\circ k}. \tag{30}$$

Thus, we obtain a map $\varphi_B : l \to \mathcal{F}(l)$ from the set of closed paths starting and ending at B to the set C_B.

Theorem 5.2. *The map $\varphi_B : l \to \mathcal{F}(l)$ descends to an epimorphism of groups $\Phi_B : \pi_1(\Gamma_B, B) \to G_B$.*

Proof. Recall that an oriented path l in a graph Γ is called *reduced* if no two successive oriented edges in l are opposite orientations of the same edge. Paths of the form $e^{-1}e$, where e is an oriented edge are called *spurs*. Paths l and l' are called *equivalent* if l' is obtained from l by a finite number of insertions and removals of spurs between successive oriented edges or at the endpoints. In these terms the fundamental group of a graph Γ can be defined as the set of equivalence classes of paths which begin and end at some vertex of Γ, equipped with the product of classes defined in an obvious way (see e.g. Section 2.1.6 of [15]).

In order to show that the map φ_B descends to a map from $\pi_1(\Gamma_B, B)$ to $C_B/\underset{B}{\sim}$, we must show that whenever closed paths l and l' in Γ_B which start and end at B are equivalent, the rational functions $\mathcal{F}(l)$ and $\mathcal{F}(l')$ are in the same equivalence class of C_B. Since any path is equivalent to its reduced form, it is enough to show that if l' is obtained from l by an insertion of a spur, then $\mathcal{F}(l) \underset{B}{\sim} \mathcal{F}(l')$. Assume that

$$l' = l_2 e^{-1} e l_1,$$

where l_1 is a path from B to B_i, and l_2 is a path from B_i to B (one of the paths l_1 and l_2 can be empty in which case $B_i = B$). Then

$$\mathcal{F}(l') = \mathcal{F}(l_2) \circ B_i^{\circ 2} \circ \mathcal{F}(l_1),$$

by (27) and (30). It follows now from (28) that

$$\mathcal{F}(l') = \mathcal{F}(l_2) \circ \mathcal{F}(l_1) \circ B^{\circ 2} = \mathcal{F}(l) \circ B^{\circ 2},$$

implying that $\mathcal{F}(l) \underset{B}{\sim} \mathcal{F}(l')$.

The above shows that the map φ_B descends to a map $\Phi_B : \pi_1(\Gamma_B, B) \to G_B$, and (27) implies that Φ_B is a homomorphism of groups. Finally, it follows from Theorem 2.5 that Φ_B is an epimorphism. Indeed, by Theorem 2.5 any $X \in C_B$ can be obtained from a chain of elementary transformations (7). Moreover, we can change if necessary each of B_i, $1 \leqslant i \leqslant s$, to any desired representative of its conjugacy class, consecutively changing U_i to $\alpha_i \circ U_i$, $1 \leqslant i \leqslant s$, for a convenient Möbius transformation α_i. Since elementary transformations $B_{i_1} \to B_{i_2}$ correspond to edges of G_B adjacent to B_{i_1} and B_{i_2}, this implies that for any $X \in C_B$ there exists a closed path l starting and ending at B such that $\mathcal{F}(l) = X$. □

Theorem 5.3. *Let A and B be equivalent rational functions. Then $G_B \cong G_A$.*

Proof. Assuming that A and B are vertices of Γ_B, take a path s from A to B in Γ_B. Since the map $\psi : l \to s^{-1} l s$, from the set of closed paths starting and ending at B to the set of closed paths starting and ending at A, descends to an isomorphism of the fundamental groups

$$\Psi : \pi_1(\Gamma_B, B) \to \pi_1(\Gamma_B, A),$$

it follows from Theorem 5.2 that we only must prove the equality

$$\Psi(\operatorname{Ker} \Phi_A) = \operatorname{Ker} \Phi_B. \tag{31}$$

Let l_0 be a path starting and ending at B such that $\mathcal{F}(l_0) = B^{\circ k}$, $k \geqslant 1$, and let $k_0 = \psi(l_0)$. Then

$$\mathcal{F}(k_0) = \mathcal{F}(s^{-1}) \circ \mathcal{F}(l_0) \circ \mathcal{F}(s) = \mathcal{F}(s^{-1}) \circ B^{\circ k} \circ \mathcal{F}(s),$$

implying by (28) and (29) that

$$\mathcal{F}(k_0) = \mathcal{F}(s^{-1}) \circ \mathcal{F}(s) \circ A^{\circ k} = A^{\circ l} \circ A^{\circ k} = A^{\circ k+l}$$

for some $k, l \geqslant 1$. This implies that

$$\Psi(\operatorname{Ker} \Phi_A) \subseteq \operatorname{Ker} \Phi_B.$$

Similarly, considering the isomorphism inverse to Ψ we obtain that

$$\Psi^{-1}(\operatorname{Ker} \Phi_B) \subseteq \operatorname{Ker} \Phi_A.$$

This proves equality (31). □

6. Examples of groups G_B

6.1. **Functions with $G_B = Aut_G(B)$.** The simplest application of Theorem 5.2 is the following result.

Theorem 6.1. *Let B be an indecomposable non-special rational function of degree at least two. Then $G_B = Aut_G(B)$. Equivalently, $X \in C_B$ if and only if $X = \mu \circ B^l$ for some $\mu \in Aut(B)$ and $l \geqslant 1$.*

Proof. Since Γ_B has a unique vertex and $|Aut(B)|$ loops corresponding to automorphisms of B (see Example 1 in Section 4), it follows from Theorem 5.2 that G_B is generated by $\boldsymbol{\mu}$, where $\mu \in Aut(B)$. This proves that $G_B \cong Aut(B)$. The second statement follows from Lemma 3.4. □

Notice that since a "random" rational function B is indecomposable and has no automorphisms, Theorem 6.1 shows that for such a function the group G_B is trivial.

Theorem 6.1 can be extended to a wide class of decomposable rational functions. Recall that a functional decomposition

$$B = U_r \circ U_{r-1} \circ \cdots \circ U_1 \tag{32}$$

of a rational function is called *maximal* if all $U_1, U_2, \ldots, U_r$ are indecomposable and of degree greater than one. The number r is called the length of the maximal decomposition (32). Two decompositions (maximal or not) having an equal number of terms

$$F = F_r \circ F_{r-1} \circ \cdots \circ F_1 \quad \text{and} \quad F = G_r \circ G_{r-1} \circ \cdots \circ G_1$$

are called equivalent if either $r = 1$ and $F_1 = G_1$, or $r \geqslant 2$ and there exist rational functions μ_i, $1 \leqslant i \leqslant r-1$, of degree 1 such that

$$F_r = G_r \circ \mu_{r-1}, \quad F_i = \mu_i^{-1} \circ G_i \circ \mu_{i-1}, \quad 1 < i < r, \quad \text{and} \quad F_1 = \mu_1^{-1} \circ G_1.$$

We say that a rational function B having a maximal decomposition (32) is *generically decomposable* if the following conditions are satisfied:

- Each of the functions

  $$B_i = (U_i \circ \cdots \circ U_2 \circ U_1) \circ (U_r \circ U_{r-1} \circ \cdots \circ U_{i+1}), \qquad 0 \leqslant i \leqslant r-1,$$

  has a unique equivalence class of maximal decompositions,
- The functions B_i, $0 \leqslant k \leqslant r-1$, are pairwise not conjugate.

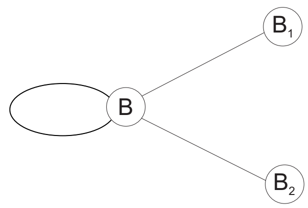

FIGURE 4

For a graph Γ_B define Γ_B^0 as a graph obtained from Γ_B by removing all loops which correspond to automorphisms. For example, for the graph Γ_B from Example 3 the graph Γ_B^0 is shown on Fig. 4. Recall that a *complete graph* is a graph in which every pair of distinct vertices is connected by a unique edge. The complete graph on n vertices is denoted by K_n.

Lemma 6.2. *Assume that a non-special rational function B having a maximal decomposition of length r is generically decomposable. Then Γ_B^0 is the complete graph K_r.*

Proof. Let (32) be a maximal decomposition of B. Since all the functions B_i, $0 \leqslant i \leqslant r-1$, are equivalent and pairwise not conjugate, the graph Γ_B contains at least r vertices. Observe now that any decomposition $B = V \circ U$ of B into a composition of two rational functions of degree at least two has the form

$$V = (U_r \circ U_{r-1} \circ \cdots \circ U_{i+1}) \circ \mu, \qquad U = \mu^{-1} \circ (U_i \circ \cdots \circ U_2 \circ U_1), \qquad 0 \leqslant i \leqslant r-1, \tag{33}$$

where μ is a Möbius transformation. Indeed, concatenating arbitrary maximal decompositions of U and V we must obtain a maximal decomposition equivalent to (32), implying that (33) holds. Therefore, any edge in Γ_B adjacent to $B_0 = B$ and not corresponding to an automorphism of B is adjacent to one of the vertices B_i, $1 \leqslant k \leqslant r-1$, and there exists exactly one edge connecting B_0 and B_i, $1 \leqslant k \leqslant r-1$. Since the same argument holds for any B_i, $0 \leqslant k \leqslant r-1$, we conclude that Γ_B^0 is the complete graph K_r. □

Lemma 6.3. *Assume that a non-special rational function B is generically decomposable, and let l be an oriented path from a vertex B_{i_1} to a vertex B_{i_2} in Γ_B. Then for any $\mu \in Aut(B_{i_1})$ there exists $\alpha(\mu) \in Aut(B_{i_2})$ such that*

$$\mathcal{F}(l) \circ \mu = \alpha(\mu) \circ \mathcal{F}(l). \tag{34}$$

Furthermore, the map

$$\mu \to \alpha(\mu) \tag{35}$$

is an isomorphism of the groups $Aut(B_{i_1})$ and $Aut(B_{i_2})$. In particular, to each vertex of Γ_B is attached the same number of loops.

Proof. In view of formula (27) it is enough to prove the lemma for the case where l is an oriented edge. If l is a loop, then by Lemma 6.2 it corresponds to a solution of (15) of the form

$$B_{i_1} = (\mu_0^{-1} \circ B_{i_1}) \circ \mu_0 = \mu_0 \circ (\mu_0^{-1} \circ B_{i_1}), \qquad \mu_0 \in Aut(B_{i_1}).$$

Thus, either $\mathcal{F}(l)=\mu_0$ or $\mathcal{F}(l)=\mu_0^{-1}\circ B_{i_1}$, and it is easy to see that in these cases equalities (34) and (35) hold for the automorphisms

$$\mu\to\mu_0\circ\mu\circ\mu_0^{-1},\qquad \mu\to\mu_0^{-1}\circ\mu\circ\mu_0,$$

correspondingly.

Assume now that l is an oriented edge from a vertex $B_{i_1}=V\circ U$ to a different vertex $B_{i_2}=U\circ V$. Clearly, for any $\mu\in Aut(B_{i_1})$ the equality

$$B_{i_1}=(\mu^{-1}\circ V)\circ(U\circ\mu) \tag{36}$$

holds. Moreover, the decompositions $B_{i_1}=V\circ U$ and (36) are equivalent, since for arbitrary maximal decompositions of U and V the induced maximal decompositions of B_{i_1} are equivalent. Therefore, for any $\mu\in Aut(B_{i_1})$ there exists a Möbius transformation $\alpha=\alpha(\mu)$ such that

$$\mu^{-1}\circ V=V\circ\alpha(\mu)^{-1},\quad U\circ\mu=\alpha(\mu)\circ U.$$

Furthermore, it is easy to see that (35) is a group homomorphism from $Aut(B_{i_1})$ to $Aut(B_{i_2})$.

Finally, if

$$\nu\to\beta(\nu)$$

is a homomorphism from $Aut(B_{i_2})$ to $Aut(B_{i_1})$, defined by the conditions

$$\nu^{-1}\circ U=U\circ\beta(\nu)^{-1},\quad V\circ\nu=\beta(\nu)\circ V,$$

then

$$V\circ U\circ\mu=V\circ\alpha(\mu)\circ U=\beta(\alpha(\mu))\circ V\circ U.$$

Since

$$V\circ U\circ\mu=\mu\circ V\circ U,$$

this implies that $\beta\circ\alpha$ is the identical mapping of $Aut(B_{i_1})$, and hence (35) is an isomorphism. □

Theorem 6.4. *Let B be a non-special generically decomposable rational function. Then $G_B=Aut_G(B)$. Equivalently, $X\in C_B$ if and only if $X=\mu\circ B^l$ for some $\mu\in Aut(B)$ and $l\geqslant 1$.*

Proof. Let (32) be a maximal decomposition of B. For convenience, define rational functions U_i for $i\geqslant r$ setting $U_i=U_{i'}$, where $i\equiv i' \bmod r$. Since any decomposition $B=V\circ U$, where U and V are functions of degree at least two, has the form (33) and a similar statement holds for all B_i, $0\leqslant i\leqslant r$, for the oriented edge e from B_{i_1} to B_{i_2} the equality

$$\mathcal{F}(e)=U_{i_2}\circ\cdots\circ U_{i_1+2}\circ U_{i_1+1}$$

holds, implying by (27) that for an arbitrary path l from B_{i_1} to B_{i_2} the equality

$$\mathcal{F}(l)=U_{i_2}\circ\cdots\circ U_{i_1+2}\circ U_{i_1+1}\circ B_{i_1}^{\circ k}$$

holds for some $k\geqslant 1$. In particular, if l is a closed path starting and ending at B and containing no loops, then $\mathcal{F}(l)=B^{\circ k}$, $k\geqslant 1$, implying that the image of l under the homomorphism Φ_B from Theorem 5.2 is the unit element. Further, if l contains a loop, then either

$$\mathcal{F}(l)=U_{kr}\circ\cdots\circ U_{i+1}\circ\nu\circ U_i\circ\cdots\circ U_1,$$

or

$$\mathcal{F}(l)=U_{kr}\circ\cdots\circ U_{i+1}\circ(\nu^{-1}\circ B_i)\circ U_i\circ\cdots\circ U_1$$

for some $\nu \in Aut(B_i)$ and $k, i \geqslant 1$, implying by Lemma 6.3 and Lemma 5.1 that either

$$\mathcal{F}(l) = \mu \circ B^{\circ k},$$

or

$$\mathcal{F}(l) = \mu \circ B^{\circ k+1}$$

for some $\mu \in Aut(B)$. Finally, if l contains several loops, then repeatedly using Lemma 6.3 and Lemma 28 we conclude that

$$\mathcal{F}(l) = \mu \circ B^{\circ l}$$

for some $\mu \in Aut(B)$ and $l \geqslant 1$. Thus, $G_B = Aut_G(B)$. □

Corollary 6.5. *Let B be a non-special rational function of degree at least two such that G_B is strictly bigger than $Aut_G(B)$. Then there exists $A \sim B$ such that either A can be represented as composition of two commuting functions of degree at least two, or A has more than one class of maximal decompositions.*

Proof. Let (32) be a maximal decomposition of B. Clearly, we only must show that the conditions of the corollary imply the second condition defining generically decomposable rational functions. Assume in contrary that say B_0 is conjugate to B_r. This means that there exists a Möbius transformation μ such that

$$(U_r \circ \cdots \circ U_{i+1}) \circ (U_i \circ \cdots \circ U_1) = \mu \circ (U_i \circ \cdots \circ U_1) \circ (U_r \circ \cdots \circ U_{i+1}) \circ \mu^{-1}.$$

However, in this case for the functions

$$L = \mu \circ (U_i \circ \cdots \circ U_1), \quad M = (U_r \circ \cdots \circ U_{i+1}) \circ \mu^{-1}$$

the equality

$$B = M \circ N = N \circ M \tag{37}$$

holds, in contradiction with the assumption. □

Notice that whenever B *is* a composition of two commuting functions of degree at least two, the group G_B is strictly bigger than $Aut_G(B)$. Indeed, equality (37) implies that the functions N and M belong to C_B. Moreover, their images in G_B are not trivial, since $\deg M < \deg B$ and $\deg N < \deg B$. On the other hand, these images do not belong to $Aut_G(B)$, since $1 < \deg M$ and $1 < \deg N$. In particular, if $B = T^{\circ s}$, where $s > 1$, the group G_B contains a cyclic group of order s whose intersection with $Aut(B)$ is trivial.

Notice also that the group G_B can be strictly bigger than $Aut_G(B)$ even if B is not a composition of commuting functions, and that the relation $A \sim B$ does not imply in general the equality $Aut_G(A) \cong Aut_G(B)$ (see Subsection 6.3).

6.2. **The group G_B for polynomial B.** Before stating the theorem describing groups G_B for polynomial B let us recall several results. First, for a non-special polynomial B of degree at least two, the set C_B consists of *polynomials*. Indeed, (1) yields that

$$B^{-1}(X^{-1}\{\infty\}) = X^{-1}\{\infty\}, \tag{38}$$

implying easily that either $X^{-1}\{\infty\} = \{\infty\}$, or $X^{-1}\{\infty\} = \{\infty, a\}$, $a \in \mathbb{C}$. In the first case X is a polynomial. On the other hand, in the second case, considering instead of B and X the commuting functions

$$X \to \mu \circ X \circ \mu^{-1}, \quad B \to \mu^{-1} \circ B \circ \mu,$$

where $\mu = z - a$, one can assume that $X^{-1}\{\infty\} = \{\infty, 0\}$. However, this implies by (38) that B is conjugate to z^n, in contradiction with the assumption that B is not special.

Second, the symmetry group $Aut(B)$ of a polynomial B of degree at least two is cyclic. Indeed, for any $\mu \in Aut(B)$ the condition $\mu^{-1}\{\infty\} = \{\infty\}$ still holds, implying that $\mu = az+b$. By conjugation, we always can assume that the coefficient of $z^{\deg B-1}$ is zero, and it easy to see that $\mu = az + b$ may commute with such B only if $b = 0$. Therefore, $Aut(B)$ is a cyclic rotation group. Furthermore, it is easy to see that $Aut(B)$ is generated by the rotation

$$\alpha : z \to \varepsilon z,$$

where ε is a primitive nth root of unity, if and only if

$$B = zR(z^n)$$

for some polynomial R which is not a polynomial in z^l for $l \geqslant 2$.

Third, a polynomial B is a special function if and only if B is conjugate to z^n or $\pm T_n$, since it is well known that a polynomial B cannot be a Lattès map.

Finally, we will need the following result (see [8], Theorem 1.3).

Theorem 6.6. *Let A and B be fixed non-special polynomials of degree at least two, and let $\mathcal{E}(A, B)$ be the set of all polynomials X such that $A \circ X = X \circ B$. Then, either $\mathcal{E}(A, B)$ is empty, or there exists $X_0 \in \mathcal{E}(A, B)$ such that a polynomial X belongs to $\mathcal{E}(A, B)$ if and only if $X = \widehat{A} \circ X_0$ for some polynomial $\widehat{A}$ commuting with A.* □

Recall that a group G is called *metacyclic* if it has a normal cyclic subgroup H such that G/H is a cyclic group.

Theorem 6.7. *Let B be a polynomial of degree at least two not conjugate to z^n or $\pm T_n$, $n \geqslant 2$. Then the group G_B is a metacyclic.*

Proof. Applying Theorem 6.6 for $A = B$ and arguing as in Lemma 3.4, we conclude that any rational function which belongs to $C_B = \mathcal{E}(B, B)$ has the form $X = \mu \circ X_0^{\circ l}$, where $\mu \in Aut(B)$ and $l \geqslant 1$.

Let $\mu \in Aut(B)$. Since the function $X_0^l \circ \mu$, $l \geqslant 1$, belongs to C_B, the equality

$$\mu' \circ X_0^l = X_0^l \circ \mu, \qquad l \geqslant 1,$$

holds for some $\mu' \in Aut(B)$, implying that

$$Aut_G B \cdot \boldsymbol{X}_0^l = \boldsymbol{X}_0^l \cdot Aut_G B, \qquad l \geqslant 1.$$

Thus, $Aut_G(B)$ is a normal subgroup in G_B. Furthermore, it is clear that G_B is generated by $Aut_G(B)$ and $\boldsymbol{X}_0$. Thus, any coset of $Aut_G(B)$ in G has the form

$$\boldsymbol{X}_0^l Aut_G(B), \qquad l \geqslant 1,$$

and hence the group $G_B/Aut_G(B)$ is cyclic. Since $Aut(B)$ is also a cyclic group, this proves that G_B is a metacyclic. □

Notice that Theorem 6.7 can be deduced from the Ritt theorem ([14], [13]) saying that any commuting non-special polynomials X and B can be represented in the form (3). Nevertheless, the Ritt theorem does not implies Theorem 6.7 immediately, since R in (3) depends on X, and the further analysis is needed.

6.3. **The group G_B for the Ritt example.** Let B be a rational function of degree at least two. Denote by $\widehat{Aut}(B)$ the group consisting of Möbius transformations μ such that

$$B \circ \mu = \nu \circ B$$

for some Möbius transformations ν. Like the group $Aut(B)$, the group $\widehat{Aut}(B)$ is a finite rotation group of the sphere (see [11], Section 4). More generally, denote by $\widehat{C}_B$ the set of rational functions X such that

$$B \circ X = Y \circ B$$

for some rational function Y. Clearly, $Aut(B)$ is a subgroup $\widehat{Aut}(B)$, and $C_B \subseteq \widehat{C}_B$.

Let

$$V = \frac{z^2+2}{z+1}, \quad U = \frac{z^2-4}{z-1}, \quad \mu = \varepsilon z,$$

where $\varepsilon^3 = 1$. In the paper [14], Ritt showed that the rational functions

$$B = V \circ U, \quad X = V \circ \mu \circ U$$

commute but no one of them is a rational function of the other. In particular, this implies that there is no R such that

$$B = \mu_1 \circ R^{\circ l_1}, \quad X = \mu_2 \circ R^{\circ l_2} \tag{39}$$

for some Möbius transformations μ_1, μ_2 and $l_1, l_2 \geqslant 1$. Ritt also observed that starting from this example one can construct infinitely many similar examples taking instead of B and X the functions

$$B' = V \circ C \circ U, \quad X' = V \circ C \circ \mu \circ U,$$

where C is any function of the form $C = zR(z^3)$, $R \in \mathbb{C}(z)$.

The Ritt statement follows from the following more general observation.

Lemma 6.8. *Let $W \in C_{U\circ V}$ but $W \notin \widehat{C}_V$. Then the functions $V \circ U$ and $V \circ W \circ U$ commute but the latter is not a rational function of the former. Furthermore, the same conclusion holds for the functions $V \circ C \circ U$ and $V \circ W \circ C \circ U$, where C is any function commuting with W.*

Proof. Indeed, we have:

$$(V \circ C \circ U) \circ (V \circ W \circ C \circ U) = V \circ C \circ (U \circ V \circ W) \circ C \circ U =$$

$$= (V \circ C \circ W \circ U) \circ (V \circ C \circ U) = (V \circ W \circ C \circ U) \circ (V \circ C \circ U).$$

On the other hand, if

$$V \circ W \circ C \circ U = R \circ V \circ C \circ U$$

for some rational function R, then

$$V \circ W = R \circ V,$$

in contradiction with the assumption that $W \notin \widehat{C}_V$. □

The Ritt statement is obtained from Lemma 6.8 for $W = \mu$. Really,

$$U \circ V = \frac{z\left(z^3-8\right)}{\left(z^3+1\right)},$$

implying that $\mu \in Aut(U \circ V)$. On the other hand, the assumption that

$$V \circ \mu = \nu \circ V \tag{40}$$

for some Möbius transformation ν leads to a contradiction. Indeed, (40) implies that $\nu(\infty) = \infty$. Therefore, $\nu = az + b$, $a, b \in \mathbb{C}$, and hence if (40) holds, then the functions V and

$$V \circ \mu = \frac{\varepsilon^2 z^2 + 2}{\varepsilon z + 1}$$

have the same set of poles. However, this is not true.

Let us calculate the group G_B. Using again a computer assistance one can check that the function

$$B = V \circ U = \frac{z^4 - 6\,z^2 - 4\,z + 18}{(z^2 + z - 5)\,(z - 1)}$$

has four critical values and the corresponding permutations in $Mon(B)$ are (13), (12)(34), (13), (12)(34), while the function

$$A = U \circ V = \frac{z\,(z^3 - 8)}{(z^3 + 1)}$$

has three critical values and the corresponding permutations in $Mon(A)$ are (12)(34), (13)(24), (14)(23). In particular, A and B are not conjugate since they have a different number of critical values. Moreover, one can check that the group $Aut(B)$ is trivial while $Aut(B_1) = \mathbb{Z}/3\mathbb{Z}$.

It is easy to see that $Mon(B)$ has a unique imprimitivity system $\{1, 3\}, \{2, 4\}$, corresponding to the decomposition $B = V \circ U$ while $Mon(A)$ has three imprimitivity systems

$$\{1, 3\}, \{2, 4\}, \qquad \{1, 2\}, \{3, 4\}, \qquad \{1, 4\}, \{2, 3\}.$$

corresponding to to the decompositions

$$B_1 = U \circ V, \qquad B_1 = (\mu^{-1} \circ U) \circ (V \circ \mu), \qquad B_1 = (\mu^{-2} \circ U) \circ (V \circ \mu^2).$$

Summing up, we see that the graph Γ_B has the form shown on Fig. 5, where the

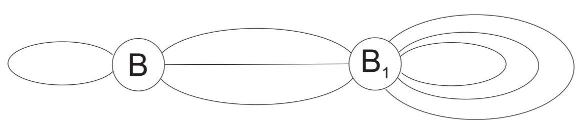

FIGURE 5

edges connecting B and B_1 correspond to the solutions

$$B = (V \circ \mu^{i-1}) \circ (\mu^{-(i-1)} \circ U), \qquad B_1 = (\mu^{-(i-1)} \circ U) \circ (V \circ \mu^{i-1}), \qquad 1 \leqslant i \leqslant 3,$$

of system (15), the loops attached to B_1 correspond to the solutions

$$B_1 = (\mu^{-(i-1)} \circ B_1) \circ \mu^{i-1} = \mu^{i-1} \circ (\mu^{-(i-1)} \circ B_1), \qquad 1 \leqslant i \leqslant 3,$$

and the loop attached to B corresponds to the solution (20).

The fundamental group of Γ_B can be easily calculated by the well known method using the spanning tree (see e. g. [15], Section 4.1.2). Namely, choosing a fixed orientation on each of edges of Γ_B as it is shown on Fig. 6, and considering the

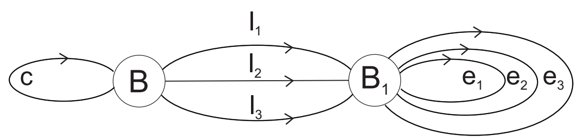

FIGURE 6

edge l_1 together with vertices B and B_1 as the spanning tree, we see that $\pi_1(\Gamma_B, B)$ is a free group of rank 6 generated by the paths

$$c, \quad l_1 l_i^{-1}, \quad 2 \leqslant i \leqslant 3, \quad l_1 e_j l_1^{-1}, \quad 1 \leqslant j \leqslant 3,$$

implying that the group G_B is generated by images of these paths under the map Φ_B. Assuming that

$$\mathcal{F}(c) = z, \quad \mathcal{F}(e_i) = \mu^{i-1}, \quad 1 \leqslant i \leqslant 3,$$

we obtain

$$\mathcal{F}(l_1 l_i^{-1}) = V \circ \mu^{i-1} \circ U, \quad 2 \leqslant i \leqslant 3, \quad \mathcal{F}(l_1 e_j l_1^{-1}) = V \circ \mu^{j-1} \circ U, \quad 1 \leqslant j \leqslant 3,$$

implying that the images of the functions

$$g_0 = z, \quad g_1 = V \circ \mu \circ U, \quad g_2 = V \circ \mu^2 \circ U \tag{41}$$

under the map Φ_B generate the group G_B. Since

$$\deg g_1 = \deg g_2 = \deg B, \tag{42}$$

and

$$g_1 \neq B, \quad g_2 \neq B, \quad g_1 \neq g_2,$$

it follows from Lemma 3.1 that g_1, g_2, g_3 represent different classes in $C_B/\underset{B}{\sim}$, so that G_B has at least three elements. On the other hand, since

$$g_1^{\circ 2} = g_2 \circ B, \quad g_1^{\circ 3} = B^{\circ 3},$$

the group G_B contains at most three elements. Therefore, $G_B = \mathbb{Z}/3\mathbb{Z}$.

In turn, the set C_B can be described as follows: $X \in C_B$ if and only if either

$$X = B^{\circ j}, \quad j \geqslant 0,$$

or

$$X = V \circ \mu \circ U \circ B^{\circ j}, \quad j \geqslant 0,$$

or

$$X = V \circ \mu^2 \circ U \circ B^{\circ j}, \quad j \geqslant 0.$$

Indeed, by Lemma 3.1, it is enough to check that functions (41) are not rational functions in B. Assume say that $g_1 = R \circ B$. Then it follows from (42) that R is a Möbius transformation, implying by Lemma 2.1 that $R \in Aut(B)$. However, since $Aut(B)$ is trivial and $g_1 \neq B$, this is impossible.

Notice that since $G_B \cong G_{B_1}$ by Theorem 5.3, and $Aut_G(B_1)$ is a cyclic group of order three,

$$G_{B_1} = Aut_G(B_1) = \mathbb{Z}/3\mathbb{Z}.$$

Notice also that since $G_B \cong G_{B_1}$, the equality $Aut(B_1) = \mathbb{Z}/3\mathbb{Z}$ by itself already implies that the group G_B is non-trivial, even though B has no automorphisms.

6.4. **Group G_B for $B = -2z^2/(z^4+1)$.** Since equality (26) implies that the function

$$W = \frac{z^2-1}{z^2+1} \tag{43}$$

commutes with B, the group G_B clearly contains a cyclic group of order two generated by $\boldsymbol{W}$. Moreover, it is easy to see that in fact $G_B = \mathbb{Z}/2\mathbb{Z}$. Indeed, providing edges of the graph Γ_B with orientations shown on Fig. 7, we see that $\pi_1(\Gamma_B, B)$ is

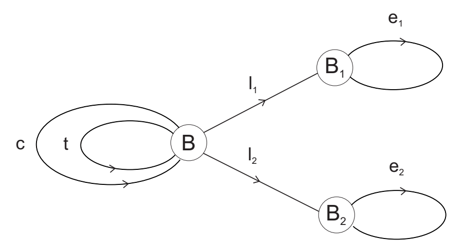

FIGURE 7

a free group of rank 4 with generators

$$c, \quad t, \quad l_i^{-1} e_i l_i, \quad i = 1, 2.$$

Assuming now that

$$\mathcal{F}(c) = \mathcal{F}(e_1) = \mathcal{F}(e_2) = z, \qquad \mathcal{F}(t) = W,$$

we see that G_B is generated by the $\boldsymbol{W}$.

Notice that switching from the function B to the function B_1 (or B_2) we can observe the same phenomena as in the Ritt example. Namely, since $G_B \cong G_{B_1}$, the group G_{B_1} is also a cyclic group of order two, and using the graph Γ_{B_1} one can show that G_{B_1} is generated by the Φ_B-image of the function

$$X = \mathcal{F}(l_1 t l_1^{-1}) = \frac{z^2+1}{z} \circ \frac{z^2-1}{z^2+1} \circ -\frac{2}{z^2-2} = \frac{16\left(z^2+2\right)^2}{\left(z^4+4\,z^2+8\right) z^2 \left(z^2+4\right)}$$

(notice that the inclusion $X \in C_B$ also follows from Lemma 6.8 for

$$V = \frac{z^2+1}{z}, \quad U = -\frac{2}{z^2-2},$$

and W defined by (43)).

On the other hand, the function X is not a rational function in B, implying that B and X cannot be represented in the form (39). Indeed, by Lemma 6.8, it is enough to show that there exists no rational function R such that

$$\frac{z^2+1}{z} \circ \frac{z^2-1}{z^2+1} = R \circ \frac{z^2+1}{z}. \tag{44}$$

Assume the inverse and let S be the rational function defined by any of the sides of equality (44). Then substituting z by $\frac{1}{z}$ in the right side of (44) we obtain that $S \circ \frac{1}{z} = S$. However, substituting z by $\frac{1}{z}$ in the left side we obtain

$$S \circ \frac{1}{z} = \frac{z^2+1}{z} \circ -\frac{z^2-1}{z^2+1} = -S.$$

The contradiction obtained shows that (44) is impossible.

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