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1 Introduction

In late 50th Selberg proved local rigidity of co-compact discrete subgroup Γ of $G = SL_n(\mathbb{R}), n \geq 3$ [S]. One of the ingredient in his proof of this result is the statement that, after a small perturbation in G, the subgroup Γ remains co-compact and discrete. While compactness can be easily proved the proof of discreteness is quite complicated. Selberg's proof of discreteness is based on the analysis of fundamental domains for the action of Γ on the symmetric space associated with G. He conjectured that the statement still true for a group acting on symmetric space with a compact fundamental domain. This conjecture was proved by A. Weil. He proved discreteness of a small perturbation of a co-compact discrete subgroup Γ of any connected Lie group G [W]. Weil's proof is very different from Selberg's proof and is based on the analysis of coverings of G/Γ by (small) open sets and the corresponding coverings of G.

The purpose of our work is to simplify and generalize Weil's proof. While we use the basic Weil's construction we prove the discreteness of small deformations of a discrete co-compact subgroup of isometries of a locally compact metric space under some natural restrictions.

Weil's theorem [W] was generalized in [A1], [A2]. Recently Weil's theorem was extended to uniform lattices of locally compact groups [GL]. We would like to note that proofs in these papers are based on the study of a fundamental domain. This is not required in our proof based on the Weil construction.

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2 Preliminaries

In order to keep the paper self-contained we start by spelling out well-known definitions, facts and well known theorems [ELL]. The subject discussed here is fundamental groups and covering spaces.

(2.1). Let X and Y be two topological spaces. The map $p : X \to Y$ is called a *covering map* (or shorter *covering*) if for very point $y \in Y$ there exists an open set $V \subset Y, y \in V$ such that $p^{-1}(V)$ is the union of disjoint open sets $U_i, i \in I$ such that the restriction $p\mid_{U_i}: U_i \to V$ is a homeomorphism. The set $p^{-1}(y) \subset X$ is called a fiber over y. Clearly, a covering map is local homeomorphism i.e. every point $x \in X$ is contained in an open set $U, U \subset X$ such that $V = p(U)$ is open in Y and the restriction $p\mid_U$ is a homeomorphism from U to V. It is easy to see that a local homeomorphism is not necessarily a covering map.

(2.2). Let f_1, f_2 be two continuous maps $f_1, f_2 : X \to Y$. These maps are *homotopic* if there exists a continuous map (*homotopy*) $H, H : X \times I \to Y$ where I is an interval $I = [0,1]$ such that $H(x,0) = f_1(x)$ and $H(x,1) = f_2(x)$ for all $x \in X$. Let $\{a(t) : t \in I\}$ be a path in X. Denote by $[a]$ the set of all paths in X with the same end points as $a(t)$ and homotopic to $a(t)$. The composition $[c]$ of two sets $[a]$ and $[b]$ is defined if the ending point of paths in$[a]$ is the starting point of paths in $[b]$ and denote $[c] = [a] \star [b]$. Let $\pi_1(X, x_0)$ be a set of all homotopic classes of all closed curves in X with x_0 as the starting and ending point.Then $\langle \pi_1(X, x_0), \star \rangle$ is a group which is called the *fundamental group* of X with a based point x_0.

(2.3). Recall that if X is path connected and $x_1, x_2 \in X$. Then there exists a natural isomorphism between $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ [ELL]. We will say that a path connected space X is *simply connected* if the fundamental group $\pi_1(X, x_0)$ is trivial for a some (hence for every) point $x_0 \in X$. A space X is called locally simple connected if X admits a basis of simply connected sets. A space X is locally path connected if X admits a basis of open path connected sets.

Remark. A path connected (resp. simply connected) space is not necessarily locally path connected (resp. locally simply connected).

Let us recall the following well known theorems [ELL].

Theorem 1. Let $p : \tilde{X} \to X$ be a covering map. Suppose that X is simply connected and locally path connected. If $\tilde{X}$ is connected then p is a homeomorphism.

Theorem 2 If a space X is path connected,locally path connected and locally simply connected then there exists a connected, simply connected space $\tilde{X}$ and a covering map $p : \tilde{X} \to X$.

The space $\tilde{X}$ is called the *universal covering* of X.

(2.4). Let G be the group of homeomorphisms of a space X. Let Γ be a subgroup of G. We will say that Γ acts on X *properly discontinuously* if for every compact set $K, K \subset X$ the set $\{\gamma \in \Gamma : \gamma K \cap K \neq \emptyset\}$ is finite. We will say that Γ acts *freely* on X if $\gamma x \neq x$ for every $\gamma \in \Gamma, \gamma \neq e$ and $x \in X$. It is not difficult to conclude that if Γ acts properly discontinuously and freely on a locally compact space X, then for every non trivial element $\gamma \in \Gamma$ and point $x \in X$ there exists a neighborhood $V, x \in V$ such that $\gamma V \cap V = \emptyset$.

Let $\tilde{X} = \{[a], a \,\text{is a path in}\, X \,\text{such that}\, a(0) = x_0\}$ be the universal covering of X and $\pi_1(X, x_0)$ be the fundamental group of X. Then the natural map $(\gamma, [a]) \to \gamma \star [a]$ define an action of $\pi_1(X, x_0)$ on $\tilde{X}$. It is well known that this action is properly discontinuous, free and the space of orbits $\Gamma \diagup \tilde{X}$ is homeomorphic to X.

(2.5.) Let Γ be subgroup of a topological group G. The representation space $R(\Gamma, G)$ the space of all homomorphisms $\Gamma \to G$ with the topology of point wise convergence. Clearly, $R(\Gamma, G)$ is a closed subset in the direct product $\amalg_{\gamma \in \Gamma} G_\gamma$ with the product topology. If Γ is finitely generated and S is a generating set then $R(\Gamma, G)$ is homeomorphic to a closed subset of $\amalg_{s \in S} G_s$ with the product topology.

3 Deformation and discreteness

Let $\tilde{X}$ be a locally compact path connected, simply connected locally compact metric space, Let ρ be a metric on $\tilde{X}$. Let Γ be a subgroup of the group of isometries $\mathrm{Isom}(\tilde{X})$ of the metric space $\tilde{X}$. Assume that Γ acts properly discontinuously, freely and the space $X = \Gamma \diagup \tilde{X}$ is compact. Thus $p : \tilde{X} \to X$ is a covering map, $\tilde{X}$ is the universal covering space for X and Γ is the fundamental group of X. Denote by $\tilde{\rho}$ the natural metric on X induced by ρ. Since X is compact there exists a compact subset K of $\tilde{X}$ such that $\Gamma K = \tilde{X}$. Let us show that

(**i**). There exists a finite set of open subsets $\{\tilde{U}_i, i \in I\}$ such that $K \subseteq \bigcup_{i \in I} \tilde{U}_i$ and the restriction $p \mid \tilde{U}_i$ is a homeomorphism for every $i \in I$.

(**ii**). For every $i, j \in I$ there exists one $\gamma_{i,j} \in \Gamma$ such that

$$\tilde{U}_i \cap \gamma_{i,j} \tilde{U}_j \neq \emptyset \tag{1}$$

In particularly $\gamma_{ii} = e$ for all $i \in I$.

(**iii**). There exists $d_1 > 0, d_1 \in \mathbb{R}$ such that if $p(\tilde{U}_i) \cap p(\tilde{U}_j) = \emptyset$ then

$$\tilde{\rho}(p(\tilde{U}_i), p(\tilde{U}_j)) > d_1 \tag{2}$$

Indeed, since Γ acts properly discontinuously and freely on $\tilde{X}$ and K is compact there exists a positive number r_0 such that $\rho(\gamma x, x) > r_0$ for all $x \in K, \gamma \in \Gamma, \gamma \neq e$. Consider a covering of K by a finite set of open balls $\tilde{U}_i = B(x_i, r), x_i \in K, r > 0, r \in \mathbb{R}, r \leq r_0/10, i \in I$.

Suppose that there exists a non trivial element $\gamma \in \Gamma$ such that for some $i \in I$ we have $\tilde{U}_i \cap \gamma\tilde{U}_i \neq \emptyset$ then $r_o \leq \rho(x_i, \gamma x_i) \leq r_0/5$. Contradiction. This proves (**i**).

Assume that there are two different elements $\gamma_1 \in \Gamma$, $\gamma_2 \in \Gamma$ and some $\tilde{U}_j$ such that $\tilde{U}_i \cap \gamma_1\tilde{U}_j \neq \emptyset$ and $\tilde{U}_i \cap \gamma_2\tilde{U}_j \neq \emptyset$. Then the $\rho(\gamma_1 x_j, \gamma_2 x_j) \leq 6r$. Hence,$\rho(\gamma_2^{-1}\gamma_1 x_j, x_j) \leq 6r \leq r_0$. Contradiction. This proves (**ii**).

Let us now show that it is possible to deform the covering $\tilde{U}_i, i \in I$ such that it also fulfills (**iii**.) The idea is to deform step by step the existing covering such that the new set of open balls still will be a covering and fulfills (**i**),(**ii**) and (**iii**.) Indeed, let $\{\leq\}$ be the lexicographic order on $I \times I$ and let $(i,j), i \leq j$ be the minimal element such that $\tilde{\rho}(\tilde{U}_i, \tilde{U}_j) = 0$. It is easy to see going alone the same way as we did in **ii** that there exists a unique $\gamma \in \Gamma$ such that $\rho(\tilde{U}_i, \gamma\tilde{U}_j) = 0$. For the covering $\tilde{U}_i, i \in I$ of K there exists a positive number d_0 such that for every point $x \in K$ there exists $\tilde{U}_{i_0}$ for some $i_0 \in I$ such that $B(x, d_0) \subset \tilde{U}_{i_0}$. Let us take $\tilde{U}_j = B(x_j, \tilde{r})$, where $\tilde{r} = r - d_0/10$. Clearly, that after a finite number of steps we will get a covering which has properties (**i**),(**ii**), (**iii**.)

Remark 1. The proof of (**i**) and (**ii**) is also valid for a locally compact but not necessarily a metric spaces $\tilde{X}$ and groups of homeomorphisms Γ of $\tilde{X}$ acting properly discontinuously and freely.

Remark 2. We can reformulate property (2),(**iii**) in the following way: assume that for $i, j \in I$ we have $\tilde{U}_i \cap \gamma\tilde{U}_j = \emptyset$ for all $\gamma \in \Gamma$. Then $\rho(\tilde{U}_i, \gamma\tilde{U}_j) \geq d_1$ for every $\gamma \in \Gamma$

Remark 3. For the covering $\{p(\tilde{U}_i), i \in I\}$ of the compact set X. there exists a constant r_1 such that for every $x \in X$ there exists $p(\tilde{U}_i)$ such that

$$B(x, r_1) \subset p(\tilde{U}_i) \qquad (4).$$

Let J be a subset of $I \times I$ such that for every element $(i,j) \in J$ there exists $\gamma_{i,j}$ such that

$$\tilde{U}_i \cap \gamma_{i,j} \tilde{U}_j \neq \emptyset \tag{5.}$$

Obviously, the property $p(\tilde{U}_i) \cap p(\tilde{U}_j) \neq \emptyset$ and (5) are equivalent. Let $u \in \tilde{U}_{i_0}, i_o \in I$ and let $\{i_0, \ldots, i_k\}$ be the set all elements of I such that $p(u) \in \bigcap_{0 \leq k \leq m} p(\tilde{U_{i_k}})$. Hence $(i_0, i_k) \in J$ for all $k, 1 \leq k \leq m$. Thus $u \in \tilde{U}_0 \cap \gamma_{i_0,i_1} \tilde{U}_{i_1} \cdots \cap \gamma_{i_0,i_m} \tilde{U}_{i_m}$. It follows from (4) that there exists $i_k, 0 \leq k \leq m$ such that

$$B(u, r_1) \subset \gamma_{i_0.i_k} \tilde{U}_k \tag{5}$$

The following construction was actually introduced by A.Weil [AW1]. Let W be the union of the disjoint open sets $\tilde{U}_i, i \in I$. Let X_* be the product $W \times \Gamma$, where Γ is provided with the discrete topology. Define a relation R as follows. Let (w_1, γ_1) and (w_2, γ_2) be two elements of X_*. Then these two elements are R-equivalent if $w_1 \in \tilde{U}_i$, $w_2 \in \tilde{U}_j$, $w_1 = \gamma_{i,j} w_2 \in \tilde{U}_i \cap \gamma_{ij} \tilde{U}_j$ and $\gamma_2 = \gamma_1 \gamma_{i,j}$. From (2) easy follows that $\gamma_{i,i} = e, \gamma_{i,j}\gamma_{j,i} = e$. If there are points $u_i \in \tilde{U}_i$, $u_j \in \tilde{U}_j$, $u_k \in \tilde{U}_k$ such that $u_i = \gamma_{i,j} u_j = \gamma_{ik} u_k$ then $u_j = \gamma_{j,i}\gamma_{i,k} u_j$. Thus $\gamma_{j,k} = \gamma_{j,i}\gamma_{i,k}$. Hence $\gamma_{i,k} = \gamma_{i,j}\gamma_{j,k}$. We conclude that R is an equivalence relation. Let $\varphi \in R(\Gamma, \text{Isom}(\tilde{X})$ be a deformation of Γ. Define a relation $\tilde{R}$ on X_* as follows: two elements (w_1, γ_1) and (w_2, γ_2) are $\tilde{R}$–equivalent if $w_1 \in \tilde{U}_i$, $w_2 \in \tilde{U}_j$, $w_1 = \varphi(\gamma_{i,j}) w_2 \in \tilde{U}_i \cap \varphi(\gamma_{i,j}) \tilde{U}_j$ and $\gamma_2 = \gamma_1 \gamma_{i,j}$. Since φ is a homomorphism we conclude that $\tilde{R}$ is an equivalence relation. Set $\tilde{X}_* = W/\tilde{R}$ and $[w, \gamma] = \{(\tilde{w}, \tilde{\gamma}) : (w, \gamma) \sim^{\tilde{R}} (\tilde{w}, \tilde{\gamma})\}$. Clearly, $\tilde{X}_*$ is a locally compact topological space.

We will say that a deformation $\varphi \in R(\Gamma, G)$ is J–**small** if

$$(i,j) \in J \Leftrightarrow \tilde{U}_i \cap \varphi(\gamma_{i,j}) \tilde{U}_j \neq \emptyset \tag{6}$$

In other words

$$\tilde{U}_i \cap \gamma_{i,j} \tilde{U}_j \neq \emptyset \Leftrightarrow \tilde{U}_i \cap \varphi(\gamma_{i,j}) \tilde{U}_j \neq \emptyset \tag{6*}$$

Indeed, from (**iii**) follows that there exists a neighbourhood U of a trivial deformation such that for $\varphi \in U$ from $\tilde{U}_i \cap \gamma_{i,j}\tilde{U}_j = \emptyset$ follows that $\tilde{U}_i \cap \varphi(\gamma_{i,j})\tilde{U}_j = \emptyset$. Clearly there exists a neighbourhood of a trivial deformation such that for $(i,j) \in J$ and $\varphi \in U$ we have $\tilde{U}_i \cap \varphi(\gamma_{i,j})\tilde{U}_j \neq \emptyset$. Therefore there exists a neighbourhood U of a trivial deformation consisting just of J–small deformations.

Remark 4. It is easy to show based on (6^*) that a J–**small** deformation is an isomorphism.

Let us show that the space $\tilde{X}_*$ is connected for any J–**small** deformation $\varphi \in R(\Gamma, G)$. It is enough to show that for any two points $x, y \in \tilde{X}_*$ there exists a connected set $Y \subset \tilde{X}_*$ such that $x, y \in Y$. Let $x = [w, \gamma]$ and $y = [u, \tilde{\gamma}]$. Without lost of generality we can and will assume that $\tilde{\gamma} = e$. Since $\tilde{X}$ is path connected, there exists a path $p(t), 0 \leq t \leq 1$ in $\tilde{X}$ such that $p(1) = \gamma w, p(0) = u$. There exist $\gamma_i, \tilde{U}_i, i = 0, 1, \ldots, m$ where $\gamma_m = \gamma, \gamma_0 = e$ such that $p(t) \subset \gamma_m \tilde{U}_m \cup \gamma_{m-1}\tilde{U}_{m-1} \cup \ldots \gamma_1 \tilde{U}_1 \cup \tilde{U}_0$ where $p(1) \in \gamma_m \tilde{U}_m, p(0) \in \tilde{U}_0$ and $\gamma_i \tilde{U}_i \cap \gamma_{i-1}\tilde{U}_{i-1} \neq \emptyset$ for all $i = 1, \ldots m$, Moreover, we can and will assume that for $i \neq j, 0 \leq i, j \leq m$ we have $\gamma_i \tilde{U}_i \nsubseteq \gamma_j \tilde{U}_j$. Obviously $\tilde{U}_i \cap \gamma_i^{-1}\gamma_{i-1}\tilde{U}_{i-1} \neq \emptyset$. It follows from (1) that $\gamma_i^{-1}\gamma_{i-1} = \gamma_{i,i-1}$ and $\tilde{U}_i \cap \gamma_{i,i-1}\tilde{U}_{i-1} \neq \emptyset$ for all $i = 1, \ldots m$. It follows from (6) that for a small deformation φ we have $\tilde{U}_i \cap \varphi(\gamma_{i,i-1})\tilde{U}_{i-1} \neq \emptyset$ for all $i = 1, \ldots m$. Therefore since $\gamma_i^{-1}\gamma_{i-1} = \gamma_{i,i-1}$ we conclude that $[\tilde{U}_i, \gamma_i] \cap [\tilde{U}_{i-1}, \gamma_{i-1}] \neq \emptyset$ for all $i = 1, \ldots m$. Therefore $Y = \bigcup_{0 \leq i \leq m} [\tilde{U}_i, \gamma_i]$ is a connected set and $x, y \in Y$.

Let $w \in \tilde{X}$. Then $B(w, r_1) \subset \gamma_{ij}\tilde{U}_j$ (see (5)). Set $\tilde{r} = r_1/2$. Then there exists a neighbourhood $\tilde{U}_{\tilde{r}} \subset U$ such that for all $\varphi \in \tilde{U}$ we have

$$B(w, \tilde{r}) \subset \varphi(\gamma_{i,j})\tilde{U}_j \qquad (7).$$

It follows from (5) that for any $x = [w, \gamma] \in \tilde{X}_*$ there exists $\tilde{U}_j$ such that

$$x \in [B(w, \tilde{r}), \gamma] \subset [\tilde{U}_j, \gamma] \qquad (8).$$

Let us show that the space $\tilde{X}_*$ is Hausdorff. Indeed, let $x, y \in \tilde{X}_*$, be two different points.

There are three cases:

(a) there exists $[\tilde{U}_i, \gamma]$ such that $x, y \in [\tilde{U}_i, \gamma]$

(b) $x \in [\tilde{U}_i, \gamma_1]$, $y \in [\tilde{U}_j, \gamma_2]$ and $[\tilde{U}_i, \gamma_1] \cap [\tilde{U}_j, \gamma_2] = \emptyset$

(c) $x \in [\tilde{U}_i, \gamma_1]$,$y \in [\tilde{U}_j, \gamma_2]$ and $[\tilde{U}_i, \gamma_1] \cap [\tilde{U}_j, \gamma_2] \neq \emptyset$.

It is clear that in cases (a) and (b) there exit two open subsets W_1 and W_2 such that $x \in W_1, y \in W_2$ and $W_1 \cap W_2 = \emptyset$. Case (c). It follows from the definition that there exists $\varphi(\gamma_{i,j}$ such that $\tilde{U}_i \cap \varphi(\gamma_{i,j})\tilde{U}_j \neq \emptyset$. Let $x = (u_i, \gamma_1)$ and $y = (u_j, \gamma_2)$. Set $T = \tilde{U}_i \cap \varphi(\gamma_{i,j})\tilde{U}_j$. Let $m = \inf_{t \in T}\{\rho(u_i, t) + \rho(\varphi(\gamma_{i,j})u_j, t)\}$. It follows from (8) that if $m = 0$ then we have (a). Thus we will assume that $m > 0$. Let W_1 be the ball $W_1 = B(u_i, m/4)$ and W_2 be the ball $W_2 = B(u_j, m/4)$. We will show that $[W_1, \gamma_1] \cap [W_2, \gamma_2] = \emptyset$. Indeed, if $[W_1, \gamma_1] \cap [W_2, \gamma_2] \neq \emptyset$ then there exists $\varphi(\gamma_{k,l})$ such that $W_1 \cap \varphi(\gamma_{k,l})W_2 \neq \emptyset$. It follows from **ii** and (6) that $\gamma_{k,l} = \gamma_{i,j}$. Thus there is a point $t \in T$ such that $t \in W_1 \cap \varphi(\gamma_{i,j})W_2$. Therefore $m \leq \rho(u_i, t) + \rho(\varphi(\gamma_{i,j})u_j, t) \leq m/2$. Contradiction. This proves the statement.

Denote by q the map $q : \tilde{X}_* \to \tilde{X}$ where $q([w, \gamma]) = \varphi(\gamma)w$. It follows from (7) and (8) that q is a covering map. Moreover, since for every point in $q([w, \gamma])$ in $q(\tilde{X}_*)$ we have $q([B(w, \tilde{r}), \gamma]) \subset q(\tilde{X}_*)$. Therefore we conclude that $q(\tilde{X}) = \tilde{X}$. Hence we have a surjective covering map q of a connected space $q(\tilde{X}_*)$ onto connected simply connected space $\tilde{X}$. Thus q is homeomorphism. Clearly, q is Γ-equivariant map. Thus we proved the following theorem.

Theorem 1. Let $\tilde{X}$ be a locally compact simply connected metric space. Let Γ be a subgroup of $\mathbf{Isom}\tilde{X}$. Suppose that Γ acts properly discontinuously and freely on $\tilde{X}$ such that the space $\Gamma \diagup \tilde{X}$ is compact. Then there exist a neighbourhood U of the inclusion $\Gamma \hookrightarrow \mathbf{Isom}\tilde{X}$ such that for every $\varphi \in U$ the group $\varphi(\Gamma)$ acts properly discontinuously on $\tilde{X}$.

Corollary 1. Let X be a compact locally simply connected path connected metric space.

Let $\tilde{X}$ be the universal covering of X and let Γ be the fundamental group of X. Then there exists a neighbourhood $U \subset \mathbf{Isom}\tilde{X}$ of the inclusion $i : \Gamma \hookrightarrow \mathbf{Isom}\tilde{X}$ such that $\varphi(\Gamma)$ is a discrete subgroup of $\mathbf{Isom}\tilde{X}$ for every $\varphi \in U$.

From theorem 1 and Selber's lemma [S, Lemma 8] follows

Corollary 2. Let $\tilde{X}$ be a locally compact simply connected metric space. Let Γ be a subgroup of $\mathbf{Isom}\tilde{X}$. Assume that there exist a faithful linear representation $\varphi : \Gamma \to GL_n(k)$ over field k. Suppose that the action of Γ on $\tilde{X}$ is properly discontinuous and the space $\Gamma \diagup \tilde{X}$ is compact. Then there exist a neighbourhood U of the inclusion $\Gamma \hookrightarrow \mathbf{Isom}\tilde{X}$ such that for every $\varphi \in U$ the group $\varphi(\Gamma)$ acts properly discontinuously on $\tilde{X}$.

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