ON GEODESIC FLOWS AND SPECTRAL RIGIDITY FOR 2–SPHERE AND REAL PROJECTIVE PLANE WITH S^1–ACTION OF ISOMETRIES

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0. Introduction

It is known that the eigenvalues of the laplacian for a compact riemannian manifold determines the set of the length of closed geodesics on it to some extent, which is due to Colin de Verdière [3], Chazarain [2], and Duistermaat-Guillemin [4]. Here and throughout the paper we consider laplacians acting on functions. In some cases these results are strong enough to show some spectral rigidity of a riemannian manifold (cf. Guillemin-Kazhdan [5], Kiyohara [6]). On the other hand, Weinstein [7] proved that if the geodesic flows of two compact riemannian manifolds are symplectically isomorphic, then the distribution of the eigenvalues of their laplacians are asymptotically similar.

In this paper we consider the geodesic flows of surfaces diffeomorphic to 2-sphere S^2 or the real projective plane $\mathbb{R}P^2$ which admit S^1-action of isometries, and prove two results. The first one is as follows: There are riemannian metrics on S^2 such that the corresponding geodesic flows are mutually symplectically isomorphic. It is well known that the geodesic flows of Zoll surfaces – surfaces of revolution all of whose geodesics are closed – are mutually symplectically isomorphic (cf. Besse [1]). Our example is a generalization of those. As a consequence, we know that for those surfaces the corresponding laplacians have asymptotically similar eigenvalues by Weinstein's result.

The second one is concerning the spectral rigidity, which may be stated as follows: Let $\{g_t\}$ be a one-parameter family of riemannian metrics on $\mathbf{R}P^2$ of the form described in section 1. Assume that the corresponding laplacians have the same eigenvalues, and also assume that each g_t satisfies some generic condition on the length of closed geodesics. Then each $(\mathbf{R}P^2, g_t)$ are mutually isometric (cf. Theorem 3.2).

In section 1 we describe the geodesic equation in terms of the polar coordinates and solve it. Then the criterion of the closedness of a geodesic and its length are given in terms of the first integral which is defined by the rotational isometries. In section 2 we give examples of symplectically isomorphic geodesic flows. In this section we also prove that under some assumption, an infinitesimal deformation of an energy function E is an image of E by an infinitesimal symplectic transformation if and only if its average over each closed geodesic is zero. This result is crucial in the next section. In section 3 we show some spectral rigidity for $\mathbf{R}P^2$. In the case of analytic metrics the assumption can be slightly weakened.

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1. Geodesics on S^2 with S^1-action of isometries

In this section we refer to Besse [1], Chapter 4. Let g be a riemannian metric on S^2 which admits non-trivial S^1-action of isometries. In this case the fixed point set under the S^1-action consists of 2 points, say N and S. By using the normal polar coordinates (r, θ) centered at N, the metric g is described as

$$g = dr^2 + a(r)^2\, d\theta^2$$

on $U = S^2 - \{N, S\} = \{(r, \theta) \mid 0 < r < L,\ \theta \in \mathbf{R}/2\pi\mathbf{Z}\}$, where the point N (resp. S) corresponds to $r = 0$ (resp. $r = L$).

Our first assumption for the metric g is:

(*) The number of $r \in [0,L]$ such that $a'(r) = 0$ is one, and if $a'(r_0) = 0$, then $a''(r_0) \neq 0$.

For example, if (S^2, g) has positive gaussian curvature, then this condition is satisfied. We put $a(r_0) = M$. The condition (*) enables us the coordinate change $r \to \theta^1$ defined by

$$a(r) = M \sin\theta^1,$$

where $r \in [0, r_0]$ (resp. $r \in [r_0, L]$) corresponds to $\theta^1 \in [0, \pi/2]$ (resp. $\theta^1 \in [\pi/2, \pi]$). Putting $\theta = \theta^2$ and $dr/d\theta^1 = M\cdot H(\cos\theta^1)$, the metric g is written as

$$g = M^2\{H(\cos\theta^1)^2(d\theta^1)^2 + (\sin\theta^1)^2(d\theta^2)^2\}$$

on $U = \{(\theta^1, \theta^2) \mid 0 < \theta^1 < \pi,\ \theta^2 \in \mathbf{R}/2\pi\mathbf{Z}\}$. In this form g represents a C^∞ riemannian metric on S^2 if and only if H is a C^∞ function on $[-1, 1]$ and $H(\pm 1) = 1$, $H > 0$ (cf. Besse [1] p. 99).

The corresponding energy function E on the cotangent bundle T^*S^2 is

$$E = \frac{1}{2M^2}\left\{ \frac{\eta_1^2}{H(\cos\theta^1)^2} + \frac{\eta_2^2}{(\sin\theta^1)^2} \right\} .$$

where $(\theta^1, \theta^2, \eta_1, \eta_2)$ are canonical coordinates. Then η_2 is a first integral for the geodesic flow, and we put

$$\eta_2 = c$$

On the unit cotangent bundle $S^*S^2 = E^{-1}(1/2)$, we have $|c| \leq M$, and the geodesic equation becomes

$$\frac{d\theta^1}{dt} = \frac{\pm\sqrt{1 - \dfrac{(c/M)^2}{(\sin\theta^1)^2}}}{M \cdot H(\cos\theta^1)}, \quad \frac{d\theta^2}{dt} = \frac{c}{M^2(\sin\theta^1)^2} .$$

If we put $i = \arcsin(|c|/M) \in [0, \pi/2]$, then we see that the geodesic with $\eta_2 = c$ oscillates between the parallels $\theta^1 = i$ and $\theta^1 = \pi - i$. Fix such a geodesic and let p_0 be a point on the geodesic with $\theta^1 = i$. Let p_1 (resp. p_2) be the first point after passing p_0 on the geodesic such that $\theta^1 = \pi - i$ (resp. $\theta^1 = i$). Then from the above equation we see that the time difference between the points p_0 and p_1 is given by

$$M \int_i^{\pi - i} \frac{H(\cos\theta^1)}{\sqrt{1 - \dfrac{(c/M)^2}{(\sin\theta^1)^2}}} \, d\theta^1 ,$$

and the difference of the values of θ^2 between these points is given by

$$\frac{c}{M}\int_{l}^{\pi-l} \frac{H(\cos\theta^1)}{(\sin\theta^1)^2\sqrt{1-\frac{(c/M)^2}{(\sin\theta^1)^2}}}\,d\theta^1 .$$

We also see that the differences of time and θ^2 between the points p_0 and p_2 are twice the corresponding differences between the points p_0 and p_1.

Now let us define the integral operators I_1 and I_2 for functions F on $[0, 1]$:

$$(I_1F)(x) = \int_{\arcsin x}^{\pi/2} \frac{F(\cos\theta)}{\sqrt{1-\frac{x^2}{(\sin\theta)^2}}}\,d\theta ,$$

$$(I_2F)(x) = x\int_{\arcsin x}^{\pi/2} \frac{F(\cos\theta)}{(\sin\theta)^2\sqrt{1-\frac{x^2}{(\sin\theta)^2}}}\,d\theta .$$

Proposition 1.1. (i) The geodesic with $\eta_2 = c$ is closed if and only if $(I_2H_{ev})(|c|/M) \in \pi\mathbb{Q}$, and if $(I_2H_{ev})(|c|/M) = (n/m)\cdot(\pi/2)$ $(n, m$ being mutually prime integers, $m > 0)$, then the length of the closed geodesic is given by $4mM\cdot(I_1H_{ev})(|c|/M)$.

(ii) $x\cdot(I_2H_{ev})'(x) = (I_1H_{ev})'(x)$.

Here $H_{ev}(x) = (1/2)\{H(x)+ H(-x)\}$, and Q is the field of rational numbers, and the prime sign means the derivative.

Proof. Let p_0, p_1, and p_2 be as above. Then it is clear that the geodesic is closed if and only if the difference of the values of θ^2 between p_0 and p_2 is 2π times a rational number, and if this difference is $2\pi n/m$, $(n, m)=1$, $m>0$, then the length of the geodesic is m times the time difference between the points p_0 and p_2. So we have (i). The second assertion is proved by a direct calculation.

We put

$$N(c) = \{w \in S^*S^2 \mid \eta_2 = c\} \ , \ |c| \le M .$$

Let $\{\xi_s\}$ be the geodesic flow on S^*S^2. If the geodesic with $\eta_2 = c$ is closed and its length is l, then ξ_{kl} is the identity on $N(c)$ for any integer k. We say that $N(c)$ is non-degenerate with respect to the mapping ξ_{kl} if for every point $z \in N(c)$ the fixed point set of the differential of ξ_{kl} at z coincides with $T_zN(c)$. The following lemma is easily obtained.

Lemma 1.2. Let l and k be as above, and assume that c is not zero nor $\pm M$. Then $N(c)$ is a connected component of the fixed point set of ξ_{kl} and is non-degenerate with respect to ξ_{kl} if and only if

$$(I_1H_{ev})'(|c|/M) \neq 0 \ .$$

2. Symplectically isomorphic geodesic flows

As we have seen in the previous section, the set of the length of closed geodesics does not depend on the odd part of H. As a matter of fact, we have a stronger result. Let

$$g_i = M^2\{ H_i(\cos\theta^1)^2(d\theta^1)^2 + (\sin\theta^1)^2(d\theta^2)^2\} \quad (i = 0, 1)$$

be two riemannian metrics on S^2.

Theorem 2.1. If $(H_0)_{ev} = (H_1)_{ev}$, then the geodesic flows of (S^2, g_i) $(i = 0, 1)$ are symplectically isomorphic, i.e., there is a homogeneous symplectic diffeomorphism ϕ of T^*S^2 - {0-section} such that $\phi^*E_1 = E_2$.

Remark. If $(H_i)_{ev} = 1$, then g_i are Zoll metrics, and the theorem is already known in this case (cf. Besse [1] p.122).

Proof. Put $H_t = (1-t)H_0 + tH_1$ $(0 \le t \le 1)$. Then $H_t > 0$, $H_t(\pm) = 1$, and $(H_t)_{ev} = (H_0)_{ev}$ for all $t \in [0, 1]$. Let E_t be the corresponding energy function:

$$E_t = \frac{1}{2M^2}\{ \frac{\eta_1^2}{H_t(\cos\theta^1)^2} + \frac{\eta_2^2}{(\sin\theta^1)^2}\} .$$

We consider the following equation for F_t:

$$X_{E_t}F_t = \dot{E}_t ,$$

where X_{E_t} is the hamiltonian vector field defined by E_t, and the dot means the derivative in t. We would like to solve this so that F_t is a C^∞ function on T^*S^2 - {0-section}, homogeneous of degree one, and also C^∞ in the variable t. If we find such F_t, then $X_{F_t}E_t = -\dot{E}_t$. Therefore if we define symplectic diffeomorphisms ϕ_t of T^*S^2 - {0-section} by

$$\frac{\partial}{\partial t}\phi_t(\omega) = (X_{F_t})_{\phi_t(\omega)}, \quad \phi_0(\omega) = \omega, \quad \omega \in T^*S^2 - \{0\text{-section}\},$$

then $(\partial/\partial t)\phi_t{}^*E_t = 0$, and the theorem will be proved.

We can describe the solution explicitly: If $(\theta^1, \theta^2) \in U = S^2 - \{N, S\}$,

$$F_t(\theta^1,\theta^2,\eta_1,\eta_2) = \frac{-1}{2M^2E_t}\int_0^A G\left(\frac{2M^2E_t - \eta_2^2 - y^2}{2M^2E_t}\right) y^2\,dy,$$

where $A = \eta_1 \sin\theta^1/H_t(\cos\theta^1)$, and if the base point of ω is N or S, then $F_t(\omega) = 0$. Here $G \in C^\infty[0, 1]$ is defined as follows: Since $\dot{H}_t = H_1 - H_0$ is an odd function and vanishes at $x = \pm 1$, we can write

$$H_1(x) - H_0(x) = x(1 - x^2)G(x^2).$$

It is easy to see that F_t is of class C^∞ and satisfies $X_{E_t}F_t = \dot{E}_t$ on $(T^*U$ - {0-section}$) \times [0, 1]$. Moreover we have

$$\int_a^b \dot{E}_t(\xi_s^t\omega)\, ds = F_t(\xi_b^t\omega) - F_t(\xi_a^t\omega)$$

for any $\omega \in T^*S^2$ with $2E_t(\omega) = 1$ and any real numbers a and b. Here $\{\xi_s^t\}$ is the geodesic flow with respect to the energy function E_t. From this formula it is now clear that $F_t(\omega)$ is the C^∞ function of $(\omega, t) \in (T^*S^2 - \{0\text{-section}\}) \times [0, 1]$.

This theorem combined with the result of Weinstein stated in Introduction yields the following

Corollary 2.2. Let g_0 and g_1 be as above, and assume that $(H_0)_{ev} = (H_1)_{ev}$. Let $\lambda_{i,k}$ $(i = 0, 1)$ be the k-th eigenvalue of the laplacian Δ_{g_i}. Then $|\lambda_{0,k} - \lambda_{1,k}|$ remains bounded when k tends to ∞.

Under some assumption the converse of the infinitesimal version of Theorem 2.1 is also true. For a riemannian metric $g = M^2\{H(\cos\theta^1)^2(d\theta^1)^2 + (\sin\theta^1)^2(d\theta^2)^2\}$ on S^2 we put

$$C_g = \{c \in (-M, M) \mid \text{geodesic of } (S^2, g) \text{ with } \eta_2 = c \text{ is closed}\}.$$

For $c \in C_g$ we denote by $l(c)$ the length of the corresponding closed geodesic. Let us consider the following condition for the metric g:

$(\#)_1$ C_g is dense in $[-M, M]$.

Proposition 2.3. Assume that the metric g satisfies the condition $(\#)_1$, and let C_0 be a subset of C_g such that C_0 is still dense in $[-M, M]$. Let

$$\dot{E} = -\frac{\dot{M}}{M^3}\left\{\frac{\eta_1^2}{H(\cos\theta^1)^2} + \frac{\eta_2^2}{(\sin\theta^1)^2}\right\} - \frac{1}{M^2}\frac{\eta_1^2\,\dot{H}(\cos\theta^1)}{H(\cos\theta^1)^3}$$

be an infinitesimal deformation of the energy function E. Suppose that

$$\int_0^l \dot{E}(\xi_t\omega)\,dt = 0$$

for every $\omega \in S^*S^2$ with $\eta_2(\omega) = c \in C_0$, where $l = l(c)$. Then $\dot{M} = 0$ and $\dot{H}$ is an odd function.

Proof. From the assumption we have

$$\int_0^l \frac{\eta_1(t)^2\,\dot{H}(\cos\theta^1(t))}{H(\cos\theta^1(t))^3}\,dt = -\dot{M}Ml ,$$

where $\xi_t\omega = (\theta^1(t), \theta^2(t), \eta_1(t), \eta_2(t))$. The left hand side of this formula is equal to

$$2mM^3\int_{\tau}^{\pi-\tau} \dot{H}(\cos\theta^1)\sqrt{1-\frac{(c/M)^2}{(\sin\theta^1)^2}}\;d\theta^1 ,$$

where m is the natural number defined in Proposition 1.1 (i), and τ =

$\arcsin(|c|/M) \in [0, \pi/2]$. Since $1 = 4mM(I_1H_{ev})(|c|/M)$, we have

$$\int_{\arcsin x}^{\pi/2} \dot{H}_{ev}(\cos\theta^1)\sqrt{1 - \frac{x^2}{(\sin\theta^1)^2}}\; d\theta^1 = -\frac{\dot{M}}{M}(I_1H_{ev})(x) \;,$$

where $x = |c|/M$.

Since this formula holds for any $c \in C_0$, and since C_0 is dense in $[-M, M]$, it follows that it holds for all $c \in [-M, M]$. So let us define an integral operator I_3 for functions F on $[0, 1]$ by

$$(I_3F)(x) = \int_0^x F(y)\sqrt{x^2 - y^2}\; dy = x^2\int_0^1 F(xy)\sqrt{1 - y^2}\; dy \;.$$

Then we have the identity

$$I_3\Big(\frac{\dot{H}_{ev}}{1-y^2}\Big)\big(\sqrt{1-x^2}\,\big) = -\frac{\dot{M}}{M}(I_1H_{ev})(x) \;, \quad x \in [0, 1] \;.$$

Note that the function $\dot{H}_{ev}(y)/(1-y^2)$ is of class C^∞ on $[0, 1]$, because $\dot{H}(\pm 1) = 0$. From the definition of I_3 we see that

$$I_3\Big(\frac{\dot{H}_{ev}}{1-y^2}\Big)(0) = 0 \;,$$

while $(I_1H_{ev})(1) = (\pi/2)\, H_{ev}(0) \neq 0$. Hence we have $\dot{M} = 0$, and

$$I_3\left(\frac{\dot{H}_{ev}}{1-y^2}\right)(x) = 0 \ , \quad x \in [0, 1] \ .$$

Therefore the proposition will be proved if we show the injectivity of the operator $I_3 : C^\infty[0, 1] \to C^\infty[0, 1]$, which will be done in the next lemma.

Lemma 2.4. The operator I_3 is injective.

Proof. Let I_4 be the operator for the functions F on $[0, 1]$ defined by

$$(I_4F)(x) = \frac{2}{\pi}\int_0^1 F(xy)\,\frac{y}{\sqrt{1-y^2}}\,dy \ .$$

We will show that $I_4 \circ (d/dx)^2 \circ I_3$ is the identity mapping on $C^\infty[0, 1]$. Observe that $(d/dx)^2 \circ I_3$ is continuous as the operator $C^2[0, 1] \to C^0[0, 1]$, and also that I_4 is continuous as the operator $C^0[0, 1] \to C^0[0, 1]$. Since polynomials exist densely in $C^2[0, 1]$, it will suffice to consider the operators for polynomials. Since the operators are linear, it is therefore enough to verify

$$I_4 \circ (d/dx)^2 \circ I_3\,(x^k) = x^k$$

for every integer $k \geq 0$. We put

$$c_k = \int_0^{\pi/2} (\sin\theta)^k \, d\theta \ .$$

Then it is easy to see that $I_3(x^k) = (c_k/(k+2))\ x^{k+2}$, $I_4(x^k) = (2/\pi)\, c_{k+1}\, x^k$, and $(k+1)\ c_k\, c_{k+1} = \pi/2$, which prove the lemma.

3. Some spectral rigidity

Let $g = g_{M,H}$ be a riemannian metric on S^2 of the form

$$M^2\{ H(\cos\theta^1)^2(d\theta^1)^2 + (\sin\theta^1)^2(d\theta^2)^2\} .$$

Let C_g' be the set of all $c \in (C_g - \{0, \pm M\})$ such that $(I_1H_{ev})'(|c|/M) \neq 0$ and that there is no c' ($\neq \pm c$) in C_g and no integer $k \geq 1$ satisfying $l(c) = k \cdot l(c')$, i.e., the fixed point set of $\xi_{l(c)}$ is just $N(c) \cup N(-c)$. We consider the following condition for g:

$(\#)_2$ C_g' is dense in $[-M, M]$.

Let $\mu_1, \mu_2, \ldots$ be the eigenvalues of the operator $\sqrt{\Delta_g + 1}$, where Δ_g is the laplacian. Set $\sigma(s) = \sum_j \delta(s-\mu_j) \in S'(\mathbf{R})$, and let $\hat{\sigma}(s)$ be its Fourier transform. Then we know that the singular support of the distribution $\hat{\sigma}$ is included in $\{ k \cdot l(c) \mid c \in C_g , k = 0, 1, 2, \ldots\}$, and it includes $\{l(c) \mid c \in C_g'\}$ (cf. Duistermaat-Guillemin [4] §4) .

Theorem 3.1. Let $g_t = g_{M_t,H_t}$ $(|t| < \varepsilon)$ be a one-parameter family of riemannian metrics on S^2 such that Δ_{g_t} are isospectral, and that each g_t satisfies the condition $(\#)_2$. Then $M_t = M_0$ and $(H_t)_{ev} = (H_0)_{ev}$. In particular, the corresponding geodesic flows are mutually symplectically isomorphic.

Proof. Let l_t be the length function with respect to the metric g_t. Let us take $c \in C_{g_0}'$ with $c > 0$ and $(I_2(H_0)_{ev})(c/M_0) = (n/m)\cdot(\pi/2)$. Since $(I_2(H_0)_{ev})'(c/M_0) \neq 0$, the mapping $(z, t) \to ((I_2(H_t)_{ev})(z), t)$ is a diffeomorphism around $(z, t) = (c/M_0, 0)$. Hence there is a unique C^∞ curve c_t for small $|t|$ such that $c_0 = c$ and $(I_2(H_t)_{ev})(c_t/M_t) = (n/m)\cdot(\pi/2)$.

We would like to show that $l_t(c_t) = l_0(c)$ if $|t|$ is sufficiently small. Assume that this is not the case. Then there is a sequence t_i $(i = 1, 2, \ldots)$ such that $|t_i| \downarrow 0$ as $i \to \infty$ and $l_{t_i}(c_{t_i}) \neq l_0(c)$. Let σ be the distribution described above defined for the metric g_0 (and hence for g_t by the assumption). Since $l_0(c)$ is isolated in the singular support of $\hat{\sigma}$, we may assume that each $l_{t_i}(c_{t_i})$ does not belong to the singular support of $\hat{\sigma}$. Therefore there is an integer $k_i \geq 1$ and $c_i' \in [0, M_{t_i}]$ with $c_i' \neq c_{t_i}$ such that $l_{t_i}(c_{t_i}) = k_i \cdot l_{t_i}(c_i')$. Since the value of $I_1(H_t)_{ev}$ $(|t| \leq |t_1|)$ is bounded both from above and from below, it follows that m_i, the denominator of $(2/\pi)\cdot(I_2(H_{t_i})_{ev})(c_i'/M_{t_i})$, and k_i $(i = 1, 2, \ldots)$ are bounded from above. Thus, taking a subsequence if necessary, we may assume that $k_i = k$, $m_i = m'$ $(i = 1, 2, \ldots)$ and c_i' converges to $c' \in [0, M_0]$ as $i \to \infty$. In case $k > 1$, we have $l_0(c) = k \cdot l_0(c')$, and this contradicts the assumption that $c \in C_g'$. In case $k = 1$, we have $c = c'$ in the same reason, and hence $m = m'$. Then we have $(I_1(H_{t_i})_{ev})(c_i'/M_{t_i}) = (I_1(H_{t_i})_{ev})(c_{t_i}/M_{t_i})$. Hence $(I_1(H_{t_i})_{ev})'(x) = 0$ for some x between c_i'/M_{t_i} and c_{t_i}/M_{t_i}, and therefore $(I_1(H_0)_{ev})'(c/M_0) = 0$, which also contradicts the assumption $c \in C_g'$. Hence we have $l_t(c_t) = l_0(c)$ for sufficiently small $|t|$.

Let $s \to \gamma_t(s)$ be the geodesic of (S^2, g_t) with unit speed which corresponds to $\eta_2 = c_t$. We take it so that $\gamma_t(s)$ is also smooth in the variable t. Then we have

$$\int_0^l g_t(\frac{\partial}{\partial s}\gamma_t(s), \frac{\partial}{\partial s}\gamma_t(s))\, ds = l \ , \quad l = l_t(c_t) = l_0(c) \ .$$

Applying $(\partial/\partial t)|_{t=0}$ to both sides,

$$\int_0^l \dot{g}_0(\frac{\partial}{\partial s}\gamma_0(s), \frac{\partial}{\partial s}\gamma_0(s))\, ds$$

$$+ 2\int_0^l g_0(\nabla_{\partial/\partial t}\frac{\partial}{\partial s}\gamma_t(s), \frac{\partial}{\partial s}\gamma_t(s))\, ds\,|_{t=0} = 0$$

where ∇ denotes the covariant derivative with respect to the metric g_0. The second term of the left hand side of this formula is

$$2\int_0^l \frac{\partial}{\partial s} g_0(\frac{\partial}{\partial t}\gamma_t(s), \frac{\partial}{\partial s}\gamma_t(s))\, ds\,|_{t=0} = 0 \ .$$

Hence we have

$$\int_0^l \dot{E}_0(\xi_s\omega)\, ds = 0$$

for every $\omega \in S^*S^2$ with $\eta_2(\omega) = c \in C_{g_0}'$ and $l = l_0(c)$. Then it follows from Proposition 2.3 that $\dot{M}_0 = 0$ and $(\dot{H}_0)_{ev} = 0$. Since this is true for every t $(|t| < \varepsilon)$, we consequently have $M_t = M_0$ and $(H_t)_{ev} = (H_0)_{ev}$.

Next we consider the case of real projective plane $\mathbf{R}P^2$. Let g be a riemannian metric on $\mathbf{R}P^2$ whose riemannian covering $\tilde{g}$ on S^2 is of the form $M^2\{H(\cos\theta^1)^2(d\theta^1)^2 + (\sin\theta^1)^2(d\theta^2)^2\}$. In this case H is an even function, i.e., $H(x) = H(-x)$. Let $C_{\tilde{g}}$ and the length function l are as before. Then the length function $L : C_{\tilde{g}} \to \mathbf{R}$ as geodesics on $\mathbf{R}P^2$ is given by

$$L(c) = \begin{cases} l(c)/2 & \text{if } m \text{ is odd} \\ l(c) & \text{if } m \text{ is even} \end{cases} .$$

where $(I_2H)(|c|/M) = (n/m)\cdot(\pi/2)$, $(n, m) = 1$, $m > 0$.

Let $\{\xi_s\}$ be the geodesic flow on $S^*\mathbf{R}P^2$ and let $\overline{N}(c)$ $(|c| < M)$ be the set of all covectors on $\mathbf{R}P^2$ with $\eta_2 = c$. As in the case of S^2, we define $C_{\tilde{g}}'$ as the set of all $c \in (C_{\tilde{g}} - \{0, \pm M\})$ such that $(I_1H)'(|c|/M) \neq 0$ and that the fixed point set of $\xi_{l(c)}$ is just $\overline{N}(c) \cup \overline{N}(-c)$. And the corresponding condition is

$(\#)_3$ $C_{\tilde{g}}'$ is dense in $[-M, M]$.

Theorem 3.2. Let g_t $(|t| < \varepsilon)$ be a one parameter family of riemannian metrics on $\mathbf{R}P^2$ such that their riemannian covering $\tilde{g}_t$ on S^2 are of the form described above. Suppose that the corresponding laplacians Δ_{g_t} are mutually isospectral and that each g_t satisfies the condition $(\#)_3$. Then $g_t = g_0$ $(|t| < \varepsilon)$.

Proof. Let E_t (resp. $\widetilde{E}_t$) and ξ_s^t (resp. $\widetilde{\xi}_s^t$) be the energy function and the geodesic flow corresponding to g_t (resp. $\widetilde{g}_t$), and let L_t and l_t be the corresponding length function. As in the proof of Theorem 3.1, we have

$$\int_0^{L_t(c)} \dot{E}_t(\xi_s^t \omega)\, ds = 0$$

for every $\omega \in S^*\mathbb{R}P^2$ such that $\eta_2(\omega) = c \in C_{g_t}'$. This implies that

$$\int_0^{l_t(c)} \dot{\widetilde{E}}_t(\widetilde{\xi}_s^t \omega)\, ds = 0 \ .$$

Since C_{g_t}' is dense in $[-M_t, M_t]$, we can apply Proposition 2.3 and obtain the theorem.

In the case of analytic metrics we can slightly refine the result.

Theorem 3.3. Let g_t $(|t| < \varepsilon)$ be a real analytic one parameter family of analytic riemannian metrics on $\mathbb{R}P^2$ such that the riemannian covering $\widetilde{g}_t$ on S^2 are of the form described before. Suppose that the corresponding laplacians Δ_{g_t} are mutually isospectral, and that C_{g_0}' is an infinite set. Then $g_t = g_0$ $(|t| < \varepsilon)$.

Proof. Let $c > 0$ be an element of C_{g_0}', and let c_t be as in the proof of Theorem 3.1. In this case c_t is an analytic function of t ($|t| < \varepsilon$) and we have $(I_2H_t)(c_t/M_t) = (I_2H_0)(c/M_0)$ and $L_t(c_t) = L_0(c)$.

As in the proof of Theorem 3.1 we have

$$\int_0^L \dot{E}_t(\xi_s^t\omega)\, ds = 0 \ ,$$

where $\eta_2(\omega) = c_t$ and $L = L_t(c_t) = L_0(c)$. Then from the proof of Proposition 2.3, we see that the equality

$$I_3\left(\frac{\dot{H}_t}{1-y^2}\right)\left(\sqrt{1-x^2}\right) = -\frac{\dot{M}_t}{M_t}(I_1H_t)(x)$$

holds for $x = c_t/M_t$. Since the number of such x is infinite, and since both sides are analytic in x over the closed interval $[0, 1]$ (note that $\dot{H}_t$ is an even function), it follows that the above equality holds for all $x \in [0, 1]$. Then the proof of Proposition 2.3 implies that $\dot{M}_t = 0$ and $\dot{H}_t = 0$.

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