MODULI OF HYPER-KÄHLERIAN

ALGEBRAIC MANIFOLDS

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Introduction

It is a well known fact that if X is a compact complex simply connected Kähler manifold with $c_1(X) = 0$, then

$$X = \Pi X_j \times \Pi y_i$$

where a) for each j $\dim H^0(X_j, \Omega^2) = 1$ and if φ_j is a non-zero holomorphic two form on X_j, and at each point $x \in X_j$ φ_j is a non-degenerate, i.e. if $\varphi_{j|U} = \Sigma(\varphi_j)_{\alpha\beta} \quad dz^\alpha \wedge dz^\beta$ then $\det((\varphi_j)_{\alpha\beta}) \in \Gamma(U, O_U^*)$. Such manifold we will call Hyper-Kählerian.

b) for each i and $0 < p < \dim_{\mathbb{C}} y_i = n$ $\dim H^0(y_i, \Omega^p) = 0$ and $\dim H^0(y_i, \Omega^n) = 1$ and $H^0(y_i, \Omega^n)$ is spanned by a holomorphic n-form which has no-zeroes and no-poles.

This fact is due to Calabi and Bogomolov. See [3]. An elegant proof based on Yau's solution of Calabi conjecture was given by M.L. Michelson. See [16].

The purpose of this article is to study the moduli space of the so called marked algebraic Hyper-Kählerian manifolds.

Definition. A tripple $(X, \gamma_1, \dots, \gamma_{b_2}; L)$ will be called a marked algebraic Hyper-Kählerian manifold if X is a Hyper-

Kählerian manifold, $\gamma_1,\ldots,\gamma_{b_2}$ is a basis of $H_2(X,\mathbb{Z})$ and L is the imaginary part as a class of cohomology of Hodge metric on X.

In this article we prove that the moduli space of marked algebraic Hyper-Kählerain manifolds exists. This is proved in § 2. More over we have an universal family of marked algebraic Hyper-Kählerain manifolds

$$X_L \xrightarrow{\pi} M_{(L;\gamma_1,\ldots,\gamma_b)}$$

The construction of the moduli space follows Burns and Rapoport. See [ ].

We have the so called period map:

$$p: M_{(L;\gamma_1,\ldots,\gamma_{b2})} \longrightarrow \mathbb{P}(H^2(X,\mathbb{Z}) \otimes \mathbb{C})$$

where

$$p(t) = (\ldots,\int_{\gamma_i} \omega(2,0),\ldots) \in \mathbb{P}(H^2(X,\mathbb{Z}) \otimes \mathbb{C})$$

where $\omega_t(2,0)$ is the unique up to a constant holomorphic two-form on $X_t = \pi^{-1}(t)$. From Bogomolov's result, that there are no obstructions to deformations and local Torelli theorem we get that the irreducible component $M_{(L;\gamma_1,\ldots,\gamma_{b_2})}$ is a non-singular manifold and $\dim_{\mathbb{C}} M_{(L;\gamma_1,\ldots,\gamma_{b_2})} = b_2 - 2$, where $b_2 = \dim H^2(X,\mathbb{C})$.

From Griffith's theory of Variations of Hodge structure we get that

$$p: M_{(L;\gamma_1,\ldots,\gamma_{b_2})} \rightarrow SO_0(2,b_2-3)/SO(2)\times SO(b_2-3) \hookrightarrow \mathbb{P}(H^2(X,\mathbb{C}))$$

is a local isomorphism.

In § 3 we prove Theorem 3. The period map

$$p: M_{(L;\gamma_1,\ldots,\gamma_{b_2})} \longrightarrow SO_0(2,b_2 - 3)/SO(2)\times SO(b_2 - 3)$$

is an embedding.

Theorem 3 is a positive answer to the so called Torelli problem, and is in some aspects a generalization of the theorem of Piatezki-Shapiro and Shafarevich about the K-3 surfaces. See [20].

In order to prove Theorem 3 we need to compactify partially the family $\chi_L \rightarrow M_{(L;\gamma_1,\ldots,\gamma_{b_2})}$ to a family $\overline{\chi}_L \rightarrow \overline{M}_{(L;\gamma_1,\ldots,\gamma_{b_2})}$ by adding singular Hyper-Kählerian algebraic manifold for which L is a very ample line bundle. Next we prove that $\overline{M}_{(L,\gamma_1,\ldots,\gamma_{b_2})}$ is a Hausdorf space and p can be extended to a proper étale map

$$p: M_{(L;\gamma_1,\ldots,\gamma_{b_2})} \rightarrow SO_0(2,b_2-3)/SO(2)\times SO(b_2-3)$$

But $SO_0(2,b_2-3)/SO(2)\times SO(b_2-3)$ is a Siegel domain of IV type so $SO_0(2,b_2-3)/SO(2)\times SO(b_2-3)$ is a simply connected manifold. From this fact and since $\overline{p}$ is a proper and étale map we get that $\overline{p}$ is a one-to-one surjective map. So we have proved both injectivity and surjectivity for algebraic Hyper-Kählerian manifolds.

So the main step of the proof of Theorem 3 is the partial compactification and this partial compactification is based on the following theorem

Theorem 1. Suppose $\pi^*: \mathcal{X}^* \to D^*$ is a family of non-singular Hyper-Kählerian manifolds such that:

a) $\pi^*: \mathcal{X}^* \to D$ has a trivial monodromy on $H_2(X_t, \mathbb{Z})$

b) $$\begin{array}{ccc} \mathcal{X}^* & \hookrightarrow & \mathbb{P}^N \times D^* \\ \downarrow & \swarrow & \\ D^* & & \end{array}$$

Then there exists a family $\pi: \mathcal{X} \to D$ such that all its fibres are non-singular Hyper-Kählerian manifolds and

$$\begin{array}{ccc} \mathcal{X}^* & \hookrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ D^* & \hookrightarrow & D \end{array}$$

This theorem is proved in § 1 and the proof is based on the existence of Calabi-Yau metric, i.e. Ricci flat metrices on Hyper-Kählerain manifolds. The existence of such metrics follows from the Yau's solution of Calabi's conjecture see [22]. More precisely the main point of the proof of Theorem 1 is based on the isometric deformations, which is an application of the existence of Ricci-flat metric. Theorem 1 gives an affirmative answer to a problem posed by Griffiths. He called this problem "the filling in problem". See [11]&[18] for counterexamples in case of surfaces of general type. Theorem 1 is a generalization of some results of Kulikov ([15]). See also [19]. Our proof is entirely different form that of Kulikov's since in my opinion the method of Kulikov works only for K3 surfaces.

The first examples of Hyper-Kählerain manifolds of dim ≥ 4 were constructed by Fujiki [12]. These examples were generalized by Beauville and Miyaoka. See [1].

It is not very difficult to prove by the method used in the proof of Theorem 1 the surjectivity of the period map for all Hyper-Kählerain manifold. This will be done in another paper.

Recently O. Debarre constructed using the so called elementary transformations introduced by Mukai in [17] two bimeromorphic but not biholomorphic non algebraic Kählerian manioflds. So the best we can hope in case of Hyper-Kählerian non-algebraic manifolds is that the Global Torelli theorem is true for bimeromorphic Hyper-Kählerian maniofolds, i.e. if X and X' have the same periods, i.e. isometric Hodge structe on $H^2(X,\mathbb{Z})$ and $H^2(X',\mathbb{Z})$, then X and X' are bimeromorphic.

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§0. SOME DEFINITIONS AND NOTATIONS

DEFINITION 0.1. Let X be a Kähler compact manifold such that:

a) $\pi_1(X) = 0$, i.e. X is a simply connected manifold

b) $\dim_{\mathbb{C}} X = 2n$

c) $\dim_{\mathbb{C}} H^\circ(X,\Omega^2) = 1$ and let $0 \neq \omega_X(2,0) \in H^\circ(X,\Omega^2)$, then $\omega_X(2,0)$ is a non-degenerate holomorphic two form on X, which means that for each point $x \in X$, there exists an open neighborhood U of x and local coordinates $z^1,\dots,z^{2n}$ such that:

$$\omega_X(2,0)\Big|_U = \Sigma\, \omega_{\alpha\beta} dz^2 \wedge dz^\beta$$

and $\det \omega_{\alpha\beta}$ is a holomorphic function in U without zeroes and poles, i.e. $\det(\omega_{\alpha\beta}) \in \Gamma(U,0_U^*)$.

If a manifold X is a Kähler one and fulfills a), b) and c) then we will called it Hyper-Kählerian manifold.

Examples of such manifolds are constructed in [12] and [1].

Some notations:

$w_X(k,0)$ will be a holomorphic k-form on X

$w_X(0,k) = \overline{w_X(k,0)}$, i.e. the autiholomorphic k-forms on X

D - will be the unit disk, i.e. $D = \{t \in \mathbb{C} \mid |t| < 1\}$

$D^* = D \setminus \{0\}$.

If $\pi : X \longrightarrow D$ is a family of manifolds, then $X_s = \pi^{-1}(s)$.

If g is a Riemannian metric on X by ∇ we will denote the Levi-Chevita connection on T^*X , where TX is the tangent bundle on X and T^*X is the cotangent bundle.
By $T^*X \otimes \mathbb{C}$, we will denote the complexified cotangent bundle.
∇ induces a covariant derivative on $\Lambda^p T^*X$ for any $p \in \mathbb{Z}$, this covariant derivative we will denote again by ∇
$\Gamma(X,\Lambda^p T^*)$ will be the global sections of the bundle $\Lambda^p T^*$

If $\varphi \in \Gamma(X,\Lambda^m(T^*X \otimes \mathbb{C}))$, then locally:

$$\varphi = \sum_{p+q=m} \varphi_{A_p,\bar{B}_q} dz^{A_p} \wedge \overline{dz^{B_q}}$$

where $A_p = (\alpha_1,\ldots,\alpha_p)$ $B_q = (\beta_1,\ldots,\beta_q)$ are multiindexes $dz^{A_p} = dz^{\alpha_1}\wedge\ldots\wedge dz^{\alpha_p}$, $dz^{\beta_y} = dz^{B_q}\wedge\ldots\wedge dz^{\beta_q}$, $z^1,\ldots,z^{2n}$ are local wordinates.

If $\varphi \in \Gamma(X,\Lambda^p T^*X)$ and $d\varphi = 0$, then by $[\varphi]$ we will denote the class of cohomology that φ defines in $H^p(X,R)$.

§1. PROOF OF THEOREM 1.

Theorem 1. Let $\pi^* : \chi^* \longrightarrow D^*$ be a family of non-singular Hyper-Kählerian manifolds such that:

a) $\pi^* : \chi^* \longrightarrow D^*$ has a trivial monodromy on $H_2(X_t,\mathbb{Z})$, i.e. if $T : H_2(X_t,\mathbb{Z}) \longrightarrow H_2(X_t,\mathbb{Z})$ is the monodromy operator, then $T = id$.

b) $$\begin{array}{ccc} \chi^* & \hookrightarrow & \mathbb{P}^N \times D^* \\ \downarrow & \swarrow & \\ D^* & & \end{array}$$

Then there exists a family $\pi : \chi \longrightarrow D$ such that:

a) $\pi^{-1}(0)$ is a non-singular Hyper-Kählerian manifold (algebraic one)

b) $$\begin{array}{ccc} \chi^* & \hookrightarrow & \chi \\ \downarrow & & \downarrow \\ D^* & \hookrightarrow & D \end{array}$$

§1.1. Marked, polarized Hyper-Kählerian manifolds and their Hodge structures of weight two

DEFINITION 1.1.1. The tripple $(X;\gamma_1,\ldots,\gamma_{b_2};L)$ we will call a marked, polarized Hyper-Kählerain manifold if X is a Hyper-Kählerian manifold; $\gamma_1,\ldots,\gamma_{p_2}$ is a basis of $H_2(X,\mathbf{Z})$ and L is the cohomology class of the imaginary part of a Kähler metric on X, i.e. $L = [g_{\alpha\bar{\beta}}]$.

Remark. Notice that two marked polarized Hyper-Kählerian manifolds $(X;\gamma_1,\ldots,\gamma_{p_2};L)$ & $(Y;\mu_1,\ldots,\mu_{p_2};L^1)$ are isomorphic iff there exists a bihomomorphic map $\varphi : X \xrightarrow{\sim} Y$ such that

a) $\varphi_*(\gamma_i) = \mu_i;\varphi_* : H_2(X,\mathbf{Z}) \longrightarrow H_2(Y,\mathbf{Z})$

b) $\varphi^*(L^1) = L;\varphi^* : H^2(Y,\mathbf{Z}) \longrightarrow H^2)X,\mathbf{Z})$

DEFINITION 1.1.2. Suppose that $\pi : \chi \longrightarrow S$ is a family of non-singular Hyper-Kählerian manifolds and suppose that the monodromy operator T induced by the action of $\pi_1(S)$ on $H_2(X_t,\mathbf{Z})$ is the identity operator. Now it is clear that if we fix a basis $\gamma_1,\ldots,\gamma_{b_2}$ of $H_2(X_t,\mathbb{Z})$, then since the monodromy operator is the trivial one we get that for every $s\in S$ $\gamma_1,\ldots,\gamma_{b_2}$ will be a basis in $H_2(X_s,\mathbb{Z})$. Now we can define the period map:

$$p : S \longrightarrow \mathbf{P}(H^2(X,\mathbb{C}))$$

in the following manner:

$$p(s) = (\ldots.,\int_{\gamma_i}\omega_S(2.0),\ldots.)$$

Now we want to see where the image of S lie in $\mathbf{P}(H^2(X,\mathbb{C}))$. So for that reason we will define a scalar product in $H^2(X,\mathbb{C})$, where X is a marked polarized Hyper-Kählerian manifold.

DEFINITION 1.1.3. The scalar product in $H^2(X,\mathbb{R})$ $<,>$ is defined as follows:

$$<w_1,w_2> = \int_X w_1\wedge w_2\wedge L^{n-2} \text{ , where } w_1,w_2\in H^2(X,\mathbb{R})$$

and L is the polarization class.

PROPOSITION 1.1.3.4. The scalar product $< , >$ has signature $(3,b_2-3)$, where $b_2 = \dim_{\mathbb{R}} H^2(X,\mathbb{R})$

Proof: Note that

$<L,L> = \int L^{2n} = \text{vol}(X) > 0$, where vol (X) is the volume of X with respect to the metric $(g_{\alpha\bar{\beta}})$, where $[g_{\alpha\bar{\beta}}] = L$. Next we will prove the following relations:

(1.1.4.) $<\omega_X(2,0),\omega_X(2,0)> = 0$

(1.1.5.) $<\omega_X(2,0),\overline{\omega_X(2,0)}> > 0$

(1.1.6) $<\omega_X(2,0),L> = 0$

Notice that (1.1.4) and (1.1.6) follow from the definition of <,> . In order to prove (1.1.5) we need the following lemma:

Lemma. If η is a primitive form of type (p,q), then

$$*\eta = \frac{(\sqrt{-1})^{p-q}}{(2n-p-q)!}(-1)^{\frac{(p+q)(p+q+1)}{2}} L^{2n-p-q}\,\bar{\eta}$$

where * is the Hodge star operator. (For the proof see [8])

From this lemma it follows that:

$$<\omega_X(2,0),\overline{\omega_X(2,0)}> = \int_X \omega_X(2,0)\wedge *\omega_X(2,0) = \|\omega_X(2,0)\|^2>0$$

So (1.1.5.) is proved.

Let $\omega_X(2,0) = \text{Re}\,\omega_X(2,0) + i\,\text{Im}\,\omega_X(2,0)$, then from (1.1.4.) and (1.1.5.) it follows that: $<\text{Re}\,\omega_X(2,0),\text{Re}\,\omega_X(2,0)> = <\text{Im}\,\omega_X(2,0),$ $\text{Im}\,\omega_X(2,0)> = \frac{1}{2}\|\omega_X(2,0)\|^2 > 0$ and $<\text{Re}\,\omega_X(2,0),\text{Im}\,\omega_X(2,0)>=0$. So we see that L, $\text{Re}\,\omega_X(2,0)$, $\text{Im}\,\omega_X(2,0)$ are three orthonormal vectors in

$H^2(X,\mathbb{R})$ such that:

$$\langle L,L\rangle > 0,\ \langle \mathrm{Re}\ \omega_X(2,0), \mathrm{Re}\omega_X(2,0)\rangle = \langle \mathrm{Im}\ \omega_X(2,0), \mathrm{Im}\ \omega_X(2,0)\rangle > 0$$

So we see that $\langle,\rangle$ has at least signature $(3,b_2-3)$. Now since $H^2(X,\mathbb{R}) = \mathbb{R}\ \mathrm{Re}\ \omega_X(2,0) + \mathbb{R}\ \mathrm{Im}\ \omega_X(2,0) + \mathbb{R}L + H^{1,1}(X,\mathbb{R})_0$ where $H^{1,1}(X,\mathbb{R})_0 = \{\omega \in H^{1,1}(X,\mathbb{R}) \mid \langle\omega,L\rangle = 0\}$, i.e. $H^{1,1}(X,\mathbb{R})_0$ are the primitive (1.1) classes in $H^2(X,\mathbb{R})$, we get that $\langle,\rangle$ has signature $(3,b_2-3)$. Indeed from the lemma used above it follows that if $\omega \in H^{1,1}(X,\mathbb{R})_0$ then $\langle\omega,\omega\rangle < 0$. It is easy to see that $\langle\omega_X(2,0),\omega\rangle = 0$ if $\omega \in H^{1,1}(X,\mathbb{R})_0$.

Q.E.D.

The scalar product (1.1.3) defines a nonsingular quadrics Q in $\mathbb{P}(H^2(X,\mathbb{C}))$ in the following way:

(1.1.7.) $$Q \overset{\mathrm{def}}{=} \{u \in \mathbb{P}(H^2(X,\mathbb{C})) \mid \langle u,u\rangle = 0\}$$

Let Ω be

(1.1.8.) $$\Omega \overset{\mathrm{def}}{=} \{u \in Q \mid \langle u,\bar{u}\rangle > 0\}$$

Ω is an open subset in Q. Let

(1.1.9.) $$\Omega(L) = \{u \in \Omega \mid \langle u,L\rangle = 0\}$$

From (1.1.4.), (1.1.5.) and (1.1.6.) and Griffith's theory [ ] we obtain that if $\chi \to S$ is a family of marked polarized Hyper-Kählerian manifolds, then $p(S) \subset \Omega(L)$, where p is the period map.

Definition 1.1.10. $\Omega(L)$ we will call the period domain of the polarized Hodge structure of weight two on Hyper-Kählerian manifolds.

Remark 1.1.11. a) If $L \in H^2(X,\mathbb{Z})$, then $\langle , \rangle$ is defined over $\mathbb{Z}$.

b) It is not difficult to see that:

$$\Omega(L) = SO_0(2,b_3-3)/U(1) \times SO(b_2-3)$$

§ 1.2. Calabi-Yau metrics and isometric deformations of Hyper-Kählerian manifolds.

Definition 1.2.1. A Kähler metric $(g_{\alpha\overline{\beta}})$ on a Hyper-Kählerian manifold will be called Calabi-Yau metric if

$$\text{Ricci } (g_{\alpha\overline{\beta}}) = \overline{\partial}\partial \log \det(g_{\alpha\overline{\beta}}) \equiv 0$$

The existence of Calabi-Yau metric follows from the deep work of Yau [22]. Notice that in the polarization class of L , there exists a unique Calabi-Yau metric $g_{\alpha\overline{\beta}}$ such that

$$[g_{\alpha\overline{\beta}}] = L$$

Let us fix the Calabi-Yau metric $g_{\alpha\overline{\beta}}$ in L. This metric induces covariant differenciation on $\Lambda^2(T^* X \otimes \mathbb{C})$. We will denote it by ∇ .

Lemma 1.2.2. $\nabla \omega_X(2,0) = \overline{\nabla} \omega_X(0,2) \equiv 0$

Proof: The following formula is proved in [14]:

Let φ be a form of type (p,q)

$$\varphi = 1/p!q! \quad \Sigma\ \varphi_{A_p,\overline{B}_q} dz^{A_p} \wedge d\overline{z}^{\overline{B}_q}$$

$A = (\alpha_1,\ldots,\alpha_p)$; $B = (\beta_1,\ldots,\beta_q)$

(1.1.2.1.) $$(\square\varphi)_{(A_p,\overline{B}_q)} = -\sum_{\alpha,\beta} g^{\overline{\beta}\alpha}\nabla_\alpha\overline{\nabla}_\beta\,\varphi_{(A_p,\overline{B}_q)} +$$

$$+\sum_{i=1}^{p}\sum_{k=1}^{q}\sum_{\tau,\sigma} R^{\tau}{}_{\alpha_i,\overline{\beta}_k}{}^{\overline{\sigma}}\,\varphi(\alpha_1,\ldots,\alpha_{i-1},\tau,\alpha_{i+1},\ldots,\alpha_p,\overline{\beta}_1,\ldots,\overline{\beta}_{k-1},\overline{\sigma},$$

$$\overline{\beta}_{k+1},\ldots,\overline{\beta}_q)$$

$$-\sum_{k=1}^{q}\sum_{\tau} R_{\overline{\beta}}{}^{\overline{\tau}}\,\varphi(A_p,\overline{\beta}_1,\ldots,\overline{\beta}_{k-1},\overline{\tau},\overline{\beta}_{k+1},\ldots,\overline{\beta}_q)$$

where $\square$ is the Laplace-Beltrami operator, $R_{\overline{\alpha}\beta,\overline{\gamma}\sigma}$ is the curvature tensor, $R_{\overline{\mu}\nu}$ is the Ricci tensor and $(g^{\overline{\beta}\alpha}) = (g_{\overline{\mu}\sigma})^{-1}$.

In our case $R_{\overline{\mu}\nu} \equiv 0$ and $\omega_X(0,2)$ is an anti-holomorphic two-form, so we obtain:

(1.2.2.2.) $$\square\omega_X(0,2) = -\sum_{\beta\ \alpha}{}^{\beta\alpha}\nabla_\alpha\nabla_\beta\omega_X(0,2) \equiv 0$$

On the other hand it is easy to see:

$$0 = \int_X \sum_{i,j}\sum_{\beta,\alpha} g^{\overline{\beta}\alpha}\nabla_\alpha\overline{\nabla}_\beta\ \omega_X(0,2))_{ij}\ \overline{(\omega_X(0,2))^{ij}}\det(g_{\alpha,\overline{\beta}})\,1/n! =$$

$$= \sum_\beta \langle\overline{\nabla}_\beta\omega_X(0,2),\overline{\nabla}_\beta\omega_X(0,2)\rangle,$$ where here $\langle\omega_1,\omega_2\rangle$ means,

that $\langle \omega_1, \omega_2 \rangle = \int_X \omega_1 \wedge {*}\, \omega_2$ ($*$ is the Hodge star operator.) So we obtain that

$$\sum_{\beta} \| \overline{\nabla}_{\beta} \omega_X(0,2) \|^2 = 0 \Rightarrow \overline{\nabla}_{\beta} \omega_X(0,2) \equiv 0 .$$

Q.E.D.

Corollary 1.2.3. If $\omega_X(2,0) = \mathrm{Re}\, \omega_X(2,0) + i\, \mathrm{Im}\omega_X(2,0)$, then

$$\nabla \mathrm{Re}\, \omega_X(2,0) \equiv \nabla \mathrm{Im}\, \omega_X(2,0) \equiv 0$$

(1.2.4) From the definition of a Kähler metric, it follows that

$$\nabla (i \sum g_{\alpha\overline{\beta}}\, dz^{\alpha} \wedge d\overline{z}^{\beta}) = \nabla (\mathrm{Im}\, g_{\alpha\overline{\beta}}) \equiv 0.$$

Re $\omega_X(2,0)$, Im $\omega_X(2,0)$ and $\mathrm{Im}(g_{\alpha\overline{\beta}})$ define a three dimensional subsapce $E_X(L)$ in $\Gamma(X, \Lambda^2 T^* X)$. Notice that $E_X(L)$ consists of two forms parallel with the respect to the connection induced by the Calabi-Yau metric $(g_{\alpha\overline{\beta}})$. Since Re $\omega_X(2,0)$, Im $\omega_X(2,0)$ are harmonic forms, we may consider $E_X(L)$ as a subspace in $H^2(X, \mathbb{R})$. We may suppose that $\langle \mathrm{Re}\omega_X(2,0), \mathrm{Re}\, \omega_X(2,0) \rangle = \langle \mathrm{Im}\, \omega_X(2,0), \mathrm{Im}\, \omega_X(2,0) \rangle = \langle \mathrm{Im}\, g_{\alpha\overline{\beta}}, \mathrm{Im}\, g_{\alpha\overline{\beta}} \rangle = 1$. On the other hand $\langle \mathrm{Re}\, \omega_X(2,0), \mathrm{Im}\, \omega_X(2,0) \rangle = \langle \mathrm{Re}\, \omega_X(2,0), \mathrm{Im}(g_{\alpha\overline{\beta}}) \rangle = \langle \mathrm{Im}\, \omega_X(2,0), \mathrm{Im}\, (g_{\alpha\overline{\beta}}) \rangle = 0$. So Re $\omega_X(2,0)$, Im $\omega_X(2,0)$ and $\mathrm{Im}(g_{\alpha\overline{\beta}})$ is an orthonormal base in $E_X(L) \subset \Gamma(X, \Lambda^2 T^*)$ with respect to the scalar product induced by $g_{\alpha\overline{\beta}}$ in $\Lambda^2 T^*$. Notice that this scalar product is the same as $\langle , \rangle$ defined by (1.1.3).

Let $\gamma = a \operatorname{Re} \omega_X(2,0) + b \operatorname{Im} \omega_X(2,0) + c \operatorname{Im}(g_{\alpha\bar{\beta}})$, where $a,b,c \in \mathbb{R}$ and $a^2 + b^2 + c^2 = 1$. Since $\gamma \in E_x(L)$, then

(*) $\nabla\gamma \equiv 0$

Locally γ can be written in the following way

$$\gamma = \sum \gamma_{\mu\nu}\, dx^\mu \wedge dx^\nu$$

If $\sum_{\tau,\nu} g_{\tau\nu} dx^\tau \otimes dx^\nu$ is the Riemannian Ricci flat metric on X defined by the Calabi-Yau metric $(g_{\alpha\bar{\beta}})$ on X, then we will define $J(\gamma)$ in the following manner

1.2.6. $J(\gamma) \in \Gamma(X, T^* \otimes T)$, where $J(\gamma)^\alpha_\beta \overset{def}{=} \sum_\tau g^{\alpha\tau}\gamma_{\tau\beta}$

Clearly $\nabla(J(\gamma)) \equiv 0$.

Lemma 1.2.7. a) $J(\gamma)$ defines a new integrable complex structure on X

b) γ is an imaginary part of a Calabi-Yau metric with respect to the new complex struture $J(\gamma)$. The Calabi-Yau metric defined by γ and $J(\gamma)$ is equivalent as a Riemannian metric to the Calabi-Yau metric $g_{\alpha\bar{\beta}}$, that we started with.

Proof: Since $\nabla J(\gamma) \equiv 0$ if we prove that in one point $x \in X$ $J(\gamma) \circ J(\gamma) = -\mathrm{id}$, then $J(\gamma)$ will define an almost complex structure globally on X. Then we will need to show that this complex struture is an integrable one.

So first we will prove that at one point $x \in X$ $J(\gamma) \circ J(\gamma) = -\mathrm{id}$. First since $\omega_X(2,0)$ is a parallel with respect to the connection induced by Calabi-Yau metric, it follows that the holonomy group of the Calabi-Yau metric is $Sp(n)$. This means that globally we can find $j \in \Gamma(X, T^* \otimes T)$ such that $\nabla j = 0$ and we have at each point x

$$T^{*1,0}_{x,X} = \mathbb{H}^n = \mathbb{C}^n + \mathbb{C}^n j$$

This splitting is global. On the other hand the Calabi-Yau metric on $T^{*1,0}_{x,X} = \mathbb{H}^n = \mathbb{R}^n + \mathbb{R}^n i + \mathbb{R}^n j + \mathbb{R}^n k$ is induced by the standart scalar product on $\mathbb{H}^n$, so from here it follows that we can find an orthonormal quaternionic base in

$$T^{*1,0}_{x,X} = \mathbb{C}^n + \mathbb{C}^n j$$

$h^1_1 = e^1_1 + e^{1+n} j,\ h^2 = e^{2+n} j, \ldots, h^n = e^n + e^{2n} j$. Then the imaginary part of Calabi-Yau metric can be written in the following way:

$$(*) \qquad \operatorname{Im}(g_{\alpha\bar{\beta}})\Big|_{T^{*1,0}_{x,X}} = i \sum_{i=1}^{2n} e^i \wedge \bar{e}^i$$

$$(**) \text{ and } \quad \omega_X(2,0)\Big|_{T^{*1,0}_{x,X}} = e^1 \wedge e^{1+n} + e^2 \wedge e^{2+n} + \ldots + e^u \wedge e^{2n} = \sum_{i=1}^{n} e^i \wedge e^{i+n}$$

Let us denote by I the original complex structure on X. Notice that $J(\operatorname{Im} g_{\alpha\bar{\beta}}) = I$. (See how we defined from γ $I(\gamma)$). Let us denote by $J = J(\operatorname{Re}\omega_X(2,0))$ and by $K = J(\operatorname{Im}\omega_X(2,0))$. From (*) and (**) we see immediately that:

$$(***) \qquad I^2 = J^2 = K^2 = -\mathrm{id} \ , \quad IJ + JI = IK + KI = JK + KJ = 0$$

So remember that $\gamma = a \operatorname{Re} \omega_X(2,0) + b\operatorname{Im} \omega_X(2,0) + cI\,m(g_{\alpha\bar\beta})$, so

$$I(\gamma) = aJ + bK + cI \quad , \quad a^2 + b^2 + c^2 = 1$$

So from (***) we get

$$I(\gamma) \circ I(\gamma) = a^2 J\circ J + b^2 K\circ K + c^2 I\circ I = -(a^2+b^2+c^2)\,\mathrm{id} = -\mathrm{id}$$

So we have proved that $I(\gamma)$ defines an almost complex structure on X. Next we must prove that the almost complex structure $J(\gamma)$ is integrable. The proof is based on the following fact:

Andreotti-Weil remark

Let ω be a n-complex valued form in a neighborhood U of a point $x \in X$, where X is a n-dimensional real manifold. Let ω satisfies:

a) $P(\omega) = 0$, where P are the Plücker relation. This means that at each point $x \in X$ $\omega_{|x\in X} = \zeta^1 \wedge \ldots \wedge \zeta^n$, $\zeta^i \in T^*_{x,X} \otimes \mathbb{C}$, so ω defines a subspace $T^{1,0}_x \subset T^*_{x,X} \otimes \mathbb{C}$ at each point $x \in V$

b) $w \wedge \bar{w} = f(x_1,\ldots,x_{2n})\, dx^1 \wedge \ldots \wedge dx^{2n}$, where $f(x_1,\ldots,x_{2n}) > 0$ in U. This means that $T^{1,0}_x + \bar{T}^{1,0} = T^*_{x,X} \otimes \mathbb{C}$ in U.

c) $dw = 0$

Notice that a) and b) means that w defines an almost complex struture in U. The condition c) means that this complex struture is integrable.

So in order to use Andreotti-Weil remark we need to construct the form w , that satisfies a),b) and c). So first we will constructa globally defined form $\omega_{J(\gamma)}(2,0)$ of type (2,0) with respect ot $J(\gamma)$ and then we will prove that:

$$\omega_{J(\gamma)}(2n,0) = \underbrace{\omega_{J(\gamma)}(2,0) \wedge \dots \wedge_{J(\gamma)}(2,0)}_{n\text{-times}}$$

fulfills the conditions of Andreotti-Weil's remark.

<u>Constructions of</u> $\omega_{J(\gamma)}\underline{(2,0)}$.

Let (α,β,γ) be an orthonormal base of $E_x(L) \subset \Gamma(X,\Lambda^2 T^*X)$ with respect to the scalar product induced by Calabi-Yau metric in $\Gamma(X,\Lambda^2 T^*X)$. We suppose that (α,β,γ) define the same orientation on $E_x(L)$ as $(\mathrm{Re}\ \omega_X(2,0), \mathrm{Im}\ \omega_X(2,0), \mathrm{Im}(g_{\alpha\bar\beta}))$.

(1.2.7.1) $$\omega_{J(\gamma)}(2,0) \stackrel{\text{def}}{=} \alpha + i\beta$$

Proposition (1.2.7.2.) $\omega_{J(\gamma)}(2,0) = \alpha + i\beta$ is a form of type (2,0) with respect to the almost complex structure on X defined by $J(\gamma)$.

Proof: Since both $\omega_{J(\gamma)}(2,0)$ and $J(\gamma)$ are paraller with respect to the connection ∇ induced by Calabi-Yau metric $(g_{\alpha\bar\beta})$, we need to check that $\omega_{J(\gamma)}(2,0)$ is a form of type (2,0) at one point x with respect to $J(\gamma)$. We will define an action of $Sp(1)$ on T^*X. Remember that the holonomy group of the Calabi-Yau metric $(g_{\alpha\overline{\beta}})$ was $Sp(n)$, so we can introduce on $T^*_{x,X}$ a quaternionie structure, i.e.

$$T^*_{x,X} = \mathbb{C} + \mathbb{C}^n j = \mathbb{H}^n \quad (\mathbb{H} \text{ is the quaternionic field})$$

$(g_{\alpha\overline{\beta}})$ is induced in $\mathbb{H}^n$ by the standart quaternionic scalar product, i.e. let $h^1 = e^1 + e^{n+1}j, \ldots, h^n = e^n + e^{2n}j$ is a quaternionic orthonormal basis in $\mathbb{H}^n$, then the restriction of Calabi-Yau's metric on $T^*_{x,X}$ is obtained from the following quaternionic product in $\mathbb{H}^n$. Let $u = \sum_{i=1} h^i u_i$ and $v = \sum_{i=1}^{n} h^i v_i$, where $v_i \in \mathbb{H}$, then

$$\langle u,v\rangle = \sum u_i \overline{v}_i .$$

Now we can identify $Sp(1) = \{A \in \mathbb{H} \mid A\overline{A} = 1\}$. Then $Sp(1)$ acts on $\mathbb{H}^n$ in the following way:

Let $A \in Sp(1)$ and let $u = \sum h^i u_i$, then

$$Au = \sum h^i u_i A , \quad \text{where } Sp(1) = \{A \in \mathbb{H} \mid \|A\|^2 = 1\}$$

Clearly $Sp(1) \subset Sp(n)$; i.e. this action of $Sp(1)$ preserves the quaternionic scalar product $\langle u,v\rangle = \Sigma u_i \overline{u}_i$.

The following remark is an easy exercise.

Remark 1. $Sp(1)$ induces an action on $\Lambda^2 T^*_{x,X}$ and $E_x(L) \subset \Gamma(X,\Lambda^2 T^*X)$ is invariant under this induced action of $Sp(1)$. More over $Sp(1)$ induces the standart $SO(3)$ action on $E_x(L)$ with respect to the Euclidean metric on $E_x(L)$ induced by the orthonormal basis $(\mathrm{Re}\,\omega_x(2,0), \mathrm{Im}\,\omega_x(2,0), \mathrm{Im}(g_{\alpha\overline{\beta}}))$. From Remark 1 it follows immediately that there exists $A \in Sp(1) \subset Sp(n)$ such that:

$$(**) \qquad A(\mathrm{Re}\ \omega_x(2,0) = \alpha,\ A(\mathrm{Im}\ \omega_x(2,0)) = \beta,\ A(\mathrm{Im}(g_{\alpha\overline{\beta}})) = \gamma.$$

So $\qquad A(\omega_x(2,0)) = \omega_{J(\gamma)}(2,0).$

On the other hand from the definition of $J(\gamma)$ we see immediately that

$$(***) \qquad J(\gamma) = AIA^t \quad (A \text{ means a matrix and } AA^t = E$$
$$\text{since } A \in Sp(1) \subset Sp(n) \subset SO(4n))$$

So from (**) and (***) we get that $\omega_{J(\gamma)}(2,0)$ is a form of type $(2,0)$ with respect to the almost complex structure $J(\gamma)$. This is so since if $\Lambda^{2,0}$ is the subspace of (2,0) vectors in $\Lambda^2(T^*_{x,X} \otimes \mathbb{C})$ with respect to I and if $J(\gamma) = AIA^t$, then $A(\Lambda^{2,0})$ is the $(2,0)$ subspace of $\Lambda^2(T^*_{x,X} \otimes \mathbb{C})$ with respect to $J(\gamma) = AIA^t$.

Q.E.D.

Now we need to show that

$$\omega_{J(\gamma)}(2n,0) = \underbrace{\omega_{J(\gamma)}(2,0) \wedge \ldots \wedge \omega_{J(\gamma)}(2,0)}_{n\text{-times}}$$

fulfills the conditions a),b) and c) of Andreotti-Weil remark. Condition a) is fulfilled since $\omega_{J(\gamma)}(2n,0)$ is a $(2n,0)$ type of form with respect to the almost complex structure operator $J(\gamma)$ acting on X and $\dim_{\mathbb{R}} X = 4n$

b) It is easy to see that $\omega_{J(\gamma)}(2n,0) \wedge \overline{\omega_{J(\gamma)}(2n,0)} = \mathrm{vol}(g_{\alpha\overline{\beta}})$ at each point $x \in X$.

c) From the definition of $\omega_{J(\gamma)}(2,0)$ it follows that

$$d\omega_{J(\gamma)}(2,0) \equiv 0$$

So $d\omega_{J(\gamma)}(2n,0) \equiv 0$.

Q.E.D.

Proof of (1.7.3.b): If $\gamma = \sum \gamma_{\mu\nu} dx^{\mu} \wedge dx^{\nu}$, then γ defines a scalar product in $T^*_{x,X}$ in the following way: Let $u = \sum u_{\alpha} dx^{\alpha}$ and $y = \sum \nu_{\beta} dx^{\beta}$, then $\langle u,v \rangle_{\gamma} = \sum u_{\alpha} \gamma_{\alpha\beta} u_{\beta}$

So if we prove that for each $u \in T^*_{x,X}$ we have:

$$\langle J(\gamma), u, u \rangle_{\gamma} > 0$$

then we will have that γ is an imaginary part of a Kähler metric on X with respect to $J(\gamma)$ since $d\gamma = 0$. So we may suppose that at $x \in X$ $(g_{\alpha\overline{\beta}}) = \delta_{\alpha\overline{\beta}}$, then:

$$J(\gamma)^{\alpha}_{\beta} = \gamma_{\alpha\beta}, \gamma_{\alpha\beta} = -\gamma_{\beta\alpha} \text{ and } \gamma_{\alpha\beta}\gamma_{\beta\mu} = -\delta_{\alpha\mu}$$

Now if $u = \sum u_{\alpha} dx^{\alpha}$, then

$$\langle J(\gamma)u,u \rangle_{\gamma} = \sum \gamma_{\mu\alpha} u_{\alpha} \gamma_{\mu\beta} = \sum u_{\alpha}(-\gamma_{\alpha\mu})\gamma_{\mu\beta} u_{\beta} =$$

$$= \sum u_{\alpha}(-\delta_{\alpha\beta}) u_{\beta} = \sum u_{\alpha}^2 > 0$$

The last calculation show that γ is an imaginary part of a Kähler metric on X with respect to the complex structure $J(\gamma)$ and this new Kähler metric is equivalent as Riemann metric to the Calabi-Yau metric we started with.

Q.E.D.

Remark 1.2.8. Lemma 1.2.8 shows that every oriented two plane $E \subset E_x(L) \subset \Gamma(X, \Lambda^2 T^*X)$ defines a new complex structure on X. So we obtain a family $\chi \to S^2$, where $S^2 = \{\gamma \in E_x(L) \mid \langle\gamma,\gamma\rangle = 1\}$. Every point $t \in S^2$ defines an oriented two plane $E_t \subset E_x(L)$ in the following manner: $E_t = \{\mathrm{Re}\ \omega_t(2,0), \mathrm{Im}\ \omega_t(2,0)\}$. Notice the conjugate complex structure on X_t defines the same $E_t \subset E_x(L)$ but with different orientation, since $\overline{\omega_t(2,0)}$ is the holomorphic two-form with respect to the conjugate complex structure and

$$\overline{\omega_t(2,0)} = \mathrm{Re}\ \omega_t(2,0) - i\mathrm{Im}\ \omega_t(2,0).$$

See also [7].

§ 1.3. Hilbert scheme of Hyper-Kählerian manifolds

Let X be a projective Hyper-Kählerian manifold embedded in $\mathbb{P}^N$. Fubbini-Schtudy metric on $\mathbb{P}^N$ in a natural way defines a class of polarization L on X. Let us denote by $\widetilde{\text{Hilb}}_{X/\mathbb{P}^N}$, the component of the Hilbert scheme that contains X. Let $\text{Hilb}_{X/\mathbb{P}^N}$ be a subscheme of $\widetilde{\text{Hilb}}_{X/\mathbb{P}^N}$ such that $\text{Hilb}_{X/\mathbb{P}^N}$ parametrizes all non-singular Hyper-Kählerian manifolds in the family $\tilde{\mathcal{X}} \to \widetilde{\text{Hilb}}_{X/\mathbb{P}^N}$. Grothendieck proved in SGA, that $\text{Hilb}_{X/\mathbb{P}^N}$ is a quasi-projective algebraic space.

Definition 1.3.1. $\Gamma_L \overset{\text{def}}{=} \{\gamma \in \text{Aut}\, H^2(X,\mathbb{Z}) \mid \langle\gamma(u),\gamma(u)\rangle = \mp \langle u,u\rangle, \gamma(L) = L\}$. Now we can define the period map $p:\text{Hilb}_{X/\mathbb{P}^N} \to \Omega(L)/\Gamma_L$. From the general Baily-Borel compactification theory, it follows that $\Omega(L)/\Gamma_L$ is a quasi-projective manifold.

Lemma 1.3.2. There exists an open Zariski set $\text{Hilb}'_{X/\mathbb{P}^N} \subset \subset \text{Hilb}_{X/\mathbb{P}^N}$ such that $p(\text{Hilb}'_{X/\mathbb{P}^N}) \overset{\text{def}}{=} W$ is an open Zariski subset in $\Omega(L)/\Gamma_L$ and every point of W corresponds to the algebraic Hyper-Kählerian manifold.

Proof: From the famous Hironaka's "resolution of singularity" theorem it follows that we can compactify $\text{Hilb}_{X/\mathbb{P}^N} \subset \widehat{\text{Hilb}}_{X/\mathbb{P}^N}$ in such a way that:

1) $\widehat{\text{Hilb}}_{X/\mathbb{P}^N}$ is a projective manifold obtained from projective manifold by successive blows up on non-singular submanifolds.

2) $\widehat{\text{Hilb}}_{X/\mathbb{P}^N} \setminus \text{Hilb}_{X/\mathbb{P}^N} = D$ is a divisor with normal crossings

Borel proved in [5] that the period map:

$$p : \mathrm{Hilb}_{X/\mathbb{P}^N} \to \Omega(L)/\Gamma_L$$

can be prolonged to a map:

$$\hat{p} : \widehat{\mathrm{Hilb}}_{X/\mathbb{P}^N} \to \overline{\Omega(L)/\Gamma_L}$$

where $\overline{\Omega(L)/\Gamma_L}$ is the Baily-Borel compactification of $\Omega(L)/\Gamma_L$. From Baily-Borel theory it follows that $\Omega(L)/\Gamma_L$ is a Zariski open set in $\overline{\Omega(L)/\Gamma_L}$, and $\overline{\Omega(L)/\Gamma_L}$ is a projective algebraic variety.

Proposition 1.3.2.1. The map $\hat{p} : \widehat{\mathrm{Hilb}}_{X/\mathbb{P}^N} \to \overline{\Omega(L)/\Gamma_L}$ is a surjective map.

Proof: First we will recall some facts about local deformation theory of Hyper-Kählerian manifolds due to Bogomolov: The Kuranishi space of any Hyper-Kählerian manifold is a non-singular manifold of dimension $h^{1,1} = \dim_{\mathbb{C}} H^1(\Omega^1)$. See [4].

For trivial reasons the local Torelli theorem is true for the period map defined in § 1.1. Beauville proved in [1] that $p(U)$ lies in the open set of the quadric Q defined by (1.1.7.) and (1.1.8.). So we may suppose that U is an open set in Q. Let U_L be defined as follows a point $t \in U_L$ iff L is a class of type (1.1) in the Hyper-Kählerian manifold X_t that corresponds to the point t. So $U_L = U \cap H_L$, where H_L is the hyperplane in $\mathbb{P}(H^2(X,\mathbb{C}))$ defined by:

$$H_L = \{u \in \mathbb{P}(H^2(X,\mathbb{C})) \mid \langle u,L\rangle = 0\} .$$

So $\dim_{\mathbb{C}} U_L = h^{1,1} - 1 = \dim \Omega(L)/\Gamma_L$. On the other hand we have a family $\begin{matrix} \mathcal{X}_L \\ \downarrow \\ U_L \end{matrix}$. Now $L_t \in H^{1,1}(X_t,\mathbb{Z})$ is a fix class so from here we obtain a line bundle L on χ_L. Now suppose that $L_{|X_t} = L_t$ is a very ample line bundle, i.e. if $\varphi_0,\ldots,\varphi_N \in H^0(X_t,L_t)$ and $(\varphi_0,\ldots,\varphi_N)$ is a basis of $H^0(X_t,L_t)$, then $\varphi_0,\ldots,\varphi_N$ define an embedding

$$X_t \hookrightarrow \mathbb{P}^N$$

By continuity argument we will get (that may be after shrinking U_L):

$$\begin{array}{ccc} \chi_L & \hookrightarrow & \mathbb{P}^N \times U_L \\ \downarrow & \swarrow & \\ U_L & & \end{array}$$

From the universal properties of $\mathrm{Hilb}_{X/\mathbb{P}^N}$ it follows that $U_L \subset \mathrm{Hilb}_{X/\mathbb{P}^N}$, so from here we get that

$$\dim_{\mathbb{C}} \hat{p}(\hat{\mathrm{Hilb}}_{X/\mathbb{P}^N}) = \dim_{\mathbb{C}} \overline{\Omega(L)}/\Gamma_L.$$

Now since $\hat{p}$ is a projective morphism and so $\hat{p}$ is proper we get that $\hat{p}(\hat{\mathrm{Hilb}}_{X/\mathbb{P}^N}) = \overline{\Omega(L)}/\Gamma_L$

Q.E.D.

Now since the map : $\hat{p} : \hat{\mathrm{Hilb}}_{X/\mathbb{P}^N} \to \overline{\Omega(L)}/L$ is a proper surjective map, then $\hat{p}(D) = \hat{p}(\hat{\mathrm{Hilb}}_{X/\mathbb{P}^N} \setminus \mathrm{Hilb}_{X/\mathbb{P}^N}) = \overline{V}$ is a proper analytic subset in $\overline{\Omega(L)}/\Gamma_L$. Let $V \neq \overline{V} \cap ((\overline{V} \cap (\overline{\Omega(L)}/\Gamma) \setminus (\Omega(L)/\Gamma))$ and let $W = \Omega(L)/\Gamma_L \setminus V$. Clearly W is a Zariski open subset in $\Omega(L)/\Gamma_L$. Now let $\mathrm{Hilb}'_{X/\mathbb{P}^N} \overset{\mathrm{def}}{=} \mathrm{Hilb}_{X/\mathbb{P}^N} \setminus (\mathrm{Hilb}_{X/\mathbb{P}^N} \cap \hat{p}^{-1}(V))$ then we will have

$$p(\mathrm{Hilb}'_{X/\mathbb{P}^N}) = W$$

So $Hilb'_{X/\mathbb{P}^N}$ is what we need.

Q.E.D.

It was proved by Bogomolov that $Hilb_{X/\mathbb{P}^N}$ is a non-singular manifold. [4]

§ 1.4. Proof of theorem 1

Since the monodromy operator:

$$T:H^2(X_t,\mathbb{Z}) \rightarrow H^2(X_t,\mathbb{Z})$$

is the identity operator, from theorem 9.5. in [13] it follows that the period map:

$$p^*:D^* \rightarrow \Omega(L) \xrightarrow{\tau} \Omega(L)/\Gamma_L$$

can be prolonged to a map

$$p:D \rightarrow \Omega(L) \xrightarrow{\tau} \Omega(L)/\Gamma_L$$

Let $p(0) = x_0 \in \Omega(L)/\Gamma_L$ $(0 \in D)$. From § 1.2. we know that there exists a proper map $\hat{p}:\hat{Hilb}_{X/\mathbb{P}^N} \rightarrow \overline{\Omega(L)/\Gamma_L}$, where $\overline{\Omega(L)/\Gamma_L}$ is the Baily-Borel compactification and $\hat{Hilb}_{X/\mathbb{P}^N}$ is obtained from the component of the Hilbert scheme $Hilb_{X/\mathbb{P}^N}$ that contains X by successive blows up along non-singular submanifolds contained in $\widetilde{Hilb}_{X/\mathbb{P}^N} \setminus Hilb_{X/\mathbb{P}^N}$. ($Hilb_{X/\mathbb{P}^N}$ is a non-singular manifold). So from Hironaka theorem it follows that we can find in this way $\hat{Hilb}_{X/\mathbb{P}^N}$ such that:

a) $\hat{Hilb}_{X/\mathbb{P}^N} \setminus Hilb_{X/\mathbb{P}^N}$ is a divisor with normal crossings

b) There exists a family $\hat{X} \rightarrow \hat{Hilb}_{X/\mathbb{P}^N}$ and it is defined in the following way, let $\hat{\pi}: \widehat{Hilb}_{X/\mathbb{P}^N} \to Hilb_{X/\mathbb{P}^N}$ be the natural map obtained by blowing down, then $\hat{\chi} \to \widehat{Hilb}_{X/\mathbb{P}^N}$ is $\hat{\pi}^* \tilde{\chi} \to \widehat{Hilb}_{X/\mathbb{P}^N}$, where $\tilde{\chi} \to \widetilde{Hilb}_{X/\mathbb{P}^N}$ is the universal family.

For each $t \in p(D^*)$ clearly $p^{-1}(t)$ consist of the orbit of x_{t_i} under the natural action of $PGL(N)$ on $Hilb_{X/\mathbb{P}^N}$, where x_{t_i} corresponds to the Hyper-Kählerian manifold $X_{t_i} \hookrightarrow \begin{matrix} \chi^* \\ \downarrow \\ D^* \end{matrix}$ and t_i are all points in D^* such that $\hat{p}(t_i) = t \in p(D^*) \subset \Omega(L)/\Gamma_L$. Suppose that

$$\widehat{Hilb}_{X/\mathbb{P}^N} \hookrightarrow \mathbb{P}^\mu$$

and D_1 is a disk in $p(D^*) \subset \Omega(L)$ such that $\overline{D_1}$ (the closure of D_1) contains x_0, i.e. $x_0 \in \overline{D_1}$. From $\widehat{Hilb}_{X/\mathbb{P}^N} \hookrightarrow \mathbb{P}^\mu \Rightarrow$ there exists a plane $\mathbb{P}^2 \subset \mathbb{P}^\mu$ such that it intersects the orbits of the Hyper-Kählerian manifolds corresponding to the points in D_1 in $Hilb_{X/\mathbb{P}^N}$ under the action of $PGL(N)$ transversally and $\mathbb{P}^2$ intersects $Hilb_{X/\mathbb{P}^N} \subset \mathbb{P}^\mu$ transversally in a point $g_0 \in \Pi^{-1}(x_0)$. It is a standart fact that such $\mathbb{P}^2$ exists. Let now $D \subset \mathbb{P}^2 \cap \widehat{Hilb}_{X/\mathbb{P}^N}$, where $g_0 \in D$ and $D \setminus \{g_0\} = D^* \subset Hilb_{X/\mathbb{P}^N}$. From the way we define D it follows that

$$p: D \hookrightarrow \Omega(L)/\Gamma_L \quad .$$

So from now on instead of the family

$$\begin{array}{ccccc} \chi & \hookrightarrow & \hat{\chi} & \hookrightarrow & \mathbb{P}^N \times \widehat{Hilb}_{X/\mathbb{P}^N} \\ \pi \downarrow & & \downarrow & \swarrow & \\ D & \hookrightarrow & \widehat{Hilb}_{X/\mathbb{P}^N} & & \end{array}$$

we will consider the family obtained from $\pi:\chi \to D$ by the pull back of the natural map $D \to D$ induced from the map : $\Omega(L) \to \Omega(L)/\Gamma_L$. We will denote this new family again by $\pi:\chi \to D$. So we will suppose from now on that the family $\pi:\chi \to D$ has the following properties:

1) $\chi^* \xrightarrow{\pi^*} D^*$ has trivial mondormy and it is a family of marked non-singular Hyper-Kählerian manifolds with a polarizatzion class L

2) $$\begin{array}{ccccc} \chi^* & \hookrightarrow & \chi & \hookrightarrow & \mathbb{P}^N \times D \\ \downarrow & & \downarrow & \swarrow & \\ D^* & \hookrightarrow & D & & \end{array}$$

3) $p:D \hookrightarrow \Omega(L)$, i.e. p is an embedding.

From now on instead of the map $p:\mathrm{Hilb}'_{X/\mathbb{P}^N} \to \Omega(L)/\Gamma_L$ we will consider the map $p:\widetilde{\mathrm{Hilb}}'_{X/\mathbb{P}^N} \to \Omega(L)$, where $\widetilde{\mathrm{Hilb}}'_{X/\mathbb{P}^N}$ is the universal convering of $\mathrm{Hilb}'_{X/\mathbb{P}^N}$. Since $\pi_1(\widetilde{\mathrm{Hilb}}'_{X/\mathbb{P}^N}) = 0$ then if we mark one fibre in the universal family

$$\chi \to \widetilde{\mathrm{Hilb}}'_{X/\mathbb{P}^N} \quad \text{(For definition of } \mathrm{Hilb}'_{X/\mathbb{P}^N} \text{ see 1.3.2.)}$$

then all the fibres will be marked and so the map

$$p:\widetilde{\mathrm{Hilb}}'_{X/\mathbb{P}^N} \to \Omega(L)$$

is correctly defined.

Let $\tau :\Omega(L) \to \Omega(L)/\Gamma_L$ be the natural map and , $V = \Omega(L)/\Gamma_L \setminus p(\mathrm{Hilb}_{X/\mathbb{P}^N})$ then $\tau^{-1}(V)$ will be an union of countable irreducible analytic closed subspaces V_i, $i = 0,1,\dots,n,\dots$ in $\Omega(L)$ (see § 1.3). Now we may suppose that $p_D(0) \in \tau^{-1}(V)$, where p_D was the map obtain from the period map: $p_{D^*}: \begin{smallmatrix} \chi^* \\ \downarrow \\ D^* \end{smallmatrix} \nearrow \Omega(L)$. Notice that if $p_D(0) \notin \tau^{-1}(V)$, then theorem 1 follows immediately.

Let $p_D(0) \in V_0$, where V_0 is one of the components of $\tau^{-1}(V)$. Let U^0 be an open polycilinder in $\Omega(L)$ such that U^0 intersects $\tau^{-1}(V)$ only on V_0 and $U^0 \supset D^*$. Let $U = U^0 \setminus (U^0 \cap V_0)$. So from the definition of U we get that

$$D^* \subset U \ , \ \dim_{\mathbb{C}} U = \dim_{\mathbb{C}} \Omega(L)$$

Lemma 1.4.1. There exists a family $\chi_U \to U$ of marked polarized Hyper-Kählerian manifolds over U (defined as above) and $\begin{matrix} \chi^* & \hookrightarrow & \chi_U \\ \downarrow & & \downarrow \\ D^* & \hookrightarrow & U \end{matrix}$. U is defined as above.

Proof: 1.4.1. Follows immediately from the existence of universal family $\chi_L \to M_L$ of marked polarized algebraic Hyper-Kählerian manifolds and the fact that $p: M_L \to \Omega(L)$ is an ètale map, i.e. p is a local isomorphism. The existence of $\chi_L \to M_L$ is proved in § 2. From these two facts and the construction $\chi \to D^*$ it follows that $\begin{matrix} \chi^* & \hookrightarrow & \chi_L \\ \downarrow & & \downarrow \\ D^* & \hookrightarrow & M_L \end{matrix}$.

Now let $\{U_i\}$ be a covering of U by polycilinders and suppose that $U_i \cap D^* \neq \emptyset$ is a disk in D^*. It is easy to see that such a covering exists (may be after we shrink) U). Now from the fact that $p: M_L \to \Omega(L)$ is a local isomorphism and $p(M_L) = \Omega(L) \setminus \tau^{-1}(V)$ (this is proved in § 2) we obtain families of marked polarized Hyper-Kählerian manifolds:

$X_i \to U_i$. Now clearly we can glue together these families along D^* and along $U_i \cap U_j$. So we will obtain the family $X_U \xrightarrow{\pi_V} U$.

Q.E.D.

Now for every point $t \in U$ we consider the isometric deformation of $X_t = \pi_U^{-1}(t)$ with respect to the Calabi-Yau metric corresponding to the polarization class L. Let us denote this family of isometric deformations by:

$$\mathbb{P}(X_t) \longrightarrow \mathbb{P}_t^1(L) \cong S^2$$

Now let us consider all isometric deformations with respect to Calabi-Yau metrics $(g_{\alpha\overline{\beta}}(t))$ corresponding in X_t for every $t \in U$ to the fixed polarization class L. So we will get a new family and we will denote it by:

$$\mathbb{P}(X_U) \longrightarrow \mathbb{P}(U)$$

Since as C^∞-family the family of isometric deformations is C^∞-diffeomorphic to $\mathbb{P}_t^1(L) \times X$ for each $t \in U$, we see that the family:

$$\mathbb{P}(X_U) \longrightarrow \mathbb{P}(U)$$

is a marked family and so the period map:

$$p: \mathbb{P}(U) \longrightarrow \Omega$$

is a well defined map. For the definition of Ω see 1.1.8.

Lemma 1.4.2. a) $p:\mathbb{P}(U) \to \Omega$ is an embedding, i.e. $\mathbb{P}(U) \hookrightarrow \Omega$.

b) $\dim_{\mathbb{C}}\mathbb{P}(U) = \dim_{\mathbb{C}}\Omega$

Proof: The proof of lemma 1.4.2. is base on the following two propositions:

1.4.3. There exists one to one map φ between the point of Ω and all two dimensional oriented vector subspaces $E \subset H^2(X,\mathbb{R})$ such that $<,>$ (defined by 1.1.3.) when restricted to E is positive, i.e. $<u,u> > 0$ for $\dot{u} \in E$. (The map φ is constructed in the following way; let $x \in \Omega \subset \mathbb{P}(H^2(X,\mathbb{Z}) \otimes \mathbb{C})$, then x defines a line $\ell_x \subset H^2(X,\mathbb{Z}) \otimes \mathbb{C}$, let ω_X be a non zero vector in ℓ_x and let $\omega_X = \mathrm{Re}\,\omega_X + i\mathrm{Im}\,\omega_X$ then $\varphi(x) = E_x$, where E_x is the two plane in $H^2(X,\mathbb{R})$ spanned $(\mathrm{Re}\,\omega_X, \mathrm{Im}\,\omega_X)$ and the orientation is defined by $\{\mathrm{Re}\,\omega_X, \mathrm{Im}\,\omega_X\}$)

Remark: From the definition of Ω it follows that if $x \in \Omega$, then

$$<x,x> = 0 \qquad <x,\bar{x}> > 0$$

So from here we get that $x \neq \bar{x}$ and so if $\omega_X \in \ell_x$, then $\mathrm{Re}\,\omega_X \neq 0$ and $\mathrm{Im}\,\omega_X \neq 0$, so φ is correctly defined. Indeed from $<\omega_X,\omega_X> = 0$ & $<\omega_X,\bar{\omega}_X> > 0$ we get that $<\mathrm{Re}\,\omega_X,\mathrm{Re}\,\omega_X> = <\mathrm{Im}\,\omega_X,\mathrm{Im}\,\omega_X> > 0$ and $<\mathrm{Re}\,\omega_X,\mathrm{Im}\,\omega_X> = 0$ and so $<,>_{|E_x}$ is strictly positive.

For the proof of 1.4.3. see [21].

Corollary 1.4.3.1. The period map $p: \begin{matrix} X \\ \downarrow \\ U \end{matrix} \nearrow \Omega$ can be defined in the following manner

$p(t) = \{\mathrm{Re}\ \omega_t(2,0), \mathrm{Im}\ \omega_t(2,0)\} = E_t = \varphi^{-1}(p(t))$

1.4.4. Proposition. Let E be a three dimensional subspace on which $<,>$ is strictly positive, then $\mathbb{P}(E \otimes \mathbb{C}) \cap Q$ will be a non-singular curve of degree two and moreover $\mathbb{P}(E \otimes \mathbb{C}) \cap Q = \mathbb{P}(E \otimes \mathbb{C}) \cap \Omega$, where $Q = \{u \in \mathbb{P}(H^2(X,\mathbb{R}) \otimes \mathbb{C}) \mid <u,u> = 0\}$ and $\Omega = \{u \in Q \mid <u,\bar{u}> > 0\}$. For the proof of 1.4.4. see [21] or [23]

Remark a) from now on $\mathbb{P}(E \otimes \mathbb{C}) \cap \Omega = \mathbb{P}(E \otimes \mathbb{C}) \cap Q \overset{def}{=} \mathbb{P}^1(E)$.
If $E = E_x(L)$ we will denote by $\mathbb{P}^1_x(L) = \mathbb{P}(E \otimes \mathbb{C}) \cap Q =$
$= \mathbb{P}(E_x(L) \otimes \mathbb{C}) \cap Q = \mathbb{P}(E \otimes \mathbb{C}) \cap \Omega$.

b) Let $\chi \to \mathbb{P}^1_t(L)$ be the isometric deformation of X_t with respect to the Calabi-Yau metric defined by L. We need to compute the image of the isometric deformation under the period map. From the definition of the isometric deformation we have the following facts:

a) $E_t(L) = \{\mathrm{Re}\ \omega_t(2,0), \mathrm{Im}\ \omega_t(2,0), \mathrm{Im}\ g_{\alpha\bar{\beta}}(t)\} \subset \Gamma(X, \Lambda^2 T^*)$

b) $E_t(L)$ is spanned by harmonic forms and so $E_t(L) \subset H^2(X,\mathbb{R})$

c) Notice that $<,>|_{E_t(L)} > 0$

We know that there is one to one map between the oriented two planes in $E_t(L)$ and the complex structures in the family of isometric deformation $\chi \to \mathbb{P}^1_t(L)$. So from here and remark 1.4.3. it follows that there is one to one map φ between the oriented two planes in $E_t(L) \subset H^2(X,\mathbb{R})$ and the points of

$\mathbb{P}(E_t(L)\otimes\mathbb{C})\cap Q = \mathbb{P}(E_t(L)\otimes\mathbb{C})\cap\Omega = \mathbb{P}^1_t(L)\subset\Omega$. The fact that $p(\mathbb{P}(U))$ lies on Ω follows from the fact that for each $t\in U$ the scalar product $<,>$ as in 1.1.3. on $E_t(L)\subset\Gamma(X_t, {}^2T^*_{X_t})$ coinside with the scalar product defined by the Calabi-Yau metric on $\Gamma(X,\Lambda^2T^*X_t)$, since

$$*\omega = \omega\wedge L^{n-2} \quad \text{and so} \quad <\omega_1,\omega_2>=\int_X\omega_1\wedge *\omega_2$$

(See [ ].)
On the other hand $*$ is defined by the Riemannian metrics coming from Calabi-Yau metric and so since all the complex structures are compatible with this fixed Riemannian metric we get that $p(\mathbb{P}(U))\subset\Omega$.

Now from local Torelli theorem and the fact that $p:U \hookrightarrow \Omega(L)$ and the definition of isometric deformation we get immediately that:

$$p:\mathbb{P}(U) \hookrightarrow \Omega .$$

Proof of 1.4.2. b): This follows immediately from local Torelli and the definition of isometric deformation.

Q.E.D.

<u>The main lemma</u> First we need some remarks.

Let $p(0) = x\in\Omega(L),(0\in D)$. Since $x\in\Omega(L)$, from 1.4.3. it follows that x corresponds to a two dimensional subspace $E_x\subset H^2(X,\mathbb{Z})$ such that $<,>|_{E_x} > 0$. From $x\in\Omega(L)\Rightarrow <E_x,L> = 0$ and since $<L,L>> 0$ it follows that the 3-dim space $E_x(L)\subset H^2(X,\mathbb{R})$ spanned by E_x & L has the following property:

$$\langle , \rangle|_{E_x(L)} > 0$$

From 1.4.4. we obtain that $(\mathbb{P}(E_x(L) \otimes \mathbb{C}) \cap \Omega = \mathbb{P}^1_x(L)$ is a complex projective non-singular curve of degree two in $\mathbb{P}(E_x(L) \otimes \mathbb{C})$.

1.4.6. Main Lemma. Let $\mathcal{X}^* \to D^*$ is the family with the properties that 1) $\overset{\mathcal{X}^*}{\downarrow} D^* \hookrightarrow \Omega(L)$ and 2) $\mathcal{X}^* \to D^*$ has a trivial monodromy, let $p: D \to \Omega(L)$ be the extended period map (this extension exists by Griffith's theorem (see [13])), let $\rho(0) = x_0 \in \Omega(L)$, then there exists a point $z_0 \in U$ such that

$$\mathbb{P}^1_{x_0}(L) \cap \mathbb{P}^1_{z_0}(L) \neq \emptyset$$

where U is defined as on p.24. (1.4.1) $\begin{array}{ccccc} \mathcal{X}^* & \hookrightarrow & \mathcal{X}_U & & \\ \downarrow & & \downarrow & & \\ D^* & \hookrightarrow & U & \hookrightarrow & \Omega(L) \end{array}$

where $\begin{array}{c}\mathcal{X}_U \\ \downarrow \\ U\end{array}$ is a family of polarized marked Hyper-Kählerian manifolds and $\dim_{\mathbb{C}} U = \dim_{\mathbb{C}} \Omega(L)$.

Proof: The proof consists of two steps:

Step 1): If $g_0 \in \mathbb{P}^1_{x_0}(L)$ and $x_0 \neq g_0 \neq \bar{x}_0$, then we will prove that there exists a plane quadric $\mathbb{P}^1_{g_0}(\omega) \subset \Omega$ such that:

a) $\mathbb{P}^1_{g_0}(\omega) \cap U \neq \emptyset$ b) $\mathbb{P}^1_{g_0}(\omega) = \overline{\mathbb{P}^1_{g_0}(\omega)}$, remember that $\Omega \subset \mathbb{P}(H^2(X,\mathbb{Z}) \otimes \mathbb{C})$, so the conjugation operator $u \to \bar{u}$ is a well defined operator.

The plane quadric $\mathbb{P}^1_{g_0}(\omega)$ is defined in the following way: Let E_{g_0} be the two dimensional plane that corresponds to g_0 given by 1.4.3. Let $\omega \in H^2(X,\mathbb{R})$ such that $\langle\omega,\omega\rangle > 0$

and $\langle\omega, E_{g_0}\rangle = 0$ and let $E_{g_0}(\omega)$ be the three dimensional subspace in $H^2(X,\mathbb{R})$ spanned by E_{g_0} and ω, then $\mathbb{P}^1_{g_0}(\omega) \overset{\text{def}}{=} \mathbb{P}(E_{g_0}(\omega) \otimes \mathbb{C}) \cap \Omega$.

Step 2. Let $\mathbb{P}^1_{g_0}(\omega) \cap U = z_0 \cup \bar{z}_0$, then we will prove that $\mathbb{P}^1_{x_0}(L) \cap \mathbb{P}^1_{z_0}(L) \neq \emptyset$, here again $\mathbb{P}^1_{z_0}(L) = \mathbb{P}(E_{z_0}(L) \otimes \mathbb{C}) \cap \Omega$.

Proof of Step 1: First we will need some definitions. Let $g_0 \in \mathbb{P}^1_{x_0}(L)$ and $g_0 \notin \Omega(L)$. From 1.4.3. follows that to g_0 there corresponds an oriented two dimensional plane $E_{g_0} \subset H^2(X,\mathbb{R})$ on which we have:

$$\langle,\rangle|E_{g_0} > 0$$

Let

$$H^{1,1}_{g_0}(\mathbb{R}) \overset{\text{def}}{=} \{u \in H^2(X,\mathbb{R}) \mid \langle u, E_{g_0}\rangle = 0\}$$

Clearly $\dim^{1,1}_{g_0}(\mathbb{R}) = b_2 - 2$ and $\langle,\rangle$ has signature $(1, b_2 - 3)$ on $H^{1,1}_{g_0}(\mathbb{R})$. Let

$$V_{g_0}(\mathbb{R}) \overset{\text{def}}{=} \{u \in H^{1,1}_{g_0}(\mathbb{R}) \mid \langle u,u\rangle > 0\}$$

Clearly since $\langle,\rangle$ on $H^{1,1}_{g_0}(\mathbb{R})$ has a signature $(1, b_2 - 3)$, then $V_{g_0}(\mathbb{R})$ will be an open cone in $H^{1,1}_{g_0}(\mathbb{R})$ and $V_{g_0}(\mathbb{R}) = V^+_{g_0} \cup V^-_{g_0}$. Let

$$E_{g_0}(\omega) \overset{\text{def}}{=} \{\text{three dim supspace in } H^2(X,\mathbb{R}) \mid \text{spanned by } E_{g_0} \text{ and } \omega \in V_{g_0}(\mathbb{R})\}.$$

From the definition of $E_{g_0}(\omega)$ it follows that

$$\langle,\rangle|_{E_{g_0}(\omega)} > 0$$

1.4.6.1. Let $K_{g_0}(\mathbb{R}) \overset{\text{def}}{=} \{$union of all $\mathbb{P}^1_{g_0}(u)$ in Ω | where $u \in V_{g_0}(\mathbb{R})\}$, then $K_{g_0}(\mathbb{R})$ is a real analytic subspace in Ω. This follows from the definition of $K_{g_0}(\mathbb{R})$ and the interpretation of Ω as Grassmannian.

1.4.6.2. Let: $V_{g_0}(\mathbb{C}) \overset{\text{def}}{=} \{u \in H^{1,1}_{g_0}(\mathbb{R}) \otimes \mathbb{C} \mid \langle u,\bar{u}\rangle > 0\}$, $(\dim_{\mathbb{C}} V_{g_0}(\mathbb{C}) = \dim_{\mathbb{C}}\Omega)$ $K_{g_0}(\mathbb{C}) = \{$the union of all $\mathbb{P}^1_{g_0}(u) = \mathbb{P}(E_{g_0})) \cap Q$ in Ω, where $E_{g_0}(u)$ is a three dimensional subspace in $H^2(X,\mathbb{R}) \otimes \mathbb{C}$, spanned by E_{g_0} and $u \in V_{g_0}(\mathbb{C})\}$. Since $\langle,\rangle|_{E_{g_0}(v)} > 0$ (if $u \in V_{g_0}(\mathbb{C})$), it follows that $\mathbb{P}(E_{g_0}(v)) \cap Q^0 = \mathbb{P}(E_{g_0}(v)) \cap \Omega$ is a projective plane curve of degree 2.

1.4.6.3. Proposition. $K_{g_0}(\mathbb{C}) \cap \Omega(L)$ contains an open set $W \subset \Omega(L)$ such that $U \subset W$ in $\Omega(L)$. (U is defined on p. 24).

Proof: H_L will be the hyperplane in $\mathbb{P}(H^2(X,\mathbb{R}) \otimes \mathbb{C})$ defined in the following manner:

$$H_L = \{u \in \mathbb{P}(H^2(X,\mathbb{R}) \otimes \mathbb{C}) \mid \langle u,L\rangle = 0\}$$

Clearly $H_L \cap \Omega = \Omega(L)$. On the other hand since $\dim_{\mathbb{C}} K_{g_0}(\mathbb{C}) = \dim_{\mathbb{C}} H^2(X,\mathbb{C}) - 2 = b_2 - 2 = \dim_{\mathbb{C}}\Omega = \dim_{\mathbb{C}} H^{1,1}(X,\mathbb{C})$ we get immediately that $\dim_{\mathbb{C}} K_{g_0}(\mathbb{C}) = \dim_{\mathbb{C}}\Omega$. If $v \in V_{g_0}(\mathbb{R})$, then

$$\overline{\mathbb{P}^1_{g_0}(v)} = \mathbb{P}^1_{g_0}(v) \quad \text{in} \quad \mathbb{P}(H^2(X,\mathbb{R}) \otimes \mathbb{C})$$

and since $H_L \cap \mathbb{P}^1_{g_0}(V) \ni z_0 \neq \emptyset$ (remember that H_L is a hyperplane in $\mathbb{P}(H^2(X,\mathbb{R}) \otimes \mathbb{C})$ and $\mathbb{P}^1_{g_0}(v)$ is a curve of degree two on the plane $\mathbb{P}^2 = \mathbb{P}(E_{g_0}(v) \otimes \mathbb{C}) \subset \mathbb{P}(H^2(X,\mathbb{R}) \otimes \mathbb{C})$), so we have that $H_L \cap \mathbb{P}^1_{g_0}(v) \neq \emptyset$).

Now let $t \in \mathbb{P}^1_{g_0}(v) \cap H_L$, from the fact that $\overline{\mathbb{P}^1_{g_0}(v)} = \mathbb{P}^1_{g_0}(v)$ $\overline{\Omega(L)} = \Omega(L)$ (since $L \in H^2(X,\mathbb{R}) \mapsto t \cup \bar{t} \in \mathbb{P}^1_{g_0}(v) \cap H_L (t \neq \bar{t})$. So we get that if $v \in V_{g_0}(\mathbb{R})$, then $\mathbb{P}^1_{g_0}(v)$ intersects $\Omega(L)$ transversally, since $\deg \mathbb{P}^1_{g_0}(v) = 2$ and $H_L \cap \mathbb{P}^1_{g_0}(v) = \Omega(L) \cap \mathbb{P}^1_{g_0}(v) = z_0 \cup \bar{z}_0$ and $z_0 \neq \bar{z}_0$. $K_{g_0}(\mathbb{R})$ intersects $\Omega(L)$ transversally and since transversality is an open condition, $\dim_{\mathbb{C}} K_g(\mathbb{C}) = \dim \Omega$ and $K_{g_0}(\mathbb{R}) \subset K_{g_0}(\mathbb{C})$ so we can find an open subset $W \subset \Omega(L)$ such that $z_0 \in \mathbb{P}^1_{g_0}(V) \cap \Omega(L) \subset U \subset W \subset K_{g_0}(\mathbb{C}) \cap \Omega(L)$.

Q.E.D

1.4.5.4. Grass $(3,b_2;\mathbb{R}) \overset{\text{def}}{=}$ {all oriented 3-dimensional subspaces $E \subset H^2(X,\mathbb{R})$ on which $\langle\ \rangle|_E > 0$} .

1.4.6.5. Grass $(3,b_2;\mathbb{C})$ = {all oriented 3-dimensional subspaces $E \subset H^2(X,\mathbb{R}) \otimes \mathbb{C}$ such that if $u \in E$, then $\langle u,\bar{u}\rangle > 0$}.

1.4.6.6. Let $\tau(E) = \bar{E}$, if $E \subset H^2(X,\mathbb{R}) \otimes \mathbb{C}$. Clearly τ acts on Grass $(3,b_2;\mathbb{C})$ and Grass $(3,b_2;\mathbb{C})^{\tau}$ = Grass $(3,b_2;\mathbb{R})$.

1.4.6.7. Let M = {all plane projective quadrics $\mathbb{P}^1_g(u)$, that are contained in Ω}. It is obvious that there exists an one-to-one map between M and Grass $(3,b_2;\mathbb{C})$.

Suppose that 1.4.6. is not true, this means that

(1.4.6.10.) $$K_{g_0}(\mathbb{R}) \cap \Omega(L) \subset V_0$$

Remember that V_0 is a proper complex analytic closed subspace in $\Omega(L)$, (For the definition of V_0 see p. 24), i.e. $\dim_{\mathbb{C}} V_0 < \dim_{\mathbb{C}} \Omega(L)$. Let

$$\mathbb{P}(V_0) \overset{\mathrm{def}}{=} \{\mathbb{P}^1_{g_0}(u) \subset K_{g_0}(\mathbb{C}) \mid \mathbb{P}^1_{g_0}(u) \cap V_0 \neq \emptyset\}$$

It is a standart fact that $\mathbb{P}(V_0)$ is a proper closed complex analytic subset in Grass $(3,b_2;\mathbb{C})$. (Use theory of elimination and $\mathbb{P}(V_0)$ = {all three dimensional subspaces E in $H^2(X,\mathbb{R}) \otimes \mathbb{C}$, such that $E \cap Z \neq \emptyset$, where Z is the cone over $V_0 \subset \mathbb{P}(H^2(X,\mathbb{R}) \otimes \mathbb{C})$} in $H^2(X,\mathbb{C})$). The same arguments show that

$$\mathbb{P}(V_{g_0}(\mathbb{R})) \overset{\mathrm{def}}{=} \{E \subset H^2(X,\mathbb{R}) \mid E \text{ is spanned by } E_{g_0}$$
and v , where $v \in V_{g_0}(\mathbb{R})$ }

is a real analytic proper subspace in $M \cong$ Grass $(3,b_2;\mathbb{C})$. Indeed $\mathbb{P}(V_{g_0}(\mathbb{R})) = \{E \in H^2(X,\mathbb{R}) \otimes \mathbb{C}) \mid E = \overline{E}$ and E contains the fixed two dimensional subspace E_{g_0} }. So from this definition it is clear that $\mathbb{P}(V_{g_0}(\mathbb{R}))$ is a proper real analytic subspace in Grass $(3,b_2;\mathbb{C})$.

Clearly that

(1.4.6.11) a) $\mathbb{P}(V_{g_0}(\mathbb{R})) = \mathbb{P}(V_{g_0}(\mathbb{C}))^{\tau}$, where

$\mathbb{P}(V_{g_0}(\mathbb{C})) = \{E \subset H^2(X,\mathbb{C}) \mid \dim_{\mathbb{C}} E = 3,$

$$\langle , \rangle |_E > 0 \quad \& \quad E \supset E_{g_0} \}$$

b) From the definition of $\mathbb{P}(V_{g_0}(\mathbb{C}))$ it follows that $\mathbb{P}(V_{g_0}(\mathbb{C}))$ is a complex analytic proper subspace in Grass $(3,b_2;\mathbb{C})$, since $\mathbb{P}(V_{g_0}(\mathbb{C})) = \{$ all three dimensional subspaces in $H^2(X,\mathbb{R}) \otimes \mathbb{C}) \mid E \supset E_{g_0}\}$.

Now we will show that (1.4.6.11) contradicts (1.4.6.10). From the definition of $\mathbb{P}(V_0)$ we get that $\mathbb{P}(V_0)$ is a proper complex analytic subspace in $\mathbb{P}(V_{g_0}(\mathbb{C}))$. From (1.4.6.10.) it follows that we have:

$$\mathbb{P}(V_{g_0}(\mathbb{C}))^{\tau} = \mathbb{P}(V_{g_0}(\mathbb{R}) \subset \mathbb{P}(V_0) \subset \mathbb{P}(V_{g_0}(\mathbb{C})$$

Since $\mathbb{P}(V_0)$ is a complex analytic subspace (proper one) in a complex analytic space $\mathbb{P}(V_{g_0}(\mathbb{C})) \subset$ Grass $(3,b_2;\mathbb{C})$ we get that localy $\mathbb{P}(V_0)$ is defined by

$$f_1(z^1,\dots,z^N) = \dots = f_K(z^1,\dots,z^N) = 0$$

where $f_1,\dots,f_N$ are complex analytic function in Grass $(3,b_2;\mathbb{C})$. From $\mathbb{P}(V_{g_0}(\mathbb{R})) \subset \mathbb{P}(V_0) \subset \mathbb{P}(V_{g_0}(\mathbb{C}))$ and since

$$\mathbb{P}(V_{g_0}(\mathbb{R})) = \mathbb{P}(V_{g_0}(\mathbb{C}))^{\tau}$$

we obtain that

$$f_1(\mathrm{Re}\, z^1,\dots,\mathrm{Re}\, z^N) = \dots = f_K(\mathrm{Re}\, z^1,\dots,\mathrm{Re}\, z^N) \equiv 0$$

on $\mathbb{P}(V_{g_0}(\mathbb{C}))$, so $f_1 = f_2 = \dots = f_N \equiv 0$ on $\mathbb{P}(V_{g_0}(\mathbb{C}))$. But this is a contradiction since $\mathbb{P}(V_0)$ is a proper subspace in $\mathbb{P}(V_{g_0}(\mathbb{C}))$, i.e. $\dim_{\mathbb{C}}\mathbb{P}(V_0) < \dim_{\mathbb{C}}\mathbb{P}(V_{g_0}(\mathbb{C}))$. So Step 1 is proved.

Q.E.D.

Proof of Step 2.

From step 1 $\Rightarrow \exists\, v \in V_{g_0}(\mathbb{R})$ such that

$$\mathbb{P}^1_{g_0}(v) \cap \Omega(L) \subset U \quad \text{(where } U \text{ is defined on p.24)}$$

Indeed we have proved, that $K_{g_0}(\mathbb{R}) \cap \Omega(L)$ is a real analytic subspace and $K_y(\mathbb{R}) \cap \Omega(L)$ not contained in V_0. Since $K_{g_0}(\mathbb{R}) \cap \Omega(L) \in g_0 \subset U^0$ open polycilinder in $\Omega(L)$) we get that $K_{g_0}(\mathbb{R}) \cap U \neq \emptyset$, where U was $U^0 \setminus V_0$ (see p. 24). So let

$$\mathbb{P}^1_{g_0}(v) \cap \Omega(L) = z_0 \cup \overline{z_0},\ z_0 \neq \overline{z_0} \quad \text{and} \quad z_0 \in U .$$

Let $E \overset{\text{def}}{=} \{$four dimensional subspace in $H^2(X,\mathbb{R})$ spanned by $E_{x_0}(L)$ and $v\}$. Since $E_{g_0} \subset E$ it follows that E_{z_0} is contained in E. From the facts that

a) $\langle , \rangle|E_{z_0}(L) > 0$, $\langle , \rangle|E_{x_0}(L) > 0$ and b) $E_{z_0}(L) \cap E_{x_0}(L) =$

$= E_{t_0} \subset E$ it follows that

i) $\dim_{\mathbb{C}} E_{t_0} = 2$ since $\dim_{\mathbb{C}} E_{x_0}(L) = E_{z_0}(L) = 3$ and $E_{x_0}(L)$ and $E_{z_0}(L)$ are contained in E; $\dim_{\mathbb{C}} E = 4$

ii) $\langle\ \rangle|E_{t_0} > 0$.

Now from 1.4.3. it follows that E_{t_0} corresponds to same point $t_0 \in \Omega$. From the fact that there is one-to-one correspondence between the points of $\mathbb{P}^1_{x_0}(L)$ and the oriented two planes in $E_{x_0}(L)$ we get that E_{t_0} corresponds to a point $t_0 \in \mathbb{P}^1_{x_0}(L)$.

Q.E.D.

1.4.7. Lemma. Let $\chi^* \to D^*$ be a family of marked polarized Hyper-Kählerian manifolds and this family fulfills the conditions 1),2) and 3) on p. 23, then

a) χ^* as C^∞ manifold is diffeomorphic to $X \times D^*$, where X is a Hyper-Kählerian manifold

b) if $\chi^* \hookrightarrow X \times D$, then $\lim_{t\to 0} \omega_t(2,0) = \omega_0(2,0)$ exists and $\omega_0(2,0)$ is a complex non-degenerate form on X.

Proof: First we see that since $\langle , \rangle|_{E_{X_0}(L)} > 0$, then $SO(3)$ acts on $E_{X_0}(L)$. From 1.4.6. it follows that there exists $z_0 \in U$ (as on p. 24) such that $E_{z_0}(L) \cap E_{x_0}(L) = E_{t_0}$, where $\dim E_{t_0} = 2$, or which is equivalent by 1.4.3., to the fact that $\mathbb{P}^1_{t_0}(L) \cap \mathbb{P}^1_{x_0} = t_0 \cup \overline{t_0}$. Now let $A \in SO(3)$ such that $A(E_{x_0}) = E_{t_0}$.

Next for each $t \in D^*$ we will define on X_t a new complex structure X_t^A in the following way:

Let $E_t(L) = \{\mathrm{Re}\, \omega_t(2,0), \mathrm{Im}\, \omega_t(2,0), \mathrm{Im}\,(g_{\alpha\bar\beta}(t))\} \subset \Gamma(X, \Lambda^2 T^*)$, where $g_{\alpha\bar\beta}(t)$ was the Calabi-Yau metric that corresponds to L. From § 1.2. we know that $\{\mathrm{Re}\, \omega_t(2,0), \mathrm{Im}\, \omega_t(2,0), \mathrm{Im}(g_{\alpha\bar\beta}(t))\}$ is an orthonormal bais of $E_t(L)$. So an action of $SO(3)$ is defined on $E_t(L)$. From § 1.2. we know that

$$AE_t \overset{\text{def}}{=} \{A \operatorname{Re} \omega_t(2,0), A\operatorname{Im} \omega_t(2,0)\} \in SO(3)$$

defines a new complex structure on X_t which we will denote by X_t^A, where

$$\omega_t^A(2,0) = A\operatorname{Re} \omega_t(2,0) + iA\operatorname{Im} \omega_t(2,0)$$

So we get a new family:

$$\mathcal{X}^{*A} \to D_A^*$$

From the definition of $\mathcal{X}^* \to D^*$ it follows that we have

$$\begin{array}{ccc} \mathcal{X}^{*A} & \hookrightarrow & \mathbb{P}(\mathcal{X}_U) \\ \downarrow & & \downarrow \\ D_A^* & \hookrightarrow & \mathbb{P}(U) \end{array}$$

(For definition of $\mathbb{P}(\mathcal{X}_U) \to \mathbb{P}(U)$ see p.)

Now since $\mathbb{P}(U) \subset \Omega$, $\mathbb{P}^1_t(L) \subset \mathbb{P}(U)$ (for each $t \in D^*$, since $D^* \subset U$) and since $\mathbb{P}^1_{z_0}(L) \cap \mathbb{P}^1_{x_0} = t_0$, where $z_0 \in U$, we get

$$(*) \qquad \lim_{t \to 0} \omega_t^A(2,0) = \omega_{t_0}(2,0)$$

Where $\omega_{t_0}(2,0)$ corresponds to some complex structure on Z_{t_0}, isometric to Calabi-Yau metric on Z_0 corresponding to L. (Here Z_0 is the marked polarized Hyper-Kählerian manifold corresponding to the point $z_0 \in U \subset \Omega(L)$). So we proved that the family

$$\mathcal{X}^{*A} \to D_A^*$$

can be embedded in a family $\mathcal{X}^A \to D_A$, where all the fibres are non-singular hyper-Kählerian manifolds. So $\mathcal{X}^A \to D_A$ as C^∞ manifold is diffeomorphic to $D \times X, X$ a Hyper-Kählerian manifold. From here we obtain, that

$$\mathcal{X}^* \cong D^* \times X$$

since $\mathcal{X}^{*A} \to D^*_A$ is the same C^∞ family as $\mathcal{X}^* \to D^*$. This follows from the definition of isometric deformation.

Q.E.D.

Proof of 1.4.7. b): From 1.4.6. it follows that there exists a point $t_0 \in \mathbb{P}^1_{x_0}(L)$ such that $t_0 = \mathbb{P}^1_{x_0}(L) \cap \mathbb{P}^1_{z_0}(L)$ where $z_0 \in U$, and so z_0 is the image under the period map of a marked Hyper-Kählerian manifold Z_0 with a polarized class L. (Remember that we have the following: a family $\begin{smallmatrix}\mathcal{X}_U\\ \downarrow\\ U\end{smallmatrix}$ is map by $p: U \hookrightarrow \Omega(L)$ $\dim_{\mathbb{C}} U = \dim_{\mathbb{C}} \Omega(L)$). Let

$$S_L = \{t \in \mathbb{P}^1_{X_0}(L) \mid E_t \text{ contains } L, E_t \text{ is the oriented two plane that corresponds to } t \text{ according to 1.4.3.}\}.$$

Clearly as C^∞ manifold $S_L \cong \{t \in \mathbb{C} \mid |t| = 1\}$. On the other hand from $\mathbb{P}^1_{x_0}(L) \cap \mathbb{P}^2_{z_0}(L) = t_0 \cup \bar{t}_0 \Rightarrow t_0 \in S_L$. From the arguments in 1.4.6. it follows that there exists an open set W_{t_0} to t_0 in S_L such that for every $t \in W_{t_0}$ $t \in \mathbb{P}^1_{x_0}(L) \cap \mathbb{P}^1_{z_t}(L)$ where $z_t \in U$. (U is defined on p. 24) Now let t_0, t_1 and t_2 are three points in $\mathbb{P}^1_{x_0}(L)$ such that: t_0, t_1 and $t_1 \in W_{t_0}$ From the way we defined W_{t_0} it follows that t_0, t_1 and t_2 are respectively in $\mathbb{P}^1_{z_0}(L)$, $\mathbb{P}^1_{z_0}(L)$ and

$\mathbb{P}^1_{z_2}(L)$, where $z_0, z_1, z_2 \in \begin{matrix} \chi_U \\ \downarrow \\ U \end{matrix}$ (See p. 24). From here and from the definition of isometric deformation it follows that t_0, t_1, t_2 corresponds to the marked Hyper-Kählerian manifold T_0, T_1, T_2 and T_0, T_1, T_2 are in the isometric families with respect to the Calabi-Yau's metrics on Z_0, Z_1, Z_2 that corresponds to L. It is clear that we can choose t_0, t_1 and t_2 in $W_{t_0} \subset S_L \subset \mathbb{P}^1_{x_0}(L)$ such that $\omega_{t_0}(2,0), \omega_{t_1}(2,0)$ and $\omega_{t_2}(2,0)$ are three linearly independent classes of cohomology in $H^2(X,\mathbb{R}) \oplus \mathbb{C}$. Since $SO(3)$ acts on $E_{X_0}(L)$ (Remember $\langle , \rangle|_{E_{X_0}(L)} > 0$) so there exist A,B and C such that $AE_{X_0} = E_{t_0}$, $BE_{X_0} = E_{t_1}$ and $CE_{X_0} = E_{t_2}$. Now we can define as in the proof of 1.4.7. a) the new families $\pi^*_A : \chi^{*A} \to D^*_A$, $\pi^*_B : \chi^{*B} \to D^*_\beta$ and $\pi^*_C : \chi^{*C} \to D^*_C$. Since we have

$$\begin{matrix} \chi^{*A} & \hookrightarrow & \mathbb{P}(\chi^*_U) \\ \downarrow & & \downarrow \\ D^*_A & \hookrightarrow & \mathbb{P}(U) \end{matrix}, \quad \begin{matrix} \chi^{*B} & \hookrightarrow & \mathbb{P}(\chi^*_U) \\ \downarrow & & \downarrow \\ D^*_B & \hookrightarrow & \mathbb{P}(U) \end{matrix} \quad \text{and} \quad \begin{matrix} \chi^{*C} & \hookrightarrow & \mathbb{P}(\chi^*_U) \\ \downarrow & & \downarrow \\ D^*_C & \hookrightarrow & \mathbb{P}(U) \end{matrix}$$

are in $\mathbb{P}(U) \subset \Omega \subset \mathbb{P}(H^2(X,\mathbb{C}))$ we get that:

$$\lim_{t\to 0}[\omega^A_t(2,0)] = [\omega_{t_0}(2,0)] \ , \ \lim_{t\to 0}[\omega^B_t(2,0)] = [\omega_{t_1}(2,0)]$$

$$\text{and} \qquad \lim_{t\to 0}[\omega^C_t(2,0)] = [\omega_{t_2}(2,0)]$$

So from here we obtain that on the level of C^∞ forms we have : $\lim_{t\to 0} \omega^A_t(2,0) = \omega_{z_0}(2,0)$, $\lim_{t\to 0} \omega^B_t(2,0) = \omega_{z_1}(2,0)$ and $\lim_{t\to 0} \omega^C_t(2,0) = \omega_{z_1}(2,0)$. Since $\omega_{t_0}(2,0) = \omega_{z_0}$, $\omega_{t_1}(2,0) = \omega_{z_1}(2,0)$ and $\omega_{t_2}(2,0) = \omega_{z_2}(2,0)$ are three linearly independent forms in $E_{t_0}(L) \oplus \mathbb{C} \subset \Gamma(X, \Lambda^2(T^*X)\mathbb{C}))$ we get that

$\omega_t^A(2,0), \omega_t^B(2,0), \omega_t^C(2,0)$ are linearly independent in each $E_t(L) \otimes \mathbb{C} \subset \Gamma(X, \Lambda^2(T^*X \otimes \mathbb{C}))$ $t \in D^*$.

So from here we have:

$$\omega_{X_t}(2,0) = a\,\omega_t^A(2,0) + b\,\omega_t^B(2,0) + c\,\omega_t^C(2,0),\quad a,b,c \in \mathbb{C}.$$

$$\lim_{t\to 0} \omega_{X_t}(2,0) = a \lim_{t\to 0} \omega_t^A(2,0) + b \lim \omega_t^B(2,0) + c \lim_{t\to 0} \omega_t^C(2,0) =$$

$$= a\,\omega_{Z_0}(2,0) + b\,\omega_{Z_1}(2,0) + c\,\omega_{Z_2}(2,0) = \omega_X(2,0) \quad \text{exists}$$

as C^∞ form and $d\,\omega_{X_0}(2,0) \equiv 0$.

Since $\det \omega_t^A(2,0) \wedge \det \overline{\omega_t^A(2,0)} = \det \omega_t(2,0) \wedge \det \omega_t(0,2)$, $\lim_{t\to 0} \omega_t^A(2,0) = \omega_{t_0}(2,0)$ and $\det \omega_{t_0}(2,0) \wedge \det \omega_{t_0}(0,2) =$ $= \det \omega_{Z_0}(2,0) \wedge \det \overline{\omega_{Z_0}(2.0)}$ (this is so because $t_0 \in \mathbb{P}^1_{Z_0}(L)$ and so T_0 is obtained from Z_0 by isometric deformation). So $\lim_{t\to 0} \det \omega_t(2,0) \wedge \det \omega_t(0,2) = \det \omega_{Z_0}(2,0) \wedge \overline{\det \omega_{Z_0}(2,0)}$ $= K \operatorname{vol}(g_{\alpha\bar\beta}(Z_0)) > 0$. This proves that $\omega_{X_0}(2,0)$ is a non-degenerate form since $\det \omega_{X_0}(2,0) = \underbrace{\omega_{X_0}(2,0) \wedge \ldots \wedge \omega_{X_0}(2,0)}_{n\text{-times}}$

Q.E.D.

In order to finish the proof of theorem 1 we need to check that $\det \omega_{X_0}(2,0)$ fulfills a),b) and c) of Andreotti-Weil remark. Clearly $d(\det \omega_{X_0}(2,0)) = 0$ and $\det \omega_{X_0}(2,0) \wedge \overline{\det \omega_X(2,0)} > 0$ so b) and c) are fulfilled.

Let P be the Plucker relation. Clearly we have

$P(\det \omega_t(2,0)) \equiv 0$ so $\lim_{t \to 0} P(\det \omega_t(2,0)) \equiv 0$.

So Theorem 1 is proved.

Q.E.D.

§ II. Construction of the moduli space of marked polarized Algebraic Hyper-Kählerian manifolds

2.1. The construction is based on the following

2.1.1. Lemma. Let g be a holomorphic automorphism of X, and suppose that $g^* = id$, where $g^*: H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{Z})$, then g induces the identity map on the Kuranishi space of X, i.e. on

$$\begin{array}{ccc} X & \hookrightarrow & \chi \\ \downarrow & & \downarrow \\ 0 & \in & U \end{array}$$

Proof: For the proof see [ ].

Q.E.D.

2.1.2. The construction of the moduli space.

Let $\begin{array}{ccc} \chi_U & \hookleftarrow & X \\ \downarrow & & \downarrow \\ U & \ni & 0 \end{array}$ be the Kuranishi family of the marked

Algebraic polarized Hyper-Kählerian manifold $(X; \gamma_1,\ldots,\gamma_{b_2}; L)$, where $\gamma_1,\ldots,\gamma_{b_2}$ is a fixed basis in $H_2(X,\mathbb{Z})$ und L is a fixed class of cohomology in $H^2(X,\mathbb{Z})$ corresponding to the to the imaginary part of a Hodge metric on X. From local Torelli theorem it follows that we may consider the following:

$$\begin{array}{ccc} X & \subset & \mathcal{X}_U \\ \downarrow & & \downarrow \\ 0 & \in & U \xhookrightarrow{p} \mathbb{P}(H^2(X,\mathbb{Z}) \otimes \mathbb{C}) \end{array}$$

where $p:U \to \mathbb{P}(H^2(X,\mathbb{Z}) \otimes \mathbb{C})$ is the period map, so from § 1.1. we may consider U as an open set in Ω this is just lemma 1.4.2.)

Let $H_L = \{x \in \mathbb{P}(H^2(X,\mathbb{Z}) \otimes \mathbb{C} | \langle x,L\rangle\}$. So from the arguments in 1.2. we get that if we restrict the Kuranishi family $\begin{array}{c}\mathcal{X}_U \\ \downarrow \\ U\end{array}$ to the family $\begin{array}{ccc}\mathcal{X}_L & \hookrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ U_L & \hookrightarrow & U\end{array}$, where $U_L = U \cap H_L$ and $U \subset \Omega \subset \mathbb{P}(H^2(X,\mathbb{Z}) \otimes \mathbb{C})$, we will get the local universal family of all Hyper-Kählerian manifold for which L corresponds to an imaginary part of a Hodge metric on X_t , for every $t \in U_L$. From 2.1.1. it follows that we can glue all families $\{\begin{array}{c}\mathcal{X}_L \\ \downarrow \\ U_L\end{array}\}$ by identifying isomorphic marked algebraic Hyper-Kählerian manifolds with fixed polarized class L. In such a way we will get an universal family $\begin{array}{c}\mathcal{X}_L \\ \downarrow \\ M_{(L;\gamma_1,\ldots,\gamma_{b_2})}\end{array}$ (since if $\varphi:X \to X$ is a biholomorphic map and $\varphi^*(L) = L$,then φ must be an isometry with respect to Yau metric and so for generic X $\varphi^* = \mathrm{id}$ on $H^2(X,\mathbb{Z})$. See [6]&[11]) of marked polarized Hyper-Kählerian manifolds with the following properties:

a) $M_{(L;\gamma_1,\ldots,\gamma_{b_2})}$ is a non-singular complex manifold of dimension $h^{1,1}-1$,

b) $$\begin{array}{ccc} \chi_L & \hookrightarrow & \mathbb{P}^N \times M(L;\gamma_1,\ldots,\gamma_{b_2}) \\ \downarrow & \swarrow & \\ M(L;\gamma_1,\ldots,\gamma_{b_2}) & & \end{array}$$ This is so since L restricted to each fibre X_t of $\begin{array}{c} X_L \\ \downarrow \\ M(L;\gamma_1,\ldots,\gamma_{b_2}) \end{array}$ corresponds to a very ample divisor D_t.

From b) it follows that $p(M_{(L;\gamma_1,\ldots,\gamma_{b_2})}$ in $\Omega(L)$ is exactly equal to $\Omega(L)\setminus\tau^{-1}(V)$, where $\tau:\Omega(L) \rightarrow \Omega(L)/\Gamma_L$ (Γ_L and V are defined in 1.2.).

$$\Gamma_L = \{\varphi \in \mathrm{Aut}\, H^2(X,\mathbb{Z}) \mid \varphi(L) = L \text{ and } \langle u,v\rangle = \langle \varphi(u),\varphi(v)\rangle\}$$

$V = p(D)$, where $D = \widehat{\mathrm{Hilb}}_{X/\mathbb{P}^N} \setminus \mathrm{Hilb}'_{X/\mathbb{P}^N}$.

§3. Torelli Problem for Hyper-Kählerian Algebraic Manifolds.

Theorem 3. Let $\pi_L:\chi_L \rightarrow M_{(L;\gamma_1,\ldots,\gamma_{b_2})}$ be the universal family of marked Hyper-Kählerian manifolds with fixed polarization class L coming from the embedding:

$$\begin{array}{ccc} \chi_L & \hookrightarrow & \mathbb{P}^N \times M_{(L;\gamma_1,\ldots,\gamma_{b_2})} \\ \downarrow & \swarrow & \\ M_{(L;\gamma_1,\ldots,\gamma_{b_2})} & & \end{array}$$

then there exists a universal partial compactification

$\overline{\pi}_L : \overline{X}_L \to \overline{M}_{(L;\gamma_1,\dots,\gamma_{b_2})}$ of the universal family of marked polarized Hyper-Kählerian manifolds definds up to an isomorphism such that:

a)
$$\begin{array}{ccccc} X_L & \hookrightarrow & \overline{X_L} & \hookrightarrow & \mathbb{P}^N \; M_{(L;\ _1,\dots,\ b_2)} \\ \downarrow & & \downarrow & & \\ M_{(L;\ _1,\dots,\ b_2)} & \hookrightarrow & \overline{M}_{(L;\ _1,\dots,\ b_2)} & & \end{array}$$

and every fibre of $\overline{\pi}:\overline{X}_L \to \overline{M}_{(L;\gamma_1,\dots,\gamma_{b_2})}$ is birationally isomorphic to a non-singular Hyper-Kählerian manifold.

b) the period map $p:M_{(L;\gamma_1,\dots,\gamma_{b_2})} \to \Omega(L)$ can be prolonged to a holomorphic isomorphism:

$$\overline{p}:\overline{M}_{(L;\gamma_1,\dots,\gamma_{b_2})} \xrightarrow{\sim} \Omega(L)$$

<u>Remark</u> $\overline{p}:\overline{M}_{(L;\gamma_1,\dots,\gamma_{b_2})}$ is defined up to a component.

Proof: First we will construct the partial compactification of

$$\pi_L : X_L \to M_{(L;\gamma_1,\dots,\gamma_{b_2})} \qquad \begin{array}{ccc} \overline{X_L} & \hookrightarrow & \mathbb{P}^N \times \overline{M}_{(L;\gamma_1,\dots,\gamma_{b_2})} \\ \downarrow & \swarrow & \\ \overline{M}_{(L;\gamma_1,\dots,\gamma_{b_2})} & & \end{array}$$

In the proof of theorem 1 we used the fact that

$$\Omega(L) \setminus p(M_{(L;\gamma_1,\dots,\gamma_{b_2})}) = V = V_0 \cup V_1 \cup \dots \cup V_K \quad \dots$$

is a countable union of analytic subsets. Now let D be a disc in $\Omega(L)$ and $D^* = D^*\setminus\{0\}$, i.e. D intersects V in one point. From the arguments on p. 22 and 23 it follows that over D^* we have a family of marked algebraic Hyper-Kählerian manifolds with polirization class L:

$$\chi^* \to D^* ,$$

and this family has the properties stated on p. 23. Now we can apply Theorem 1 to $X^* \to D^*$ and we will get a family $\pi:\chi \to D$, where all the fibres are non-singular Hyper-Kählerian manifolds. So from here it follows the existence of a family of non-singular Hyper-Kählerian marked manifolds $\tilde{X}_L \to \overline{M}_{(L;\gamma_1,\dots,\gamma_{b_2})}$ such that

a)
$$\begin{array}{ccc} X_L & \hookrightarrow & \tilde{X}_L \\ \downarrow & & \downarrow \\ M_{(L;\gamma_1,\dots,\gamma_{b_2})} & \hookrightarrow & \overline{M}_{(L;\gamma_1,\dots,\gamma_{b_2})} \end{array}$$

b) the period map

$$p:\overline{M}_{(L;\gamma_1,\dots,\gamma_{b_2})} \to \Omega(L)$$

is a surjective map and étale map.

3.1.1. Lemma. There exists meromorphic map

$$\begin{array}{lll} \tilde{\varphi}:\tilde{X}_L & \rightarrow & \mathbb{P}^N \times \overline{M}_{(L;\gamma_1,\dots,\gamma_{b_2})} \\ \downarrow & \swarrow & \\ \overline{M}_{(L;\gamma_1,\dots,\gamma_{b_2})} & & \end{array}$$

such that:

a) the restriction of $\tilde{\varphi}$ on $\chi_L \rightarrow M_{(L;\gamma_1,\dots,\gamma_{b_2})}$ gives the embedding

$$\begin{array}{lll} \tilde{\varphi}:\chi_L & \hookrightarrow & \mathbb{P}^N \times M_{(L;\gamma_1,\dots,\gamma_{b_2})} \\ \downarrow & \swarrow & \\ M_{(L;\gamma_1,\dots,\gamma_{b_2})} & & \end{array}$$

b) for each $t \in \overline{M}_{(L;\ _1,\dots,\ _{b_2})} \setminus M_{(L;\ _1,\dots,\ _{b_2})}$ the map $\tilde{\varphi}$ defines a holomorphic map

$$\varphi_t : X_t \rightarrow \overline{X}_t$$

where $\overline{X}_t$ is the closure of the fibre X_t in $\mathbb{P}^N$ under the map $\tilde{\varphi}_t$ and $\tilde{\varphi}_t$ is a birational map.

<u>Proof</u>: We know that:

a) $\overline{M}_{(L;\gamma_1,\dots,\gamma_{b_2})} \setminus M_{(L;\gamma_1,\dots,\gamma_{b_2})}$ is a countable union of closed analytic subsets

b) $$\begin{array}{lll} \chi_L & \hookrightarrow & \mathbb{P}^N \times M_{(L;\gamma_1,\dots,\gamma_{b_2})} \\ \downarrow & \swarrow & \\ M_{(L;\gamma_1,\dots,\gamma_{b_2})} & & \end{array}$$

So from a) & b) it follows that it is enough to prove the lemma for a family $\pi: \mathcal{X} \to D$, where $D \hookrightarrow \overline{M}_{(L;\gamma_1,\ldots,\gamma_{b_2})}$ and $D^* \hookrightarrow M_{(L;\gamma_1,\ldots,\gamma_{b_2})}$. Since $D^* \hookrightarrow D \hookrightarrow \Omega(L)$, from the arguments on p.24 it follows that the family $\pi^*: \mathcal{X}^* \to D^*$ has the following property:

(*) there exists an embedding
$$\begin{array}{ccccc} \mathcal{X}^* & \hookrightarrow & \mathcal{X}_1 & \hookrightarrow & \mathbb{P}^N \times D \\ \downarrow & & \downarrow & \swarrow & \\ D^* & \hookrightarrow & D & & \end{array}$$

Now let $\{\varphi_0(t),\ldots,\varphi_N(t)\}$ $(t \in D^*)$ are the section of the line bundle L^*, that gives the embedding
$$\begin{array}{ccc} \mathcal{X}^* & \hookrightarrow & \mathbb{P}^N \times D^*. \\ \downarrow & \swarrow & \\ D^* & & \end{array}$$

From the fact that we have

$$\begin{array}{ccccc} \mathcal{X}^* & \hookrightarrow & \mathcal{X}_1 & \hookrightarrow & \mathbb{P}^N \times D \\ \downarrow & & \downarrow & \swarrow & \\ D^* & \hookrightarrow & D & & \end{array}$$

it follows that we can continue $\{\varphi_0(t),\ldots,\varphi_N(t)\}$ to sections in $\pi^{-1}(0) = X_0$, where X_0 is the zero fibre of the family of the non-singular Hyper-Kählerian manifolds $\begin{array}{c}\mathcal{X}\\ \downarrow\\ D\end{array}$. So from here we get that there exists a birational map between

$$\begin{array}{c} \mathcal{X} \\ \downarrow \\ D \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{X}_1 & \hookrightarrow & \mathbb{P}^N \times D \\ \pi_1 \downarrow & \swarrow & \\ D & & \end{array}$$

since if $(\varphi_0(t),\dots,\varphi_N(t))_{t\in D}$ have fixed point then these fixed point are in X_0 so the set of fixed points of the linear system $(\varphi_0(t),\dots,\varphi_N(t))$ can be at most a divisor in X_0, and so has codimension ≥ 2 in X. So from here we obtain that $\begin{matrix} \chi^1 & \hookrightarrow & \mathbb{P}^N\times D \\ \downarrow & & \downarrow \\ D_1 & \longrightarrow & D \end{matrix}$ is a birational map. Even more we will prove that there exists a holomorphic map

$$\varphi_0 : X_0 \to X_0^1 \hookrightarrow \mathbb{P}^N \qquad X_0^1 = \pi_1^{-1}(0)$$

which is induced by the birational isomorphism between X_0 and X_0^1

Proof: Let H be the closure of the very ample divisor H^* that difines L^* in χ. Let $L = \mathcal{O}(H)$ and let $L_0 = L|_{X_0}$. we will prove that L_0 gives us

$$\varphi_0 : X_0 \to X_0^1 \hookrightarrow \mathbb{P}^N$$

Fist it is easy to see that on $X_1 \setminus \operatorname{Sing}(X_0^1)$ there exists a Kähler metric; this is the restriction of Fublini-Study metric $+\, dt \otimes d\bar{t}$ on $X_1 \setminus A$, $A = \operatorname{Sing}(X_0^1)$. For each $t \in D^*$ the restriction of the imaginary part of this Kähler metric gives the Chern class of $L|_{X_t}$. Notice that codim $A \geq 2$ in X^1. Let $\{W_p\}$ be a covering of X such that

$$i((\Sigma\, g^e_{i\bar{j}}(t)\, dz^i \wedge d\bar{z}^j + dt \wedge d\bar{t}))|_{(W_e \setminus (W_e \cap A))} = i\partial\bar{\partial} u_e$$

where u_e is a plurihsubharmonic function. From a theorem about the continuation of plurisubharmonic functions proved in [9] it follows that we can continue u_e in W_e and we will have

$$i\,\partial\bar{\partial}\,u_e \geq 0$$

From this fact we get:

For every effective analytic cycle $C \subset X_0$ dim $C = k$ we have

(*) $$\int_C c_1(L_0) \wedge \ldots \wedge c_1(L_0) \geq 0$$

(*) is equivalent to the following inequality

(**) $$\langle H_0^{2n-k}, C\rangle \geq 0$$

where $H_0 = H|_{X_0}$. (*,*) means that the linear system $|H_0|$ gives a holomorphic map:

$$\varphi_0 : X_0 \to \mathbb{P}^N$$

This is Kleinman-Moishezon criterion [14]. So this proves lemma 3.1.1.

Q.E.D.

Now we can define the family $\pi : \overline{X}_L \to \overline{M}_{(L;\gamma_1,\ldots,\gamma_{b_2})}$ in the following way: $\overline{X}_L \to \overline{M}_{(L;\gamma_1,\ldots,\gamma_{b_2})}$ is the closure of the fibres of the image of the family $^2\ \tilde{X}_L \to \overline{M}_{(L;\gamma_1,\ldots,\gamma_{b_2})}$

in $\mathbb{P}^N \times \overline{M}_{(L;\gamma_1,\ldots,\gamma_{b_2})}$

Lemma 3.1.2. Suppose that:

a) $\pi_1^*:\chi_1^* \to D^*$ and $\pi_2^*:\chi_2^* \to D^*$ are two isomorphic families of marked polarized Hyper-Kählerian algebraic manifolds with trivial monodromy.

b) Let $\pi_1:\chi_1 \to D_1$ and $\pi_2:\chi_2 \to D_2$ are obtained from $\pi_1^*:\chi_1^* \to D_1^*$ and $\pi_2^*:\chi_2^* \to D_2^*$ in the following way:

$$\begin{array}{ccccccc} \chi_i^* & \hookrightarrow & \chi_1 & \hookrightarrow & \overline{\chi_L} & \hookrightarrow & \mathbb{P}^N \times \overline{M}_{(L;\gamma_1,\ldots,\gamma_{b_2})} \\ \downarrow & & \downarrow & & \downarrow & \swarrow & \\ D_i^* & \hookrightarrow & D_i & \hookrightarrow & \overline{M}_{(L;\gamma_1,\ldots,\gamma_{b_2})} & & \end{array}$$

where $\overline{\chi_L} \to \overline{M}_{(L;\gamma_1,\ldots,\gamma_{b_2})}$ is defined on p. 49.

Then the two families $\chi_1 \to D_1$ and $\chi_2 \to D_2$ are biholomorphically isomorphic

<u>Proof:</u> Let $\varphi : \begin{array}{ccc} \chi_1^* & \to & \chi_2^* \\ \downarrow & & \downarrow \\ D^* & = & D^* \end{array}$ be a holomorphic isomorphism between those two marked polarized families of algebraic Hyper-Kählerian manifolds. From the definition of φ it follows that:

1) $\varphi^*(L_2) = L_1$, where L_i is the polarization class on $\pi_i^*:\chi_i^* \to D^*$

2) $\varphi^*:H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{Z})$ is the identity map.

Since $\begin{array}{ccc} X_t^* & \hookrightarrow & \mathbb{P}^N \times D^* \\ \downarrow & \swarrow & \\ D^* & & \end{array}$ and L_i is the restriction of the Fubini-Study metric on $\mathbb{P}^N \times D^*$ and from 1) and 2) we get that $\varphi : \begin{array}{ccc} X_1^* & \to & X_2^* \\ \downarrow & & \downarrow \\ D^* & = & D^* \end{array}$ is induced by a biholomorphic map

$\Psi^* : \begin{array}{ccc} \mathbb{P}^N \times D^* & \to & \mathbb{P}^N \times D^* \\ \downarrow & & \downarrow \\ D^* & = & D^* \end{array}$. Indeed φ is given by the sections of the line bundle $\mathcal{O}(H) \otimes D^*|_{X_1^*}$, where H is the hyperplane section. Let Γ_{Ψ^*} be the graph of the map Ψ^* in $(\mathbb{P}^N \times D^*) \times_{D^*} (\mathbb{P}^N \times D^*) = \mathbb{P}^N \times \mathbb{P}^N \times D^*$. Since Ψ^* induces the identity map $H_*(\mathbb{P}^N, \mathbb{Z})$, Bishop criterium and the fact that $(\mathbb{P}^N \times D) \times_D (\mathbb{P}^N \times D) = \mathbb{P}^N \times \mathbb{P}^N \times D$ is a Kähler manifold we get that Γ_{Ψ^*} can be prolonged to Γ_Ψ in $\mathbb{P}^N \times \mathbb{P}^N \times D$. The arguments are exactly the same as Proposition 3.1. of [23]. Since Ψ^* is given by $\mathcal{O}_{\mathbb{P}^N}(1) \otimes_{\mathcal{O}_{D^*}} \mathcal{O}_{D^*}$ and Γ_{Ψ^*} can be prolonged to Γ_Ψ in $\mathbb{P}^N \times \mathbb{P}^N \times D^N$ we get that the sections of $\Gamma(\mathbb{P}^N \times D^*, \mathcal{O}_{\mathbb{P}^N}(1) \otimes_{\mathcal{O}_{D^*}} \mathcal{O}_{D^*})$ can be prolonged to meromorphic sections of $\Gamma(\mathbb{P}^N\ D, \mathcal{O}_{\mathbb{P}^N}(1) \otimes_{\mathcal{O}_D} \mathcal{O}_D)$ can be prolonged to meromorphic section of $\Gamma(\mathbb{P}^N \times D, \mathcal{O}_{\mathbb{P}^N}(1) \otimes_{\mathcal{O}_D} \mathcal{O}_D)$ so this sections can have poles along $\pi^{-1}(0) = \mathbb{P}^N$, where

$$\pi : \mathbb{P}^N \times D \to D .$$

From here we get that if we multiply each section $\varphi_i(t)$ by t^{n_i} then we will get a section $t^{n_i}\varphi_i \in \Gamma(\mathbb{P}^N \times D, \mathcal{O}_{\mathbb{P}^N} \times_{\mathcal{O}_D} \mathcal{O}_D)$ abd even more $t^{n_i}\varphi_i \neq 0$ on $\pi^{-1}(0)$.

So from here directly lemma 3.1.2. follows, because we can prolong Ψ^* to an isomorphism

$$\Psi : \begin{array}{ccc} \mathbb{P}^N \times D & \to & \mathbb{P}^N \times D \\ \downarrow & & \downarrow \\ D & = & D \end{array}$$

The end of the proof of Theorem 3.

From 3.1.2. it follows that $\overline{\pi}:\overline{X}_L \to \overline{M}_{(L;\gamma_1,\ldots,\gamma_{b_2})}$ is a unique family up to an isomorphism and so it induces a Hausdorf topology on $\overline{M}_{(L;\gamma_1,\ldots,\gamma_{b_2})}$. We know that the period map

$$\overline{p}:\overline{M}_{(L;\gamma_1,\ldots,\gamma_{b_2})} \longrightarrow \Omega(L)$$

is a surjective map. From local Torelli theorem and the way we constructed $\overline{X}_L \to \overline{M}_{(L;\gamma_1,\ldots,\gamma_{b_2})}$ we get that $\overline{p}$ is an étale map. Now if we prove that $\overline{p}$ is a proper map, since

$$\Omega(L) \cong SO_0(2,b_2-3)/SO(2)\times SO(b_2-3)$$

and so simply connected Theorem will follow. So we need to check that $\overline{p}$ is a proper map. So we need to use the valuative criterium of Grothendieck of a properness, [SGA], so we need to prove that if

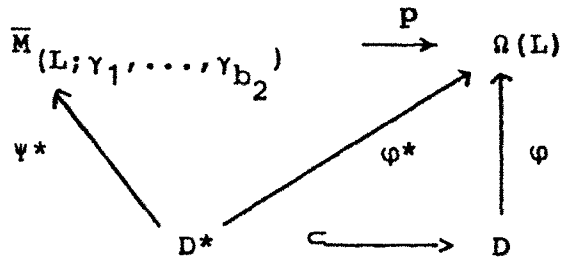
$$x \in \Omega(L)$$

and if $\varphi:D \to \Omega(L)$ is a holomorphic map from any disc such that:

a) $\varphi(0) = x$

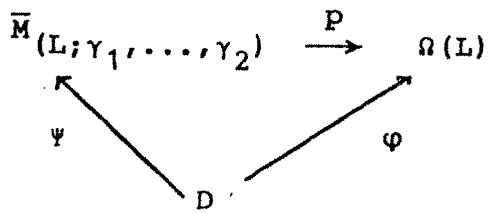
b) the following diagramm is commutative

(*)



then $\Psi\cdot$ can be prolonged to a map $\Psi : D \to \overline{M}_{(L;\gamma_1,\ldots,\gamma_{b_2})}$ such that the diagram is commutative:

(**)



If we prove this (which is exactly Grothendieck's criterion of properness) the map $p:\overline{M}_{(L;\gamma_1,\ldots,\ \gamma_{b_2})} \longrightarrow \Omega(L)$ will be an êtale and proper. On the other hand we know that

$$\Omega(L) \cong SO(2,b_2-3)/SO(2)\quad SO(b_2-3)$$

is Siegel domain of IV type and so $\Omega(L)$ is a simply connected manifold. From this fact it follows that

$$\overline{p}:\overline{M}_{(L;\gamma_1,\ldots,\gamma_{b_2})} \longrightarrow \Omega(L)$$

is a biholomorphic map. This will prove theorem 3. So we need to prove the valuative criterium of Grothendieck, i.e. we showed that the map $\varphi^*:D^* \to \overline{M}_{(L;\gamma_1,\ldots,\gamma_{b_2})}$ of the commutative diagram can be prolonged to a map

$\Psi : D \to \overline{M}_{(L;\gamma_1,\dots,\gamma_{b_2})}$ so that the diagramm $(**)$ must be commutative one. See [ ]. We must consider two cases:

a) Let $\Psi^* : D^* \to \begin{array}{c} X_L \\ \downarrow \\ M_{(L;\gamma_1,\dots,\gamma_{b_2})} \end{array}$. In this case we have a family $X^* \to D^*$ of marked polarized Hyper-Kählerian manifolds. The condition that the map $p : D^* \to \Omega(L)$ can be continued to the map $p : D \to \Omega(L)$ means that the monodromy of the family $X^* \to D^*$ is trivial. This follows from theorem 9.5. proved by Griffiths in [13]. Then Theorem 1 says that we can embeded $\begin{array}{ccc} X^* & \hookrightarrow & X \\ \downarrow & & \downarrow \pi \\ D^* & \hookrightarrow & D \end{array}$ in a family $\pi : X \to D$, where all fibres are non-singular Hyper-Kählerian manifolds. Now lemma 3.1.1. shows that Grothendieck's criterium is fulfilled.

b) Let $\Psi^*(\Delta^*) \subset \overline{M}_{(L;\gamma_1,\dots,\gamma_{b_2})} \setminus M_{(L;\gamma_1,\dots,\gamma_{b_2})}$. Since $\overline{M}_{(L;\gamma_1,\dots,\gamma_{b_2})} \setminus M_{(L;\gamma_1,\dots,\gamma_{b_2})}$ is a union of closed complex analytic subsets and the period map $p : \begin{array}{c} X^* \\ \downarrow \\ D \end{array} \to \Omega(L)$ can be continued to a map $p : D \to \Omega(L)$ it follows that we can find a disc D_1 such that

1) $D_1^* \subset M_{(L;\gamma_1,\dots,\gamma_{b_2})}$

2) $p : D_1^* \to \Omega(L)$ can be continued to a map $p : D_1 \to \Omega(L)$ and $p(0_1) = p(0)$, where $0_1 \in D_1$ and $0 \in D$.

3) D and D_1 are contained in U, where $U = p^{-1}(U)$, U is a policynder $\dim_{\mathbb{C}} U = \dim_{\mathbb{C}} \Omega(L)$ such that $p(D) \in U$. Then everything follows from a.

Theorem 3 is proved.

Q.E.D.

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