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THE FROBENIUS MORPHISM ON HOMOGENEOUS SPACES, I

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Abstract. In this paper, we continue the study of cohomology groups of the first term of p–filtration on the sheaf of differential operators with divided powers in positive characteristic. Higher cohomology vanishing for these should lead to localization theorems for the category of $\mathcal{D}$–modules with vanishing p–curvature. We prove such a cohomology vanishing for the Hilbert scheme $\mathrm{Hilb}^2(\mathbb{P}^2)$ in odd characteristic, and for the flag variety of exceptional group of type $\mathbf{G}_2$ for $p = 7$. This also allows to construct a tilting bundle on the flag variety $\mathbf{G}_2/\mathbf{B}$. We emphasize the relation to the problem of cohomology vanishing of line bundles on flag varieties. Finally, we discuss further applications of the key proposition from [22], and give a geometric proof of it, based on deformation to the normal cone.

1. Introduction

This paper continues the study of the cohomology groups of sheaves of differential operators on flag varieties in positive characteristic that we begun in a series of papers [23] – [25].

Recall briefly the setting. Fix an algebraically closed field k of characteristic $p > 0$. Consider a simply connected semisimple algebraic group $\mathbf{G}$ over k, and let $\mathbf{G}_n$ be the n–the Frobenius kernel. Our main interest is in understanding the vector bundle $\mathsf{F}^n_*\mathcal{O}_{\mathbf{G}/\mathbf{B}}$ on the flag variety $\mathbf{G}/\mathbf{B}$, where $\mathsf{F}^n : \mathbf{G}/\mathbf{B} \to \mathbf{G}/\mathbf{B}$ is the (n–iteration of) Frobenius morphism. The bundle in question is homogeneous, and one would like to understand its $\mathbf{G}_n\mathbf{B}$–structure (this question goes back to H.H. Andersen's paper [3] from 1979). However, such a description seems to be missing so far in general.

A related question is about the $\mathbf{G}_n\mathbf{B}$–structure of associated endomorphism bundle $\mathcal{E}nd(\mathsf{F}^n_*\mathcal{O}_{\mathbf{G}/\mathbf{B}})$. For groups of type $\mathbf{A}_2$ and $\mathbf{B}_2$ (and $n = 1$ in the latter case) this structure was described in [13] and [6], respectively. One of the reasons why these questions are sensible is the following: $\mathcal{E}nd(\mathsf{F}^n_*\mathcal{O}_{\mathbf{G}/\mathbf{B}})$ are related to the sheaf of differential operators with divided powers $\mathcal{D}_{\mathbf{G}/\mathbf{B}}$. Specifically, recall that $\mathcal{D}_{\mathbf{G}/\mathbf{B}} = \bigcup_n \mathcal{E}nd(\mathsf{F}^n_*\mathcal{O}_{\mathbf{G}/\mathbf{B}})$. Higher cohomology vanishing for sheaves of differential operators is essential for localization type theorems, of which the Beilinson–Bernstein localization [8] is the prototype. It is known that, unlike the characteristic zero case, in positive characteristic localization does not hold in general [19]. However, one could hope for a weaker, but still interesting statement. Namely, consider the sheaf $\mathcal{D}^{(1)}_{\mathbf{G}/\mathbf{B}} = \mathcal{E}nd(\mathsf{F}_*\mathcal{O}_{\mathbf{G}/\mathbf{B}})$ (the first term of the p–filtration on $\mathcal{D}_{\mathbf{G}/\mathbf{B}}$ on $\mathbf{G}/\mathbf{B}$). The category of sheaves of modules over $\mathcal{D}^{(1)}_{\mathbf{G}/\mathbf{B}}$ is precisely that of $\mathcal{D}_{\mathbf{G}/\mathbf{B}}$–modules with vanishing p–curvature, which is equivalent to the category of coherent sheaves via Cartier equivalence. The question is then whether localization theorem holds for the category $\mathcal{D}^{(1)}_{\mathbf{G}/\mathbf{B}}$–mod.

In this guise this problem has first appeared in [14]. Independently, at around the same time we started to be interested in constructing specific equivalences for derived categories of coherent sheaves on homogeneous spaces (e.g., flag varieties) using the Frobenius morphism. Localization statement, when formulated in these terms, is equivalent to that the bundle $\mathsf{F}^n_*\mathcal{O}_{\mathbf{G}/\mathbf{B}}$ is tilting, and in [25] we conjectured that it was the case when p is greater than the Coxeter number of $\mathbf{G}$. Our previous partial results towards this statement can be found in [23] – [25]. The present paper aims to provide more evidence.

Throughout the paper we emphasize the relation between the higher cohomology vanishing of $\mathcal{D}^{(1)}_{\mathbf{G}/\mathbf{B}}$ and cohomology vanishing of line bundles on $\mathbf{G}/\mathbf{B}$. A long–standing difficult problem is to find uniform patterns of cohomology vanishing of line bundles and to describe the module structure on cohomology groups. The first part is known only for groups of rank ≤ 2 by the works of H.H.Andersen from the early 80's (while several gaps in the exceptional case $\mathbf{G}_2$ have been filled in a very recent paper [7]). As for the second part, the only non–trivial case when the complete answer is known is that of $\mathbf{A}_2$ and due to S. Donkin [11].

The present work has definitely a technical flavour, and should be considered more as a report on the work in progress. Most of the results stated here are still in their preliminary form. Major part is devoted to calculation of cohomology groups $\mathrm{H}^i(\mathbf{G}/\mathbf{B}, \mathcal{E}nd(\mathsf{F}^n_*\mathcal{O}_{\mathbf{G}/\mathbf{B}}))$, where $\mathbf{G}$ is of type $\mathbf{G}_2$. Using the same considerations as in [24] in the case of $\mathbf{B}_2$, one shows $\mathrm{H}^i(\mathbf{G}/\mathbf{B}, \mathcal{E}nd(\mathsf{F}^n_*\mathcal{O}_{\mathbf{G}/\mathbf{B}})) = 0$ for $i > 1$. The real difficulty happens while attempting to prove the vanishing of H^1, and so far we are able to provide an argument valid only for $p \leq 7$. It would have had very little interest, if any, had it not been for the case $p = 7$, which is the first sensible prime for $\mathbf{G}_2$ if one keeps in mind applications to localization theorems. Namely, the Coxeter number h of $\mathbf{G}_2$ being 6, the vanishing result for $p = 7$ allows to construct a derived equivalence (see Subsection 4.5). Our hope that the vanishing holds for $p > 7$ is also supported by a remark from Humphreys' treatise on modular representations [15, Chapter 18]: "In line with Lusztig's Conjecture, the representation theory of $\mathbf{G}_2$ shows considerable uniformity when $p \geq 6$". In a subsequent paper [26], we discuss the vanishing of the first cohomology group $\mathrm{H}^1(\mathbf{G}/\mathbf{B}, \mathcal{E}nd(\mathsf{F}^n_*\mathcal{O}_{\mathbf{G}/\mathbf{B}}))$ for generic primes.

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Notation. Throughout we fix an algebraically closed field k of characteristic $p > 0$. Let $\mathbf{G}$ be a semisimple algebraic group over k. Let $\mathbf{T}$ be a maximal torus of $\mathbf{G}$, and $\mathbf{T} \subset \mathbf{B}$ a Borel subgroup containing it. Denote $\mathrm{X}(\mathbf{T})$ the weight lattice, R – the root lattice and $\mathrm{S} \subset \mathrm{R}$ the set of simple roots. The Weyl group $\mathcal{W} = \mathrm{N}(\mathbf{T})/\mathbf{T}$ acts on $X(\mathbf{T})$ via the dot–action: if $w \in \mathcal{W}$, and $\lambda \in \mathrm{X}(\mathbf{T})$, then $w \cdot \lambda = w(\lambda + \rho) - \rho$. For a simple root $\alpha \in \mathrm{S}$ denote $\mathbf{P}_\alpha$ the minimal parabolic subgroup of $\mathbf{G}$. For a weight $\lambda \in \mathrm{X}(\mathbf{T})$ denote $\mathcal{L}_\lambda$ the corresponding line bundle on $\mathbf{G}/\mathbf{B}$.

2. Recollections

2.1. **Differential operators.** Let X be a smooth algebraic variety over k, and $\mathcal{D}_X$ the sheaf of differential operators on X. Recall (e.g., [25, Section 1.3.]) that there exists a filtration $\mathcal{D}_X^{(p^n)}$ on $\mathcal{D}_X$ such that

$$\mathcal{D}_X^{(p^n)} = \mathcal{E}nd_{\mathcal{O}_X^{p^n}}(\mathcal{O}_X), \tag{2.1}$$

and

$$\mathcal{D}_X = \bigcup_{n\geq 1} \mathcal{E}nd_{\mathcal{O}_X^{p^n}}(\mathcal{O}_X). \tag{2.2}$$

The filtration above has the name of *p–filtration.* By definition of the Frobenius morphism one has $\mathrm{H}^i(X, \mathcal{E}nd_{\mathcal{O}_X^p}(\mathcal{O}_X)) = \mathrm{H}^i(X', \mathcal{E}nd(\mathsf{F}_*\mathcal{O}_X))$.

2.2. **A short exact sequence.** Assume given a smooth variety S and a locally free sheaf $\mathcal{E}$ of rank 2 on S. Let $X = \mathbb{P}_S(\mathcal{E})$ be the projective bundle over S and $\pi : X \to S$ the projection. Denote $\mathcal{O}_\pi(-1)$ the relative invertible sheaf. One has $\pi_*\mathcal{O}_\pi(1) = \mathcal{E}^*$.

The statement below is proven in [23, Lemma 2.4].

Lemma 2.1. *For any $n \geq 1$ there is a short exact sequence of vector bundles on X:*

$$0 \to \pi^*\mathsf{F}_*^n\mathcal{O}_S \to \mathsf{F}_*^n\mathcal{O}_X \to \pi^*(\mathsf{F}_*^n(\mathsf{D}^{p^n-2}\mathcal{E}\otimes \det\, \mathcal{E})\otimes \det\, \mathcal{E}^*)\otimes \mathcal{O}_\pi(-1) \to 0. \tag{2.3}$$

Here $\mathsf{D}^k\mathcal{E} = (\mathsf{S}^k\mathcal{E}^*)^*$ is the k-th divided power of $\mathcal{E}$.

The above sequence is typically applied to $\mathbb{P}^1$–fibrations of $\mathbf{G}/\mathbf{B}$ associated to simple roots; associated to a simple root $\alpha \in \mathrm{S}$ are a minimal parabolic subgroup $\mathbf{P}_\alpha$ and a $\mathbb{P}^1$–bundle $\mathbf{G}/\mathbf{B} \to \mathbf{G}/\mathbf{P}_\alpha$.

Remark 2.1. M. Kaneda informed us [17] that for fibrations $\mathbf{G}/\mathbf{B} \to \mathbf{G}/\mathbf{P}_\alpha$ sequence (2.3) is induced from a short exact sequence of $\mathbf{G}_n\mathbf{B}$–modules.

2.3. **Cohomology of line bundles on G/B.** We collect here several results on cohomology groups of line bundles on $\mathbf{G}/\mathbf{B}$ that are due to H.H.Andersen.

2.3.1. *First cohomology group of a line bundle.* Let α be a simple root, and denote s_α a corresponding reflection in $\mathcal{W}$. One has $s_\alpha \cdot \chi = s_\alpha(\chi) - \alpha$. Thre is a complete description [2, Theorem 3.6] of (non)–vanishing of the first cohomology group of a line bundle $\mathcal{L}_\lambda$.

Theorem 2.1. $\mathrm{H}^1(\mathbf{G}/\mathbf{B}, \mathcal{L}_\chi) \neq 0$ *if and only if there exist a simple root α such that one of the following conditions is satisfied:*

- $-p \leq \langle \chi, \alpha^\vee\rangle \leq -2$ *and $s_\alpha \cdot \chi = s_\alpha(\chi) - \alpha$ is dominant.*
- $\langle \chi, \alpha^\vee\rangle = -ap^n - 1$ *for some $a, n \in \mathbf{N}$ with $a < p$ and $s_\alpha(\chi) - \alpha$ is dominant.*
- $-(a+1)p^n \leq \langle \chi, \alpha^\vee\rangle \leq -ap^n - 2$ *for some $a, n \in \mathbf{N}$ with $a < p$ and $\chi + ap^n\alpha$ is dominant.*

Some bits of Demazure's proof of the Bott theorem in characteristic zero are still valid in positive characteristic [2, Corollary 3.2]:

Theorem 2.2. *Let χ be a weight. If either $\langle \chi, \alpha^\vee \rangle \geq -p$ or $\langle \chi, \alpha^\vee \rangle = -ap^n - 1$ for some $a, n \in \mathbf{N}$ and $a < p$ then*

$$\mathrm{H}^i(\mathbf{G}/\mathbf{B}, \mathcal{L}_\chi) = \mathrm{H}^{i-1}(\mathbf{G}/\mathbf{B}, \mathcal{L}_{s_\alpha \cdot \chi}). \tag{2.4}$$

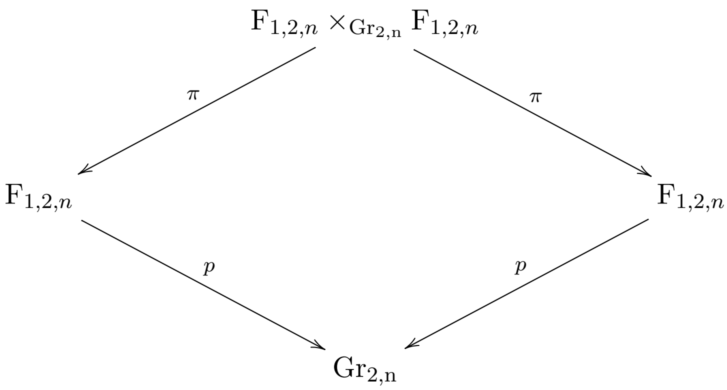
Further, Theorem 2.3 of [3] states:

Theorem 2.3. *If χ is a weight such that for a simple root α one has $0 \leq \langle \chi + \rho, \alpha^\vee \rangle \leq p$ then*

$$\mathrm{H}^i(\mathbf{G}/\mathbf{B}, \mathcal{L}_\chi) = \mathrm{H}^{i+1}(\mathbf{G}/\mathbf{B}, \mathcal{L}_{s_\alpha \cdot \chi}). \tag{2.5}$$

3. Hilbert scheme of two points on $\mathbb{P}^2$

Let V be a vector space over k of dimension n. In this section we assume the characteristic of k to be odd. Denote $\mathrm{Gr}_{2,\mathrm{n}}$ the grassmannian of 2–dimensional subspaces in V and $\mathrm{F}_{1,2,n}$ the variety of partial flags of type $(1,2)$ in V. Consider the fibered square:



Denote X_n the fibered product $\mathrm{F}_{1,2,n} \times_{\mathrm{Gr}_{2,\mathrm{n}}} \mathrm{F}_{1,2,n}$. The projection $X_n \to \mathbb{P}^n \times \mathbb{P}^n$ presents X_n as the blow-up of $\mathbb{P}^n \times \mathbb{P}^n$ along the diagonal $\Delta \subset \mathbb{P}^n \times \mathbb{P}^n$. The group $\mathbb{Z}/2\mathbb{Z}$ acts on X_n, and the quotient variety is identified with the Hilbert scheme of two points on $\mathbb{P}^n$.

Consider the case $n = 3$. The flag variety $\mathrm{F}_{1,2,3}$ is isomorphic to $\mathbf{SL}_3/\mathbf{B}$.

Theorem 3.1. *One has $\mathrm{H}^i(X_3, \mathcal{E}nd(\mathsf{F}_*\mathcal{O}_{X_3})) = 0$ for $i > 0$.*

Proof. The proof essentially follows the argument in the proof of [24, Theorem 2], and we therefore skip several details that can be found in *loc.cit.*.

Denote $\mathcal{U}_2$ the rank two tautological bundle on $\mathrm{Gr}_{2,3} = (\mathbb{P}^2)^\vee$ (in fact, $\mathcal{U}_2$ is isomorphic to $\Omega^1(1)$). Let $\mathcal{L}_{\omega_1}, \mathcal{L}_{\omega_2}$ be the line bundles on $\mathbf{SL}_3/\mathbf{B}$ corresponding to the fundamental weights ω_1, ω_2. The projection p is then isomorphic to projectivization of $\mathcal{U}_2$, the relative tautological bundle being $\mathcal{L}_{-\omega_1}$. Similarly, the projection π is isomorphic to projectivization of the bundle $p^*\mathcal{U}_2$.

Denote S the flag variety $\mathbf{SL}_3/\mathbf{B}$. The short exact sequence from Lemma 2.1 with respect to the projection π looks as follows:

$$0 \to \pi^*\mathsf{F}_*\mathcal{O}_S \to \mathsf{F}_*\mathcal{O}_{X_3} \to \pi^*(\mathsf{F}_*(\mathsf{S}^{p-2}p^*\mathcal{U}_2 \otimes \mathcal{L}_{-\omega_2}) \otimes \mathcal{L}_{\omega_2}) \otimes \mathcal{O}_\pi(-1) \to 0; \tag{3.1}$$

one has $\pi_*\mathcal{O}_\pi(1) = p^*\mathcal{U}_2^*$. Recall the resolution of the diagonal for S [18, Proposition 4.17]:

$$0 \to \mathcal{L}_{-\omega_1} \boxtimes \mathcal{L}_{-\omega_1-2\omega_2} \oplus \mathcal{L}_{-\omega_2} \boxtimes \mathcal{L}_{-2\omega_1-\omega_2} \to \Psi_{1,1} \boxtimes \mathcal{L}_{-\omega_1-\omega_2} \to \\ \to \Psi_{1,0} \boxtimes \mathcal{L}_{-\omega_1} \oplus \Psi_{0,1} \boxtimes \mathcal{L}_{-\omega_2} \to \mathcal{O}_{S\times S} \to \mathcal{O}_\Delta \to 0. \tag{3.2}$$

Here $\Psi_{i,j}$ are vector bundles (terms of the dual exceptional collection). Arguing as in the proof of Theorem 2 of [24], we see that it is sufficient to show that the vanishing of $\mathrm{E}xt$–groups:

$$\mathrm{E}xt^i_S(\mathsf{F}^n_*(\mathsf{S}^{p-2}p^*\mathcal{U}_2 \otimes \mathcal{L}_{-\omega_2}) \otimes \mathcal{L}_{\omega_2}, \mathsf{F}_*\mathsf{S}^p p^*\mathcal{U}_2^*) = 0 \tag{3.3}$$

for $i > 0$. Recall that the canonical sheaf ω_S is isomorphic to $\mathcal{L}_{-2(\omega_1+\omega_2)}$. By [24, Corollary 1], the above group is isomorphic to

$$\mathrm{H}^i(S \times S, (\mathsf{F} \times \mathsf{F}^*)(i_*\mathcal{O}_\Delta) \otimes (\mathsf{S}^p p^*\mathcal{U}_2^* \boxtimes (\mathsf{S}^{p-2}p^*\mathcal{U}_2^* \otimes \mathcal{L}_{(2p-2)\omega_1+(2p-4)\omega_2}))). \tag{3.4}$$

The sheaves $\Psi_{1,0}, \Psi_{0,1}, \Psi_{1,1}$ have right resolutions consisting of direct sums of line bundles $\mathcal{L}_{\omega_1}, \mathcal{L}_{\omega_2}, \mathcal{L}_{\omega_1+\omega_2}$, respectively. Explicitly, tensoring resolution (3.2) with $\mathcal{L}_{\omega_1}, \mathcal{L}_{\omega_2}, \mathcal{L}_{\omega_1+\omega_2}$ respectively, and pushing it down onto the first component, one obtains:

$$0 \to \Psi_{1,0} \to \mathrm{H}^0(\mathcal{L}_{\omega_1}) \otimes \mathcal{O}_{X_3} \to \mathcal{L}_{\omega_1} \to 0, \tag{3.5}$$

$$0 \to \Psi_{0,1} \to \mathrm{H}^0(\mathcal{L}_{\omega_2}) \otimes \mathcal{O}_{X_3} \to \mathcal{L}_{\omega_2} \to 0, \tag{3.6}$$

$$0 \to \Psi_{1,1} \to \Psi_{1,0} \otimes \mathrm{H}^0(\mathcal{L}_{\omega_2}) \oplus \Psi_{0,1} \otimes \mathrm{H}^0(\mathcal{L}_{\omega_1}) \to \mathrm{H}^0(\mathcal{L}_{\omega_1+\omega_2}) \otimes \mathcal{O}_{X_3} \to \mathcal{L}_{\omega_1+\omega_2} \to 0. \tag{3.7}$$

Arguing as in [24, Lemma 5], we see that it is sufficient to show:

(i) $\mathrm{H}^i(S \times S, \mathsf{S}^p p^*\mathcal{U}_2^* \boxtimes (\mathsf{S}^{p-2}p^*\mathcal{U}_2^* \otimes \mathcal{L}_{(2p-2)\omega_1+(2p-4)\omega_2})) = 0$ for $i > 0$;

(ii) $\mathrm{H}^i(S\times S, (\mathsf{F}^*(\Psi_{1,0}\oplus\Psi_{0,1})\otimes\mathsf{S}^p p^*\mathcal{U}_2^*)\boxtimes(\mathsf{S}^{p-2}p^*\mathcal{U}_2^*\otimes\mathcal{L}_{(2p-2)\omega_1+(2p-4)\omega_2}\otimes(\mathcal{L}_{-p\omega_1}\oplus\mathcal{L}_{-p\omega_2})) = 0$ for $i > 1$;

(iii) $\mathrm{H}^i(S \times S, , (\mathsf{F}^*\Psi_{1,1} \otimes \mathsf{S}^p p^*\mathcal{U}_2^*) \boxtimes (\mathsf{S}^{p-2}p^*\mathcal{U}_2^* \otimes \mathcal{L}_{(2p-2)\omega_1+(2p-4)\omega_2} \otimes \mathcal{L}_{-p(\omega_1+\omega_2)})) = 0$ for $i > 2$;

(iv) $\mathrm{H}^i(S \times S, (\mathcal{L}_{-p\omega_1} \otimes \mathsf{S}^p p^*\mathcal{U}_2^*) \boxtimes ((\mathsf{S}^{p-2}p^*\mathcal{U}_2^* \otimes \mathcal{L}_{(2p-2)\omega_1+(2p-4)\omega_2} \otimes \mathcal{L}_{-p(\omega_1+2\omega_2)}) = 0$ for $i > 3$;

(v) $\mathrm{H}^i(S \times S, (\mathcal{L}_{-p\omega_2} \otimes \mathsf{S}^p p^*\mathcal{U}_2^*) \boxtimes ((\mathsf{S}^{p-2}p^*\mathcal{U}_2^* \otimes \mathcal{L}_{(2p-2)\omega_1+(2p-4)\omega_2} \otimes \mathcal{L}_{-p(2\omega_1+\omega_2)}) = 0$ for $i > 3$.

Using resolutions (3.6), (3.5), and (3.7), and the Kempf vanishing theorem, one immediately obtains (i)–(iii).

As for (iv) and (v) we see, using the Künneth formula, that it is sufficient to show:

(1) $\mathrm{H}^i(S, \mathcal{L}_{-p\omega_1} \otimes \mathsf{S}^p p^*\mathcal{U}_2^*) = 0$ for $i \neq 2$, and $\mathrm{H}^i(S, \mathsf{S}^{p-2}p^*\mathcal{U}_2^* \otimes \mathcal{L}_{(p-2)\omega_1-4\omega_2}) = 0$ for $i \neq 1$;

(2) $\mathrm{H}^i(S, \mathcal{L}_{-p\omega_2} \otimes \mathsf{S}^p p^*\mathcal{U}_2^*) = 0$ and $\mathrm{H}^i(S, \mathsf{S}^{p-2}p^*\mathcal{U}_2^* \otimes \mathcal{L}_{(p-4)\omega_2-2\omega_1}) = 0$ for $i \neq 1$.

Indeed, consider the first group in (1). Tensor the short exact sequence

$$0 \to \mathsf{F}^*p^*\mathcal{U}_2^* \to \mathsf{S}^p p^*\mathcal{U}_2^* \to \mathsf{S}^{p-2}p^*\mathcal{U}_2^* \otimes \mathcal{L}_{\omega_2} \to 0 \tag{3.8}$$

with $\mathcal{L}_{-p\omega_1}$. Further, there is a short exact sequence:

$$0 \to \mathcal{L}_{-p\omega_1} \to \mathsf{F}^*p^*\mathcal{U}_2 \to \mathcal{L}_{p(\omega_1-\omega_2)} \to 0. \tag{3.9}$$

Taking the dual of it, and tensoring the dual sequence with $\mathcal{L}_{-p\omega_1}$, one gets:

$$0 \to \mathcal{L}_{p(\omega_2-2\omega_1)} \to \mathsf{F}^*p^*\mathcal{U}_2^* \otimes \mathcal{L}_{-p\omega_1} \to \mathcal{O}_S \to 0. \tag{3.10}$$

The top cohomology groups of both $\mathcal{L}_{p(\omega_2-2\omega_1)}$ and $\mathcal{O}_S$ vanish (the first group vanishes by the Kempf vanishing). Hence, $\mathrm{H}^3(S, \mathsf{F}^*p^*\mathcal{U}_2^* \otimes \mathcal{L}_{-p\omega_1}) = 0$.

On the other hand, the Euler sequence on $(\mathbb{P}^2)^\vee$ gives:

$$0 \to \mathcal{L}_{-\omega_2} \to \mathsf{V}^* \otimes \mathcal{O}_S \to \mathcal{U}_2^* \to 0. \tag{3.11}$$

Pulling it back to S and taking its p-th symmetric power, we get:

$$0 \to \mathcal{L}_{-\omega_2} \otimes \mathsf{S}^{p-1}\mathsf{V}^* \to \mathsf{S}^p\mathsf{V}^* \otimes \mathcal{O}_S \to \mathsf{S}^p p^*\mathcal{U}_2^* \to 0. \tag{3.12}$$

Tensoring it with $\mathcal{L}_{-p\omega_1}$, we see that the line bundle $\mathcal{L}_{-\omega_2-p\omega_1}$ is acyclic, while $\mathcal{L}_{-p\omega_1}$ has non–vanishing cohomology group only in the degree 2 (since $\mathcal{L}_{-p\omega_1}$ is pulled back from $\mathbb{P}^2$). Thus, $\mathrm{H}^i(S, \mathcal{L}_{-p\omega_1} \otimes \mathsf{S}^p p^*\mathcal{U}_2^*) = 0$ for $i \neq 2$.

We restrict ourselves to the case $p > 3$ (for $p = 3$ the cohomology vanishing of some the groups below is different, but it does not affect the statements). Considering the $(p-2)$–th symmetric power of the Euler sequence, and tensoring it with $\mathcal{L}_{(p-2)\omega_1-4\omega_2}$, we see that the cohomology groups of $\mathsf{S}^{p-2}p^*\mathcal{U}_2^* \otimes \mathcal{L}_{(p-2)\omega_1-4\omega_2}$ are squeezed in between the cohomology groups of $\mathcal{L}_{(p-2)\omega_1-4\omega_2}$ and $\mathcal{L}_{(p-2)\omega_1-5\omega_2}$. The cohomology groups of both line bundles vanish except H^1. Hence the statement of (1).

Consider the statements of (2). From the projection formula we see that $\mathrm{H}^i(S, \mathcal{L}_{-p\omega_2} \otimes \mathsf{S}^p p^*\mathcal{U}_2^*) = \mathrm{H}^i(S, \mathcal{L}_{p(\omega_1-\omega_2)})$. By Theorem 2.2 we conclude that $\mathrm{H}^i(S, \mathcal{L}_{p(\omega_1-\omega_2)}) = \mathrm{H}^{i-1}(S, \mathcal{L}_{p(\omega_1-\omega_2)+(p-1)\alpha_2}) = 0$ for $i \neq 1$, the weight $p(\omega_1-\omega_2)+(p-1)\alpha_2$ being dominant.

Finally, that the group $\mathrm{H}^3(S, \mathsf{S}^{p-2}p^*\mathcal{U}_2^* \otimes \mathcal{L}_{(p-4)\omega_2-2\omega_1})$ vanishes can once again be seen from sequence (3.12), the groups $\mathrm{H}^i(S, \mathcal{L}_{(p-4)\omega_2-2\omega_1})$ being non–trivial only for $i \neq 1$. □

Corollary 3.1. $\mathrm{H}^i((\mathbb{P}^2)^{[2]}, \mathcal{E}nd(\mathsf{F}_*\mathcal{O}_{((\mathbb{P}^2)^{[2]}}) = 0$ *for* $i > 0$.

Remark 3.1. We expect the bundle $\mathsf{F}_*\mathcal{O}_{(\mathbb{P}^2)^{[2]}}$ to be a generator in the derived category of coherent sheaves on $(\mathbb{P}^2)^{[2]}$; in other words, the bundle $\mathsf{F}_*\mathcal{O}_{(\mathbb{P}^2)^{[2]}}$ should be tilting, and an equivalence of derived categories holds (cf. Section 6 of [25]). The proof will be given in [26].

4. The flag variety of $\mathbf{G}_2$

4.1. **Geometric preliminaries.** Let $\mathbf{G}$ be a group of type $\mathbf{G}_2$. Its root system has two simple roots α and β, the root β being the long root. Associated to the simple roots are two minimal parabolic subgroups $\mathbf{P}_\alpha$ and $\mathbf{P}_\beta$. The homogeneous space $\mathbf{G}/\mathbf{P}_\beta$ is isomorphic to the 5-dimensional quadric Q_5 and $\mathbf{G}/\mathbf{P}_\alpha$ is a five-dimensional variety X, a subvariety of the grassmannian $\mathrm{Gr}_{2,\mathsf{W}}$. Here W is the smallest irreducible (for odd p) representation of $\mathbf{G}_2$ of dimension 7 that gives rise to an embedding $\mathbf{G}_2 \hookrightarrow \mathbf{SO}_{2,7}$. Denote q and π the two projections of $\mathbf{G}/\mathbf{B}$ onto Q_5 and X. The line bundles $\mathcal{L}_{\omega_\alpha}$ and $\mathcal{L}_{\omega_\beta}$ on $\mathbf{G}/\mathbf{B}$ are isomorphic to $q^*\mathcal{O}_{\mathsf{Q}_5}(1)$ and $\pi^*\mathcal{O}_X(1)$, respectively. The canonical line bundle $\omega_{\mathbf{G}/\mathbf{B}}$ is isomorphic to $\mathcal{L}_{-2\rho} = \mathcal{L}_{-2(\omega_\alpha+\omega_\beta)}$. Both projections π and q are $\mathbb{P}^1$–bundles over $\mathbf{G}/\mathbf{B}$ associated to rank two vector bundles $\pi_*\mathcal{L}_{\omega_\alpha}$ and $q_*\mathcal{L}_{\omega_\beta}$, respectively. The latter bundle of rank 2 on Q_5 is also known as the *Cayley bundle* (cf. [21]) that we denote $\mathcal{K}$. There is a short exact sequence:

$$0 \to \mathcal{K} \to \mathcal{T}_{\mathbb{P}(\mathsf{W})}(-2) \otimes \mathcal{O}_{\mathsf{Q}_5} \to \mathrm{S} \to 0, \tag{4.1}$$

where S is the spinor bundle on Q_5. One has det $\mathcal{K} = \mathcal{L}_{-3\omega_\alpha}$. The relative Euler sequence for the projection q looks as follows:

$$0 \to \mathcal{L}_{-\omega_\beta} \to q^*\mathcal{K} \to \mathcal{L}_\beta \to 0. \tag{4.2}$$

From Lemma 2.1 one obtains:

$$0 \to q^*\mathsf{F}_*\mathcal{O}_{\mathsf{Q}_5} \to \mathsf{F}_*\mathcal{O}_{\mathbf{G}/\mathbf{B}} \to q^*(\mathsf{F}_*(\mathsf{S}^{p-2}\mathcal{K} \otimes \mathcal{L}_{-3\omega_\alpha}) \otimes \mathcal{L}_{3\omega_\alpha}) \otimes \mathcal{L}_{-\omega_\beta} \to 0. \tag{4.3}$$

4.2. **Clifford algebra of $\mathcal{T}_{\mathsf{Q}_5}(-1)$.** Recall some facts from [18]. If $\mathcal{E}$ is an orthogonal vector bundle over a base variety Y then one defines a sheaf of graded algebras $A(\mathcal{E}) = \oplus A_i(\mathcal{E})$ over Y; here $A_i(\mathcal{E})$ is a vector bundle isomorphic to $\oplus_{k\geq 0} \wedge^{i-2k} (\mathcal{E})$. There is a short exact sequence of vector bundles over Q_n [18, Proposition 4.2]:

$$0 \to A_{i-1}(\mathcal{T}_{\mathsf{Q}_n}(-1)) \otimes \mathcal{O}_{\mathsf{Q}_n}(-1) \to \Psi_i \to A_i(\mathcal{T}_{\mathsf{Q}_n}(-1)) \to 0. \tag{4.4}$$

One defines the Clifford algebra $Cl(\mathcal{E})$ of $\mathcal{E}$ to be $A(\mathcal{E})/(h-1)A(\mathcal{E})$ (where h is a free generator of degree two in the graded Clifford algebra). The Clifford algebra of $\mathcal{T}_{\mathsf{Q}_n}(-1)$ is a sheaf of matrix algebras $\mathcal{E}nd(\mathrm{S}_+ \oplus \mathrm{S}_-)$ for even-dimensional quadrics and the sum of two sheaves of matrix algebras $\mathcal{E}nd(\mathrm{S}) \oplus \mathcal{E}nd(\mathrm{S})$ for odd-dimensional quadrics. Here S is the spinor bundle. There is the short exact sequence for S:

$$0 \to \mathrm{S} \to \mathsf{U} \otimes \mathcal{O}_{\mathsf{Q}_n} \to \mathrm{S}^* \to 0, \tag{4.5}$$

the vector space U being the spinor representation.

4.3. **Left dual exceptional collection on a quadric.** By [18, Proposition 4.11], the set of exceptional bundles $\mathrm{S}(-n), \Psi_{n-1}, \dots, \Psi_1, \mathcal{O}_{\mathsf{Q}_n}$ on an odd-dimensional quadric Q_n is a full exceptional collection. One has $\Psi_i = \mathrm{S}^{\oplus 2^{\frac{n+1}{2}}}$ for $i \geq n$. There are short exact sequences connecting the bundles Ψ_i than can be obtained from the resolution of the diagonal:

$$0 \to \Omega^i_{\mathbb{P}(\mathsf{W})}(i) \otimes \mathcal{O}_{\mathsf{Q}_n} \to \Psi_i \to \Psi_{i-2} \to 0. \tag{4.6}$$

In particular, one has:

$$0 \to \mathcal{T}_{\mathbb{P}(\mathsf{W})}(-2) \otimes \mathcal{O}_{\mathsf{Q}_n} \to \mathrm{S}^{\oplus 2^{\frac{n+1}{2}}} \to \Psi_{n-2} \to 0, \tag{4.7}$$

and

$$0 \to \mathcal{O}_{\mathsf{Q}_n}(-1) \to \mathrm{S}^{\oplus 2^{\frac{n+1}{2}}} \to \Psi_{n-1} \to 0. \tag{4.8}$$

The goal of this section is prove the following statement:

Theorem 4.1. *Let $p \leq 7$. Then*

$$\mathrm{H}^i(\mathbf{G}/\mathbf{B}, \mathcal{E}nd(\mathsf{F}_*\mathcal{O}_{\mathbf{G}/\mathbf{B}})) = 0 \tag{4.9}$$

for $i > 0$.

Remark 4.1. The unfortunate restriction on characteristic of k that we are forced to impose at the moment is explained by non–standard cohomology vanishing in type $\mathbf{G}_2$. Specifically, in the course of the proof one has to deal with cohomology groups of a particular line bundle (see Proposition 4.3 below); according to Andersen and Kaneda's recent rectification of Andersen's old results in [4] on cohomology vanishing patterns in type $\mathbf{G}_2$ (see [7]) this line bundle acquires an extra non–vanishing cohomology group for $p \geq 11$. There is sufficient evidence that the non–vanishing of this group does not prevent nevertheless the sought–for higher cohomology vanishing of $\mathrm{H}^i(\mathbf{G}/\mathbf{B}, \mathcal{E}nd(\mathsf{F}_*\mathcal{O}_{\mathbf{G}/\mathbf{B}}))$ for $p \geq 11$. We relegate the further study to a subsequent paper [26].

4.4. **Computations.** The goal of this section is to provide necessary statements for Theorem 4.1. The non–vanishing cohomology group for $p \geq 11$ appears in Proposition 4.3. It results eventually in a non–trivial differential in the spectral sequence computing the group from Lemma 4.1 (for $p = 7$ the group in question vanishes, and the spectral sequence immediately collapses, thus giving the statement from Theorem 4.1).

Since the essence of the argument is again the same as in [24, Theorem 2], we immediately skip to non–trivial part of the proof, just reminding that the higher cohomology vanishing of $\mathcal{E}nd(\mathsf{F}_*\mathcal{O}_{\mathsf{Q}_5})$ for odd primes follows from either [20] or [25], and from [1] for $p = 2$.

Lemma 4.1. $\mathrm{E}xt^i(q^*(\mathsf{F}_*(\mathsf{S}^{p-2}\mathcal{K} \otimes \mathcal{L}_{-3\omega_\alpha}) \otimes \mathcal{L}_{3\omega_\alpha}) \otimes \mathcal{L}_{-\omega_\beta}, \mathsf{F}_*\mathcal{O}_{\mathbf{G}/\mathbf{B}}) = 0$ *for $i > 0$.*

Proof. We have to show that $\mathrm{H}^i(\mathsf{Q}_5, \mathsf{F}^*\mathsf{F}_*\mathsf{S}^p\mathcal{K}^* \otimes \mathsf{S}^{p-2}\mathcal{K}^* \otimes \mathcal{L}_{(2p-2)\omega_\alpha}) = 0$ for $i > 0$. One has an isomorphism

$$\begin{aligned} \mathrm{H}^i(\mathsf{Q}_5, \mathsf{F}^*\mathsf{F}_*\mathsf{S}^p\mathcal{K}^* \otimes \mathsf{S}^{p-2}\mathcal{K}^* \otimes \mathcal{L}_{(2p-2)\omega_\alpha}) = \\ \mathrm{H}^i(\mathsf{Q}_5 \times \mathsf{Q}_5, (\mathsf{F} \times \mathsf{F}^*)(i_*\mathcal{O}_\Delta) \otimes \mathsf{S}^p\mathcal{K}^* \boxtimes (\mathsf{S}^{p-2}\mathcal{K}^* \otimes \mathcal{L}_{(2p-2)\omega_\alpha})). \end{aligned} \tag{4.10}$$

Recall the resolution of the diagonal for Q_5:

$$0 \to \mathrm{S} \boxtimes (\mathrm{S} \otimes \mathcal{L}_{-4\omega_\alpha}) \to \Psi_4 \boxtimes \mathcal{L}_{-4\omega_\alpha} \to \cdots \to \Psi_1 \boxtimes \mathcal{L}_{-\omega_\alpha} \to \mathcal{O}_{\mathsf{Q}_3} \boxtimes \mathcal{O}_{\mathsf{Q}_3} \to i_*\mathcal{O}_\Delta \to 0, \tag{4.11}$$

Denote $\mathrm{C}^\bullet$ the complex, whose terms are $\mathrm{C}^j = \mathsf{F}^*\Psi_{-j} \boxtimes \mathsf{F}^*\mathcal{L}_{-j\omega_\alpha}$ for $j = -4, \ldots, -1, 0$ and $\mathrm{C}^{-5} = \mathsf{F}^*\mathrm{S} \boxtimes \mathsf{F}^*(\mathrm{S} \otimes \mathcal{L}_{-4\omega_\alpha})$. Tensor $\mathrm{C}^\bullet$ with the bundle $\mathsf{S}^p\mathcal{K}^* \boxtimes (\mathsf{S}^{p-2}\mathcal{K}^* \otimes \mathcal{L}_{(2p-2)\omega_\alpha})$. Then the complex $\mathrm{C}^\bullet \otimes (\mathsf{S}^{p^n}\mathcal{U}_2^* \boxtimes \mathsf{S}^{p^n-2}\mathcal{U}_2^*(2p^n-2))$ computes the cohomology group in the right hand side of (4.10).

As in [24, Lemma 5], it will be sufficient to ensure that $\mathrm{H}^i(\mathsf{Q}_3 \times \mathsf{Q}_3, \mathrm{C}^j \otimes (\mathsf{S}^{p^n}\mathcal{U}_2^* \boxtimes \mathsf{S}^{p^n-2}\mathcal{U}_2^*(2p^n - 2))) = 0$ for $i > -j$. The proof of Lemma 4.1 is broken up into a series of propositions below. □

Proposition 4.1. $\mathrm{H}^i(\mathsf{Q}_5, \mathsf{S}^{p-2}\mathcal{K}^* \otimes \mathcal{L}_{-2\omega_\alpha}) = 0$ *for $i \neq 1$.*

Proof. One has $\mathrm{H}^i(\mathsf{Q}_5, \mathsf{S}^{p-2}\mathcal{K}^* \otimes \mathcal{L}_{-2\omega_\alpha}) = \mathrm{H}^i(\mathbf{G}/\mathbf{B}, \mathcal{L}_{p\omega_\beta} \otimes \omega_{\mathbf{G}/\mathbf{B}})$. Therefore, by Serre duality this group is isomorphic to $\mathrm{H}^{6-i}(\mathbf{G}/\mathbf{B}, \mathcal{L}_{-p\omega_\beta})$. This group is zero unless $i \neq 1$. Note also that

$\chi = p\omega_\beta - 2\rho = s_\alpha \cdot (p-3)\omega_\beta$. Finally, one obtains:

$$\mathrm{H}^i(\mathsf{Q}_5, \mathsf{S}^{p-2}\mathcal{K}^* \otimes \mathcal{L}_{-2\omega_\alpha}) = \nabla_{(p-3)\omega_\beta}.$$

□

Proposition 4.2. $\mathrm{H}^i(\mathsf{Q}_5, \mathsf{F}^*\Psi_2 \otimes \mathsf{S}^p\mathcal{K}^*) = 0$ *for* $i > 1$.

Tensoring resolution (4.11) with $p_2^*\mathcal{L}_{-3\omega_\alpha}$ and taking its pushforward onto the first component, one obtains a left resolution of Ψ_2:

$$0 \to \mathrm{S} \otimes \mathrm{H}^0(\mathsf{Q}_5, \mathrm{S}^* \otimes \mathcal{L}_{\omega_\alpha})^* \to \Psi_4 \otimes \mathrm{H}^0(\mathsf{Q}_5, \mathcal{L}_{\omega_{2\alpha}})^* \to \Psi_3 \otimes \mathrm{H}^0(\mathsf{Q}_5, \mathcal{L}_{\omega_\alpha})^* \to \Psi_2 \to 0. \tag{4.12}$$

Tensor it with $\mathsf{S}^p\mathcal{K}^*$. Propositions 4.4, 4.6, and 4.7 below show that all the terms of the resolution have vanishing cohomology groups H^i for $i > 1$, hence the statement.

Proposition 4.3. $\mathrm{H}^i(\mathsf{Q}_5, \mathsf{S}^{p-2}\mathcal{K}^* \otimes \mathcal{L}_{-(p+2)\omega_\alpha}) = 0$ *for* $i \neq 2$ *if* $p = 7$ *and* $i \neq 2, 3$ *for* $p \geq 11$.

Proof. One has isomorphisms

$$\mathrm{H}^i(\mathsf{Q}_5, \mathsf{S}^{p-2}\mathcal{K}^* \otimes \mathcal{L}_{-(p+2)\omega_\alpha}) = \mathrm{H}^i(\mathbf{G}/\mathbf{B}, \mathcal{L}_{p(\omega_\beta - \omega_\alpha)} \otimes \omega_{\mathbf{G}/\mathbf{B}}) = \mathrm{H}^{6-i}(\mathbf{G}/\mathbf{B}, \mathcal{L}_{-p(\alpha+\beta)})^*.$$

The weight $p(\alpha+\beta) - 2\rho$ lies in the H^2–chamber, and would classically have only non–vanishing H^2. However, according to Figure 12 from [7], this weight exhibits non–standard vanishing for $p \geq 11$ (the corresponding alcove is labeled by 3).

Let us track down this non–standard vanishing behaviour via more direct arguments. First, that the group H^1 vanishes follows from Andersen's criterion: $\langle \chi, \alpha^\vee \rangle = -p-2$ and $\langle \chi, \beta^\vee \rangle = p-2$. Only the last condition can be fulfilled for $a = n = 1$, but the weight $\chi + p\alpha$ is not dominant since $\langle \chi, \beta^\vee \rangle = -2$.

There is a short exact sequence:

$$0 \to \mathcal{L}_{-p\omega_\alpha} \to \pi^*\mathsf{F}^*\mathcal{U}_X \to \mathcal{L}_{-p(\alpha+\beta)} \to 0. \tag{4.13}$$

Clearly, $\mathrm{H}^i(\mathbf{G}/\mathbf{B}, \mathcal{L}_{-p\omega_\alpha}) = 0$ for $i \neq 5$. One has $\mathrm{H}^i(X, \mathsf{F}^*\mathcal{U}_X) = \mathrm{H}^{5-i}(X, \mathsf{F}^*\mathcal{U}_X^* \otimes \mathcal{L}_{-3\omega_\beta})^*$, by Serre duality. Taking the dual of the above sequence and tensoring it with $\mathcal{L}_{-3\omega_\beta}$, we get:

$$0 \to \mathcal{L}_{p(\alpha+\beta)-3\omega_\beta} \to \pi^*(\mathsf{F}^*\mathcal{U}_X^* \otimes \mathcal{L}_{-3\omega_\beta}) \to \mathcal{L}_{p\omega_\alpha - 3\omega_\beta} \to 0. \tag{4.14}$$

Let $\chi_1 = p(\alpha+\beta) - 3\omega_\beta$ and $\chi_2 = p\omega_\alpha - 3\omega_\beta$. One has $\langle \chi_1, \alpha^\vee \rangle = -p$ and $\langle \chi_2, \beta^\vee \rangle = -3$. Therefore, $s_\alpha \cdot \chi_1 = (p-2)\omega_\alpha - 2\omega_\beta$. Further, $\langle s_\alpha \cdot \chi_1, \beta^\vee \rangle = -2 > -p$, and $s_\beta \cdot s_\alpha \cdot \chi_1 = (p-5)\omega_\alpha$. By Theorem 2.2 one has a chain of isomorphisms:

$$\mathrm{H}^i(\mathbf{G}/\mathbf{B}, \mathcal{L}_{\chi_1}) = \mathrm{H}^{i-1}(\mathbf{G}/\mathbf{B}, \mathcal{L}_{s_\alpha \cdot \chi_1}) = \mathrm{H}^{i-2}(\mathbf{G}/\mathbf{B}, \mathcal{L}_{s_\beta \cdot s_\alpha \cdot \chi_1}) = 0 \tag{4.15}$$

for $i \neq 2$ by the Kempf vanishing, the latter weight in this chain being dominant. Similarly, $s_\beta \cdot \chi_2 = (p-6)\omega_\alpha + \omega_\beta$, therefore

$$\mathrm{H}^i(\mathbf{G}/\mathbf{B}, \mathcal{L}_{\chi_2}) = \mathrm{H}^{i-1}(\mathbf{G}/\mathbf{B}, \mathcal{L}_{s_\beta \cdot \chi_2}) = 0 \tag{4.16}$$

for $i \neq 1$ by the same reason as above. Hence one gets the exact sequence:

$$0 \to \mathrm{H}^1(X, \mathsf{F}^*\mathcal{U}_X^* \otimes \mathcal{L}_{-3\omega_\beta}) \to \nabla_{(p-6)\omega_\alpha + \omega_\beta} \xrightarrow{c} \nabla_{(p-5)\omega_\alpha} \to \mathrm{H}^2(X, \mathsf{F}^*\mathcal{U}_X^* \otimes \mathcal{L}_{-3\omega_\beta}) \to 0. \tag{4.17}$$

By Serre duality this implies $\mathrm{H}^i(X,\mathsf{F}^*\mathcal{U}_X)=0$ for $i\neq 3,4$. If $\mathrm{H}^3(X,\mathsf{F}^*\mathcal{U}_X)=0$ then remembering the arguments at the beginning, we obtain $\mathrm{H}^i(\mathbf{G}/\mathbf{B},\mathcal{L}_{-p\omega_\alpha+p\omega_\beta-2\rho})=0$ for $i\neq 2$. We have a homomorphism between two Weyl modules:

$$\nabla_{(p-6)\omega_\alpha+\omega_\beta}\stackrel{c}{\to}\nabla_{(p-5)\omega_\alpha} \tag{4.18}$$

For $p=7$ the map c is surjective, thus $\mathrm{H}^3(X,\mathsf{F}^*\mathcal{U}_X)=0$.

□

Proposition 4.4. $\mathrm{H}^i(\mathsf{Q}_5,\mathsf{F}^*\Psi_3\otimes\mathsf{S}^p\mathcal{K}^*)=0$ *for* $i>1$.

Proof. From Section 4.3 we have the short exact sequence:

$$0\to\mathcal{T}_{\mathbb{P}(\mathsf{W})}(-2)\otimes\mathcal{O}_{\mathsf{Q}_5}\to\mathsf{S}\otimes\mathsf{U}\to\Psi_3\to 0. \tag{4.19}$$

Using Proposition 4.2, we see that the statement will follow from $\mathrm{H}^i(\mathsf{Q}_5,\mathsf{F}^*\mathcal{T}_{\mathbb{P}(\mathsf{W})}(-2)\otimes\mathsf{S}^p\mathcal{K}^*)=0$ for $i>2$ (recall that $\mathcal{O}_{\mathsf{Q}_5}(-1)=\mathcal{L}_{-\omega_\alpha}$). One has a short exact sequence

$$0\to\mathcal{T}_{\mathsf{Q}_5}(-2)\to\mathcal{T}_{\mathbb{P}(\mathsf{W})}\otimes\mathcal{O}_{\mathsf{Q}_5}(-2)\to\mathcal{O}_{\mathsf{Q}_5}\to 0, \tag{4.20}$$

and the restriction of the Euler sequence on $\mathbb{P}^6$ to Q_5 tensored with $\mathcal{O}_{\mathsf{Q}_5}(-1)$:

$$0\to\mathcal{O}_{\mathsf{Q}_5}(-2)\to\mathsf{W}\otimes\mathcal{O}_{\mathsf{Q}_5}(-1)\to\mathcal{T}_{\mathbb{P}(\mathsf{W})}\otimes\mathcal{O}_{\mathsf{Q}_5}(-2)\to 0. \tag{4.21}$$

The long exact cohomology sequence gives:

$$\begin{aligned}\dots\to\mathsf{F}^*\mathsf{W}\otimes\mathrm{H}^i(\mathsf{Q}_5,\mathsf{S}^p\mathcal{K}^*\otimes\mathcal{L}_{-p\omega_\alpha})\to\mathrm{H}^i(\mathsf{Q}_5,\mathsf{F}^*\mathcal{T}_{\mathbb{P}(\mathsf{W})}(-2)\otimes\mathsf{S}^p\mathcal{K}^*)\to\\ \to\mathrm{H}^{i+1}(\mathsf{Q}_5,\mathsf{S}^p\mathcal{K}^*\otimes\mathcal{L}_{-2p\omega_\alpha})\to\dots.\end{aligned} \tag{4.22}$$

Firstly, $\mathrm{H}^i(\mathsf{Q}_5,\mathsf{S}^p\mathcal{K}^*\otimes\mathcal{L}_{-p\omega_\alpha})=\mathrm{H}^i(\mathbf{G}/\mathbf{B},\mathcal{L}_{p(\omega_\beta-\omega_\alpha)})$. Further, $p(\omega_\beta-\omega_\alpha)=p(\alpha+\beta)$. One has $\langle p(\alpha+\beta),\alpha^\vee\rangle=-p$ and $s_\alpha\cdot(p(\alpha+\beta))=p\omega_\alpha-\alpha\in X_+$. By Theorem 2.2 we get $\mathrm{H}^i(\mathbf{G}/\mathbf{B},\mathcal{L}_{p(\omega_\beta-\omega_\alpha)})=0$ for $i\neq 1$.

Hence, the vanishing of $\mathrm{H}^i(\mathsf{Q}_5,\mathsf{F}^*(\mathcal{T}_{\mathbb{P}(\mathsf{W})}\otimes\mathcal{O}_{\mathsf{Q}_5}(-2))\otimes\mathsf{S}^p\mathcal{K}^*))=0$ for $i>2$ will follow from the vanishing $\mathrm{H}^i(\mathbf{G}/\mathbf{B},\mathcal{L}_{p(\omega_\beta-2\omega_\alpha)})$ for $i>3$. To this end, consider a short exact sequence (the relative Euler sequence with respect to projection π):

$$0\to\mathcal{L}_{-\omega_\alpha}\to\pi^*\mathcal{U}_X\to\mathcal{L}_{\omega_\alpha-\omega_\beta}\to 0. \tag{4.23}$$

Applying F^* to it, we see that it is sufficient to show $\mathrm{H}^4(\mathbf{G}/\mathbf{B},\pi^*\mathsf{F}^*\mathcal{U}_X^*\otimes\mathcal{L}_{-p\omega_\alpha})=0$. Indeed, let $\chi=p(\omega_\beta-2\omega_\alpha)$. By Serre duality

$$\mathrm{H}^i(\mathbf{G}/\mathbf{B},\mathcal{L}_\chi)=\mathrm{H}^{6-i}(\mathbf{G}/\mathbf{B},\mathcal{L}_{-\chi-2\rho})^*. \tag{4.24}$$

One has $\langle-\chi-2\rho,\beta^\vee\rangle=-p-2$ and $\langle-\chi-2\rho+p\beta,\alpha^\vee\rangle=-p-2$. By Theorem 2.1 $\mathrm{H}^1(\mathbf{G}/\mathbf{B},\mathcal{L}_{-\chi-2\rho})=0$.

Applying $\mathrm{R}^{\bullet}\pi_*$ to $\pi^*\mathsf{F}^*\mathcal{U}_X^*\otimes\mathcal{L}_{-p\omega_\alpha}$, and using an isomorphism $\mathrm{R}^{\bullet}\pi_*\mathcal{L}_{-p\omega_\alpha}=\mathsf{S}^{p-2}\mathcal{U}_X\otimes\mathcal{L}_{-\omega_\beta}[-1]$, we get:

$$\mathrm{H}^4(\mathbf{G}/\mathbf{B},\pi^*\mathsf{F}^*\mathcal{U}_X^*\otimes\mathcal{L}_{-p\omega_\alpha})=\mathrm{H}^3(X,\mathsf{F}^*\mathcal{U}_X^*\otimes\mathsf{S}^{p-2}\mathcal{U}_X\otimes\mathcal{L}_{-\omega_\beta})= \\ =\mathrm{H}^3(X,\mathsf{F}^*\mathcal{U}_X\otimes\mathsf{S}^{p-2}\mathcal{U}_X^*\otimes\mathcal{L}_{\omega_\beta}), \tag{4.25}$$

the last isomorphism coming from $\det\mathcal{U}_X=\mathcal{L}_{-\omega_\beta}$. Applying the Serre duality on X, we get:

$$\mathrm{H}^3(X,\mathsf{F}^*\mathcal{U}_X\otimes\mathsf{S}^{p-2}\mathcal{U}_X^*\otimes\mathcal{L}_{\omega_\beta})=\mathrm{H}^2(X,\mathsf{F}^*\mathcal{U}_X^*\otimes\mathsf{S}^{p-2}\mathcal{U}_X\otimes\mathcal{L}_{-4\omega_\beta})^* \tag{4.26}$$

the sheaf ω_X being isomorphic to $\mathcal{L}_{-3\omega_\beta}$. Remembering that $\mathsf{S}^{p-2}\mathcal{U}_X\otimes\mathcal{L}_{-4\omega_\beta}=\mathrm{R}^1\pi_*\mathcal{L}_{-p\omega_\alpha-3\omega_\beta}$, and using again the Leray spectral sequence, we get:

$$\mathrm{H}^2(X,\mathsf{F}^*\mathcal{U}_X^*\otimes\mathsf{S}^{p-2}\mathcal{U}_X\otimes\mathcal{L}_{-4\omega_\beta})=\mathrm{H}^3(\mathbf{G}/\mathbf{B},\pi^*\mathsf{F}^*\mathcal{U}_X^*\otimes\mathcal{L}_{-p\omega_\alpha-3\omega_\beta}). \tag{4.27}$$

Consider the sequence

$$0\to\mathcal{L}_{(p-3)\omega_\beta-2p\omega_\alpha}\to\pi^*\mathsf{F}^*\mathcal{U}_X^*\otimes\mathcal{L}_{-p\omega_\alpha-3\omega_\beta}\to\mathcal{L}_{-3\omega_\beta}\to 0. \tag{4.28}$$

One has $\mathrm{H}^i(\mathbf{G}/\mathbf{B},\mathcal{L}_{-3\omega_\beta})=\mathrm{H}^i(X,\omega_X)=0$ for $i\neq 5$. At the left end we have the weight $\chi=-2p\omega_\alpha+(p-3)\omega_\beta$ with $\langle\chi,\beta^\vee\rangle=p-3$. So, $s_\beta\cdot\chi=\chi-(p-2)\beta$. By Theorem 2.3 one has:

$$\mathrm{H}^3(\mathbf{G}/\mathbf{B},\mathcal{L}_{(p-3)\omega_\beta-2p\omega_\alpha})=\mathrm{H}^4(\mathbf{G}/\mathbf{B},\mathcal{L}_{s_\beta\cdot\chi}). \tag{4.29}$$

Further, $\langle s_\beta\cdot\chi,\alpha^\vee\rangle=p-6$ and $\langle s_\beta\cdot\chi,\beta^\vee\rangle=-p+1$, so $s_\beta\cdot\chi=(p-6)\omega_\alpha-(p-1)\omega_\beta$. By Serre duality one has:

$$\mathrm{H}^4(\mathbf{G}/\mathbf{B},\mathcal{L}_{s_\beta\cdot\chi})=\mathrm{H}^2(\mathbf{G}/\mathbf{B},\mathcal{L}_{-s_\beta\cdot\chi-2\rho})^*, \tag{4.30}$$

and $-s_\beta\cdot\chi-2\rho=-(p-4)\omega_\alpha+(p-3)\omega_\beta$. Its label at $\alpha^\vee$ is equal to $-p+4$. Hence, by Theorem 2.2:

$$\mathrm{H}^2(\mathbf{G}/\mathbf{B},\mathcal{L}_{-s_\beta\cdot\chi-2\rho})^*=\mathrm{H}^1(\mathbf{G}/\mathbf{B},\mathcal{L}_{s_\alpha\cdot(-s_\beta\cdot\chi-2\rho)})^*. \tag{4.31}$$

However, the weight $s_\alpha\cdot(-s_\beta\cdot\chi-2\rho)=-(p-4)\omega_\alpha+(p-3)\omega_\beta+(p-5)\alpha\in X_+(\mathbf{T})$ as its labels at $\alpha^\vee$ and $\beta^\vee$ are equal to $(p-6)$ and 2, respectively. Hence $\mathrm{H}^1(\mathbf{G}/\mathbf{B},\mathcal{L}_{s_\alpha\cdot(-s_\beta\cdot\chi-2\rho)})=0$ by the Kempf vanishing, and the statement follows. □

Proposition 4.5. $\mathrm{H}^i(\mathsf{Q}_5,\mathsf{S}^{p-2}\mathcal{K}^*\otimes\mathcal{L}_{-(2p+2)\omega_\alpha})=0$ *for* $i\neq 4$.

Proof. One has $\mathrm{H}^i(\mathsf{Q}_5,\mathsf{S}^{p-2}\mathcal{K}^*\otimes\mathcal{L}_{-(2p+2)\omega_\alpha})=\mathrm{H}^i(\mathbf{G}/\mathbf{B},\mathcal{L}_{p(\omega_\beta-2\omega_\alpha)}\otimes\omega_{\mathbf{G}/\mathbf{B}})$. By Serre duality one has:

$$\mathrm{H}^i(\mathbf{G}/\mathbf{B},,\mathcal{L}_{p(\omega_\beta-2\omega_\alpha)}\otimes\omega_{\mathbf{G}/\mathbf{B}})=\mathrm{H}^{6-i}(\mathbf{G}/\mathbf{B},\mathcal{L}_{p(2\omega_\alpha-\omega_\beta)})^*.$$

The weight in question is $\chi=2p\omega_\alpha-p\omega_\beta$. One checks that $s_\alpha\cdot s_\beta\cdot\chi=\chi+(p-1)\beta+(p-4)\alpha=(p-5)\omega_\alpha+2\omega_\beta\in X_+(\mathbf{T})$, and applying Theorem 2.2 twice one gets the statement.

□

Proposition 4.6. $\mathrm{H}^i(\mathsf{Q}_5,\mathsf{F}^*\Psi_4\otimes\mathsf{S}^p\mathcal{K}^*)=0$ *for* $i>1$.

Proof. Similar to Proposition 4.4, we have the short exact sequence:

$$0 \to \mathcal{O}_{\mathsf{Q}_5}(-1) \to \mathrm{S} \otimes \mathsf{U} \to \Psi_4 \to 0. \tag{4.32}$$

Applying F^* to it, tensoring with $\mathsf{S}^p\mathcal{K}^*$ and using Proposition 4.2, we get the statement. □

Proposition 4.7. $\mathrm{H}^i(\mathsf{Q}_5, \mathsf{F}^*\mathrm{S} \otimes \mathsf{S}^p\mathcal{K}^*) = 0$ *for* $i > 1$.

Proof. From sequence 4.5 we see that the statement will follow from $\mathrm{H}^i(\mathsf{Q}_5, \mathsf{F}^*\mathrm{S}^* \otimes \mathsf{S}^p\mathcal{K}^*) = 0$ for $i > 0$. Tensor the sequence (4.1) with $\mathcal{L}_{\omega_\alpha}$:

$$0 \to \mathcal{K} \otimes \mathcal{L}_{\omega_\alpha} \to \mathcal{T}_{\mathbb{P}(\mathsf{W})} \otimes \mathcal{L}_{-\omega_\alpha} \to \mathrm{S}^* \to 0. \tag{4.33}$$

Consider the Euler sequence restricted to Q_5:

$$0 \to \mathcal{L}_{-\omega_\alpha} \to \mathsf{W} \otimes \mathcal{O}_{\mathsf{Q}_5} \to \mathcal{T}_{\mathbb{P}(\mathsf{W})} \otimes \mathcal{L}_{-\omega_\alpha} \to 0. \tag{4.34}$$

Taking the pullback of this sequence to $\mathbf{G}/\mathbf{B}$, applying F^* to it and finally tensoring with $\mathcal{L}_{\omega_\beta}$, we get:

$$0 \to \mathcal{L}_{p(\omega_\beta-\omega_\alpha)} \to \mathsf{F}^*\mathsf{W} \otimes \mathcal{L}_{p\omega_\beta} \to q^*\mathsf{F}^*\mathcal{T}_{\mathbb{P}(\mathsf{W})} \otimes \mathcal{L}_{p(\omega_\beta-\omega_\alpha)} \to 0. \tag{4.35}$$

Two claims below finish the proof. □

Claim 4.1. $\mathrm{H}^i(\mathbf{G}/\mathbf{B}, \mathcal{L}_{p(\omega_\beta-\omega_\alpha)}) = 0$ *for* $i \neq 1$.

Proof. One has $p(\omega_\beta-\omega_\alpha) = p(\alpha+\beta)$, and $s_\alpha\cdot(p(\alpha+\beta)) = p\omega_\alpha-\alpha \in X_+(\mathbf{T})$. Hence, $s_\alpha\cdot(p\omega_\alpha-\alpha) = p(\alpha+\beta) \in s_\alpha \cdot X_+(\mathbf{T})$, and $\langle p(\alpha+\beta), \alpha^\vee\rangle = -p$. We are done by Theorem 2.2. □

Recall that $\mathcal{K}^* = \mathcal{K} \otimes \mathcal{L}_{3\omega_\alpha}$. Let us show that $\mathrm{H}^i(\mathsf{Q}_5, \mathsf{F}^*(\mathcal{K} \otimes \mathcal{L}_{\omega_\alpha}) \otimes \mathsf{S}^p\mathcal{K}^*) = 0$ for $i > 1$. Given the previous isomorphism, these groups are isomorphic to $\mathrm{H}^i(\mathsf{Q}_5, \mathsf{F}^*(\mathcal{K}^* \otimes \mathcal{L}_{-2\omega_\alpha}) \otimes \mathsf{S}^p\mathcal{K}^*)$. Consider the short exact sequence:

$$0 \to \mathcal{L}_{3p\omega_\alpha} \to \mathsf{F}^*\mathcal{K}^* \otimes \mathsf{S}^p\mathcal{K}^* \to \mathsf{S}^{2p}\mathcal{K}^* \to 0. \tag{4.36}$$

Tensoring it with $\mathcal{L}_{-2p\omega_\alpha}$, we get:

$$0 \to \mathcal{L}_{p\omega_\alpha} \to \mathsf{F}^*(\mathcal{K}^* \otimes \mathcal{L}_{-2\omega_\alpha}) \otimes \mathsf{S}^p\mathcal{K}^* \to \mathsf{S}^{2p}\mathcal{K}^* \otimes \mathcal{L}_{-2p\omega_\alpha} \to 0. \tag{4.37}$$

One has $\mathrm{H}^i(\mathsf{Q}_5, \mathsf{S}^{2p}\mathcal{K}^* \otimes \mathcal{L}_{-2p\omega_\alpha}) = \mathrm{H}^i(\mathbf{G}/\mathbf{B}, \mathcal{L}_{2p(\omega_\beta-\omega_\alpha)})$.

Claim 4.2. $\mathrm{H}^i(\mathbf{G}/\mathbf{B}, \mathcal{L}_{2p(\omega_\beta-\omega_\alpha)}) = 0$ *for* $i \neq 1$.

Proof. Let $\chi = 2p(\omega_\beta - \omega_\alpha)$. Theorem 2.1 shows that $\mathrm{H}^1(\mathbf{G}/\mathbf{B}, \mathcal{L}_\chi) \neq 0$. Indeed, $\langle \chi, \alpha^\vee\rangle = -2p$ and $\chi+p\alpha$ is dominant. On the other hand, $s_\alpha\cdot\chi = \chi+(2p-1)\alpha = (2p-2)\omega_\alpha+\omega_\beta \in X_+(\mathbf{T})$ and, therefore, $s_\alpha \cdot (\chi + (2p-1)\alpha) = \chi \in s_\alpha \cdot X_+(\mathbf{T})$. By [7], the weight χ exhibits standard vanishing, hence the cliam. □

Proposition 4.8. $\mathrm{H}^i(\mathsf{Q}_5, \mathsf{F}^*(\mathrm{S} \otimes \mathcal{L}_{-2\omega_\alpha}) \otimes \mathsf{S}^{p-2}\mathcal{K}^* \otimes \mathcal{L}_{-2\omega_\alpha}) = 0$ *for* $i > 4$.

Proof. We only need to show $\mathrm{H}^5(\mathsf{Q}_5, \mathsf{F}^*\mathrm{S}(-2) \otimes \mathsf{S}^{p-2}\mathcal{K}^*(-2)) = 0$. Tensor the sequence (4.1) with $\mathcal{L}_{-2\omega_\alpha}$:

$$0 \to \mathcal{K} \otimes \mathcal{L}_{-2\omega_\alpha} \to \mathcal{T}_{\mathbb{P}(\mathsf{W})} \otimes \mathcal{L}_{-4\omega_\alpha} \to \mathrm{S} \otimes \mathcal{L}_{-2\omega_\alpha} \to 0. \tag{4.38}$$

Applying F^* and tensoring with $\mathsf{S}^{p-2}\mathcal{K}^*(-2)$, we see that it is sufficient to prove

$$\mathrm{H}^5(\mathsf{Q}_5, \mathsf{F}^*(\mathcal{T}_{\mathbb{P}(\mathsf{W})} \otimes \mathcal{L}_{-4\omega_\alpha}) \otimes \mathsf{S}^{p-2}\mathcal{K}^* \otimes \mathcal{L}_{-2\omega_\alpha}) = 0.$$

Considering the Euler sequence tensored with $\mathcal{L}_{-4\omega_\alpha}$ and apply F^* to it, we obtain:

$$0 \to \mathcal{L}_{-4p\omega_\alpha} \to \mathsf{F}^*\mathsf{W} \otimes \mathcal{L}_{-3p\omega_\alpha} \to \mathsf{F}^*(\mathcal{T}_{\mathbb{P}(\mathsf{W})} \otimes \mathcal{L}_{-4\omega_\alpha}) \to 0. \tag{4.39}$$

Consequently, it is sufficient to show

$$\mathrm{H}^5(\mathsf{Q}_5, \mathcal{L}_{-3p\omega_\alpha} \otimes \mathsf{S}^{p-2}\mathcal{K}^*\mathcal{L}_{-2\omega_\alpha}) = \mathrm{H}^5(\mathbf{G}/\mathbf{B}, \mathcal{L}_{p(\omega_\beta-3\omega_\alpha)} \otimes \omega_{\mathbf{G}/\mathbf{B}}).$$

By Serre duality we have:

$$\mathrm{H}^5(\mathbf{G}/\mathbf{B}, \mathcal{L}_{p(\omega_\beta-3\omega_\alpha)} \otimes \omega_{\mathbf{G}/\mathbf{B}}) = \mathrm{H}^1(\mathbf{G}/\mathbf{B}, \mathcal{L}_{p(3\omega_\alpha-\omega_\alpha)})^*.$$

The corresponding weight is $3p\omega_\alpha - p\omega_\beta$. Using Theorem 2.1, we get:

$$\langle 3p\omega_\alpha - p\omega_\beta, \beta^\vee \rangle = -p.$$

Hence H^1 would not vanish if the weight $s_\beta \cdot (3p\omega_\alpha - p\omega_\beta) = 3p\omega_\alpha - \beta$ was dominant. This is not the case as its label at $\beta^\vee$ is equal to negative 2. This proves the claim.

□

At this stage we have a non-vanishing $\mathrm{H}^5(\mathsf{Q}_5 \times \mathsf{Q}_5, (\mathsf{F}^*\Psi_4 \otimes \mathsf{S}^p\mathcal{K}^*) \boxtimes \mathsf{S}^{p-2}\mathcal{K}^* \otimes (\mathcal{L}_{-(2p+2)\omega_\alpha}))$.

Proposition 4.9. *The map c*

$$\begin{aligned} &\mathrm{H}^1(\mathsf{Q}_5, \mathsf{F}^*\mathrm{S} \otimes \mathsf{S}^p\mathcal{K}^*) \otimes \mathrm{H}^4(\mathsf{Q}_5, \mathsf{F}^*\mathrm{S}(-2) \otimes \mathsf{S}^{p-2}\mathcal{K}^* \otimes \mathcal{L}_{-2\omega_\alpha}) \to \\ &\to \mathrm{H}^1(\mathsf{Q}_5, \mathsf{F}^*\Psi_4 \otimes \mathsf{S}^p\mathcal{K}^*) \otimes \mathrm{H}^4(\mathsf{Q}_5, \mathsf{S}^{p-2}\mathcal{K}^* \otimes \mathcal{L}_{-(2p+2)\omega_\alpha}) \end{aligned} \tag{4.40}$$

is surjective.

Proof. First, Proposition 4.6 implies that the map

$$\mathrm{H}^1(\mathsf{Q}_5, \mathsf{F}^*\mathrm{S} \otimes \mathsf{U} \otimes \mathsf{S}^p\mathcal{K}^*) \to \mathrm{H}^1(\mathsf{Q}_5, \mathsf{F}^*\Psi_4 \otimes \mathsf{S}^p\mathcal{K}^*) \tag{4.41}$$

is surjective. Further, the map c is the composition

$$\mathrm{H}^1(\mathsf{Q}_5, \mathsf{F}^*\mathrm{S} \otimes \mathsf{S}^p\mathcal{K}^*) \otimes \mathrm{H}^4(\mathsf{Q}_5, \mathsf{F}^*\mathrm{S}(-2) \otimes \mathcal{L}_{(p-2)\omega_\beta-2\omega_\alpha}) \to \tag{4.42}$$

$$\to \mathrm{H}^1(\mathsf{Q}_5, \mathsf{F}^*\mathrm{S} \otimes \mathsf{S}^p\mathcal{K}^*) \otimes \mathrm{H}^4(\mathsf{Q}_5, \mathsf{U} \otimes \mathsf{S}^{p-2}\mathcal{K}^* \otimes \mathcal{L}_{-(2p+2)\omega_\alpha}) \simeq \tag{4.43}$$

$$\begin{aligned} &\simeq \mathrm{H}^1(\mathsf{Q}_5, \mathsf{F}^*\mathrm{S} \otimes \mathsf{U} \otimes \mathsf{S}^p\mathcal{K}^*) \otimes \mathrm{H}^4(\mathsf{Q}_5, \mathsf{S}^{p-2}\mathcal{K}^* \otimes \mathcal{L}_{-(2p+2)\omega_\alpha}) \to \\ &\to \mathrm{H}^1(\mathsf{Q}_5, \mathsf{F}^*\Psi_4 \otimes \mathsf{S}^p\mathcal{K}^*) \otimes \mathrm{H}^4(\mathsf{Q}_5, \mathsf{S}^{p-2}\mathcal{K}^* \otimes \mathcal{L}_{-(2p+2)\omega_\alpha}). \end{aligned} \tag{4.44}$$

One has therefore to show that the map

$$\mathrm{H}^4(\mathsf{Q}_5, \mathsf{F}^*\mathrm{S}(-2) \otimes \mathcal{L}_{(p-2)\omega_\beta-2\omega_\alpha}) \to \mathrm{H}^4(\mathsf{Q}_5, \mathsf{U} \otimes \mathsf{S}^{p-2}\mathcal{K}^* \otimes \mathcal{L}_{-(2p+2)\omega_\alpha}) \tag{4.45}$$

is surjective.

Tensoring the sequence (4.5) with $\mathcal{L}_{-2\omega_\alpha}$, applying F^* and finally tensoring with $\mathsf{S}^{p-2}\mathcal{K}^* \otimes \mathcal{L}_{-2\omega_\alpha}$, we see that this map is the one on H^4 from the associated cohomology sequence:

$$\begin{aligned} &\cdots \to \mathrm{H}^4(\mathsf{Q}_5, \mathsf{F}^*\mathrm{S}(-2) \otimes \mathcal{L}_{(p-2)\omega_\beta-2\omega_\alpha}) \to \mathrm{H}^4(\mathsf{Q}_5, \mathsf{U} \otimes \mathsf{S}^{p-2}\mathcal{K}^* \otimes \mathcal{L}_{-(2p+2)\omega_\alpha}) \to \\ &\to \mathrm{H}^4(\mathsf{Q}_5, \mathsf{F}^*\mathrm{S}(-1) \otimes \mathcal{L}_{(p-2)\omega_\beta-2\omega_\alpha}) \to \dots. \end{aligned} \tag{4.46}$$

To prove surjectivity of the map (4.45) it is sufficient to show $\mathrm{H}^4(\mathsf{Q}_5, \mathsf{F}^*\mathrm{S}(-1)\otimes\mathcal{L}_{(p-2)\omega_\beta-2\omega_\alpha}) = 0$. The Koszul complex associated to (4.1) gives:

$$\cdots \to \mathrm{S}^* \otimes \mathsf{S}^{p-3}(\Omega^1_{\mathbb{P}(\mathsf{W})}(2)) \otimes \mathcal{L}_{-2\omega_\alpha} \to \mathsf{S}^{p-2}(\Omega^1_{\mathbb{P}(\mathsf{W})}(2)) \otimes \mathcal{L}_{-2\omega_\alpha} \to \mathsf{S}^{p-2}\mathcal{K}^* \otimes \mathcal{L}_{-2\omega_\alpha} \to 0. \tag{4.47}$$

Further, the $(p-2)$-th symmetric power of the Euler sequence gives:

$$0 \to \mathsf{S}^{p-3}\mathsf{W} \otimes \mathcal{L}_{-(p-1)\omega_\alpha} \to \mathsf{S}^{p-2}\mathsf{W} \otimes \mathcal{L}_{-(p-2)\omega_\alpha} \to \mathsf{S}^{p-2}(\mathcal{T}_{\mathbb{P}(\mathsf{W})}(-2)) \to 0. \tag{4.48}$$

The rest of the proof is broken up into two claims.

Claim 4.3. *One has* $\mathrm{H}^4(\mathsf{Q}_5, \mathsf{S}^{p-2}(\Omega^1_{\mathbb{P}(\mathsf{W})}(2)) \otimes \mathcal{L}_{-2\omega_\alpha} \otimes \mathsf{F}^*\mathrm{S}(-1)) = 0$.

Proof. By Serre duality one obtains:

$$\mathrm{H}^4(\mathsf{Q}_5, \mathsf{S}^{p-2}(\Omega^1_{\mathbb{P}(\mathsf{W})}(2))\otimes\mathcal{L}_{-2\omega_\alpha}\otimes\mathsf{F}^*\mathrm{S}(-1)) = \mathrm{H}^1(\mathsf{Q}_5, \mathsf{S}^{p-2}(\mathcal{T}_{\mathbb{P}(\mathsf{W})}(-2))\otimes\mathsf{F}^*\mathrm{S}^*(1)\otimes\mathcal{L}_{-3\omega_\alpha})^* \tag{4.49}$$

From the sequence (4.48) it is sufficient to show injectivity of the map from the long exact cohomology sequence :

$$\cdots \to \mathrm{H}^2(\mathsf{Q}_5, \mathsf{S}^{p-3}\mathsf{W}^* \otimes \mathsf{F}^*\mathrm{S}^* \otimes \mathcal{L}_{-2\omega_\alpha}) \to \mathrm{H}^2(\mathsf{Q}_5, \mathsf{S}^{p-2}\mathsf{W}^* \otimes \mathsf{F}^*\mathrm{S}^* \otimes \mathcal{L}_{-\omega_\alpha}) \to \dots. \tag{4.50}$$

A small notational disclaimer. To prove injectivity of the above map, we are going to consider a general hyperplane section of Q_5, which is a smooth quadric Q_4 of dimension four, In the previous notation we wrote $\mathcal{L}_{\omega_\alpha}$ identifying it with $\mathcal{O}_{\mathsf{Q}_5}(1)$ to emphasize the fact that all the cohomology computations are eventually done on the flag variety $\mathbf{G}/\mathbf{B}$. Since the computation below appeals only to the quadric Q_5 and its hyperplane section, and involves restrictions of the line bundle $\mathcal{O}_{\mathsf{Q}_5}(1)$ to Q_4, in what follows we return to conventional notation $\mathcal{O}_{\mathsf{Q}_5}(1)$ (hence, assuming that for a vector bundle $\mathcal{E}$ on Q_5 the twist $\mathcal{E}(1)$ is $\mathcal{E}\otimes\mathcal{O}_{\mathsf{Q}_5}(1)$).

Denote $i : \mathsf{Q}_4 \hookrightarrow \mathsf{Q}_5$ the embedding, and consider the short exact sequence:

$$0 \to \mathcal{O}_{\mathsf{Q}_5}(-2) \to \mathcal{O}_{\mathsf{Q}_5}(-1) \to i_*\mathcal{O}_{\mathsf{Q}_4}(-1) \to 0, \tag{4.51}$$

Tensor this sequence with $\mathsf{F}^*\mathrm{S}^*$. Let us first show injectivity of the map on H^2 of the above sequence. This is equivalent to $\mathrm{H}^1(\mathsf{Q}_5, \mathsf{F}^*\mathrm{S}^* \otimes i_*\mathcal{O}_{\mathsf{Q}_4}(-1)) = \mathrm{H}^1(\mathsf{Q}_4, i^*\mathsf{F}^*\mathrm{S}^* \otimes \mathcal{O}_{\mathsf{Q}_4}(-1))$. The restriction of the spinor bundle on Q_5 to Q_4 splits into direct sum of the spinor bundles on Q_4: $i^*\mathrm{S}^* = \mathcal{U}_2^* \oplus \mathsf{V}/\mathcal{U}_2$ (see [20]). By *loc.cit.*, the bundle $\mathsf{F}_*\mathcal{O}_{\mathsf{Q}_4}(-1)$ on Q_4 decomposes as follows:

$$\mathsf{F}_*\mathcal{O}_{\mathsf{Q}_4}(-1) = \mathcal{O}_{\mathsf{Q}_4}(-1)^{\oplus p_1} \oplus \mathcal{O}_{\mathsf{Q}_4}(-2)^{\oplus p_2} \oplus \mathcal{U}_2(-2)^{\oplus s_1} \oplus (\mathsf{V}/\mathcal{U}_2)^*(-2)^{\oplus s_2} \oplus \mathcal{O}_{\mathsf{Q}_4}(-3)^{\oplus p_3}, \tag{4.52}$$

where p_i are certain polynomials depending only on p. One has $\mathrm{H}^k(\mathsf{Q}_4, \mathcal{U}_2^* \otimes \mathcal{O}_{\mathsf{Q}_4}(-i)) = \mathrm{H}^i(\mathsf{Q}_4, (\mathsf{V}/\mathcal{U}_2)\otimes\mathcal{O}_{\mathsf{Q}_4}(-i)) = 0$ for all k and $i = -1, -2, -3$. Therefore one needs to check that the first cohomology group of

$$\mathcal{U}_2 \otimes \mathcal{U}_2(-1),\ \mathcal{U}_2 \otimes (\mathsf{V}/\mathcal{U}_2)^*(-1),\ (\mathsf{V}/\mathcal{U}_2)\otimes\mathcal{U}_2(-2),\ (\mathsf{V}/\mathcal{U}_2)\otimes(\mathsf{V}/\mathcal{U}_2)^*(-2) \tag{4.53}$$

is zero. Using the sequence on Q_4

$$0 \to \mathcal{U}_2 \to \mathsf{V}\otimes\mathcal{O}_{\mathsf{Q}_4} \to \mathsf{V}/\mathcal{U}_2 \to 0, \tag{4.54}$$

and its dual

$$0 \to (\mathsf{V}/\mathcal{U}_2)^* \to \mathsf{V}^*\otimes\mathcal{O}_{\mathsf{Q}_4} \to \mathcal{U}_2^* \to 0, \tag{4.55}$$

we get:

- $\mathrm{H}^\bullet(\mathsf{Q}_4,\mathcal{U}_2\otimes\mathcal{U}_2(-1))=0$ (since $\mathrm{H}^\bullet(\mathsf{Q}_4,\mathsf{S}^2\mathcal{U}_2(-1))=0$).
- $\mathrm{H}^\bullet(\mathsf{Q}_4,\mathcal{U}_2\otimes(\mathsf{V}/\mathcal{U}_2)^*(-1))=0$ (since $\mathrm{H}^\bullet(\mathsf{Q}_4,\mathsf{S}^2\mathcal{U}_2)=0$).
- $\mathrm{H}^\bullet(\mathsf{Q}_4,(\mathsf{V}/\mathcal{U}_2)\otimes\mathcal{U}_2(-2))=0$ (since $\mathrm{H}^\bullet(\mathsf{Q}_4,\mathsf{S}^2\mathcal{U}_2(-2))=0$).
- $\mathrm{H}^i(\mathsf{Q}_4,(\mathsf{V}/\mathcal{U}_2)\otimes(\mathsf{V}/\mathcal{U}_2)^*(-2))=0$ for $i\neq 2$ (the only non-zero cohomology group being isomorphic to $\mathrm{H}^1(\mathsf{Q}_4,\mathcal{T}_{\mathsf{Q}_4}(-2))=k$).

□

Claim 4.4. $\mathrm{H}^5(\mathsf{Q}_5,\mathrm{S}^*\otimes\mathsf{S}^{p-3}(\Omega^1_{\mathbb{P}(\mathsf{W})}(2))\otimes\mathcal{O}_{\mathsf{Q}_5}(-2)\otimes\mathsf{F}^*\mathrm{S}(-1))=0.$

Proof. By Serre duality we have:

$$\begin{aligned}\mathrm{H}^5(\mathsf{Q}_5,\mathrm{S}^*\otimes\mathsf{S}^{p-3}(\Omega^1_{\mathbb{P}(\mathsf{W})}(2))\otimes\mathcal{O}_{\mathsf{Q}_5}(-2)\otimes\mathsf{F}^*\mathrm{S}(-1))&=\\ =\mathrm{H}^0(\mathsf{Q}_5,\mathrm{S}\otimes\mathsf{S}^{p-3}(\mathcal{T}_{\mathbb{P}(\mathsf{W})}(-2))\otimes\mathcal{O}_{\mathsf{Q}_5}(-3)\otimes\mathsf{F}^*\mathrm{S}^*(1))^*.&\end{aligned}\tag{4.56}$$

Take the (4.48) for $(p-3)$:

$$0\to\mathsf{S}^{p-4}\mathsf{W}\otimes\mathcal{O}_{\mathsf{Q}_5}(-p+2)\to\mathsf{S}^{p-3}\mathsf{W}\otimes\mathcal{O}_{\mathsf{Q}_5}(-p+3)\to\mathsf{S}^{p-3}(\mathcal{T}_{\mathbb{P}(\mathsf{W})}(-2))\to 0.\tag{4.57}$$

Let us show that $\mathrm{H}^0(\mathsf{Q}_5,\mathsf{F}^*\mathrm{S}^*\otimes\mathrm{S})=\mathrm{H}^1(\mathsf{Q}_5,\mathsf{F}^*\mathrm{S}^*\otimes\mathrm{S}(-1))=0$. Tesnoring the sequence (4.5) with $\mathcal{O}_{\mathsf{Q}_5}(-1)$ and then tensoring it with $\mathsf{F}^*\mathrm{S}^*(1)$, we get

$$0\to\mathsf{F}^*\mathrm{S}^*\otimes\mathrm{S}(-1)\to\mathsf{U}\otimes\mathsf{F}^*\mathrm{S}^*\otimes\mathcal{O}_{\mathsf{Q}_5}(-1)\to\mathsf{F}^*\mathrm{S}^*\otimes\mathrm{S}\to 0.\tag{4.58}$$

Using the decomposition for $\mathsf{F}_*\mathcal{O}_{\mathsf{Q}_5}(-1)$ one checks that $\mathrm{H}^i(\mathsf{Q}_5,\mathsf{F}^*\mathrm{S}^*\otimes\mathcal{O}_{\mathsf{Q}_5}(-1))=0$ for $i=0,1$. It is sufficient therefore to prove that $\mathrm{H}^0(\mathsf{Q}_5,\mathsf{F}^*\mathrm{S}^*\otimes\mathrm{S})=0$. By Proposition 5.2 of [20] the Frobenius pushforward $\mathsf{F}_*\mathrm{S}$ contains a unique twist $\mathrm{S}(-t)$ of S and the number t must be strictly positive. Further, line bundles in the decomposition of $\mathsf{F}_*\mathcal{O}_{\mathsf{Q}_5}(-1)$ are also negative since S does not have global sections. This shows $\mathrm{H}^0(\mathsf{Q}_5,\mathsf{F}^*\mathrm{S}^*\otimes\mathrm{S})=0$, hence the statement. □

□

4.5. **Derived equivalence.** Let $\mathbf{G}$ be an arbitrary semisimple algebraic group over k. Recall [25, Lemma 14], that if p is greater than the Coxeter number of $\mathbf{G}$, then the bundle $\mathsf{F}_*\mathcal{O}_{\mathbf{G}/\mathbf{B}}$ is tilting. Since the Coxeter number of $\mathbf{G}_2$ is equal to 6, we obtain that $\mathsf{F}_*\mathcal{O}_{\mathbf{G}_2/\mathbf{B}}$ is tilting bundle on the flag variety $\mathbf{G}_2/\mathbf{B}$ for $p=7$. It can be lifted formally to characteristic zero, since deformation theory for almost exceptional objects is trivial (see [9]), and the property of being a generator in a triangulated category is open in families.

5. Homogeneous spaces of $\mathbf{SL}_4$

This section is supposed to further illustrate the relation between non–standard vanishing of cohomology groups of line bundles on flag varieties and higher cohomology vanishing of the endomorphism bundle $\mathcal{E}nd(\mathsf{F}_*\mathcal{O}_{\mathbf{G}/\mathbf{B}})$. We chose a particular example of a homogeneous spaces of the group $\mathbf{SL}_4$ – the variety of partial flags $\mathrm{F}_{1,2,4}$. One easily obtains the vanishing of $\mathrm{H}^i(\mathcal{E}nd(\mathsf{F}_*\mathcal{O}_{\mathrm{F}_{1,2,4}}))$ for $i>1$ as shown below (in fact, similar statement holds as well for the full flag variety $\mathbf{SL}_4/\mathbf{B}$). However, because of non–standard vanishing behaviour (see Remark 5.1 below) the spectral sequence related to the cohomology $\mathrm{H}^i(\mathcal{E}nd(\mathsf{F}_*\mathcal{O}_{\mathrm{F}_{1,2,4}}))$ becomes non–trivial, and vanishing of the first cohomology group $\mathrm{H}^1(\mathcal{E}nd(\mathsf{F}_*\mathcal{O}_{\mathrm{F}_{1,2,4}}))$ requires more work. We discuss approaches to proving the vanishing of $\mathrm{H}^1(\mathcal{E}nd(\mathsf{F}_*\mathcal{O}_{\mathbf{G}_2/\mathbf{B}}))$ in a subsequent paper [26].

5.1. **The partial flag variety** $\mathrm{F}_{1,2,4}$**.** Let $\mathbf{G}$ be a semisimple algebraic group of type $\mathbf{A}_3$. Denote $\omega_1, \omega_2, \omega_3$ the fundamental weights, and let $\alpha_1, \alpha_2, \alpha_3$ be the simple roots. For each simple root α_i let $\mathbf{P}_{\hat{\alpha}_i} \supset \mathbf{B}$ denote the corresponding minimal parabolic subgroup. Homogeneous spaces $\mathbf{G}/\mathbf{P}_{\hat{\alpha}_i}$ can then be identified with varieties of partial flags $0 \subset \mathrm{V}_1 \subset \mathrm{V}_{i-1} \subset \mathrm{V}_{i+1} \subset \mathrm{V}$.

Denote X the partial flag variety $\mathrm{F}_{1,2,4} = \mathbf{G}/\mathbf{P}_{\hat{\alpha}_3}$. It is a $\mathbb{P}^1$–bundle, the projectivization of rank 2 vector bundle $\mathcal{U}_2$ over $\mathrm{Gr}_{2,4}$; denote q the corresponding projection. On the other hand, $\mathrm{F}_{1,2,4}$ is a $\mathbb{P}^2$–bundle over $\mathbb{P}^3$; denote π this projection.

The pullback $q^*\mathcal{U}_2$ to X is an extension of two line bundles:

$$0 \to \mathcal{L}_{-\omega_1} \to q^*\mathcal{U}_2 \to \mathcal{L}_{\omega_1-\omega_2} \to 0. \tag{5.1}$$

Consider the short exact sequence:

$$0 \to q^*\mathsf{F}^n_*\mathcal{O}_{\mathsf{Q}_4} \to \mathsf{F}^n_*\mathcal{O}_X \to q^*(\mathsf{F}^n_*(\mathsf{D}^{p^n-2}\mathcal{U}_2 \otimes \mathcal{L}_{-\omega_2}) \otimes \mathcal{L}_{\omega_2}) \otimes \mathcal{L}_{-\omega_1} \to 0. \tag{5.2}$$

As in [24], one has $\mathrm{E}xt^i(\mathsf{F}^n_*\mathcal{O}_{\mathsf{Q}_4}, \mathsf{F}^n_*\mathcal{O}_{\mathsf{Q}_4}) = 0$ for $i > 0$ and $n \geq 1$.

Proposition 5.1.

$$\mathrm{E}xt^i(q^*(\mathsf{F}^n_*(\mathsf{D}^{p^n-2}\mathcal{U}_2 \otimes \mathcal{L}_{-\omega_2}) \otimes \mathcal{L}_{\omega_2}) \otimes \mathcal{L}_{-\omega_1}, \mathsf{F}^n_*\mathcal{O}_X) = 0 \tag{5.3}$$

for $i > 0$ *and* $n \geq 1$.

There is the universal short exact sequence:

$$0 \to \mathcal{U}_2 \to \mathsf{V} \otimes \mathcal{O}_{\mathrm{Gr}_{2,4}} \to (\mathcal{U}_2^\perp)^* \to 0. \tag{5.4}$$

Recall the resolution of the sheaf $i_*\mathcal{O}_\Delta$ (see [16]):

$$0 \to \mathcal{L}_{-2\omega_2} \boxtimes \mathcal{L}_{-2\omega_2} \to \wedge^3(\mathcal{U}_2 \boxtimes \mathcal{U}_2^\perp) \to \wedge^2(\mathcal{U}_2 \boxtimes \mathcal{U}_2^\perp) \to \mathcal{U}_2 \boxtimes \mathcal{U}_2^\perp \to \mathcal{O}_{\mathrm{Gr}_{2,4}\times\mathrm{Gr}_{2,4}} \to i_*\mathcal{O}_\Delta \to 0. \tag{5.5}$$

There is an isomorphism $\wedge^3(\mathcal{U}_2 \boxtimes \mathcal{U}_2^\perp) = (\mathcal{U}_2^* \boxtimes (\mathsf{V}/\mathcal{U}_2)) \otimes (\mathcal{L}_{-2\omega_2} \boxtimes \mathcal{L}_{-2\omega_2})$, the bundle $\mathcal{U}_2 \boxtimes \mathcal{U}_2^\perp$ being of rank four and its determinant being isomorphic to $\mathcal{L}_{-2\omega_2} \boxtimes \mathcal{L}_{-2\omega_2}$. Taking into account isomorphisms $\mathcal{U}_2 = \mathcal{U}_2^* \otimes \mathcal{L}_{-\omega_2}$ and $\mathcal{U}_2^\perp = \mathsf{V}/\mathcal{U}_2 \otimes \mathcal{L}_{-\omega_2}$, we obtain:

$$\wedge^3(\mathcal{U}_2 \boxtimes \mathcal{U}_2^\perp) = (\mathcal{U}_2 \otimes \mathcal{L}_{-\omega_2}) \boxtimes (\mathcal{U}_2^\perp \otimes \mathcal{L}_{-\omega_2}).$$

Further, by the Cauchy formula, one obtains, for odd p:

$$\wedge^2(\mathcal{U}_2 \boxtimes \mathcal{U}_2^\perp) = \mathcal{L}_{-\omega_2} \boxtimes \mathsf{S}^2\mathcal{U}_2^\perp \oplus \mathsf{S}^2\mathcal{U}_2 \boxtimes \mathcal{L}_{-\omega_2}.$$

For $p = 2$ there is a non–split filtration on $\wedge^2(\mathcal{U}_2 \boxtimes \mathcal{U}_2^\perp)$ with graded factors isomorphic to $\mathcal{L}_{-\omega_2} \boxtimes \mathsf{S}^2\mathcal{U}_2^\perp$ and $\mathsf{S}^2\mathcal{U}_2 \boxtimes \mathcal{L}_{-\omega_2}$.

As in the proof of Lemma 3 of [24] we have:

$$\begin{aligned}\mathrm{E}xt^i(q^*(\mathsf{F}^n_*(\mathsf{D}^{p^n-2}\mathcal{U}_2 \otimes \mathcal{L}_{-\omega_2}) \otimes \mathcal{L}_{\omega_2}) \otimes \mathcal{L}_{-\omega_1}, \mathsf{F}^n_*\mathcal{O}_X) = \\ \mathrm{H}^i(\mathrm{Gr}_{2,4}, \mathsf{F}^*\mathsf{F}_*\mathsf{S}^{p^n}\mathcal{U}_2^* \otimes \mathsf{S}^{p^n-2}\mathcal{U}_2^* \otimes \mathcal{L}_{(3p^n-3)\omega_2}).\end{aligned} \tag{5.6}$$

Claim 5.1. *Let $\mathcal{E}$ and $\mathcal{L}$ be a vector bundle of arbitrary rank and a line bundle, respectively, on a smooth variety X. Then there is an isomorphism:*

$$\mathsf{F}^*\mathsf{F}_*\mathsf{S}^p\mathcal{E}\otimes\mathcal{L}^p = \mathsf{F}^*\mathsf{F}_*\mathsf{S}^p(\mathcal{E}\otimes\mathcal{L}). \tag{5.7}$$

Proof. By the projection formula one obtains:

$$\mathsf{F}^*\mathsf{F}_*\mathsf{S}^p\mathcal{E}\otimes\mathcal{L}^p = \mathsf{F}^*(\mathsf{F}_*\mathsf{S}^p\mathcal{E}\otimes\mathcal{L}) = \mathsf{F}^*(\mathsf{F}_*(\mathsf{S}^p\mathcal{E}\otimes\mathcal{L}^p)) = \mathsf{F}^*\mathsf{F}_*\mathsf{S}^p(\mathcal{E}\otimes\mathcal{L}). \tag{5.8}$$

□

In particular, we obtain:

$$\mathsf{F}^*\mathsf{F}_*\mathcal{O}_X\otimes\mathcal{L}^p = \mathsf{F}^*(\mathsf{F}_*\mathcal{O}_X\otimes\mathcal{L}) = \mathsf{F}^*(\mathsf{F}_*\mathsf{F}^*\mathcal{L}) = \mathsf{F}^*\mathsf{F}_*\mathcal{L}^p. \tag{5.9}$$

Therefore, the cohomology groups from (5.6) are isomorphic to:

$$\mathrm{H}^i(\mathrm{Gr}_{2,4},\mathsf{F}^*\mathsf{F}_*(\mathsf{S}^{p^n}(\mathcal{U}_2^*\otimes\mathcal{L}_{\omega_2}))\otimes\mathsf{S}^{p^n-2}\mathcal{U}_2^*\otimes\mathcal{L}_{(2p^n-3)\omega_2}) \tag{5.10}$$

Applying Lemma (?) we obtain:

$$\begin{aligned} &\mathrm{H}^i(\mathrm{Gr}_{2,4},\mathsf{F}^*\mathsf{F}_*(\mathsf{S}^{p^n}(\mathcal{U}_2^*\otimes\mathcal{L}_{\omega_2}))\otimes\mathsf{S}^{p^n-2}\mathcal{U}_2^*\otimes\mathcal{L}_{(2p^n-3)\omega_2}) = \\ &= \mathrm{H}^i(\mathrm{Gr}_{2,4}\times\mathrm{Gr}_{2,4},(\mathsf{F}^n\times\mathsf{F}^n)^*(i_*\mathcal{O}_\Delta)\otimes(\mathsf{S}^{p^n}((\mathcal{U}_2^*\otimes\mathcal{L}_{\omega_2})))\boxtimes(\mathsf{S}^{p^n-2}\mathcal{U}_2^*\otimes\mathcal{L}_{(2p^n-3)\omega_2})). \end{aligned} \tag{5.11}$$

Apply $\mathsf{F}^{n*}\times\mathsf{F}^{n*}$ to the resolution (5.5). Denote $\mathrm{C}^\bullet$ the complex with the following terms: $\mathrm{C}^j = \wedge^j(\mathsf{F}\times\mathsf{F})^*(\mathcal{U}_2\boxtimes\mathcal{U}_2^\perp)$ for $j=-4,\dots,0$. Then the complex

$$\mathrm{C}^\bullet\otimes\mathsf{S}^{p^n}(\mathcal{U}_2^*\otimes\mathcal{L}_{\omega_2})\boxtimes(\mathsf{S}^{p^n-2}\mathcal{U}_2^*\otimes\mathcal{L}_{(2p^n-3)\omega_2}) \tag{5.12}$$

computes the cohomology group in the right hand side of (5.6).

Claim 5.2. $\mathrm{H}^i(\mathrm{Gr}_{2,4}\times\mathrm{Gr}_{2,4},\mathsf{S}^{p^n}(\mathcal{U}_2^*\otimes\mathcal{L}_{\omega_2})\boxtimes(\mathsf{S}^{p^n-2}\mathcal{U}_2^*\otimes\mathcal{L}_{(2p^n-3)\omega_2}))=0$ *for* $i>0$.

Proof. Immediate. □

Claim 5.3. $\mathrm{H}^i(\mathrm{Gr}_{2,4}\times\mathrm{Gr}_{2,4},\mathrm{C}^1\otimes\mathsf{S}^{p^n}(\mathcal{U}_2^*\otimes\mathcal{L}_{\omega_2})\boxtimes(\mathsf{S}^{p^n-2}\mathcal{U}_2^*\otimes\mathcal{L}_{(2p^n-3)\omega_2}))=0$ *for* $i>1$.

Proof. The statement will follow from:

- $\mathrm{H}^i(\mathrm{Gr}_{2,4},\mathsf{F}^*\mathcal{U}_2\otimes\mathsf{S}^{p^n}(\mathcal{U}_2^*\otimes\mathcal{L}_{\omega_2}))=0$ for $i>0$;
- $\mathrm{H}^i(\mathrm{Gr}_{2,4},\mathsf{F}^*\mathcal{U}_2^\perp\otimes(\mathsf{S}^{p^n-2}\mathcal{U}_2^*\otimes\mathcal{L}_{(2p^n-3)\omega_2}))=0$ for $i>1$.

Indeed, the first group is isomorphic to $\mathrm{H}^i(\mathrm{Gr}_{2,4},\mathsf{F}^*\mathcal{U}_2^*\otimes\mathsf{S}^{p^n}\mathcal{U}_2^*)$ since $\mathcal{U}_2^*=\mathcal{U}_2\otimes\mathcal{L}_{\omega_2}$. We conclude as in [24, Propostion 7].

Further, taking into account an isomorphism $\mathsf{V}/\mathcal{U}_2=\mathcal{U}_2^\perp\otimes\mathcal{L}_{\omega_2}$, we obtain:

$$\mathrm{H}^i(\mathrm{Gr}_{2,4},\mathsf{F}^*\mathcal{U}_2^\perp\otimes(\mathsf{S}^{p^n-2}\mathcal{U}_2^*\otimes\mathcal{L}_{(2p^n-3)\omega_2}))=\mathrm{H}^i(\mathrm{Gr}_{2,4},\mathsf{F}^*(\mathsf{V}/\mathcal{U}_2)\otimes\mathsf{S}^{p^n-2}\mathcal{U}_2^*\otimes\mathcal{L}_{(p^n-3)\omega_2}). \tag{5.13}$$

There is a short exact sequence:

$$0\to\mathcal{L}_{\omega_2-\omega_3}\to q^*(\mathsf{V}/\mathcal{U}_2)\to\mathcal{L}_{\omega_3}\to 0. \tag{5.14}$$

Applying F^* to it, and tensoring with $\mathcal{L}_{(p-2)\omega_1+(p-3)\omega_2}$, we see that the cohomology groups in question are comprised in between those of line bundles $\mathcal{L}_{(p-2)\omega_1+(2p-3)\omega_2-p\omega_3}$ and $\mathcal{L}_{(p-2)\omega_1+(p-3)\omega_2+p\omega_3}$, hence the statement.

□

Claim 5.4. $\mathrm{H}^i(\mathrm{Gr}_{2,4}\times\mathrm{Gr}_{2,4}, \mathrm{C}^2\otimes\mathsf{S}^{p^n}(\mathcal{U}_2^*\otimes\mathcal{L}_{\omega_2})\boxtimes(\mathsf{S}^{p^n-2}\mathcal{U}_2^*\otimes\mathcal{L}_{(2p^n-3)\omega_2})) = 0$ *for* $i>2$.

Proof. It is sufficient to show the following:

- $\mathrm{H}^i(\mathrm{Gr}_{2,4},\mathsf{S}^p\mathcal{U}_2^*)=0$ for $i>0$;
- $\mathrm{H}^i(\mathrm{Gr}_{2,4},\mathsf{F}^*\mathsf{S}^2\mathcal{U}_2^{\perp}\otimes\mathsf{S}^{p^n-2}\mathcal{U}_2^*\otimes\mathcal{L}_{(2p^n-3)\omega_2})$ =0 for $i>2$;
- $\mathrm{H}^i(\mathrm{Gr}_{2,4},\mathsf{F}^*\mathsf{S}^2\mathcal{U}_2\otimes\mathsf{S}^p\mathcal{U}_2^*\otimes\mathcal{L}_{p\omega_2})=0$ for $i>2$;
- $\mathrm{H}^i(\mathrm{Gr}_{2,4},\mathsf{S}^{p^n-2}\mathcal{U}_2^*\otimes\mathcal{L}_{(p^n-3)\omega_2})=0$ for $i>0$.

The first and the last one statements follow immediately. The second group is isomorphic to $\mathrm{H}^i(\mathrm{Gr}_{2,4},\mathsf{F}^*\mathsf{S}^2(\mathsf{V}/\mathcal{U}_2)\otimes\mathsf{S}^{p-2}\mathcal{U}_2^*\otimes\mathcal{L}_{-3\omega_2})$. Consider the second symmetric power of the universal sequence:

$$0\to\mathcal{L}_{-p\omega_2}\to\mathsf{F}^*(\mathcal{U}_2\otimes\mathsf{V})\to\mathsf{F}^*\mathsf{S}^2\mathsf{V}\otimes\mathcal{O}_{\mathrm{Gr}_{2,4}}\to\mathsf{F}^*\mathsf{S}^2(\mathsf{V}/\mathcal{U}_2)\to 0. \tag{5.15}$$

One first obtains $\mathrm{H}^i(\mathrm{Gr}_{2,4},\mathsf{S}^{p-2}\mathcal{U}_2^*\otimes\mathcal{L}_{-3\omega_2})=0$ for $i\neq 2$. Indeed, by the projection formula the latter group is isomorphic to $\mathrm{H}^i(\mathbf{G}/\mathbf{B},\mathcal{L}_{(p^n-2)\omega_1-3\omega_2})$ (recall that $q_*\mathcal{L}_{k\omega_1}=\mathsf{S}^k\mathcal{U}_2^*$ for $k\geq 0$). Considering the weight $\lambda=(p^n-2)\omega_1-3\omega_2$ we see that $s_{\alpha_3}\cdot s_{\alpha_2}\cdot\lambda=(p^n-4)\omega_1$ is dominant, while all the intermediate weights satisfy the conditions of Theorem 2.2. Hence, the only non–vanishing cohomology group is $\mathrm{H}^2(\mathbf{G}/\mathbf{B},\mathcal{L}_{(p^n-2)\omega_1-3\omega_2})=\mathrm{H}^0(\mathbf{G}/\mathbf{B},\mathcal{L}_{(p^n-4)\omega_1})=\nabla_{(p^n-4)\omega_1}$. If $n=1$ it is an irreducible representation of $\mathbf{SL}_4$ and isomorphic to $\mathsf{S}^{p-4}\mathsf{V}$.

We could also argue as follows. Recall that $\omega_X=\mathcal{L}_{-2\omega_1-3\omega_2}$. By Serre duality one has

$$\mathrm{H}^i(X,\mathcal{L}_{(p^n-2)\omega_1-3\omega_2})=\mathrm{H}^{5-i}(X,\mathcal{L}_{-p^n\omega_1}))^*=\mathrm{H}^{5-i}(\mathbb{P}^3,\mathcal{O}_{\mathbb{P}^3}(-p^n))^*=\mathrm{H}^{i-2}(\mathbb{P}^3,\mathcal{O}_{\mathbb{P}^3}(p^n-4)). \tag{5.16}$$

Further, let us show that $\mathrm{H}^4(\mathrm{Gr}_{2,4},\mathsf{F}^*\mathcal{U}_2\otimes\mathsf{S}^{p-2}\mathcal{U}_2^*\otimes\mathcal{L}_{-3\omega_2})=0$. By Serre duality one has: $\mathrm{H}^4(\mathrm{Gr}_{2,4},\mathsf{F}^*\mathcal{U}_2\otimes\mathsf{S}^{p-2}\mathcal{U}_2^*\otimes\mathcal{L}_{-3\omega_2})=\mathrm{H}^0(\mathrm{Gr}_{2,4},\mathsf{F}^*\mathcal{U}_2^*\otimes\mathsf{S}^{p-2}\mathcal{U}_2\otimes\mathcal{L}_{-\omega_2})$. The bundle in question is an extention of two line bundle corresponding to non–dominant weights, hence the statement.

Similarly, for the third group consider the second wedge power of the universal sequence:

$$0\to\mathsf{F}^*\mathsf{S}^2\mathcal{U}_2\to\mathsf{F}^*\mathcal{U}_2\otimes\mathsf{V}\to\mathsf{F}^*\wedge^2\mathsf{V}\otimes\mathcal{O}\to\mathcal{L}_{p\omega_2}\to 0. \tag{5.17}$$

Tensoring it with $\mathsf{S}^p\mathcal{U}_2^*\otimes\mathcal{L}_{p\omega_2}$ we see that all the terms of this right resolution of $\mathsf{F}^*\mathsf{S}^2\mathcal{U}_2$ have vanishing higher cohomology, hence the statement. □

Claim 5.5. $\mathrm{H}^i(\mathrm{Gr}_{2,4}\times\mathrm{Gr}_{2,4}, \mathrm{C}^3\otimes\mathsf{S}^{p^n}(\mathcal{U}_2^*\otimes\mathcal{L}_{\omega_2})\boxtimes(\mathsf{S}^{p^n-2}\mathcal{U}_2^*\otimes\mathcal{L}_{(2p^n-3)\omega_2})) = 0$ *for* $i>4$.

Proof. This in turn will follow from:

- $\mathrm{H}^i(\mathrm{Gr}_{2,4},\mathsf{F}^*\mathcal{U}_2\otimes\mathsf{S}^p\mathcal{U}_2^*)=0$ for $i\neq 2$;
- $\mathrm{H}^i(\mathrm{Gr}_{2,4},\mathsf{F}^*\mathcal{U}_2^{\perp}\otimes\mathsf{S}^{p^n-2}\mathcal{U}_2^*\otimes\mathcal{L}_{(p^n-3)\omega_2}))=0$ for $i>2$.

Indeed, apllying F^* to sequence (5.1) and tensoring it with $\mathcal{L}_{p\omega_1}$ we get:

$$0\to\mathcal{O}_X\to q^*\mathsf{F}^*\mathcal{U}_2\otimes\mathcal{L}_{p\omega_1}\to\mathcal{L}_{p(2\omega_1-\omega_2)}\to 0. \tag{5.18}$$

The cohomology groups of the bundle in the middle are precisely those of the bundle $\mathsf{F}^*\mathcal{U}_2\otimes\mathsf{S}^p\mathcal{U}_2^*$. One has $\mathrm{H}^2(X,\mathcal{L}_{p(2\omega_1-\omega_2)})=\mathrm{H}^0(X,\mathcal{L}_{(p+1)\omega_1+(p-3)\omega_3})$. Hence, $\mathrm{H}^2(\mathrm{Gr}_{2,4},\mathsf{F}^*\mathcal{U}_2\otimes\mathsf{S}^p\mathcal{U}_2^*)=\nabla_{(p+1)\omega_1+(p-3)\omega_3}$.

The interesting group here is $\mathrm{H}^2(\mathrm{Gr}_{2,4}, \mathsf{F}^*\mathcal{U}_2^{\perp} \otimes \mathsf{S}^{p^n-2}\mathcal{U}_2^* \otimes \mathcal{L}_{(p^n-3)\omega_2})$, which we will show to be isomorphic to k (see Lemma 5.1 below). Observe first an isomorphism:

$$\mathsf{F}^*\mathcal{U}_2^{\perp} \otimes \mathsf{S}^{p-2}\mathcal{U}_2^* \otimes \mathcal{L}_{(p-3)\omega_2} = \mathsf{F}^*(\mathsf{V}/\mathcal{U}_2) \otimes \mathsf{S}^{p-2}\mathcal{U}_2^* \otimes \mathcal{L}_{-3\omega_2}). \tag{5.19}$$

By Serre duality one obtains:

$$\mathrm{H}^i(\mathrm{Gr}_{2,4}, \mathsf{F}^*\mathcal{U}_2 \otimes \mathsf{S}^{p-2}\mathcal{U}_2^* \otimes \mathcal{L}_{-3\omega_2}) = \mathrm{H}^{4-i}(\mathrm{Gr}_{2,4}, \mathsf{F}^*\mathcal{U}_2^* \otimes \mathsf{S}^{p-2}\mathcal{U}_2 \otimes \mathcal{L}_{-\omega_2})^*. \tag{5.20}$$

The bundle $\mathsf{F}^*\mathcal{U}_2^* \otimes \mathsf{S}^{p-2}\mathcal{U}_2 \otimes \mathcal{L}_{-\omega_2}$ is isomorphic to $\mathsf{F}^*\mathcal{U}_2^* \otimes \mathsf{S}^{p-2}\mathcal{U}_2^* \otimes \mathcal{L}_{-(p-1)\omega_2}$. Tensoring the short exact sequence

$$0 \to \mathcal{L}_{p(\omega_2-\omega_1)} \to q^*\mathsf{F}^*\mathcal{U}_2^* \to \mathcal{L}_{p\omega_1} \to 0 \tag{5.21}$$

with $\mathcal{L}_{(p-2)\omega_1-(p-1)\omega_2}$ we see that the cohomology groups in question are those in the middle of the above sequence. At the ends we get the following line bundles: $\mathcal{L}_{-2\omega_1+\omega_2}$ and $\mathcal{L}_{(2p-2)\omega_1-(p-1)\omega_2}$. The first bundle has non–vanishing H^1 as $\langle -2\omega_1+\omega_2+\alpha_1, \alpha_i^{\vee}\rangle = 0$ for all i. On the other hand, the only non–vanishing cohomology group of $\mathcal{L}_{(2p-2)\omega_1-(p-1)\omega_2}$ is

$$\mathrm{H}^2(\mathbf{SL_4}/\mathbf{B}, \mathcal{L}_{(2p-2)\omega_1-(p-1)\omega_2}) = \mathrm{H}^0(\mathbf{SL_4}/\mathbf{B}, \mathcal{L}_{p\omega_1+(p-4)\omega_2}) = \nabla_{p\omega_1+(p-4)\omega_3}. \tag{5.22}$$

We finally obtain:

$$\begin{aligned} 0 \to \mathrm{H}^1(\mathcal{L}_{-2\omega_1+\omega_2}) = k \to \mathrm{H}^3(\mathrm{Gr}_{2,4}, \mathsf{F}^*\mathcal{U}_2 \otimes \mathsf{S}^{p-2}\mathcal{U}_2^* \otimes \mathcal{L}_{-3\omega_2})^* \to 0 \to 0 \to \\ \to \mathrm{H}^2(\mathrm{Gr}_{2,4}, \mathsf{F}^*\mathcal{U}_2 \otimes \mathsf{S}^{p-2}\mathcal{U}_2^* \otimes \mathcal{L}_{-3\omega_2})^* \to \nabla_{p\omega_1+(p-4)\omega_3} \to 0. \end{aligned} \tag{5.23}$$

□

Remark 5.1. The following calculation elucidates non–vanishing nature of $\mathrm{H}^1(\mathcal{L}_{-p\alpha})$ for a non–dominant simple root α. One has:

$$\mathrm{H}^i(\mathrm{Gr}_{2,4}, \mathsf{F}^*\mathcal{U}_2 \otimes \mathsf{S}^{p-2}\mathcal{U}_2^* \otimes \mathcal{L}_{-3\omega_2}) = \mathrm{H}^i(X, \mathsf{F}^*q^*\mathcal{U}_2 \otimes \mathcal{L}_{p\omega_1} \otimes \omega_X). \tag{5.24}$$

By Serre duality on X:

$$\mathrm{H}^i(X, \mathsf{F}^*q^*\mathcal{U}_2 \otimes \mathcal{L}_{p\omega_1} \otimes \omega_X) = \mathrm{H}^{5-i}(X, \mathsf{F}^*q^*\mathcal{U}_2^* \otimes \mathcal{L}_{-p\omega_1})^*. \tag{5.25}$$

From sequence (5.21) we obtain:

$$0 \to \mathcal{L}_{p(\omega_2-2\omega_1)} \to q^*\mathsf{F}^*\mathcal{U}_2^* \otimes \mathcal{L}_{-p\omega_1} \to \mathcal{O}_X \to 0. \tag{5.26}$$

We have $\omega_2 - 2\omega_1 = -\alpha_1$. This group has a standard non–vanishing cohomology group in degree 3: $\mathrm{H}^3(\mathbf{SL}_4/\mathbf{B}, \mathcal{L}_{-p\alpha_1}) = \nabla_{p\omega_1+(p-4)\omega_3}$. On the other hand, it always has non–vanishing first cohomology group (see [11]), which is isomorphic to k. We thus obtain:

$$\begin{aligned} 0 \to \mathrm{H}^0(X, \mathcal{O}_X) \simeq \mathrm{H}^1(\mathcal{L}_{-p\alpha_1}) \to 0 \to 0 \to \mathrm{H}^2(\mathcal{L}_{-p\alpha_1}) \to \mathrm{H}^2(X, \mathsf{F}^*q^*\mathcal{U}_2^* \otimes \mathcal{L}_{-p\omega_1}) \to 0 \\ \to \mathrm{H}^3(\mathbf{SL}_4/\mathbf{B}, \mathcal{L}_{-p\alpha_1}) = \nabla_{p\omega_1+(p-4)\omega_3} \simeq \mathrm{H}^2(\mathrm{Gr}_{2,4}, \mathsf{F}^*\mathcal{U}_2 \otimes \mathsf{S}^{p-2}\mathcal{U}_2^* \otimes \mathcal{L}_{-3\omega_2}) \to 0. \end{aligned} \tag{5.27}$$

The first coboundary map should be an isomorphism since otherwise we would have a non–zero $\mathrm{H}^4(\mathrm{Gr}_{2,4}, \mathsf{F}^*\mathcal{U}_2 \otimes \mathsf{S}^{p-2}\mathcal{U}_2^* \otimes \mathcal{L}_{-3\omega_2}))$, which is actually trivial. The second cohomology group $\mathrm{H}^2(\mathcal{L}_{-p\alpha_1})$ survives to compensate for the non–vanishing of $\mathrm{H}^1(\mathcal{L}_{-p\alpha_1})$, the Euler characteristic of $\mathcal{L}_{-p\alpha_1}$ being preserved.

Claim 5.6. $\mathrm{H}^i(\mathrm{Gr}_{2,4}\times\mathrm{Gr}_{2,4},\mathrm{C}^4\otimes\mathsf{S}^{p^n}(\mathcal{U}_2^*\otimes\mathcal{L}_{\omega_2})\boxtimes(\mathsf{S}^{p^n-2}\mathcal{U}_2^*\otimes\mathcal{L}_{(2p^n-3)\omega_2}))=0$ *for* $i>4$.

Proof. Let us show that

- $\mathrm{H}^i(\mathrm{Gr}_{2,4},\mathcal{L}_{-p\omega_2}\otimes\mathsf{S}^p\mathcal{U}_2^*)=0$ for $i\neq 2$;
- $\mathrm{H}^i(\mathrm{Gr}_{2,4},\mathsf{S}^{p-2}\mathcal{U}_2^*\otimes\mathcal{L}_{-3\omega_2})=0$ for $i\neq 2$.

□

Lemma 5.1. $\mathrm{H}^2(\mathrm{Gr}_{2,4},\mathsf{F}^*(\mathsf{V}/\mathcal{U}_2)\otimes\mathsf{S}^{p-2}\mathcal{U}_2^*\otimes\mathcal{L}_{-3\omega_2})=k$.

Proof. We compute the group in question by two slightly different ways. First, by the projection formula one obtains:

$$\mathrm{H}^i(\mathrm{Gr}_{2,4},\mathsf{F}^*(\mathsf{V}/\mathcal{U}_2)\otimes\mathsf{S}^{p-2}\mathcal{U}_2^*\otimes\mathcal{L}_{-3\omega_2})=\mathrm{H}^i(X,q^*(\mathsf{F}^*(\mathsf{V}/\mathcal{U}_2))\otimes\mathcal{L}_{-3\omega_2}\otimes\mathcal{L}_{(p-2)\omega_1}).$$

Applying F^* to it, and tensoring with $\mathcal{L}_{(p-2)\omega_1-3\omega_2}$, we obtain:

$$0\to\mathcal{L}_{p(\omega_2-\omega_3)+(p-2)\omega_1-3\omega_2}\to q^*\mathsf{F}^*(\mathsf{V}/\mathcal{U}_2)\otimes\mathcal{L}_{-3\omega_2}\otimes\mathcal{L}_{(p-2)\omega_1}\to\mathcal{L}_{p\omega_3+(p-2)\omega_1-3\omega_2}\to 0. \tag{5.28}$$

One finds $\mathrm{H}^1(\mathbf{SL}_4/\mathbf{B},\mathcal{L}_{p\omega_3+(p-2)\omega_1-3\omega_2})=\nabla_{(p-4)\omega+1+\omega_2+(p-2)\omega_3}$. On the other hand, one finds $\mathrm{H}^2(\mathbf{SL}_4/\mathbf{B},\mathcal{L}_{p(\omega_2-\omega_3)+(p-2)\omega_1-3\omega_2})=\nabla_{(p-3)(\omega_1+\omega_3)}$. Finally, the sought–for group fits into a short exact sequence:

$$\nabla_{(p-4)\omega_1+\omega_2+(p-2)\omega_3}\to\nabla_{(p-3)(\omega_1+\omega_3)}\to\mathrm{H}^2(\mathrm{Gr}_{2,4},\mathsf{F}^*(\mathsf{V}/\mathcal{U}_2)\otimes\mathsf{S}^{p-2}\mathcal{U}_2^*\otimes\mathcal{L}_{-3\omega_2})\to \tag{5.29}$$

On the other hand, from the universal exact sequence we see that our group fits as well into a short exact sequence:

$$\begin{aligned}\nabla_{p\omega_1+(p-4)\omega_3}\to\mathsf{F}^*\mathsf{V}\otimes\nabla_{(p-4)\omega_1}\to\mathrm{H}^2(\mathrm{Gr}_{2,4},\mathsf{F}^*(\mathsf{V}/\mathcal{U}_2)\otimes\mathsf{S}^{p-2}\mathcal{U}_2^*\otimes\mathcal{L}_{-3\omega_2})\to\\ \to\mathrm{H}^3(\mathrm{Gr}_{2,4},\mathsf{F}^*\mathcal{U}_2\otimes\mathsf{S}^{p-2}\mathcal{U}_2^*\otimes\mathcal{L}_{-3\omega_2})\simeq k\to 0.\end{aligned} \tag{5.30}$$

It follows that the group $\mathrm{H}^2(\mathrm{Gr}_{2,4},\mathsf{F}^*(\mathsf{V}/\mathcal{U}_2)\otimes\mathsf{S}^{p-2}\mathcal{U}_2^*\otimes\mathcal{L}_{-3\omega_2})$ fits into a short exact sequence, the module $\nabla_{(p-4)\omega_1}$ being irreducible:

$$0\to\oplus\nabla_{(p-4)\omega_1}\to\mathrm{H}^2(\mathrm{Gr}_{2,4},\mathsf{F}^*(\mathsf{V}/\mathcal{U}_2)\otimes\mathsf{S}^{p-2}\mathcal{U}_2^*\otimes\mathcal{L}_{-3\omega_2})\to k\to 0. \tag{5.31}$$

On the other hand, from sequence (5.29) we see that the module $\nabla_{(p-3)(\omega_1+\omega_3)}$ surjects onto our $\mathrm{H}^2(\mathrm{Gr}_{2,4},\mathsf{F}^*(\mathsf{V}/\mathcal{U}_2)\otimes\mathsf{S}^{p-2}\mathcal{U}_2^*\otimes\mathcal{L}_{-3\omega_2})$. However, $\mathcal{H}om_{\mathbf{SL}_4}(\nabla_{(p-3)(\omega_1+\omega_3)},\nabla_{(p-4)\omega_1})=0$.

A theorem of Carter and Lusztig [10] says that if there is a non–zero homomorphism between two Weyl modules $\nabla_{\lambda_1},\nabla_{\lambda_2}$ that correspond to partitions λ_1,λ_2, then $\lambda_1\leq\lambda_2$ in a standard partial ordering. Namely, $\lambda_1\leq\lambda_2$ means that one can obtain the partition diagram of λ_1 from that of λ_2 by a sequence of steps, at each step getting a new partition diagram by raising a square from the end of one row to the end of another. Now, the weight $(p-3)(\omega_1+\omega_3)$ corresponds to partition $(2p-6,p-3,p-3,0)$, while the corresponding partition for the weight $(p-4)\omega_1$ is $(p-4,0,0,0)$. We conclude that the space of homomorphisms is zero, hence the statement.

We finally obtain a map:

$$\mathrm{H}^2(\mathrm{Gr}_{2,4},\mathcal{L}_{-p\omega_2}\otimes\mathsf{S}^p\mathcal{U}_2^*)\otimes\mathrm{H}^2(\mathrm{Gr}_{2,4},\mathsf{S}^{p-2}\mathcal{U}_2^*\otimes\mathcal{L}_{-3\omega_2})\to \tag{5.32}$$

$$\to\mathrm{H}^2(\mathrm{Gr}_{2,4},\mathsf{F}^*\mathcal{U}_2\otimes\mathsf{S}^p\mathcal{U}_2^*)\otimes\mathrm{H}^2(\mathrm{Gr}_{2,4},\mathsf{F}^*\mathcal{U}_2^{\perp}\otimes\mathsf{S}^{p^n-2}\mathcal{U}_2^*\otimes\mathcal{L}_{(p^n-3)\omega_2})), \tag{5.33}$$

or, taking into account the above identifications,

$$\nabla_{\omega_1+(p-3)\omega_3} \otimes \nabla_{(p-4)\omega_1} \to \nabla_{(p+1)\omega_1+(p-3)\omega_3} \otimes k. \tag{5.34}$$

□

Finally, denote C the cokernel of the map from (5.34), we obtain:

$$\mathcal{H}om(\mathsf{F}_*\mathcal{O}_X, \mathsf{F}_*\mathcal{O}_X) \to \mathcal{H}om(\mathsf{F}_*\mathcal{O}_{\mathrm{Gr}_{2,4}}, \mathsf{F}_*\mathcal{O}_{\mathrm{Gr}_{2,4}}) \to \mathsf{C} \to \mathcal{E}xt^1(\mathsf{F}_*\mathcal{O}_X, \mathsf{F}_*\mathcal{O}_X) \to 0, \tag{5.35}$$

and all the higher $\mathcal{E}xt^i(\mathsf{F}_*\mathcal{O}_X, \mathsf{F}_*\mathcal{O}_X)$ vanish.

6. **Truncated symmetric powers**

If one tries to prove cohomology vanishing for a single sheaf and there is a deformation of this sheaf at one's disposal, then one may try to compute the cohomology on the special fiber of the family. Provided that cohomology vanishing holds on the special fiber, the vanishing of cohomology for close fibers can be obtained from semicontinuity.

Let X be a scheme, and $\mathcal{E}$ a locally free sheaf over X of rank n. Consider the symmetric algebra $\mathsf{S}^\bullet(\mathcal{E})$, and let Spec $\mathsf{S}^\bullet(\mathcal{E}) =: S(\mathcal{E})$ be the cone defined by $\mathcal{E}$. The symmetric algebra is then the sheaf of functions on $S(\mathcal{E})$ polynomial along the fibers of the projection $\pi : S(\mathcal{E}) \to X$; in other words, $\pi_*\mathcal{O}_{S(\mathcal{E})} = \mathsf{S}^\bullet(\mathcal{E})$.

We have the zero section embedding $s : X \to S(\mathcal{E})$ splitting the projection π. Let $\mathcal{I}$ be the sheaf of ideals of X inside $S(\mathcal{E})$. If $e_1, e_2, \dots, e_n$ is a local basis of $\mathcal{E}$, then $\mathcal{I}$ is locally given by the ideal $(e_1, e_2, \dots, e_n)$. One has $\mathcal{O}_X = \mathcal{O}_{S(\mathcal{E})}/\mathcal{I}$.

Consider the Frobenius power $\mathcal{I}^{[p]}$ of $\mathcal{I}$; locally it is generated by $(e_1^p, e_2^p, \dots, e_n^p)$. The subscheme given by the sheaf of ideals $\mathcal{I}^{[p]}$ is called the *Frobenius neighborhood* of X; one has $\mathcal{O}_{X^{[p]}} = \mathcal{O}_{S(\mathcal{E})}/\mathcal{I}^{[p]} = \mathsf{F}^*s_*\mathcal{O}_X$, and $X \subset X^{[p]} \subset S(\mathcal{E})$ is an embedding of schemes.

Definition 6.1. *Put* $\tau(\mathcal{E}) := \pi_*\mathcal{O}_{X^{[p]}}$.

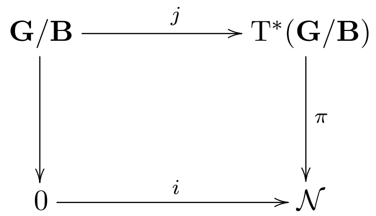
Since X is smooth, the Frobenius morphism is flat, and computation in local coordinates shows that $\tau(\mathcal{E})$ is locally free over $\mathcal{O}_X$ of the rank p^n (in a local basis $(e_1, e_2, \dots, e_n)$ as above the fiber of $\tau(\mathcal{E})$ is $\mathsf{S}^\bullet(\mathcal{E})/(e_1^p, e_2^p, \dots, e_n^p)$, also called the `algebra of truncated symmetric powers`).

6.0.1. *More on truncated symmetric powers.* Let V be a vector space over k. The truncated symmetric power functor $\mathbb{T}^k\mathsf{V}$ can also be defined as the image of the canonical map:

$$\mathbb{T}^k\mathsf{V} = \mathrm{Im}\ (\mathsf{S}^k\mathsf{V} \to \mathsf{D}^k\mathsf{V}). \tag{6.1}$$

Here $\bigoplus \mathsf{D}^k\mathsf{V}$ is the divided powers algebra. The notion of truncated symmetric powers can obviously be sheafified. Thus, for a scheme X and a vector bundle $\mathcal{E}$ over X one has the sheaf of truncated symmetric powers $\mathbb{T}^\bullet\mathcal{E}$ of $\mathcal{E}$.

Consider the Grothendieck–Springer resolution:



Remark 6.1. In [26] we show that Lemma 4.1, and hence the vanishing of $\mathrm{H}^1(\mathbf{G}_2/\mathbf{B}, \mathcal{E}nd(\mathsf{F}_*\mathcal{O}_{\mathbf{G}_2/\mathbf{B}}))$ follow from $\mathrm{H}^1(\mathbf{G}/\mathbf{B}, \mathbb{T}^\bullet\mathcal{T}_{\mathbf{G}_2/\mathbf{B}}) = 0$.

Example 6.1. Let X be a point and $\mathcal{E}$ is a locally free sheaf of rank 1 over X. The cone defined by $\mathcal{E}$ is then isomorphic to $\mathbb{A}^1 = \mathrm{Spec}\ k[x]$. The sheaf $\mathcal{O}_X$ is the skyscraper sheaf $\delta_0 = k[x]/(x)$, and $\pi_*\mathcal{O}_{X^{[p]}}$ is given by the k-module $k[x]/(x^p)$. It is a finite-dimensional vector space over k with a basis $1, x, \dots, x^{p-1}$.

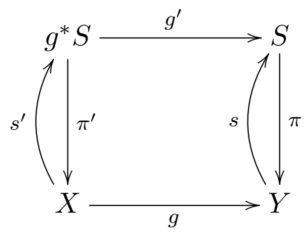
Example 6.2. Let X be a scheme and $\mathcal{L}$ a line bundle over X. Then X is a divisor in $S(\mathcal{L})$, and $X^{[p]}$ is given by the sheaf of ideals $\mathcal{I}^p$; in this case the Frobenius neighborhood coincides with the p-th infinitesimal neighborhood. The sheaf $\mathcal{O}_X/\mathcal{I}^p$ has a filtration with graded factors isomorphic to $\mathcal{I}^k/\mathcal{I}^{k+1} = \mathsf{S}^k(\mathcal{I}/\mathcal{I}^2)$ for $0 \leq k \leq p-1$. Note that $\mathcal{I}/\mathcal{I}^2$ is the conormal sheaf to X in $S(\mathcal{L})$; it is isomorphic to $\mathcal{L}$. Hence, in $\mathsf{K}^0(X)$ one has

$$[\tau(\mathcal{L})] = [\mathcal{O}_X] + [\mathcal{L}] + \cdots + [\mathcal{L}^{p-1}].$$

Claim 6.1. *Let $g : X \to Y$ be a morphism of smooth schemes, and $\mathcal{E}$ a locally free sheaf on Y. Then*

$$g^*(\tau(\mathcal{E})) \simeq \tau(g^*(\mathcal{E})).$$

Proof. Let S be the cone defined by $\mathcal{E}$. Consider the diagram:



Here $s : Y \to S$ and $s' : X \to g^*S$ are the zero section embeddings. Using the base change along the above diagram, and commutativity of the Frobenius morphism with arbitrary morphisms, one has:

$$\mathrm{L}g^*\pi_*\mathsf{F}^*s_*\mathcal{O}_Y = \pi'_*\mathrm{L}g'^*\mathsf{F}^*s_*\mathcal{O}_Y \tag{6.2}$$

Applying the functor $\mathcal{H}^0 = \tau_{\leq 0}\tau_{\geq 0}$ to the above isomorphism, we get:

$$g^*\pi_*\mathsf{F}^*s_*\mathcal{O}_Y = \pi'_*g'^*\mathsf{F}^*s_*\mathcal{O}_Y. \tag{6.3}$$

Therefore,

$$g^*\tau(\mathcal{E}) = g^*\pi_*\mathsf{F}^*s_*\mathcal{O}_Y = \pi'_*g'^*\mathsf{F}^*s_*\mathcal{O}_Y = \pi'_*\mathsf{F}^*g'^*s_*\mathcal{O}_Y. \tag{6.4}$$

On the other hand,

$$\tau(g^*\mathcal{E}) = \pi'_*\mathsf{F}^*s'_*\mathcal{O}_X,$$

and the statement follows from an isomorphism $g'^*s_*\mathcal{O}_Y = s'_*\mathcal{O}_X$ (for instance, one can use the Koszul resolution $\bigwedge^\bullet(\pi^*\mathcal{E}) = s_*\mathcal{O}_Y$, and

$$g'^*\overset{\bullet}{\bigwedge}(\pi^*\mathcal{E}) = \overset{\bullet}{\bigwedge}(g'^*\pi^*\mathcal{E}) = \overset{\bullet}{\bigwedge}(\pi'^*g^*\mathcal{E}) = s'_*\mathcal{O}_X.$$

□

Remark 6.2. Proposition 2.6 of [22] states the equality $\tau(\mathcal{E}) = \theta^p(\mathcal{E})$ in $\mathsf{K}^0(X)$. Here $\theta^p(\mathcal{E})$ is the Bott element, which shows up in the Adams–Riemann-Roch formula. Proposition 3.2 of *loc.cit.* allows to deduce the Adams–Riemann-Roch theorem.

7. Frobenius neighborhood of the diagonal

In this section we show that Proposition 3.2 from [22] is in fact just yet another instance of deformation to the normal cone. For the purposes of computing of cohomology groups using degeneration, it seems instructive to give a proof of this statement that would highlight the degeneration construction.

Proposition 7.1. *Let* $x \in \mathsf{K}^0(X)$. *Then* $\mathsf{F}^*\mathsf{F}_*(x) = \tau(\Omega^1_X)\cdot x$.

Proof. Let $\Delta \subset X \times X$ be the diagonal, and p_1, p_2 the two projections. First, one has:

Claim 7.1.

$$\mathsf{F}^*\mathsf{F}_*(x) = p_{1*}([\mathsf{Fr}(\Delta)]\cdot p_2^*(x)) = p_{2*}([\mathsf{Fr}(\Delta)]\cdot p_1^*(x)), \tag{7.1}$$

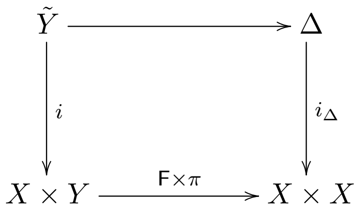
where $[\mathsf{Fr}(\Delta)]$ *is the class of the structure sheaf of the Frobenius neighborhood of* Δ.

Proof. This claim follows from Lemma 2.1 of [23]. For convenience of the reader we recall the proof.

Let $\pi : Y \to X$ be a morphism. Consider the cartesian square:

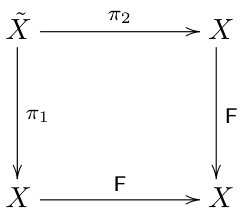
$$\begin{CD} \tilde{Y} @>{p_2}>> Y \\ @V{p_1}VV @VV{\pi}V \\ X @>{\mathsf{F}}>> X \end{CD}$$

We first observe that the fibered product $\tilde{Y}$ is isomorphic to the left uppermost corner in the cartesian square:



where Δ is the diagonal in $X \times X$. If π is flat then one has an isomorphism of sheaves $i_*\mathcal{O}_{\tilde{Y}} = (\mathsf{F} \times \pi)^*(i_{\Delta *}\mathcal{O}_\Delta)$. Indeed, the isomorphism of two fibered products follows from the definition of fibered product. The isomorphism of sheaves follows from flatness of the Frobenius morphism and from flat base change.

In particular, consider the cartesian square:



Denote $\tilde{i} : \tilde{X} \to X \times X$ the embedding. Then:

$$\tilde{i}_*\mathcal{O}_{\tilde{X}} = (\mathsf{F} \times \mathsf{F})^*(i_{\Delta *}\mathcal{O}_\Delta) = \mathsf{Fr}(\Delta). \tag{7.2}$$

By flat base change one gets an isomorphism of functors, the Frobenius morphism F being flat:

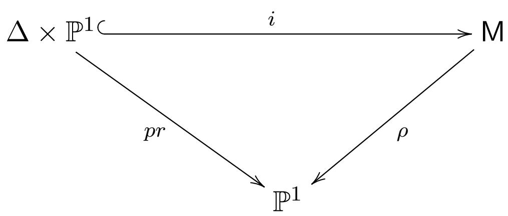
$$\mathsf{F}^*\mathsf{F}_* = \pi_{1*}{\pi_2}^*. \tag{7.3}$$

Note that all the functors $\mathsf{F}_*, \mathsf{F}^*, \pi_{1*}, {\pi_2}^*$ are exact, the Frobenius morphism F being affine. One has $\pi_1 = p_1 \circ \tilde{i}$, $\pi_2 = p_2 \circ \tilde{i}$. Hence, for $x \in \mathsf{K}^0(X)$, one has:

$$\mathsf{F}^*\mathsf{F}_*(x) = p_{1*}(p_2^*(x) \cdot [\tilde{i}_*\mathcal{O}_{\tilde{X}}]) = p_{1*}(p_2^*(x) \cdot [\mathsf{Fr}(\Delta)]), \tag{7.4}$$

the other isomorphism following from symmetry. □

Consider deformation of the diagonal embedding $\Delta \subset X \times X$ to the normal cone ([12], Chapter 5):



For any value of the parameter $t \neq \infty \in \mathbb{P}^1$, the restriction i_t of the embedding i to $\Delta \times t$ is isomorphic to the diagonal embedding $\Delta \times t \subset X \times X$, while the restriction i_∞ of the embedding

i to $pr^{-1}(\infty)$ is isomorphic to the embedding of X (identified with Δ) to the normal cone, that is the cone defined by the sheaf of Kähler differentials Ω^1_X.

Consider the structure sheaf $i_*\mathcal{O}_{\Delta\times\mathbb{P}^1}$ of the big diagonal, and its Frobenius pullback $\mathsf{F}^*i_*\mathcal{O}_{\Delta\times\mathbb{P}^1}$. The total space M of the deformation, by construction, is equipped with two projections $p_1, p_2 : \mathsf{M} \to X \times \mathbb{P}^1$, such that $p_i|_{\mathsf{M}_t} = \pi_i : X \times X \to X$ for $t \neq \infty$, and $p_1|_{\mathsf{M}_\infty} = p_2|_{\mathsf{M}_\infty} = \pi : \mathsf{M}_\infty = \mathrm{Spec}\,(\mathsf{S}^\bullet(\Omega^1_X)) \to X$. Consider the diagram:

$$\begin{array}{ccc} \mathsf{M}_t & \xhookrightarrow{\quad i_t \quad} & \mathsf{M} \\ {\scriptstyle \pi_t}\downarrow & & \downarrow{\scriptstyle p_1} \\ X \times \{t\} & \xhookrightarrow{\quad i_t \quad} & X \times \mathbb{P}^1 \end{array}$$

We can assume that a class $x \in \mathsf{K}^0(X)$ is represented by a vector bundle on X. Denote $\mathcal{E}$ the pullback of that vector bundle to $X \times \mathbb{P}^1$, and consider the complex of sheaves on $X \times \mathbb{P}^1$ obtained as follows:

$$\mathsf{F}(\mathcal{E}) = \mathrm{R}^\bullet p_{1*}(\mathsf{F}^*i_*\mathcal{O}_{\Delta\times\mathbb{P}^1} \otimes p_2^*\mathcal{E}). \tag{7.5}$$

We first observe that $\mathsf{F}(\mathcal{E})$ is actually a sheaf; this follows from $\mathrm{R}^i p_{1*}(\mathsf{F}^*i_*\mathcal{O}_{\Delta\times\mathbb{P}^1} \otimes p_2^*\mathcal{E}) = 0$ for $i > 0$. Indeed, p_1 is obtained by base change from the Frobenius morphism, which is affine, hence the vanishing of higher direct images. Furthermore, $\mathsf{F}(\mathcal{E})$ is locally free, i.e. a vector bundle. To see this, it is sufficient to show that the rank of $\mathsf{F}(\mathcal{E})$ is constant at all closed points $(x,t) \in X \times \mathbb{P}^1$. Since the sheaf $\mathsf{F}^*i_*\mathcal{O}_{\Delta\times\mathbb{P}^1}$ if flat over $X \times \mathbb{P}^1$, the Euler characteristic $\chi = \sum(-1)^i h^i(p_1^{-1}(x,t), \mathsf{F}^*i_*\mathcal{O}_{\Delta\times\mathbb{P}^1} \otimes p_2^*(\mathcal{E})|_{p_1^{-1}(x,t)})$ is constant, and by the vanishing of higher direct images we see that $\mathsf{F}(\mathcal{E})$ has constant rank. Hence it is a locally free sheaf.

Using the base change around the above diagram, we obtain, the morphism p_1 being flat:

$$\mathrm{L}i_t^*\mathsf{F}(\mathcal{E}) = \mathrm{L}i_t^*\mathrm{R}^\bullet p_{1*}(\mathsf{F}^*i_*\mathcal{O}_{\Delta\times\mathbb{P}^1} \otimes p_2^*\mathcal{E}) = \mathrm{R}^\bullet \pi_{t*}\mathrm{L}i_t^*(\mathsf{F}^*i_*\mathcal{O}_{\Delta\times\mathbb{P}^1} \otimes p_2^*\mathcal{E}), \tag{7.6}$$

and

$$\mathrm{R}^\bullet \pi_{t*}\mathrm{L}i_t^*(\mathsf{F}^*i_*\mathcal{O}_{\Delta\times\mathbb{P}^1} \otimes p_2^*\mathcal{E}) = \pi_{t*}\mathrm{L}i_t^*(\mathsf{F}^*i_*\mathcal{O}_{\Delta\times\mathbb{P}^1} \otimes p_2^*\mathcal{E}). \tag{7.7}$$

The last equality follows from affinity of π_t that is obtained by base change from the affine morphism p_1. On the other hand, the sheaf $\mathsf{F}(\mathcal{E})$ is locally free; hence higher inverse images $\mathrm{L}^k i_t^*\mathsf{F}(\mathcal{E})$ vanish, and we can simply write:

$$i_t^*\mathsf{F}(\mathcal{E}) = \pi_{t*}i_t^*(\mathsf{F}^*i_*\mathcal{O}_{\Delta\times\mathbb{P}^1} \otimes p_2^*\mathcal{E}). \tag{7.8}$$

For any $t_1, t_2 \in \mathbb{P}^1$ the classes $[\mathrm{L}i_{t_1}^*\mathsf{F}(\mathcal{E})] = [i_{t_1}^*\mathsf{F}(\mathcal{E})]$ and $[\mathrm{L}i_{t_2}^*\mathsf{F}(\mathcal{E})] = [i_{t_2}^*\mathsf{F}(\mathcal{E})]$ are equal in $\mathsf{K}^0(X \times \{t\}) = \mathsf{K}^0(X)$. One has for $t \neq \infty$:

$$\pi_{t*}i_t^*(\mathsf{F}^*i_*\mathcal{O}_{\Delta\times\mathbb{P}^1} \otimes p_2^*\mathcal{E}) = \pi_{t*}(\mathsf{F}^*i_{t*}\mathcal{O}_{\Delta_t} \otimes \pi_2^*\mathcal{E}_t), \tag{7.9}$$

and by Claim 7.1 the last group is isomorphic to $\mathsf{F}^*\mathsf{F}_*\mathcal{E}_t$. On the other hand, for $t = \infty$ one obtains:

$$\pi_{\infty*}i_\infty^*(\mathsf{F}^*i_*\mathcal{O}_{\Delta\times\mathbb{P}^1} \otimes p_2^*\mathcal{E}) = \pi_*(\mathsf{F}^*i_\infty^*i_*\mathcal{O}_{\Delta\times\mathbb{P}^1} \otimes \pi^*\mathcal{E}_\infty) = \pi_*(\mathsf{F}^*s_{\infty*}\mathcal{O}_X \otimes \pi^*\mathcal{E}_\infty), \tag{7.10}$$

where $s_\infty = i_\infty^* i$ is the zero section embedding of X into the normal cone $\mathrm{Spec}(\mathsf{S}^\bullet(\Omega_X^1))$. By the projection formula the last group is isomorphic to $\tau(\Omega_X^1) \otimes \mathcal{E}$. Hence the statement of the lemma. □

References

[1] P. Achinger, *Frobenius push-forwards on quadrics*, arxiv.org/abs/1005.0594

[2] H.H.Andersen, *The first cohomology group of a line bundle on* $\mathbf{G}/\mathbf{B}$, Invent. Math. **51** (1979), no. 3, 287–296

[3] H.H.Andersen, *Cohomology of line bundles on* $\mathbf{G}/\mathbf{B}$, Ann. scient. Ec. Norm. Sup., **12** (1979), pp. 85–100

[4] H.H.Andersen, *On the structure of the cohomology of line bundles on* $\mathbf{G}/\mathbf{B}$, J. of Algebra, **71** (1981), pp. 245–258

[5] H. H. Andersen, *On the generic structure of cohomology modules for semisimple algebraic groups*, Trans. of the Amer. Math. Soc., **295** 1 (1986), pp. 397–415

[6] H. H. Andersen, M. Kaneda, *On the* D*-affinity of the flag variety in type* B_2, Mansucripta Math. 103 (2000), 393–399

[7] H.H.Andersen, M. Kaneda, *Cohomology of line bundles on the flag variety for type* $\mathbf{G}_2$, arXiv:1107.3055v2

[8] A. Beilinson, J. Bernstein, *Localisation de* $\mathfrak{g}$*-modules*, C.R.Acad. Sci. Paris, Ser. I 292 (1981), 15–18

[9] R. Bezrukavnikov, D. Kaledin, *McKay equivalence for symplectic resolutions of quotient sngularities*, Proc. Steklov Inst. Math., **246** (2004), pp. 13–33

[10] R.W. Carter, G. Lusztig, *On the modular representations of the general linear and symmetric groups*, Math. Z. 136 (1974), 193-242

[11] S. Donkin, *The cohomology of line bundles on the three-dimensional flag variety*, J. of Algebra **307** (2007), pp. 570–613

[12] W. Fulton, *Intersection Theory*, (1984)

[13] B. Haastert, *Über Differentialoperatoren und* $\mathbb{D}$*-Moduln in positiver Charakteristik*, Manuscripta Math., 58 (1987), no. 4, 385–415

[14] Y. Hashimoto, M. Kaneda, D. Rumynin, *On localization of* $\overline{D}$*-modules*, in "Representations of algebraic groups, quantum groups, and Lie algebras", pp. 43–62, Contemp. Math., **413**, Amer. Math. Soc., Providence, RI, (2006)

[15] J.E.Humphreys, *Modular representations of finite groups of Lie type*, LMS Lecture Notes, **326** (2005)

[16] M. Kaneda, *Kapranov's tilting sheaf on the grassmannian in positive characteristic*, Alg. Rep. Th., **11** (2008), no. 4, pp. 347-354,

[17] M. Kaneda, *letter to the author*, (2011)

[18] M. Kapranov, *On the derived categories of coherent sheaves on some homogeneous spaces*, Invent. Math. **92** (1988), pp. 479 – 508

[19] M. Kashiwara, N. Lauritzen, *Local cohomology and* $\mathcal{D}$*-affinity in positive characteristic*, C. R. Acad. Sci. Paris, Ser. I, 335 (2002), 993–996

[20] A. Langer, D*-affinity and Frobenius morphism on quadrics*, Int. Math. Res. Not., (2008), no. 1, Art. ID rnm 145, 26 pp.

[21] G. Ottaviani, *On Cayley bundles on the five-dimensional quadric*, Boll. Un. Mat. Ital. A (7) 4 (1990), no. 1, 87–100

[22] R. Pink, D.Rössler, *On the Adams-Riemann-Roch theorem in positive characteristic*, arXiv:0812.0254v2

[23] A. Samokhin, *Tilting bundles via the Frobenius morphism*, arXiv:0904.1235v2

[24] A. Samokhin, *On the* D*-affinity of flag varieties in positive characteristic*, J. Algebra, (6) **4** (2010), pp. 1435–1446

[25] A. Samokhin, *A vanishing theorem for differential operators in positive characteristic*, Transform. Groups **15** (2010), no. 1, pp. 227–242

[26] A. Samokhin, *The Frobenius morphism on homogeneous spaces,II*, in preparation

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