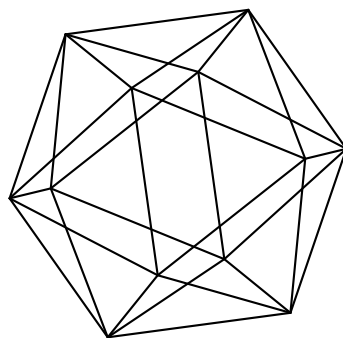# Max-Planck-Institut für Mathematik Bonn

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by

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ALMOST TOTALLY COMPLEX POINTS ON ELLIPTIC CURVES

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ABSTRACT. Let F/F_0 be a quadratic extension of totally real number fields, and let E be an elliptic curve over F which is isogenous to its Galois conjugate over F_0. A quadratic extension M/F is said to be almost totally complex (ATC) if all archimedean places of F but one extend to a complex place of M. The main goal of this note is to provide a new construction of a supply of Darmon-like points on E, which are conjecturally defined over certain ring class fields of M. These points are constructed by means of an extension of Darmon's ATR method to higher dimensional modular abelian varieties, from which they inherit the following features: they are algebraic provided Darmon's conjectures on ATR points hold true, and they are explicitly computable, as we illustrate with a detailed example that provides certain numerical evidence for the validity of our conjectures.

Contents

1. Introduction

Let E be an elliptic curve defined over a number field F and, for any field extension K/F, let $L(E/K, s)$ denote the Hasse-Weil L-function of the base change of E to K, which is known to converge on the half-plane $\{s \in \mathbb{C} : \operatorname{Re}(s) > \frac{3}{2}\}$.

The Mordell-Weil theorem asserts that the abelian group $E(K)$ of K-rational points on E is finitely generated, that is to say,

$$E(K) \simeq T \times \mathbb{Z}^r,$$

where T is a finite group and $r = r(E/K) \geqslant 0$ is a non-negative integer, which is called the *Mordell-Weil rank* of E/K.

There are two conjectures which stand out as cornerstones in the arithmetic of elliptic curves:

Conjecture (MOD). *The elliptic curve E/K is modular: there exists an automorphic representation π of $\operatorname{GL}_2(\mathbb{A}_K)$ such that $L(E/K, s - \frac{1}{2}) = L(\pi, s)$. In particular, $L(E/K, s)$ can be analytically continued to an entire function on the complex plane and it satisfies a functional equation relating the values at s and $2 - s$.*

Conjecture (BSD). Assume that (MOD) holds for E/K and let $r_{an}(E/K) = \mathrm{ord}_{s=1} L(E/K, s)$ denote the order of vanishing of $L(E/K, s)$ at $s = 1$, which we call the *analytic rank* of E/K. Then

$$r(E/K) \overset{?}{=} r_{an}(E/K).$$

Conjecture (MOD) is nowadays known to hold, under mild hypothesis, when F is totally real and K/F is Galois with solvable Galois group, thanks to the work of Wiles, Skinner-Wiles, Langlands and others. More precisely, when F is totally real, E is known to be modular by [Wi], [BCDT], [SW], unconditionally if the base field is $F = \mathbb{Q}$ and under some technical conditions on the reduction type at the primes of F above 3 when $[F : \mathbb{Q}] > 1$. In this setting, this amounts to saying that there exists a Hilbert modular eigenform f_E of parallel weight 2 over F such that $L(E/F, s)$ is equal to the L-function $L(f_E, s)$ associated with that form. If K/F is solvable, then (MOD) follows from the modularity of E over F by applying Langlands's cyclic base change. If $F = \mathbb{Q}$ and K is a totally real Galois number field, recent work of Dieulefait [Di] proves (MOD) under simple local assumptions on K, and one can expect that similar techniques may lead in the future to a similar result for arbitrary totally real fields F.

In light of these results, we assume throughout that F is totally real and E is modular. Let $\mathfrak{N}$ denote the conductor of E, an integral ideal of F, which for simplicity we assume to be square-free.

Thanks to the work of Kolyvagin, Gross-Zagier and Zhang, Conjecture (BSD) is then known to hold when K is either F or a totally imaginary extension of F, $(\mathfrak{N}, \mathrm{disc}(K/F)) = (1)$, $r_{an}(E/K) \leqslant 1$ and the *Jacquet-Langlands (JL) hypothesis* holds:

(JL) Either $[F : \mathbb{Q}]$ is odd or $\mathfrak{N} \neq (1)$.

In particular, when K is a totally imaginary extension of F and $r_{an}(E/K) = 1$, the above result implies that if (JL) is satisfied, there exists a non-torsion point in $E(K)$. Precisely when (JL) holds, such a point P_K, a so-called *Heegner point*, can be manufactured by means of the theory of complex multiplication on Shimura curves, and it is Gross-Zagier [GZ] and Zhang [Zh] who showed that the hypothesis $r_{an}(E/K) = 1$ implies that P_K is not torsion. Finally, Koyvagin's method [Ko] of Euler systems is the device which permits to show that in fact there are no points in $\mathbb{Q} \otimes E(K)$ which are linearly independent of P_K, thereby showing (BSD). This is made possible thanks to the existence, along with the point P_K, of a system

$$\{P_c \in E(H_c), c \geqslant 1, (c, \mathrm{disc}(K/F)) = 1\}$$

of rational points on E over the *ring class field* H_c/K, the abelian extension of K associated by class field theory to the Picard group $\mathrm{Pic}(\mathcal{O}_c)$ of invertible ideals in the order $\mathcal{O}_c \subset K$ of conductor c of K.

That this supply of points should exist can be predicted using Conjecture (BSD), even if K is not totally imaginary, as we now explain. Let K/F be *any* quadratic field extension such that $(\mathfrak{N}, \mathrm{disc}(K/F)) = 1$. Write

$$\mathfrak{N} = \mathfrak{N}^+ \cdot \mathfrak{N}^-, \tag{1}$$

where $\mathfrak{N}^+$ (resp. $\mathfrak{N}^-$) is the product of the prime divisors of $\mathfrak{N}$ which split (resp. remain inert) in K.

A character $\chi\colon \mathrm{Gal}(K^{ab}/K) \to \mathbb{C}^\times$ is called a *ring class character* if it corresponds, via the identification provided by class field theory, to a character of $\mathbb{A}_K^\times$ whose restriction to $\mathbb{A}_F^\times$ is trivial. They are precisely the characters factoring through $\mathrm{Gal}(H/K)$ for some ring class field H. Let $\chi : \mathrm{Gal}(K^{ab}/K) \to \mathbb{C}^\times$ be a ring class character of finite order and conductor relatively prime to $\mathfrak{N}$. Let $r_1(K/F)$ and $r_2(K/F)$ be the number of archimedean places of F which extend to a couple of real (resp. to a complex) place(s) of K, so that $[F : \mathbb{Q}] = r_1(K/F) + r_2(K/F)$. Then the sign of the functional equation of the L-function $L(E/K, \chi, s)$ of E/K twisted by χ is

$$\mathrm{sign}(E/K) = \mathrm{sign}(E/K, \chi) = (-1)^{r_2(K/F) + \sharp\{\wp | \mathfrak{N}^-\}}, \tag{2}$$

independently of the choice of χ.

For any ring class field H/K, let $\hat{\mathrm{Gal}}(H/K) = \mathrm{Hom}(\mathrm{Gal}(H/K), \mathbb{C}^\times)$ denote the group of characters of $\mathrm{Gal}(H/K)$. The L-function of the base change of E to H factors as

$$L(E/H, s) = \prod_{\chi \in \hat{\mathrm{Gal}}(H/K)} L(E/K, \chi, s).$$

The Birch and Swinnerton-Dyer conjecture (BSD) in combination with (2) gives rise to the following:

Conjecture 1.1. *Assume* $\mathrm{sign}(E/K) = -1$ *and let* H *be a ring class field extension of* K, *unramified at the primes dividing* $\mathfrak{N}$. *Then*

$$(3) \qquad\qquad \mathrm{rank}\, E(H) \stackrel{?}{=} [H:K],$$

if and only if $L'(E/K, \chi, 1) \neq 0$ *for all* $\chi \in \mathrm{Hom}(\mathrm{Gal}(H/K), \mathbb{C}^\times)$.

No proven result is known about Conjecture 1.1 beyond the achievements of Gross-Zagier, Kolyvagin and Zhang in the case $r_2(K/F) = [F : \mathbb{Q}]$ mentioned above. In spite of this, a plethora of *conjectural* constructions of points have been proposed so far in various settings beyond the classical one. These points are commonly called *Stark-Heegner points*, or *Darmon points*, as it was H. Darmon in [Da2] who first introduced them.

Since then, several authors [Das1], [Gr], [LRV], [DL], [Ga1] have proposed variations of Darmon's theme, always giving rise to a recipe that allows to attach, to a given ring class field extension H/K satisfying the hypothesis of Conjecture 1.1, a point

$$(4) \qquad\qquad P_H \in E(H_v),$$

rational over the completion H_v of H at some finite or archimedean place v of H, which is conjectured to satisfy the following properties:

(SH1) $P_H \stackrel{?}{\in} E(H)$,
(SH2) For any character $\chi : \mathrm{Gal}(H/K) \to \mathbb{C}^\times$, the point

$$P_\chi := \sum_{\sigma \in \mathrm{Gal}(H/K)} \chi(\sigma)^{-1} \sigma(P_H) \in E(H) \otimes_\mathbb{Z} \mathbb{C}$$

is non-zero if and only if $L'(E/K, \chi, 1) \neq 0$, and
(SH3) there is a reciprocity law describing the action of $\mathrm{Gal}(H/K)$ on P_H in terms of ideal theory.

The main result of this paper is a new, computable construction of a supply of Darmon-like points in a setting that was not computationally accessible before. Before describing our contribution in more detail, and being the constructions of Darmon points dispersed in the literature, we take the chance to report on the state of the art of this question. Namely, explain which cases of Conjecture 1.1 are already covered by the union of those constructions, and which ones remain intractable.

Keep the above notations and the assumptions of conjecture 1.1, and assume that H is the narrow ring class field associated with some order in K. Then:

a) If $r_1(K/F) = 0$, $r_2(K/F) = [F : \mathbb{Q}]$, then assumption $\mathrm{sign}(E/K) = -1$ implies that (JL) holds, and conjecture 1.1 holds thanks to [GZ], [Ko] and [Zh].

b) If $\sharp\{\wp \mid \mathfrak{N}^-\} \geqslant 1$, points $P_H \in E(H_\wp)$ have been constructed in [Da2], [Gr] and [LRV], for which conditions (SH1), (SH2) and (SH3) above have been conjectured.
   Some theoretical evidence has been provided for them when $F = \mathbb{Q}$ in [BD], [GSS] and [LV]. Numerical evidence has been given in [Da2] when $F = \mathbb{Q}$ and $N^- = 1$.

c) If $r_1(K/F) \geqslant 1$, $r_2(K/F) \geqslant 1$ let us distinguish two possibilities:

   c1) If $r_2(K/F) = 1$, K/F is called an *almost totally real* (ATR) quadratic extension and we let v denote the unique archimedean place of F which extends to a complex place of K. Then $H_v = \mathbb{C}$ for any place of H above it and points $P_H \in E(H_v)$ have been constructed in [Da1, Ch. VIII], for which conditions (SH1), (SH2) and (SH3) above have been conjectured. These conjectures have been tested numerically in [DL].

   c2) J. Gartner has extended the idea of Darmon [Da1, Ch. VIII] to any K/F with $1 \leqslant r_2(K/F) < [F : \mathbb{Q}]$: in this more general setting, he constructs points $P_H \in E(H_v)$ and again conjectures that (SH1), (SH2) and (SH3) hold true. His method does not appear to be amenable to explicit calculations and as a consequence no numerical evidence has been provided for these conjectures.

Note that a), b), c) cover all cases contemplated in Conjecture 1.1. Indeed, the only case not covered by b) arises when $\sharp\{\wp \mid \mathfrak{N}^-\} = 0$, that is, all primes $\wp \mid \mathfrak{N}$ split in K. But then assumption $\mathrm{sign}(E/K) = -1$ implies that $r_2(K/F)$ is odd, hence $r_2(K/F) \geqslant 1$. Then a) and c) cover respectively the case in which $r_1(K/F) = 0$ and $r_1(K/F) > 0$.

The main contribution of this article is an explicitly computable, construction of a supply of points $P_M \in E(\mathbb{C})$ in a setting which lies within c2), but which is completely different to the one proposed by Gartner. It only works under the following restrictive hypothesis:

- F contains a field F_0 with $[F : F_0] = 2$,
- E/F is F-isogenous to its Galois conjugate over F_0, and
- M is an almost totally complex quadratic extension of F, that is to say, $r_2(M/F) = [F : \mathbb{Q}] - 1$.

While this setting is obviously much less general than the one considered in [Ga1], it enjoys the following features:

- Numerical approximations to the points P_M are computable, as we illustrate with a fully detailed explicit example in §3.3.

  As we explain in §3.1, our construction relies on the computation of certain ATR cycles on Hilbert modular varieties. To the best of our knowledge, at present there is available an algorithm for computing such ATR cycles only when the level is *trivial* (see §2.4 for more details). However, in our setting the level is always nontrivial, and so far this stands as the single issue which prevents our method from being completely automatized. In the example of §3.3 we circumvent the lack of a general algorithm with an ad hoc computation.
- We prove that the points P_M belong to $E(M)$ and that they are non-torsion if and only if $L'(E/M, 1) \neq 0$ provided (SH1), (SH2) and (SH3) hold true for ATR extensions of F_0: see Theorem 3.9 for the precise statement. This is worth remarking, as the conjectures for ATR extensions can be tested numerically in practice: see §2.4 for a sketch of the algorithm, and [DL], [GM] for explicit numerical examples.

The main source of inspiration for the construction presented here is the previous work [DRZ] of two of the authors with Henri Darmon, in which Heegner points on quotients of the modular curve $X_1(N)$ were used to manufacture ATR points on elliptic curves.

2. Quadratic points on modular abelian varieties

The basis of the main construction of this note –which we explain in §3– lies in Darmon's conjectural theory of points on modular elliptic curves over almost totally real (ATR) quadratic extensions of a totally real number field.

In a recent article, Darmon's theory has been generalized by Gartner [Ga2] by considering *quaternionic* modular forms with respect to not necessarily split quaternion algebras over the base field. Although we do not exploit Gartner's construction here, our points do lie in a theoretical setting which is also covered by him and therefore the natural question arises of whether Gartner's points are equal to ours when both constructions are available. We address this issue in §3.2, where we point out that Conjecture (BSD) implies that one is a non-zero multiple of the other; the difference between them is that ours are numerically accessible, and this stands as the main motivation of this article.

This section is devoted to review the work of Darmon and Gartner, settling on the way the notations that shall be in force for the rest of this note. As Gartner's exposition [Ga1], [Ga2] is already an excellent account of the theory, we choose here to reword it in the classical language of Hilbert modular forms, under the simplifying hypothesis that the narrow class number of the base field F_0 is 1.

In doing so, we take the chance to contribute to the theory with a few novel aspects. To name one, it will be convenient for our purposes to work with the natural, relatively straight-forward extension of the theory to the setting of eigenforms with not necessarily trivial nebentypus and whose eigenvalues generate a number field of arbitrarily large degree over $\mathbb{Q}$. This will lead us to a construction of rational points on higher-dimensional modular abelian varieties of GL_2-type.

2.1. **Quadratic extensions and L-functions.** Let $F_0 \subset \mathbb{R}$ be a totally real number field, together with a fixed embedding into the field of real numbers. Write $d = [F_0 : \mathbb{Q}]$ for its degree over $\mathbb{Q}$ and let $R_0 \subset F_0$ denote its ring of integers. In order to keep our notations simple, we assume that the narrow class number of F_0 is 1.

Let N be a square-free integral ideal of F_0 and let ψ be a Hecke character of conductor N. Let $f_0 \in S_2(N, \psi)$ be a normalized Hilbert eigenform of parallel weight 2, level N and nebentypus ψ. Let $\mathbb{Q}_{f_0}$ denote the number field generated by the eigenvalues of the Hecke operators acting on f_0, which we regard as embedded in the algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$ in the field $\mathbb{C}$ of complex numbers; for each $\sigma \in \mathrm{Hom}(\mathbb{Q}_{f_0}, \bar{\mathbb{Q}})$, there exists a unique normalized eigenform $^\sigma f_0$ whose family of eigenvalues is equal to the family of eigenvalues of f_0 conjugated by σ.

The following standard conjecture is a generalized form of the Eichler–Shimura philosophy:

Conjecture 2.1. *There exists an abelian variety $A = A_{f_0}/F_0$ of dimension $g = [\mathbb{Q}_{f_0} : \mathbb{Q}]$ and conductor N^g such that $\mathbb{Q} \otimes \mathrm{End}_{F_0}(A) \simeq \mathbb{Q}_{f_0}$, and whose L-series factors as*

$$(5) \qquad L(A, s) = \prod_{\sigma \in \mathrm{Hom}(\mathbb{Q}_{f_0}, \bar{\mathbb{Q}})} L(^\sigma f_0, s).$$

Note that, if such an A exists, it is well-defined only up to isogenies.

Conjecture 2.1 is known to hold when (JL) is satisfied. When (JL) fails it is not even known whether there exists a motive M_{f_0} over F whose L-function is (5) and one certainly does not expect the motive $h^1(E)$ to arise in the cohomology of any (quaternionic) Hilbert variety (cf. [BR] and for more details). See [De] for the numerical verification of Conjecture 2.1 in several instances in which (JL) fails.

We shall *assume for the remainder of this section* that Conjecture 2.1 holds true.

Let K/F_0 be a quadratic extension such that $(\mathrm{disc}(K/F_0), N) = 1$ and $r_2(K/F_0) \geqslant 1$. Label the set of embeddings of F_0 into the field $\mathbb{R}$ of real numbers as

$$\{v_1, v_2, ..., v_r, v_{r+1}, ..., v_d : F_0 \hookrightarrow \mathbb{R}\}, \quad 1 \leqslant r \leqslant d$$

in such a way that
- v_1 is the embedding fixed at the outset that we use to identify F_0 as a subfield of $\mathbb{R}$,
- each of the places $v_2, ..., v_r$ extends to a pair of real places of K, which by a slight abuse of notation we denote v_j and v_j' for each $j = 2, ..., r$, and
- each of the places $v_1, v_{r+1}, ..., v_d$ extends to a complex place on K, that we still denote with the same letter; we use v_1 to regard K as a subfield of $\mathbb{C}$.

Definition 2.2. If $r = 1$, the set $\{v_2, ..., v_r\}$ is empty and K/F_0 is a CM-field extension.
If $r = 2$ we call K/F_0 an almost totally complex (ATC) extension.
If $r = d$ we have $\{v_1, v_{r+1}, ..., v_d\} = \{v_1\}$ and we say that K/F_0 is almost totally real (ATR).

Letting ε_K denote the quadratic Hecke character of F_0 associated with the extension K/F_0, the L-function of the base change of A to K is

$$L(A/K, s) = L(A, s) \cdot L(A, \varepsilon_K, s) = \prod_{\sigma \in \mathrm{Hom}(\mathbb{Q}_{f_0}, \bar{\mathbb{Q}})} L(^\sigma f_0, s) \cdot L(^\sigma f_0, \varepsilon_K, s).$$

It extends to an entire function on $\mathbb{C}$ and satisfies a functional equation relating the values at s with $2 - s$. Assume that the sign of the functional equation of $L(f_0/K, s) = L(f_0, s) \cdot L(f_0, \varepsilon_K, s)$ is -1. This is equivalent to saying that the set

$$(6) \qquad \{v_{r+1}, ..., v_d\} \cup \{\wp \mid N, \wp \text{ inert in } K\}$$

has *even* cardinality.

Let B be the (unique, up to isomorphism) quaternion algebra over F_0 whose set of places of ramification is $\mathrm{Ram}(B) = (6)$. In particular we have $B \otimes_{F_0,v_j} \mathbb{R} \simeq \mathrm{M}_2(\mathbb{R})$ for $j = 1, ..., r$, and the choice of such isomorphisms gives rise to an embedding

$$(7) \qquad (v_1, ..., v_r) : B^\times \hookrightarrow \mathrm{GL}_2(\mathbb{R}) \times \overset{(r)}{...} \times \mathrm{GL}_2(\mathbb{R}) \subset (B \otimes_{\mathbb{Q}} \mathbb{R})^\times.$$

Let N^+ be the product of primes in F_0 such that divide N and are split in K, and N^- the product of primes that divide N and remain inert in K. Choose an Eichler order $\mathcal{O}$ of level N^+ in B together with, for each prime $\wp \mid N^+$, isomorphisms $i_\wp : B \otimes F_{0,\wp} \simeq M_2(F_{0,\wp})$ such that

$$i_\wp(\mathcal{O}) = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right), \wp \mid c \} \subseteq \mathrm{M}_2(R_{0,\wp}).$$

Definition 2.3. Let F_0^+ denote the subgroup of $F_0^\times$ of totally positive elements and B^+ be the subgroup of elements in $B^\times$ whose reduced norm lies in F_0^+. Define the congruence subgroups

$$\Gamma_0 = \Gamma_0^{N^-}(N^+) = \mathcal{O}^\times \cap B^+ \quad \text{and}$$

$$\Gamma_1 = \Gamma_1^{N^-}(N^+) = \{\gamma \in \Gamma_0, i_\wp(\gamma) \cong \left(\begin{smallmatrix} 1 & \star \\ 0 & 1 \end{smallmatrix} \right), \wp \mid N^+\} \subset \Gamma_0.$$

Through (7), Γ_1 acts on the cartesian product $\mathcal{H}^r = \mathcal{H}_1 \times ... \times \mathcal{H}_r$ of r copies of Poincaré's upper-half plane and we let $X_{\mathbb{C}} = \Gamma_1 \backslash \mathcal{H}^r$ denote its quotient, which has a natural structure of analytic manifold with finitely many isolated singularities.

Definition 2.4. Let F_0^{gal} denote the galois closure of F_0 in $\mathbb{C}$ and view the places v_i as elements of the Galois group $G = \mathrm{Gal}(F_0^{\mathrm{gal}}/\mathbb{Q})$, so that $v_1 = \mathrm{Id}$. The *reflex field of B* is the subfield $F_0^\star$ of F_0^{gal} fixed by the subgroup of those $\sigma \in G$ such that $\sigma \cdot \{v_1, ..., v_r\} = \{v_1, ..., v_r\}$.

The cases one encounters most often in the literature arise when either $r = 1$, where $F_0^\star = F_0$, or when $r = d$, in which case $F_0^\star = \mathbb{Q}$.

Let

$$X = X_1^{N^-}(N^+)/F_0^\star$$

denote Shimura's canonical model over $F_0^\star$ of $X_{\mathbb{C}}$, as introduced e.g. in [Mi1, §12]. If $\mathrm{Ram}(B) \neq \emptyset$, $X_{\mathbb{C}}$ is compact and X is projective over $F_0^\star$, while if $\mathrm{Ram}(B) = \emptyset$ then $B = \mathrm{M}_2(F_0)$ and $X_{\mathbb{C}}$ admits a canonical compactification by adding a finite number of cusps; by an abuse of notation, we continue to denote X the resulting projective model.

2.2. **Oda-Yoshida's conjecture.** Let $\Sigma = \{\pm 1\}^{r-1}$ and for each $\epsilon = (\epsilon_2, ..., \epsilon_r) \in \Sigma$, let $\gamma_\epsilon \in \mathcal{O}^\times$ be an element such that $v_j(n(\gamma_\epsilon)) = \det(v_j(\gamma_\epsilon)) > 0$ if $j = 1$ or $\epsilon_j = +1$, and $v_j(n(\gamma_\epsilon)) < 0$ if $\epsilon_j = -1$. Such elements exist thanks to our running assumption that the narrow class number of F_0 is 1. For $\tau_j \in \mathcal{H}_j$, set

$$\tau_j^\epsilon = \begin{cases} v_j(\gamma_\epsilon)\tau_j & \text{if } j = 1 \text{ or } \epsilon_j = +1, \\ v_j(\gamma_\epsilon)\bar{\tau}_j & \text{if } \epsilon_j = -1. \end{cases}$$

For each $0 \leqslant i \leqslant 2r$, let $H_i(X_{\mathbb{C}}, \mathbb{Z}) = Z_i(X_{\mathbb{C}}, \mathbb{Z})/B_i(X_{\mathbb{C}}, \mathbb{Z})$ denote the i-th Betti homology group of $X_{\mathbb{C}}$. Attached to f_0 there is the natural holomorphic r-form on $\mathcal{H}^r$ given by

$$\omega_{f_0} = (2\pi i)^r f_0(\tau_1, ..., \tau_r) d\tau_1 ... d\tau_r,$$

which is easily shown to be Γ_1-invariant (and to extend to a smooth form on the cusps, if $B = \mathrm{M}_2(F_0)$), giving rise to a regular differential r-form $\omega_{f_0} \in H^0(X_{\mathbb{C}}, \Omega^r)$.

Label the set $\mathrm{Hom}(\mathbb{Q}_{f_0}, \mathbb{C}) = \{\sigma_1, ..., \sigma_g\}$ of embeddings of $\mathbb{Q}_{f_0}$ into the field of complex numbers. The set $\{\sigma_1(\omega_{f_0}), ..., \sigma_g(\omega_{f_0})\}$ is then a basis of the f_0-isotypical component of $H^0(X, \Omega^r)$.

Definition 2.5. [Da1, (8.2)], [Ga1, §2] Let d_0 be a totally positive generator of the different ideal of F_0 and let $\beta : \Sigma \to \{\pm 1\}$ be a character. The differential r-form $\omega_{f_0}^\beta$ on X associated with f_0 and β is

$$\omega_{f_0}^\beta := |d_0|^{-1/2}(2\pi i)^r \sum_{\epsilon \in \Sigma} \beta(\epsilon) f_0(\tau_1^\epsilon, ..., \tau_r^\epsilon) d\tau_1^\epsilon ... d\tau_r^\epsilon.$$

If $r = 1$, note that the only choice for β is the trivial one and in this case one recovers the usual holomorphic 1-form ω_{f_0} on the Shimura curve X/F_0. On the other hand, when $r > 1$, the differential form $\omega_{f_0}^{\beta}$ is not holomorphic anymore for any choice of β, including the trivial one.

Definition 2.6. The *lattice of periods* of $\omega_{f_0}^{\beta}$ is

$$\Lambda_{f_0}^{\beta} = \{(\int_{\tilde{Z}} \sigma_1(\omega_{f_0}^{\beta}), ..., \int_{\tilde{Z}} \sigma_g(\omega_{f_0}^{\beta})), \tilde{Z} \in H_r(X_{\mathbb{C}}, \mathbb{Z})\} \subseteq \mathbb{C}^g.$$

In addition to that, under the running assumption that conjecture 2.1 holds true, we can also introduce another lattice as follows. For each $j = 1, ..., r$, let $A_j = A \times_{F_0, v_j} \mathbb{C}$ denote the base change of A to the field of complex numbers via the embedding $F_0 \overset{v_j}{\hookrightarrow} \mathbb{R} \subset \mathbb{C}$. Since we identify v_1 with the identity embedding, A_1 is identified with A. Let $H_1(A_j, \mathbb{Z})^{\pm}$ be the $\mathbb{Z}$-submodule of $H_1(A_j, \mathbb{Z})$ on which complex conjugation acts as $+1$ (resp. -1). Since $\mathbb{Q}_{f_0} \simeq \mathbb{Q} \otimes \mathrm{End}_{F_0}(A)$, there is a natural action of $\mathbb{Q}_{f_0}$ on $H_1(A_j, \mathbb{Q})^{\pm}$ and in fact the latter is a free module of rank 1 over the former.

Similarly, the space $H^0(A, \Omega^1)$ of global regular differential 1-forms on A is an F_0-vector space of dimension g equipped with a F_0-linear action of $\mathbb{Q}_{f_0}$ inherited from the isomorphism $\mathbb{Q}_{f_0} \simeq \mathbb{Q} \otimes \mathrm{End}_{F_0}(A)$.

Recall that R_0 stands for the ring of integers of F_0. Make the following choices:

- A regular differential $\omega_A \in H^0(A, \Omega^1)$ which extends to a smooth differential on the Néron model of A over R_0 and generates $H^0(A, \Omega^1)$ as a $\mathbb{Q}_{f_0}$-module.
- For each $j = 1, ..., r$, generators c_j^+, c_j^- of $H_1(A_j, \mathbb{Q})^+$ and $H_1(A_j, \mathbb{Q})^-$ as $\mathbb{Q}_{f_0}$-modules.

Given these choices, define

$$\Omega_j^+ = \int_{c_j^+} v_j(\omega_A) \in \mathbb{C}, \quad \Omega_j^- = \int_{c_j^-} v_j(\omega_A) \in \mathbb{C}, \text{ for } j = 1, ..., r \quad \text{and}$$

$$\Omega_{\beta} = \Omega_2^{\beta_2(-1)} \cdot ... \cdot \Omega_r^{\beta_r(-1)}.$$

Definition 2.7. Let R_{f_0} denote the ring of integers of $\mathbb{Q}_{f_0}$ and define

$$\Lambda_0^{\beta} := \Omega_{\beta} \cdot (\mathbb{Z}\Omega_1^+ + \mathbb{Z}\Omega_1^-) \subset \mathbb{C}, \quad \Lambda_A^{\beta} := \Lambda_0^{\beta} \otimes_{\mathbb{Z}} R_{f_0} \subseteq \mathbb{C} \otimes_{\mathbb{Z}} \mathbb{Q}_{f_0} \simeq \mathbb{C}^g.$$

Let us now analyze how these lattices depend on the above choices. Note that ω_A is well-defined only up to multiplication by units $u \in R_0^{\times}$ and non-zero endomorphisms $t \in \mathbb{Q}_{f_0}^{\times}$. If we replace ω_A by $u \cdot \omega_A$, we obtain

$$\Lambda_0^{\beta}(u \cdot \omega_A) = \langle \int_{c_1^+} v_1(u \cdot \omega_A) \cdot \prod_{j=2}^{r} \int_{c_j^{\beta_j(-1)}} v_j(u \cdot \omega_A), \int_{c_1^-} v_1(u \cdot \omega_A) \cdot \prod_{j=2}^{r} \int_{c_j^{\beta_j(-1)}} v_j(u \cdot \omega_A) \rangle =$$

$$= \mathrm{N}_{F_0/\mathbb{Q}}(u)\Lambda_0^{\beta}(\omega_A) = \Lambda_0^{\beta}(\omega_A),$$

because $\mathrm{N}_{F_0/\mathbb{Q}}(u) = \pm 1$, and thus also $\Lambda_A^{\beta}(u \cdot \omega_A) = \Lambda_A^{\beta}(\omega_A)$.

If instead we replace ω_A by $t \cdot \omega_A$ for some $t \in \mathbb{Q}_{f_0}^{\times}$, then

$$\Lambda_A^{\beta}(t\omega_A) = \{\Omega_{\beta} \int_{c_1^+} v_1(t^*\omega_A) \otimes s, \Omega_{\beta} \int_{c_1^-} v_1(t^*\omega_A) \otimes s, s \in R_{f_0}\} =$$

$$= \{\Omega_{\beta} \int_{c_1^+} v_1(\omega_A) \otimes st, \Omega_{\beta} \int_{c_1^-} v_1(\omega_A) \otimes st, s \in R_{f_0}\}$$

and therefore $\mathbb{Q} \otimes_{\mathbb{Z}} \Lambda_A^{\beta}(t\omega_A) = \mathbb{Q} \otimes_{\mathbb{Z}} \Lambda_A^{\beta}(\omega_A)$. We reach to the same conclusion if we take different choices of homotopically equivalent paths c_j^+ or c_j^-. Hence the $\mathbb{Q}$-submodule $\mathbb{Q} \otimes_{\mathbb{Z}} \Lambda_A^{\beta}$ of $\mathbb{C}^g$ is determined uniquely independently of the choices made.

Conjecture 2.8 (Oda, Yoshida). [Od] *The lattices $\Lambda_{f_0}^{\beta}$ and Λ_A^{β} are commensurable, that is to say,*

$$\mathbb{Q} \otimes_{\mathbb{Z}} \Lambda_{f_0}^{\beta} = \mathbb{Q} \otimes_{\mathbb{Z}} \Lambda_A^{\beta},$$

and therefore there exists an isogeny of abelian varieties

$$\eta_\beta : \mathbb{C}^g/\Lambda_{f_0}^\beta \xrightarrow{\sim} \mathbb{C}^g/\Lambda_A^\beta \simeq A(\mathbb{C}) = \mathbb{C}^g/\Lambda_1,$$

where the last isomorphism is given by multiplication by Ω_β^{-1}.

Note that, consistently with Conjecture 2.1, the above Conjecture 2.8 only concerns the isogeny class of the abelian variety A.

Remark 2.9. If $r = 1$ and (JL) holds, Conjecture 2.8 holds true: the abelian variety A may be constructed explicitly as a constituent of the Jacobian of the Shimura curve X and it follows from the very construction that the lattices $\Lambda_{f_0}^\beta$ and Λ_A^β are commensurable.

2.3. **Darmon points.** Let $\mathcal{Z}_{r-1}(X_\mathbb{C})$ denote the set of null-homologous cycles of real dimension $r-1$ in $X_\mathbb{C}$. For each character β as above, Conjecture 2.8 allows us to define the *topological Abel-Jacobi map*

$$(8) \qquad \begin{array}{rccc} \mathrm{AJ}^\beta: & \mathcal{Z}_{r-1}(X_\mathbb{C}) & \longrightarrow & A(\mathbb{C}) \\ & T & \mapsto & \eta_\beta\left(\int_{\tilde{T}} \omega_{f_0}^\beta\right), \end{array}$$

where $\tilde{T} \in C_r(X_\mathbb{C}, \mathbb{Z})$ is any r-dimensional chain satisfying $\partial \tilde{T} = T$. Observe that $\tilde{T}$ is determined up to elements in $H_r(X_\mathbb{C}, \mathbb{Z})$, so that the quantity $\int_{\tilde{T}} \omega_{f_0}^\beta \in \mathbb{C} \otimes_\mathbb{Z} \mathbb{Q}_{f_0}$ is a well-defined element in $\mathbb{C}^g/\Lambda_{f_0}^\beta$ and AJ^β is thus a well-defined map.

Remark 2.10. If $r = 1$ and (JL) holds, the map AJ^β is nothing else but the classical *algebraic* Abel-Jacobi map of curves $X_\mathbb{C} \longrightarrow A(\mathbb{C})$, which factors through the jacobian of $X_\mathbb{C}$. This was one of Darmon's motivations for extending the rule to the general case, though the reader must be warned that when $r > 1$ the maps AJ^β are *not* algebraic.

Let now $c \subseteq R_0$ be an integral ideal of F_0 relatively coprime with N and let $R_c := R_0 + cR_K \subseteq R_K$ be the order of conductor c in the ring of integers of K. Let η be the homomorphism

$$\eta \colon \mathcal{O} \longrightarrow R_0/N^+ R_0$$

sending an element $x \in \mathcal{O}$ to the upper-left hand entry of its image in $\mathcal{O} \otimes_{R_0} R_{0,N^+} \simeq \mathrm{M}_2(R_{0,N^+})$, taken modulo $N^+ R_{0,N^+}$.

Definition 2.11. Fix a factorization of ideals $N^+ R_K = \mathfrak{N}^+ \cdot \bar{\mathfrak{N}}^+$. An embedding of R_0-algebras $\varphi \colon R_c \hookrightarrow \mathcal{O}$ is said to be *optimal* if $\varphi(R_c) = \varphi(K) \cap \mathcal{O}$. We say that φ is *normalized* (with respect to $\mathfrak{N}^+$) if it satisfies the following conditions:

(1) φ acts on $u = (\tau_1, 1) \in \mathbb{C}^2$ as $\varphi(a)_1 \cdot u = v_1(a) \cdot u$ for all $a \in R_c$, where $\varphi(a)_1$ denotes the image of $\varphi(a)$ in $B \otimes_{F_0,v_1} \mathbb{R}$.
(2) The kernel of $\eta \circ \varphi$ is equal to $\mathfrak{N}^+$.

We denote by $\mathcal{E}(R_c, \mathcal{O})$ the set of normalized optimal embeddings.

Recall that v_1 extends to a complex place of K and that $v_2, \ldots, v_r$ extend to real places. Given $\varphi \in \mathcal{E}(R_c, \mathcal{O})$, the action of $K^\times$ on $\mathbb{C}$ by fractional linear transformations via the composition of φ and the isomorphism $(B \otimes_{F_0,v_1} \mathbb{R})^\times \simeq \mathrm{GL}_2(\mathbb{R})$ has a unique fixed point $z_1 \in \mathcal{H}_1$. For $j = 2, \ldots, r$ it has two fixed points $\tau_j, \tau_j' \in \mathbb{R} = \partial \mathcal{H}_j$ under the isomorphism $(B \otimes_{F_0,v_j} \mathbb{R})^\times \simeq \mathrm{GL}_2(\mathbb{R})$. Let γ_j be the geodesic path joining τ_j and τ_j' in $\mathcal{H}_j$.

Definition 2.12. We denote by T_φ the $(r-1)$-real dimensional cycle in $X_\mathbb{C}$ given by the image of the region

$$R_\varphi = \{z_1\} \times \gamma_2 \times \cdots \times \gamma_r \subset \mathcal{H}_1 \times \overset{r)}{\cdots} \times \mathcal{H}_r$$

under the natural projection map $\mathcal{H}^r \longrightarrow X_\mathbb{C}$.

Note that the stabilizer of R_φ in Γ_1 is the subgroup $\Gamma_\varphi = \varphi(K) \cap \Gamma_1$ and therefore there is a natural homeomorphism $T_\varphi \simeq \Gamma_\varphi \backslash R_\varphi$. As an application of the Matsushima–Shimura Theorem [MS], it is easy to show (cf. [Ga2, Proposition 4.3.1]) that the class of T_φ has finite order in $H_{r-1}(X_\mathbb{C}, \mathbb{Z})$. In particular, if e denotes the order of T_φ then eT_φ is null-homologous. This allows the following definition.

Definition 2.13. The *Darmon point* attached to φ and β is

$$P_\varphi^\beta := \frac{1}{e} \mathrm{AJ}^\beta(eT_\varphi) \in A_1(\mathbb{C}).$$

Darmon points are conjectured to be rational over certain number fields, with the Galois action given by an explicit reciprocity law. This is the content of Conjecture 2.14. Next we define the number fields and the actions involved in the conjecture.

Define

$$U_c := \hat{R}_0^\times (1 + c\hat{R}_K) \subset \hat{K}^\times = (K \otimes_\mathbb{Z} \hat{\mathbb{Z}})^\times.$$

For every character β of Σ, let H_c^β denote the abelian extension of K corresponding by class field theory to the open compact subgroup $K_\infty^\beta \times U_c$, where

$$(9) \qquad K_\infty^\beta := \prod_{\substack{j=1,r+1,\ldots,d}} \mathbb{C}^\times \times \prod_{\substack{j=2,\ldots,r \\ \beta_j(-1)=+1}} \mathbb{R}^\times \times \prod_{\substack{j=2,\ldots,r \\ \beta_j(-1)=-1}} \mathbb{R}_{>0}.$$

Recall that ψ denotes the Nebentypus of f_0. For $\varphi\colon R_c \hookrightarrow \mathcal{O}$ a normalized optimal embedding define

$$U_\varphi^+ = \ker(U_c \longrightarrow (R_K/\mathfrak{N}^+ R_K)^\times \simeq (R_0/N^+ R_0)^\times \overset{\psi}{\to} \mathbb{C}^\times),$$

$$U_\varphi^- = \ker(U_c \longrightarrow (R_K/\bar{\mathfrak{N}}^+ R_K)^\times \simeq (R_0/N^+ R_0)^\times \overset{\psi}{\to} \mathbb{C}^\times).$$

Denote by $L_\varphi^\beta/H_c^\beta$ (resp. $L_\varphi'^\beta/H_c^\beta$) the abelian extension of the ring class field of conductor c associated to $K_\infty^\beta \times U_\varphi^+$ (resp. $U_\varphi^- \times K_\infty^\beta$). Let also $U_\varphi = U_\varphi^+ \cap U_\varphi^-$ and let $H_\varphi^\beta/H_c^\beta$ be the extension associated to $K_\infty^\beta \times U_\varphi \subset \mathbb{A}_K^\times$.

Observe that we can extend ψ to a character on $\mathcal{O}^\times$ by composing with η. Then we define

$$\Gamma_1 \subseteq \Gamma_\psi := \Gamma_\psi^{N^-}(N^+) := \{\gamma \in \Gamma_0 \,:\, \psi(\eta\gamma) = 1\} \subseteq \Gamma_0.$$

The group Γ_0 acts on $\mathcal{E}(R_c,\mathcal{O})$ by conjugation, and we denote by $\mathcal{E}(R_c,\mathcal{O})/\Gamma_0$ the set of conjugacy classes. Any element $W_\varepsilon \in \Gamma_0\backslash\Gamma_\psi$ defines an involution on $\mathcal{E}(R_c,\mathcal{O})/\Gamma_0$ which interchanges the preimages of the natural projection $\mathcal{E}(R_c,\mathcal{O})/\Gamma_\psi \to \mathcal{E}(R_c,\mathcal{O})/\Gamma_0$. In addition to W_ε, there is also an Atkin–Lehner involution acting naturally on the set of embeddings, although it does not preserve the normalization. To be more precise, let ω_N be an element in B such that

- for every $\wp \mid N^-$, ω_N generates the single two-sided ideal of $\mathcal{O} \otimes R_{0,\wp}$ of norm $\wp$, and
- for every $\wp \mid N^+$, $\iota_\wp(\omega_N) = \begin{pmatrix} 0 & -1 \\ \pi_\wp & 0 \end{pmatrix}$, where $\pi_\wp$ is any uniformizer in $R_{0,\wp}$.

Let us denote by $\bar{\mathcal{E}}(R_c,\mathcal{O})$ the set of optimal embeddings normalized with respect to $\bar{\mathfrak{N}}^+$. Then the map $\varphi \mapsto W_N(\varphi) := \omega_N \varphi \omega_N^{-1}$ is a bijection between $\mathcal{E}(R_c,\mathcal{O})$ and $\bar{\mathcal{E}}(R_c,\mathcal{O})$. From now on denote by $W_N(P_\varphi^\beta)$ the point $P_{W_N(\varphi)}^\beta$.

Finally, there is also a natural action of $\hat{K}^\times$ on $\mathcal{E}(R_c,\mathcal{O})$, which works as follows. Pick a finite idèle $x \in \hat{K}^\times$ and an embedding φ in $\mathcal{E}(R_c,\mathcal{O})$. Since the class number of $\mathcal{O}$ is $h(\mathcal{O}) = h(F_0^*) = 1$ by [Vi, Cor. 5.7 bis], the fractional ideal $I_x = \varphi(x)\hat{\mathcal{O}} \cap B$ is principal, generated by some $\gamma_x \in B^\times$ with $n(\gamma_x) \in F_0^+$. Moreover, we can choose γ_x such that $a_x = \varphi(x_{\mathfrak{N}^+} x_{\bar{\mathfrak{N}}^+})^{-1} \cdot \gamma_x$ lies in the kernel of $\psi\eta$. (Indeed, note that, locally at the primes $\wp \mid N^+$, we have $\varphi(x_{\mathfrak{N}^+} x_{\bar{\mathfrak{N}}^+})^{-1} I_{x,\wp} = \mathcal{O}_\wp$ and thus a_x belongs to $\mathcal{O}_\wp^\times$. It hence makes sense to consider its image under $\psi\eta$. We can assume γ_x is as claimed by replacing it by a suitable unit in $\mathcal{O}^\times$.) We define $x \star \varphi := \gamma_x^{-1} \circ \varphi \circ \gamma_x$. Observe that U_φ^+ acts trivially on $\mathcal{E}(R_c,\mathcal{O})$.

For $y \in K_\infty^\times$ and a character $\beta\colon \Sigma \to \{\pm 1\}$, set

$$\beta(y) = \prod_{j=2}^r \beta(\mathrm{sign}(\prod_{w|v_j} y_w)).$$

The following statement collects, in a precise form, the conjectures (SH1), (SH2), (SH3) that were somewhat vaguely formulated in the introduction for Darmon points over abelian extensions of K.

Conjecture 2.14.   *(1) If $\varphi \in \mathcal{E}(R_c, \mathcal{O})$ then $P_\varphi^\beta \overset{?}{\in} A_1(L_\varphi^\beta)$.*
    (2) For any character $\chi : \mathrm{Gal}(L_\varphi^\beta/K) \to \mathbb{C}^\times$, the point

$$P_\chi := \sum_{\sigma \in \mathrm{Gal}(L_\varphi^\beta/K)} \chi(\sigma)^{-1} \sigma(P_\varphi^\beta) \in A_1(L_\varphi^\beta) \otimes_\mathbb{Z} \mathbb{C}$$

   is non-zero if and only if $L'(f_0/K, \chi, 1) \neq 0$.
    (3) For any $a = (a_\infty, a_f) \in \mathbb{A}_K^\times$ we have that $\mathrm{rec}(a)P_\varphi^\beta = \beta(a_\infty)P_{a_f \star \varphi}$. In addition, for any $\tau \in \mathrm{Gal}(H_\varphi^\beta/F_0)$ whose restriction to K is not trivial, there exists an element $\sigma \in \mathrm{Gal}(H_\varphi^\beta/K)$ such that

$$\tau(P_\varphi^\beta) = W_N(\sigma(P_\varphi^\beta)) \pmod{A_1(H_\varphi^\beta)_{\mathrm{tors}}}.$$

Here $\mathrm{rec} : \mathbb{A}_K^\times \longrightarrow \mathrm{Gal}(K^{\mathrm{ab}}/K)$ is Artin's reciprocity map, normalized so that uniformizers correspond to geometric Frobenius elements. Note that the three statements of Conjecture 2.14 are the translation to the current context of (SH1), (SH2), (SH3) given in the introduction.

2.4. **Darmon-Logan's algorithm for the computation of ATR points.** One naturally wonders whether Darmon points, as introduced in Definition 2.13, can be computed effectively in explicit examples. A positive answer would allow us to test Conjecture 2.14 numerically, leading to an explicit construction of rational points on elliptic curves over number fields which were not accessible before.

However, the image of T_φ under the Abel-Jacobi map AJ^β of (8) can only be computed provided we are able to write down an explicit candidate for a region $\tilde{T}_\varphi$ having T_φ as boundary and we can integrate it against the differential form $\omega_{f_0}^\beta$. The latter only seems possible when there is available a natural, explicit description of $\omega_{f_0}^\beta$. And this is precisely the case when the following Gross-Zagier assumption holds:

Assumption 2.15. $r = d$ and all the primes dividing N are split in K/F_0.

Indeed, when this is the case we have that K/F_0 is an ATR extension, $B \simeq \mathrm{M}_2(F_0)$ and X is a d-dimensional Hilbert modular variety over $F_0^\star = \mathbb{Q}$. In addition, and most importantly, the form $\omega_{f_0}^\beta$ admits a natural fourier expansion around the cusp at infinity, and there exist algorithms which allow to compute it up to a given precision: cf. e.g. [DV].

If this hypothesis does not hold true, we are at a loss to compute numerical approximations to the points P_φ^β. We impose Assumption 2.15 for the remainder of this section, that we devote to sketch Darmon-Logan's algorithm for computing an explicit chain $\tilde{T}_\varphi$ whose boundary is T_φ. We adapt it to our slightly more general setting in which $[\mathbb{Q}_{f_0} : \mathbb{Q}] \geqslant 1$, so that we can also make use of it later. To simplify the exposition, and since this is the case encountered in the numerical example described in §3.3, let us assume also that $[F_0 : \mathbb{Q}] = 2$.

The key point in Darmon–Logan's approach is the definition of certain 3-limit integrals of $\omega_{f_0}^\beta$, allowed by the following interpretation of the homology groups of X. Let Γ denote the quotient of Γ_ψ by the normal closure of the subgroup generated by the elliptic and parabolic elements. Let I_Γ be the augmentation ideal, which sits in the exact sequence

$$0 \longrightarrow I_\Gamma \longrightarrow \mathbb{Z}[\Gamma] \longrightarrow \mathbb{Z} \longrightarrow 0.$$

For a Γ-module M we denote by $M_\Gamma = M/I_\Gamma M$ its ring of Γ-coinvariants. Tensoring the above sequence by I_Γ and taking the group homology exact sequence we obtain

$$(10) \qquad 0 \longrightarrow H_1(\Gamma, I_\Gamma) \longrightarrow (I_\Gamma \otimes_\mathbb{Z} I_\Gamma)_\Gamma \overset{\partial}{\longrightarrow} (Z[\Gamma] \otimes_\mathbb{Z} I_\Gamma)_\Gamma \longrightarrow (I_\Gamma)_\Gamma \longrightarrow 0,$$

where ∂ is the natural map induced by the inclusion $I_\Gamma \subset Z[\Gamma]$. There are canonical isomorphisms $(I_\Gamma)_\Gamma \simeq H_1(\Gamma, \mathbb{Z})$ and $H_1(\Gamma, I_\Gamma) \simeq H_2(\Gamma, \mathbb{Z})$. Therefore, in view of the natural isomorphisms $H_1(\Gamma, \mathbb{Z}) \simeq H_1(X, \mathbb{Z})$ and $H_2(\Gamma, \mathbb{Z}) \simeq H_2(X, \mathbb{Z})$ one can identify (10) with the exact sequence

$$(11) \qquad 0 \longrightarrow Z_2(X, \mathbb{Z}) \longrightarrow C_2(X, \mathbb{Z}) \overset{\partial}{\longrightarrow} Z_1(X, \mathbb{Z}) \longrightarrow H_1(X, \mathbb{Z}) \longrightarrow 0,$$

where δ is the topological boundary map.

Recall that integrals of $\omega_{f_0}^{\beta}$ satisfy the following invariance property:

$$\int_x^y \int_z^t \omega_{f_0}^{\beta} = \int_{\gamma x}^{\gamma y} \int_{\gamma z}^{\gamma t} \omega_{f_0}^{\beta}, \quad \text{for all } \gamma \in \Gamma \text{ and } x, y, z, t \in \mathcal{H}.$$

We remark that in this expression γ is acting on the outer limits (resp. inner limits) of the integral through v_1 (resp. v_2). By choosing base points $z_1 \in \mathcal{H}_1$ and $z_2 \in \mathcal{H}_2$ one obtains then a group homomorphism

$$I_{z_1, z_2}^{\beta} : \quad \begin{array}{ccc} (I_\Gamma \otimes_{\mathbb{Z}} I_\Gamma)_\Gamma & \longrightarrow & \mathbb{C} \\ (\gamma_1 - 1) \otimes (\gamma_2 - 1) & \longmapsto & \int_{z_1}^{\gamma_1 z_1} \int_{z_2}^{\gamma_2 z_2} \omega_{f_0}^{\beta}, \end{array}$$

which can be identified with the map

$$\begin{array}{ccc} C_2(X, \mathbb{Z}) & \longrightarrow & \mathbb{C} \\ T & \longmapsto & \int_T \omega_{f_0}^{\beta}. \end{array}$$

Observe that the identification $H_1(\Gamma, I_\Gamma) \simeq Z_2(X, \mathbb{Z})$ yields then an explicit description of the lattice $\Lambda_{f_0}^{\beta}$; indeed $\Lambda_{f_0}^{\beta} \simeq I_{z_1, z_2}^{\beta}(H_1(\Gamma, I_\Gamma))$.

Suppose now that $1 \otimes (\gamma_2 - 1) \in \mathbb{Z}[\Gamma] \otimes I_\Gamma$ is such that $e(1 \otimes (\gamma_2 - 1))$ lies in the image of δ for some integer e. That would correspond in (11) to a cycle T such that eT is null homologous. Following [DL] one defines

$$(12) \qquad \int_{z_2}^{z_1} \int_{z_2}^{\gamma_2 z_2} \omega_{f_0}^{\beta} := \frac{1}{e} I_{z_1, z_2}^{\beta}(\partial^{-1}(e \cdot (1 \otimes (\gamma_2 - 1)))) \in \mathbb{C}/\Lambda_{f_0}^{\beta}.$$

This is indeed a well-defined quantity in $\mathbb{C}/\Lambda_{f_0}^{\beta}$, because any two preimages of $e(1 \otimes (\gamma_1 - 1))$ by δ differ by an element of $Z_2(X, \mathbb{Z})$. It is sometimes convenient to use expressions such as $\int_y^x \int_y^z \omega_{f_0}^{\beta}$, but we warn the reader that they only make sense if $z = \gamma y$ for some $\gamma \in \Gamma$ and $e \cdot (1 \otimes \gamma) \in \text{im}(\delta)$ for some e. It follows from the definitions that the 3-limit integrals of (12) enjoy the following properties:

$$(13) \qquad \int_y^x \int_y^z \omega_{f_0}^{\beta} = \int_{\gamma y}^{\gamma x} \int_{\gamma y}^{\gamma z} \omega_{f_0}^{\beta} \quad \text{for all } \gamma \in \Gamma,$$

$$(14) \qquad \int_y^x \int_y^z \omega_{f_0}^{\beta} = \int_y^x \int_y^t \omega_{f_0}^{\beta} + \int_t^x \int_t^z \omega_{f_0}^{\beta},$$

$$(15) \qquad \int_t^y \int_t^z \omega_{f_0}^{\beta} - \int_t^x \int_t^z \omega_{f_0}^{\beta} = \int_x^y \int_t^z \omega_{f_0}^{\beta}.$$

Now let K/F_0 be a quadratic ATR extension and let $\varphi : R_c \hookrightarrow \mathcal{O}$ be a normalized optimal embedding of conductor c. Denote by z_1 the unique fixed point of K acting on $\mathcal{H}_1$ through v_1. The stabilizer Γ_φ of z_1 in Γ is an abelian group or rank 1 (cf. [DL, Proposition 1.4]). Call γ_φ one of its generators. Let $z_2, z_2' \in \partial \mathcal{H}_2$ denote the two fixed points of K acting through v_2. Then we have that

$$\int_{\tilde{T}_\varphi} \omega_{f_0}^{\beta} = \int^{z_1} \int_{z_2}^{z_2'} \omega_{f_0}^{\beta} = \int^{z_1} \int_{z_2}^{\gamma_\varphi z_2} \omega_{f_0}^{\beta}.$$

Using properties (13), (14) and (15) it is easy to check that the last integral does not depend on z_2. Therefore, we see that

$$(16) \qquad \int_{\tilde{T}_\varphi} \omega_{f_0}^{\beta} = \int^{z_1} \int_x^{\gamma_\varphi x} \omega_{f_0}^{\beta}$$

for any $x \in \mathcal{H}_2 \cup \mathbb{P}^1(F_0)$. If $N = 1$ an algorithm for computing 3-limit integrals as the one in (16) is given in [DL, §4], by means of the *continued fractions trick*. To the best of our knowledge, for arbitrary level N at the moment no generalization of this algorithm is known (cf. also [Ga1, Annexe A2]).

3. Almost totally complex points

This section is devoted to the main construction of the article. It is an explicitly computable construction of points on certain elliptic F_0-curves. By granting conjectures of §2 over ATR extensions, these points are shown to be rational over ATC fields. Recall that for a field extension F/F_0, an elliptic curve E/F is said to be an *elliptic F_0-curve* if it is isogenous over F to all of its $\mathrm{Gal}(F/F_0)$-conjugates.

The construction of ATC points on F_0-curves is given in 3.1. In 3.2 we compare our ATC points with Gartner's Darmon points, and conjecture a precise relation between them. Finally, in 3.3 we carry out an explicit calculation of such an ATC point for a particular elliptic curve. At the same time of giving the details of how explicit computations can be handled, we numerically verify that the obtained point satisfies the conjectures of §2, which provides certain evidence for their validity.

3.1. **Construction of ATC points.** Let F_0 be a totally real number field of narrow class number 1 and degree r. We denote by $v_1, \ldots, v_r$ the embeddings of F_0 into $\mathbb{C}$, and we regard F_0 as a subfield of $\mathbb{C}$ via v_1. We will also regard all extensions of F_0 as embedded in $\mathbb{C}$ via a fixed extension of v_1 to $\overline{F}_0$, which we denote by v_1 as well. Let $F = F_0(\sqrt{N_0})$ be a totally real quadratic extension and let E/F be an elliptic F_0-curve without complex multiplication.

We denote by $A = \mathrm{Res}_{F/F_0} E$ the variety over F_0 obtained by restriction of scalars. If E is not isogenous to the base change of an elliptic curve defined over F_0, then A/F_0 is simple and $\mathbb{Q} \otimes \mathrm{End}_{F_0}(A)$ is isomorphic to a quadratic field. From now on we restrict to the following setting.

Assumption 3.1. $\mathbb{Q} \otimes \mathrm{End}_{F_0}(A)$ is a quadratic imaginary field.

We shall also make the following assumption, which is a consequence of the generalized Shimura–Taniyama Conjecture for abelian varieties of GL_2-type.

Assumption 3.2. There exists a normalized Hilbert modular form f_0 over F_0 of parallel weight 2 such that A is isogenous to A_{f_0} (where we recall that A_{f_0} is the modular abelian variety attached to f_0 by means of the generalized Eichler–Shimura construction, cf. Assumption 2.1).

Therefore we can suppose that $A = A_{f_0}$. Observe that, since E is an F_0-curve, we have that $A \sim_F E^2$. Denote by N and ψ the level and the nebentypus of f_0 respectively and, for an ideal $\mathfrak{m}$ of F_0, denote by $a_{\mathfrak{m}}$ the Fourier coefficient of f_0 corresponding to $\mathfrak{m}$.

Lemma 3.3. *The character ψ is quadratic and F is the field corresponding by class field theory to the kernel of ψ.*

Proof. Denote by F_ψ the field cut by the kernel of ψ. Let $G = \mathrm{Gal}(\overline{\mathbb{Q}}/F_0)$, $H = \mathrm{Gal}(\overline{\mathbb{Q}}/F)$ and $H_\psi = \mathrm{Gal}(\overline{\mathbb{Q}}/F_\psi)$. It is enough to show that $H = H_\psi$ (the fact that ψ is quadratic follows from this because $[F : F_0] = 2$). Let ℓ be a prime number that splits in $\mathbb{Q}_{f_0}$, say as $\ell = \lambda\lambda'$, and denote by $V_\ell = T_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ the ℓ-adic Tate module of A. There is an isomorphism of $\mathbb{Q}_\ell[G]$-modules $V_\ell = V_\lambda \times V_{\lambda'}$, where $V_\lambda = E_\lambda \otimes_{E \otimes \mathbb{Q}_\ell} V_\ell$ and $V_{\lambda'} = E_{\lambda'} \otimes_{E \otimes \mathbb{Q}_\ell} V_\ell$.

Denote by ρ_λ and ρ'_λ the representations of G afforded by V_λ and $V_{\lambda'}$ respectively, which are irreducible because E is not CM. Since A is the variety attached to f_0 by the Eichler–Shimura construction, and relabeling λ and λ' if necessary, we can suppose that:

$$(17) \qquad \mathrm{Tr}(\rho_\lambda(\mathrm{Frob}_{\mathfrak{p}})) = a_{\mathfrak{p}} \quad \text{and} \quad \mathrm{Tr}(\rho_{\lambda'}(\mathrm{Frob}_{\mathfrak{p}})) = \overline{a}_{\mathfrak{p}}, \quad \text{for all primes } \mathfrak{p} \nmid N,$$

where the bar denotes complex conjugation. By [Sh, Theorem 2.5] the nebentypus ψ is characterized by the fact that $a_{\mathfrak{p}} = \overline{a}_{\mathfrak{p}}\psi(\mathfrak{p})$ for primes $\mathfrak{p} \nmid N$. Therefore V_λ and $V_{\lambda'}$ are isomorphic as $\mathbb{Q}_\ell[H_\psi]$ representations, so that $\mathrm{End}_{\mathbb{Q}_\ell[H_\psi]} V_\ell \simeq \mathrm{M}_2(\mathbb{Q}_\ell)$. Moreover, H_ψ is the largest subgroup of G for which this is true. On the other hand, we have that $\mathrm{End}_F(A) \otimes \mathbb{Q} \simeq \mathrm{End}_F(E^2) \otimes \mathbb{Q} \simeq \mathrm{M}_2(\mathbb{Q})$. By the case of Tate's Conjecture proven by Faltings this implies that $\mathrm{End}_{\mathbb{Q}_\ell[H]} V_\ell \simeq \mathrm{End}_F^0(A) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \simeq \mathrm{M}_2(\mathbb{Q}_\ell)$, from which we deduce that necessarily $H = H_\psi$. $\square$

Observe that, as a consequence of the conductor-discriminant formula, F has discriminant N over F_0. For simplicity we assume from now on that N is not divisible by any dyadic prime, and thus squarefree.

Let $M = F(\sqrt{\alpha})$ be a quadratic ATC extension of F. Recall that ATC stands for *almost totally complex*, and it means in this case that M has exactly two real places. We suppose that M is real under the place v_1. We aim to give an explicitly computable construction of points in $E(M)$, by making use of the conjectural constructions of Section 2.

Write $\mathrm{Gal}(F/F_0) = \{1, \tau\}$ and let $M' = F(\sqrt{\alpha^\tau})$. Clearly M is not Galois over F_0, and its Galois closure $\mathcal{M}$ is the composition of M and M'. It is easily seen that $\mathrm{Gal}(\mathcal{M}/F_0) \simeq D_{2\cdot4}$, the dihedral group of order 8. The field $K = F_0(\sqrt{\alpha\alpha^\tau})$ is contained in $\mathcal{M}$, and there exist fields L and L' such that the diagram of subfields of $\mathcal{M}/F_0$ is given by:

(18)
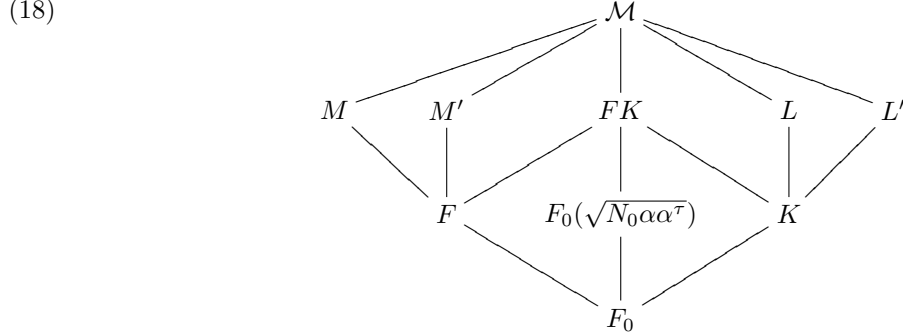
Our construction relies on the fact that K is ATR. Indeed, we have the following lemma.

Lemma 3.4. *The field K is ATR and it is complex under v_1. The fields L and L' are totally imaginary.*

Proof. The first assertion follows immediately from the definitions. The property about L comes from the fact that it can be identified with $K(\sqrt{\alpha} + \sqrt{\alpha^\tau}) = K(\sqrt{\alpha + \alpha^\tau + 2\sqrt{\alpha\alpha^\tau}})$, and similarly for L'. Since M is ATC, under a complex embedding of L the image of either $\sqrt{\alpha}$ or $\sqrt{\alpha^\tau}$ does not lie in $\mathbb{R}$. $\square$

Since K is an ATR extension which is complex under v_1 we are in the setting of §2.4. Let $c \subset R_0$ be an integral ideal and let R_c be the order of conductor c in R_K. Let $\mathcal{O}$ be the Eichler order of level N in $M_2(R_0)$ consisting on matrices which are upper triangular modulo N, and let $\varphi\colon R_c \hookrightarrow \mathcal{O}$ be an optimal embedding. Observe that the points P_φ^β constructed in Section 2 are explicitly computable in this case, because Assumption 2.15 holds true. Moreover, granting Conjecture 2.14, they belong to $A(H_\varphi^\beta)$. The key point is that, as we shall see in Proposition 3.8, for suitable choices of c and β the field M is contained in H_φ^β. Therefore, points in $E(M)$ can be constructed by projecting P_φ^β via the isogeny $A \sim_F E^2$, and then taking trace over M.

Before stating and proving Proposition 3.8 we need some preliminary results. Let $\chi_M, \chi_{M'}\colon \mathbb{A}_F^\times \to \{\pm1\}$ and $\chi_L, \chi_{L'}\colon \mathbb{A}_K^\times \to \{\pm1\}$ denote the quadratic Hecke characters corresponding to the fields M, M', L and L'. Similarly, let $\varepsilon_F, \varepsilon_K\colon \mathbb{A}_{F_0}^\times \to \{\pm1\}$ be the ones corresponding to F and K. Recall that $\varepsilon_F = \psi$ by Lemma 3.3.

Lemma 3.5. (1) $\chi_L\chi_{L'} = \psi \circ \mathrm{Nm}_{K/F_0}$.
   (2) *The central character of χ_L is ψ.*
   (3) *We have that $\mathrm{Ind}_F^{F_0} \chi_M \simeq \mathrm{Ind}_F^{F_0} \chi_{M'} \simeq \mathrm{Ind}_K^{F_0} \chi_L \simeq \mathrm{Ind}_F^{F_0} \chi_{L'}$ are isomorphic as representations of $\mathrm{Gal}(\mathcal{M}/F_0)$.*

Proof. Assertion (1) follows from the fact that $\chi_L\chi_{L'}$ is the quadratic character associated with the extension FK/K, which is $\psi \circ \mathrm{Nm}_{K/F_0}$. If we let σ denote the generator of $\mathrm{Gal}(K/F_0)$, then we have that $\chi_L(x^\sigma) = \chi_{L'}(x)$. Then from (1) we see that χ_L restricted to $\mathrm{Nm}_{K/F_0}\mathbb{A}_K^\times$ is equal to ψ. Then by class field theory the central character of χ_L is either ψ or $\psi\varepsilon_K$. But it cannot be $\psi\varepsilon_K$: let $u = (-1, 1, \cdots, 1) \in \mathbb{A}_{F_0,\infty}^\times$ (where the first position corresponds to the place v_1). Then $\psi\varepsilon_K(u) = -1$, but $\chi_L(u) = 1$ because v_1 extends to a complex embedding of K. Finally, (3) follows from the fact that the group $D_{2\cdot4}$ has a unique 2-dimensional irreducible representation. $\square$

Proposition 3.6. *Let* $\mathfrak{d}_{L/K}$ *denote the discriminant of* L/K. *Then* $\mathfrak{d}_{L/K} = c \cdot \mathfrak{N}$, *where* c *is an ideal of* F_0 *and* $\mathrm{Nm}_{K/F_0}\mathfrak{N} = N$.

Proof. By the conductor-discriminant formula $\mathfrak{d}_{L/K}$ equals the conductor of χ_L. Then the proposition is a consequence of the fact that the central character of χ_L is ψ, which has conductor N. We give the precise statements from which Proposition 3.6 follows as Lemma 3.7 below. $\square$

Lemma 3.7. *(1) If* $p \subset F_0$ *is a prime such that* $p \mid N$, *then either* p *splits or ramifies in* K. *In both cases, exactly one of the primes above* p *exactly divides the conductor of* χ_L.
(2) Let $p \subset F_0$ *be a prime such that* $p \nmid N$ *and* $\mathfrak{p}^e$ *divides exactly the conductor of* χ_L *for some prime* $\mathfrak{p} \subset K$ *above* p. *Then either* p *is inert in* K *or splits as* $p \cdot R_K = \mathfrak{p}\mathfrak{p}'$ *and* $(\mathfrak{p}')^e$ *divides exactly the conductor of* χ_L.

Proof. To prove (1), let p be a prime of F_0 dividing N. If p splits as $\mathfrak{p}\mathfrak{p}'$ in K then by Lemma 3.5 the composition

$$R_{0,p}^\times \longrightarrow R_{K,\mathfrak{p}}^\times \times R_{K,\mathfrak{p}'}^\times \overset{\chi_{L,\mathfrak{p}}\cdot\chi_{L,\mathfrak{p}'}}{\longrightarrow} \{\pm 1\}$$

equals ψ_p. Since by assumption p is not dyadic and N is squarefree, ψ_p is the unique character or order 2 of $R_{0,p}^\times/(1+p)$. Since $R_{K,\mathfrak{p}}^\times/(1+\mathfrak{p}) \simeq R_{K,\mathfrak{p}'}^\times/(1+\mathfrak{p}') \simeq R_{0,p}^\times/(1+p)$ we see that the character

$$\begin{array}{ccc}
R_{0,p}^\times/(1+p) \times R_{0,p}^\times/(1+p) & \overset{\chi_{L,\mathfrak{p}}\cdot\chi_{L,\mathfrak{p}'}}{\longrightarrow} & \{\pm 1\} \\
(x,x) & \longmapsto & \chi_{L,\mathfrak{p}}(x) \cdot \chi_{L,\mathfrak{p}'}(x)
\end{array}$$

has order 2. This implies that exactly one of $\chi_{L,\mathfrak{p}}$ or $\chi_{L,\mathfrak{p}'}$ is trivial. Suppose that $\chi_{L',\mathfrak{p}}$ is trivial and $\chi_{L,\mathfrak{p}}$ has order 2. Then $\mathfrak{p}$ divides exactly the conductor of χ_L and $\mathfrak{p}'$ does not divide it.

Suppose now that $p \mid N$ is ramified in K so that $pR_K = \mathfrak{p}^2$. Then by Lemma 3.5 the composition

$$R_{0,p}^\times \longrightarrow R_{K,\mathfrak{p}}^\times \overset{\chi_{L,\mathfrak{p}}}{\longrightarrow} \{\pm 1\}$$

equals ψ_p, which is a character of order 2 factorizing through $R_{0,p}^\times/(1+p)$. This implies that $\chi_{L,\mathfrak{p}}$ necessarily factorizes through $R_{K,\mathfrak{p}}^\times/(1+\mathfrak{p})$, because $R_{K,\mathfrak{p}}^\times/(1+\mathfrak{p}) \simeq R_{0,p}^\times/(1+p)$. Therefore $\mathfrak{p}$ divides exactly the conductor $\mathfrak{d}_{L/K}$ of χ_L.

Suppose now that $p \mid N$ is inert in K, so that $pR_K = \mathfrak{p}$. Again by Lemma 3.5 the character ψ_p equals

$$(19) \qquad\qquad R_{0,p}^\times \longrightarrow R_{K,\mathfrak{p}}^\times \overset{\chi_{L,\mathfrak{p}}}{\longrightarrow} \{\pm 1\},$$

the composition of the natural inclusion with $\chi_{L,\mathfrak{p}}$. But the map in (19) is trivial. Indeed, in this case $\mathbb{F}_p^\times = R_{0,p}^\times/(1+p)$ is strictly contained in $\mathbb{F}_{\mathfrak{p}}^\times = R_{K,\mathfrak{p}}^\times/(1+\mathfrak{p})$. Then $\chi_{L,\mathfrak{p}}$ is the unique quadratic character of $\mathbb{F}_{\mathfrak{p}}^\times$, and such character is always trivial on $\mathbb{F}_p^\times$. The fact that ψ_p is trivial contradicts the fact that $p \mid N$, so this case does not occur.

To prove (2) we use again that the localization at p of the composition

$$(20) \qquad\qquad \mathbb{A}_{F_0}^\times \longrightarrow \mathbb{A}_K^\times \overset{\chi_L}{\longrightarrow} \{\pm 1\}$$

coincides with ψ_p, and therefore it is trivial because in this case $p \nmid N$. But $\chi_{L,\mathfrak{p}}$ has order 2, so that in particular it is not trivial. Suppose that $\chi_{L,\mathfrak{p}}$ has conductor $\mathfrak{p}^e$ for some $e \geqslant 1$. Observe that now, since $\mathfrak{p}$ can be dyadic, the exponent e may be greater than 1 (in fact, it is equal to 1 except if $\mathfrak{p}$ is dyadic, in which case it may also be 2 or 3). In any case, the localization of (20) at $\mathfrak{p}$ is trivial only in one of the following situations:

i) The inclusion $\mathbb{A}_{F_0}^\times \longrightarrow \mathbb{A}_K^\times$ localizes to a strict inclusion $R_{0,p}^\times/(1+p^e) \hookrightarrow R_{K,\mathfrak{p}}^\times/(1+\mathfrak{p}^e)$.
ii) The map in (20) localizes to

$$\begin{array}{ccccc}
R_{0,p}^\times/(1+p^e) & \longrightarrow & R_{0,p}^\times/(1+p^e) \times R_{0,p}^\times/(1+p^e) & \overset{\chi_{L,\mathfrak{p}}\cdot\chi_{L,\mathfrak{p}'}}{\longrightarrow} & \{\pm 1\} \\
x & \longmapsto & (x,x) & \longmapsto & \chi_{L,\mathfrak{p}}(x) \cdot \chi_{L,\mathfrak{p}'}(x)
\end{array}$$

and $\chi_{L,\mathfrak{p}} = \chi_{L,\mathfrak{p}'}$.

In the first case p is inert in K. In the second case p splits as $p \cdot R_K = \mathfrak{p}\mathfrak{p}'$ and $(\mathfrak{p}')^e$ divides exactly the conductor of χ_L.

<div align="right">□</div>

Proposition 3.8. *Let $\varphi \colon R_c \hookrightarrow \mathcal{O}$ be a normalized optimal embedding, with c as in Proposition 3.6. The field L_φ^β contains L if and only if $\beta_j(-1) = -1$ for $j = 2, \ldots, r$.*

Proof. Recall that

$$U_c = \hat{R}_0^\times (1 + c\hat{R}_K) \subset \hat{K}^\times$$

and that

$$U_\varphi^+ = \{\beta \in U_c \text{ such that } (\beta)_\mathfrak{N} \in \ker(\psi) \subset (R_0/NR_0)^\times\},$$

where ψ is the nebentypus of f_0 and also the character corresponding to the quadratic extension F/F_0. Here $(\beta)_\mathfrak{N}$ denotes the image of the local term of the idèle β in the quotient $R_{K,\mathfrak{N}}^\times/(1 + \mathfrak{N} \cdot R_{K,\mathfrak{N}}) \simeq (R_0/NR_0)^\times$. The field L_φ^β is defined by

$$\mathrm{Gal}(L_\varphi^\beta/K) \simeq \mathbb{A}_K^\times/K^\times U_c^+ K_\infty^\beta,$$

where K_∞^β is as in (9). Now let $\chi_L \colon \mathbb{A}_K^\times \to \{\pm 1\}$ be the quadratic character corresponding to L. Observe that by class field theory $L \subset L_\varphi^\beta$ if and only if $U_\varphi^+ K_\infty^\beta \subset \ker \chi_L$.

Let $\chi_L = \prod_v \chi_{L,v}$ be the decomposition of χ_L as a product of local characters. By the conductor–discriminant formula the conductor of χ_L is equal to $\mathfrak{d}_{L/K} = c \cdot \mathfrak{N}$. This means that $\chi_{L,f} = \prod_{\mathfrak{p} \nmid \infty} \chi_\mathfrak{p}$ factorizes through a character

$$\chi_{L,f} \colon R_{K,c \cdot \mathfrak{N}}^\times/(1 + c \cdot \mathfrak{N} R_{K,c \cdot \mathfrak{N}}) \to \{\pm 1\}.$$

First of all we check that $\chi_L(U_\varphi^+ \cap R_{K,c \cdot \mathfrak{N}}^\times) = 1$. Let $a = (a_\mathfrak{p})_\mathfrak{p}$ be an element in $U_\varphi^+ \cap R_{K,c \cdot \mathfrak{N}}^\times$. We write it as $a = a_c \cdot a_\mathfrak{N}$, where $a_c = \prod_{\mathfrak{p}|c} a_\mathfrak{p}$ and $a_\mathfrak{N} = \prod_{\mathfrak{p}|\mathfrak{N}} a_\mathfrak{p}$.

If $\mathfrak{p} \mid c$ then $\chi_{L,\mathfrak{p}}(a_\mathfrak{p}) = 1$ by the very definition of U_φ^+. Namely, if $e = v_\mathfrak{p}(c)$ then $\chi_{L,\mathfrak{p}}$ has conductor $\mathfrak{p}^e$ so it can be regarded as a character

$$\chi_{L,\mathfrak{p}} \colon R_{K,\mathfrak{p}}^\times/(1 + \mathfrak{p}^e R_{K,\mathfrak{p}}) \to \{\pm 1\}.$$

But $a_\mathfrak{p}$ belongs to $(1 + \mathfrak{p}^e R_{K,\mathfrak{p}})$ by the definition of U_φ^+, so that $\chi_{L,p}(a_\mathfrak{p}) = 1$. Since this is valid for any $\mathfrak{p} \mid c$ we see that $\chi_L(a_c) = 1$.

Since $\mathfrak{N}$ has norm N and N is squarefree we have that $R_{K,\mathfrak{N}}^\times/(1 + \mathfrak{N} \cdot R_{K,\mathfrak{N}}) \simeq (R_0/NR_0)^\times$. Therefore the image of $a_\mathfrak{N}$ via the map $\mathbb{A}_K^\times \to R_{K,\mathfrak{N}}^\times/(1 + \mathfrak{N} \cdot R_{K,\mathfrak{N}})$ can be regarded as the image of some $b \in \mathbb{A}_{F_0}^\times$ via the map $\mathbb{A}_{F_0}^\times \to \mathbb{A}_K^\times \to R_{K,\mathfrak{N}}^\times/(1 + \mathfrak{N} \cdot R_{K,\mathfrak{N}})$. By Lemma 3.5 we have that $\chi_{L|\mathbb{A}_{F_0}} = \psi$. Therefore, by the definition of U_φ^+ we see that $\chi_L(a_\mathfrak{N}) = \psi(a_\mathfrak{N}) = 1$.

Since we have seen that $U_\varphi^+ \subseteq \ker \chi_L$, we have that $L \subseteq L_\varphi^\beta$ if and only if $\chi_L(K_\infty^\beta) = 1$. It is clear that for the character β such that $\beta_j(-1) = -1$ for $j = 2, \ldots, r$ this is true, because then any character of $\mathbb{A}_{K,\infty}^\times$ is trivial when restricted to K_∞^β. Suppose now that β is such that $\beta_j(-1) = 1$ for some j. Then the j-th component of K_∞^β is equal to $\mathbb{R}^\times$, and χ_L is not trivial restricted to this component because, by Lemma 3.4, the field L is totally imaginary so the real place v_j extends to a complex place of L. □

Now we let c be as in Proposition 3.8, and we take $\beta \colon \Sigma \to \{\pm 1\}$ to be the character such that $\beta_j(-1) = -1$ for $j = 2, \ldots, r$. Moreover we let $\varphi \colon R_c \hookrightarrow \mathcal{O}$ be an optimal embedding normalized with respect to $\mathfrak{N}$, with $\mathfrak{N}$ as in Proposition 3.6. From now on we grant Conjecture 2.14 so that $P_\varphi^\beta \in A(L_\varphi^\beta)$. Thanks to Proposition 3.8 we can set

$$P_{A,L} = \mathrm{Tr}_{L_\varphi^\beta/L}(P_\varphi^\beta) \in A(L).$$

If we denote by $C_L = \mathrm{rec}^{-1}(\mathrm{Gal}(L_\varphi^\beta/L))$, then by the reciprocity law of Conjecture 2.14 $P_{A,L}$ can be computed as

$$P_{A,L} = \sum_{a \in C_L} (P_{a \star \varphi}^\beta) \in A(L).$$

Observe that in Diagram (18) complex conjugation takes L to L'. Therefore the point

$$P_{A,M} := P_{A,L} + \overline{P_{A,L}}$$

lies in $A(M)$. Finally, we define

$$P_M = \pi(P_{A,M}) \in E(M),$$

where $\pi\colon A \to E$ is the natural projection, an algebraic map defined over F.

Theorem 3.9. *Assume Conjecture 2.14 holds true for the ATR extension K/F. Suppose also that the sign of the functional equation of $L(E/F, s)$ is $+1$ and that of $L(E/M, s)$ is -1. Then P_M is non-torsion if and only if $L'(E/M, 1) \neq 0$.*

Proof. If $L'(E/M, s) \neq 0$ then $L(E/M, s)$ vanishes with order 1 at $s = 1$. Since

$$L(E/M, s) = L(E/F, s)L(E/F, \chi_M, s)$$

we see that $L(E/F, \chi_M, s)$ vanishes with order 1 at $s = 1$. By Lemma 3.5 we have that $\mathrm{Ind}_F^{F_0} \chi_M \simeq \mathrm{Ind}_K^{F_0} \chi_L$. Then

$$
\begin{aligned}
L(E/F, \chi_M, s) &= L(f_0/F \otimes \chi_M, s) = L(f_0 \otimes \mathrm{Ind}_F^{F_0}\chi_M, s) \\
&= L(f_0 \otimes \mathrm{Ind}_K^{F_0}\chi_L, s) = L(f_0/K \otimes \chi_L, s) \\
&= L(f_0/K, \chi_L, s),
\end{aligned}
$$

and therefore $L(f_0/K, \chi_L, s)$ vanishes with order 1 at $s = 1$. If we denote by $\chi\colon \mathrm{Gal}(L_\varphi^\beta/K) \to \mathbb{C}$ the induction of χ_L, then part (2) of 2.14 implies that the point

$$P_\chi = \sum_{\sigma \in \mathrm{Gal}(L_\varphi^\beta/K)} \chi(\sigma)^{-1}\sigma(P_\varphi^\beta) \in A(L_\varphi^\beta)$$

is non-torsion.

In order to apply the reciprocity law, let us view for a moment the fields K, L and L_φ^β as subfields of $\mathbb{C}$ via a place of $\overline{\mathbb{Q}}$ extending v_j, for a fixed $j \in \{2, \ldots, r\}$. Since K is real under v_j and L is complex, we see that complex conjugation induces an element in $s \in \mathrm{Gal}(L_\varphi^\beta/K)$ that restricts to a generator of $\mathrm{Gal}(L/K)$. But s corresponds under the reciprocity map to the idèle

$$(21) \qquad \xi_j := (\xi_\infty, \xi_f) = (1, \ldots, 1, \overset{j)}{-1}, 1, \ldots, 1) \times (1, 1, \ldots) \in K_\infty^\times \times \hat{K}^\times,$$

so by part (3) of 2.14 we have that $s(P_\varphi^\beta) = \beta(\xi_\infty)P_\varphi^\beta = -P_\varphi^\beta$. Then we have that

$$
\begin{aligned}
P_\chi &= \sum_{\sigma \in \mathrm{Gal}(L_\varphi^\beta/L)} \sigma(P_\varphi^\beta) + \sum_{\sigma \in \mathrm{Gal}(L_\varphi^\beta/L)} \chi(\sigma s)\sigma s(P_\varphi^\beta) \\
&= \sum_{\sigma \in \mathrm{Gal}(L_\varphi^\beta/L)} \sigma(P_\varphi^\beta) + \sum_{\sigma \in \mathrm{Gal}(L_\varphi^\beta/L)} \chi_L(s)\sigma(-P_\varphi^\beta) \\
&= 2 \cdot \mathrm{Tr}_{L_\varphi^\beta/L}(P_\varphi^\beta) = 2 \cdot P_{A,L},
\end{aligned}
$$

which implies that $P_{A,L}$ is non-torsion. Moreover, as $s(P_\varphi^\beta) = -P_\varphi^\beta$ we have that $P_{A,L} \in A(L)^{\chi_L}$. Then $P_{A,M} = P_{A,L} + \overline{P_{A,L}}$ belongs to $A(M)$ and is non-torsion as well. Since the projection $\pi\colon A \to E$ is defined over F and $A(M) \simeq E^2(M)$, we see that $P_M = \pi(P_A(M))$ belongs to $E(M)$ and it is of infinite order. $\qquad \square$

Let W_N denote the Atkin-Lehner involution on $S_2(\Gamma_\psi(N))$ corresponding to the ideal N. By abuse of notation we also denote by W_N the involution that it induces on A. Then the splitting of the variety A over F is accomplished by the action of W_N. More precisely we have that

$$A \sim_F (1 + W_N)A \times (1 - W_N)A.$$

Let λ_N be the pseudoeigenvalue of f_0 corresponding to N; that is, the complex number satisfying that $W_N(f_0) = \lambda_N \cdot \overline{f}_0$. Observe that the modular form

$$\alpha_{f_0}^\beta := \frac{1}{1 + \lambda_N}(f_0 + W_N(f_0))$$

is normalized. In view of Conjecture 2.8 the lattice of E can be computed as

$$(22) \qquad \Lambda_E = (\Omega_2^- \cdots \Omega_r^-)^{-1} \cdot \langle \int_Z \alpha_{f_0}^\beta \rangle,$$

where $Z \in H_2(X_\psi(\mathbb{C}), \mathbb{Z})$ runs over the cycles such that $\int_Z (\omega_{f_0}^\beta - W_N(\omega_{f_0}^\beta)) = 0$. From this we obtain the following explicit analytic formula for the points P_M.

Theorem 3.10. *Let*

$$(23) \qquad J_\mathcal{M} = (\Omega_2^- \cdots \Omega_r^-)^{-1} \cdot \left(\sum_{a \in C_L} \int_{\tilde{T}_{a \star \varphi}} \alpha_{f_0}^\beta \right).$$

Then the point P_M can be computed as

$$(24) \qquad P_M = \eta \left(J_\mathcal{M} + \overline{J}_\mathcal{M} \right),$$

where η is the Weierstrass parametrization $\eta \colon \mathbb{C}/\Lambda_E \to E(\mathbb{C})$ and the bar denotes complex conjugation.

Proof. The Atkin–Lehner involution W_N is defined over F and $FL = \mathcal{M}$, so $\eta(J_\mathcal{M})$ belongs to $E(\mathcal{M})$. We recall that we are viewing $\mathcal{M}$ as a subfield of $\mathbb{C}$ by means of v_1. Under this embedding $\mathcal{M}$ is complex and M is real and therefore

$$P_M = \mathrm{Tr}_{\mathcal{M}/M}(\eta(J_\mathcal{M})) = \eta(J_\mathcal{M}) + \overline{\eta(J_\mathcal{M})}.$$

Since E is defined over F and $F \subseteq \mathbb{R}$ we have that Weierstrass map commutes with complex conjugation, and (24) follows. $\qquad \square$

Remark 3.11. Observe that $W_N(P_L) = \mathrm{Tr}_{L'^\beta_\varphi/L'}(W_N(P_\varphi^\beta))$ belongs to $A(L')$. Since complex conjugation does not fix K, by part *(3)* of Conjecture 2.14 we see that

$$\overline{P_{A,L}} = W_N(\sigma(P_{A,L})) + P_t$$

for some $\sigma \in \mathrm{Gal}(L/K)$ and some $P_t \in A(L')_{\mathrm{tors}}$. If σ turns out to be trivial and P_t belongs to $A(F_0)_{\mathrm{tors}}$, then the point $P_{A,L} + W_N(P_{A,L})$ is already defined over M. In this case $\eta(J_\mathcal{M})$ lies in $E(M)$ and P_M coincides, up to torsion, with $2 \cdot \eta(J_\mathcal{M})$. As we will see, this is the situation encountered in the example of §3.3.

Remark 3.12. Observe that the integral appearing in the formula of Theorem 3.10 is completely explicit. Indeed, in the case where f_0 has trivial nebentypus, an algorithm for determining the chains $\tilde{T}_\varphi$ is worked out in [DL], based on the approach taken in [Das2]. As we showed in §2.4, Darmon–Logan's method adapts to provide an explicit description of $\tilde{T}_\varphi$ also in the current setting, in which f_0 has quadratic nebentypus.

3.2. **Comparison with Gartner's ATC points.** Let us keep the notations of the previous section 3.1; in particular E/F is an elliptic curve defined over the totally real field F and M/F is an ATC quadratic extension. The curve E is modular: its isogeny class corresponds by the Eichler-Shimura construction to the Hilbert modular form f that one obtains from f_0 by base-change to F, in such a way that

$$L(E, s) = L(f, s)$$

as in (5). Write $N_E \subseteq R_F$ for the conductor of E, that is to say, the level of f. It is related to the level N of f_0 by the formula

$$(25) \qquad \mathrm{Norm}_{F/F_0}(N_E) \cdot \mathrm{disc}(F/F_0)^2 = N^2.$$

We place ourselves under the hypothesis of Theorem 3.9, so that we assume N_E is square-free, the sign of the functional equation of $L(E/F, s)$ is $+1$ and that of $L(E/M, s)$ is -1.

As discussed in §1, our point P_M in $E(M)$ is expected to coexist with another point P_M^{Gar} ([Ga2, §5.4]), provided Conjecture 2.14 for the abelian extensions of M holds true. This point can be manufactured by applying the machinery of §2.1, 2.2, 2.3, setting M/F to play the role of the extension K/F_0 of loc. cit.

Let us sketch the details: let B be the quaternion algebra over F which ramifies precisely at all the archimedean places of F but v_1, v_2 (over which M is complex) and at the prime ideals $\wp \mid N_E$ which remain inert in M. That this is a set of even cardinality is guaranteed by the sign of the functional equation of $L(E/M, s)$. Let $\mathcal{O}$ be an Eichler order in B of square-free level, divisible exactly by those primes $\wp \mid N_E$ which split in M.

Let R_M denote the ring of integers of M and fix a normalized optimal embedding $\varphi_M \in \mathcal{E}(R_M, \mathcal{O})$. In the notations of §2.1 and 2.2 we have $r = 2$ and $\Sigma = \{\pm 1\}$. Take β to be the trivial character and, granting Conjecture 2.14, let $P_\varphi \in E(L_\varphi^\beta)$ denote the Darmon point associated with this choice. Set

$$(26) \qquad\qquad P_M^{\text{Gar}} = \text{Tr}_{L_\varphi^\beta/M}(P_\varphi^\beta) \in E(M).$$

It is expected that the Néron-Tate height of P_M^{Gar} should be related to $L'(E/M, 1)$ while the Néron-Tate height of P_M constructed in this paper should be connected to $L'(E/F, \chi_M, 1)$. Hence from the basic quality

$$L'(E/M, 1) = L(E/F, 1)L'(E/F, \chi_M, 1),$$

we propose the following conjecture about the relation between P_M and P_M^{Gar}. Let

$$\Omega_{E/F} = \frac{\prod_{\tau: F \hookrightarrow \mathbb{R}} c_{E^\tau}}{\sqrt{\text{disc}(F)}}$$

where c_{E^τ} is either the real period or twice the real period of $E^\tau = E \times_\tau \mathbb{R}$, depending on whether $E^\tau(\mathbb{R})$ is connected or not.

Conjecture 3.13. *The point P_M^{Gar} is of infinite order if and only if P_M is of infinite order and $L(E/F, 1) \neq 0$. Moreover,*

$$P_M^{\text{Gar}} = 2^s \ell \cdot P_M,$$

where s is an integer which depends on M and $\ell \in \mathbb{Q}^\times$ satisfies $\ell^2 = \frac{L(E/F,1)}{\Omega_{E/F}}$.

3.3. **A numerical example.** In this section we give the details for the computation of an ATC point on a particular elliptic curve. We used Sage [Sage] for all the numerical calculations. We begin by describing the elliptic curve and the corresponding Hilbert modular form f_0, which we will take to be the base change of a modular form f over $\mathbb{Q}$.

3.3.1. *The curve and the modular form.* Let f be the (unique up to Galois conjugation) classical newform over $\mathbb{Q}$ of level 40 and nebentypus $\varepsilon(\cdot) = \left(\frac{10}{\cdot}\right)$. It corresponds to the third form of level 40 in the table 4.1 of the appendix to [Qu]. We see from this table that the modular abelian variety A_f has dimension 4. Moreover, it breaks as the fourth power of an elliptic curve E/F, where $F = \mathbb{Q}(\sqrt{2}, \sqrt{5})$. Jordi Quer computed an equation for E using the algorithms of [GL]; a global minimal model of E is given by:

$$(27) \qquad\qquad y^2 + b_1 xy + b_3 y = x^3 + b_2 x^2 + b_4 x + b_6,$$

where

$$\begin{aligned}
b_1 &= 1 - 9/2\sqrt{2} + 3\sqrt{5} - 1/2\sqrt{10}, \\
b_2 &= -15/2 + 13/2\sqrt{2} - 9/2\sqrt{5} + 5/2\sqrt{10}, \\
b_3 &= -11/2 - 27/2\sqrt{2} + 17/2\sqrt{5} + 3/2\sqrt{10}, \\
b_4 &= 41/2 + 8\sqrt{2} - 15/2\sqrt{5} - 8\sqrt{10}, \\
b_6 &= 525/2 + 8\sqrt{2} - 13/2\sqrt{5} - 84\sqrt{10}.
\end{aligned}$$

Let $F_0 = \mathbb{Q}(\sqrt{2})$ and let v_1 (resp. v_2) be the embedding taking $\sqrt{2}$ to the positive (resp. negative) square root of 2. Since E is a $\mathbb{Q}$-curve, it is also an F_0-curve. If we set $\alpha = \sqrt{10} + \sqrt{5} + \sqrt{2}$ then $M = F(\sqrt{\alpha})$ is an ATC extension of F. Since the conductor of E/F is equal to 1 the sign of the

functional equation of $L(E/F, s)$ is $+1$, and the sign of $L(E/M, s)$ is -1. The point $P_{\mathrm{nt}} \in E(M)$ whose x coordinate is given by

$$x = \frac{-3259 + 2126\sqrt{\alpha} - 8957\sqrt{\alpha}^2 + 5297\sqrt{\alpha}^3 - 4989\sqrt{\alpha}^4 + 1954\sqrt{\alpha}^5 - 743\sqrt{\alpha}^6 + 39\sqrt{\alpha}^7}{72}$$

is a generator or the Mordell-Weil group of $E(M)$. Conjecture 3.9 predicts that the point P_M coincides, up to torsion, with a multiple of P_{nt}. We computed an approximation to $J_{\mathcal{M}} \in \mathbb{C}/\Lambda_E$ with an accuracy of 30 decimal digits using formula (23). Let $J_{\mathrm{nt}} \in \mathbb{C}/\Lambda_E$ be a preimage of P_{nt} by Weierstrass's uniformization map. Then the following relation

$$(28) \qquad 7 \cdot J_{\mathcal{M}} - 14 \cdot J_{\mathrm{nt}} \in \Lambda_E,$$

holds up to the computed numerical precision of 30 digits. The torsion group $E(M)_{\mathrm{tors}}$ is isomorphic to $\mathbb{Z}/14\mathbb{Z}$. Observe that this gives numerical evidence for the fact that $\eta(J_M)$ is already a non-torsion point in $E(M)$ in this case. We find a similar relation for $P_M = \eta(J_{\mathcal{M}} + \overline{J_{\mathcal{M}}})$:

$$7 \cdot (J_{\mathcal{M}} + \overline{J_{\mathcal{M}}}) - 28 \cdot J_{\mathrm{nt}} \in \Lambda_E.$$

In the rest of the section provide the details about the computation of $J_{\mathcal{M}}$, beginning with those related to compute the Hilbert modular form attached to E over F_0.

Let f_0 be the base change of f to F_0. Denote by N the level of f, and let $A = \mathrm{Res}_{F/F_0} E$, which is a GL_2-variety over F_0. By Milne's formula [Mi2, Proposition 1] it has conductor $\mathrm{cond}(A/F_0) = (25)$. By the Shimura–Taniyama conjecture for GL_2-type varieties A is isogenous to A_{f_0}, which has conductor N^2. Then we see that $N = (5)$ and that f_0 belongs to $S_2(\Gamma_\psi(N))$, where ψ is the restriction of ε to $\mathrm{Gal}(\overline{\mathbb{Q}}/F_0)$. By identifying ε with a character $\mathbb{A}_{\mathbb{Q}}^\times \to \{\pm 1\}$ by means of class field theory, ψ can be identified with the idèle character $\varepsilon \circ \mathrm{Nm}_{F_0/\mathbb{Q}} \colon \mathbb{A}_{F_0}^\times \to \{\pm 1\}$.

The Fourier coefficients of $f = \sum_{n \geq 1} c_n q^n$ can be explicitly computed in Sage. Let us see how to compute the coefficients of f_0 in terms of the c_n's. The field $\mathbb{Q}_f = \mathbb{Q}(\{c_n\})$ turns out to be $\mathbb{Q}(\sqrt{2}, \sqrt{-3})$. Let $\mathrm{Gal}(\mathbb{Q}_f/\mathbb{Q}) = \{1, \sigma, \tau, \sigma\tau\}$, where σ denotes the automorphism that fixes $\mathbb{Q}(\sqrt{-3})$ and τ the one that fixes $\mathbb{Q}(\sqrt{2})$. The inner twists of f are given by

$$\chi_\sigma = \varepsilon_{\mathbb{Q}(\sqrt{5})}, \ \chi_\tau = \varepsilon_{\mathbb{Q}(\sqrt{10})}, \ \chi_{\sigma\tau} = \chi_\sigma \chi_\tau = \varepsilon_{\mathbb{Q}(\sqrt{2})},$$

where $\varepsilon_{\mathbb{Q}(\sqrt{a})}$ denotes the Dirichlet character corresponding to $\mathbb{Q}(\sqrt{a})/\mathbb{Q}$. Recall that inner twists are defined by the relations $f^\rho = \chi_\rho \otimes f$. This is also equivalent to say that $c_p^\rho = \chi_\rho(p) c_p$ for all p not dividing the level of f (see [Ri] for more details).

Lemma 3.14. $L(f_0, s) = L(f, s) L(f^{\sigma\tau}, s)$.

Proof. Indeed f_0 is the base change of f to $F_0 = \mathbb{Q}(\sqrt{2})$. Then,

$$L(f, s) = L(f, s) L(f \otimes \varepsilon_{\mathbb{Q}(\sqrt{2})}, s) = L(f, s) L(f \otimes \chi_{\sigma\tau}, s) = L(f, s) L(f^{\sigma\tau}, s).$$

$\square$

The L-series of f_0 is of the form

$$(29) \qquad L(f_0, s) = \prod_{\mathfrak{p} \nmid N} (1 - a_\mathfrak{p} \mathrm{Nm}(\mathfrak{p})^{-s} + \psi(\mathfrak{p}) \mathrm{Nm}(\mathfrak{p})^{1-2s})^{-1} \prod_{\mathfrak{p} | N} (1 - a_\mathfrak{p} \mathrm{Nm}(\mathfrak{p})^{-s})^{-1},$$

for some coefficients $a_\mathfrak{p}$, indexed by the primes in F_0.

Lemma 3.15. *Let $\mathfrak{p}$ be a prime in F_0, and let $p = \mathfrak{p} \cap \mathbb{Z}$. Then*

$$a_\mathfrak{p} = \begin{cases} c_p & \text{if } \varepsilon_{\mathbb{Q}(\sqrt{2})}(p) = 1 \text{ and } p \neq 5, \\ c_p^2 - 2\varepsilon(p)p & \text{if } \varepsilon_{\mathbb{Q}(\sqrt{2})}(p) = -1 \text{ and } p \neq 5, \\ c_p^2 & \text{if } p = 5, \\ c_p + c_p^{\sigma\tau} & \text{if } p = 2. \end{cases}$$

Proof. For a rational prime p let $L_p(f_0, s) = \prod_{\mathfrak{p}|p} L_{\mathfrak{p}}(f, s)$ denote the product of local factors for the primes $\mathfrak{p} \mid p$. For $p \neq 2, 5$ Lemma 3.14 gives that

$$(30) \qquad L_p(f_0, s) = (1 - c_p p^{-s} + \varepsilon(p)p^{1-2s})^{-1}(1 - c_p^{\sigma\tau} p^{-s} + \varepsilon(p)p^{1-2s})^{-1}.$$

If $\varepsilon_{\mathbb{Q}(\sqrt{2})}(p) = 1$ then p splits in F_0 so there are two primes $\mathfrak{p}_1, \mathfrak{p}_2$ dividing p, each one having norm p. On the other hand, $c_p^{\sigma\tau} = c_p$ (because $\chi_{\sigma\tau} = \varepsilon_{\mathbb{Q}(\sqrt{2})}(p)$), and $\psi(\mathfrak{p}_i) = \varepsilon(\mathrm{Nm}(\mathfrak{p}_i)) = \varepsilon(p)$. Comparing (30) and (29) we see that $a_{\mathfrak{p}_i} = c_p$.

If $\varepsilon_{\mathbb{Q}(\sqrt{2})}(p) = -1$ then there is only one prime $\mathfrak{p}$ dividing p, and $\psi(\mathfrak{p}) = \varepsilon(\mathrm{Nm}(\mathfrak{p})) = 1$. On the other hand $c_p^{\sigma\tau} = -c_p$, so

$$
\begin{aligned}
L_p(f_0, s) &= (1 - c_p p^{-s} + \varepsilon(p)p^{1-2s})^{-1}(1 + c_p p^{-s} + \varepsilon(p)p^{1-2s})^{-1} \\
&= 1 + 2\varepsilon(p)p^{1-2s} - c_p^2 p^{-2s} + p^{2-4s} \\
&= 1 + (2\,\varepsilon(p)\,p - c_p^2)\mathrm{Nm}(\mathfrak{p})^{-s} + \mathrm{Nm}(\mathfrak{p})^{1-2s},
\end{aligned}
$$

and we see that $a_p = c_p^2 - 2\,\varepsilon(p)\,p$.

If $p = 5$, then $\chi_{\sigma\tau}(p) = -1$ so $c_p^{\sigma\tau} = -c_p$. Since 5 divides the level of f we have that

$$L_p(f_0, s) = (1 - c_p p^{-s})^{-1}(1 + c_p p^{-s})^{-1} = (1 - c_p^2 \mathrm{Nm}(\mathfrak{p})^{-s})^{-1},$$

so that $a_{\mathfrak{p}} = c_p^2$.

Finally, if $p = 2$ then $(p) = \mathfrak{p}^2$ in F_0. But $\mathfrak{p}$ does not divide the level of f_0 and $\psi(\mathfrak{p}) = -1$ (because $\mathfrak{p}$ is inert in F), so $L_{\mathfrak{p}}(f_0, s) = L_p(f_0, s)$ is of the form

$$(31) \qquad L_{\mathfrak{p}}(f_0, s) = (1 - a_{\mathfrak{p}} p^{-s} - p^{1-2s})^{-1}.$$

On the other hand, p divides the level of f, so that

$$(32) \qquad L_{\mathfrak{p}}(f_0, s) = (1 - c_p p^{-s})(1 - c_p^{\sigma\tau} p^{-s}) = (1 - (c_p + c_p^{\sigma\tau})p^{-s} + c_p c_p^{\sigma\tau} p^{-2s}).$$

It turns out that $c_p c_p^{\sigma\tau} = -p$, so (31) and (32) match and we see that $a_{\mathfrak{p}} = c_p + c_p^{\sigma\tau}$. □

3.3.2. *Computation of the ATC point.* Let $e = \sqrt{2} - 1$ be a fundamental unit of F_0. Observe that $e_1 = v_1(e) > 0$ and $e_2 = v_2(e) < 0$. Let $\beta \colon \{\pm 1\} \to \{\pm 1\}$ be the nontrivial character. The differential $\omega_{f_0}^{\beta}$ is then the one corresponding to

$$\omega_{f_0}^{\beta} = \frac{-4\pi^2}{\sqrt{8}}\left(f_0(z_1, z_1)dz_1 dz_2 - f_0(e_1 z_1, e_2 \overline{z}_2)d(e_1 z_1)d(e_2 \overline{z}_2)\right).$$

As for $W_N(\omega_{f_0}^{\beta})$, it is easy to compute because $W_N(f_0) = \lambda_N \overline{f}_0$, where the pseudoeigenvalue λ_N is equal to $a_{(N)}/N = \frac{-1+2\sqrt{-6}}{5}$. Therefore

$$W_N(\omega_{f_0}^{\beta}) = \frac{(4 - 8\sqrt{-6})\pi^2}{5\sqrt{8}}\left(\overline{f}_0(z_1, z_1)dz_1 dz_2 - \overline{f}_0(e_1 z_1, e_2 \overline{z}_2)d(e_1 z_1)d(e_2 \overline{z}_2)\right).$$

and we have completely determined $\alpha_{f_0}^{\beta} = \omega_{f_0}^{\beta} + W_N(\omega_{f_0}^{\beta})$.

Recall that M is not Galois over F_0, and that the diagram of subfields of its Galois closure $\mathcal{M}$ is the one given in (18). The ATR field K is easily computed to be $K = F_0(\omega)$, where $\omega^2 + (\sqrt{2}+1)\omega + 3\sqrt{2} + 4 = 0$. Here we remark that K is complex under the embeddings extending v_1, and it is real under the embeddings extending v_2. The discriminant of L/K is an ideal $\mathfrak{N}$ which in this case satisfies that $\mathrm{Nm}_{K/F_0}(\mathfrak{N}) = N$. Therefore the ideal c of Proposition 3.8 is equal to 1 for this example. Let $\varphi \colon R_K \hookrightarrow \mathcal{O}$ be the optimal embedding of the maximal order R_K into the Eichler order of conductor N of $M_2(F_0)$ given by

$$\varphi(\omega) = \begin{pmatrix} -\sqrt{2}+2 & -2 \\ 5 & -3 \end{pmatrix}.$$

By Proposition 3.8 we see that L is contained in L_{φ}^{β}. But L_{φ}^{β} is a quadratic extension of the narrow Hilbert class field of K. Since K turns out to have narrow class number 1, we see that L_{φ}^{β} is a quadratic

extension of K, hence equals L. This means that $H_\varphi^\beta = \mathcal{M}$, so that according to Conjecture 2.14 the point P_φ^β is defined over $\mathcal{M}$.

The fixed point of $K^\times$ under φ (with respect to v_1) is

$$z_1 \simeq 0.358578643762691 + 0.520981147679366 \cdot i$$

The unit

$$e_K = (-10\sqrt{2} + 14)w + 7\sqrt{2} - 11$$

satisfies that $\mathrm{Nm}_{K/F_0}(e_K) = 1$ and generates the group of such units, so that

$$\gamma_\varphi = \varphi(e_K) = \begin{pmatrix} -27\sqrt{2} + 37 & 20\sqrt{2} - 28 \\ -50\sqrt{2} + 70 & 37\sqrt{2} - 53 \end{pmatrix}$$

and

$$\gamma_\varphi \cdot \infty = \frac{-27\sqrt{2} + 37}{-50\sqrt{2} + 70} = \frac{4\sqrt{2} + 11}{10}.$$

To compute $J_\mathcal{M}$ we need to evaluate the 3-limits integral

$$(33) \qquad J_\mathcal{M} = \int^{z_1} \int_\infty^{\gamma_\varphi \cdot \infty} \alpha_{f_0}^\beta = \int^{z_1} \int_\infty^{\frac{4\sqrt{2}+11}{10}} \alpha_{f_0}^\beta.$$

The next step is to use properties (13), (14), and (15) to transform (33) into a sum of usual 4-limit integrals, because they can be numerically computed by integrating (a truncation of) the Fourier series of $\alpha_{f_0}^\beta$. Observe that $\alpha_{f_0}^\beta$ is invariant under $W_N = W_{(5)}$, so we have the following additional invariance property:

$$(34) \qquad \int^x \int_y^z \alpha_{f_0}^\beta = \int^{\frac{-1}{5x}} \int_{\frac{-1}{5y}}^{\frac{-1}{5z}} \alpha_{f_0}^\beta.$$

We will also use the following matrices, both belonging to $\Gamma_\psi(N)$:

$$G = \begin{pmatrix} 4\sqrt{2} + 11 & -3\sqrt{2} + 5 \\ 10 & -6\sqrt{2} + 9 \end{pmatrix}, \qquad H = \begin{pmatrix} -15\sqrt{2} + 21 & -\sqrt{2} - 1 \\ -35\sqrt{2} + 50 & 1 \end{pmatrix}.$$

Since $\gamma_\varphi \cdot \infty = G \cdot \infty$ and $G \cdot 0 = \frac{-3\sqrt{2}+5}{-6\sqrt{2}+9} = \sqrt{2}/3 + 1$, we have that

$$(35) \qquad \begin{aligned} \int^{z_1} \int_\infty^{\gamma_\varphi \cdot \infty} \alpha_{f_0}^\beta &= \int^{z_1} \int_\infty^{G \cdot \infty} \alpha_{f_0}^\beta = \int^{z_1} \int_\infty^{G \cdot 0} \alpha_{f_0}^\beta + \int^{z_1} \int_{G \cdot 0}^{G \cdot \infty} \alpha_{f_0}^\beta \\ &= \int^{z_1} \int_\infty^{\sqrt{2}/3+1} \alpha_{f_0}^\beta + \int^{G^{-1} \cdot z_1} \int_0^\infty \alpha_{f_0}^\beta. \end{aligned}$$

Now, since $H \cdot \infty = \frac{-15\sqrt{2}+21}{-35\sqrt{2}+50} = \frac{-3\sqrt{2}}{10}$ and $H \cdot 0 = -\sqrt{2} - 1$ we have that

$$
\begin{aligned}
\int^{z_1} \int_{\infty}^{\sqrt{2}/3+1} \alpha_{f_0}^{\beta} &= \int^{z_1-1} \int_{\infty}^{\sqrt{2}/3} \alpha_{f_0}^{\beta} = \int^{\frac{-1}{5(z_1-1)}} \int_{0}^{\frac{-3\sqrt{2}}{10}} \alpha_{f_0}^{\beta} \\
&= \int^{\frac{-1}{5(z_1-1)}} \int_{0}^{H\cdot\infty} \alpha_{f_0}^{\beta} = \int^{\frac{-1}{5(z_1-1)}} \int_{0}^{H\cdot 0} \alpha_{f_0}^{\beta} + \int^{\frac{-1}{5(z_1-1)}} \int_{H\cdot 0}^{H\cdot\infty} \alpha_{f_0}^{\beta} \\
&= \int^{\frac{-1}{5(z_1-1)}} \int_{0}^{-\sqrt{2}-1} \alpha_{f_0}^{\beta} + \int^{H^{-1}\cdot\frac{-1}{5(z_1-1)}} \int_{0}^{\infty} \alpha_{f_0}^{\beta} \\
&= \int^{\frac{-1}{5(z_1-1)}} \int_{0}^{\infty} \alpha_{f_0}^{\beta} + \int^{\frac{-1}{5(z_1-1)}} \int_{\infty}^{-\sqrt{2}-1} \alpha_{f_0}^{\beta} + \int^{H^{-1}\cdot\frac{-1}{5(z_1-1)}} \int_{0}^{\infty} \alpha_{f_0}^{\beta} \\
&= \int^{\frac{-1}{5(z_1-1)}} \int_{0}^{\infty} \alpha_{f_0}^{\beta} + \int^{\frac{-1}{5(z_1-1)}+\sqrt{2}+1} \int_{\infty}^{0} \alpha_{f_0}^{\beta} + \int^{H^{-1}\cdot\frac{-1}{5(z_1-1)}} \int_{0}^{\infty} \alpha_{f_0}^{\beta} \\
&= \int^{\frac{-1}{5(z_1-1)}}_{\frac{-1}{5(z_1-1)}+\sqrt{2}+1} \int_{0}^{\infty} \alpha_{f_0}^{\beta} + \int^{H^{-1}\cdot\frac{-1}{5(z_1-1)}} \int_{0}^{\infty} \alpha_{f_0}^{\beta}
\end{aligned}
\tag{36}
$$

Now, putting together (35) and (36) we have that

$$
\begin{aligned}
\int^{z_1} \int_{\infty}^{\gamma_\varphi\cdot\infty} \alpha_{f_0}^{\beta} &= \int^{G^{-1}\cdot z_1} \int_{0}^{\infty} \alpha_{f_0}^{\beta} + \int^{\frac{-1}{5(z_1-1)}}_{\frac{-1}{5(z_1-1)}+\sqrt{2}+1} \int_{0}^{\infty} \alpha_{f_0}^{\beta} + \int^{H^{-1}\cdot\frac{-1}{5(z_1-1)}} \int_{0}^{\infty} \alpha_{f_0}^{\beta} \\
&= \int^{\frac{-1}{5G^{-1}\cdot z_1}} \int_{\infty}^{0} \alpha_{f_0}^{\beta} + \int^{\frac{-1}{5(z_1-1)}}_{\frac{-1}{5(z_1-1)}+\sqrt{2}+1} \int_{0}^{\infty} \alpha_{f_0}^{\beta} + \int^{H^{-1}\cdot\frac{-1}{5(z_1-1)}} \int_{0}^{\infty} \alpha_{f_0}^{\beta} \\
&= \int^{\frac{-1}{5(z_1-1)}}_{\frac{-1}{5(z_1-1)}+\sqrt{2}+1} \int_{0}^{\infty} \alpha_{f_0}^{\beta} + \int^{H^{-1}\cdot\frac{-1}{5(z_1-1)}}_{\frac{-1}{5G^{-1}\cdot z_1}} \int_{0}^{\infty} \alpha_{f_0}^{\beta}
\end{aligned}
\tag{37}
$$

Now both of these integrals can be easily computed, because for $x, y \in \mathcal{H}$ one has that

$$
\int_{x}^{y} \int_{0}^{\infty} = \int_{x}^{y} \int_{0}^{i/\sqrt{5}} + \int_{x}^{y} \int_{i/\sqrt{5}}^{\infty} = \int_{\frac{-1}{5x}}^{\frac{-1}{5y}} \int_{\infty}^{i/\sqrt{5}} + \int_{x}^{y} \int_{i/\sqrt{5}}^{\infty},
$$

which are integrals with all of their limits lying in $\mathcal{H}$ and they can be computed by integrating term by term the Fourier expansion.

Let Λ_1 and Λ_2 be the period lattices of E with respect to v_1 and v_2, and denote by Ω_1^+, Ω_2^+ the real periods and Ω_1^-, Ω_2^- the imaginary periods. Using the above limits we integrated the truncation of the Fourier expansion of $\alpha_{f_0}^{\beta}$ up to ideals of norm 160000 obtaining

$$
\begin{aligned}
J_{\mathcal{M}} = \left(\Omega_2^-\right)^{-1} \int^{z_1} \int_{\infty}^{\gamma_\varphi\infty} \alpha_{f_0}^{\beta} \quad &\simeq \quad 6.12100695194721053022236902 35 \\
&+ \quad i \cdot 5.438190302948632068621199446 0.
\end{aligned}
$$

Recall that J_{nt} stands for the logarithm of P_{nt} in $\mathbb{C}/\Lambda_E$. The actual value is

$$
J_{\mathrm{nt}} \simeq 3.38350550589702494601408880 86 + i \cdot 2.71909515147431603431059972 32.
$$

We have that

$$
7 \cdot J_{\mathcal{M}} - 14 \cdot J_{\mathrm{nt}} + \Omega_1^+ \simeq 3.742356 \cdot 10^{-27} - i \cdot 3.23117 \cdot 10^{-27},
$$

which is the numerical evidence for the fact that relation (28) holds and that, up to torsion, $\eta(J_{\mathcal{M}})$ equals $2P_{\mathrm{nt}}$.

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