THE STRONG ANICK CONJECTURE IS TRUE

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Abstract. Recently Umirbaev has proved the long-standing Anick conjecture, that is, there exist wild automorphisms of the free associative algebra $K\langle x,y,z\rangle$ over a field K of characteristic 0. In particular, the well-known Anick automorphism is wild. In this article we obtain a stronger result (the Strong Anick Conjecture that implies the Anick Conjecture). Namely, we prove that there exist wild coordinates of $K\langle x,y,z\rangle$. In particular, the two nontrivial coordinates in the Anick automorphism are both wild. We establish a similar result for several large classes of automorphisms of $K\langle x,y,z\rangle$. We also find a large new class of wild automorphisms of $K\langle x,y,z\rangle$ which is not covered by the results of Umirbaev. Finally, we study the lifting problem for automorphisms and coordinates of polynomial algebras, free metabelian algebras and free associative algebras and obtain some interesting new results.

1. Introduction

Let K be a field of characteristic 0 and let $X = \{x_1,\ldots,x_n\}$, $n \geq 2$, be a finite set. We denote by $K[X]$ the polynomial algebra in the set of variables X and by $K\langle X\rangle$ the free associative algebra (or the algebra of polynomials in the set X of noncommuting variables) with the same set of free generators. For small n we shall denote the free generators also with x, y, etc. We write the automorphisms of $K[X]$ and $K\langle X\rangle$ as n-tuples of the images of the coordinates, and $\varphi = (f_1,\ldots,f_n)$ means that $\varphi(x_j) = f_j(X) = f_j(x_1,\ldots,x_n)$, $j = 1,\ldots,n$. Also, the product $\varphi\psi$ of the automorphisms φ and ψ acts on $u(X)$ by $(\varphi\psi)(u) = \varphi(\psi(u))$. An automorphism of $K[X]$ or $K\langle X\rangle$ is called *elementary*, if it is of the

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form

$$(x_1,\dots,x_{j-1},\alpha x_j+f(x_1,\dots,x_{j-1},x_{j+1},\dots,x_n),x_{j+1},\dots,x_n),\quad \alpha\in K^*,$$

and the polynomial $f(x_1,\dots,x_{j-1},x_{j+1},\dots,x_n)$ does not depend on the variable x_j. The automorphisms belonging to the group generated by elementary automorphisms are called *tame*, otherwise they are *wild*. A polynomial $p\in K[X]$ is called a *coordinate* if it is an automorphic image of x_1. Moreover, a coordinate $p\in K[X]$ is called *tame* if there exists a tame automorphism $\varphi\in \mathrm{Aut}K[X]$ such that $\varphi(x_1)=p$, otherwise p is called a *wild coordinate*. One defines in a similar way the coordinates, tame and wild coordinates of $K\langle X\rangle$ and other relatively free algebras. In noncommutative algebra coordinates are often called *primitive elements*. Obviously, the existence of wild coordinates implies the existence of wild automorphisms, but not vice versa in general.

Problems concerning automorphisms of free objects are similar for free groups, polynomial algebras, free associative and free Lie algebras, for relatively free groups and algebras, see the recent book [MSY] by Mikhalev, Shpilrain, and Yu. One of the most important problems in the theory of automorphisms of polynomial and free associative algebras is whether all automorphisms of $K[X]$ and $K\langle X\rangle$ are tame. The answer is in the affirmative for $K[x,y]$ (Jung–van der Kulk [J, K]) and for $K\langle x,y\rangle$ (Czerniakiewicz–Makar-Limanov [Cz, ML1, ML2]). In 1970, Nagata [N] constructed his famous automorphism of $K[x,y,z]$ which is wild as an automorphism fixing z and conjectured that it is wild also as a usual automorphism of $K[x,y,z]$. More than 30 years later Shestakov and Umirbaev [SU1, SU2, SU3], using methods of Poisson brackets, degree estimations and peak reduction, proved the Nagata Conjecture, in particular, they proved that the Nagata automorphism is wild. Umirbaev and J.-T. Yu [UY] proved the Strong Nagata Conjecture, namely, there exist wild coordinates of $K[x,y,z]$. In particular, the two nontrivial Nagata coordinates are both wild. There were also attempts, unfortunately unsuccessful, to lift the Nagata automorphism to an automorphism of $K\langle x,y,z\rangle$, see, for instance, our paper with Gutierrez [DGY]. (A lifting of the Nagata automorphism would provide immediately a wild automorphism of $K\langle x,y,z\rangle$.) There is another automorphism of $K\langle x,y,z\rangle$, suggested by Anick,

$$(x+y(xy-yz),y,z+(xy-yz)y)\in \mathrm{Aut}K\langle x,y,z\rangle,$$

see the book by Cohn [C2], p. 343, which was suspected to be wild, although it induces a tame automorphism of $K[x,y,z]$. Exchanging the places of y and z in the above automorphism, we replace it with the automorphism

$$\omega = (x + z(xz - zy), y + (xz - zy)z, z).$$

It has the property that fixes z and $\omega(x)$ and $\omega(y)$ are linear in x and y. From now on, we shall refer to ω as the *Anick automorphism.*

Conjecture 1.1. (Anick Conjecture) *There exist wild automorphisms in* $\mathrm{Aut}K\langle x,y,z\rangle$*. In particular, the Anick automorphism is wild.*

We established [DY3] that the Anick automorphism is wild in the class of automorphisms fixing the variable z, and very recently Umirbaev [U2] has proved the Anick Conjecture, that is, the wildness of the Anick automorphism as an automorphism of $K\langle x,y,z\rangle$.

The work of Nagata [N] motivated the study of automorphisms fixing a variable. More generally, we introduce another set $Z = \{z_1,\ldots,z_m\}$ and consider the algebras $K[X,Z]$ and $K\langle X,Z\rangle$, freely generated by the set $X \cup Z$. Studying automorphisms fixing the set Z, we use the same notation $\varphi = (f_1,\ldots,f_n,z_1,\ldots,z_m)$, meaning that $\varphi(x_j) = f_j(X,Z)$, $j = 1,\ldots,n$, and $\varphi(z_i) = z_i$, $i = 1,\ldots,m$, and denote the corresponding automorphism group by $\mathrm{Aut}_Z K[X,Z]$ and $\mathrm{Aut}_Z K\langle X,Z\rangle$, respectively. Again, an automorphism is Z-elementary if it is of the form $\varphi = (x_1,\ldots,\alpha x_j + f(X,Z),\ldots,x_n,z_1,\ldots,z_m)$, and $f(X,Z)$ does not depend on x_j. The automorphism is tame in the class of automorphisms fixing Z (or Z-tame) if it belongs to the group generated by Z-elementary automorphisms. In the general case, this group may be smaller than the group generated by Z-triangular and Z-affine automorphisms because some Z-affine automorphisms may be not products of Z-elementary ones.

An important consequence of [SU1, SU2, SU3, DY1, DY2] is the result that every z-wild automorphism of $K[x,y,z]$ is a wild automorphism of $K[x,y,z]$. A way to construct a large class of such automorphisms was given by us in [DY2].

The next step of studying automorphisms of free algebras is to study *coordinates*, or automorphic images of x_1. In noncommutative algebra coordinates are also called *primitive elements*. There are algorithms to recognize coordinates of $K[x,y]$ (Shpilrain and Yu [SY]) and z-coordinates of $K[x,y,z]$, see our paper [DY2] as well as our survey

[DY1]. As a consequence of their proof of the Strong Nagata Conjecture, Umirbaev and J.-T. Yu [UY] established that if $f(x,y,z)$ is a z-wild coordinate in $K[x,y,z]$, then it is immediately a wild coordinate in $K[x,y,z]$.

As a common sense, results for (commutative) polynomial algebras inspire problems on free associative algebras as there is a natural surjective homomorphism from $K\langle X\rangle$ to $K[X]$. which induces a natural (not necessarily surjective) homomorphism from $\mathrm{Aut}K\langle X\rangle$ to $\mathrm{Aut}K[X]$. In this paper we shall be interested in the following problem motivated by [UY]:

Problem 1.2. *If $f(X,Z)\in K\langle X,Z\rangle$ is an image of x_1 under a Z-wild automorphism, is there a tame automorphism (maybe not fixing Z) which also sends x_1 to $f(X,Z)$?*

If such a tame automorphism does not exist, then we call $f(X,Z)$ a *wild coordinate* of $K\langle X,Z\rangle$.

As an analog of the Strong Nagata Conjecture in [UY], we state

Conjecture 1.3. (Strong Anick Conjecture) *There exist wild coordinates in $K\langle x,y,z\rangle$. In particular, the two nontrivial coordinates of the Anick automorphism are both wild.*

The Anick automorphism has the property that it fixes z and $\omega(x)$ and $\omega(y)$ are linear in x and y. In our paper [DY3] we showed that such an automorphism of $K\langle x,y,z\rangle$ is z-tame if and only if certain 2×2 matrix with entries from $K[z_1,z_2]$ is a product of elementary matrices. This idea has been used by Umirbaev [U2] in the final step of his proof of the the wildness of the Anick automorphism. Now we show that if $f(x,y,z)$ is a wild z-coordinate in $K\langle x,y,z\rangle$, and $f(x,y,z)$ is linear in x,y, then $f(x,y,z)$ is also wild in the sense of Problem 1.2. This is one of the main results of the paper. It immediately gives an affirmative answer to the Strong Anick Conjecture.

The class of wild automorphisms of $K\langle x,y,z\rangle$ discovered by Umirbaev [U2] is larger than the class of z-wild automorphisms (f,g,z) such that the polynomials f,g are linear in x,y. Our method also gives that all automorphisms of the class of Umirbaev have the property that at least two of their coordinates are wild. The same result holds for another large class of automorphisms of $K\langle x,y,z\rangle$ which is not covered by Umirbaev [U2].

Our main result suggests an algorithm deciding whether a polynomial $f(x,y,z) \in K\langle x,y,z\rangle$ which is linear in x and y, is a tame coordinate. If it is, then the algorithm shows how to find a product of z-elementary automorphisms which sends x to $f(x,y,z)$. (Of course, in all algorithmic considerations we assume that the ground field K is constructive, and we may perform calculations there.) In this part of the paper we use the approach and the results of Umirbaev [U2], combined with our approach from [DY3].

On the other hand, we show that the situation is completely different in the case of the free metabelian algebra $M(x,y,z)$. We construct an automorphism which fixes y and z and cannot be lifted to an automorphism of $K\langle x,y,z\rangle$. The proof is based on a test recognizing some classes of endomorphisms which are not automorphisms. This test is originally from group theory, see Bryant, Gupta, Levin and Mochizuki [BGLM] and was addapted to algebras by Bryant and Drensky [BD].

In addition, we show that an automorphism of $K[X,Z]$ or $K\langle X,Z\rangle$ which is Z-wild cannot be lifted to a Z-automorphism of the absolutely free algebra $K\{X,Z\}$. (As a consequence of a result of Kurosh [Ku], all automorphisms of the absolutely free algebra are tame.) This is equivalent to the fact that there exist no Z-wild automorphisms of $K\{X,Z\}$.

2. Proof of main results

Dicks and Lewin [DL] introduced the Jacobian matrix of an endomorphism of $K\langle X\rangle$. This is an $n\times n$ matrix with entries from the tensor product $K\langle X\rangle \otimes_K K\langle X\rangle^{\text{op}}$ of the free algebra $K\langle X\rangle$ and its opposite algebra (or its anti-isomorphic algebra) $K\langle X\rangle^{\text{op}}$. For $n=2$ they proved that the Jacobian matrix is invertible over $K\langle x,y\rangle \otimes_K K\langle x,y\rangle^{\text{op}}$ if and only if the endomorphism is an automorphism. The general case of any n was established by Schofield [Sc], which is the Jacobian Conjecture for free associative algebras. The partial derivatives and the Jacobian matrix of Dicks and Lewin can be defined as follows:

$$\frac{\partial x_i}{\partial x_i} = 1, \quad \frac{\partial x_j}{\partial x_i} = 0,\ j \neq i,$$

and, for a monomial $w = x_{i_1}\cdots x_{i_m} \in K\langle X\rangle$

$$\frac{\partial w}{\partial x_i} = \sum_{k=1}^{m} (x_{i_1}\cdots x_{i_{k-1}}) \otimes (x_{i_{k+1}}\cdots x_{i_m}) \frac{\partial x_{i_k}}{\partial x_i},$$

where $x_{i_1}\cdots x_{i_{k-1}}\in K\langle X\rangle$ and $x_{i_{k+1}}\cdots x_{i_m}\in K\langle X\rangle^{\mathrm{op}}$. Then, as usual,

$$J(\varphi)=\left(\frac{\partial\varphi(x_j)}{\partial x_i}\right),\quad \varphi\in \mathrm{End}K\langle X\rangle.$$

We need the following lemma.

Lemma 2.1. *The only automorphisms of $K\langle X\rangle$ fixing $x_2,\dots,x_n$ are the tame automorphisms of the form*

$$\tau=(\alpha x_1+f(x_2,\dots,x_n),x_2,\dots,x_n),\quad \alpha\in K^*,\ f(x_2,\dots,x_n)\in K\langle x_2,\dots,x_n\rangle.$$

Proof. The shortest way to establish the lemma is to use the invertibility of the Jacobian matrix. Let $\tau=(g(X),x_2,\dots,x_n)\in\mathrm{Aut}K\langle X\rangle$ fix $x_2,\dots,x_n$. Then the matrix

$$J(\tau)=\begin{pmatrix}\frac{\partial g}{\partial x_1} & 0 & \dots & 0\\ \frac{\partial g}{\partial x_2} & 1 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\partial g}{\partial x_n} & 0 & \dots & 1\end{pmatrix}$$

is invertible over $K\langle X\rangle\otimes_K K\langle X\rangle^{\mathrm{op}}$ and this implies that $\partial g/\partial x_1$ is equal to a nonzero constant α. Hence the only term of $g(X)$ depending on x_1 is αx_1. $\square$

For $K\langle x,y,z\rangle$, the endomorphisms which fix z and are linear in x and y are of the form $\rho=(f(x,y,z),g(x,y,z),z)$, where

$$f(x,y,z)=\sum_{p,q\geq 0}\alpha_{pq}z^pxz^q+\sum_{p,q\geq 0}\beta_{pq}z^pyz^q+f_0(z),$$

$$g(x,y,z)=\sum_{p,q\geq 0}\gamma_{pq}z^pxz^q+\sum_{p,q\geq 0}\delta_{pq}z^pyz^q+g_0(z),$$

$\alpha_{pq},\beta_{pq},\gamma_{pq},\delta_{pq}\in K$, and $f_0(z),g_0(z)$ are polynomials in z. Applying the Jacobian matrix of Dicks and Lewin in this concrete case, in [DY3] we obtained:

Proposition 2.2. (i) *The endomorphism $\rho=(f(x,y,z),g(x,y,z),z)$ which fixes z and is linear in x and y is an automorphism if and only if the 2×2 matrix*

$$J_z(\rho)=\begin{pmatrix}\sum_{p,q\geq 0}\alpha_{pq}z_1^pz_2^q & \sum_{p,q\geq 0}\gamma_{pq}z_1^pz_2^q\\ \sum_{p,q\geq 0}\beta_{pq}z_1^pz_2^q & \sum_{p,q\geq 0}\delta_{pq}z_1^pz_2^q\end{pmatrix}$$

with entries from $K[z_1,z_2]$ is invertible. All such automorphisms induce z-tame automorphisms of $K[x,y,z]$.

(ii) *The automorphism ρ is z-tame if and only if the matrix $J_z(\rho)$ belongs to the group generated by elementary matrices with entries from $K[z_1, z_2]$.*

For example, for the Anick automorphism,

$$J_z(\omega) = \begin{pmatrix} 1 + z_1 z_2 & z_2^2 \\ -z_1^2 & 1 - z_1 z_2 \end{pmatrix}$$

and by a result of Cohn [C1], the matrix $J_z(\omega)$ cannot be presented as a product of elementary matrices.

Let $Z = \{z_1, \dots, z_m\}$. We denote by $GE_2(K[Z])$ the subgroup of $GL_2(K[Z])$ generated by the diagonal and by the elementary matrices

$$\begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}, \quad \begin{pmatrix} 1 & f(Z) \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ f(Z) & 1 \end{pmatrix}$$

with entries from $K[Z]$. There is an algorithm deciding whether a matrix in $GL_2(K[Z])$ belongs to $GE_2(K[Z])$. It was suggested by Tolhuizen, Hollmann, and Kalker [THK] for the partial ordering by degree and, independently, by Park [P1, P2] for any monomial ordering on $K[Z]$. One applies Gaussian elimination process on the matrix based on the Euclidean division algorithm for $K[Z]$. The matrix belongs to $GE_2(K[Z])$ if and only if this procedure brings it to the identity matrix. For our purposes, we need the following version of the Euclidean algorithm. If $a(Z), b(Z)$ are two nonzero polynomials with homogeneous components of maximal degree $\overline{a(Z)}, \overline{b(Z)}$, respectively, then the Euclidean algorithm can be applied to $a(Z)$ and $b(Z)$ if $\overline{a(Z)} = \overline{b(Z)}q(Z)$ for some $q(Z) \in K[Z]$ (or $\overline{b(Z)} = \overline{a(Z)}q(Z)$) when we replace $a(Z)$ with $a(Z) - b(Z)q(Z)$ (or, respectively, we replace $b(Z)$ with $b(Z) - a(Z)q(Z)$). In matrix form, these operations correspond, respectively, to

$$\begin{pmatrix} a(Z) - b(Z)q(Z) \\ b(Z) \end{pmatrix} = \begin{pmatrix} 1 & -q(Z) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a(Z) \\ b(Z) \end{pmatrix}, \tag{1}$$

$$\begin{pmatrix} a(Z) \\ b(Z) - a(Z)q(Z) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -q(Z) & 1 \end{pmatrix} \begin{pmatrix} a(Z) \\ b(Z) \end{pmatrix}. \tag{2}$$

For us, the most convenient form of the result in [P1, P2, THK] is as stated in [THK].

Proposition 2.3. *Let $a(Z), b(Z)$ be two polynomials in $K[Z]$. Then there exist $c(Z), d(Z) \in K[Z]$ such that the matrix*

$$G = \begin{pmatrix} a(Z) & c(Z) \\ b(Z) & d(Z) \end{pmatrix}$$

belongs to $GE_2(K[Z])$ if and only if we can bring the pair $(a(Z), b(Z))$ to $(\alpha, 0)$, $0 \neq \alpha \in K$, using the Euclidean algorithm only.

Clearly, in this case the equations (1) and (2) give the decomposition of G as a product of elementary matrices.

We need a description of the free metabelian associative algebra and a short exposition of the results of Umirbaev. Recall that the free metabelian algebra

$$M(X) = K\langle X\rangle/([t_1, t_2][t_3, t_4])^T$$

is the relatively free algebra of rank n in the variety of associative algebras defined by the polynomial identity $[t_1, t_2][t_3, t_4] = 0$. In order to define partial derivatives and the Jacobian matrix of an endomorphism, we need two more sets of variables $U = \{u_1, \dots, u_n\}$ and $V = \{v_1, \dots, v_n\}$ of the same cardinality as X. We consider the polynomial algebra $K[U, V]$. Changing a little the notation of Umirbaev [U1], we define formal partial derivatives $\partial_M/\partial_M x_i$ assuming that

$$\frac{\partial_M x_i}{\partial_M x_i} = 1, \quad \frac{\partial_M x_j}{\partial_M x_i} = 0,\ j \neq i,$$

and, for a monomial $w = x_{i_1} \cdots x_{i_m} \in M(X)$

$$\frac{\partial_M w}{\partial_M x_i} = \sum_{k=1}^{m} u_{i_1} \cdots u_{i_{k-1}} v_{i_{k+1}} \dots v_{i_m} \frac{\partial_M x_{i_k}}{\partial_M x_i}.$$

These are the homomorphic images of the partial derivatives of Dicks and Lewin under the natural homomorphism $K\langle X\rangle \otimes_K K\langle X\rangle^{\mathrm{op}} \to K[U, V]$ which sends $x_i \otimes 1$ and $1 \otimes x_j$ to u_i and v_j, respectively. A polynomial $f(X) \in M(X)$ belongs to the commutator ideal of $M(X)$, i.e., to the kernel of the natural homomorphism $M(X) \to K[X]$, if and only if

$$\sum_{i=1}^{n} (u_i - v_i) \frac{\partial_M f}{\partial_M x_i} = 0.$$

The Jacobian matrix of an endomorphism φ of $M(X)$ is

$$J_M(\varphi) = \left(\frac{\partial_M \varphi(x_j)}{\partial_M x_i}\right),$$

which is a matrix with entries from $K[U,V]$. One of the main results in [U1] is that the Jacobian matrix $J_M(\varphi)$ is invertible (as a matrix with entries from $K[U,V]$) if and only if φ is an automorphism of $M(X)$. Clearly, the invertibility of $J_M(\varphi)$ is equivalent to $0\neq\det(J_M(\varphi))\in K$. In this section we shall work with free algebras of rank 3 only and shall assume that the sets X,U,V are, respectively,

$$X=\{x,y,z\},\quad U=\{x_1,y_1,z_1\},\quad V=\{x_2,y_2,z_2\}.$$

Let $T(K\langle x,y,z\rangle)$, $T(M(x,y,z))$ and $T(K[x,y,z])$ be, respectively, the groups of tame automorphisms of $K\langle x,y,z\rangle$, $M(x,y,z)$, and $K[x,y,z]$. There is a natural homomorphism

$$T(K\langle x,y,z\rangle)\to T(M(x,y,z))\to T(K[x,y,z]).$$

Let $\mathrm{Ker}(\pi)$ be the kernel of $\pi:T(M(x,y,z))\to T(K[x,y,z])$. Further developing the methodology in [SU1, SU2, SU3], Umirbaev [U2] discovered the defining relations of $T(K[x,y,z])$. As a consequence of that, he proved the following.

Proposition 2.4. *As a normal subgroup of $T(M(x,y,z))$, the kernel of π is generated by the automorphisms*

$$\psi=(x+f(y,z),y,z),\quad f(y,z)=\sum_{p,q,r,s\geq 0}\alpha_{pqrs}y^pz^q[y,z]y^rz^s,\ \alpha_{pqrs}\in K.$$

Moreover, any tame automorphism ϑ from $\mathrm{Ker}(\pi)$ has a Jacobian matrix which is a product of elementary matrices. The next key observation of Umirbaev is the following. Let ϑ be any automorphism from the kernel of the natural homomorphism $\mathrm{Aut}M(x,y,z)\to\mathrm{Aut}K[x,y,z]$. Then

$$J_M(\vartheta)=J_M(\vartheta)(x_1,y_1,z_1,x_2,y_2,z_2)$$

is a 3×3 matrix with entries from $K[x_1,y_1,z_1,x_2,y_2,z_2]$. If we replace x_1,y_1,x_2,y_2 with zeros, then the matrix $J_M(\vartheta)(0,0,z_1,0,0,z_2)$ will be of the form

$$\overline{J_M(\vartheta)}(z_1,z_2)=\begin{pmatrix}1+w_{11}&w_{12}&w_{13}\\w_{21}&1+w_{22}&w_{23}\\0&0&1\end{pmatrix},$$

where the polynomials $w_{ij}=w_{ij}(z_1,z_2)$ have no constant terms. Define the 2×2 matrix

$$J_2(\vartheta)(z_1,z_2)=\begin{pmatrix}1+w_{11}(z_1,z_2)&w_{12}(z_1,z_2)\\w_{21}(z_1,z_2)&1+w_{22}(z_1,z_2)\end{pmatrix}.$$

Proposition 2.5. (Umirbaev [U2]) *If $\vartheta \in \mathrm{Ker}(\pi)$, then $J_2(\vartheta)(z_1, z_2)$ is a product of elementary matrices with entries from $K[z_1, z_2]$.*

Note that the matrix $J_2(\overline{\rho})$ of the automorphism $\overline{\rho}$ of $M(x, y, z)$ induced by the automorphism ρ of $K\langle x, y, z\rangle$ coincides with the matrix $J_z(\rho)$, the Jacobian matrix of $(\rho(x), \rho(y))$, when ρ fixes z and is linear with respect to x, y.

Now we are ready to prove the main results in this article.

Theorem 2.6. *Let K be a field of characteristic 0 and let the polynomial $f(x, y, z) \in K\langle x, y, z\rangle$ be linear in x, y. If there exists a wild automorphism of $K\langle x, y, z\rangle$ which fixes z and sends x to $f(x, y, z)$, then every automorphism of $K\langle x, y, z\rangle$ which sends x to $f(x, y, z)$ is also wild. So, $f(x, y, z)$ is a wild coordinate of $K\langle x, y, z\rangle$.*

Proof. Let $\sigma = (f(x, y, z), h(x, y, z), z)$ be a wild automorphism of $K\langle x, y, z\rangle$ which fixes z and sends x to $f(x, y, z)$. We write $f(x, y, z)$ in the form

$$f(x, y, z) = \sum_{p,q\geq 0} \alpha_{pq} z^p x z^q + \sum_{p,q\geq 0} \beta_{pq} z^p y z^q + f_0(z),$$

where $\alpha_{pq}, \beta_{pq} \in K$, and $f_0(z)$ is a polynomial in z. Let

$$a(z_1, z_2) = \sum_{p,q\geq 0} \alpha_{pq} z_1^p z_2^q,$$

$$b(z_1, z_2) = \sum_{p,q\geq 0} \beta_{pq} z_1^p z_2^q.$$

First we shall show that the polynomials $a(z_1, z_2), b(z_1, z_2)$ cannot constitute the first column of a matrix from $GE_2(K[z_1, z_2])$. Suppose on the contrary,

$$J = \begin{pmatrix} a(z_1, z_2) & c(z_1, z_2) \\ b(z_1, z_2) & d(z_1, z_2) \end{pmatrix} \in GE_2(K[z_1, z_2]),$$

$$c(z_1, z_2) = \sum_{p,q\geq 0} \gamma_{pq} z_1^p z_2^q,$$

$$d(z_1, z_2) = \sum_{p,q\geq 0} \delta_{pq} z_1^p z_2^q.$$

Consider the polynomial

$$g(x, y, z) = \sum_{p,q\geq 0} \gamma_{pq} z^p x z^q + \sum_{p,q\geq 0} \delta_{pq} z^p y z^q.$$

By Proposition 2.2 (ii), the automorphism $\rho = (f(x,y,z), g(x,y,z), z)$ is tame in the group of automorphisms fixing z. Hence the automorphism $\rho^{-1}\sigma$ is also wild. But

$$\rho^{-1}\sigma = (x, k(x,y,z), z)$$

for some $k(x,y,z) \in K\langle x,y,z\rangle$. This contradicts to Lemma 2.1.

Hence $a(z_1,z_2), b(z_1,z_2)$ cannot constitute the first column of a matrix from $GE_2(K[z_1,z_2])$.

The next step is to produce a wild automorphism of $K\langle x,y,z\rangle$ which induces the identity automorphism of $K[x,y,z]$.

Let $\sigma = (f(x,y,z), h(x,y,z), z)$ be the above wild automorphism of $K\langle x,y,z\rangle$ which fixes z and sends x to $f(x,y,z)$, and let $h_1(x,y,z)$ be the component of h which is linear with respect to x, y. Then $\tau = (f(x,y,z), h_1(x,y,z), z)$ is also a wild automorphism of $K\langle x,y,z\rangle$ which induces a z-tame automorphism of $K[x,y,z]$. (The automorphism τ is wild since $\sigma^{-1}\tau$ sends x to x and z to z, then by Lemma 2.1 $\sigma^{-1}\tau$ is tame. The induced automorphism is z-tame by Proposition 2.2.) Let ψ be the corresponding z-tame, linear in x, y automorphism of $K\langle x,y,z\rangle$. Then $\widetilde{\tau} = \psi^{-1}\tau$ is still wild and induces the identity automorphism of $K[x,y,z]$.

Now, let φ be any tame automorphism of $K\langle x,y,z\rangle$ which sends x to $f(x,y,z)$. Replacing φ with $\widetilde{\varphi} = \psi^{-1}\varphi$, we obtain a tame automorphism for which $\widetilde{\varphi}(x) = \widetilde{\tau}(x)$.

The automorphism $\widetilde{\varphi}$ induces a tame automorphism of $K[x,y,z]$ which fixes x. By results in [DY1, DY2, SU1, SU3], such an automorphism is tame in the class of automorphisms fixing x and we can lift it to an x-tame automorphism θ of $K\langle x,y,z\rangle$. So we obtain a tame automorphism $\widehat{\varphi} = \widetilde{\varphi}\theta^{-1}$ which induces the identity automorphism of $K[x,y,z]$ and $\widehat{\varphi}(x) = \widetilde{\tau}(x)$.

Let ξ be the automorphism of $M(x,y,z)$ induced by $\widehat{\varphi}$. It is in the kernel of the homomorphism π of $\mathrm{Aut}M(x,y,z) \to \mathrm{Aut}K[x,y,z]$. The first columns of the matrices $J_2(\xi)$ and $J_2(\pi(\widetilde{\tau}))$ coincide. As we remarked above this column cannot be a column of a matrix from $GE_2(K[z_1,z_2])$ since $\widetilde{\tau}$ is wild. On the other hand by Proposition 2.5 it is a column of a matrix from $GE_2(K[z_1,z_2])$. This contradiction completes the proof. $\square$

Theorem 2.6 and Proposition 2.3 give an algorithm deciding whether a polynomial $f(x,y,z) \in K\langle x,y,z\rangle$ which is linear in x and y, is a tame coordinate. If it is, then the algorithm finds a product of z-elementary automorphisms which sends x to $f(x,y,z)$.

The following consequence of Theorem 2.6 gives the affirmative answer to the Strong Anick Conjecture.

Theorem 2.7. *The Strong Anick Conjecture is true. Namely, there exist wild coordinates in $K\langle x,y,z\rangle$. In particular, the two nontrivial coordinates $x+z(xz-zy)$ and $y+(xz-zy)z$ of the Anick automorphism*

$$\omega=(x+z(xz-zy),y+(xz-zy)z,z)$$

are both wild.

Proof. The partial derivatives of $f(x,y,z)=\omega(x)=x+z(xz-zy)$ are

$$a(z_1,z_2)=\frac{\partial f}{\partial x}=1+z_1z_2,\quad b(z_1,z_2)=\frac{\partial f}{\partial y}=-z_1^2.$$

Since we cannot apply the Euclidean algorithm to $a(z_1,z_2)$ and $b(z_1,z_2)$, Theorem 2.6 gives that $f(x,y,z)$ is a wild coordinate. □

We call an automorphism $\varphi=(f(x,y,z),g(x,y,z),z)$ of $K\langle x,y,z\rangle$ *Anick-like* if $f(x,y,z)$ and $g(x,y,z)$ are linear in x,y and the matrix $J_z(\varphi)$ does not belong to $GE_2(K[z_1,z_2])$. The following corollary is an analogue of a result from [UY].

Corollary 2.8. *The two nontrivial coordinates $f(x,y,z),g(x,y,z)$ of any Anick-like automorphism*

$$\varphi=(f(x,y,z),g(x,y,z),z)$$

of $K\langle x,y,z\rangle$ are wild.

Proof. Let

$$\frac{\partial f}{\partial x}=a(z_1,z_2),\quad \frac{\partial f}{\partial y}=b(z_1,z_2).$$

We cannot apply the Euclidean algorithm to bring the pair $(a(z_1,z_2),b(z_1,z_2))$ to $(\alpha,0)$, $0\neq\alpha\in K$, because $J_z(\varphi)\notin GE_2(K[z_1,z_2])$. Hence Theorem 2.6 gives that $f(x,y,z)$ is a wild coordinate. Similar arguments work for $g(x,y,z)$. □

In the spirit of the above results, we obtain the following theorem which is much stronger.

Theorem 2.9. *Let $f(x,y,z)$ be a z-coordinate of $K\langle x,y,z\rangle$ without terms depending only on z (i.e. $f(0,0,z)=0$). If the linear part (with respect to x and y) $f_1(x,y,z)$ of $f(x,y,z)$ is a z-wild coordinate, then $f(x,y,z)$ itself is also a wild coordinate of $K\langle x,y,z\rangle$.*

Proof. Since $f(x,y,z)$ is a z-coordinate of $K\langle x,y,z\rangle$, there exists a z-automorphism $\sigma = (f(x,y,z), g(x,y,z), z)$ of $K\langle x,y,z\rangle$. Obviously we may assume $g(0,0,z) = 0$ (otherwise just replace $g(x,y,z)$ by $(g(x,y,z) - g(0,0,z))$. Let $\sigma_1 = (f_1(x,y,z), g_1(x,y,z), z)$ be the automorphism which is the linear part of σ. By assumption σ_1 is a wild automorphism. We have to prove the wildness of all automorphisms $\varphi = (f(x,y,z), u(x,y,z), v(x,y,z))$ of $K\langle x,y,z\rangle$ with first coordinate equal to $f(x,y,z)$. Consider the automorphisms $\overline{\sigma} = (\overline{f}, \overline{g}, z)$ and $\overline{\varphi} = (\overline{f}, \overline{u}, \overline{v})$ of $K[x,y,z]$ induced by σ and φ, respectively. If $\overline{\sigma}$ is wild, then, by the theorem of Umirbaev and Yu [UY], $\overline{f}$ is a wild coordinate of $K[x,y,z]$. Hence $\overline{\varphi}$ is a wild automorphism of $K[x,y,z]$. This implies that φ is a wild automorphism of $K\langle x,y,z\rangle$ and therefore $f(x,y,z)$ is a wild coordinate. Hence we may assume that $\overline{\sigma}$ is a tame automorphism of $K[x,y,z]$. Now we suppose that the automorphism φ is tame and repeat the main steps of the proof of Theorem 2.6. Since $\overline{\sigma}$ is tame, by [DY1, DY3, SU1, SU3] it is also z-tame. Let ψ be some z-tame automorphism of $K\langle x,y,z\rangle$ which induces $\overline{\sigma}$ and let ψ_1 be the linear part of ψ. Replacing σ with $\widetilde{\sigma} = \psi^{-1}\sigma$ and φ with $\widetilde{\varphi} = \psi^{-1}\varphi$, we obtain that the tame automorphism $\widetilde{\varphi}$ fixes x modulo the commutator ideal of $K\langle x,y,z\rangle$. Since $\widetilde{\sigma}$ is a composition of the z-automorphisms ψ^{-1} and σ, its linear part $(\widetilde{\sigma})_1$ is also a z-automorphism which is equal to the composition $\psi_1^{-1}\varphi_1$ of the linear components of ψ_1^{-1} and φ_1. Hence $(\widetilde{\sigma})_1$ is wild and we may reduce our considerations to the case when $\widetilde{\sigma}(x) = f(x,y,z)$ is congruent to x modulo the commutator ideal of $K\langle x,y,z\rangle$. Since $\widetilde{\varphi}$ induces a tame automorphism of $K[x,y,z]$, by [DY1, DY3, SU1, SU3] again, the induced automorphism is also x-tame and we can lift it to an x-tame automorphism θ of $K\langle x,y,z\rangle$. The tame automorphism $\widehat{\varphi} = \widetilde{\varphi}\theta^{-1}$ induces the identity automorphism of $K[x,y,z]$ and $\widehat{\varphi}(x) = f(x,y,z)$. Now, as in Theorem 2.6, the proof is completed with considerations in the free metabelian algebra $M(x,y,z)$. □

Remark 2.10. The restriction $f(0,0,z) = 0$ is essential for the proof of Theorem 2.9 (Note that obviously we may assume $g(0,0,z) = 0$, otherwise just replace $g(x,y,z)$ by $g(x,y,z) - g(0,0,z)$). We use it when, modifying simultaneously the automorphisms $\sigma = (f(x,y,z), g(x,y,z), z)$ and $\varphi = (f(x,y,z), u(x,y,z), v(x,y,z))$ of $K\langle x,y,z\rangle$, we bring σ and φ to automorphisms which send x to the same element congruent to x modulo the commutator ideal, still keeping the property that the linear component of the image of x is wild. Nevertheless, it seems very unlikely to have a wild automorphism (f,g,z) with $f(0,0,z)=0$ such that $f(x,y,z)+a(z)$ is a tame coordinate for some polynomial $a(z)$ in view of the next theorem.

Theorem 2.11. *Let (f,g,z) be an automorphism of $K\langle x,y,z\rangle$ and let the linear part (with respect to x and y) of it, (f_1,g_1,z), be a z-wild automorphism. Then (f,g,z) is also a wild automorphism of $K\langle x,y,z\rangle$.*

Proof. Let $f(x,y,z)=f'(x,y,z)+f_0(z)$, $g(x,y,z)=g'(x,y,z)+g_0(z)$, where f',g' do not contain monomials depending on z only. Define the automorphism $\tau=(x-f_0(z),y-g_0(z),z)$. Then the automorphism $\sigma=(f,g,z)$ is tame (or z-tame) if and only if $\sigma\tau=(f',g',z)$ is tame (or z-tame). Since the polynomials f,f' and g,g' have the same linear components f_1 and g_1, we apply Theorem 2.9. $\square$

Remark 2.12. The above theorem is much stronger than the main result in [U2] where only the automorphisms linear with respect to x and y are dealt.

The following example gives a large class of wild automorphisms and wild coordinates. It is based on the polynomial $xz-zy$ which appears in the Anick automorphism.

Example 2.13. Let $h(t,z)\in K\langle t,z\rangle$ and let $h(0,0)=0$. Then

$$\sigma_h=(x+zh(xz-zy,z),y+h(xz-zy,z)z,z)$$

is an automorphism of $K\langle x,y,z\rangle$ fixing $xz-zy$. If the linear component (with respect to x,y) $h_1(xz-zy,z)$ of $h(xz-zy,z)$ is not equal to 0, then this automorphism belongs to the class of wild automorphisms in Theorem 2.9: As $(\sigma_h)_1=(x+zh_1(xz-zy,z),y+h_1(xz-zy,z)z,z)$ is an automorphism of $K\langle x,y,z\rangle$ and its matrix $J_z((\sigma_h)_1)$ is

$$J_z((\sigma_h)_1)=\begin{pmatrix}1+q(z_1,z_2)z_1z_2 & q(z_1,z_2)z_2^2\\ -q(z_1,z_2)z_1^2 & 1-q(z_1,z_2)z_1z_2\end{pmatrix}$$

for some nonzero polynomial $q(z_1,z_2)\in K[z_1,z_2]$, it is easy to see that this matrix does not belong to $GL_2(K[z_1,z_2])$ because we cannot apply the Euclidean algorithm to its first column.

Example 2.14. A minor modification of the Anick automorphism is the automorphism of $K\langle x,y,z\rangle$

$$\omega_m=(x+z(xz-zy)^m,y+(xz-zy)^mz,z).$$

Note that the automorphisms ω_m, $m > 1$, are not covered by Theorem 2.9, as the polynomials $z(xz - zy)^m$ and $(xz - zy)^m z$ have no linear components with respect to x and y.

Theorem 2.15. *The above automorphisms ω_m are wild for all $m \geq 1$.*

Proof. Consider the automorphism $\tau = (x+1, y, z)$ of $K\langle x, y, z\rangle$. Clearly, ω_m is wild if and only if $\omega_m\tau$ is wild. Direct calculations show that the linear part of the z-automorphism

$$\omega_m\tau = x + 1 + z((x + 1)z - zy)^m, y + ((x + 1)z - zy)^m z, z)$$

is equal to

$$(\omega_m\tau)_1 = \left(x + z\sum_{i=0}^{m-1} z^i(xz - zy)z^{m-1-i}, y + \sum_{i=0}^{m-1} z^i(xz - zy)z^{m-1-i}z, z\right).$$

Hence the matrix $J_z((\omega_m\tau)_1)$ has the form

$$J_z((\omega_m\tau)_1) = \begin{pmatrix} 1 + q(z_1, z_2)z_1z_2 & q(z_1, z_2)z_2^2 \\ -q(z_1, z_2)z_1^2 & 1 - q(z_1, z_2)z_1z_2 \end{pmatrix},$$

where $q(z_1, z_2) = z_1^{m-1} + z_1^{m-2}z_2 + \cdots + z_2^{m-1}$. As in Example 2.13, the automorphism $(\omega_m\tau)_1$ is wild. Hence ω_m is also wild by Corollary 2.11. $\square$

It seems plausible that the nontrivial coordinates of ω_m, $m > 1$, are wild. However, our methods and the the methods in [U2] are not applicable here.

Problem 2.16. *Are the two nontrivial coordinates of the above automorphism ω_m, $m > 1$, both wild?*

Remark 2.17. The most general form of the result of Umirbaev [U2] gives that the automorphism $\vartheta = (f, g, h)$ of the free metabelian algebra $M(x, y, z)$ is wild, if it induces the identity automorphism of $K[x, y, z]$ and the matrix $J_2(\vartheta)(z_1, z_2)$ cannot be presented as a product of elementary matrices with entries from $K[z_1, z_2]$, see Proposition 2.5. Hence the classes of wild automorphisms and wild coordinates in Theorem 2.9, Example 2.13 and Example 2.14 are not covered by Umirbaev [U2].

Now we are going to show that at least two coordinates of the automorphisms of the class of Umirbaev are wild.

Theorem 2.18. *Let $\vartheta = (f, g, h)$ be an automorphism of the free metabelian algebra $M(x, y, z)$ which induces the identity automorphism of $K[x, y, z]$ and the matrix $J_2(\vartheta)(z_1, z_2)$ does not belong to $GE_2(K[z_1, z_2])$. Then the two coordinates $f(x, y, z)$ and $g(x, y, z)$ are both wild.*

Proof. We repeat the main steps of the proof of Theorem 2.6. The polynomial $f(x, y, z) \in M(x, y, z)$ is equal to x modulo the commutator ideal of $M(x, y, z)$ and has the form

$$f = \sum_{p,q\geq 0} \alpha_{pq} z^p x z^q + \sum_{p,q\geq 0} \beta_{pq} z^p y z^q + \sum_{k\geq 2} f_k(x, y, z),$$

where f_i is the homogeneous component of degree i in x, y (and $f_0 = 0$). Let

$$a(z_1, z_2) = \sum_{p,q\geq 0} \alpha_{pq} z_1^p z_2^q, \quad b(z_1, z_2) = \sum_{p,q\geq 0} \beta_{pq} z_1^p z_2^q.$$

The polynomials $a(z_1, z_2), b(z_1, z_2)$ constitute the first column of the matrix $J_2(\vartheta)(z_1, z_2)$ which does not belong to $GE_2(K[z_1, z_2])$. By Proposition 2.3 $a(z_1, z_2), b(z_1, z_2)$ cannot be reduced to $(\alpha, 0)$, $0 \neq \alpha \in K$, by the Euclidean algorithm only.

Now, let $\varphi = (f(x, y, z), u(x, y, z), v(x, y, z))$ be any tame automorphism which sends x to $f(x, y, z)$. Clearly, φ induces the tame automorphism

$$\overline{\varphi} = (\overline{f}, \overline{u}, \overline{v}) = (x, \overline{u}, \overline{v})$$

of the polynomial algebra $K[x, y, z]$. Since $\overline{\varphi}$ fixes x, the results in [DY1, DY2, SU1, SU2, SU3] give that $\overline{\varphi}$ is tame also in the class of automorphisms fixing x. So, as in the proof of Theorem 2.6, we may replace φ with a tame automorphism $\xi = (f(x, y, z), u_1(x, y, z), v_1(x, y, z))$ of $M(x, y, z)$ such that ξ is in the kernel of the natural homomorphism $\mathrm{Aut}M(x, y, z) \to \mathrm{Aut}K[x, y, z]$. The tameness of ξ implies that $J_2(\xi) \in GE_2(K[z_1, z_2])$. Since the first column of $J_2(\xi)$ consists of $a(z_1, z_2)$ and $b(z_1, z_2)$, this contradicts to Proposition 2.5. The considerations for the other coordinate g of ϑ are similar. $\square$

Remark 2.19. Any automorphism $\phi \in \mathrm{Aut}K\langle x, y, z\rangle$ which induces an automorphism in $\mathrm{Aut}M(x, y, z)$ of the type in Theorem 2.18 (in other words, any automorphism in $\mathrm{Aut}K\langle x, y, z\rangle$ obtained by lifting an automorphism in $\mathrm{Aut}M(x, y, z)$ of the type in Theorem 2.18) is a wild automorphism containing at least two wild coordinates.

The above results suggest the following problems.

Problem 2.20. *Is it true that the two nontrivial coordinates of a wild automorphism of $K\langle x,y,z\rangle$ fixing z are both wild?*

Problem 2.21. *Is it true that every wild automorphism of $K\langle x,y,z\rangle$ contains at least two wild coordinates?*

3. Special wild automorphisms of the free metabelian algebra

In this section we shall construct a wild automorphism τ of the free metabelian algebra $M(x,y,z)$ over any field K of arbitrary characteristic with the following properties:

(i) $\tau=(f(x,y,z),y,z)$ fixes two of the variables. (Hence Lemma 2.1 does not hold for $M(x,y,z)$.)

(ii) The Jacobian matrix $J_M(\tau)$ is a product of elementary matrices.

(iii) It cannot be lifted to an automorphism of $K\langle x,y,z\rangle$.

Recall the definition of the Fox derivatives of the free algebra $K\langle X\rangle$, see e.g. [MSY]. If

$$f(X)=\sum_{i=1}^{n}x_if_i(X)+\alpha,\quad \alpha\in K,\quad f_i(X)\in K\langle X\rangle,$$

then the *right Fox derivatives* of $f(X)$ are

$$\frac{\partial_r f}{\partial_r x_i}=f_i(X),\quad i=1,\dots,n.$$

Similarly, if

$$f(X)=\sum_{i=1}^{n}f_i(X)x_i+\alpha,\quad \alpha\in K,\quad f_i(X)\in K\langle X\rangle,$$

then the *left Fox derivatives* of $f(X)$ are

$$\frac{\partial_l f}{\partial_l x_i}=f_i(X),\quad i=1,\dots,n.$$

The right and left Jacobian matrices of an endomorphism φ of $K\langle X\rangle$ are, respectively,

$$J_r(\varphi)=\left(\frac{\partial_r\varphi(x_j)}{\partial_r x_i}\right),\quad J_l(\varphi)=\left(\frac{\partial_l\varphi(x_j)}{\partial_l x_i}\right).$$

The chain rule gives that if φ is an automorphism, then $J_r(\varphi)$ and $J_l(\varphi)$ are invertible (but the oposite is not true in the general case).

We need some machinery from [BGLM] and [BD]. We state it in the case of three variables only. We define an equivalence relation $\sim$ on

$K\langle x,y,z\rangle$. We say that two monomials u and v are equivalent, if they can be obtained from each other by cyclic permutation (i.e., $u\sim v$ if and only if $u=w_1w_2$ and $v=w_2w_1$ for some monomials w_1,w_2) and then extend $\sim$ to $K\langle x,y,z\rangle$ by linearity.

Proposition 3.1. [BGLM, BD] *Let σ be an endomorphism of $K\langle x,y,z\rangle$ which is equal to the identity of $K\langle x,y,z\rangle$ modulo the k-th degree of the augmentation ideal, i.e.*

$$\sigma=(x+f_k+\cdots+f_m,y+g_k+\cdots+g_m,z+h_k+\cdots+h_m),$$

where f_i,g_i,h_i are the homogeneous components of degree i of $\sigma(x),\sigma(y),\sigma(z)$, respectively. If σ is an automorphism and $k\geq 2$, then the homogeneous component of degree $k-1$ of the trace of the right Jacobian matrix

$$\frac{\partial_r f_k}{\partial_r x}+\frac{\partial_r g_k}{\partial_r y}+\frac{\partial_r h_k}{\partial_r z}$$

is equivalent to 0. Similar statement holds for the trace of the left Jacobian matrix.

Theorem 3.2. *The endomorphism*

$$\tau=(x+x^2[y,z],y,z)$$

of the free metabelian algebra $M(x,y,z)$ is a wild automorphism which cannot be lifted to an automorphism of $K\langle x,y,z\rangle$. Its Jacobian matrix

$$J_M(\tau)=\begin{pmatrix}1&0&0\\ x_1^2(z_2-z_1)&1&0\\ x_1^2(y_1-y_2)&0&1\end{pmatrix}$$

is a product of two elementary matrices.

Proof. Obviously τ is an automorphism and $\tau^{-1}=(x-x^2[y,z],y,z)$. Also, its Jacobian matrix $J_M(\tau)$ is a product of elementary matrices. Now, let τ be lifted to an automorphism σ of $K\langle x,y,z\rangle$. Then

$$\sigma=(x+x^2[y,z]+f(x,y,z),y+g(x,y,z),z+h(x,y,z)),$$

where $f(x,y,z),g(x,y,z),h(x,y,z)$ belong to the T-ideal generated by the polynomial identity $[x_1,x_2][x_3,x_4]=0$. Hence f,g,h have no homogeneous components of degree ≤ 3 and

$$f=f_4+\cdots+f_m,\quad g=g_4+\cdots+g_m,\quad h=h_4+\cdots+h_m,$$

where f_i,g_i,h_i are homogeneous of degree i. Clearly, the components f_4,g_4,h_4 are linear combinations of products of two commutators of the variables. By Proposition 3.1,

$$\frac{\partial_r(x^2[y,z]+f_4)}{\partial_r x}+\frac{\partial_r g_4}{\partial_r y}+\frac{\partial_r h_4}{\partial_r z}\sim 0, \tag{3}$$

$$\frac{\partial_l(x^2[y,z]+f_4)}{\partial_l x}+\frac{\partial_l g_4}{\partial_l y}+\frac{\partial_l h_4}{\partial_l z}\sim 0. \tag{4}$$

Since $x^2[y,z]=x^2yz-x^2zy$, we obtain that

$$\frac{\partial_r x^2[y,z]}{\partial_r x}=x[y,z]\sim xyz-xzy, \quad \frac{\partial_l x^2[y,z]}{\partial_l x}=0. \tag{5}$$

The components of (3) and (4) which are multilinear in x,y,z are equivalent to 0. The components of the Fox derivatives

$$\frac{\partial_r f_4}{\partial_r x},\quad \frac{\partial_r g_4}{\partial_r y},\quad \frac{\partial_r h_4}{\partial_r z}$$

which are multilinear in x,y,z come, respectively, from

$$f_4'=\alpha_1[x,y][x,z]+\beta_1[x,z][x,y],$$
$$g_4'=\alpha_2[x,y][y,z]+\beta_2[y,z][x,y],$$
$$h_4'=\alpha_3[x,z][y,z]+\beta_3[y,z][x,z].$$

Direct calculations give that

$$\frac{\partial_r f_4'}{\partial_r x}+\frac{\partial_r g_4'}{\partial_r y}+\frac{\partial_r h_4'}{\partial_r z}$$

$$\sim(\alpha_1y[x,z]+\beta_1z[x,y])+(-\alpha_2x[y,z]+\beta_2z[x,y])-(\alpha_3x[y,z]+\beta_3y[x,z])$$
$$\sim(-\alpha_1+\beta_1-\alpha_2+\beta_2-\alpha_3+\beta_3)(xyz-xzy).$$

Together with (5) this implies that

$$-\alpha_1+\beta_1-\alpha_2+\beta_2-\alpha_3+\beta_3+1=0. \tag{6}$$

Similarly,

$$\frac{\partial_l f_4'}{\partial_l x}+\frac{\partial_l g_4'}{\partial_l y}+\frac{\partial_l h_4'}{\partial_l z}$$

$$\sim-(\alpha_1[x,y]z+\beta_1[x,z]y)+(-\alpha_2[x,y]z+\beta_2[y,z]x)+(\alpha_3[x,z]y+\beta_3[y,z]x)$$
$$\sim(-\alpha_1+\beta_1-\alpha_2+\beta_2-\alpha_3+\beta_3)(xyz-xzy)\sim 0$$

in virtue of (5). Hence

$$-\alpha_1+\beta_1-\alpha_2+\beta_2-\alpha_3+\beta_3=0. \tag{7}$$

Clearly, (6) and (7) contradict to each other. Hence τ cannot be lifted to an automorphism of $K\langle x,y,z\rangle$ and, therefore, is a wild automorphism of $M(x,y,z)$. $\square$

Problem 3.3. *Is the polynomial $x+x^2[y,z]$ a wild coordinate of $M(x,y,z)$? Can it be lifted to a coordinate of $K\langle x,y,z\rangle$?*

Problem 3.4. *Do there exist wild automorphisms and wild coordinates of the free metabelian algebra $M(X)$ of rank $n>3$? Are there wild automorphisms similar to the automorphism τ constructed above?*

4. Lifting of automorphisms fixing variables

The considerations in this section work over an arbitrary field of any characteristic.

Let $G(X)$ be the free group generated by the finite set X. The theorem of Nielsen [Ni] states that every automorphism of $G(X)$ is a product of the elementary automorphisms $(x_1^{-1},x_2,\dots,x_n)$, $(x_1x_2,x_2,\dots,x_n)$, and $(x_{\sigma(1)},\dots,x_{\sigma(n)})$, where σ belongs to the symmetric group S_n. The proof of Nielsen gives also an algorithm which finds such a decomposition. The theorem of Schreier [Sch] states that every subgroup of the free group with any number of generators is also free.

There are several important varieties of algebras over a field with free objects which share the above properties of free groups. The variety $\mathfrak{V}$ of algebras is called *Schreier* if the subalgebras of the relatively free algebras $F(\mathfrak{V})$ are again relatively free, where $F(\mathfrak{V})$ is freely generated by a set of any cardinality. The variety $\mathfrak{V}$ is *Nielsen* if all automorphisms of the free algebras $F_n(\mathfrak{V})$ of finite rank are tame. A theorem of Lewin [L] gives that over an infinite field K the two notions coincide, i.e., $\mathfrak{V}$ is Nielsen if and only if it is Schreier. The same holds over an arbitrary field K, provided that the variety $\mathfrak{V}$ is defined by a multilinear system of polynomial identities. See the book [MSY] for more details about examples of Schreier varieties, and the properties of the subalgebras and the automorphisms of their free objects.

The variety of all (not necessarily associative) algebras is Schreier, by the theorem of Kurosh [Ku]. Recall that the absolutely free algebra $K\{X\}$ consists of all polynomials in the set of noncommuting and nonassociative variables X, e.g. $(xx)x\neq x(xx)$. One of the key moments of the proof of Kurosh (and of all other proofs that some varieties are Schreier) is the following, see [MSY], Theorem 11.1.1. For a nonzero polynomial $f\in K\{X\}$ we denote by $\bar{f}$ the homogeneous component of maximal degree of f.

Proposition 4.1. (i) *Any finite set S of $K\{X\}$ can be transformed into a set of free generators of the subalgebra generated by S by a finite* *sequence of elementary transformations (with cancellation of possible zeros).*

(ii) *If* $F = \{f_1, \dots, f_n\}$ *is a set of free generators of* $K\{X\}$*, and* $g \in K\{X\}$*, then* $\bar{g}$ *belongs to the subalgebra of* $K\{X\}$ *generated by* $\overline{f_1}, \dots, \overline{f_n}$*.*

For an automorphism $\varphi = (f_1, \dots, f_n)$ of $K\{X\}$ we define the degree of φ as the sum of the degrees of the coordinates f_i:

$$\deg(\varphi) = \sum_{i=1}^{n} \deg(f_i).$$

Clearly, $\deg(\varphi) \geq n$. The following consequence of Proposition 4.1 can be used effectively to decompose an automorphism of $K\{X\}$ as a product of elementary automorphisms.

Corollary 4.2. *Let* $\varphi = (f_1, \dots, f_n) \in \mathrm{Aut}K\{X\}$ *with* $\deg(\varphi) > n$*. Then there exists an integer* i *and a polynomial* $g(y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n)$*, such that*

$$\overline{f_i} = g(\overline{f_1}, \dots, \overline{f_{i-1}}, \overline{f_{i+1}}, \dots, \overline{f_n}).$$

Let τ *be the elementary automorphism of* $K\{X\}$ *defined by*

$$\tau = (x_1, \dots, x_{i-1}, x_i - g(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n), x_{i+1}, \dots, x_n).$$

Then $\deg(\varphi\tau) < \deg(\varphi)$*.*

Now we are able to prove the following.

Theorem 4.3. *Let*

$$\varphi = (f_1(X, Z), \dots, f_n(X, Z), z_1, \dots, z_m) \in \mathrm{Aut}_Z K\{X, Z\}$$

be an automorphism of $K\{X, Z\}$ *fixing the variables* Z*. Then* φ *is tame in the class of* Z*-automorphisms.*

Proof. Let us consider φ as an automorphism of $K\{X, Z\}$ in the usual sense. The total degree of φ is

$$\deg(\varphi) = \sum_{i=1}^{n} \deg(f_i(X, Z)) + \sum_{j=1}^{m} \deg(z_j) = \sum_{i=1}^{n} \deg(f_i) + m.$$

Since φ is a Z-automorphism, each polynomial $f_1, \dots, f_n$ essentially depends on X.

If $\deg(\varphi) = n + m$, then all polynomails $f_i(X, Z)$ are of total degree equal to 1 and φ is affine. We replace φ with the product $\psi = \varphi\tau_0$, where τ_0 is the translation

$$\tau_0 = (x_1 - f_1(0, 0), \dots, x_n - f_n(0, 0), z_1, \dots, z_m).$$

Clearly, τ_0 is a product of Z-elementary automorphisms and ψ is a linear automorphism. Its matrix, as a linear operator of the vector space with basis $X \cup Z$, is

$$\begin{pmatrix} A & 0 \\ B & E_m \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} & 0 & 0 \dots & 0 \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} & 0 & 0 \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \ddots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nn} & 0 & 0 \dots & 0 \\ \beta_{11} & \beta_{12} & \dots & \beta_{1n} & 1 & 0 \dots & 0 \\ \beta_{21} & \beta_{22} & \dots & \beta_{2n} & 0 & 1 \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \ddots & \vdots \\ \beta_{m1} & \beta_{m2} & \dots & \beta_{mn} & 0 & 0 \dots & 1 \end{pmatrix},$$

and $A = (\alpha_{pq})$, $B = (\beta_{rs})$ are, respectively, $n \times n$ and $m \times n$ matrices with entries in K, E_m is the $m \times m$ identity matrix, and A is invertible. Since we work over a field, A is a product of elementary matrices and this implies that, multiplying ψ by a product of elementary linear automorphisms fixing Z, we bring it to the automorphism

$$\tau_1 = (x_1 + g_1(Z), \dots, x_n + g_n(Z), z_1, \dots, z_m),$$

which is a product of elementary automorphisms fixing Z.

Now, let $\deg(\varphi) > n + m$. Then, at least one of the polynomials $f_i(X, Z)$ is not linear. The leading components of the $n+m$ coordinates are

$$\overline{f_1(X, Z)}, \dots, \overline{f_n(X, Z)}, \ \overline{z_1} = z_1, \dots, \overline{z_m} = z_m.$$

By Corollary 4.2, one of these homogeneous components is expressed by the others. Obviously, z_j cannot be expressed as a polynomial of $\overline{f_1}, \dots, \overline{f_n}$ and the other $z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_m$. Hence, some $\overline{f_i}$ is a polynomial of $\overline{f_1}, \dots, \overline{f_{i-1}}, \overline{f_{i+1}}, \dots, \overline{f_n}$ and $z_1, \dots, z_m$. This gives that the elementary automorphism τ of $K\{X, Z\}$ prescribed by Corollary 4.2, is of the form

$$\tau = (x_1, \dots, x_{i-1}, x_i - g(X, Z), x_{i+1}, \dots, x_n, z_1, \dots, z_m),$$

where $g(X, Z)$ does not depend on x_i. Then $\deg(\varphi\tau) < \deg(\varphi)$ and the proof is completed by obvious induction on the degree of φ. $\square$

The theorem below is an immediate concequence of Theorem 4.3.

Theorem 4.4. *Let φ be an automorphism of $K[X, Z]$ or $K\langle X, Z\rangle$ which fixes Z. If φ is wild as a Z-automorphism, then it cannot be lifted to an automorphism of $K\{X, Z\}$ which also fixes Z.*

Remark 4.5. The Nagata automorphism is wild as a z-automorphism of $K[x,y,z]$, [N], as well as wild in the usual sense, [SU1, SU3]. Hence it cannot be lifted to any automorphism of $K\{x,y,z\}$. On the other hand, by the theorem of Smith [Sm], automorphisms of $K[X]$ of a large class become tame as automorphisms of $K[X,t]$, if we extend them to act identically t. In particular, the extension of the Nagata automorphism

$$\nu' = (x - 2y(y^2+xz) - z(y^2+xz)^2, y + z(y^2+xz), z, t)$$

is tame as an automorphism of $K[x,y,z,t]$. It is easy to see that it is wild in the class of automorphisms of $K[x,y,z,t]$ fixing z and t. Hence, Theorem 4.4 gives that ν' cannot be lifted to an automorphism of $K\{x,y,z,t\}$ which fixes z and t.

Similarly, the automorphism of Anick is wild as an automorphism fixing a variable [DY3] and even wild in the usual sense [U2]. But it becomes tame extended to an automorphism of $K\langle x,y,z,t\rangle$. The technique of [DY3] gives that the extension of the Anick automorphism

$$(x + z(xz - zy), y + (xz - zy)z, z, t)$$

is wild in the group of automorphisms of $K\langle x,y,z,t\rangle$ which fix z,t, although this automorphism is tame in the usual sense. Hence, our theorem gives that it cannot be lifted to an automorphism of $K\{x,y,z,t\}$ which fixes z,t.

We shall conclude this section with several open problems.

Problem 4.6. (i) *If φ is an automorphism of $K[X]$, can it be lifted to an automorphism of $K\langle X\rangle$? (If φ is wild, and nevertheless the answer is positive, this would mean that it is not "too wild".)*

(ii) *If $\varphi \in \mathrm{Aut}_Z K[X,Z]$, can it be lifted to a Z-automorphism of $K\langle X,Z\rangle$? Are Z-wild automorphisms wild also in the usual sense?*

Problem 4.7. *How far can be lifted the automorphisms of $K[X]$? Describe the varieties of algebras $\mathfrak{V}$ with the property that every automorphism of $K[X]$ can be lifted to an automorphism of the relatively free algebra $F_n(\mathfrak{V})$ of rank $n = |X|$.*

For example, a theorem of Umirbaev [U1] gives that every automorphism of $K[X]$ can be lifted to an automorphism of the free metabelian algebra $M(X)$.

Problem 4.8. (i) *If $p(X)$ is a coordinate of $K[X]$, can it be lifted to a coordinate of $K\langle X\rangle$?*

(ii) *How far can be lifted the coordinates of $K[X]$? Describe the varieties of algebras $\mathfrak{V}$ with the property that every coordinate of $K[X]$ can be lifted to a coordinate of $F_n(\mathfrak{V})$.*

(iii) *Can the two nontrivial Nagata coordinates $x - 2y(y^2 + xz) - z(y^2 + xz)^2$ and $y + z(y^2 + xz)$ be lifted to coordinates of $K\langle x, y, z\rangle$?*

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