Geometrical approach to the free solvable groups

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Abstract

We give a topological interpretation of the free metabelian group, following the plan described in [Ver1],[Ver2]. This interpretation is based on considering the Cayley graph of a finitely generated group as a one-dimensional complex, its homology group with induced action of the group, and the corresponding extension of the homology group determined by the canonical cocycle. We also define satellite solvable groups. Our method can be applied to arbitrary solvable groups.

1 Introduction

In this paper we suggest a useful topological model of free solvable groups of given degree of solvability (mainly of degree two, i.e., the metabelian free group). Using simple topological tools, we obtain a recursive (with respect to the degree of solvability) description of free solvable groups. The key idea (see [Ver1, Ver2]) is to consider the Cayley graph of a finitely generated group as a one-dimensional complex and use classical topological results.

We start from the fact that the (right) Cayley graph of any countable group as one-dimensional complex is homotopic to a bouquet of 1-spheres, and hence its fundamental group either is trivial (for the Cayley graph of a free group with standard generators), or is a free group. On the other hand, if a given group G is represented as a quotient of a free group, the corresponding normal subgroup of the free group is also free, i.e., isomorphic to the fundamental group of the Cayley graph of the group G.

The group G acts on its Cayley graph by left shifts; at the same time, it acts on the normal subgroup by inner automorphisms of the whole free group. An important statement (Theorem 1) asserts that we can establish an equivariant isomorphism between the normal subgroup and the fundamental group, or, in other words, an isomorphism of these groups regarded as G-spaces.

The next step is much more instructive — we use the classical topological result, the Gurevich theorem, which asserts that the first homology group is the quotient of the fundamental group over its commutant. Abelianizing the normal subgroup N (i.e., factorizing over its commutant) gives the first homology group of the Cayley graph and thus the interpretation of the quotient of the free group

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over the commutant N' as an extension of the homology of the Cayley graph by the group G (Theorem 2).

In particular, if the normal subgroup is the nth commutant of the free group with d generators, then the quotient group is the free solvable group of degree n with d generators; defining a canonical 2-cocycle on the free solvable group $\mathrm{Sol}(n-1,d)$ of smaller degree with values in the homology groups, we obtain (Theorem 3) a recursive interpretation of the free solvable group: this is simply the extension of the homology group of the Cayley graph of the previous solvable group by the previous group $\mathrm{Sol}(n-1,d)$ itself with canonical cocycle. For the metabelian group, we obtain an especially simple interpretation (see [Ver1]): the free metabelian group is the extension of the homology group of a lattice by the lattice itself with canonical 2-cocycle.

The key technical point is the study of the sets of 2-cocycles of the group $\mathbf{Z}^d$ with values in $H_1(\mathbf{E}^d)$ and the corresponding second cohomology group $H^2(\mathbf{Z}^d; H_1(\mathbf{E}^d))$. We prove that this group is isomorphic to $\mathbf{Z}$; its generator is the 2-cocycle that determines the metabelian free group; we also describe extensions that correspond to other 2-cocycles.

The free solvable groups and their properties had been discussed in the algebraic literature for a long time; see the books by Magnus et al. [MagKarSol], Lindon [LySch], Kurosh [Kur], articles by Malcev [Malc], Shmel'kin [Shmel] on the connection with wreath products, and so on. In the papers [Sok, Eg, Ber, Petr], the growth of the Golod–Shafarevich series for such groups was studied.

Nevertheless, there are many open problems concerning numerical characteristics of free solvable groups, especially problems related to random walks on these groups, Poisson–Furstenberg boundaries (see [Ver1]), connections with wreath products, normal forms, etc. We hope that the suggested model will help in the solution of such questions.

There are some applications of the constructed model, which will be considered elsewhere.

In what follows we will use the notation

$$U \leftthreetimes_{\tilde{c}} V$$

for the extension of a group V by a group U determined by action of the group U on the group V as a group of automorphisms, and by 2-cocycle $\tilde{c}$ of the group U with values in the group V which will be described in each particular case.

2 Fundamental and homology groups of Cayley graphs

2.1 The fundamental group of the Cayley graph of a finitely generated group

Let G be a finitely generated group with a fixed set of generators $S = \{x_1, \ldots, x_d\}$. We do not include the inverses of generators into the set S, but in what follows we will assume that they are included into the alphabet. Let $\mathrm{Cal}(G)$ be the right Cayley graph of the group G in these generators, i.e., the graph whose vertices are elements of G, with two vertices g_1, g_2 connected by a (directed) edge whenever $g_1^{-1}g_2 \in S$ is a generator[1].

Let F_d be the free group with d free generators, which we will denote by $X_1, \ldots, X_d$. There exists a normal subgroup $N \lhd F_d$ such that the sequence

$$0 \longrightarrow N \xrightarrow{i} F_d \xrightarrow{\phi} G \longrightarrow 0$$

is exact.

Here $i : N \to F_d$ is the inclusion map and the homomorphism $\phi : F_d \to G$ is defined by the rule $X_i \to x_i$.

Thus $G \simeq F_d/N$ and G acts on the group N by inner automorphisms, i.e., given $a \in N$,

$$a^{x_i} = X_i a X_i^{-1} \in N.$$

The normal subgroup N is a group with an action of the group G, or, in short, a G-space.

The key point of our further considerations (as well as the main idea of [Ver1, Ver2]) is to consider the Cayley graph not as an abstract graph, but as a one-dimensional topological complex in a natural sense. Define $\mathrm{Cal}(G)$ as the space $G \times S \times [0,1]$ with the following identifications: 1) for a fixed g and all $i = 1, \ldots, d$, all the points $(g, x_i, 0)$ are identified with each other and denoted by $(g, 0)$; 2) the triples $(g, x_i, 1)$ are identified with $(gx_i, 0)$. We denote this one-dimensional complex by the same symbol $\mathrm{Cal}(G)$. The group G acts on the graph $\mathrm{Cal}(G)$ from the right by shifts, and this action is an isomorphism of the Cayley graph. This action induces a right action of G on the fundamental group $\pi_1(\mathrm{Cal}(G))$ and homology group $H_1(\mathrm{Cal}(G); \mathbf{Z})$ of the complex $\mathrm{Cal}(G)$; hence these groups are G-spaces too.

More precisely, let $\pi_1(\mathrm{Cal}(G))$ be the fundamental group with basepoint at the identity element $e \in G$, i.e., the group of homotopy classes of loops on $\mathrm{Cal}(G)$ that start and end at the point e.

An action of the group G on $\pi_1(\mathrm{Cal}(G))$ is defined as follows. Let x_i be one of the generators of G, and let $\bar{\lambda} \in \pi_1(\mathrm{Cal}(G))$ be the homotopy class containing a loop λ. Then x_i acts on $\bar{\lambda}$ according to the rule

$$x_i(\bar{\lambda}) = \bar{\lambda}^{x_i},$$

[1] This means that we multiply an element by some generator from the right.

where λ^{x_i} is the loop that first goes from the basepoint to the vertex $x_i \in \mathrm{Cal}(G)$ along the edge corresponding to x_i, then passes the loop λ shifted to this point, and then goes back to the basepoint along the same edge.

Now we will give a natural definition of a mapping γ that maps elements of the free group F_d to directed paths on $\mathrm{Cal}(G)$ starting at the basepoint.

Let $w \in F_d$ be some element of F_d. It has a unique representation as a word in the alphabet $X_1, \dots, X_d, X_1^{-1}, \dots, X_d^{-1}$ that does not contain trivial cancellations.

Definition 1. The path $\gamma(w)$ on $\mathrm{Cal}(G)$ is constructed as follows: it starts at the point $e \in \mathrm{Cal}(G)$, goes with unit velocity along the edges corresponding to the generators, and ends at the point $(w, 0) \in \mathrm{Cal}(G)$.

In a similar way, given a path that starts at the basepoint and consists of edges of $\mathrm{Cal}(G)$, we can construct a word in the alphabet $X_1, \dots, X_d, X_1^{-1}, \dots, X_d^{-1}$.

Lemma 1. *An element $w \in F_d$ belongs to the normal subgroup $N \triangleleft F_d$ if and only if $\gamma(w)$ is a loop.*

Proof. Write an element w as a word without cancellations:

$$w = X_{k_1}^{\delta_1} X_{k_2}^{\delta_2} \cdots X_{k_n}^{\delta_n},$$

where $\delta_i = \pm 1$.

This element belongs to N if and only if $x_{k_1}^{\delta_1} x_{k_2}^{\delta_2} \cdots x_{k_n}^{\delta_n} = e$ in G; this means that the last edge of $\gamma(a)$ connects the vertex $x_{k_1}^{\delta_1} x_{k_2}^{\delta_2} \cdots x_{k_{n-1}}^{\delta_{n-1}}$ with the basepoint, and thus $\gamma(w)$ is a loop. □

Now we can define a mapping $\bar{\gamma} : N \to \pi_1(\mathrm{Cal}(G))$ that maps an element $a \in N$ to the homotopy class $\bar{\gamma}(a)$ of the loop $\gamma(a)$.

Lemma 2. *The mapping $\bar{\gamma} : N \to \pi_1(\mathrm{Cal}(G))$ is a group isomorphism. Moreover, if $g \in G$ and $a \in N$, then*

$$\bar{\gamma}(a^g) = g(\bar{\gamma}(a)).$$

Proof. First of all, let us check that this mapping is a group homomorphism. Let $u, v \in N$; then the path $\gamma(u)\gamma(v)$ is the concatenation of the paths $\gamma(u)$ and $\gamma(v)$. On the other hand, the word representing uv differs from the concatenation of the words representing u and v by trivial cancellations, which lead to homotopically trivial subloops. Thus the loop $\gamma(uv)$ is homotopic to $\gamma(u)\gamma(v)$, and $\bar{\gamma}(uv) = \bar{\gamma}(u)\bar{\gamma}(v)$.

Note that every homotopy class from $\pi_1(\mathrm{Cal}(G))$ contains at least one loop that consists of edges of $\mathrm{Cal}(G)$ and hence, by Lemma 1, corresponds to some element of N. Thus $\bar{\gamma}$ is an epimorphism.

Now let us prove that if $\gamma(a)$ is homotopic to the trivial loop, then $a = e$. This follows immediately from the fact that $\mathrm{Cal}(G)$ is an infinite bouquet of one-dimensional spheres and N is a free group, and, consequently, a homotopically trivial loop corresponds to the identity element of G.

The condition $\bar{\gamma}(a^g) = g(\bar{\gamma}(a))$, $a \in N$, $g \in G$, follows directly from the definition of the action of G on $\pi_1(\mathrm{Cal}(G))$ and the construction of γ. □

Now we can deduce the following theorem.

Theorem 1. *The mapping $\bar{\gamma}$ defined above gives a canonical isomorphism of the G-spaces N and $\pi_1(\mathrm{Cal}(G))$, which commutes with the action of G, i.e.,*

$$\bar{\gamma}(a^g) = g(\bar{\gamma}(a))$$

for any $g \in G$ and $a \in N$.

Note that both groups N and $\pi_1(\mathrm{Cal}(G))$ are free, the first one due to the Nilsen-Schreier theorem (see [Kur]), and the second one since $\mathrm{Cal}(G)$ is homotopically equivalent to a bouquet of one-dimensional spheres. The theorem asserts that they are isomorphic as G-spaces.

2.2 The homology group $H_1(\mathrm{Cal}(G); \mathbf{Z})$

As was mentioned in the previous section, the first homology group $H_1(\mathrm{Cal}(G); \mathbf{Z})$ can naturally be considered as a G-space.

Note that since $\mathrm{Cal}(G)$ is a one-dimensional complex, the group $H_1(\mathrm{Cal}(G); \mathbf{Z})$ is the group of one-dimensional cycles on $\mathrm{Cal}(G)$, i.e., a subgroup of the free abelian group on all edges of $\mathrm{Cal}(G)$ consisting of elements with trivial boundary, where the boundary of an edge $((g, x_i, 0), (g, x_i, 1))$ is the null-dimensional chain $(g, x_i, 0) - (g, x_i, 1)$.

The group $H_1(\mathrm{Cal}(G); \mathbf{Z})$ is abelian, and we will write its group law additively.

Note that G acts on $H_1(\mathrm{Cal}(G); \mathbf{Z})$ shifting cycles by elements of G.

Let N' be the commutant of the group N, i.e., the normal subgroup of N generated by commutators, elements of the form $[x, y] = xyx^{-1}y^{-1}$, $x, y \in N$.

Note that if N is a normal subgroup of F_d, then the commutant N' of N is also normal in F^d. Thus the group N/N' is a normal subgroup of F_d/N', and the sequence

$$0 \longrightarrow N/N' \xrightarrow{i'} F_d/N' \xrightarrow{\phi'} G \longrightarrow 0$$

is exact. Here i' is again the inclusion map and ϕ' is defined by the rule $\pi(X_i) \mapsto x_i$, where $\pi : F_d \to F_d/N'$ is the natural projection.

Again N/N' has the natural structure of a G-space.

Theorem 2. *There is a canonical isomorphism of the G-spaces N/N' and $H_1(\mathrm{Cal}(G), \mathbf{Z})$.*

Proof. Indeed, since $\mathrm{Cal}(G)$ is a linearly connected complex, the Gurevich theorem (see [FuchsRokh]) implies

$$\pi_1(\mathrm{Cal}(G))/\pi(\mathrm{Cal}(G))' \simeq H_1(\mathrm{Cal}(G); \mathbf{Z}).$$

According to Theorem 1, there is a canonical isomorphism of G-spaces $\bar{\gamma} : N \to \pi_1(\mathrm{Cal}(G))$. Obviously, $\bar{\gamma}(N') = \pi_1(\mathrm{Cal}(G))'$. Since the action of G maps elements of N' to N', this isomorphism induces a canonical isomorphism of the G-spaces N/N' and $H_1(\mathrm{Cal}(G); \mathbf{Z})$. □

2.3 The cocycle

The group F_d/N' can be represented as an extension of the G-space $H_1(\mathrm{Cal}(G);\mathbf{Z})$ by the group G. Now we will describe the cohomology class of cocycles corresponding to this extension.

For every element $g \in G$, fix a representation of g as a word in the alphabet of generators $\{x_1, \ldots, x_d\} = S$ and their inverses. There are many such representations, and we will see that different choices lead to cohomologous cocycles.

To fix a representation for every element means that for every $g \in G$, we fix a path ω_g on $\mathrm{Cal}(G)$ connecting the basepoint with the vertex $(g, 0)$.

Let $g_1, g_2 \in G$. Consider the loop λ_{g_1,g_2} on $\mathrm{Cal}(G)$ constructed as follows: λ_{g_1,g_2} starts at the basepoint, then passes to the vertex $(g_1, 0)$ along the path ω_{g_1}, then passes to the vertex $(g_1g_2, 0)$ along the path $g_1(\omega_{g_2})$ (i.e., the path ω_{g_2} shifted by the element g_1), and then returns to the basepoint along the path $\omega^{-1}_{g_1 \cdot g_2}$.

Denote by $\tilde{\lambda}_{g_1,g_2}$ the class in the homology group $H_1(\mathrm{Cal}(G);\mathbf{Z})$ that contains the loop λ_{g_1,g_2}.

Definition 2. Define a mapping $c : G \times G \to H_1(\mathrm{Cal}(G);\mathbf{Z})$ by the rule

$$c(g_1, g_2) = \tilde{\lambda}_{g_1,g_2}.$$

Lemma 3. *The mapping c is a 2-cocycle of the group G with values in $H_1(\mathrm{Cal}(G);\mathbf{Z})$.*

Proof. We must prove that the mapping c has trivial coboundary, i.e., for any $g_1, g_2, g_3 \in G$,

$$(\partial c)(g_1, g_2, g_3) = c(g_1, g_2) + c(g_1 \cdot g_2, g_3) - c(g_1, g_2 \cdot g_3) - g_1(c(g_2, g_3)) = 0.$$

Indeed, the cycle $(\partial c)(g_1, g_2, g_3)$ corresponds to the product of four loops λ_{g_1,g_2}, $\lambda_{g_1\cdot g_2,g_3}$, $\lambda^{-1}_{g_1,g_2\cdot g_3}$, and $g_1(\lambda_{g_2,g_3})^{-1}$, where

$$\begin{aligned}
\lambda_{g_1,g_2} &= \omega_{g_1} \cdot g_1(\omega_{g_2}) \cdot \omega^{-1}_{g_1g_2}, \\
\lambda_{g_1\cdot g_2,g_3} &= \omega_{g_1g_2} \cdot (g_1g_2)(\omega_{g_3}) \cdot \omega^{-1}_{g_1g_2g_3}, \\
\lambda^{-1}_{g_1,g_2\cdot g_3} &= \omega_{g_1g_2g_3} \cdot g_1(\omega_{g_2g_3})^{-1} \cdot \omega^{-1}_{g_1}, \\
g_1(\lambda_{g_2,g_3})^{-1} &= g_1(\omega_{g_2g_3})^{-1} \cdot g_1(g_2(\omega_{g_3}))^{-1} \cdot g_1(\omega_{g_2})^{-1}.
\end{aligned}$$

(here the product of paths means their (coordinatewise) sum as functions on the edges of $\mathrm{Cal}(G)$).

Note that $g_1(g_2(\omega_{g_3})) = (g_2g_1)(\omega_{g_3})$ and each path occurs in the sum twice with opposite directions. Hence $(\partial c)(g_1, g_2, g_3) = 0$ in $H_1(\mathrm{Cal}(G);\mathbf{Z})$. $\square$

Our definition of the cocycle c depends on the choice of the set of paths $\{\omega_g\}_{g\in G}$. The next lemma asserts that cocycles constructed from different sets of paths are cohomologous.

Lemma 4. *Let $\{\omega_g\}_{g\in G}$ and $\{\omega'_g\}_{g\in G}$ be two sets of paths. Then the corresponding cocycles $c = c(\{\omega_g\})$ and $c' = c'(\{\omega'_g\})$ are cohomologous.*

Conversely, let $c = c(\{\omega_g\})$ be the cocycle constructed from a set of paths $\{\omega_g\}_{g\in G}$, and let c' be a cocycle cohomologous to c. Then there exists a set of paths $\{\omega_g\}_{g\in G}$ such that the cocycle $c' = c'(\{\omega_g\})$ is constructed from $\{\omega_g\}_{g\in G}$.

Proof. Cocycles c and c' are cohomologous if and only if there exists a mapping $\alpha : G \to H_1(\mathrm{Cal}(G); \mathbf{Z})$ such that for any $g_1, g_2 \in G$,

$$c(g_1, g_2) - c'(g_1, g_2) = (\partial\alpha)(g_1, g_2) = \alpha(g_1) + g_1(\alpha(g_2)) - \alpha(g_1 g_2).$$

Now, if c and c' are constructed from the sets $\{\omega_g\}_{g\in G}$ and $\{\omega'_g\}_{g\in G}$, then the mapping $\alpha(g)$ that maps g to the loop $\tilde{\xi}_g$, where ξ_g goes from the basepoint to $(g, 0)$ along ω_g and then returns to the basepoint along $(\omega'_g)^{-1}$ satisfies this condition.

To prove the converse claim, construct an appropriate set of paths $\{\omega_g\}_{g\in G}$ using α and ω_g. □

Definition 3. Let $\tilde{c}$ be the cohomology class that consists of all cocycles obtained from sets of paths as described above.

Theorem 3. *Let F_d be the free group with d generators, N be some normal subgroup of F_d, and N' be the commutant of N. Let $\mathrm{Cal}(F_d/N)$ be the right Cayley graph of the quotient group F_d/N associated with the generators obtained from the standard generators of F_d through the canonical projection.*

Then there is a canonical isomorphism

$$F_d/N' \simeq F_d/N \leftthreetimes_{\tilde{c}} H_1(\mathrm{Cal}(F_d/N); \mathbf{Z}),$$

where F_d/N acts on $H_1(\mathrm{Cal}(F_d/N); \mathbf{Z})$ from the right by shifts and the extension is determined by the cohomology class $\tilde{c}$ described above.

Proof. Let $G = F_d/N$. It suffices to prove that the cohomology class that determines F_d/N' as an extension of N/N' by G coincides with the class described above in this section.

Indeed, let $g \in G$. Denote by $u(g) \in F_d/N'$ some representative of the class of $(F_d/N')/(N/N')$ corresponding to g.

The cocycle C that determines the extension is given by the formula

$$u(g_1)u(g_2) = u(g_1 g_2)C(g_1, g_2),$$

where $C(g_1, g_2) \in N/N'$.

Fixing any representations of the elements $u(g)$ as words in the alphabet $S \cup S^{-1}$ and considering the canonical image of $C(g_1, g_2)$ in $H_1(\mathrm{Cal}(G); \mathbf{Z})$, we arrive at the cocycle c constructed from the set of paths determined by the elements $\{u(g)\}_{g\in G}$. □

In the next section we will consider the special case when $N = F_d^{(n-1)}$ and $N' = F_d^{(n)}$.

3 Free solvable groups

3.1 The free solvable group $\mathrm{Sol}(n, d)$

In this section we will apply Theorem 2 to the special case when N is the $(n-1)$th commutant of the free group F_d.

Definition 4. Let $\mathrm{Sol}(n, d)$ be the free solvable group of degree n with d generators, i.e., the free object in the category of all solvable groups of degree n with d generators.

The group $\mathrm{Sol}(n, d)$ is isomorphic to the quotient of the free group with d generators F_d over its nth commutant $F_d^{(n)}$:

$$\mathrm{Sol}(n, d) \simeq F_d / F_d^{(n)}.$$

Let $\mathrm{Met}(d) = \mathrm{Sol}(2, d)$ be the free metabelian group, i.e., the free solvable group of degree 2 with d generators.

Let us take the symmetric set of generators $S = \{x_1, \ldots, x_d\}$ in $\mathrm{Sol}(n, d)$, where x_i corresponds to the standard generator X_i of the free group.

Applying Theorem 3 to $N = F_d^{(n-1)}$, we arrive at the following theorem.

Theorem 4. *There is a canonical isomorphism*

$$\mathrm{Sol}(n, d) \simeq \mathrm{Sol}(n-1, d) \leftthreetimes_{\tilde{c}} H_1(\mathrm{Cal}(\mathrm{Sol}(n-1, d))),$$

where the extension in the right-hand side is determined by the right action of $\mathrm{Sol}(n-1, d)$ *on* $H_1(\mathrm{Cal}(\mathrm{Sol}(n-1, d)))$ *by shifts and by the cohomology class* $\tilde{c}$ *defined in the same way as in Section 2.3.*

Proof. Indeed, this theorem is an immediate consequence of Theorem 3 with $N = F_d^{(n-1)}$. In this case $N' = F_d^{(n)}$ and $G \simeq F_d / F_d^{(n)} \simeq \mathrm{Sol}(n-1, d)$. □

In particular, consider the case $n = 2$.

Note that $\mathrm{Sol}(1, d)$ is the free abelian group $\mathbf{Z}^d$ and its Cayley graph $\mathrm{Cal}(\mathbf{Z}^d)$ is the d-dimensional integer lattice in $\mathbf{R}^d$, i.e., the vertices of $\mathrm{Cal}(\mathbf{Z}^d)$ are points of $\mathbf{R}^d$ with integer coordinates and its edges are unit edges connecting these points. As a one-dimensional complex, $\mathrm{Cal}(\mathbf{Z}^d)$ is the union of all lines in $\mathbf{R}^d$ parallel to axes and crossing integer points. Let us denote this complex by $\mathbf{E}^d$.

As an immediate consequence of the previous theorem, we obtain the following theorem, which was stated in [Ver1].

Theorem 5. *There is a canonical isomorphism*

$$\mathrm{Met}(d) \simeq \mathbf{Z}^d \leftthreetimes_{\tilde{c}} H_1(\mathbf{E}^d),$$

where $\mathbf{Z}^d$ *acts on cycles by shifts from the right and the cohomology class is defined as before.*

In the next section we will describe this isomorphism more precisely.

3.2 The free metabelian group $\mathrm{Met}(d)$

Let $\bar{m} = (m_1, \dots, m_d) \in \mathbf{Z}^d$ be an integer vector and $1 \le i, j \le d$. Consider the unit square $P(\bar{m}, i, j)$ in $\mathbf{R}^d$ with coordinates of vertices

$$\begin{gathered}(m_1, \dots, m_i, \dots, m_j, \dots, m_d),\\ (m_1, \dots, m_i + 1, \dots, m_j, \dots, m_d),\\ (m_1, \dots, m_i + 1, \dots, m_j + 1, \dots, m_d),\\ (m_1, \dots, m_i, \dots, m_j + 1, \dots, m_d),\end{gathered}$$

i.e., $P(\bar{m}, i, j)$ lies in the plane parallel to the coordinate plane defined by the ith and jth axes, and the point of $P(\bar{m}, i, j)$ closest to the origin has coordinates $\bar{m}$.

Definition 5. Let $p(\bar{m}, i, j) \in H_1(\mathbf{E}^d))$ be the oriented boundary of $P(\bar{m}, i, j)$, where the orientation is induced by the order of coordinates i, j. Elements of these form will be called *plaquettes*.

Lemma 5. *The group $H_1(\mathbf{E}^d)$ is generated by plaquettes $p(\bar{m}, i, j)$, where $\bar{m} \in \mathbf{Z}$, $1 \le i < j \le d$.*

For $d = 2$, plaquettes generate the group $H_1(\mathbf{E}^d)$ freely.

For $d \ge 3$, the set of relations on plaquettes is generated by relations of the form

$$\begin{gathered}p(\bar{m}, i, j) + p(\bar{m}, i, k) + p(\bar{m}, j, k)\\ -p(\bar{m} + e_k, i, j) - p(\bar{m} + e_j, i, k) + p(\bar{m} + e_i, j, k) = 0,\end{gathered}$$

where $1 \le i < j < k \le d$, $\bar{m} \in \mathbf{Z}^d$, and e_i, e_j, e_k are unit coordinate vectors of the corresponding axes.

In other words, the relations on the set of generators $\{p(\bar{m}, i, j)\}$ are generated by the "cube" relation: for every unit 3-dimensional cube with integer vertices, the sum of the oriented boundaries of its six faces is equal to zero.

Proof. Consider the cell structure on $\mathbf{R}^d$ with k-dimensional cells being unit k-dimensional cubes with integer vertices. Let C_k be the free abelian group on the set of all k-dimensional cells. Defining the boundary ∂ in a standard way, we have the exact sequence

$$0 \xleftarrow{\partial} C_0 \xleftarrow{\partial} C_1 \xleftarrow{\partial} \cdots \xleftarrow{\partial} C_d \xleftarrow{\partial} 0.$$

Note that $H_1(\mathbf{E}^d) \simeq Z^1(\mathbf{E}^d)$, where $Z^1(\mathbf{E}^d)$ is the kernel of the boundary in C^1.

Since this sequence is exact,

$$H_1(\mathbf{E}^d) \simeq C_2/\partial(C_3),$$

but C_2 is generated by the elements $P(\bar{m}, i, j)$, $\bar{m} \in \mathbf{Z}^d$, $1 \le i, j \le d$. Hence $H_1(\mathbf{E}^d)$ is generated by the boundaries of these elements, which are plaquettes. The relations are defined by elements of $\partial(C_3)$ that are generated by the sums of faces of unit 3-dimensional cubes. In the case $d = 2$, we have $\partial(C_3) = 0$, and hence plaquettes are free generators.

The theorem can also be proved directly using induction and simple geometrical considerations. ∎

Now let $\mathbf{p}(\bar{m}, i, j) \in \mathrm{Met}(d)$ be the element of the free metabelian group defined by the formula

$$\mathbf{p}(\bar{m}, i, j) = x_1^{m_1} x_2^{m_2} \cdots x_d^{m_d} \cdot [x_i, x_j] x_d^{-m_d} \cdots x_2^{-m_2} x_1^{-m_1}.$$

Fix the set $\{\omega_{\bar{m}}\}_{\bar{m} \in \mathbf{Z}^d}$ of paths defined by the formula

$$\omega_{\bar{m}} = \gamma(X_1^{m_1} X_2^{m_2} \cdots X_d^{m_d})$$

(i.e., the path $\omega_{\bar{m}}$ first goes along the axis X_1, then along X_2, etc., until it reaches the point $\bar{m}$).

Consider the cocycle $c : \mathbf{Z}^d \times \mathbf{Z}^d \to H_1(\mathbf{E}^d)$ defined as in Section 1.3 with this choice of $\omega_{\bar{m}}$. Let $\tilde{c}$ be the cohomology class containing c.

Consider the extension $\mathbf{Z}^d \leftthreetimes_{\tilde{c}} H_1(\mathbf{E}^d)$ determined by the action of $\mathbf{Z}^d$ on plaquettes defined by the formula

$$\bar{m}(p(\bar{n}, i, j)) = p(\bar{n} + \bar{m}, i, j)$$

and the class $\tilde{c}$. Obviously, this extension is generated by elements of the form $(e_k, p(\bar{m}, i, j))$, $\bar{m} \in \mathbf{Z}^d$, $k, i < j \in Z$.

Theorem 6. *There is a canonical isomorphism*

$$\mathrm{Met}(d) \simeq \mathbf{Z}^d \leftthreetimes_{\tilde{c}} H_1(\mathbf{E}^d)$$

defined on the generators by the rule

$$\alpha((e_k, p(\bar{m}, i, j))) = x_k \mathbf{p}(\bar{m}, i, j).$$

Proof. The theorem follows directly from Theorem 4. $\square$

3.3 The cohomology group $H^2(\mathbf{Z}^2, H_1(\mathbf{E}^2))$

Let $H^2(\mathbf{Z}^2, H_1(\mathbf{E}^2))$ be the cohomology group with respect to a fixed right action of $\mathbf{Z}^2$ on $H_1(\mathbf{E}^2)$ by shifts.

In this section we will prove that this group is isomorphic to the group of integers $\mathbf{Z}$, and under this isomorphism, the class $\tilde{c}$ that determines the free metabelian group $\mathrm{Met}(2)$ corresponds to the identity element of $\mathbf{Z}$.

Let $\tilde{y} \in H^2(\mathbf{Z}^2, H_1(\mathbf{E}^2))$ be a cohomology class, and let $y : \mathbf{Z}^2 \times \mathbf{Z}^2 \to H_1(\mathbf{E}^2)$ be some representative of this class.

Let G be the extension of $H_1(\mathbf{E}^2)$ by $\mathbf{Z}^2$ determined by the class $\tilde{y}$; in particular, we consider the representation determined by the cocycle y. Every element of G can be written as a pair $((m, n), \gamma)$, where $m, n \in \mathbf{Z}$ and $\gamma \in H_1(\mathbf{E}^2)$ is a cycle.

The group law in G is determined by the formula

$$((m, n), \gamma_1) \cdot ((k, l), \gamma_2) = ((m + k, n + l), \gamma_1^{(k,l)} + \gamma_2 + y((m, n), (k, l))),$$

where, as before, $\gamma_1^{(k,l)}$ denotes the cycle γ_1 shifted by the vector (k, l).

Denote the vector $(0,1)$ by e_1 and the vector $(1,0)$ by e_2. We will write 0 for the origin $(0,0) \in \mathbf{Z}^2$.

In every cohomology class there is at least one normalized element; thus we may assume that the cocycle y is normalized, i.e., $y(0,(m,n)) = y((m,n),0) = 0$ for every integer vector (m,n).

Consider the commutator of the elements $(e_1,0)$ and $(e_2,0)$ in the group G:

$$\begin{aligned}[(e_1,0),(e_2,0)] &= (e_1,0)(e_2,0)(-e_1,-y(e_1,-e_1))(-e_2,-y(e_2,-e_2)) \\ &= (0, y(e_1,e_2)^{-e_1-e_2} - y(e_1,-e_1)^{-e_2} + y(e_1+e_2,-e_1)^{-e_2}).\end{aligned}$$

Let

$$\gamma_y = y(e_1,e_2)^{-e_1-e_2} - y(e_1,-e_1)^{-e_2} + y(e_1+e_2,-e_1)^{-e_2}.$$

Definition 6. Set

$$\beta(y) = S(\gamma_y),$$

where $S(\gamma_y)$ is the algebraic area of γ_y, i.e., the sum of the winding numbers of all integer points in the domain bounded by γ_y with respect to γ_y.

As follows from Lemma 5, the cycle γ_y has a unique representation as a sum of plaquettes:

$$\gamma_y = \sum_{(m,n)} k_{(m,n)} p(m,n),$$

where the sum is taken over all integer points $(m,n) \in \mathbf{Z}^2$, $k_{(m,n)} \in Z$ are integer coefficients (only finitely many of them are not equal to zero), and $p(m,n) = p((m,n),1,2)$ are plaquettes as defined in Section 3.2.

Then

$$\beta(y) = S(\gamma_y) = \sum_{(m,n)} k_{(m,n)}.$$

The latter equality can also be taken as a definition of $S(\gamma_y)$.

Theorem 7. *The constructed mapping β is constant on cohomology classes, i.e., if cocycles y and y' are cohomologous, then $\beta(y) = \beta(y')$.*

The well-defined homomorphism

$$\beta : H^2(\mathbf{Z}^2, H_1(\mathbf{E}^2)) \simeq \mathbf{Z}$$

is a group isomorphism, and

$$\beta(\tilde{c}) = 1,$$

where the class $\tilde{c}$ determines the extension isomorphic to the free metabelian group Met(2).

Proof. Let $y' \in \tilde{y}$. We have to show that $S(\gamma_{y'}) = S(\gamma_y)$.

Indeed, since y and y_1 are cohomologous, there exists a mapping $u : \mathbf{Z}^2 \to H_1(\mathbf{E}^2)$ such that for any (m,n) and (k,l) from $\mathbf{Z}^2$,

$$y'((m,n),(k,l)) = y((m,n),(k,l)) + u(k,l) + u(m,n)^{(k,l)} - u(m+k,n+l).$$

Then

$$\gamma_{y'} = \gamma_y + u(e_1)^{e_2 - e_1} - u(e_1)^{-e_1} + u(e_2)^{-e_1} - u(e_2). \qquad (*)$$

It is obvious that the algebraic area of the difference $\gamma_{y'} - \gamma_y$ is equal to zero.

By definition, β is a group homomorphism. Now we will show that β is an epimorphism. Note that if $\tilde{c}$ is the cocycle defined in Section 2.1, then $\gamma_c = p(0, -1)$, and thus $\beta(\tilde{c}) = 1$. Then for every $n \in \mathbf{Z}$ we have

$$\beta(\tilde{nc}) = n,$$

and β is an epimorphism.

It suffices to show that β is a monomorphism. Assume that $\tilde{y} \in H_1(\mathbf{E}^2)$ is a cohomology class, $y \in \tilde{y}$ is its representative, and $\beta(\tilde{y}) = S(\gamma_y) = 0$.

If $S(\gamma_y) = 0$, then, using equation (*), we can find a mapping $u : \mathbf{Z}^2 \to H_1(\mathbf{E}^2)$ such that $\gamma_{y'} = 0$. This means that in the extension G of $H_1(\mathbf{E}^2)$ by $\mathbf{Z}^2$ determined by y', the elements $(e_1, 0)$ and $(e_2, 0)$ commute.

Let $H \subset G$ be the subgroup of G generated by the elements $(e_1, 0), (e_2, 0)$, and let $N \triangleleft G$ be the normal subgroup of G consisting of all elements of the form $(0, \gamma)$, $\gamma \in H_1(\mathbf{E}^2)$. Then $H \simeq \mathbf{Z}^2$, $N \simeq H_1(\mathbf{E}^2)$.

Obviously, $H \cap N = \emptyset$ and $HN = G$. Then G is a split extension, and the mapping y is zero. $\square$

3.4 The structure of other extensions

As was proved in the previous section, the cohomology group $H^2(\mathbf{Z}^2, H_1(\mathbf{E}^2))$ is isomorphic to the group $\mathbf{Z}$, and the free metabelian group Met(2) corresponds to the generator $1 \in \mathbf{Z}$. It is interesting to see what is the structure of extensions corresponding to other integers.

Let $k \in \mathbf{Z}$ be an integer. Consider the group $\mathrm{Met}_k(2)$ constructed as follows.

The group $\mathrm{Met}_k(2)$ has three generators x, y, z. As before, let

$$z^g = gzg^{-1}, \qquad g \in \mathrm{Met}_k(2).$$

Then the group G_k is determined by the relations

$$\begin{aligned} [x, y] &= z^k, & (1)\\ z^{x^m y^n} \cdot z^{x^{m'} y^{n'}} &= z^{x^{m'} y^{n'}} \cdot z^{x^m y^n} \qquad \forall m, n \in \mathbf{Z}. & (2) \end{aligned}$$

Note that $\mathrm{Met}_1(2) = \mathrm{Met}(2)$.

Consider the normal subgroup N of $\mathrm{Met}_k(2)$ generated by the element z (in the sense that N is the smallest normal subgroup of $\mathrm{Met}_k(2)$ that contains z); in other words, N consists of elements of the form $\{z^{x^m y^n}\}_{m,n \in \mathbf{Z}}$ and their products.

Note that for $k = \pm 1$, the subgroup N is equal to the commutant $\mathrm{Met}_k(2)'$, but for other values of k, the commutant is strictly included into M.

Theorem 8. *1) There is a group isomorphism* $\mathrm{Met}_k(2)/N \simeq \mathbf{Z}^2$.

2) The group N is isomorphic to $H_1(\mathbf{E}^2; \mathbf{Z})$ as a group with operators from $\mathbf{Z}^2$.

3) Let $\tilde{c}_k \in H^2(\mathbf{Z}^2, H_1(\mathbf{E}^2))$ be the class of 2-cocycles that determines the group $\mathrm{Met}_k(2)$ as an extension of the group $N \simeq H_1(\mathbf{E}^2)$ by the group $\mathrm{Met}_k(2)/N \simeq \mathbf{Z}^2$. Then

$$\beta(\tilde{c}_k) = k,$$

where the mapping β is defined in Section 3.3.

Proof. 1) This follows from the fact that $[x, y] = e \mod N$ and there are no other relations on the elements x and y modulo N.

2) Indeed, the isomorphism is given by the correspondence

$$z^{x^m y^n} \mapsto p(m, n),$$

where $p(m, n) \in H_1(\mathbf{E}^2)$ is the plaquette defined in the previous section.

3) Let $c_k \in \tilde{c}_k$ be the cocycle corresponding to the following set of representatives:

$$(m, n) \mapsto x^m y^n.$$

Then the value of the cocycle $c_k((m, n), (m', n'))$ is a cycle corresponding to the word $x^m y^n x^{m'} y^{n'} y^{-n-n'} x^{-m-m'}$.

Thus $c_k((1, 0), (1, 0)) = 0$, $c_k((1, 0), (0, 1)) = 0$, and the value of $c_k((1, 1), (-1, 0))$ corresponds to the word $[x, y] = z^k$, i.e,

$$c_k((1, 1), (-1, 0)) = kp(0, 0),$$

and

$$\begin{aligned} \gamma_{c_k} &= c_k((1,0),(0,1))^{(-1,-1)} - c_k((1,0),(-1,0))^{(0,-1)} + c_k((1,1),(-1,0))^{(0,-1)} \\ &= kp(0, -1); \end{aligned}$$

thus

$$\beta(\tilde{c}_k) = S(\gamma_{c_k}) = k.$$

□

Remeber that set of placuettes $p(n, m)$ is the set of free generators of $H_1(E^2)$, so each cycle is a finite sum $h = \sum k(n, m)p(n, m)$. Let M be the normal subgroup of N generated by z^k. In other words, M consists of elements of the form $z^{kx^m y^n}$, $m, n \in \mathbf{Z}$, and their products. Theorem 8 implies the following assertions.

Theorem 9. *1) The groups N and M are isomorphic to $H_1(\mathbf{E}^2)$. Let us fix isomorphism between N and $H_1(\mathbf{E}^2)$ then the group M goes to subgroup of $H_1(\mathbf{E}^2)$ generated by all the cycles multiplicities of which with respect to each placuettes belong to the group $k\mathbf{Z}$,*

2)

$$N/M \simeq \oplus_{\mathbf{Z}^2} \mathbf{Z}_{|k|},$$

$$\mathrm{Met}_k(2)/M \simeq \mathbf{Z}^2 \rightthreetimes \oplus_{\mathbf{Z}^2} \mathbf{Z}_{|k|},$$

where $\mathbf{Z}_{|k|}$ is the finite cyclic group of order $|k|$. and $\rightthreetimes$ denotes the semidirect product with respect to the action of $\mathbf{Z}^2$ on $\oplus_{\mathbf{Z}^2}\mathbf{Z}_{|k|}$ by shifts of coordinates.

3) 2-cocycles of the class $\tilde{c}_k$ have values in the group M as subgroup of $H_1(\mathbf{E}^2)$ (see item 1))

4) The commutant $\mathrm{Met}_k(2)'$ of the group $\mathrm{Met}_k(2)$ is subgroup of $N \subset H_1(\mathbf{E}^2)$ and has the following description:

$$M \subset \mathrm{Met}_k(2)' = \{h = \sum k(n,m)p(n,m) \in H_1(\mathbf{E}^2) : \sum_{n,m} k(n,m) \in k\mathbf{Z}\} \subset N$$

so the corresponding quotient is the following:

$$\mathrm{Met}_k(2)/\mathrm{Met}_k(2)' = \mathbf{Z}^2 \oplus \mathbf{Z}_{|k|}$$

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