PUSH-OUT OF SCHEMES AND SOME APPLICATIONS

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0. Introduction.

Let $p : Z \to X$ and $q : Z \to Y$ be morphisms of S-schemes. A commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{p} & X \\ {\scriptstyle q}\downarrow & & {\scriptstyle f}\downarrow \\ Y & \xrightarrow{g} & T \end{array} \tag{1}$$

over S is called a *push-out* if for any S-scheme T' and any S-morphisms $f' : X \to T'$ and $g' : Y \to T'$ such that $f' \circ p = g' \circ q$, there is a unique morphism $\phi : T \to T'$ such that $f' = \phi \circ f$ and $g' = \phi \circ g$. (in this case we also say T is a push-out of p and q). This is the dual concept of pull-back (i.e. fiber product) in the category of schemes. However, unlike pull-back, push-out usually does not exist.

But in many cases we need to see if a push-out exists. For example, let G be a group scheme over S, X be an S-scheme and $\rho : G \times_S X \to X$ be an action. Then the quotient X/G of ρ, if it exists, is just the push-out of ρ and $\mathrm{pr}_2 : G \times_S X \to X$. Another example comes from the construction of moduli schemes: a technical method is to construct a "catalogue scheme" first, and then descent to the moduli scheme by shrinking each locus corresponding to one equivalence class to one point. In other words, this is "dividing by an equivalence relation", which is also a kind of push-out. One can also use push-out to show the existence of a "descent" of a morphism.

Of course the general answer to the existence of a push-out is hard. There have been many works about this, for example see [2], [3] or [20]. However, people have several quite different philoshphy along this line.

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In §1 we first give a criterion of push-out: for any two morphisms $X \to S$ and $Y \to S$ of finite type, if one of them is f.f. and the other is an epimorphism, then their pull-back is also a push-out. Then we give a necessary and sufficient condition for the exitence of push-out (1) when Z is a closed subscheme of $X \times_S Y$ and p, q are both f.f. and relatively projective. We call this "flat projective case". In this case the push-out is very "nice", i.e. it is universal and geometric.

The non-flat or non-projevtive case is usually more complicated, and usually we can not hope the push-out to be very nice. We only have some partial results, which are included in §2. In particular, this gives a criterion of the existence of "rational push-out".

As an application, in §3 we use the results of §1 to give a construction of Picard scheme in general case (i.e. for any flat family of projective varieties). In particular, when $S = \mathrm{Spec}(K)$ for a field K, this method does not need a field extension of K. We also give some other applications in §1 and §2.

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1. Push-out and some criteria.

Terminology: A morphism $f : X \to Y$ is called an *epimorphism* if f is set-theoretically onto and $O_Y \to f_*O_X$ is a monomorphism.

Definition. Let $p : Z \to X$, $q : Z \to Y$ be two morphisms of schemes. If there is a scheme T together with two morphisms $f : X \to T$ and $g : Y \to T$ satisfying

a) The following diagram is commutative:

$$\begin{array}{ccc} Z & \xrightarrow{p} & X \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & T \end{array} \tag{1}$$

b) If there is another scheme T' together with morphisms $f' : X \to T'$, $g' : Y \to T'$ such that

$$\begin{array}{ccc} Z & \xrightarrow{p} & X \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow{g'} & T' \end{array}$$

is commutative, then there is a unique morphism $\psi : T \to T'$ such that $f' = \psi \circ f$ and $g' = \psi \circ g$,

then we say T (or (1), or (T, f, g)) is a (categorical) *push-out* of p and q. The push-out is called *geometric* if the induced morphism $Z \to X \times_T Y$ is epimorphic; called *universal* if for any morphism $T' \to T$, the following diagram

$$\begin{array}{ccc} Z \times_T T' & \xrightarrow{p} & X \times_T T' \\ {\scriptstyle q}\downarrow & & {\scriptstyle f}\downarrow \\ Y \times_T T' & \xrightarrow{g} & T' \end{array}$$

is also a push-out.

Push-out is the dual concept of pull-back (i.e. fiber product) in the category of schemes. However, unlike pull-back, push-out usually does not exist.

Remark 1. Note that if (1) is a push-out, then

$$\begin{array}{ccc} X \times_T Y & \xrightarrow{\mathrm{pr}_1} & X \\ {\scriptstyle \mathrm{pr}_2}\downarrow & & {\scriptstyle f}\downarrow \\ Y & \xrightarrow{g} & T \end{array}$$

is also a push-out, hence we will often consider a push-out which is also a pull-back; furthermore, if $\psi : Z' \to Z$ is an epimorphism, then

$$\begin{array}{ccc} Z' & \xrightarrow{p\circ\psi} & X \\ {\scriptstyle q\circ\psi}\downarrow & & {\scriptstyle f}\downarrow \\ Y & \xrightarrow{g} & T \end{array}$$

is also a push-out.

Example 1. Let $G \to S$ be a group scheme, $X \to S$ be a scheme and $\rho : G \times_S X \to X$ be an action of G on X. An S-morphism $p : X \to Y$ is called a *categorical quotient* of ρ if the following diagram is a push-out

$$\begin{array}{ccc} G \times_S X & \xrightarrow{\rho} & X \\ {\scriptstyle \mathrm{pr}_2}\downarrow & & {\scriptstyle p}\downarrow \\ X & \xrightarrow{p} & Y \end{array}$$

In this case we denote $Y = X/\rho$ (or $Y = X/G$ if there is no confusion).

Example 2. More generally, let $X \to S$ be a scheme and $W \subset X \times_S X$ be a locally closed subscheme. We say W gives an *equivalence relation* $\sim_W$ of X over S if

i) ("reflexivity") $\Delta(X) \subset W$;
ii) ("symmetricity") Let $\iota : X \times_S X \to X \times_S X$ be the morphism by switching factors, then $\iota(W) = W$;
iii) ("transitivity") Let W_{ij} $(i, j = 1 \text{ or } 2)$ be the pull-back of $\mathrm{pr}_i : W \to X$ and $\mathrm{pr}_j : W \to X$, viewed as a subscheme of $X \times_S X \times_S X$, then $W_{11} = W_{12}$.

In this case, an S-morphism $p : X \to Y$ is called a *categorical equilizer* of W (or a *categorical quotient* of $\sim_W$, denoted by $Y = X/\sim_W$) if the following diagram is a push-out

$$\begin{tikzcd} W \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"'] & X \arrow[d, "p"] \\ X \arrow[r, "p"] & Y \end{tikzcd} \tag{2}$$

and Y (or p) is called a *geometric equilizer* of W (or a *geometric quotient* of $\sim_W$) if in addition that $X \times_Y X = W$ as subschemes of $X \times_S X$ (this is equivalent to that (2) is a geometric push-out; note that $X \times_Y X$ is a closed subscheme of $X \times_S X$ when Y is separated over S). A quotient $Y = X/\sim_W$ is called *universal* if for any morphism $Y' \to Y$, the equivalence relation of $X' = X \times_Y Y'$ over S given by $W' = W \times_Y Y' \subset X \times_Y X \times_Y Y' \subset X' \times_S X'$ has a quotient $X'/\sim_{W'} \cong (X/\sim_W) \times_Y Y'$ (this is equivalent to that (2) is a universal push-out).

Suppose $W \subset X \times_S X$ gives an equivalence relation on X such that there is a geometric quotient $X/\sim_W$. If X is separated over $X/\sim_W$, then obviously W is a closed subscheme of $X \times_S X$. (In fact we mostly consider the case when W is a closed subscheme of $X \times_S X$.)

Lemma 1. *Let S be a noetherian scheme, $f : X \to S$ and $g : Y \to S$ be morphisms of finite type, where g is f.f. and f is an epimorphism. Let $p : X \times_S Y \to X$ and $q : X \times_S Y \to Y$ be projections. Then S is a geometric push-out of p and q, and is universal if f is also flat.*

Proof. Let $f' : X \to S'$ and $g' : Y \to S'$ be morphisms such that $f' \circ p = g' \circ q$. For any $y \in Y$, let $s = g(y)$, then there exists $x \in X$ such that $f(x) = s$ (since f is an epimorphism); for any $y' \in g^{-1}(s)$ there exists $z \in X \times_S Y$ such that $p(z) = x$ and $q(z) = y'$, hence all of the points in $g^{-1}(s)$ maps to the same point

$s' = g'(y) \in S'$; similarly all of the points in $f^{-1}(s)$ maps to s', therefore there is a (unique) induced map of sets $\phi : S \to S'$ satisfying set-theoretic equations $\phi \circ g = g'$ and $\phi \circ f = f'$. Since g is an open map (of topological spaces), we see ϕ is a continuous map of topological spaces.

Let $U' = \mathrm{Spec}R' \subset S'$ be an affine open subscheme. For any $s \in \phi^{-1}(U')$ and any $x \in X$, $y \in Y$ such that $f(x) = g(y) = s$, there exists an affine open neighborhood $U = \mathrm{Spec}R \subset \phi^{-1}(U')$ of s, an affine open neighborhood $V = \mathrm{Spec}A \subset f^{-1}(U)$ of x and an affine open neighborhood $W = \mathrm{Spec}B \subset g^{-1}(U)$ of y such that $g(W) = U$ and $f^* : R \to A$ is a monomorphism (because $f^* : O_{s,S} \to O_{x,X}$ is a monomorphism). Note that B is f.f. over R, i.e. $g^* : R \to B$ is monomorphic and $M = B/g^*(R)$ is a flat R-module, hence we have a commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 \to R & \xrightarrow{g^*} & B & \xrightarrow{\lambda} & M & \to 0 \\
\downarrow f^* & & \downarrow q^* & & \downarrow \eta & \\
0 \to A & \xrightarrow{p^*} & A \otimes_R B & \xrightarrow{\mu} & A \otimes_R M & \to 0
\end{array}$$

Since η is a monomorphism and $\eta \circ \lambda \circ g'^* = \mu \circ q^* \circ g'^* = \mu \circ p^* \circ f'^* = 0 : R' \to A \otimes_R M$, we have $\lambda \circ g'^* = 0 : R' \to M$, hence there is a (unique) induced homomorphism $\phi^* : R' \to R$ such that $f^* \circ \phi^* = f'^*$ and $g^* \circ \phi^* = g'^*$. Therefore we can define ϕ as a morphism and we have equations of morphisms $\phi \circ g = g'$, $\phi \circ g = g'$. Q.E.D.

Example 3. Let S be a noetherian scheme, X be an S-scheme of finite type and G be a group scheme of finite type over S. An action $\rho : G \times_S X \to X$ of G on X is called *transitive* if $(\rho, \mathrm{pr}_2) : G \times_S X \to X \times_S X$ is an epimorphism. If ρ has a geometric quotient X/G which is a locally closed subscheme of S, then ρ is transitive by definition; on the other hand, if $X \to S$ is f.f. and ρ is transitive, then there is a geometric quotient $X/G \cong S$ by Lemma 1 and Remark 1.

Proposition 1. *Let S be a noetherian scheme, $X \to S$ be a relatively quasi-projective morphism and $W \subset X \times_S X$ be a locally closed subscheme which gives an equivalence relation of X over S. Suppose that $\mathrm{pr}_1 : W \to X$ is relatively projective and flat. Then there is a universal geometric quotient $Y = X/\sim_W$ which is relatively quasi-projective (and relatively projective if $X \to S$ is so) over S. Furthermore, $X \to Y$ is f.f. and relatively projective.*

Proof. Let X_1 be a connected component of X and $W_1 = \mathrm{pr}_1^{-1}(X_1) \subset W$. Let $X_2 = \mathrm{pr}_2(W_1)$ (this makes sense since pr_2 is proper). Then by the definition of equivalence relation we see that $\mathrm{pr}_1^{-1}(X_2) = W \cap X_2 \times_S X_2$ which gives the restriction of $\sim_W$ on X_2. Hence for simplicity we may assume $X = X_2$.

Take a relatively projective scheme $\bar{X} \to S$ which contains X as an open subscheme such that $X \to \bar{X}$ is dominant, and fix an $O_{\bar{X}}(1)$. Then W is a closed subscheme of $\bar{X} \times_S X$ since $W \to X$ is proper. The above assumption guarantees that the fibers of $\mathrm{pr}_2 : W \to X$ all have the same Hilbert polynomial χ. Let $\mathcal{H} = \mathcal{H}ilb^{\chi}_{\bar{X}/S}$ be the Hilbert scheme representing the following functor:

$$
\begin{aligned}
&((S\text{-schemes})) \to ((\text{sets}))\\
&\qquad T \mapsto \{\text{closed subschemes of } \bar{X} \times_S T, \text{ flat over } T \text{ with } \chi \text{ as}\\
&\qquad\qquad \text{the Hilbert polynomial of each fiber over } T\}
\end{aligned}
$$

and $\mathcal{Z} \subset \bar{X} \times_S \mathcal{H}$ be the universal subscheme. Then W induces an S-morphism $\phi : X \to \mathcal{H}$ such that $W = \mathcal{Z} \times_{\mathcal{H}} X$ (as closed subschemes of $\bar{X} \times_S X$). Let $\bar{Y} \subset \mathcal{H}$ be the closed subscheme defined by the ideal sheaf $\ker(O_{\mathcal{H}} \to \phi_* O_X)$, and $\bar{Z} = \mathcal{Z} \times_{\mathcal{H}} \bar{Y} \subset \bar{X} \times_S \bar{Y}$. Then $W = \bar{Z} \times_{\bar{Y}} X$. Denote by $q : X \to \bar{Y}$, $p_1 : \bar{Z} \to \bar{X}$, $p_2 : \bar{Z} \to \bar{Y}$ and $p : W \to \bar{Z}$ the projections. Then p_2 is f.f. and q is domoiant, hence p is dominant. Note that $\mathrm{pr}_1 = p_1 \circ p : W \to \bar{X}$, hence p_1 is surjective (because it is dominant and proper).

The pull-back of $\bar{Z} \to \bar{Y}$ and $q \circ \mathrm{pr}_2 : W \to \bar{Y}$ is equal to the pull-back of pr_2 and pr_2 (i.e. W_{22}, see Example 2), while the pull-back of $\bar{Z} \to \bar{Y}$ and $q \circ \mathrm{pr}_1 : W \to \bar{Y}$ is equal to the pull-back of pr_1 and pr_2 (i.e. W_{12}). Since $W_{12} = W_{22}$, by the universality of $\mathcal{H}$ we have $q \circ \mathrm{pr}_1 = q \circ \mathrm{pr}_2 : W \to \bar{Y}$. Hence

$$q \circ p_1 \circ p = q \circ \mathrm{pr}_1 = q \circ \mathrm{pr}_2 = p_2 \circ p \tag{3}$$

Let $Z = p_1^{-1}(X)$. By abuse of notation we still denote by $p_1 : Z \to X$, $p_2 : Z \to \bar{Y}$ and $p : W \to Z$ the projections. Since $p : W \to Z$ is an epimorphism (because it is proper and dominant), (3) gives

$$q \circ p_1 = p_2 : Z \to \bar{Y} \tag{4}$$

Hence $p : W \cong Z \times_{\bar{Y}} X \to Z$ has a section. Since $p_2 : \bar{Z} \to \bar{Y}$ is an open map we can define $Y = p_2(Z) = q(X)$ which is an open dense subscheme of $\bar{Y}$. Furthermore we have $W \subset X \times_Y X$, hence in $X \times_S X \times_S X$ we have

$$
\begin{aligned}
Z \times_Y W &\subset Z \times_Y (X \times_Y X) = W \times_Y X = W \times_Z (Z \times_Y X)\\
&= W \times_Z W \subset W \times_X W = Z \times_Y W
\end{aligned}
$$

i.e. $Z \times_Y W = Z \times_Y (X \times_Y X)$, thus $W = X \times_Y X$ because Z is f.f. over Y. Therefore

$$Z \times_Y W = W_{21} = W_{12} = W \times_Y Z = X \times_Y X \times_Y Z = X \times_Y W \tag{5}$$

Now (5)$\times_W Z$ (via the section $Z \to W$ of p) gives

$$Z \times_Y Z \cong X \times_Y Z \tag{6}$$

hence $p_1 : Z \to X$ is an isomorphism because Z is f.f. over Y.

Finally, let $Z' = p_2^{-1}(Y)$, then $W = Z' \times_Y X$, hence W is f.f. over Z'. This shows $Z' = Z \cong X$, hence $W = X \times_Y X$. We see that Y is a universal geometric quotient of $\sim_W$ by Lemma 1. Q.E.D.

Let X, Y be S-schemes and $W \subset X \times_S Y$ be a locally closed subscheme. If $\mathrm{pr}_1 : W \to X$ and $\mathrm{pr}_2 : W \to Y$ has a geometric push-out Z, then by definition $W \xrightarrow{\cong} X \times_Z Y$, hence

$$W \times_X W \times_Y W = W \times_Y W \times_X W \subset X \times_S X \times_S Y \times_S Y \tag{7}$$

because both sides are equal to $X \times_Z X \times_Z Y \times_Z Y$.

Proposition 2. *Let S be a noetherian scheme, $X \to S$ and $Y \to S$ be relatively quasi-projective morphisms and $W \subset X \times_S Y$ be a locally closed subscheme satisfying (7). Suppose that $\mathrm{pr}_1 : W \to X$ $\mathrm{pr}_2 : W \to Y$ are f.f. and relatively projective. Then there is a universal geometric push-out Z of pr_1 and pr_2 which is relatively quasi-projective (and relatively projective if $X \to S$ and $Y \to S$ are so) over S. Furthermore, $X \to Z$ and $Y \to Z$ are f.f. and relatively projective.*

Proof. In the language of presheaves (for simplicity denote $\underline{X}(T) = \mathrm{Mor}_S(T, X)$ etc.), (7) can be expressed as: For any S-scheme T and any $x, x' \in \underline{X}(T)$ and $y, y' \in \underline{Y}(T)$, if $(x, y), (x, y'), (x', y') \in \underline{W}(T)$, then $(x', y) \in \underline{W}(T)$. We will use this language in the following.

Let $X_1 = W \times_X W$, viewed as a (relatively projective) scheme over $Y_2 = Y \times_S Y$. Let $W_1 = X_1 \times_Y W$. Then it is easy to see (by the language of presheaves) that

$$W_1 = X_1 \times_{Y_2} X_1 \tag{8}$$

hence gives an equivalence relation on X_1 over Y_2. Furthermore, $\mathrm{pr}_1 : W_1 \to X_1$ is f.f. and relatively projective, hence by Proposition 1 there is a universal geometric quotient $Y_3 = X_1/\sim_{W_1}$, and $X_1 \to Y_3$ is f.f. and relatively projective. By (8) we have

$$X_1 \times_{Y_3} X_1 = W_1 = X_1 \times_{Y_3} (Y_3 \times_{Y_2} X_1) \tag{9}$$

hence $X_1 \cong Y_3 \times_{Y_2} X_1$ because X_1 is f.f. over Y_3, by the same reason we see $\Delta : Y_3 \to Y_3 \times_{Y_2} Y_3$ is an isomorphism. This shows $Y_3 \to Y_2$ is a closed immersion (a proper morphism $T \to T'$ is a closed immersion iff $\Delta : T \to T \times_{T'} T$ is an isomorphism). Furthermore, the following commutative diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{\mathrm{pr}_2} & W \\ \downarrow & & \downarrow \\ Y_3 & \xrightarrow{\mathrm{pr}_2} & Y \end{array} \tag{10}$$

is a pull-back, i.e. $X_1 \cong Y_3 \times_Y W$, this comes from

$$X_1 \times_{Y_3} X_1 \cong W_1 = X_1 \times_Y W \cong X_1 \times_{Y_3} (Y_3 \times_Y W) \tag{11}$$

(and that $X_1 \to Y_3$ is f.f.). Hence $\mathrm{pr}_2 : Y_3 \to Y$ is f.f. and relatively projective (since the composition $Y_3 \hookrightarrow Y_2 \to Y$ is quasi-projective and proper).

We now show that Y_3 gives an equivalence relation of Y over S. The commutativity is obvious. For the reflexivity, note that the composition

$$W \overset{\Delta}{\to} W \times_X W = X_1 \to Y_3 \to Y_2$$

is equal to the composition $W \to Y \overset{\Delta}{\to} Y_2$, where $W \to Y$ is an epimorphism. For the transitivity, we need to show $Y_{311} = Y_{312} \subset Y \times_S Y \times_S Y$, where Y_{3ij} $(i, j = 1$ or $2)$ is the pull-back of $\mathrm{pr}_i : Y_3 \to Y$ and $\mathrm{pr}_j : Y_3 \to Y$. It is enough to show that $X_1 \times_{Y_3} Y_{311} = X_1 \times_{Y_3} Y_{312}$ in $X \times_S Y \times_S Y \times_S Y$. But by (10) we have

$$X_1 \times_{Y_3} Y_{311} \cong X_1 \times_Y Y_3 \cong W \times_X W \times_Y Y_3 \cong W \times_X X_1 \cong W \times_X W \times_X W$$

similarly we have $X_1 \times_{Y_3} Y_{311} \cong W \times_X W \times_X W$, hence we get an isomorphism $X_1 \times_{Y_3} Y_{311} \to X_1 \times_{Y_3} Y_{312}$ compatible with the inclusions to $X \times_S Y \times_S Y \times_S Y$.

By Proposition 1 there is a universal geometric quotient $Z = Y/\sim_{Y_3}$ which is relatively quasi-projective over S, and $Y \to Z$ is f.f. and relatively projective. By (10) we see that $W \to Z$ equilizes pr_1 and $\mathrm{pr}_2 : X_1 \to W$, hence $W \to Z$ factors through $W/\sim_{X_1} \cong X$. Since $W \to X$ and $W \to Z$ are both f.f. and relatively projective, we see that $X \to Z$ is f.f. and relatively projective. Furthermore,

$$\begin{aligned} W \times_X (X \times_Z Y) &\cong W \times_Z Y \cong W \times_Y (Y \times_Z Y) \\ &\cong W \times_Y Y_3 \cong X_1 \cong W \times_X W \end{aligned} \tag{12}$$

Hence $W \to X \times_Z Y$ is an isomorphism. Q.E.D.

2. Some results for non-flat case.

Proposition 1. *Let S be a noetherian scheme, $X \to S$ and $Y \to S$ be relatively quasi-projective morphisms and $W \subset X \times_S Y$ be a locally closed subscheme satisfying (7) of §1. Suppose that $\mathrm{pr}_1 : W \to X$ and $\mathrm{pr}_2 : W \to Y$ are relatively projective, $\mathrm{pr}_1 : W \to X$ is f.f. and $\mathrm{pr}_2 : W \to Y$ is an epimorphism. Then $\mathrm{pr}_2 : W \to Y$ factors through an epimorphism $W \to Z$, where $Z \to Y$ is finite and set-theoretically one to one, such that there is a geometric push-out*

$$\begin{array}{ccc} W & \xrightarrow{\mathrm{pr}_2} & Z \\ \Big\downarrow{\scriptstyle \mathrm{pr}_1} & & \Big\downarrow \\ X & \longrightarrow & T \end{array} \tag{1}$$

where T is relatively quasi-projective (and relatively projective if $X \to S$ and $Y \to S$ are so) over S, $X \to T$ and $Z \to T$ are relatively projective, $Z \to T$ is f.f., and $X \to T$ is an epimorphism.

Proof. Take a relatively projective scheme $\bar{Y} \to S$ which contains Y as an open subscheme such that $Y \to \bar{Y}$ is dominant, and fix an $O_{\bar{Y}}(1)$. Then W is a closed subscheme of $X \times_S \bar{Y}$ since $W \to X$ is proper. Let $\mathcal{H} = \mathcal{H}ilb_{\bar{Y}/S}$ be the Hilbert scheme representing the following functor:

$$\begin{aligned} &((S\text{-schemes})) \to ((\text{sets})) \\ &\qquad S' \mapsto \{\text{closed subschemes of } \bar{Y} \times_S S', \text{ flat over } S'\} \end{aligned}$$

and $\mathcal{Z} \subset \bar{Y} \times_S \mathcal{H}$ be the universal subscheme. Then W induces an S-morphism $\phi : X \to \mathcal{H}$ such that $W = X \times_{\mathcal{H}} \mathcal{Z}$ (as closed subschemes of $X \times_S \bar{Y}$). Let $\bar{T} \subset \mathcal{H}$ be the closed subscheme defined by the ideal sheaf $\ker(O_{\mathcal{H}} \to \phi_* O_X)$, and $\bar{Z} = \mathcal{Z} \times_{\mathcal{H}} \bar{T} \subset \bar{Y} \times_S \bar{T}$. Then $W = X \times_{\bar{T}} \bar{Z}$. Denote by $q : X \to \bar{T}$, $p_1 : \bar{Z} \to \bar{Y}$, $p_2 : \bar{Z} \to \bar{T}$ and $p : W \to \bar{Z}$ the projections. Then p_2 is f.f. and q is dominant, hence p is dominant. Note that $\mathrm{pr}_2 = p_1 \circ p : W \to \bar{Y}$, hence p_1 is surjective (because it is dominant and proper). Let $Z = p_1^{-1}(Y)$, then $Z \to Y$ is relatively projective (because $\bar{Z} \to \bar{Y}$ is so), and $W \to Z$ is an epimorphism (because it is proper and dominant).

The pull-back of $\bar{Z} \to \bar{T}$ and $q \circ \mathrm{pr}_1 : W \to \bar{T}$ is equal to $Z \times_{\bar{T}} W \cong (Z \times_{\bar{T}} X) \times_X W \cong W \times_X W$, hence the pull-back of $\bar{Z} \to \bar{T}$ and $q \circ \mathrm{pr}_1 \circ \mathrm{pr}_1 : W \times_Y W \to \bar{T}$ is equal to $W \times_X W \times_Y W$, and the pull-back of $\bar{Z} \to \bar{T}$ and $q \circ \mathrm{pr}_1 \circ \mathrm{pr}_2 : W \times_Y W \to \bar{T}$ is equal to $W \times_Y W \times_X W$. By abstract nonsense (and the assumption that $W \times_X W \times_Y W = W \times_Y W \times_X W$), we see the following diagram is commutative:

$$\begin{array}{ccc} W \times_Y W & \xrightarrow{\mathrm{pr}_2} & W \\ \Big\downarrow{\scriptstyle \mathrm{pr}_1} & & \Big\downarrow \\ W & \longrightarrow & \bar{T} \end{array} \tag{2}$$

This induces a closed immersion $W \times_Y W \to W \times_{\bar{T}} W$.

We now show that $W \times_Z W \to W \times_Y W$ is an isomorphism. Since $W \to X$ is f.f., it is enough to show that $W \times_Z W \times_X W \cong W \times_Y W \times_X W$. Note that $W \times_Z W \times_X W \cong W \times_{\bar{T}} W$, hence we have a closed immersion

$$\eta : W \times_Y W \times_X W \hookrightarrow W \times_{\bar{T}} W \times_X W \overset{\simeq}{\to} W \times_Z W \times_X W \times_X W \tag{3}$$

and $\mathrm{pr}_1 \circ \eta = \mathrm{pr}_1$, $\mathrm{pr}_3 \circ \eta = \mathrm{pr}_2$, $\mathrm{pr}_4 \circ \eta = \mathrm{pr}_3$: $W \times_Y W \times_X W \to W$. Furthermore, we have $\mathrm{pr}_2 \circ \eta = \mathrm{pr}_2$ because $\mathrm{pr}_1 \circ \mathrm{pr}_2 \circ \eta = \mathrm{pr}_1 \circ \mathrm{pr}_2$: $W \times_Y W \times_X W \to X$ and $\mathrm{pr}_2 \circ \mathrm{pr}_2 \circ \eta = \mathrm{pr}_2 \circ \mathrm{pr}_1 = \mathrm{pr}_2 \circ \mathrm{pr}_2$: $W \times_Y W \times_X W \to Y$. Thus we have a commutative diagram

$$\begin{array}{ccc} W \times_Y W \times_X W & \xrightarrow{\eta} & W \times_Z W \times_X W \times_X W \\ \Big\downarrow{\scriptstyle \zeta} & & \Big\downarrow \\ W \times_Z W \times_X W & \xrightarrow{\mathrm{id}_W \times_Z \Delta \times_X \mathrm{id}_W} & W \times_Z W \times_X W \times_X W \end{array} \tag{4}$$

where $\Delta : W \to W \times_X W$ is the diagonal morphism. Hence ζ is a closed immersion. It is easy to see that the composition of ζ with the closed immersion $W \times_Z W \times_X W \to W \times_Y W \times_X W$ is equal to the identity morphism of $W \times_Z W \times_X W$, hence ζ is an isomorphism.

For a closed point $t : \mathrm{Spec}(K) \hookrightarrow T$ (where K is a field), denote by W_t (resp. X_t, Z_t) the fiber of W (resp. X, Z) over t. Then $W_t \cong X_t \times_K Z_t$, and $W_t \times_Y W_t \subset W \times_Y W_t = W \times_Z W_t = W_t \times_Z W_t$, hence $W_t \times_Y W_t = W_t \times_Z W_t = W_t \times_{Z_t} W_t$. Note that the following commutative diagram

$$\begin{array}{ccc} W_t \times_{Z_t} W_t & \xrightarrow{\simeq} & W_t \times_Y W_t \\ \Big\downarrow & & \Big\downarrow \\ Z_t & \xrightarrow{\Delta} & Z_t \times_Y Z_t \end{array} \tag{5}$$

is a pull-back, and $W_t \times_Y W_t \to Z_t \times_Y Z_t$ is f.f., we see $\Delta : Z_t \to Z_t \times_Y Z_t$ is an isomorphism, hence $Z_t \to Y$ is a closed immersion. This shows that $p_1 : Z \to Y$ is set-theoretically one to one, hence finite.

The other statements hold by Lemma 1.1. Q.E.D.

Remark 1. In many cases we can take $Z = Y$, for example when W is flat over Y (see Proposition 2). This is also the case when W is reduced and Y is normal, as shown in the following corollary.

Corollary 1. *Under the conditions of Proposition 1, if in addition that W is reduced and Y is normal, then $Z \cong Y$.*

Proof. The reducedness of W implies that Z and T are reduced. Let $t \in T$ be a generic point. Then by the argument in the proof of Proposition 1 we see that the diagonal morphism $Z_t \to Z_t \times_Y Z_t$ is an isomorphism. Let $\mathcal{I}$ be the ideal sheaf of $\Delta : Z \to Z \times_Y Z$, then $\mathcal{I}_t = 0$, hence there is an open neighborhood $U_t \subset T$ of t such that $\mathcal{I}|_{Z' \times_Y Z'} = 0$, where $Z' = p_2^{-1}(U_t)$ (because $\mathcal{I}$ is coherent). Let $Y' = Y - p_1(Z - Z')$ (where $p_1 : Z \to Y$ is as in the proof of Proposition 1), then Y' is an open subscheme of Y and $p_1^{-1}(Y') = Z'$. Since the diagonal morphism $Z' \to Z' \times_{Y'} Z'$ is an isomorphism, we have $Z' \cong Y'$.

Therefore for each irreducible component Z_0 of Z, we have $K(Z_0) \cong K(p_1(Z_0))$. But $Z \to Y$ is finite and Y is normal by assumption, we see $Z_0 \to p_1(Z_0)$ is an isomorphism, hence $Z \cong Y$. Q.E.D.

Remark 2. In Proposition 1.1, it is essential to assume that $W \to X$ is both relatively projective and flat. However, without these assumptions, at least $\sim_W$ has a "rational quotient" when X is reduced, as shown in the following corollary.

Corollary 2. *Let S be a noetherian scheme, X be a reduced scheme, $X \to S$ be a relatively quasi-projective morphism and $W \subset X \times_S X$ be a locally closed subscheme defining an equivalence relation. Then there is an open dense subscheme $X' \subset X$ such that the equivalence relation of X' given by $W \cap X' \times_S X'$ has a universal geometric quotient. Furthermore, $X'/\sim_{W'} \to S$ is relatively quasi-projective and $X' \to X'/\sim_{W'}$ is f.f. and relatively quasi-projective.*

Proof. For simplicity we may assume $X \to S$ is relatively projective and W is closed. Hence there is an open dense subscheme $X' \subset X$ such that $W|_{X'}$ is relatively projective and flat over X'. As in the proof of Proposition 1.1, we may assume that under a choice of $O_X(1)$, this induces a morphism of finite type $\phi : X' \to \mathcal{H} = \mathcal{H}ilb^{\chi}_{X/S}$ for some Hilbert polynomial χ. By further shrinking X' we may assume ϕ maps X' onto a locally closed subscheme $Y' \subset \mathcal{H}$. Let $W' = W \cap X' \times_S X'$. Then by the proof of Proposition 1 we have $W' = X' \times_{Y'} X'$ and $X' \to Y'$ is f.f., hence Y' is a universal geometric quotient of $\sim_{W'}$ by Lemma

1.1. Q.E.D.

Example 1. Let k be an algebraically closed field, A be an abelian variety over k, X be a variety over k, and $\rho : A \times_k X \to X$ be an action. If ρ is free, then it has a universal geometric quotient by Proposition 1.1. More generally, let H be the pull-back of

$$\begin{tikzcd} H \arrow[r] \arrow[d] & X \arrow[d, "\Delta"] \\ A \times_k X \arrow[r, "(\rho,\mathrm{pr}_2)"] & X \times_k X \end{tikzcd}$$

which is a closed subgroup scheme of $A \times_k X$ over X (called the *stabilizer* of ρ). If H is flat over X, then ρ has a universal geometric quotient by Proposition 1.1.

If H is not flat over X, at least ρ has a rational quotient by Corollary 2.

3. An application: A construction of Picard schemes

Let S be a noetherian scheme and $\pi : X \to S$ be a morphism of finite type. A closed subscheme $D \subset X$ is called an *S-effective divisor* if D is flat over S and the ideal sheaf $\mathcal{I}$ of D is an invertible sheaf. In this case denote $\mathcal{L}(D) = \mathcal{I}^{-1}$.

Suppose π is f.f. and relatively projective with a fixed $O_X(1)$, D is an S-effective divisor and $\mathcal{L} = \mathcal{L}(D)$. If S is connected, then the fibers of D over S all have the same Hilbert polynomial χ. By Castelnuovo and Mumford (see [12]), there exists an integer N, depending only on χ, such that for any $m > N$, $\pi_*\mathcal{L}(m)$ is locally free of rank $\chi(m)$ and generate $\mathcal{L}(m)$.

For any Hilbert polynomial χ, there is an open subscheme $\mathcal{D}iv^{\chi}_{X/S}$ of $\mathcal{H}ilb^{\chi}_{X/S}$ representing the following functor:

$$\begin{aligned} &((\text{locally noetherian } S\text{-schemes}))\to((\text{sets})) \\ &\qquad T \mapsto \{T\text{-effective divisors } D \subset X \times_S T \text{ whose fibers have Hilbert polynomial } \chi\} \end{aligned}$$

Let $S = \mathrm{Spec}(R)$, where R is a noetherian local ring, and $X \to S$ be a f.f. relatively projective morphism with geometrically integral fibers. Let X_0 be the fiber of X over the closed point $s_0 \in S$. Fix an $O_X(1)$. Let ξ be a numerical class of X_0. For simplicity we use the following terminology: For any S-scheme T whose closed points are all over s_0 and any invertible sheaf $\mathcal{L}$ on $X \times_S T$, if for any closed point $t \in T$ the fiber $\mathcal{L}_t$ is in the numerical class ξ, then we say $\mathcal{L}$ is in the numerical class ξ.

Lemma 1. *Let S be a noetherian local scheme, $X \to S$ be a f.f. relatively projective morphism with geometrically integral fibers, and X_0 be the fiber of X over the closed point of S. Then for any numerical class ξ_0 of X_0, there exists $n > 0$ such that for $\xi = \xi_0(n)$,*

($*$) for any noetherian S-scheme T and any invertible sheaf $\mathcal{L}$ on $X \times_S T$ in the numerical class ξ, $\mathcal{E} = \mathrm{pr}_{2*}\mathcal{L}$ is finitely generated locally free sheaf of O_T-modules which generates $\mathcal{L}$, and for any $t \in T$ we have $\mathcal{E}_t \cong \Gamma(X \times_S \{t\}, \mathcal{L}_t)$.

We omit the proof which needs some criterion of ampleness and semicontinuity theory (cf. [20]).

Suppose $\pi : X \to S$ is f.f. and relatively projective with geometrically integral fibers, $\mathcal{E} = \pi_*\mathcal{L}$ is locally free generating $\mathcal{L}$, and $\mathcal{E}_s \cong \pi_*\mathcal{L}_s$ for any $s \in S$. Let $O_S \to \mathcal{E}$ be an O_S-monomorphism such that $\mathcal{E}/f(O_S)$ is locally free, then f induces an O_X-monomorphism $O_X\mathcal{L}$, hence induces an O_X-monomorphism $f' : \mathcal{L}^{-1} \to O_X$. By local criterion of flatness, we see f' is a monomorphism and $O_X/\mathrm{im}(f')$ is O_S-flat, hence defines an S-effective divisor D such that $\mathcal{L}(D) \cong \mathcal{L}$. Conversely, if $D \subset X$ is an S-effective divisor satisfying $\mathcal{L}(D) \cong \mathcal{L}$, then D gives an O_X-monomorphism $f' : \mathcal{L}^{-1} \to O_X$ which is equivalent to an O_X-monomorphism $O_X \to \mathcal{L}$, hence induces an O_S-monomorphism $f : O_S \to \mathcal{E}$. By local criterion of flatness, we see that $\mathcal{E}/f(O_S)$ is O_S-locally free. Note that such an f is equivalent to a section of $\mathbb{P}(\mathcal{E}^\vee) \to S$, we get:

Lemma 2. *Let π and $\mathcal{L}$ be as above. Then $\mathbb{P}(\mathcal{E}^\vee)$ represents the following functor:*

$$\begin{aligned}&((\text{locally noetherian } S\text{-schemes}))\to((\text{sets}))\\ &\qquad (\phi : T \to S) \mapsto \{T\text{-effective divisors } D \subset X \times_S T\\ &\qquad\qquad \text{such that } \mathcal{L}(D) \cong (\mathrm{id}_X \times_S \phi)^*\mathcal{L}\}\end{aligned}$$

Lemma 3 ([13, Ⅲ.10. Proposition]). *Let S be a noetherian scheme and $\pi : X \to S$ be an f.f. relatively projective morphism with geometrically reduced and connected fibers. Let $\mathcal{L}$ be an invertible sheaf on X. Then there exists a closed subscheme $S_0 \subset S$ representing the following functor:*

$$\begin{aligned}&((\text{locally noetherian schemes}))\to((\text{sets}))\\ &\qquad T \mapsto \{\text{morphisms } T \to S \text{ such that } (\mathrm{id}_X \times_S f)^*\mathcal{L}\\ &\qquad\qquad \text{is locally trivial over } T\}\end{aligned}$$

Proof. Let X_0 be a reduced connected projective scheme over a field K, and $\mathcal{L}_0$ be an invertible sheaf on X_0. Then it is easy to see that $\mathcal{L}_0 \cong O_{X_0}$ iff $\Gamma(X_0, \mathcal{L}_0)$ and $\Gamma(X_0, \mathcal{L}_0^{-1})$ are both non-zero.

For simplicity we may assume $S = \text{Spec}R$. Let

$$\tilde{S}_0 = \{s \in S | \Gamma(X_s, \mathcal{L}_s) \neq 0, \Gamma(X_s, \mathcal{L}_s^{-1}) \neq 0\}$$

Then $\tilde{S}_0$ is a closed subset of S by semicontinuity theory. Furthermore, there is an R-homomorphism of finitely generated free R-modules $\phi : M_1 \to M_2$ such that for any morphism $f : T = \text{Spec}A \to S$ we have

$$\Gamma(X \times_S T, (\text{id}_X \times_S f)^* \mathcal{L}) \cong \ker(\phi \otimes_R \text{id}_A : M_1 \otimes_R A \to M_2 \otimes_R A)$$

Let $M = \text{coker}(\phi^\vee : M_2^\vee \to M_1^\vee)$, then $\ker(\phi \otimes_R \text{id}_A) \cong Hom_R(M, A)$. In particular, for any maximal ideal $P \subset R$ in $\tilde{S}_0$, we have $M/PM \cong R/P$, hence there is an open neighborhood $U \subset S$ of $\tilde{S}_0$ such that M is locally generated by one element on U. Note that if $(\text{id}_X \times_S f)^* \mathcal{L}$ is locally trivial over T, then f maps the points of T into $\tilde{S}_0$. Hence by localization we may assume $M \cong R/I$ for some ideal $I \subset R$.

We check that $S_0 = \text{Spec}(R/I) = V(I) \subset S$ satisfies our requirement. Indeed, for any f as above, let $J = \{\psi(1) | \psi \in Hom_R(R/I, A)\}$, then J is an ideal of A. If $J \neq A$, take a maximal ideal $P \supset J$ in A, then $Hom_R(R/I, A) \to Hom_R(R/I, A/P)$ is zero homomorphism, hence $(\text{id}_X \times_S f)^* \mathcal{L}$ is not trivial near P. Note that $J = A$ iff $f^*(I) = 0$, and in this case $Hom_R(R/I, A) \cong A$, which induces $(\text{id}_X \times_S f)^* \mathcal{L} \cong O_{X \times_S T}$. Q.E.D.

Lemma 4. *Let S be a noetherian scheme, $\pi : X \to S$ be a smooth relatively projective morphism and $D \subset X$ be a closed subscheme which is flat over S. Suppose there is an open dense subscheme $U \subset S$ such that $D \times_S U \subset X \times_S U$ is a U-effective divisor. Then $D \subset X$ is an S-effective divisor.*

Proof. Case 1: S is regular. For simplicity assume $S = \text{Spec}R$. Let $x \in X$ be a closed point and $s = \pi(x)$ corresponding to maximal ideal $m \subset R$. Then $A = O_{x,X}$ is a regular local ring, hence a UFD. Let I be the stalk of the ideal sheaf of $D \subset X$ at x, and a be the greatest common divisor of the elements in I. Then $a \notin mA$ since $I \not\subset mA$. Hence there is an open neighborhood $U' \subset X$ of x such that (a) defines a divisor D_1 in U', flat over S. Let $D_2 \subset U'$ be the closed subscheme defined by $J = (I : a)$. By the exact sequence

$$0 \to J \to A \xrightarrow{a\cdot} A/I \to A/(a) \to 0$$

we see A/J is R-flat, i.e. D_2 is flat over S. By assumption it is easy to see that $D_2 \times_S U$ is a divisor of $U' \times_S U$, hence any non-empty fiber of D_2 over U has dimension equal to the relative dimension of π minus 1; On the other hand, the fiber of D_2 over $\pi(x)$ has dimension less than the relative dimension of π minus 1. Since the fiber dimension of a flat morphism equals the relative dimension, the only possibility is $D_2 = \emptyset$, i.e. $I = (a)$. Hence D is a divisor in X.

Case 2: S is of finite type over $\mathbb{Z}$. Again assume $S = \mathrm{Spec} R$. By local criterion of flatness, we need only show that for any $s \in S$, the fiber D_s is a divisor in X_s. Let K be the residue field at s. We can take an integral one-dimensional subscheme $V \subset S$ containing s such that $V \cap U \neq \emptyset$. Let V' be the normalization of V, then V' is regular. By Case 1 we see that $D \times_S V'$ is a divisor in $X \times_S V'$. Hence there exists a finite extension $K' \supset K$ such that $D_s \otimes_K K'$ is a divisor in $X_s \otimes_K K'$. Therefore the ideal sheaf of $D_s \subset X_s$ is a flat O_{X_s}-module, i.e. D_s is a divisor in X_s.

Case 3: General case. We may assume $S = \mathrm{Spec} R$ and S is connected. Take a closed immersion $X \subset \mathbb{P}^n_S$ for some n. Let χ_0 and χ be the Hilbert polynomials of the fibers of D and X over S respectively. Then there is an induced morphism from S to the closed subscheme of $\mathcal{H}ilb^{\chi_0}_{\mathbb{P}^n} \times \mathcal{H}ilb^{\chi}_{\mathbb{P}^n}$ representing the following functor

$$\begin{aligned} ((\text{locally noetherian schemes})) &\to ((\text{sets})) \\ T &\mapsto \{\text{pairs of closed subschemes } D_0 \subset X_0 \subset \mathbb{P}^n_T, \\ &\qquad \text{flat over } T, \text{ with } \chi_0 \text{ and } \chi \text{ as Hilbert} \\ &\qquad \text{polynomials of fibers respectively}\} \end{aligned}$$

This gives a finitely generated $\mathbb{Z}$-subalgebra $R' \subset R$ such that $D \subset X$ is defined over R', i.e. there are projective schemes $D' \subset X'$ over $S' = \mathrm{Spec} R'$ whose pull-back to S is isomorphic to $D \subset X$. We may choose R' such that $X' \to S'$ is smooth and D' is a divisor over an open dense subscheme of S'. Hence $D' \subset X'$ is a divisor by Case 2. Therefore $D \subset X$ is a divisor. Q.E.D.

Note that $\mathcal{H}ilb^{\chi}_{X/S}$ is relatively projective over S, we get:

Corollary 1. *If $\pi : X \to S$ is relatively projective and smooth, then $\mathcal{D}iv^{\chi}_{X/S}$ is relatively projective over S.*

For any scheme X denote by $\mathrm{Pic}(X)$ the Picard group of X (i.e. the group of isomorphism classes of invertible sheaves on X). For any morphism $\pi : X \to S$ there is an induced homomorphism $\pi^* : \mathrm{Pic}(S) \to \mathrm{Pic}(X)$. If π is f.f. and relatively projective, then $\mathrm{Pic}(S)$ can be viewed as a subgroup of $\mathrm{Pic}(X)$ via π^*. We call $\mathrm{Pic}(X/S) := \mathrm{Pic}(X)/\mathrm{Pic}(S)$ the *relative Picard group* of π. Furthermore, if in addition that S is local and $X \to S$ has geometrically integral fibers, letting X_0 be the fiber of X over the closed point of S, then for any numerical class ξ of X_0 we denote by $\mathrm{Pic}^{\xi}(X/S)$ the subset of $\mathrm{Pic}(X/S)$ consisting of invertible sheaves with numerical class ξ. Similarly we can also use the notation $\mathrm{Div}^\xi(X/S)$.

Theorem. *Let $\pi : X \to S$ be a f.f. relatively projective morphism with geometrically integral fibers, and with a section $\zeta : S \to X$. Then there exists a locally quasi-projective scheme $\mathcal{P}ic(X/S)$ over S representing the following functor:*

$$\begin{aligned}\mathfrak{Pic}(X/S)\colon ((\text{locally noetherian } S\text{-schemes}))&\to((\text{sets}))\\ T&\mapsto \mathrm{Pic}(X\times_S T/T)\end{aligned}$$

Furthermore, if π is smooth, then $\mathcal{P}ic(X/S)$ is locally projective over S.

Proof. For any S-scheme T and any invertible sheaf $\mathcal{L}_0$ on $X\times_S T$, the invertible sheaf $\mathcal{L} = \mathcal{L}_0 \otimes_{O_{X\times_S T}} (((\zeta\circ\pi)\times_S \mathrm{id}_T)^*\mathcal{L}_0^{-1}$ satisfies $(\zeta\times_S \mathrm{id}_T)^*\mathcal{L}\cong O_T$, Hence every element of $\mathrm{Pic}(X\times_S T/T)$ can be represented by an invertible sheaf $\mathcal{L}$ on $X\times_S T$ satisfying $(\zeta\times_S \mathrm{id}_T)^*\mathcal{L}\cong O_T$. In this way $\mathrm{Pic}(X\times_S T/T)$ is embedded into $\mathrm{Pic}(X\times_S T)$ as a subgroup. In the following $\mathrm{Pic}(X\times_S T/T)$ will be understood as this.

First consider the case when S is local, with closed point s_0. Let X_0 be the fiber of X over the closed point of S. Let ξ be a numerical class of X_0 satisfying $(*)$. Let $Y_1 = \mathcal{D}iv^\xi_{X/S}$ representing divisors with numerical class ξ (which is an open and closed subscheme of some $\mathcal{D}iv^\chi_{X/S}$). By Lemma 3 there is a closed subscheme $Y_2\subset Y_1\times_S Y_1$ representing the following functor:

$$\begin{aligned}\left(\left(\begin{matrix}\text{locally noetherian } S\text{-schemes}\\ \text{whose closed fibers are over } s_0\end{matrix}\right)\right)&\to((\text{sets}))\\ (\phi : T\to S)&\mapsto \{\text{pairs } D_1, D_2\in \mathrm{Div}^\xi(X\times_S T/T)\\ &\qquad\text{such that } \mathcal{L}(D_1)\cong\mathcal{L}(D_2)\}\end{aligned}$$

and $\mathrm{pr}_1 : Y_2\to Y_1$ is a projective space bundle by Lemma 2. Clearly Y_2 gives an equivalence relation of Y_1 over S. Hence by Proposition 1.1, pr_1 and $\mathrm{pr}_2 : Y_2\to Y_1$ have an equilizer $p : Y_1\to Y = Y_1/\sim_{Y_2}$ which is f.f. and relatively projective, where Y is relatively quasi-projective over S, and $Y_2 = Y_1\times_Y Y_1$. We are going to show that Y represents the following functor:

$$\begin{aligned}\mathfrak{Pic}^\xi(X/S)\colon \left(\left(\begin{matrix}\text{locally noetherian } S\text{-schemes}\\ \text{whose closed fibers are over } s_0\end{matrix}\right)\right)&\to((\text{sets}))\\ T&\mapsto \mathrm{Pic}^\xi(X\times_S T/T)\end{aligned}$$

Let $Y_3 = Y_2\times_{Y_1} Y_2\cong Y_1\times_Y Y_1\times_Y Y_1$ (each copy of Y_2 is viewed as a Y_1-scheme via pr_2). Let $D_1\subset X\times_S Y_1$ be the universal divisor, $\zeta_1 = \zeta\times_S \mathrm{id}_{Y_1} : Y_1\to X\times_S Y_1$, and $\mathcal{L}_1 = \mathcal{L}(D_1)\otimes_{O_{Y_1}} (\zeta_1\circ\mathrm{pr}_2)^*\mathcal{L}(D_1)^{-1}$. Then $\zeta_1^*\mathcal{L}_1\cong O_{Y_1}$ (i.e. $\mathcal{L}_1\in \mathrm{Pic}^\xi(X\times_S Y_1/Y_1)$) and $\mathcal{L}_2 := \mathrm{pr}_{12}^*\mathcal{L}_1\cong \mathrm{pr}_{13}^*\mathcal{L}_1$. Let $\mathcal{E}_1 = \mathrm{pr}_{2*}\mathcal{L}_1$ and $\mathcal{E}_2 = \mathrm{pr}_{2*}\mathcal{L}_2$.

Then ζ induces a canonical epimorphism $\eta_1 : \mathcal{E}_1 \to \zeta_1^*\mathcal{L}_1 \cong O_{Y_1}$ and similarly a canonical epimorphism $\eta_2 : \mathcal{E}_2 \to O_{Y_2}$. By Lemma 2 and $(*)$ $Y_2 \cong \mathbb{P}(\mathcal{E}_1^\vee)$, we see η_1 gives a canonical hyperplane bundle $H_2 \subset Y_2$ in the projective space bundle $Y_2 \to Y_1$. Note that the pull-backs of η_1 via pr_1 and $\mathrm{pr}_2 : Y_2 \to Y_1$ both equal to $\eta_2 : \mathcal{E}_2 \to O_{Y_2}$, we have $H_3 := \mathrm{pr}_{12}^{-1}(H_2) = \mathrm{pr}_{13}^{-1}(H_2) \subset Y_3$. Obviously pr_{12} and pr_{13} induce projective space bundles $q_1 : H_3 \to H_2$ and $q_2 : H_3 \to H_2$ respectively, and $H_3 \subset H_2 \times_{Y_1} H_2$, it is easy to check that H_3 gives an equivalence relation on H_2 (induced by the equivalence relation on Y_2 given by Y_3), hence by Proposition 1.1 q_1 and q_2 has a geometric equilizer $H_2 \to H_1 = H_2/\sim_{H_3}$ over Y_1, and we have an induced morphism $H_2 \to H_1 \times_{Y_1} Y_2$. By

$$H_2 \times_{H_1} H_2 = H_3 = H_2 \times_{Y_1} Y_2 \cong H_2 \times_{H_1} (H_1 \times_{Y_1} Y_2)$$

we have $H_2 \cong H_1 \times_{Y_1} Y_2$, hence $H_1 \to Y_1$ is a closed immersion because its pull-back via the f.f. morphism pr_2 is $H_2 \to Y_2$. Since $Y_1 \to Y$ is f.f. we see $H_1 \subset Y_1$ is a Y-effective divisor, and $\mathcal{L}(H_1)$ gives $p : Y_1 \to Y$ a projective space bundle structure.

For any open subscheme $U \subset Y$ and any local section $\mu : U \to Y_1$ of p, the invertible sheaf $\mathcal{L}_\mu = (\mathrm{id}_X \times_S \mu)^*\mathcal{L}_1$ on $X \times_S U$ satisfies

$$(\mathrm{id}_X \times_S p|_U)^*\mathcal{L}_\mu \cong \mathcal{L}_1|_{X\times_S p^{-1}(U)}$$

Hence up to isomorphism $\mathcal{L}_\mu$ is independent of the choice of μ. Furthermore, since $\mathcal{E}nd_{O_{X\times_S U}}(\mathcal{L}_\mu) \cong O_{X\times_S U}$ and $\mathrm{pr}_{2*}O_{X\times_S U} \cong O_U$ (because π has geometrically integral fibers), for any other section $\mu' : U \to Y_1$ there is a unique isomorphism $\mathcal{L}_\mu \cong \mathcal{L}_{\mu'}$ whose pull-back via $\zeta \times_S \mathrm{id}_U$ is the identity isomorphism of O_U. Thus all of such $\mathcal{L}_\mu$'s glue to give an invertible sheaf $\mathcal{L}$ on Y such that $(\mathrm{id}_X \times_S p)^*\mathcal{L} \cong \mathcal{L}_1$.

We now show that $(Y, \mathcal{L})$ represents $\mathfrak{Pic}^\xi(X/S)$. For any S-scheme T and any $\mathcal{L}' \in \mathrm{Pic}^\xi(X \times_S T/T)$, by $(*)$ $\mathcal{E}' = \mathrm{pr}_{2*}\mathcal{L}'$ is locally free. Over any open subscheme $U \subset T$, any local section $\mu' : \mathcal{E}'^\vee|_U \twoheadrightarrow O_U$ of $\mathcal{E}'$ gives a U-effective divisor $D \subset X \times_S U$ such that $\mathcal{L}(D) \cong \mathcal{L}'_U := \mathcal{L}'|_{X\times_S U}$, this induces an S-morphism $\phi : U \to Y_1$ such that $(\mathrm{id}_X \times_S \psi)^*\mathcal{L} \cong \mathcal{L}'_U$, where $\psi = p \circ \phi$. If there is another S-mrphism $\psi' : U \to Y$ such that $(\mathrm{id}_X \times_S \psi')^*\mathcal{L} \cong \mathcal{L}'_U$, by taking a local section μ of p (for simplicity we take U small enough so that some μ is defined on $\psi'(U)$), we get a divisor $D' \subset X \times_S U$ such that $\mathcal{L}(D') \cong \mathcal{L}'_U$, this gives a morphism $\lambda : U \to Y_2$ such that $\mathrm{pr}_1 \circ \lambda = \phi$ and $\mathrm{pr}_2 \circ \lambda = \mu \circ \psi'$, hence

$$\psi' = p \circ \mu \circ \psi' = p \circ \mathrm{pr}_2 \circ \lambda = p \circ \mathrm{pr}_1 \circ \lambda = p \circ \phi = \psi$$

This shows that Y represents $\mathfrak{Pic}^\xi(X/S)$, thus we can denote $Y = \mathcal{P}ic^\xi(X/S)$.

For any numerical class ξ_0, one can take $\xi = \xi_0(n)$ for some $n \gg 0$ such that ξ satisfies $(*)$ by Lemma 1. If $(Y, \mathcal{L})$ represents $\mathfrak{Pic}^{\xi}(X/S)$, then clearly $(Y, \mathcal{L}(-n))$ represents $\mathfrak{Pic}^{\xi_0}(X/S)$.

For general S and each point $s \in S$, denote by $S_{(s)}$ and $X_{(s)}$ the localizations at s. The by abstract nonsense, all of the $\mathcal{P}ic^{\xi}(X_{(s)}/S_{(s)})$ (for all s and all ξ) glue together to give a locally quasi-projective scheme $\mathcal{P}ic^{\xi}(X/S)$ representing $\mathfrak{Pic}(X/S)$. Also by abstract nonsense, $\mathcal{P}ic(X/S)$ has a group scheme structure.

Finally, if π is smooth, then by Corollary 1 and Proposition 1.1 we see that $\mathcal{P}ic(X/S) \to S$ is locally projective. Q.E.D.

Remark 1. From the above proof we see that without the given section ζ, we can still construct $\mathcal{P}ic(X/S)$ (but may not construct a universal invertible sheaf), and there is a natural transformation $\theta : \mathfrak{Pic}(X/S) \to \underline{\mathcal{P}ic(X/S)}$. Note that for any morphism $T \to S$ we have $\mathcal{P}ic(X \times_S T/T) \cong \overline{\mathcal{P}ic(X/S)} \times_S T$, hence $\theta(T) : \mathrm{Pic}(X \times_S T/T) \to \mathrm{Mor}_S(T, \mathcal{P}ic(X/S))$ is a monomorphism for any S-scheme T, and is a one to one correspondence when $\mathrm{pr}_2 : X \times_S T \to T$ has a section (at least $\mathrm{pr}_2 : X \times_S X \to X$ has a section). Therefore $\mathcal{P}ic(X/S)$ is a coarse moduli scheme of $\mathfrak{Pic}(X/S)$ if π has no section.

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