The solutions of discrete symmetry equations for four-dimensional self-dual system in the case of arbitrary semisimple Gauge algebra

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Abstract

It is shown that the equations of discrete symmetry of the four-dimensional self-dual theory may be solved in the determinantal form for arbitrary semisimple gauge algebra similar to the known case of A^1 algebra. The essential difference compare with A_1 case consists in the fact that in the general case it arises the r independent linear systems of equations in terms of which the solution of the self-dual system may be expressed (r is the rank of the semisimple algebra).

1 Introduction

The first attempt to apply the concept of Bäcklund transformations to the problem of self-dual Yang-Mills equations was made about twenty years ago in the well known paper of Corrigan, Fairlie, Goddard and Yates [1], (see also [2]). In the case of the A^1 gauge algebra they constructed a hierarchy of

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explicit solutions in determinantal form in terms of a known solution of free four-dimensional d'Alembert equation.

Now it has become clear that this situation is generic to all integrable systems [3]. The equations of all such systems are invariant with respect to a specific form of nonlinear transformation which permits the construction of new solutions from previously known ones. In a sense the transformation employed may be considered as particular case of a Bäcklund transformation in the usual interpretation of this term. But in contradiction to the Bäcklund case this transformation does not contain any adjustable parameters and so the number of arbitrary parameters (functions) are the same for the whole hierarchy of solutions as the given initial one.

To emphasize this property we use the term discrete transformation (discrete substitution) in spite of its obvious connection with the idea of Bäcklund. The discrete transformation is always invertible and so from an algebraic point of view may be considered as an element of the infinite-dimensional cyclic group (Z) which of course possesses some additional properties [4].

The importance of the investigation of discrete transformations in the case of the four-dimensional self-dual system is connected with the conjecture of R.S. Ward [5] that all integrable systems may be obtained as a reduction of this self-dual system on to spaces of lower dimensions. If this hypothesis is true, in whole or in part, then it will be possible to obtain the discrete transformation for all such integrable systems (which satisfy Ward's conjecture) by the corresponding reduction from the discrete transformation of the self-dual one.

The general form of the discrete transformation for the self-dual system with an arbitrary semisimple gauge algebra was established in paper [6]. In the case of A^1 a solution of the equation of discrete transformation was obtained which generalized that of the paper quoted above [1]. The generalization to the case of N-extended supersymmetric self-duality equations was presented in [7], where the reader can find all the necessary background information for understanding the material of the present paper.

It is remarkable that all results concerning the A^1 case admit generalization to the case of an arbitrary semisimple gauge algebra from which it is possible to obtain a solution of the equations of the discrete transformation in explicit determinantal form using some appropriate boundary conditions similar to the A^1 case. The goal of the present paper is to make known to the reader the results of corresponding calculations.

Now we describe briefly the strategy of our calculations. In section 2 we present notations and explicit form of discrete transformation in terms of equations on unknown algebra and group-valued functions. In section 3 we remind the reader about some universal A^1 algebra embedding into arbitrary semi-simple one. The gradation of semisimple algebras arising in this way will be intensively used in our calculations. In section 4 the equation for group valued function is solved in explicit form. In section 5 this problem is solved for the algebra-valued function.

In both cases the solution is represented algebraicaly in terms of an arbitrary known solution of the self-dual system and its 3 nonlocal integrals. In section 6 we represent the final form of direct and inverse discrete transformations, show its symmetry with respect to group multiplication from left and right and discuss briefly possibilities arising after its many-times application to some given solution of the self-dual system. In section 7 a reduced self-dual system is introduced and its additional symmetry (auto-Bäcklund transformation) is represented in the form of infinite chain of equations. This is the main point because solution of initial self-dual system will be possible to represent by help of discrete transformation via solution of reduced system interrupted by appropriate boundary conditions. In section 8 further necessary properties of "maximal root" embedding of section 3 are investigated. This material will be necessary for understanding the representation of discrete transformation in invariant form. In section 9 two steps of the programme above are realised in explicit form. Solution of self-dual system is represented in explicit form in terms of solution of reduced self-dual system r_0, r_1, r_2, p_0, p_1 on the first step and $r_0, r_1, r_2, r_3, r_4, p_0, p_1, p_2$ on the second one via rational functions of variables enumerated above. In section 10 the concrete example of the A^{n+1} algebra is considered. In component form all formulae of the previous section may be represented in determinantal form. In section 11 solution of reduced self-dual system represented in explicit form. Possible way of interrupting of infinite reduced self-dual chain by appropriate boundary conditions are discuused in section 12.

2 Notations and discrete transformation of four-dimensional self-dual system

Let us write the self-duality equations for the elements G,f with values in a semisimple Lie group and algebra respectively in the form:

$$G_{\bar{z}}G^{-1} = f_y, \quad G_{\bar{y}}G^{-1} = -f_z$$

$$G^{-1}G_z = \bar{f}_{\bar{y}}, \quad G^{-1}G_y = -\bar{f}_{\bar{z}} \tag{2.1}$$

where $y, \bar{y}, z\bar{z}$ are the four independent variables of the problem. As a direct consequence of (2.1) we obtain from one side the single equation for the element G in usual Yang's formalism

$$(G_{\bar{z}}G^{-1})_z + (G_{\bar{y}}G^{-1})_y = 0, \quad (G^{-1}G_z)_{\bar{z}} + (G^{-1}G_y)_{\bar{y}} = 0$$

and on the other side equations for algebra valued functions f and $\bar{f}$

$$f_{y,\bar{y}} + f_{z,\bar{z}} = [f_y, f_z], \quad \bar{f}_{y,\bar{y}} + \bar{f}_{z,\bar{z}} = [\bar{f}_{\bar{z}}, \bar{f}_{\bar{y}}] \tag{2.2}$$

Equation (2.2) may be rewritten in the divergence ("conservation law") form [8],[9]

$$(f_{\bar{y}} - \frac{1}{2}[f, f_z])_y + (f_{\bar{z}} - \frac{1}{2}[f_y, f])_z = 0 \tag{2.3}$$

which will be used in further calculations.

The following holds [6],[7]:

There exists such an element S taking the values in the gauge group such that

$$\begin{aligned} S^{-1}\frac{\partial S}{\partial y} &= \frac{1}{f_-}[X^+_M, \frac{\partial f}{\partial y}] + \frac{\partial}{\partial \bar{z}}(\frac{1}{f_-})X^+_M, \\ S^{-1}\frac{\partial S}{\partial z} &= \frac{1}{f_-}[X^+_M, \frac{\partial f}{\partial z}] - \frac{\partial}{\partial \bar{y}}(\frac{1}{f_-})X^+_M, \end{aligned} \tag{2.4}$$

Here X^+_M is the element of the algebra corresponding to its maximal root, divided by its norm, i.e.

$[X^+_M, X^-_M] = H$, $[H, X^\pm_M] = \pm 2X^\pm_M$,

f_- is the coefficient in the decomposition of f of the element X^-_M corresponding to the maximal negative root of the algebra. Now define the element F taking values in the algebra by the following relations:

$$\frac{\partial F}{\partial y} = S\frac{\partial f}{\partial y}S^{-1} + \frac{\partial S}{\partial \bar{z}}S^{-1}, \qquad \frac{\partial F}{\partial z} = S\frac{\partial f}{\partial z}S^{-1} - \frac{\partial S}{\partial \bar{y}}S^{-1}. \qquad (2.5)$$

then the algebra-valued function F satisfy the same equations (2.2) as f. The corresponding discrete transformation of the group and algebra-valued functions G and $\bar{f}$ are the following:

$$\tilde{G} = SG, \quad \bar{F} = \frac{G^{-1}X^+_M G}{f_-} + \bar{f} \qquad (2.6)$$

The proof of this proposition can be found in the papers cited above. Our goal here is the explicit solution of (2.4).

3 A_1 algebra embedding of maximal root and its properties

We will begin with a brief description of the properties of a universal A_1 algebra embedding into an arbitrary semisimple one. Let us consider three-dimensional subalgebra constructed from generators of maximal root $X_\pm, H$ (henceforth we omit the index M in definition of the generators of the maximal root). In this embedding the structure constants of H commuted with the generators of the semisimple algebra take only values $s = 0, \pm 1, \pm 2$: $[H, X^\pm_r] = \pm s X^\pm_r$. This means that the semisimple algebra may be considered as a graded one. The spaces with $s = \pm 2$ are one-dimensional and consist of the single elements $X^\pm$ - the generators of the highest root of the algebra. The generators of the divisors of the highest roots $X^\pm_\alpha$: $[X^\pm_\alpha, X^\pm_\beta] = \pm\delta_{\alpha+\beta,M}X^\pm$ belong to subspaces with degree $s = \pm 1$. All other generators are contained in the subspace with degree zero.

In other words with respect to such an embedding all generators of the algebra are decomposed into multiplets with "orbital quantum numbers" equal to $0, \frac{1}{2}$ and 1. The multiplet with $l = 1$ represents the generators of the A^1 subalgebra. The spinor multiplets are constructed from the divisors of the maximal root: $X^+_\alpha, [X^+_\alpha, X^-]$. All the generators of the space with the zero degree are singlets. In connection with this grading an arbitrary algebra-valued function (e.g. the solution of the equation (2.1)) may be represented as the sum of components with different degree

$$f = f_- X^- + f_-^1 + f_0 + f_+^1 + f_+ X^+ \tag{3.1}$$

In what follows symbols $\dot{}$ and $'$ will mean differentiation with respect to two pairs of the independent arguments $(y, \bar{z})$ and $(z, -\bar{y})$ and consequently each equation in this notations is really the pair of equations after corresponding substitution.

In this notation the system of self-dual equations in the "conservation law" form is the following

$$\dot{R}_- = f_-' + \dot{\omega} f_- + \frac{1}{2}([f_-^1, \dot{f}_-^1]X_+) \quad \dot{R}_+ = f_+' - \dot{\omega} f_+ + \frac{1}{2}([f_+^1, \dot{f}_+^1]X_-)$$

$$\dot{P}_-^1 = (f_-^1)' + [f_-^1, \dot{f}^0] + f_-[X_-, \dot{f}_+^1], \quad \dot{P}_+^1 = (f_+^1)' + [f_+^1, \dot{f}^0] + f_+[X_+, \dot{f}_-^1] \tag{3.2}$$

$$\dot{R}_0 = f_0' + \frac{1}{2}[f_+^1, \dot{f}_-^1] + \frac{1}{2}[f_-^1, \dot{f}_+^1] + \frac{1}{2}(f_+\dot{f}_- - f_-\dot{f}_+)H + \frac{1}{2}[f_0, \dot{f}_0]$$

The values $R_\pm, R_0, P^1\pm$ introduced above we will call nonlocal integrals of the corresponding graded space,

It is not difficult to check that system (3.2) is invariant with respect to the following change of unknown functions

$$\begin{gathered} f_+ \to -f_-, \quad f_+^1 \to [X_+, f_-^1], \quad f_-^1 \to -[X_-, f_+^1], \\ f_0 - \frac{\omega}{2} \to f_0 - \frac{\omega}{2}, \quad \omega \to -\omega, \quad f_- \to f_+. \end{gathered} \tag{3.3}$$

This symmetry is a direct consequence of an inner authomorphism of the semisimple algebra connected with Weyl reflection of its maximal root. We conserve this notation for symmetry (3.3).

4 Explicit expression for the group-valued function S

Now let us return to equation (2.4) and note that its algebra-valued right hand side contains the generators of maximal positive root X^+, all its divisors and the Cartan element H. This $2j+2$ ($2j$ is the number of the divisors of the maximal root) generators realise a solvable algebra, the "diagonal" part of which coincides with the generator H, the nilpotent part coincides with the Heisenberg algebra in j-dimensional space, where the roles of the generalised coordinates X^+_k and momenta X^+_{M-k} are taken by the divisors of the maximal root:$[X^+_k, X^+_{M-k}] = X^+_M (\equiv X^+)$; the generator X^+_M plays the role of the ideal of the Heisenberg algebra which commutes with all its other elements. An arbitrary element of such an algebra may be written in the form

$$F = f_0 H + X + P + f_+ X^+$$

where "position" X and "momentum" P subspaces are each commutative.

As it follows from (2.4) the group-valued element S belongs to the corresponding solvable group and hence may be represented in the form:

$$S = \exp \tau H \exp A \exp B \exp \alpha X^+ \tag{4.1}$$

The commutation relations between the different elements taking part in the last equality are as follows:

$$[H, A] = A, \quad [H, B] = B, \quad [X^+, B] = [X^+, B] = 0,$$

$$[B, B] = [A, A] = 0, \quad [A, B] = X^+ \mathrm{or} 0$$

In the notations of the end of the last section equation (2.4) takes the form

$$S^{-1}\dot{S} = \dot{\tau}(H + A + B - [B, A] + 2\alpha X^+) + \dot{A} + \dot{B} - [B, \dot{A}] + \dot{\alpha} X^+ =$$

$$\frac{1}{f_-}([X^+, f^1_-] + \dot{f}_- H) - (\frac{\dot{\omega}}{f_-} + \frac{f'_-}{f^2_-}) X^+ \tag{4.2}$$

where $[X^+, f_0] = -\omega X^+$, $\omega = Sp(Hf_0)$. From (3.1) comparing the generators of the same graded subspaces we obtain

$$\tau = \ln f_-, \quad (A + B) f_- = [X^+, f^1_-]$$

$$(\alpha + \frac{1}{2}[A, B]\dot{)} f^2_-) X^+ - \frac{f^2_-}{2}([B, \dot{A}] - [\dot{B}, A]) = -(\dot{\omega} f_- + f'_-) X^+ \tag{4.3}$$

Using the Weyl formula $\exp A \exp B = \exp(A + B + \frac{1}{2}[A, B])$ (in the case when $[A, B]$ commutes with both operators A and B) we can represent S (4.1) in the form

$$S = \exp \tau H \exp(A + B) \exp(\alpha + \frac{1}{2}(X^-[A, B]))X^+ \tag{4.4}$$

Conserving for the sum $\alpha + \frac{1}{2}(X^-[A, B])$ the same notation α (only this combination will be necessary for further calculations) we rewrite equation (4.3) finally in the form

$$(\dot{\alpha f_-^2}) + \frac{1}{2}([f_-^1, \dot{f}_-^1]X_+) = -(\dot{\omega} f_- + f_-') \tag{4.5}$$

So we see that in order to obtain an explicit expression for element S it is necessary to solve the pair of equations (4.5), which are self-consistent (as it will be seen from the further consideration).

By the same technique we obtain

$$S'S^{-1} = (\alpha' f_-^2) - \frac{1}{2}([f_-^1, (f_-^1)']X^+))X^+ + f_-(A + B)' + \frac{f_-'}{f_-}H \tag{4.6}$$

5 Explicit solution of the discrete transformation for algebra-valued function F

Now we substitute (3.1) , (4.4) and (4.6) into equation of the discrete transformation (2.5) and compare the algebra-valued functions in the subspaces with equal degree.

The space $s = -2$ is one-dimensional and we obtain immediately

$$F_- = -\frac{1}{f_-} \tag{5.1}$$

In the case of subspace $s = -1$ the following equality arises:

$$\dot{F}_-^1 = \frac{1}{f_-}(\dot{f}_-^1 + \frac{[[X^+, f_-^1]X^-]}{f_-}\dot{f}_-)$$

Keeping in mind that

$$[X^-, f_-^1] = 0, \quad [X^+, X^-] = H, \quad [H, f_-^1] = -f_-^1$$

we obtain finally

$$F^1_- = \frac{f^1_-}{f_-} \tag{5.2}$$

In the case zero-graded subspace it is necessary to take into account the input from terms $S'S^{-1}$. We obtain successively (for simplicity we introduce notation $[X^+, f^1_-] = \theta$)

$$\dot{F}_0 = \dot{f}_0 + \frac{1}{2}(\frac{[\theta, \dot{f}^1_-]}{f_-}) - \frac{1}{2}\frac{[X^+[\dot{f}^1_-, f^1_-]]}{f_-} + (\alpha\dot{f}_- + \frac{f'_-}{f_-})H \tag{5.3}$$

So for all generators of the zero-graded subspace which are ortogonal to H we obtain

$$F_0 - \frac{\Omega}{2}H = f_0 - \frac{\omega}{2}H + \frac{1}{2}\frac{[\theta, f^1_-]}{f_-} \tag{5.4}$$

The projection of (5.3) on H gives

$$\frac{\dot{\Omega}}{2} = \frac{\dot{\omega}}{2} + \frac{1}{2f_-}([f^1_-, \dot{f}^1_-]X^+) + \alpha\dot{f}_- + \frac{f'_-}{f_-} \tag{5.5}$$

Comparing the last equation with (4.5) we come to the conclusion that

$$\alpha = -\frac{\Omega + \omega}{2f_-}, \quad (\Omega = Sp(FH_0)) \tag{5.6}$$

Substituting this expression in (5.5) we obtain

$$(\dot{\frac{\Omega + \omega}{2}f_-}) = f'_- + \dot{\omega}f_- + \frac{1}{2}([f^1_-, \dot{f}^1_-]X^+) \equiv \dot{R}_- \tag{5.7}$$

where is R_- is the nonlocal integral of the degree -2 subspace of equation (2.3). After this (5.7) can be rewritten also in the form

$$(\frac{\Omega + \omega}{2})f_- = R_-, \quad \alpha = -\frac{R_-}{f^2_-} \tag{5.8}$$

Notice that in the case $f^1_- = 0$ the equation (5.5) is the same as in the previously known case of the A_1 algebra [6],[7].

The subspace of unit degree gives the equation

$$\dot{F}_{+}^{1}=\frac{1}{3!}\frac{\dot{f}_{-}}{f_{-}^{2}}[\theta[\theta[\theta,X^{-}]]]+\frac{1}{2!}\frac{1}{f_{-}}[\theta[\theta,\dot{f}_{-}^{1}]]+[\theta,\dot{f}_{0}+\alpha\dot{f}_{-}H]+\dot{f}_{+}^{1}f_{-}+\dot{\theta}\alpha f_{-}+(\frac{\theta}{f_{-}})'f_{-} \tag{5.9}$$

The further evolution of this equality is connected with the following obvious identities which are the direct consequence of the determination of the grading operator H given above and some purely algebraic operations:

$$[\theta,X^{-}]=f_{-}^{1},\quad [\theta,[\dot{\theta},f_{-}^{1}]]=3[\theta,[\theta,\dot{f}_{-}^{1}]]+3(\dot{f}_{-}^{1},\theta)\theta$$

Substituting all these expressions into the last equality in (5.9) we obtain

$$\dot{F}_{+}^{1}=-\frac{1}{3!}(\frac{[\theta,[\theta,[\dot{\theta},X_{M}^{-}]]]}{f_{-}})+(\theta\dot{\alpha}f_{-})+\dot{f}_{+}^{1}f_{-}+\theta'+\dot{\omega}\theta+[\theta,\dot{f}_{0}] \tag{5.10}$$

The equation for algebra-valued function θ may be obtained on commuting the degree -1 component of equation (2.1) with the generator of the maximal positive root of the algebra and has the form

$$\theta_{y\bar{y}}+\theta_{z\bar{z}}=(f_{-})_{z}(f_{+}^{1})_{y}-(f_{-})_{y}(f_{+}^{1})_{z}+\omega_{y}\theta_{z}-\omega_{z}\theta_{y}+[\theta_{z},f_{y}^{0}]-[\theta_{y},f_{z}^{0}]$$

This equation may be written in the divergence form, which can be partially resolved by introduction nonlocal conserved quantities p_1

$$(p_1)_y=(f_{+}^{1})_y f_{-}+\theta_{\bar{z}}+\omega_y\theta+[\theta,f_y^0]$$

$$(p_1)_z=(f_{+}^{1})_z f_{-}-\theta_{\bar{y}}+\omega_z\theta+[\theta,f_z^0]$$

Bearing all this in mind we obtain finally

$$F_{+}^{1}=-\frac{1}{3!}\frac{[\theta,[\theta,[\theta,X^{-}]]]}{f_{-}}-\frac{R_{-}\theta}{f_{-}}+p_1 \tag{5.11}$$

It only remains to calculate the function F_{+} of the one-dimensional $+2$ graded space. We have

$$\dot{F}_{+}=\frac{1}{4!}\frac{\dot{f}_{-}}{f_{-}^{2}}[\theta[\theta[\theta,f_{-}^{1}]]]+\frac{1}{3!}\frac{1}{f_{-}}[\theta[\theta[\theta,\dot{f}_{-}^{1}]]]+\frac{1}{2!}[\theta[\theta,\dot{f}_{0}]]+f_{-}[\theta,\dot{f}_{+}^{1}]+ \tag{5.12}$$

$$X^+(f_-^2\dot{f}_+ + \alpha^2 f_-^2 \dot{f}_- - \alpha f_-^2\dot{\omega} + \alpha' f_-^2) + \frac{1}{2}(f_-^1\dot{\theta}))$$

Let us use the following notation $f_- \equiv R_0, R_- \equiv R_1$. Remembering the determination of R_1 with help of equation (5.7) we can calculate the d'Alembertian of it and the represent this result in the form of a conservation law. We obtain in this way

$$\dot{R}_2 = R_1' + \dot{\omega}R_1 + \frac{1}{2}(f_-^1\dot{p}) + R_0^2\dot{f}_+ - \frac{1}{2}f_-(\dot{f}_+^1 f_-^1) \tag{5.13}$$

Substituting (5.7),(5.10) and (5.13) into (5.12) we obtain finally

$$F_+ = -R_2 + \frac{R_1^2}{R_0} - \frac{1}{4!}\frac{([\theta, f_-^1][\theta, f_-^1])}{R_0} \tag{5.14}$$

6 The final form of direct and inverse discrete transformation

As it is possible to see from the explicit form of the discrete transformation of the last section, its double application to a given solution f returns to the same solution excluding possible trivial additional terms to f depending only on arguments $\bar{y}, \bar{z}$ (constants of integration). By this reason if we want to obtain new solutions from a given one by help of multiple application of our discrete transformation it is necessary first to perform some point like transformation on f with respect to which self-dual system (3.2) is invariant. As such kind of transformation we will use reflection of maximal root of algebra (3.3). As a result of consecutive application of these two transformations to the solution f of self-dual system the corresponding formulas of the last section take the form

$$F_- = \frac{1}{f_+}, \quad F_-^1 = \frac{[X_-, f_+^1]}{f_+}$$

$$F_0 - \frac{\Omega}{2}H = f_0 - \frac{\omega}{2}H + \frac{1}{2}\frac{[f_+^1[f_+^1, X_-]]}{f_+}$$

$$\frac{\Omega}{2} = \frac{\omega}{2} + \frac{R_1}{f_+}, \quad \dot{R}_1 = f_+' - \dot{\omega}f_+ + \frac{1}{2}([f_+^1, \dot{f}_+^1]X_-), \quad \alpha = -\frac{R_1}{f_+^2}$$

$$
F^1_+ = -\frac{1}{3!}\frac{[f^1_+,[f^1_+,[f^1_+,X^-]]]}{f_+} + \frac{R_1 f^1_+}{f_+} - P_1, \quad \dot{P}_1 = [X^+,\dot{f}^1_-]f_+ + f'^1_+ + [f^1_+,\dot{f}^0]
$$

$$
F_+ = R_2 - \frac{R_1^2}{R_0} + \frac{1}{4!}\frac{([f^1_+,[f^1_+,X_-]])^2}{R_0}
$$

$$
\dot{R}_2 = R'_1 - \dot{\omega}R_1 + \frac{1}{2}(X^-[f^1_+\dot{P}_1]) + f^2_+\dot{f}_- + \frac{1}{2}f_+(\dot{f}^1_- f^1_+)
$$

$$
F_- = \frac{1}{f_+}, \quad F^1_- = \frac{[X_-,f^1_+]}{f_+}
$$

$$
F_0 - \frac{\Omega}{2}H = f_0 - \frac{\omega}{2}H + \frac{1}{2}\frac{[f^1_+[f^1_+,X_-]]}{f_+}
$$

$$
\frac{\Omega}{2} = \frac{\omega}{2} + \frac{R_1}{f_+}, \quad \dot{R}_1 = f'_+ - \dot{\omega}f_+ + \frac{1}{2}([f^1_+,\dot{f}^1_+]X_-), \quad \alpha = -\frac{R_1}{f^2_+} \tag{6.1}
$$

$$
F^1_+ = -\frac{1}{3!}\frac{[f^1_+,[f^1_+,[f^1_+,X^-]]]}{f_+} + \frac{R_1 f^1_+}{f_+} - P_1, \quad \dot{P}_1 = [X^+,\dot{f}^1_-]f_+ + f'^1_+ + [f^1_+,\dot{f}^0]
$$

$$
F_+ = R_2 - \frac{R_1^2}{R_0} + \frac{1}{4!}\frac{([f^1_+,[f^1_+,X_-]])^2}{R_0}
$$

$$
\dot{R}_2 = R'_1 - \dot{\omega}R_1 + \frac{1}{2}(X^-[f^1_+\dot{P}_1]) + f^2_+\dot{f}_- + \frac{1}{2}f_+(\dot{f}^1_- f^1_+) \tag{6.2}
$$

Very complicated at first sight, the expressions for the graded components of the discrete transformation become simpler after rewriting (6.2) in the form

$$
F = f^0 - P_{+1} + R_2 X_+ + R_0^{-1}\exp(R_1 X_+ - P_0)X_- \exp -(R_{+1}X_+ - P_0) \tag{6.3}
$$

From the last expression it is clear that discrete transformation for algebra-valued function f determined at whole by non negative components of initial solution $f_+ \equiv R_0, f^1_+ \equiv P_0, f^0$ and nonlocal integrals R_1, R_2 of $+2$ graded subspace and P_1 of the $+1$ graded ones. If the reader remembers that (6.3) describe discrete transformation for the case of arbitrary simisimple algebra then he must agree that the form of the final result (6.3) is astonishingly simple.

The discrete transformation (6.3) is invertible. As a result of resolving (6.3) the "old" solution f may be represented in terms of the "new" one F as

$$f = F^0 - P_{-1} + \tilde{R}_2 X_+ + (\tilde{R}_0)^{-1} \exp(-R_{-1}X_- + \tilde{P}_0) X_+ \exp(R_{-1}X_- - \tilde{P}_0) \quad (6.4)$$

where $F_0, \tilde{R}_0 = F_-, \tilde{P}_0$ are nonpositive degree components of solution F, R_{-1}, R_{-2}, P_{-1} are the conserved quantities introduced above (up to a sign change). Equations for them may be obtained from equations for conserved values with positive indexes (6.2) by means of changing the "old" function f by the new one F.

It may seem that the discrete transformation for the self-dual system is asymmetrical with respect to multiplication of its unknown group-valued function only from left $G = Sg$. But this is not so. The same trick it is possible to repeat with the pair $g, \bar{f}$ as it was done before with pair g, f. All formulae for this case may be obtained from corresponding formulae of this section by operation of complex cojugation with simultaneously changing of independent arguments $y \to \bar{y}, z \to \bar{z}$ and visa versa. To distinguish these kinds of discrete transformation we will use terms left (S_l) and right (S_r) ones for them in what follows. So by help of multiple application of direct (6.3) or inverse (6.4) discrete transformation it is possible to construct a new solution from an arbitrary given one. There are some obvious possibilities which may be arise in this process. It may happen that this process is unlimited in both directions and we will obtain infinite number "new"solutions all of which are in some sense equivalent to the initial one. For instance if we have deal with general solution of Cauchy problem for self-dual system that it is obvious that after discrete transformation we will have the same solution only with possible change of the initial function of the Cauchy problem. The other possibility consists in assuming that after a definite number of discrete transformation we will come back to the initial solution or to a solution connected with it by some other transformation (not a discrete one). In this case we will have some periodicity in the infinite chain of "new" solutions. Finally it may happen that after a definite number of steps we will come to a solution to which further application of the discrete transformation is meaningless ($f_\pm = 0$ in the case under consideration). Precisely this case will be the subject of our further consideration. In the case of integrable systems in $(1+1)$ and $(1+2)$ such a situation always arose in consideration of multi-soliton solutions [4]. So it is possible to postulate that in the case of the four-dimensional self-dual system this last possibility may in future be connected with instanton and monopole problems

7 The self-dual system in the case of "solvable" algebra

Now let us assume that the terms from degree -1 and -2 subspaces are absent in the solution of the self-dual system (3.2). In other words we assume that the algebra-valued functions f may be decomposed on the elements of $0, +1, +2$ graded subspaces only. In this sense we have used the term "solvable" in the tittle of this section. The arising system of equations for unknown functions of degree +2,+1 and zero spaces takes the form (in conservation law form)

$$\begin{aligned} \dot{r}_+ &= f'_+ - \dot{\omega} f_+ + \frac{1}{2}([f^1_+, \dot{f}^1_+] X_-) \\ \dot{p}_1 &= (f^1_+)' + [f^1_+, \dot{f}^0] \\ (f_0)_{y\bar{y}} + (f_0)_{z\bar{z}} &= [(f_0)_y, (f_0)_z] \end{aligned} \tag{7.1}$$

We see that the equation for the degree zero subspace is the usual self-dual system with gauge group $G^{in}/SL(2,R)$. As a direct consequence of (7.1) it follows that the function ω satisfies the free d'Alembert equation

$$(\omega)_{y\bar{y}} + (\omega)_{z\bar{z}} = 0$$

The remarkable property of the reduced self-dual system (7.1) (we use small-case letters for its nonlocal conserved quantities) consists in the fact that it possess some additional symmetry compared with the initial self-dual system (3.2). Namely each set of functions r_{2n}, p_n, f_0 taking values in the subspaces with graded indexes +2,+1 and 0 respectively from the following infinite-dimensional system

$$\dot{r}_{2n} = r'_{2n-1} - \dot{\omega} r_{2n-1} + \frac{1}{2}(X^-[p_{n-1}, \dot{p}_n]), \quad \dot{r}_{2n+1} = r'_{2n} - \dot{\omega} r_{2n} + \frac{1}{2}(X^-[p_n, \dot{p}_n])$$

$$\dot{p}_{n+1} = p'_n + [p_n, \dot{f}_0] \tag{7.2}$$

are the solutions of system (7.1). The validity of this proposition can be verified by directly. In terms of the solution of infinite-dimensional system it will be possible to represent in explicit form result of multiple application of discrete transformation (6.2) to initial solution of reduced self-dual system (7.1) which in its turn satisfy the self-dual system (3.2).

8 Universal A^2-algebra embedding into arbitrary semisimple one

In this section we investigate the further remarkable properties of A^1 embedding of the maximal root of the third section. Let us consider arbitrary element of $+1$ graded subspace p and construct by help of it the following algebra-valued elements belonging to subspaces with degrees $+1$, 0 and -1

$$p, \quad \nu = [p[p[p, X_-]]], \quad s = [p[p, X_-], \quad [p, X_-], \quad [[p, X_-][p[p, X_-]]]$$

These five elements together with the generators of the highest root $X_\pm, H$ realize the closed eight-dimensional algebra which is isomorphic to A_2 $(SU(3, R))$ algebra (if $(s^2) \neq 0$).

To prove this assertion let us consider the following obvious equality

$$[p[p[p[p, X_-]]]] = cX_+$$

Indeed the algebra-valued function from the left side of the last equality belongs to $+2$ graded subspace which is one-dimensional and so may differ from X_+ only my numerous multiplicator c. The values of it may determined by multiplication of the last equality on X_- and taking the trace from both sides. The result is $c = (s, s) \equiv Trs^2 \equiv N^2(s)$. Commuting the same equality with the generator X_- we obtain two equal terms and as a corollary important for further calculations relation

$$[[p, X_-][p[p[p, X_-]]]] = [p[[p, X_-][p[p, X_-]]]] = \frac{(s, s)}{2} H \qquad (8.1)$$

With the help of (8.1) one can convinced that the set of the following six generators

$$h_1 + h_2 = H, \quad h_2 - h_1 = \frac{\sqrt{6}}{\sqrt{(s, s)}} s$$

$$X_+^1=\theta_1(\sqrt{\frac{3}{2}}Np+\nu),\quad X_+^2=\theta_2(-\sqrt{\frac{3}{2}}Np+\nu),\quad \theta_1\theta_2\sqrt{6}N^3(s)=1$$

$$X_-^1=\theta_2(-\sqrt{\frac{3}{2}}N[p,X_-]+[\nu,X_-]),\quad X_-^2=-\theta_1(\sqrt{\frac{3}{2}}N[p,X_-]+[\nu,X_-])$$

satisfy the system of commutation relations for generators of the simple roots and corresponding Cartan elements of A^2 algebra. The remaining generators of the maximal root coincide with $X_\pm$:

$$X_+^{12}=[X_+^1,X_+^2]=X_+,\quad X_+^{12}=[X_-^2,X_-^1]=X_-$$

All usual conditions of normalization

$$N(H)=2,\quad N(h_1-h_2)=6,\quad (X_+^\alpha,X_-^\beta)=\delta_{\alpha,\beta}$$

are also satisfied.

As a direct corollary of the above consideration we have the following:

If the "norm" of element p of $+1$ subspace $Tr([p[p,X^-]])^2)\equiv 4!\delta$ is not equal to zero identically then it by itself and element $\tilde{\nu}\equiv\frac{[p[p[p,X^-]]]}{3!\sqrt{\delta}}$ constructed from it and also belongs to $+1$ graded subspace may be represented in the form

$$p=tg_0(X_1^+-X_2^+)g_0,\quad \tilde{\nu}=tg_0(X_1^++X_2^+)g_0 \tag{8.2}$$

where $X_{1,2}^+$ are the simple roots of A^2 algebra, t some constant and g_0 the element of the group algebra of which coincides with the subspace of zero graded index.

9 Explicit expressions for solution of discrete transformation

We now want to choose as the initial solution of reduced self-dual system (7.1) and obtain the explicit expression for solution of self-dual system after multi-time application of discrete transformation (6.3). We introduce also the following notation for the coefficient functions $f_+=r_0, f_+^1=p_0$ of the initial solution.

9.1 The zero step

We represent here the explicit expressions for algebra $f, \bar{f}$ and group g_0 valued functions of self-dual system (2.1)

$$f = r_0 X_+ + p_0 + \phi_0$$

$$f_- = 0, \quad f_-^1 = 0, \quad f_0 = \phi_0, \quad f_+^1 = p_0, \quad f_+ = r_0$$

$$g_0 = \exp(r_{-1} X_+ + p_{-1})\Theta$$

where r_{-1} and p_{-1} are solution of infinite chain (7.2); $\Theta'\Theta^{-1} = \dot{\phi}_0$.

$$\bar{f} = \bar{\phi}_0 + \Theta^{-1}(r_{-2} X_+ + p_{-2})\Theta$$

where $\Theta^{-1}\dot{\Theta} = \dot{\bar{\phi}}_0$.

9.2 The first step

$$f_- = \frac{1}{r_0}, \quad f_-^1 = \frac{[X_-, p_0]}{r_0}, \quad f_0 = \phi_0 + \frac{r_1}{r_0} H + \frac{1}{2}\frac{s_0}{r_0}$$

$$f_+^1 = -\frac{\nu_0}{3! r_0} + \frac{r_1}{r_0} p_0 - p_1, \quad f_+ = r_2 - \frac{r_1^2 - \delta_0}{r_0},$$

$$s_0 = [p_0[p_0, X_-]], \quad \nu_0 \equiv [p_0[p_0[p_0, X_-]]], \quad \delta_0 \equiv \frac{1}{4!}([p_0[p_0, X_-]]^2).$$

$$\bar{f} = \bar{\phi} + \Theta^{-1}(\frac{1}{r_0} X_- - \frac{[p_{-1}, X_-]}{r_0} - \frac{r_{-1}}{r_0} H + \frac{1}{2}\frac{s_{-1}}{r_0} +$$

$$(p_{-2} - \frac{1}{3!}\frac{\nu_{-1}}{r_0} - \frac{r_{-1}}{r_0} p_{-1}) + (r_{-2} - \frac{r_{-1}^2 - \delta_{-1}}{r_0}) X_+)\Theta$$

$$s_{-1} = [p_{-1}[p_{-1}, X_-]], \quad \nu_{-1} \equiv [p_{-1}[p_{-1}[p_{-1}, X_-]]], \quad \delta_{-1} \equiv \frac{1}{4!}([p_{-1}[p_{-1}, X_-]]^2).$$

9.3 The second step

At first we represent the explicit expressions for integrals of the motions of self-dual system R_3, R_4, P_2 in terms of corresponding expressions of reduced self-dual chain (7.2) $r_0, r_1, r_2, r_3, r_4, p_0, p_1, p_2$. We use the notation $P_1^{(m}, R_{1,2}^{(m}$ with the purpose to emphasize that this quatities coincide with nonlocal conserved values after m steps of discrete transformation.

$$P_1^{(2} = -p_2 + \frac{p_0}{r_0}(r_2 - \frac{r_1^2+\delta}{r_0}) + \frac{p_1 r_1}{r_0} + \frac{2}{3!}\frac{\nu_0 r_1}{r_0^2} - \frac{1}{2}\frac{[p_1, s_0]}{r_0}$$

$$R_1^{(2} = r_3 - \frac{r_1}{r_0}(r_2 - \frac{1}{2}([p_0,p_1],X_-)) - \frac{r_1}{r_0}(r_2 - \frac{r_1^2-\delta_0}{r_0}) - \frac{1}{2!3!}\frac{([\nu_0,p_1]X_-)}{r_0}$$

$$R_2^{(2} = \frac{1}{4}\frac{(s_0,s_1)}{r_0} + \frac{([p_0,p_1],X_-)}{r_0}(r_2 - \frac{r_1^2+\delta_0}{r_0}) + \frac{1}{3}\frac{r_1}{r_0^2}([\nu_0,p_1]X_-) +$$

$$\delta_0(\frac{r_2}{r_0} + 2\frac{r_1^2}{r_0^3}) - \frac{\delta_0^2}{r_0^3}$$

$$\delta^{(2} = \frac{([p_0,p_1],X_-)}{r_0}(r_2^2 - r_3 r_1) + \frac{1}{3!}\frac{r_3}{r_0}([\nu_0,p_1]X_-) -$$

$$\frac{1}{3!3!}\frac{([\nu_0,\nu_1]X_-)}{r_0} - \frac{1}{3!}\frac{r_1}{r_0}([\nu_1,p_0]X_-) + \frac{1}{4}\frac{r_2}{r_0}(s_0,s_1).$$

Now we represent the explicit expressions for solution of self-dual system after substitution of the last expressions for nonlocal integrals into general farmulae (6.3)

$$f_- = \frac{r_0}{r_2 r_0 - r_1^2 + \delta_0}, \quad f_-^1 = \frac{\frac{1}{3!}[\nu_0, X_-] - r_1[p_0, X_-] + r_0[p_1, X_-]}{r_0 r_2 - r_1^2 + \delta_0}$$

$$f_0 = \phi_0 + \frac{r_3 r_0 - r_1 r_2 + \frac{1}{2} r_1([p_0,p_1]X_-) - \frac{1}{2!3!}([\nu_0,p_1]X_-)}{r_0 r_2 - r_1^2 + \delta_0} H +$$

$$\frac{1}{2}\frac{s_0 r_2 + s_1 r_0 + \frac{1}{3!}([\nu_0[p_1,X_-]] + [p_1[\nu_0,X_-]]) - r_1([p_0[p_1,X_-]] + [p_1[p_0,X_-]]}{r_0 r_2 - r_1^2 + \delta_0}$$

$$(r_0 r_2 - r_1^2 + \delta_0) f_+^1 = p_2(r_0 r_2 - r_1^2 + \delta_0) + r_3(r_1 p_0 - r_0 p_1 - \frac{\nu_0}{3!}) - \frac{1}{2} r_1 [p_0, s_1] +$$

$$\frac{1}{2} r_2 [p_1, s_0] + r_0 \frac{\nu_1}{3!} + \frac{1}{2!3!}[\nu_0, s_1] - p_0 r_2^2 + p_1(r_1(r_2 - ([p_0,p_1]X_-)) + \frac{1}{3!}([\nu_0,p_1]X_-))$$

$$f_+(r_2r_0-r_1^2+\delta_0)=r_4(r_0r_2-r_1^2)-r_2^3-r_3^2r_0+2r_1r_2r_3+\delta_1r_0+\frac{1}{4}(s_0s_1)r_2+\delta_0r_4+$$
$$\frac{1}{3!}([\nu_0,p_1]X_-)r_3+\frac{1}{3!}([\nu_1,p_0]X_-)r_1+(X_-[p_0,p_1])(r_2^2-r_1r_3)-\frac{1}{3!3!}([\nu_0,\nu_1]X_-)$$

10 The gauge algebras of A^n series

In the case of A^1 algebra all components of the self-dual field may be expressed as a fractions of principle minors of some definite infinite-dimensional matrix constructed via known solution of reduced self-dual system [6]. To observe this dependence from the solution of the previous section is not very simple for the general case and so to have some experience we consider at first the gauge algebras of Unitary series A^n which as always the most simplest for concrete calculations.

10.1 The case of A^2 algebra

In this case the ± 1 graded subspaces are two-dimensional. Basis vectors of them are $X_1^{\pm},X_2^{\pm}$. The basis of zero-graded subspace are two Cartan elements of A^2 algebra h_1,h_2. ± 2 graded subspaces are as always one-dimensional. We choose solution of the zero step in component form as

$$f^{(0}=\tau h_1+\rho h_2+\beta_0X_1^++\gamma_0X_2^++r_0X^+$$

Substituting this expression into formulae of the first step of we obtain

$$f^{(1}=\frac{1}{r_0}X^-+\frac{\beta_0X_2^--\gamma_0X_1^-}{r_0}+(\tau+\frac{r_1^+}{r_0})h_1+(\rho+\frac{r_1^-}{r_0})h_2+$$

$$\frac{Det\begin{pmatrix}\beta_0 & \beta_1\\ r_0 & r_1^+\end{pmatrix}}{r_0}X_1^++\frac{Det\begin{pmatrix}\gamma_0 & \gamma_1\\ r_0 & r_1^-\end{pmatrix}}{r_0}X_2^++\frac{Det\begin{pmatrix}r_0 & r_1^+\\ r_1^- & r_2\end{pmatrix}}{r_0}X^+ \qquad (10.1)$$

where $r_1^{\pm}=r_1\pm\frac{1}{2}\beta_0\gamma_0$.

We omit the expressions for negative components of the second step which can be obtained from the last formulae by help of only algebraic operations and represent the explicit expressions for nonnegative ones

$$f_0^{(2}=(\tau+\frac{Det\begin{pmatrix}r_0 & r_1^+\\ r_2^- & r_3^-\end{pmatrix}}{\Delta})h_1+(\rho+\frac{Det\begin{pmatrix}r_0 & r_1^-\\ r_2^+ & r_3^+\end{pmatrix}}{\Delta})h_2+$$

$$f^{(2}_{+1} = \frac{Det\begin{pmatrix}\beta_0 & \beta_1 & \beta_2\\ r_0 & r_1^+ & r_2^+\\ r_1^- & r_2 & r_3^+\end{pmatrix}}{\Delta}X_1^+ + \frac{Det\begin{pmatrix}\gamma_0 & \gamma_1 & \gamma_2\\ r_0 & r_1^- & r_2^-\\ r_1^+ & r_2 & r_3^-\end{pmatrix}}{\Delta}X_2^+ \tag{10.2}$$

$$f^{(2}_{+} = \frac{Det\begin{pmatrix}r_0 & r_1^+ & r_2^+\\ r_1^- & r_2 & r_3^+\\ r_2^- & r_3^- & r_4\end{pmatrix}}{\Delta}X^+$$

where

$$r_1^{\pm} = r_1 \pm \frac{1}{2}\beta_0\gamma_0 \quad , r_3^{\pm} = r_3 \pm \frac{1}{2}\beta_1\gamma_1$$

$$r_2^+ = r_2 + \beta_1\gamma_0, \quad r_2^- = r_2 - \beta_0\gamma_1, \quad \Delta = Det\begin{pmatrix}r_0 & r_1^+\\ r_1^- & r_2\end{pmatrix}$$

if in the above formulae put $\beta_i = \gamma_i = 0$ we return to known before expressions of A^1 algebra case.

10.2 The case of arbitrary n

We will use $(n+1)$-dimensional representation of A^n algebra - so called its first fundamental one. The arbitrary element of $+1$ graded space p_k may be represented in matrix form as

$$p_k = \begin{pmatrix}0 & a_k & 0\\ 0 & 0 & b_k\\ 0 & 0 & 0\end{pmatrix}$$

where a_k (n-1)-dimensional row vector, b_k (n-1)-dimensional column vector. Without any difficulties for all other values taking part in the formulae of two first steps of discrete transformation we obtain

$$s_k = \begin{pmatrix}(a_k b_k) & 0 & 0\\ 0 & -2 \parallel b_k^i a_k^j \parallel & 0\\ 0 & 0 & (a_k b_k)\end{pmatrix}, \quad \delta_k = \frac{1}{2}(a_k b_k)$$

where $(a_r b_k) = \sum_{i=1}^{n-1} a_r^i b_k^i$ and $\parallel b_k^i a_k^j \parallel$- $(n-1)\times(n-1)$ matrix with corresponding matrixes elements.

$$\tilde{\nu}_k \equiv \frac{\nu_k}{3!\sqrt{\delta_k}} = \begin{pmatrix}0 & -a_k & 0\\ 0 & 0 & b_k\\ 0 & 0 & 0\end{pmatrix},$$

$$p_k^+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & b_k \\ 0 & 0 & 0 \end{pmatrix}, \quad p_k^- = \begin{pmatrix} 0 & a_k & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

As a direct corollary of the above formulae the following relations take place

$$\frac{1}{2}[p_1, s_0] = (a_0 b_1) p_0^+ - (a_1 b_0) p_0^- + \frac{1}{2}(b_0 a_0)(p_1^+ - p_1^-)$$

$$\frac{1}{2}[\frac{\nu_0}{3!}, s_1] = \sqrt{\delta_0 \delta_1}(p_0^+ + p_0^-) + \sqrt{\delta_0}((b_0 a_1) p_1^+ + (b_1 a_0) p_1^-)$$

The following functions having two indexes i, j taking values in subspace of zero graded index will be important for compact form writing of the result of m-th times application of discrete transformation

$$I_{i,j}^{(m} = [p_i^+ [p_j^-, X^-]] + [p_i^- [p^+, X^-]], \quad 0 \leq i, j \leq (m-1)$$

The substitution of all exppressions above in corresponding formulae of the previous section leads to the following solution of self-dual system on the first step

$$f^{(1} = \frac{1}{r_0} X^- + \frac{[X_-, p_0]}{r_0} + (\tau + \frac{r_1}{r_0}) H + (\rho + \frac{1}{2} \frac{[p_0 [p_0, X^-]]}{r_0})$$

$$\frac{Det \begin{pmatrix} p_0^- & p_1^- \\ r_0 & r_1^+ \end{pmatrix}}{r_0} + \frac{Det \begin{pmatrix} p_0^+ & p_1^+ \\ r_0 & r_1^- \end{pmatrix}}{r_0} + \frac{Det \begin{pmatrix} r_0 & r_1^+ \\ r_1^- & r_2 \end{pmatrix}}{r_0} X^+ \tag{10.3}$$

Nonnegative components of the second step are the following ones

$$f_0^{(2} = (\tau + \frac{1}{2}(\frac{Det \begin{pmatrix} r_0 & r_1^+ \\ r_2^- & r_3^- \end{pmatrix}}{\Delta} + \frac{Det \begin{pmatrix} r_0 & r_1^- \\ r_2^+ & r_3^+ \end{pmatrix}}{\Delta})) H +$$

$$(\rho + \frac{1}{2} Tr(I^{(2} D^{-1})$$

where D^{-1} is 2×2 matrix invertible to matrix

$$D = \begin{pmatrix} r_0 & r_1^- \\ r_1^+ & r_2 \end{pmatrix}$$

$$f_{+1}^{(2} = \frac{Det \begin{pmatrix} p_0^- & p_1^- & p_2^- \\ r_0 & r_1^+ & r_2^+ \\ r_1^- & r_2 & r_3^+ \end{pmatrix}}{\Delta} + \frac{Det \begin{pmatrix} p_0^+ & p_1^+ & p_2^+ \\ r_0 & r_1^- & r_2^- \\ r_1^+ & r_2 & r_3^- \end{pmatrix}}{\Delta} \tag{10.4}$$

$$f_+^{(2} = \frac{Det\begin{pmatrix} r_0 & r_1^+ & r_2^+ \\ r_1^- & r_2 & r_3^+ \\ r_2^- & r_3^- & r_4 \end{pmatrix}}{\Delta}$$

where

$$r_1^\pm = r_1 \pm \frac{1}{2}(a_0 b_0), \quad r_3^\pm = r_3 \pm \frac{1}{2}(a_1 b_1),$$

$$r_2^+ = r_2 + (a_1 b_0), \quad r_2^- = r_2 - (a_0 b_1), \quad \Delta = Det\begin{pmatrix} r_0 & r_1^+ \\ r_1^- & r_2 \end{pmatrix}$$

It is not difficult, not only to generalize the expressions above to an arbitrary step of the discrete transformation, but by direct calculation also to check (by use of methods similar those used in [7]) their validity. Corresponding calculations are not very simple and straightforward. So we present here only the final result of discrete transformation after the m-th step for nonnegative components of self-dual field. The main role will play the following infinite-dimensional matrix constructed from solution of reduced self-dual system

$$D = \begin{pmatrix} 0 & p_0^- & p_1^- & p_2^- & \cdots \\ p_0^+ & r_0 & r_1^+ & r_2^+ & \cdots \\ p_1^+ & r_1^- & r_2 & r_3^+ & \cdots \\ p_2^+ & r_2^- & r_3^- & r_4 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \end{pmatrix} \tag{10.5}$$

For finite $n \times n$ matrix (counting from the left upper corner) arising from D after interrupting its $i, j, ..$ rows and $k, l, ..$ columns we will use notation $\parallel (D^{i,j,..}_{k,l,..})_n \parallel$. In this notation we obtain

$$f_0^{(m} = (\tau + \frac{1}{2}\frac{Det \parallel (D^1_{1,m-1})_m \parallel + Det \parallel (D_1^{1,m-1})_m \parallel}{Det \parallel (D_1^1)_m \parallel} H +$$

$$\rho + Tr(I^{(m}(\parallel (D^1_1)_m \parallel)^{-1}) \tag{10.6}$$

$$f_{+1}^{(m} = \frac{Det \parallel (D^1)_{m+1} \parallel + Det \parallel (D_1)_{m+1} \parallel}{Det \parallel (D^1_1)_m \parallel}$$

$$f_+^{(m} = \frac{Det \parallel (D^1_1)_{m+1} \parallel}{Det \parallel (D^1_1)_m \parallel}$$

11 Solution of the reduced self-dual system by the methods of matrix Riemann problem

It is well known that the general solution of self-dual system is equivalent to the matrix Riemann problem, the equation for which in this case has the form

$$\exp F(y+\lambda\bar{z}, z-\lambda\bar{y}, \lambda) = G_+^{-1}(\lambda)G_-(\lambda) \tag{11.1}$$

where $F(y+\lambda\bar{z}, z-\lambda\bar{y}, \lambda)$ is an arbitrary algebra-valued function, $G_+(\lambda), G_-(\lambda)$ are limiting values on some contour C in the complex λ plane of two group-valued functions $G_1(\lambda), G_2(\lambda)$ analytical outside and inside the circle C. We take the point $\lambda = \inf$ to be outside C. The boundary condition for (11.1) at the point $\lambda \to \inf$ is the following one

$$G_1(\lambda) = 1 + \lambda^{-1}f + ..$$

where f takes values in the algebra.

The proof is very simple. Let us act on both sides of equation (11.1) by two operators of differentiation $D_1 = \frac{\partial}{\partial\bar{z}} - \lambda\frac{\partial}{\partial y}, D_1 = \frac{\partial}{\partial\bar{y}} + \lambda\frac{\partial}{\partial z}$. Keeping in mind that both these operators annihilate the function of the left hand side of equality (11.1) we come to the following relation on circle C

$$(D_{1,2}G_+(\lambda))G_+^{-1}(\lambda) = (D_{1,2}G_-(\lambda))G_-^{-1}(\lambda) \tag{11.2}$$

From the latter relation, as usual for Riemann problem methods we conclude that we have deal with single on analitical function which has no singularities in the whole complex plane and so by Liouville's theorem is constant. Using the boundary conditions we come to equalities in the notations of the section 3 (see (3.1) and below)

$$G_1'(\lambda)G_1^{-1}(\lambda) = G_2'(\lambda)G_2^{-1}(\lambda) = \dot{f} \tag{11.3}$$

And this is exactly equation of self-dual system (compare with (2.1)).

Unfortunely explicit form of solution of matrix Riemann problem is unknown. But in the case of solvable algebra under additional condition that solution of Riemann problem is known for its solvable part the whole solution may be reconstructed in explicit form. This situation is arised as we will see below in the case of reduced self-dual system (7.2). In this case using the grading of section 3 it is possible after some obvious transformation to rewrite (11.1) in the form

$$\exp \alpha X^+ \exp A \exp B \exp \tau H \exp F_0 = G_+^{-1}(\lambda) G_-(\lambda) \qquad (11.4)$$

where functions scalar functions α, τ, A, B taking values in the degree $+1$ subspace, F_0 (0 graded subspace, $[X^\pm, F_0] = 0$)) are arbitrary functions of three independent arguments $(y + \lambda \bar{z}, z - \lambda \bar{y}, \lambda)$. The algebraic properties of functions A, B are described in section 3 (see (4.1) and below).

Now let us assume that solution of the Riemann problem under the choice of its homogeneous coefficient function as $\exp F_0$ is known, i.e.,

$$\exp F_0 = t_+^{-1}(\lambda) t_-(\lambda)$$

Substituting this expression for $\exp F_0$ in (11.4) and removing the term $t_+^{-1}(\lambda)$ on the first from the left place after some trivial regrouping of terms we will have

$$\exp \alpha X^+ \exp t_+ A t_+^{-1} \exp t_+ B t_+^{-1} \exp \tau H = t_+ G_+^{-1}(\lambda) G_-(\lambda) t_-^{-1}$$

The next step of transformation of the last equation is as follows: we represent the function τ in the form $\tau = \tau^+ - \tau^-$ and recalling the necessary commutation relations remove the term $\exp \tau^+ H$ on the first (from the left place) with the result

$$\exp(\exp -2\tau^+ \alpha) X^+ \exp(\exp -\tau^+ t_+ A t_+^{-1}) \exp(\exp -\tau^+ t_+ B t_+^{-1}) =$$

$$\exp -\tau^+ H t_+ G_+^{-1}(\lambda) G_-(\lambda) t_-^{-1} \exp \tau^- H$$

The trick of the same kind it is possible to repeat with all others terms of the last product. Indeed the subspaces A and B commutative and so any difficulties will not be met this way. Finally we come to equation of the form

$$1 = S_+^{-1}(\lambda) S_-(\lambda)$$

with the single solution $S = 1$ from which follows the explicit expression for $G_1(\lambda)$

$$G_1(\lambda) = \exp -[(\alpha \exp -2\tau^+ - [\tilde{B}^+, \tilde{A}])^+ + \frac{1}{2}[\tilde{B}^+, \tilde{A}^+] X^+$$

$$\exp \tilde{\tau}(A+B)^{+} \exp \tau^{+} H t_1(\lambda) \tag{11.5}$$

Asymptotic value of the last group element gives the explicit expressions for solution of reduced self-dual system in the form

$$f = (\int d\lambda \alpha(\lambda) \exp -2\tau^{+}(\lambda) + \int d\lambda \int d\lambda' \frac{1}{2} \frac{[\beta^1(\lambda), \beta^1(\lambda')]}{\lambda - \lambda'}) X^{+} +$$

$$\int d\lambda \beta^1(\lambda) + f_0 \tag{11.6}$$

where the function of $+1$ graded space β^1 is defined as $\beta^1(\lambda) = \exp -\tau^{+} t_{+}(A+B)t_{+}^{-1}$ and f_0 is the solution of self-dual system the gauge algebra of which conicide with the subspace of zero graded index of the initial algebra. the single solution $S = 1$ from which follows the explicit expression for $G_1(\lambda)$

$$G_1(\lambda) = \exp -[(\alpha \exp -2\tau^{+} - [\tilde{B}^{+}, \tilde{A}])^{+} + \frac{1}{2}[\tilde{B}^{+}, \tilde{A}^{+}] X^{+}$$

$$\exp \tilde{\tau}(A+B)^{+} \exp \tau^{+} H t_1(\lambda) \tag{11.7}$$

Asymptotic value of the last group element gives the explicit expressions for solution of reduced self-dual system in the form

$$f = (\int d\lambda \alpha(\lambda) \exp -2\tau^{+}(\lambda) + \int d\lambda \int d\lambda' \frac{1}{2} \frac{[\beta^1(\lambda), \beta^1(\lambda')]}{\lambda - \lambda'}) X^{+} +$$

$$\int d\lambda \beta^1(\lambda) + f_0 \tag{11.8}$$

where the function of $+1$ graded space β^1 is defined as $\beta^1(\lambda) = \exp -\tau^{+} t_{+}(A+B)t_{+}^{-1}$ and f_0 is the solution of self-dual system the gauge algebra of which conicide with the subspace of zero graded index of the initial algebra.

Some additional explanation is neccessary to understand and use the final formulas (11.8). In the process of its obtaining we have used the fact that homogeneous function of Riemann problem is annihilated by the pair of differential operators $D_1 = \frac{\partial}{\partial \bar{z}} - \lambda \frac{\partial}{\partial y}, D_1 = \frac{\partial}{\partial \bar{y}} + \lambda \frac{\partial}{\partial z}$. So if the parameter λ as function of independent arguments of the problem satisfy the pair of equations

$$\frac{\partial \lambda}{\partial \bar{z}} = \lambda \frac{\partial \lambda}{\partial y}, \quad \frac{\partial \lambda}{\partial \bar{y}} = \lambda \frac{\partial \lambda}{\partial z} \tag{11.9}$$

then all formulae of the present section will hold without any changes. So the integral in (11.8) is necessary to understand in the continual sense

$$\int d\lambda \to \sum F_s(\lambda_s)$$

where functions with different s are distinguished from each other and all λ_s are distinct solutions of the (in implicit form) exactly integrable system of the pair Monge equations (11.9). In all concrete applications connected with solving some problem with given boundary conditions (connected with the interruption of the infinite chain) there appear only sums instead of integrals.

12 The conditions of interrupting of reduced self-dual chain

All solutions constructed up to now by help of discrete transformation are some partial solution of self-dual system. From the point of view of physical applications the most interesting ones are those which satisfy additional conditions of "reality" in the form

$$G = G^\dagger, \quad \bar{f} = f^\dagger, \quad \bar{y} = x_1 - ix_2 = y*, \quad \bar{z} = x_0 - ix_3 = z*$$

where x_j are real coordinates of four-dimensional space-time and $^\dagger, *$ are the signs of Hermitian and complex conjugations. The interest to solution of this kind arised something about 20 years ago in connection with attempts to understand situation in field Yang-Mills gauge theories. The circle of these problems may be called shortly as monopole and instanton problems. It is obvious that self-dual system (2.1) is invariant with respect to transformation (inner automorphism)

$$G \to G^\dagger, \quad f \to (\bar{f})^\dagger, \quad y \to \bar{y}, \quad z \to \bar{z} \tag{12.1}$$

To find the place of monopole and instanton configurations among those obtainable by means of discrete transformation, let us consider more carefully the result of application of discrete transformation from left and right (see section 6) to a solution G_0 which is invariant with respect to transformation (12.1). We have consequently

$$G^{(+1} = S_l G_0, \quad G^{(-1} = G_0 S_r$$

But from the comments of the section it follows that $S_l \dot{=} S_r^\dagger$ if $G_0 = G_0^\dagger$ and so

$$G^{(+1} = (G^{(-1})^\dagger$$

Repeating the same operation m times we come to conclusion that

$$G^{(+m} = (G^{(-m})^\dagger$$

where the notation denotes the m-th application on the left $(+m)$ or right $(-m)$ of the discrete transformation to aan initial solution G_0 satisfying the condition of reality (12.1). So if starting from a solution of the reduced self-dual system (7.2), after m steps of left and subsequently m steps of right discrete transformations, we return to a solution of the reduced self-dual system (for the function $\bar{f}$) then in the middle of this chain we will have a solution for which the condition of reality is satisfied. This means that this solution allows some physical interpretation.

The result of $2m$-th application of the discrete transformation is known (at least for the Unitary series (10.6)) in explicit form and boundary condition on the second end of the chain restrict the choice of initial solution of reduced self-dual system of the last section.

The self-dual system possesses a rich class of inner automorphisms and with each of them it is possible by means of a discrete transformation to construct a solution invariant under the corresponding automorphism. The simplest example of direct solution (without using the general results of section 12) of the problem of interrupting of self-dual chain on both sides, the reader can find in Appendix II.

13 Conclusion

By means of rather complicated calculations we came to the remarkably simple form of the final result which we conjecture to be true in the case of arbitrary semisimple algebras (10.6). This means only that methods which we have used are not adequate for the problem under consideration and integrable substitution by itself must be the subject of independent consideration. We know now that this is some kind of discrete group with not simple inner structure and only on having available a representation theory of discrete groups of such kind will it be possible to know why we have obtaind such a simple final result after the comparatively cumbersome calculations in this paper.

14 Appendix I

Here we represent without comments and proofs some formulas useful for concrete calculations of section 8 which can be checked using the results of section 7. We conserve also all notations of this section.

$$[p,\nu]=(s,s)X_+,\quad [\nu,s]=\frac{3}{2}(s,s)p,\quad [\nu[\nu,X_-]]=-\frac{3}{2}(s,s)s$$

In all relations below $\dot{}$ is the differentiation with respect to arbitrary argument.

$$\dot\nu=3([p[p[\dot p,X_-]]]+([p,\dot p],X_-)p)=3[p[\dot p[p,X_-]]],\quad [p,\ldots]=\frac{2}{3}\dot\nu-([p,\dot p],X_-)$$

$$[\nu,\dot s]=\frac{3}{2}(s,\dot s)+(s,s)\dot p-([p,\dot p],X_-),\quad [\dot p,\nu]=\frac{1}{4}(s,\dot s)X_+,\quad [p,\dot\nu]=\frac{3}{4}(s,\dot s)X_-$$

15 Appendix II

Here we want to demonstrate briefly on the simplest example how the problem of interrupting of the chain may be resolved directly. Let us consider the case when the infinite chain (7.2) is interrupted on the first step. In other words our initial solution is of the form $f^{(0}=(0,0,f_0,p_0,r_0)$ and we want to have the final solution as $f^{(1}=(f_-^{(1},f_{-1}^{(1},f_0^{(1},0,0)$. To solve this problem it is necessary to remember the explicit formulas of the first step of the previous section and arising from its system of equations

$$f_{+1}^{(1}=0\to p_1=-\frac{\nu_0}{3!r_0}+\frac{r_1}{r_0}p_0,\quad f_+^{(1}=0\to r_2=\frac{r_1^2-\delta_0^2}{r_0} \tag{15.1}$$

and system (7.2) for the case of $n=0,1$

$$\dot r_2=r_1'-\dot\omega r_1+\frac{1}{2}(X^-[p_0,\dot p_1]),\quad \dot r_1=r_0'-\dot\omega r_0+\frac{1}{2}(X^-[p_0,\dot p_0])$$

$$\dot p_1=p_0'+[p_0,\dot f_0] \tag{15.2}$$

By the series of transformation from the latter systems (15.1) and (??) it is possible to separate the the following system of equations for two functions $u = \frac{r_1}{r_0}, v = \frac{\sqrt{\delta}}{r_0}$

$$u\dot{u} + v\dot{v} = u', \quad v\dot{u} + u\dot{v} = v'$$

which is equivalent to two independent Monge equations for functions $w_\pm = u \pm v$

$$w_\pm \dot{w}_\pm = w'_\pm \tag{15.3}$$

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