Alexander Polynomials for Projective Hypersurfaces

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In a sequence of papers A. Libgober has obtained interesting results on the Alexander polynomial Δ_C of a (complex, irreducible) curve $C \subset \mathbb{P}^2$, following an explicit suggestion due to D. Mumford, see the Introduction to [4].

This polynomial can be identified to the characteristic polynomial of the monodromy operator acting on the first cohomology group of the associated Milnor fiber, as shown by R. Randell [R]. Using this identification and results of H. Esnault [E], the Alexander polynomial Δ_C has been computed in cohomological terms by F. Loeser and M. Vaquié [LV], without the irreducibility assumption.

The Alexander polynomial Δ_V for an arbitrary dimensional complex hypersurface $V \subset \mathbb{P}^n$ has been defined by A. Libgober in [L3], where some of its properties are stated.

In this paper we develop a new point of view on these Alexander polynomials Δ_V. First we identify the polynomial Δ_V as the first in a sequence of $s+2$ polynomials ($s = \dim V_{sing}$) defined as characteristic polynomials of the monodromy operator on various cohomology groups of the associated Milnor fibers.

Unlike the method used by Randell in the curve case (which uses explicit generators and relations for the fundamental groups involved [R]), our identification is more geometric, being based on Zariski–Lefschetz type theorems (see Prop. (1.8)). This point of view offers also a new interpretation of the Alexander polynomial at infinity P_∞ of V, see Cor. (1.10).

Next we restrict to the case of isolated singularities $(s = 0)$, but <u>without</u> the additional restriction that V is a $\mathbb{Q}$–manifold which is used throughout in [L3]. In this more general setting, we prove stronger versions of the main results in [L3] related to the Alexander polynomial Δ_V.

Our approach is based on using explicit meromorphic differential forms in studying the topology of a hypersurface $V \subset \mathbb{P}^n$ and this paper is a natural continuation of our results in [D2].

To prove the computational power of this approach, we end with several explicit examples (nodal hypersurfaces and Cayley–Bacharach type hypersurfaces).

We like to thank Professor A. Libgober for drawing our attention on relations between his work and our preprint [D2] and to Professor D. Barlet for explaining to us a stimulating related result (see [B], Appendice 2) during a pleasant visit to Nancy.

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§ 1. Definition of the Alexander Polynomials

First we fix some notations. Let $V : f = 0$ be our hypersurface in $\mathbb{P}^n$ and assume that $s = \dim V_{sing} < n-1$, i.e. V is reduced. Let $F : f-1 = 0$ be the associated Milnor fiber which is a smooth hypersurface in $\mathbb{C}^{n+1}$. Let $h : F \longrightarrow F$, $h(x) = \lambda \cdot x$ for $\lambda = \exp(2\pi i/N)$ be the monodromy operator, where $N = \deg(f) = \deg(V)$.
Let $X = \{x \in \mathbb{C}^{n+1};\ f(x) = 0\}$ and note that the fibration

$$(1.1) \qquad F \longrightarrow \mathbb{C}^{n+1} \setminus X \xrightarrow{f} \mathbb{C}^*$$

is essentially the Milnor fibration of f [M1], having as monodromy operator exactly h. Let S be a sphere of sufficiently large radius centered at the origin of $\mathbb{C}^{n+1}$ and note that the inclusion $S \setminus K \longrightarrow \mathbb{C}^{n+1} \setminus X$ is a homotopy equivalence, where $K = X \cap S$ is the associated link.

Next we give a general definition for the Alexander polynomials, compare with [LV]

(1.2) Definition

Let M be a connected, locally contractible topological space having the same homotopy type as a CW–complex of dimension m and let $\varphi : \pi_1(M) \longrightarrow \mathbb{Z}$ be an epimorphism. Let $\tilde{M} \longrightarrow M$ be the covering space associated to the normal subgroup $\ker(\varphi)$ and let $T : \tilde{M} \longrightarrow \tilde{M}$ be a generator for the group of covering transformations.

Then we define the Alexander polynomials

$$\Delta^k(M,\varphi)(t) = \det(t \cdot \mathrm{Id} - T^* \mid H^k(\tilde{M}))$$

for $k = 1,\ldots,m$. Here and in the sequel cohomology is considered with $\mathbb{C}$–coefficients and by convention $\det(0 \longrightarrow 0) = 1$.

It is assumed that the vector spaces $H^k(\tilde{M})$ have finite dimension and then the above definition makes sense and moreover $\Delta^k(M,\varphi) \in \mathbb{Z}[t]$ for all k .

(1.3) Lemma

Let (M_1,φ_1) and (M_2,φ_2) be two pairs of objects as above. Let $j : M_1 \longrightarrow M_2$ be a map which is a p–equivalence and such that $\varphi_2 \circ \pi_1(j) = \varphi_1$. Then $\Delta^k(M_1,\varphi_1) = \Delta^k(M_2,\varphi_2)$ for any $k < p$ and $\Delta^p(M_2,\varphi_2)$ divides $\Delta^p(M_1,\varphi_1)$.

Proof

We recall first that j is said to be a p–equivalence if $\pi_k(j)$ are isomorphisms for $k < p$ and $\pi_p(j)$ is an epimorphism, see [Sp], p. 404. The map j has a lifting $\tilde{j} : \tilde{M}_1 \longrightarrow \tilde{M}_2$ (use its second property) and one has $\tilde{T}_2 \circ \tilde{j} = \tilde{j} \circ \tilde{T}_1$. Moreover, by the fibration homotopy exact sequence it follows that $\tilde{j}$ is again a p–equivalence. Next by Whitehead Theorem [Sp], p. 399 it follows that $H^k(\tilde{j})$ are isomorphisms for $k < p$ and is a monomorphism for $k = p$, which clearly implies the result.

(1.4) Remark

One can define another (infinite) series of Alexander polynomials for (M,φ) by setting

$$\hat{\Delta}^k(M,\varphi)(t) = \det(t\,\mathrm{Id} - \pi_k(T) \otimes 1_{\mathbb{Q}} \mid \pi_k(\tilde{M}) \otimes \mathbb{Q})$$

for all $k \geq 1$.

If M is q–connected for some $q \geq 1$, it follows by Hurewicz Theorem [Sp], p. 398 that

$$\Delta^{q+1}(M,\varphi) = \hat{\Delta}^{q+1}(M,\varphi)$$

i.e. the first (possibly) nontrivial polynomials in the two series considered coincide.

To define the Alexander polynomials of the hypersurface V, we take in the above definition $M_V = \mathbb{C}^{n+1} \setminus X$ and

$$\varphi_V = \pi_1(f) : \pi_1(M_V) \longrightarrow \pi_1(\mathbb{C}^*) = \mathbb{Z} \ .$$

The corresponding covering $\tilde{M}_V \longrightarrow M_V$ can then be described explicitly as follows. Let $\exp : \mathbb{C} \longrightarrow \mathbb{C}^*$ be the standard universal covering of $\mathbb{C}^*$. Then $\tilde{M}_V \longrightarrow M_V$ is the pull–back of the covering $\mathbb{C} \longrightarrow \mathbb{C}^*$ via the map f. It follows that $\tilde{M}_V \simeq F \times \mathbb{C}$ and the action of the covering transformation T on $\tilde{M}_V$ corresponds to the action of the monodromy operator h on F. Hence one can restate Definition (1.2) in this case as follows.

(1.5) <u>Definition</u>

The Alexander polynomials of a hypersurface $V \subset \mathbb{P}^n$ with $s = \dim V_{sing}$ are defined by the next formulas

$$\Delta_V^k(t) = \Delta^{n-s-2+k}(M_V, \varphi_V) =$$

$$= \det(t \, \mathrm{Id} - h^* \,|\, H^{n-s-2+k}(F))$$

for $k = 1, \ldots, s+2$.

The shift $k \longrightarrow n-s-2+k$ is due to the fact that F being $(n-s-2)$–connected [KM], Δ_V^1 is the first polynomial which may be different from 1. Moreover F is a CW–complex of dimension n [M2] and hence $H^m(F) = 0$ for all $m > n$ and $k = s+2$ is thus the last interesting value.

These (s+2) Alexander polynomials are not completely independent.

(1.6) Proposition

$$\prod_{k=1,s+2} \Delta_V^k(t)^{(-1)^{n-s+k}} = \left[\frac{(t^N-1)}{t-1}^{\chi(F)/N} \right]^{(-1)^n}$$

where $\chi(F)$ is the Euler characteristic of F. When V has only isolated singularities $a_1,\dots,a_p$ one has

$$\chi(F) = 1 + (-1)^n [(N-1)^{n+1} - N \sum_{i=1,p} \mu(V,a_i)]$$

where $\mu(V,a_i)$ denotes the Milnor number.

Proof

This is just a restatement of Prop. (3.13) and Remark (3.14) in [D1]. See also [O1]. In general $\chi(F)$ is a much easier to compute invariant and hence we are left with the problem of computing $s+1$ Alexander polynomials.

(1.7) Example

(i) V smooth ($s = -1$) :

The using (1.6) above we get

$$\Delta_V^1(t) = (t^N-1)^{(N-1)^{n+1}+(-1)^n/N}(t-1)^{(-1)^{n+1}} .$$

(ii) An example with nonisolated singularities: Let $V : x^2z + y^3 + xyt = 0$ be the cubic surface in $\mathbb{P}^3$ considered in (4.3) [D1]. Then it follows from the computations done there that

$$\Delta_V^1(t) = 1 \ , \ \Delta_V^2(t) = t^2 + t+1 \ \text{ and } \ \Delta_V^3(t) = 1 \ .$$

In fact Theorem A stated in the Introduction to [D1] offers a purely algebraic way to compute all the Alexander polynomials Δ_V^k, but this method is quite difficult to work out in practice, as the above example already shows.

Now we intend to relate our definition (1.5) to the definition given by Libgober in [L3]. Let $H : \ell = 0$ a hyperplane in $\mathbb{P}^n$ and let $\tilde{H}$ be the corresponding affine hyperplane in $\mathbb{C}^{n+1}$ given by the equation $\ell = 1$ ($\tilde{H}$ is well–defined up to translation !).

The obvious identification $\mathbb{P}^n \setminus H = \tilde{H}$ gives an identification of $N_V = \mathbb{P}^n \setminus (H \cup V)$ to the hyperplane section $\tilde{H} \cap M_V$. Hence we get an inclusion
$j_V : N_V = \tilde{H} \cap M_V \longrightarrow M_V$.

(1.8) Lemma

For a generic hyperplane H, the map j_V is an n–equivalence.

Proof

Use the Zariski Theorem of Lefschetz type in [HL] or the Affine Lefschetz Theorem in [H] (Thm. 2 plus the following comments).

It follows from (1.3) that the Alexander polynomials

$$\Delta^k(N_V,\pi_1(f \circ j_V)) \ \text{ and } \ \Delta^k(M_V,\varphi_V)$$

coincide for $k < n$ and according to the remarks above this covers all the interesting cases. From now on the first Alexander polynomial Δ_V^1 will be denoted simply by Δ_V and using Remark (1.4) it follows that our definition of Δ_V coincide with Libgober definition (Δ_V is in his notation simply P).

We have the next result similar to (1.8) (same proof).

(1.9) Proposition

For a generic hyperplane H in $\mathbb{C}^{n+1}$ passing through the origin, the inclusion $j_F : F \cap H \longrightarrow F$ is an (n–1)–equivalence.

(1.10) Corollary (Lefschetz Theorem for Alexander polynomials)

If H is a generic hyperplane in $\mathbb{P}^n$, then $\Delta^k_{V\cap H} = \Delta^k_V$ for all $k \leq s$ and the polynomial Δ_V^{s+1} divides the polynomial $\Delta^{s+1}_{V\cap H}$.

Note that in the case $s = 0$ the polynomial $\Delta_{V\cap H}$ can be identified to the so called Alexander polynomial at infinity considered in [L3] (where is denoted by P_∞) and hence (1.10) gives a proof of Theorem 2, (2) in [L3]. Indeed, if $T(H)$ is a small tubular neighbourhood of H in $\mathbb{P}^n$ one can identify its border $\partial T(H)$ with the sphere S_R in $\mathbb{C}^n$ centered at the origin and with a large enough radius R. And the space $\partial T(H)\setminus V$ has the same homotopy type as $M_{V\cap H}$, where in this case $V \cap H$ is smooth. This remark combined with (1.7i) gives the explicit formulas for P_∞ which appear in [L2] (3.4) and [L3], bottom of p. 50. Using inductively (1.10) one gets the next

(1.11) Corollary

With the above notations

$$\Delta_V = \Delta_{V\cap H_1\cap\ldots\cap H_s}$$

where H_i are generic hyperplanes in $\mathbb{P}^n$.

Note that in this case $V \cap H_1 \cap \ldots \cap H_s$ has only isolated singularities and hence the computation of the first Alexander polynomial Δ_V can be reduced to the case $s = 0$.

(1.12) Example

Consider the cubic surface V from (1.7.ii). A general plane section $V \cap H$ is a nodal cubic in $H \cong \mathbb{P}^2$ and hence we get again in this way

$$\Delta_V(t) = \Delta_{V \cap H}(t) = 1$$

since all the nodal curves in $\mathbb{P}^2$ have trivial Alexander polynomials by Deligne–Fulton result [De], [F].

§ 2. Local Alexander polynomials and their reduced versions

Let $Y : g = 0$ be an isolated hypersurface singularity at the origin of $\mathbb{C}^n$, let F_Y denote the corresponding Milnor fiber and $h_Y : F_Y \longrightarrow F_Y$ the monodromy operator [M1]. We assume throughout that $n \geq 2$. Let $\Delta_{Y,0}$ be the characteristic polynomial $\det(t\,\mathrm{Id} - h_Y^* \,|\, H^{n-1}(F_Y))$ which may (in an obvious way) be regarded as an Alexander polynomial as defined in (1.2).

However we will be more interested in a divisor of $\Delta_{Y,0}$, namely the reduced Alexander polynomial of the singularity $(Y,0)$ defined as follows

$$\tilde{\Delta}_{Y,0}(t) = \prod_{\lambda} (t-\lambda)^{a(\lambda)} \ . \tag{2.1}$$

Here the product is over all the eigenvalues λ of $h_Y^* \,|\, H^{n-1}(F_Y)$ and the multiplicities $a(\lambda)$ are defined as

$$a(\lambda) = \dim \ker(h_Y^* - \lambda \mathrm{Id} \,|\, H^{n-1}(F_Y)) \ . \tag{2.2}$$

Note that indeed $\tilde{\Delta}_{Y,0}$ is a factor of $\Delta_{Y,0}$ and moreover one has equality $\tilde{\Delta}_{Y,0} = \Delta_{Y,0}$ if and only if h_Y^* is diagonalizable. We give next a more geometric description of these multiplicities $a(\lambda)$.

(2.3) Lemma

Let B_0 be a small open ball centered at the origin of $\mathbb{C}^n$. Then $a(1) = \dim H^n(B_0 \setminus Y)$.

Proof

Let $S_0 = \partial \bar{B}_0$ be the corresponding small sphere and $K_Y = S_0 \cap Y$ the corresponding link. The Wang sequence associated to the Milnor fibration [M1] reads as follows

$$0 \longrightarrow H^{n-1}(S_0\backslash K_Y) \longrightarrow H^{n-1}(F_Y) \xrightarrow{h_Y^*-\mathrm{Id}} H^{n-1}(F_Y) \longrightarrow H^n(S_0\backslash K_Y) \longrightarrow 0 \ .$$

Since $S_0\backslash K_Y$ is homotopy equivalent to $B_0\backslash Y$ [M1] this ends the proof.

To interpret the multiplicity $a(\lambda)$ for $\lambda \neq 1$ one may proceed as follows. Write $\lambda = \exp\left[\frac{2\pi i b}{d}\right]$ for some integer $b \in [1,d-1]$. Let G be the multiplicative group $\{\alpha \in \mathbb{C}^*;\ \alpha^d = 1\}$ and note that the dual group of characters

$$G' = \{\chi : G \longrightarrow \mathbb{C}^*;\ \chi \text{ homomorphism}\}$$

may be identified to $\mathbb{Z}/d\mathbb{Z}$ via the correspondence

$$\mathbb{Z}/d\mathbb{Z} \ni c \longmapsto \chi_c \in G' \ \text{ defined by } \ \chi_c(\alpha) = \alpha^c\,.$$

Let $\tilde{Y} : g + t^d = 0$ in $\mathbb{C}^{n+1}$ be the d–suspension of the singularity $(Y,0)$. Then $(\tilde{Y},0)$ is again an isolated singularity.

Let $\tilde{B}_0$ denote a small ball centered at the origin in $\mathbb{C}^{n+1}$ and note that G acts on the set $\tilde{B}_0\backslash\tilde{Y}$ by the formula $\alpha \cdot (\chi,t) = (\chi,\alpha t)$. Then there is an induced action of G on $H^{n+1}(\tilde{B}_0\backslash\tilde{Y})$ and hence a direct sum decomposition

$$H^{n+1}(\tilde{B}_0\backslash\tilde{Y}) = \bigoplus_{\chi} H^{n+1}(\tilde{B}_0\backslash\tilde{Y})^{\chi} \tag{2.4}$$

For any G–module E, we define the eigenspace E^{χ} corresponding to the character

$\chi \in G'$ by

$$E^{\chi} = \{e \in E;\ \alpha \cdot e = \chi(\alpha)e \text{ for all } \alpha \in G\}) \ .$$

(2.5) Lemma

$$a(\exp\left[\left[\frac{2\pi ib}{d}\right]\right]) = \dim H^{n+1}(\tilde{B}_0 \backslash \tilde{Y})^{\chi_b}$$

Proof

According to the Thom–Sebastiani formula [TS], the cohomology $H^n(F_{\tilde{Y}})$ of the Milnor fiber for $\tilde{Y}$ can be identified to the tensor product $H^{n-1}(F_Y) \otimes \tilde{H}^0(F_d)$, where $F_d : t^d - 1 = 0$ is regarded as the Milnor fiber of a 0–dimensional singularity. The corresponding monodromy operator h_p^* acting on $\tilde{H}^0(F_d)$ is diagonalizable and has as characteristic polynomial the polynomial

$$\frac{t^d-1}{t-1} \ .$$

Moreover the monodromy operator $h_{\tilde{Y}}^*$ corresponds to the tensor product $T_Y \otimes T_d$ [TS], while the action of the generator $\lambda_1 = \exp\left[\frac{2\pi i}{d}\right]$ of G on $H^n(F_{\tilde{Y}})$ corresponds to $1 \otimes T_d$.

Recalling the Wang sequence from the proof of (2.3) (but applied this time to the singularity $(\tilde{Y},0)$), we get that $H^{n+1}(\tilde{B}_0 \backslash \tilde{Y})^{\chi_{d-b}}$ is spanned by elements of the form $v_i \otimes w$, where $\{v_i\}$ is a basis for $\ker(h_Y^* - \lambda\ \mathrm{Id})$ and w is the unique eigenvector in $\tilde{H}^0(F_d)$ corresponding to the eigenvalue λ_1^{d-b}.

Since all of our representations are real, it follows that

$$\dim H^{n+1}(\tilde{B}_0\backslash\tilde{Y})^{\chi_b} = \dim H^{n+1}(\tilde{B}_0\backslash\tilde{Y})^{\chi_{d-b}}$$

since $\bar{\chi}_b = \chi_{d-b}$, and this clearly ends of the proof. The next result is a stronger version of Thm. 2 (1) in [L3].

(2.6) Theorem

Let $V \subset \mathbb{P}^n$ be a hypersurface which is singular exactly at the points $a_1,\dots,a_p$. Then the Alexander polynomial Δ_V of V divides the product $\prod_{i=1,p} \tilde{\Delta}_{V,a_i}$ of all the reduced local Alexander polynomials.

Proof

We know already by (1.10) and (1.7.i) that any root λ of Δ_V is an element in the group $G_N = \{\alpha \in \mathbb{C}^*;\ \alpha^N = 1\}$. Let $A(\lambda)$ be the multiplicity of λ as a root in Δ_V. There are again two cases to discuss

Case 1 ($\lambda = 1$)

Let $U = \mathbb{P}^n\backslash V$ and note that U is just the quotient F/G_N, where G_N acts on F via the monodromy transformation h.
It follows easily that

$$A(1) = \dim H^{n-1}(F)^{\mathrm{Fix}(h^*)} = \dim H^{n-1}(U) = H^n_0(V) \ ,$$

where $H^n_0(V) = \operatorname{coker}(H^n(\mathbb{P}^n) \xrightarrow{j^*} H^n(V))$ $j : V \longrightarrow \mathbb{P}^n$ may be called the primitive n–cohomology of V.

Next consider the basic exact sequence

$$(2.7)\qquad H^n(U) \xrightarrow{\ \theta\ } \bigoplus_{a \in S} H^n(B_a \setminus V) \longrightarrow H_0^n(V) \longrightarrow 0$$

where B_a is a small ball around the point a in $\mathbb{P}^n$, $S = V_{sing} = \{a_1,\dots,a_p\}$ and θ is induced by the obvious inclusions, see for details [D2]. By (2.3) it follows that the multiplicity of $\lambda = 1$ as a root in $\Delta_{V,a}$ is exactly $\dim H^n(B_a \setminus V)$ and (2.7) implies that the sum of these multiplicities is greater or equal to $A(1)$.

Case 2 ($\lambda \neq 1$)

Assume that $\lambda = \exp\left[\frac{2\pi ib}{N}\right]$ and let $\tilde{V}$ be the hypersurface defined by $\tilde{f}(x,t) = f(x) + t^N = 0$ in $\mathbb{P}^{n+1}$. It is clear that $\pi : \tilde{V} \longrightarrow \mathbb{P}^n$ $\pi(x,t) = x$ is the unique covering of $\mathbb{P}^n$ ramified along V and of order N (with normal total space).

Let $\tilde{U} = \mathbb{P}^{n+1} \setminus \tilde{V}$ and note that the group G_N acts on all the cohomology groups $H^\cdot(\tilde{U})$, $H^\cdot(\tilde{V})$, $H^\cdot(\tilde{B}_a \setminus \tilde{V})$ which appear in the exact sequence (2.7) corresponding to $\tilde{V}$. It follows that for each character $\chi \in G_N'$ we get an exact sequence

$$(2.8)\qquad H^{n+1}(\tilde{U})^\chi \longrightarrow \bigoplus_{a \in S} H^{n+1}(\tilde{B}_a \setminus \tilde{V})^\chi \longrightarrow H_0^{n+1}(\tilde{V})^\chi \longrightarrow 0\ .$$

Using again a result of Thom–Sebastiani type for $\tilde{f}(x,t)$ (if necessary have a look in [D1], Section 1) it follows that

$$A(\lambda) = \dim H^n(\tilde{U})^{\chi_b} = \dim H_0^{n+1}(\tilde{V})^{\chi_b}\ .$$

Using Lemma (2.5), the result follows.

(2.9) <u>Remark</u>

Libgober in [L3] put the restriction on V of being a $\mathbb{Q}$–manifold (equivalently 1 is not a root of the polynomial $\prod_{a \in S} \tilde{\Delta}_{V,a}$) since he works there essentially with coverings of $\mathbb{P}^n$ along V (of various orders) and in this way one cannot keep track of the multiplicities A(1) and a(1) for the various (V,a) .

The next result shows that in general Δ_V is just a small factor of $\prod_{a \in S} \tilde{\Delta}_{V,a}$ and can be regarded as a generalization of the M. Oka result in [O2].

(2.10) <u>Proposition</u>

Let $V \subset \mathbb{P}^n$ be a nodal hypersurface (i.e. V has only isolated singularities of type A_1) with $\deg V = N$. Then

(i) $\Delta_V(t) = 1$ if nN is odd;

(ii) $\Delta_V(t) = [t + (-1)^{n+1}]^{\operatorname{def}(\mathscr{A})}$ if nN is even, where def($\mathscr{A}$) is the defect of the linear system $\mathscr{A}$, defined as follows.

Let S_k denote the vector space of homogeneous polynomials of degree k in $\mathbb{C}[x_0,\ldots,x_n]$ and let

$$\mathscr{A} = \{h \in S_D;\ h \mid V_{sing} = 0\} \quad \text{for } D = nN/2-n-1 .$$

Then $\operatorname{def}(\mathscr{A}) = \#\,S\text{–codim}\ \mathscr{A}$, where the codimension is taken in S_D . (It is clear that the number def($\mathscr{A}$) measures the degree of the independancy of the singular points of V with respect to polynomials in S_D).

<u>Proof</u>

This is exactly a restatement of our result (3.6) in [D2] (to which we refer for the details), in view of the next remark.

In the case n odd and N even, the only possible root for Δ_V is -1 by our Theorem (2.6) above, since we have $\Delta_{V,a}(t) = t+1$ for all $a \in V_{sing}$ in this case.

We also remark that the main idea in the proof of (3.6) in [D2] is used also with more details in the proof of the result in the next section.

§ 3. Cayley–Bacharach Hypersurfaces

The result in this section was suggested by (and is a strong generalization of) the Corollary in [L3], p. 51.

First we define the class of hypersurfaces we are interested in here.

Let $f_i \in \mathbb{C}[x_0,\dots,x_n]$ be a homogeneous polynomial of degree $d_i \geq 1$ for $i = 1,\dots,n$ and let N be a common multiple of the degrees d_i such that:

(i) $S' = \{x \in \mathbb{P}^n;\ f_1(x) = \dots = f_n(x) = 0\}$ consists of exactly $d_1 \dots d_n$ points;

(ii) $e_i = N \cdot d_i^{-1} > 1$ for all $i = 1,\dots,n$

(iii) the hypersurface $V = V(f_1,\dots,f_n;N)$ defined by $F = f_1^{e_1} + \dots + f_n^{e_n} = 0$ in $\mathbb{P}^n$ has only isolated singularities (clearly $S = V_{sing} \supset S'$).

We say that the hypersurface V defined above is of Cayley–Bacharach type.
It is easy to see that such hypersurfaces exist (take for instance $f_i = x_0^{d_i} + x_i^{d_i}$ and note that $V \cap \{x_0 = 0\}$ is smooth!)
For any point $a \in S'$, the germ (V,a) is a hypersurface singularity of Brieskorn–Pham type given in local coordinates by

$$(3.1) \qquad (V,a) : g = u_1^{e_1} + \dots + u_n^{e_n} = 0 \ .$$

Our interest in this class of hypersurfaces comes from the next result.

(3.2) Proposition

Any hypersurface V of Cayley–Bacharach type has a nontrivial Alexander polynomial Δ_V.

Proof

There is a unique pair of positive integers (k,s) such that $k = sN - d_1 - \dots - d_n$ and $k < N$. Again there are two cases to consider.

Case 1 $(k = 0)$

Then in each cohomology group $H^n(B_a \backslash V)$ for $a \in S'$ one has a nonzero element given in the local coordinates $(u_1,\dots,u_n)$ used in (3.1) by the meromorphic form

$$\omega_a = \frac{du_1 \wedge \dots \wedge du_n}{g^s}$$

see [D1] (3.6).

To show the map θ in (2.7) is not surjective (which would imply that 1 is a root for Δ_V), it is enough to show that the map

$$H^n(U) \longrightarrow \bigoplus_{a \in S} H^n(B_a \backslash V) \longrightarrow \bigoplus_{a \in S'} H^n(B_a \backslash V) \longrightarrow \bigoplus_{a \in S'} \mathbb{C} < \omega_a >$$

is not surjective (the second and the third map being the obvious projections).

It is known that $\mathbb{C} < \omega_a > = F_H^{n-s+1} H^n(B_a \backslash V)$ where F_H denotes the Hodge filtration on $H^n(B_a \backslash V)$ (see [D1], [D2], [D3]) and hence it remains to show that the induced mapping

$$\vartheta : F_H^{n-s+1} H^n(U) \longrightarrow \bigoplus_{a \in S'} \mathbb{C} < \omega_a >$$

is not surjective.

But using Thm. (2.2) in [D1], any element in $F_H^{n-s+1}H^n(U)$ can be written in the form $h\,\Omega/F^s$ where $\Omega = \sum_{i=0,n} (-1)^i x_i dx_0 \wedge \ldots \wedge \hat{dx_i} \wedge \ldots \wedge dx_n$ and h is a homogeneous polynomial in S_{sN-n-1}. It follows that $\mathrm{im}(\mathcal{V})$ is contained in $\mathrm{im}(E)$ where E is the <u>evaluation</u> map

$$E : S_{sN-n-1} \longrightarrow \bigoplus_{a \in S'} \mathbb{C} <\omega_a> = \mathbb{C}^{S'} \ , \ E(h) = (h(a))_{a \in S'} \ .$$

Since $sN = d_1 + \ldots + d_n$, it follows from the generalized version of Cayley–Bacharach Theorem (see [GH], p. 671) that E is not surjective.

<u>Case 2</u> $(k \neq 0)$

Then we consider again the hypersurface $\tilde{V} : F(x) + t^N = 0$ in $\mathbb{P}^{n+1}$. Note that for any point $a \in S'$, the germ $(\tilde{V},a)$ is given in suitable local coordinates by an equation

$$(3.3) \qquad (\tilde{V},a) : \tilde{g}(u,t) = u_1^{e_1} + \ldots + u_n^{e_n} + t^N = 0 \ .$$

Consider the element $\tilde{\omega}_a$ in $H^{n+1}(\tilde{B}_a \backslash \tilde{V})$ given by

$$\tilde{\omega}_a = \frac{t^{k-1} du_1 \wedge \ldots \wedge du_n \wedge dt}{\tilde{g}^s} \ .$$

Using a similar argument as above, one can again apply the generalized Cayley–Bacharach Theorem and deduce that the Alexander polynomial Δ_V is divisible by the product

$$\left[t - \exp\left[\frac{2\pi i k}{N}\right]\right] \left[t - \exp\left[\frac{2\pi i(-k)}{N}\right]\right] \ .$$

Hence in any case $\Delta_V \neq 1$, which proves our statement.

(3.4) Remark

Let $V \subset \mathbb{P}^n$ be a hypersurface with isolated singularities and let $\pi : \hat{V} \longrightarrow V$ be a resolution of singularities. Since $H^n(V)$ has a pure Hodge structure of weight n [S] , it follows that $H^n(\pi)$ is injective (in general $\ker H^n(\pi) = W_{n-1}H^n(V)$ as explained in [Df]).

Hence one gets in this way

$$h^{p,q}(H^n(\hat{V})) \geq h^{p,q}(H^n(V)) \text{ for all } p+q=n \ .$$

In the above proof we have shown that for a hypersurface V of Cayley–Bacharach type one has either $h^{n-s+1,s-1}(V) \neq 0$ (case $k=0$) or $h^{n-s+2,s-1}(\hat{\tilde{V}}) \neq 0$ (case $k \neq 0$). In this way one may get nonvanishing (or lower bounds) results for the Hodge numbers $h^{p,q}(\hat{V})$ or $h^{p,q}(\hat{\tilde{V}})$ similar to Theorem 3 in [L3].

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