MIXED HODGE STRUCTURES OF CONFIGURATION SPACES

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Let X be a smooth projective variety over $\mathbb{C}$. The configuration space $F(X,n)$ of X is the complement of the diagonals in $T(X,n) = X^n$:

$$F(X,n) = \{(z_1,\dots,z_n) \in X^n \mid z_i \neq z_j \text{ for } i \neq j\}.$$

The symmetric group $\mathbb{S}_n$ acts freely on $F(X,n)$; in this paper, we study the induced action of the symmetric group $\mathbb{S}_n$ on $H^{p,q}(F(X,n))$. As an application of our results, we calculate the $\mathbb{S}_n$-equivariant Hodge polynomial of the Fulton-MacPherson compactification of $F(X,n)$.

In a sequel to this paper, we extend our results to the relative setting: this is technically more difficult, requiring Saito's theory of mixed Hodge modules. As an application of the relative theory, we will calculate the $\mathbb{S}_n$-equivariant Hodge polynomials of the projective varieties $\overline{\mathcal{M}}_{1,n}$. (We have calculated the $\mathbb{S}_n$-equivariant Hodge polynomials of the projective varieties $\overline{\mathcal{M}}_{0,n}$ in [6].)

Let X be a complex quasi-projective variety. The Serre polynomial $e(X)$ of X is the polynomial in variables u and v, satisfying (and indeed characterized by) the following axioms:

i) if X is projective and smooth, $e(X)$ is the Hodge polynomial[1]

$$e(X) = \sum_{p,q=0}^{\infty} (-u)^p (-v)^q \dim H^{p,q}(X,\mathbb{C});$$

ii) if Z is a closed subvariety of X, then $e(X) = e(X \setminus Z) + e(Z)$.

A formula for the Serre polynomial follows from mixed Hodge theory (Deligne [4]):

$$e(X) = \sum_{p,q=0}^{\infty} u^p v^q \chi\big(H_c^\bullet(X,\mathbb{C})^{p,q}\big), \tag{0.1}$$

where if (V,F,W) is a mixed Hodge structure over $\mathbb{C}$,

$$V^{p,q} = F^p \operatorname{gr}^W_{p+q} V \cap \bar{F}^q \operatorname{gr}^W_{p+q} V,$$

and $\chi(V)$ denotes the Euler characteristic of the finite-dimensional graded vector space V. This formula was introduced by Danilov and Khovanskiĭ, although they do not give it a name: earlier, Serre had observed that the Weil conjectures, together with resolution of singularities, implied the existence of such a polynomial.

The Serre polynomial represents the class of $H_c^\bullet(X,\mathbb{C})$ in the Grothendieck group of mixed Hodge structures over $\mathbb{C}$. It is a "character" on varieties, since it satisfies the Künneth formula:

$$e(X \times Y) = e(X)\, e(Y).$$

We borrow from Manin [15] the notation $\mathbb{L}$ for the Serre polynomial uv of $\mathbb{C}(-1) = H^2(\mathbb{CP}^1,\mathbb{C})$.

[1] Here, we have modified the usual conventions, replacing u and v by $-u$ and $-v$. This will lead to cleaner formulas later.

Denote by S^nX the nth symmetric power of the variety X, that is, the quotient of X^n by the symmetric group $\mathbb{S}^n$. It follows from the formula of Macdonald [14] that the generating function $\sigma_t(X)$ for the Serre polynomials $e(S^nX)$ has the formula

$$\sigma_t(X) = \sum_{n=0}^{\infty} t^n\, e(S^nX) = \prod_{p,q}(1-tu^pv^q)^{-\chi(H_c^\bullet(X,\mathbb{C})^{p,q})}.$$

(This is the Hodge analogue of the zeta function of a variety over a finite field.)

If a finite group G acts on X, define the equivariant Serre polynomial by the formula

$$e_g(X) = \sum_{p,q=0}^{\infty} u^pv^q \sum_i (-1)^i \operatorname{Tr}(g|H_c^i(X,\mathbb{C})^{p,q}).$$

Let $F(\mathbb{C},n)$ be the configuration space of n ordered points in $\mathbb{C}$. By a formula of Lehrer and Solomon [12] (see (2.5)), we see that if $\sigma \in \mathbb{S}_n$ has n_j cycles of length j, then

$$e_\sigma(F(\mathbb{C},n)) = \prod_{j=1}^{\infty} \alpha_j\big(\alpha_j - jt^j\big)\dots\big(\alpha_j-(n_j-1)jt^j\big),$$

where $\alpha_j = \sum_{d|j} \mu(j/d)L^d$. (For another proof of this formula, see [6].) In this paper, we prove the following generalization of this formula.

Theorem. *Let X be a quasi-projective variety, and let* $\alpha_j(X) = \sum_{d|j} \mu(j/d)\, e(X;u^d,v^d)$. *If* $\sigma \in \mathbb{S}_n$ *is a permutation with* n_j *cycles of length* j, *then*

$$e_\sigma(F(X,n)) = \prod_{j=1}^{\infty} \alpha_j(X)\big(\alpha_j(X)-j\big)\dots\big(\alpha_j(X)-(n_j-1)j\big).$$

Given a $\mathbb{S}_n$-module V, consider the local system $\mathbb{V} = F(X,n)\times_{\mathbb{S}_n} V$ over $F(X,n)/\mathbb{S}_n$. This local system has a Serre polynomial

$$e(X,V) = \sum_{p,q=0}^{\infty} u^pv^q\chi\big(H_c^\bullet(X,\mathbb{V})^{p,q}\big),$$

related to the $\mathbb{S}_n$-equivariant Serre polynomial of $F(X,n)$ by the formula

$$e(X,V) = \frac{1}{n!}\sum_{\sigma\in\mathbb{S}_n} \operatorname{Tr}(\sigma|V)\, e_\sigma(F(X,n)).$$

A special case of this is the trivial representation $\mathbb{1}$, for which $e(F(X,n),\mathbb{1}) = e(F(X,n)/\mathbb{S}_n)$: we will prove that

$$\sum_{n=0}^{\infty} x^n\, e(F(X,n)/\mathbb{S}_n) = \frac{\sigma_t(X)}{\sigma_{t^2}(X)} = \prod_{p,q}\Big(\frac{1-t^2u^pv^q}{1-tu^pv^q}\Big)^{\chi(H_c^\bullet(X,\mathbb{C})^{p,q})}.$$

The analogue of this formula for $X = \operatorname{spec}(\mathbb{Z})$ is the Dirichlet series

$$\frac{\zeta(s)}{\zeta(2s)} = \sum_{n=1}^{\infty}\frac{\mu(n)^2}{n^s},$$

where $\mu(n)^2$ is the arithmetic function which is 1 if n is square-free and 0 otherwise: the analogy with configuration spaces is clear.

Although the above formulas involve de Rham cohomology, they may be proved with no greater difficulty for motivic cohomology, using the recent results of Gillet and Soulé [8]. Let Mot be the category of (pure effective rational) Chow motives, with Grothendieck group $K_0(\mathrm{Mot})$. Given a quasi-projective variety X over $\mathbb{C}$, let $e(X) \in K_0(\mathrm{Mot})$ be the

virtual motive associated to X by Gillet and Soulé. Then $K_0(\mathsf{Mot})$ is a λ-ring, with σ-operations satisfying

$$\sigma_n(e(X)) = e(X^n/\mathbb{S}_n),$$

and λ-operations satisfying

$$\lambda_n(e(X)) = e(F(X,n),\varepsilon),$$

where ε is the sign representation of $\mathbb{S}_n$. For each irreducible representation V_λ of $\mathbb{S}_n$, we will construct a natural transformation Φ_λ of the category of λ-rings such that

$$\Phi_\lambda(e(X)) = e(F(X,n),V_\lambda),$$

where V_λ is the irreducible $\mathbb{S}_n$-module associated to the partition λ. Here is a table of Φ_λ for small λ:

Φ_1	σ_1
Φ_2	$\sigma_2 - \sigma_1$
Φ_{1^2}	σ_{1^2}
Φ_3	$\sigma_3 - \sigma_2 - \sigma_{1^2}$
Φ_{21}	$\sigma_{21} - \sigma_2 - \sigma_{1^2} + \sigma_1$
Φ_{1^3}	σ_{1^3}
Φ_4	$\sigma_4 - \sigma_3 - \sigma_{21} + \sigma_{1^2}$
Φ_{31}	$\sigma_{31} - \sigma_3 - 2\sigma_{21} - \sigma_{1^3} + 2\sigma_2 + \sigma_{1^2} - \sigma_1$
Φ_{2^2}	$\sigma_{2^2} - \sigma_3 - \sigma_{21} + \sigma_2 + 2\sigma_{1^2}$
Φ_{21^2}	$\sigma_{21^2} - \sigma_{21} - \sigma_{1^3} + \sigma_2 + \sigma_{1^2} - \sigma_1$
Φ_{1^4}	σ_{1^4}

(Here, σ_λ denotes the operation on λ-rings associated to the Schur polynomial s_λ.) For the generating function of the operations Φ_λ, see (2.6).

The organization of this paper is as follows. In Section 1, we recall some of the theory of symmetric functions in an infinite number of variables, and its relationship to the theory of λ-rings. In Section 2, we introduce the completion of a λ-ring: this is used in the proof of our main result, where we work with generating (symmetric) functions, which lie in such a completion. We also introduce the Φ-operations, which enter into the statement of our formula for $e(F(X,n),V_\lambda)$.

In Section 3, we introduce a class of categories, Karoubian rrings (sic), which have many of the properties of the category of modules over a ring. In particular, we prove the Peter-Weyl Theorem: any representation of a finite group in a Karoubian rring over a field of characteristic zero is completely reducible. Similar results have been obtained by del Baño Rollin [1]. In Section 4, we apply this result to construct a λ-ring structure on the Grothendieck group of a Karoubian rring (over a field of characteristic zero). Section 5 contains our main result, Theorem (5.6), which is a theorem about Serre functors; this is the name we give to a sequence $\{\mathcal{X} \mapsto E^n(\mathcal{X}) \mid n \geq 0\}$ of functors from the category of quasi-projective varieties to a Karoubian rring $\mathcal{R}$, satisfying appropriate analogues of the Meyer-Vietoris and the Künneth theorems. Examples of Serre functors are the de Rham cohomology and the cohomology of the weight complex of Gillet and Soulé [8].

In Section 6, we give a simple application of these results, to the calculation of the $\mathbb{S}_n$-equivariant Hodge polynomial of the Fulton-Macpherson compactification of the configuration space $F(X,n)$.

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1. Symmetric functions and λ-rings

(1.1). **Symmetric functions.** In this section, we recall some results on symmetric functions and representations of S_n which we need later. For the proofs of these results, we refer to Macdonald [13].

The ring of symmetric functions is the inverse limit

$$\Lambda = \varprojlim \mathbb{Z}[x_1, \dots, x_k]^{S_k}.$$

It is is a polynomial ring in the complete symmetric functions

$$h_n = \sum_{i_1 \le \cdots \le i_n} x_{i_1} \dots x_{i_n}.$$

The power sums (also known as Newton polynomials)

$$p_n = \sum_i x_i^n$$

form a set of generators of the polynomial ring $\Lambda_{\mathbb{Q}} = \Lambda \otimes \mathbb{Q}$. This is shown by means of the elementary formula

$$P_t = t\frac{d}{dt} \log H_t, \tag{1.2}$$

where

$$H_t = \sum_{n=0}^{\infty} t^n h_n = \prod_i (1 - tx_i)^{-1} \quad \text{and} \quad P_t = \sum_{n=0}^{\infty} t^n p_n = \sum_i (1 - tx_i)^{-1}.$$

Written out explicitly, we obtain Newton's formula relating the two sets of generators:

$$nh_n = p_n + h_1 p_{n-1} + \cdots + h_{n-1} p_1.$$

We may also invert (1.2), obtaining the formula

$$H_t = \exp\Big(\sum_{n=1}^{\infty} \frac{t^n p_n}{n}\Big). \tag{1.3}$$

A partition λ is a decreasing sequence $(\lambda_1 \ge \cdots \ge \lambda_\ell)$ of positive integers; we write $\lambda \vdash n$, where $n = \lambda_1 + \cdots + \lambda_\ell$, and denote the length of λ by $\ell(\lambda)$. Identifying Λ with the ring of characters of the Lie algebra $\mathfrak{gl}_\infty = \varprojlim \mathfrak{gl}_k$, we see that partitions correspond to dominant weights, and thus Λ has a basis of consisting of the characters of the irreducible representations of $\mathfrak{gl}_\infty$. These characters, given by the Weyl character formula

$$s_\lambda = \lim_{k \to \infty} \frac{\det(x_i^{\lambda_j + k - j})_{1 \le i,j \le k}}{\det(x_i^{k-j})_{1 \le i,j \le k}},$$

are known as the Schur functions. In terms of the polynomial generators h_n, they are given by the Jacobi-Trudy formula $s_\lambda = \det\big(h_{\lambda_i - i + j}\big)_{1 \le i,j \le \ell(\lambda)}$.

There is a non-degenerate integral bilinear form on Λ, denoted $\langle f, g \rangle$, for which the Schur functions s_λ form an orthonormal basis. (This is sometimes called the Hall inner product.) The adjoint of multiplication by $f \in \Lambda$ with respect to this inner product is

denoted $D(f)$. Written in terms of the power sums p_n, the operator $D(f)$ has the formula $D(p_n) = n\partial/\partial p_n$, while the inner product $\langle f, g\rangle$ has the formula

$$\langle f, g\rangle = D(f)g\Big|_{p_n=0, n\geq 1}.$$

(1.4). **Pre-λ-rings.** A pre-λ-ring is a commutative ring R, together with a morphism of commutative rings $\sigma_t : R \to R[\![t]\!]$ such that $\sigma_t(a) = 1 + ta + O(t^2)$. Expanding σ_t in a power series

$$\sigma_t(a) = \sum_{n=0}^{\infty} t^n \sigma_n(a),$$

we obtain endomorphisms σ_n of R such that $\sigma_0(a) = 1$, $\sigma_1(a) = a$, and

$$\sigma_n(a+b) = \sum_{k=0}^{n} \sigma_{n-k}(a)\sigma_k(b).$$

There are also operations $\lambda_k(a) = (-1)^k \sigma_k(-a)$, with generating function

$$\lambda_t(a) = \sum_{n=0}^{\infty} t^n \lambda_n(a) = \sigma_{-t}(a)^{-1}. \tag{1.5}$$

The λ-operations are polynomials in the σ-operations with integral coefficients, and vice versa. In this paper, we take the σ-operations to be more fundamental; nevertheless, following custom, the structure they define is called a pre-λ-ring.

Given a pre-λ-ring R, there is a bilinear map $\Lambda \otimes R \to R$, which we denote $f \circ a$, defined by the formula

$$(h_{n_1} \dots h_{n_k}) \circ a = \sigma_{n_1}(a) \dots \sigma_{n_k}(a).$$

The image of the power sum p_n under this map is the operation on R known as the Adams operation ψ_n. We denote the operation corresponding to the Schur function s_λ by σ_λ. Note that (1.3) implies the relation

$$\sigma_t(a) = \exp\Big(\sum_{n=1}^{\infty} \frac{t^n \psi_n(a)}{n}\Big).$$

The following formula (I.4.2 of [13]) is known as Cauchy's formula:

$$H_t(\dots, x_i y_j, \dots) = \prod_{i,j} (1 - t x_i y_j)^{-1} = \sum_{\lambda \vdash n} s_\lambda(x) \otimes s_\lambda(y) = \exp\Big(\sum_{k=1}^{\infty} \frac{p_k(x) \otimes p_k(y)}{k}\Big). \tag{1.6}$$

From it, the following result is immediate.

Proposition (1.7). *If R and S are pre-λ-rings, their tensor product $R \otimes S$ is a pre-λ-ring, with σ-operations*

$$\sigma_n(a \otimes b) = \sum_{\lambda \vdash n} \sigma_\lambda(a) \otimes \sigma_\lambda(b),$$

and Adams operations $\psi_n(a \otimes b) = \psi_n(a) \otimes \psi_n(b)$.

For example, $\sigma_2(a \otimes b) = \sigma_2(a) \otimes \sigma_2(b) + \lambda_2(a) \otimes \lambda_2(b)$.

(1.8). **λ-rings.** The polynomial ring $\mathbb{Z}[x]$ is a pre-λ-ring, with σ-operations characterized by the formula $\sigma_n(x^i) = x^{ni}$. Taking tensor powers of this pre-λ-ring with itself, we see that the polynomial ring $\mathbb{Z}[x_1, \ldots, x_k]$ is a pre-λ-ring. The λ-operations on this ring are equivariant with respect to the permutation action of the symmetric group $\mathbb{S}_k$ on the generators, hence the ring of symmetric functions $\mathbb{Z}[x_1, \ldots, x_k]^{\mathbb{S}_k}$ is a pre-λ-ring. Taking the limit $k \to \infty$, we obtain a pre-λ-ring structure on Λ.

Definition (1.9). A λ-ring is pre-λ-ring such that if $f, g \in \Lambda$ and $x \in R$,

$$f \circ (g \circ x) = (f \circ g) \circ x. \tag{1.10}$$

By definition, the pre-λ-ring Λ is a λ-ring; in particular, the operation $f \circ g$, called plethysm, is associative.

The following result (see Knutson, [10]) is the chief result in the theory of λ-rings.

Theorem (1.11). *Λ is the universal λ-ring on a single generator h_1.*

This theorem makes it straighforward to verify identities in λ-rings: it suffices to verify them in Λ. As an application, we have the following corollary.

Corollary (1.12). *The tensor product of two λ-rings is a λ-ring.*

Proof. We need only verify this for $R = \Lambda$. A torsion-free pre-λ-ring whose Adams operations are ring homomorphisms which satisfy $\psi_m(\psi_n(a)) = \psi_{mn}(a)$ is a λ-ring. It is easy to verify these conditions for $\Lambda \otimes \Lambda$, since $\psi_n(a \otimes b) = \psi_n(a) \otimes \psi_n(b)$. □

In the definition of a λ-ring, it is usual to adjoin the axiom

$$\sigma_n(xy) = \sum_{\lambda \vdash n} \sigma_\lambda(a) \otimes \sigma_\lambda(y).$$

However, this formula follows from our definition of a λ-ring: by universality, it suffices to check it for $R = \Lambda \otimes \Lambda$, $x = h_1 \otimes 1$ and $y = 1 \otimes h_1$, for which it is evident.

2. Complete λ-rings

A filtered λ-ring R is a λ-ring with decreasing filtration

$$R = F^0R \supset F^1R \supset \ldots,$$

such that

i) $\bigcap_k F^kR = 0$ (the filtration is discrete);
ii) $F^mR \cdot F^nR \subset F^{m+n}R$ (the filtration is compatible with the product);
iii) $\sigma_m(F^nR) \subset F^{mn}R$ (the filtration is compatible with the λ-ring structure).

The completion of a filtered λ-ring is again a λ-ring; define a complete λ-ring to be a λ-ring equal to its completion. For example, the universal λ-ring Λ is filtered by the subspaces $F^n\Lambda$ of polynomials vanishing to order $n-1$, and its completion is the λ-ring of symmetric power series, whose underlying ring is the power series ring $\mathbb{Z}[\![h_1, h_2, h_3, \ldots]\!]$.

The tensor product of two filtered λ-rings is again a filtered λ-ring, when furnished with the filtration

$$F^n(R \otimes S) = \sum_{k=0}^{n} F^{n-k}R \otimes F^kS.$$

If R and S are filtered λ-rings, denote by $R\hat{\otimes}S$ the completion of $R \otimes S$.

Let $\mathcal{R}$ be a Karoubian rring over a field of characteristic zero, and consider the complete λ-ring $\Lambda\hat{\otimes}K_0(\mathcal{R})$, where $K_0(\mathcal{R})$ has the discrete filtration. This λ-ring has a natural realization, as the Grothendieck group of the Karoubian rring

$$[\![\mathbb{S},\mathcal{R}]\!] = \prod_{n=0}^{\infty} [\mathbb{S}_n,\mathcal{R}],$$

whose objects are the $\mathbb{S}$-modules in $\mathcal{R}$. In this rring, the sum and product are given by the same formulas as in the rring $[\mathbb{S},\mathcal{R}]$ of bounded $\mathbb{S}$-modules.

Without assuming the existence of infinite sums in $\mathcal{R}$, plethysm does not extend to a monoidal structure on $[\![\mathbb{S},\mathcal{R}]\!]$. However, $\mathcal{X}\circ\mathcal{Y}$ is well-defined in $[\![\mathbb{S},\mathcal{R}]\!]$ under either of the following two hypotheses:

$$\text{i) } \mathcal{X} \text{ is bounded, or ii) } \mathcal{Y}(0) = 0.$$

The first of these situations allows us to construct a λ-ring structure on the Grothendieck group of $[\![\mathbb{S},\mathcal{R}]\!]$, by the same method as for $[\mathbb{S},\mathcal{R}]$, while the second will be needed in the proof of our main theorem. Introducing the notation ${}^k[\![\mathbb{S},\mathcal{R}]\!]$ for the subcategory of $[\![\mathbb{S},\mathcal{R}]\!]$ consisting of $\mathbb{S}$-modules $\mathcal{X}$ such that $\mathcal{X}(n) = 0$ for $n < k$, we see that plethysm extends to a symmetric monoidal structure on ${}^1[\![\mathbb{S},\mathcal{R}]\!]$.

Denote the Grothendieck group of the full subcategory ${}^1[\![\mathbb{S},\mathcal{R}]\!] \subset [\![\mathbb{S},\mathcal{R}]\!]$ by $\check{K}_0^{\mathbf{S}}(\mathcal{R})$. Since $\check{K}_0^{\mathbf{S}}(\mathcal{R})$ is a (non-unital) λ-ring, we may define a bilinear operation

$$\circ : \hat{K}_0^{\mathbf{S}}(\mathsf{Proj}) \otimes \check{K}_0^{\mathbf{S}}(\mathcal{R}) \longrightarrow \check{K}_{\mathbf{S}}(\mathcal{R}),$$

satisfying (1.10). This operation may be extended to a bilinear operation (which we denote by the same symbol),

$$\circ : \hat{K}_0^{\mathbf{S}}(\mathcal{R}) \otimes \check{K}_0^{\mathbf{S}}(\mathcal{R}) \longrightarrow \check{K}_{\mathbf{S}}(\mathcal{R}),$$

using the Peter-Weyl Theorem: to define $x\circ y$, we expand x in a series $x = \sum_\lambda x_\lambda \cdot s_\lambda$, and define

$$x\circ y = \sum_\lambda x_\lambda \cdot \sigma_\lambda(y).$$

The interest of this operation lies in the following lemma, which is a simple consequence of the definition of $x\circ y$.

Lemma (2.1). *If $\mathcal{X}$ and $\mathcal{Y}$ are objects of $[\![\mathbb{S},\mathcal{R}]\!]$ and ${}^1[\![\mathbb{S},\mathcal{R}]\!]$ respectively, $[\mathcal{X}\circ\mathcal{Y}] = [\mathcal{X}]\circ[\mathcal{Y}]$.*

If R is a complete λ-ring, the operation

$$\mathrm{Exp}(a) = \sum_{n=0}^{\infty} \sigma_n(a) : R \longrightarrow 1 + F_1R$$

is an analogue of exponentiation, whose logarithm is given by a formula of Cadogan [2].

Proposition (2.2). *On a complete filtered λ-ring R, the operation $\mathrm{Exp} : R \to 1 + F_1R$ has inverse*

$$\mathrm{Log}(1+a) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log(1+\psi_n(a)).$$

Proof. Expanding $\mathrm{Log}(1+a)$, we obtain

$$\mathrm{Log}(1+a) = -\sum_{n=1}^{\infty} \frac{1}{n} \sum_{d|n} \mu(d)\psi_d(-a)^{n/d} = \sum_{n=1}^{\infty} \mathrm{Log}_n(a).$$

Let χ_n be the character of the cyclic group C_n equalling $e^{2\pi i/n}$ on the generator of C_n. The characteristic of the $\mathbb{S}_n$-module $\mathrm{Ind}_{C_n}^{\mathbf{S}_n}\chi_n$ equals

$$\frac{1}{n}\sum_{k=0}^{n-1} e^{2\pi ik/n} p_{(k,n)}^{n/(k,n)} = \frac{1}{n}\sum_{d|n}\mu(d)p_d^{n/d},$$

while the characteristic of the $\mathbb{S}_n$-module $\mathrm{Ind}_{C_n}^{\mathbf{S}_n}\chi_n\otimes\varepsilon_n$, where ε_n is the sign representation of $\mathbf{S}_n$, equals

$$\frac{1}{n}\sum_{d|n}\mu(d)\big((-1)^{d-1}p_d\big)^{n/d} = \frac{(-1)^n}{n}\sum_{d|n}\mu(d)(-p_d)^{n/d}.$$

It follows that $(-1)^{n-1}\,\mathrm{Log}_n$ is the operation associated to the $\mathbb{S}_n$-module $\mathrm{Ind}_{C_n}^{\mathbf{S}_n}\chi_n\otimes\varepsilon_n$, and hence defines a map from F_1R to F_nR.

To prove that Log is the inverse of Exp, it suffices to check this for $R=\Lambda$ and $x=h_1$. We must prove that

$$\mathrm{Exp}\left(\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\log(1+p_n)\right) = 1+h_1.$$

The logarithm of the expression on the left-hand side equals

$$\exp\Big(\sum_{k=1}^{\infty}\frac{p_k}{k}\Big)\circ\Big(\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\log(1+p_n)\Big) = \sum_{n=1}^{\infty}\sum_{d|n}\mu(d)\frac{\log(1+p_n)}{n} = \log(1+p_1),$$

and the formula follows. □

Example (2.3). If $a\in F^1R$ is a line bundle in the complete λ-ring R (that is, $\sigma_n(a)=a^n$ for all $n\geq 0$), we see that

$$\mathrm{Exp}(a) = \frac{1}{1-a}.$$

In particular, this shows that $\mathrm{Exp}(t^n)=(1-t^n)^{-1}$, and that

$$\mathrm{Exp}(t-t^2) = \frac{\mathrm{Exp}(t)}{\mathrm{Exp}(t^2)} = \frac{1-t^2}{1-t} = 1+t.$$

It follows that $\mathrm{Log}(1-t)=t$ and that $\mathrm{Log}(1+t)=t-t^2$.

We now introduce the operations on λ-rings which will arise in the calculation of the Serre polynomials of the local systems $F(X,n)\times_{\mathbf{S}_n}V_\lambda$. We start by considering the case $X=\mathbb{C}$.

Proposition (2.4).

$$\sum_{\lambda} s_\lambda\otimes e(F(\mathbb{C},n),V_\lambda) = \prod_{k=1}^{\infty}(1+p_k)^{\frac{1}{k}\sum_{d|k}\mu(k/d)L^d} \in \Lambda\hat{\otimes}\mathbb{Z}[L]$$

Proof. It is proved in Lehrer-Solomon [12] that

$$\sum_{n=0}^{\infty}\sum_{i=0}^{\infty}(-x)^i\,\mathrm{ch}_n(H^i(F(\mathbb{C},n),\mathbb{C})) = \prod_{k=1}^{\infty}(1+x^kp_k)^{\frac{1}{k}\sum_{d|k}\mu(k/d)x^{-d}}, \tag{2.5}$$

where $H^i(F(\mathbb{C},n),\mathbb{C})$ is the $\mathbb{S}_n$-module associated to the de Rham cohomology of degree i. By Poincaré duality, we see that

$$\sum_{n=0}^{\infty}\sum_{i=0}^{\infty}(-x)^i\,\mathrm{ch}_n(H_c^i(F(\mathbb{C},n),\mathbb{C})) = \prod_{k=1}^{\infty}(1+p_k)^{\frac{1}{k}\sum_{d|k}\mu(k/d)x^d}.$$

But the mixed Hodge structure on the cohomology group $H^i_c(F(\mathbb{C},n),\mathbb{C})$ is pure of weight $2i$, and indeed $H^i_c(F(\mathbb{C},n),\mathbb{C}) = H^i_c(F(\mathbb{C},n),\mathbb{C})^{i,i}$, proving the result. □

Motivated by this proposition, we define operations Φ_λ in a λ-ring R, parametrized by partitions λ, by means of the generating function

$$\Phi(x) \equiv \sum_\lambda s_\lambda \otimes \Phi_\lambda(x) = \prod_{k=1}^{\infty}(1+p_k)^{\frac{1}{k}\sum_{d|k}\mu(k/d)\psi_d(x)} \in \Lambda\hat{\otimes}R. \tag{2.6}$$

Theorem (2.7). *We have the formula* $\Phi(x) = \mathrm{Exp}(\mathrm{Log}(1+p_1)x)$. *In particular, the operations* Φ_λ *are defined on any* λ*-ring.*

Proof. Applying Log to the definition of $\Phi(x)$, we obtain

$$\begin{aligned}
\mathrm{Log}(\Phi(x)) &= \sum_{n=1}^{\infty}\frac{\mu(n)}{n}\psi_n\log(\Phi(x)) \\
&= \sum_{n=1}^{\infty}\frac{\mu(n)}{n}\psi_n\sum_{k=1}^{\infty}\frac{1}{k}\sum_{d|k}\mu(k/d)\log(1+p_k)\psi_d(x) \\
&= \sum_{n,d,e=1}^{\infty}\frac{\mu(n)\mu(e)}{nde}\psi_{nd}(x)\log(1+p_{nde}) \\
&= \sum_{e=1}^{\infty}\frac{\mu(e)}{e}\log(1+p_e)x,
\end{aligned}$$

by Möbius inversion. On applying Exp, we obtain the desired formula. □

Using this theorem, we can prove more explicit formulas for Φ_n and Φ_{1^n}.

Corollary (2.8).

$$\sum_{n=0}^{\infty}t^n\Phi_n(y) = \frac{\sigma_t(y)}{\sigma_{t^2}(y)} \text{ and } \Phi_{1^n}(y) = \lambda_n(y)$$

Proof. We obtain $\sum_{n=0}^{\infty}t^n\Phi_n(y)$ from $\Phi(x)$ by replacing p_n by t^n. By Theorem (2.7), it follows that

$$\sum_{n=0}^{\infty}t^n\Phi_n(y) = \mathrm{Exp}(\mathrm{Log}(1+t)x) = \mathrm{Exp}((t-t^2)x) = \frac{\sigma_t(x)}{\sigma_{t^2}(x)},$$

since $\mathrm{Log}(1+t) = t - t^2$ by Example (2.3). The proof of the second formula is similar, except that we replace p_n by $(-t)^n$, and apply the formula $\mathrm{Log}(1-t) = -t$. □

3. Representations of finite groups in Karoubian rrings

Let $(\mathcal{R},\otimes,\mathbb{1})$ be a symmetric monoidal category with coproducts, denoted $X\oplus Y$. We say that $\mathcal{R}$ is a **rring** (this is our abbreviation for the usual term *ring category*) if there are natural isomorphisms

$$(X\oplus Y)\otimes Z \cong (X\otimes Z)\oplus(Y\otimes Z) \quad \text{and} \quad X\otimes 0 \cong 0$$

which describe the distributivity of the tensor product over the sum, satisfying the coherence axioms of Laplaza [11]. If $\otimes$ is the categorical product, we say that $\mathcal{R}$ is a Cartesian rring.

The Grothendieck group $K_0(-)$ is a functor from rrings to commutative rings. Given an object X of a rring $\mathcal{R}$, denote by $[X]$ its isomorphism class; then $K_0(\mathcal{R})$ is generated as an abelian group by the isomorphism classes of objects, with the relation

$$[X] + [Y] = [X \oplus Y].$$

The product on $K_0(\mathcal{R})$ is given by the formula $[X] \cdot [Y] = [X \otimes Y]$. (Here, we suppose that the isomorphism classes of objects of $\mathcal{R}$ form a set; this hypothesis will always be fulfilled in this paper.)

If $\mathcal{R}$ and $\mathcal{S}$ are two rrings, $\mathcal{R} \otimes \mathcal{S}$ is a rring whose objects are formal sums of tensor products $X \otimes Y$, where X and Y are objects of $\mathcal{R}$ and $\mathcal{S}$ respectively; note that $K_0(\mathcal{R} \otimes \mathcal{S}) \cong K_0(\mathcal{R}) \otimes K_0(\mathcal{S})$.

Recall that an additive category over a commutative ring R is a category $\mathcal{R}$ such that the set of morphisms $\mathcal{R}(X, Y)$ is a R-module for all objects X and Y, the composition maps $\mathcal{R}(Y, Z) \otimes_K \mathcal{R}(X, Y) \to \mathcal{R}(X, Z)$ are R-linear, and every finite set of objects has a direct sum. A **Karoubian category** over a ring R is an additive category over R such that every idempotent has an image, denoted $\mathrm{Im}(p)$. (Karoubian categories are also sometimes known as pseudo-abelian categories.)

Definition (3.1). A **Karoubian rring** $\mathcal{R}$ is a rring which is a Karoubian category, and whose sum $X \oplus Y$ is the direct sum.

An example of a Karoubian rring is the category Proj of finitely generated projective R-modules.

If $\mathcal{R}$ is a Karoubian rring and G is a group, let $[G, \mathcal{R}]$ be the Karoubian rring of G-modules in $\mathcal{R}$, that is, functors from G to $\mathcal{R}$. If X and Y are objects of $[G, \mathcal{R}]$, the R-module of morphisms $\mathcal{R}(X, Y)$ carries a natural $R[G]$-module structure, given by the formula $f^g = g^{-1} \cdot f \cdot g$.

There is a natural bifunctor $V \boxtimes X$, the external tensor product, from $[G, \mathsf{Proj}] \times [G, \mathcal{R}]$ to $[G, \mathcal{R}]$, characterized by the identity of $R[G]$-modules

$$\mathcal{R}(V \boxtimes X, Y) \cong V \otimes \mathcal{R}(X, Y).$$

For the finitely generated free module $R[G]^n$, we have

$$R[G]^n \boxtimes X = \bigoplus_{g \in G} X^{\oplus n}.$$

For general V, we realize V as the image of an idempotent p in a free module $R[G]^n$, and define $V \boxtimes X$ to be the image of the corresponding idempotent in $R[G]^n \boxtimes X$. Using the external tensor product, we may embed $[G, \mathsf{Proj}]$ into $[G, \mathcal{R}]$ by the functor $V \mapsto V \boxtimes \mathbb{1}$.

If G is a group whose order is invertible in R, the functor $(-)^G$ of G-invariants from $[G, \mathcal{R}]$ to $\mathcal{R}$ is defined by taking the image of the idempotent automorphism of $\mathcal{R}$

$$p = \frac{1}{|G|} \sum_{g \in G} g.$$

From now on, we restrict attention to groups satisfying this condition.

If H is a subgroup of G, the induction functor $\mathrm{Ind}_H^G : [H, \mathcal{R}] \to [G, \mathcal{R}]$ is defined by the formula

$$\mathrm{Ind}_H^G X = (R[G] \boxtimes X)^H.$$

Here, we use the $G \times H$-module structure of $R[G]$, where G acts on the left and H acts on the right.

The following is a generalization of the Peter-Weyl theorem to Karoubian categories.

Theorem (3.2) (Peter-Weyl). *If $\mathcal{R}$ is a Karoubian rring over a commutative ring R and G is a group whose order is invertible in R, the composition*

$$[\mathsf{G}, \mathsf{Proj}] \otimes \mathcal{R} \hookrightarrow [\mathsf{G}, \mathsf{Proj}] \otimes [\mathsf{G}, \mathcal{R}] \xrightarrow{\boxtimes} [\mathsf{G}, \mathcal{R}]$$

is an equivalence of categories.

Proof. Since the order of G is invertible in R, the group algebra $\mathsf{R}[\mathsf{G}]$ is semi-simple, and may be written

$$\mathsf{R}[\mathsf{G}] \cong \bigoplus_{\alpha} \mathrm{End}(V_\alpha) \cong \bigoplus_{\alpha} V_\alpha \otimes V_\alpha^*,$$

where we sum over the isomorphism classes of irreducible representations $\{V_\alpha\}$ of G. This permits us to rewrite the induction functor as

$$\mathrm{Ind}_{\mathsf{H}}^{\mathsf{G}} X = (\mathsf{R}[\mathsf{G}] \otimes X)^{\mathsf{H}} \cong \bigoplus_{\alpha} V_\alpha \otimes (V_\alpha^* \boxtimes X)^{\mathsf{H}}.$$

Taking $\mathsf{H} = \mathsf{G}$, and recalling that $\mathrm{Ind}_{\mathsf{G}}^{\mathsf{G}}$ is equivalent to the identity functor, we obtain the desired equivalence between $[\mathsf{G}, \mathsf{Proj}] \otimes \mathcal{R}$ and $[\mathsf{G}, \mathcal{R}]$. □

4. $\mathbb{S}$-modules in Karoubian rrings

Let $\mathbb{S}$ be the category of permutations $\coprod_{n=0}^{\infty} \mathbb{S}_n$ and let $\mathcal{R}$ be a rring. A bounded $\mathbb{S}$-module in $\mathcal{R}$ is an object $\mathcal{X}$ of

$$[\mathbb{S}, \mathcal{R}] = \bigoplus_{n=0}^{\infty} [\mathbb{S}_n, \mathcal{R}],$$

in other words, a sequence $\{\mathcal{X}(n) \mid n \geq 0\}$ of $\mathbb{S}_n$-modules in $\mathcal{R}$ such that $\mathcal{X}(n) = 0$ for $n \gg 0$. Let $\mathbb{1}_n$ denote the $\mathbb{S}$-module such that $\mathbb{1}_n(n)$ is the trivial $\mathbb{S}_n$-module, while $\mathbb{1}_n(k) = 0$ for $k \neq n$.

The category $[\mathbb{S}, \mathcal{R}]$ is itself a rring:

i) the sum of two $\mathbb{S}$-modules is $(\mathcal{X} \oplus \mathcal{Y})(n) = \mathcal{X}(n) \oplus \mathcal{Y}(n)$;

ii) the product of two $\mathbb{S}$-modules is defined using induction:

$$(\mathcal{X} \otimes \mathcal{Y})(n) = \bigoplus_{j+k=n} \mathrm{Ind}_{\mathbf{S}_j \times \mathbf{S}_k}^{\mathbf{S}_n} \mathcal{X} \otimes \mathcal{Y};$$

iii) the unit of the product is $\mathbb{1}_0$.

Denote the Grothendieck group of the rring $[\mathbb{S}, \mathcal{R}]$ by $K_0^{\mathbb{S}}(\mathcal{R})$.

There is another monoidal structure $\mathcal{X} \circ \mathcal{Y}$ on $[\mathbb{S}, \mathcal{R}]$, called plethysm. If λ is a partition of n, let $\mathbb{S}_\lambda = \mathbb{S}_{\lambda_1} \times \cdots \times \mathbb{S}_{\lambda_{\ell(\lambda)}} \subset \mathbb{S}_n$, and let $\mathsf{N}(\mathbb{S}_\lambda)$ be the normalizer of $\mathbb{S}_\lambda$ in $\mathbf{S}_n$. The quotient $W(\mathbb{S}_\lambda) = \mathsf{N}(\mathbb{S}_\lambda)/\mathbb{S}_\lambda$ may be identified with

$$\{\sigma \in \mathbf{S}_{\ell(\lambda)} \mid \lambda_{\sigma(i)} = \lambda_i \text{ for } 1 \leq i \leq \ell(\lambda)\} \subset \mathbb{S}_{\ell(\lambda)}.$$

Given bounded $\mathbb{S}$-modules $\mathcal{X}$ and $\mathcal{Y}$, we obtain an action of $\mathsf{N}(\mathbb{S}_\lambda)$ on the tensor product

$$\mathcal{X}(\ell(\lambda)) \otimes \bigotimes_{1 \leq i \leq \ell(\lambda)} \mathcal{Y}(\lambda_i).$$

Plethysm is the monoidal structure (not symmetric) defined by the formula

$$(\mathcal{X} \circ \mathcal{Y})(n) = \bigoplus_{\lambda \vdash n} \bigoplus_{k=0}^{\infty} \mathrm{Ind}_{\mathsf{N}(\mathbf{S}_\lambda)}^{\mathbf{S}_n} \left(\mathcal{X}(\ell(\lambda) + k) \otimes \bigotimes_{1 \leq i \leq \ell(\lambda)} \mathcal{Y}(\lambda_i) \otimes \mathcal{Y}(0)^{\otimes k} \right)^{\mathbf{S}_k},$$

and with unit $\mathbb{1}_1$.

Lemma (4.1). *Let $\mathcal{R}$ be a Karoubian rring over a field of characteristic zero. The Grothendieck group $K_0^{\mathbf{S}}(\mathcal{R})$ is a pre-λ-ring, with σ-operations characterized by the formula*

$$\sigma_n([\mathcal{X}]) = [\mathbb{\Pi}_n \circ \mathcal{X}],$$

where $\mathcal{X}$ is a bounded $\mathbb{S}$-module.

Proof. We must prove that for bounded $\mathbb{S}$-modules $\mathcal{X}$ and $\mathcal{Y}$,

$$\sigma_n([\mathcal{X}] + [\mathcal{Y}]) = \sum_{i=0}^{n} \sigma_i([\mathcal{X}]) \cdot \sigma_{n-i}([\mathcal{Y}]). \tag{4.2}$$

Observe that $\mathbb{\Pi}_n \circ (\mathcal{X} \oplus \mathcal{Y})$ equals

$$\bigoplus_{i=0}^{n} \bigoplus_{\lambda \vdash i} \bigoplus_{\mu \vdash n-i} \bigoplus_{j,k=0}^{\infty} \mathrm{Ind}_{\mathbf{N}(\mathbf{S}_\lambda)\times\mathbf{N}(\mathbf{S}_\mu)}^{\mathbf{S}_n} \Big(\bigotimes_{1\le i\le \ell(\lambda)} \mathcal{X}(\lambda_i) \otimes \mathcal{X}(0)^{\otimes j} \otimes \bigotimes_{1\le i\le \ell(\mu)} \mathcal{Y}(\mu_i) \otimes \mathcal{Y}(0)^{\otimes k} \Big)^{\mathbf{S}_j \times \mathbf{S}_k}$$

Since

$$\mathrm{Ind}_{\mathbf{N}(\mathbf{S}_\lambda)\times\mathbf{N}(\mathbf{S}_\mu)}^{\mathbf{S}_n} V \otimes W = \mathrm{Ind}_{\mathbf{S}_i \times \mathbf{S}_{n-i}}^{\mathbf{S}_n} \Big(\mathrm{Ind}_{\mathbf{N}(\mathbf{S}_\lambda)}^{\mathbf{S}_i} V \otimes \mathrm{Ind}_{\mathbf{N}(\mathbf{S}_\mu)}^{\mathbf{S}_{n-i}} W \Big),$$

it follows that

$$\mathbb{\Pi}_n \circ (\mathcal{X} \oplus \mathcal{Y}) \cong \bigoplus_{i=0}^{n} (\mathbb{\Pi}_i \circ \mathcal{X}) \otimes (\mathbb{\Pi}_{n-i} \circ \mathcal{Y}),$$

proving (4.2) for elements of $K_0^{\mathbf{S}}(\mathcal{R})$ of the form $[\mathcal{X}]$ and $[\mathcal{Y}]$. The definition of the sigma operations on virtual elements $[\mathcal{X}_0] - [\mathcal{X}_1]$ is now forced by (1.5):

$$\sigma_n([\mathcal{X}_0] - [\mathcal{X}_1]) = \sum_{k=0}^{\infty} \sum_{i+j_1+\cdots+j_\ell=n,\ j_\ell>0} (-1)^k \sigma_i([\mathcal{X}_0]) \sigma_{j_1}([\mathcal{X}_1]) \ldots \sigma_{j_k}([\mathcal{X}_1]). \quad \square$$

Lemma (4.3). *There is an isomorphism of λ-rings $K_0^{\mathbf{S}}(\mathsf{Proj}) \cong \Lambda$.*

Proof. The pre-λ-ring $K_0^{\mathbf{S}}(\mathsf{Proj})$ is the sum of abelian groups $K_0^{\mathbf{S}}(\mathcal{R}) = \bigoplus_{n=0}^{\infty} R(\mathbb{S}_n)$, where $R(\mathbb{S}_n) = K_0([\mathbb{S}_n, \mathsf{Proj})$ is the abelian group underlying the virtual representation ring of $\mathbb{S}_n$. The identification of $K_0^{\mathbf{S}}(\mathsf{Proj})$ with Λ is via the Frobenius characteristic $\mathrm{ch} : R(\mathbb{S}) \to \Lambda$, which sends the irreducible representation V_λ associated to the partition λ to the Schur function s_λ. The Frobenius characteristic is given by the explicit formula

$$\mathrm{ch}_n(V) = \frac{1}{n!} \sum_{\sigma \in \mathbf{S}_n} \mathrm{Tr}_V(\sigma) p_\sigma,$$

where p_σ is the monomial in the power sums obtained by taking one factor p_k for each cycle of σ of length k. For the proof that $\mathrm{ch}(\ldots)$ is a map of λ-rings, see Knutson [10] or Appendix A of Macdonald [13]. $\square$

Using these lemmas and the Peter-Weyl Theorem, we will show that $K_0^{\mathbf{S}}(\mathcal{R})$ is a λ-ring for any Karoubian rring over a field of characteristic zero. First, we prove some simple lemmas which are of interest in their own right.

Plethysm is distributive on the left with respect to sum.

Lemma (4.4). $(\mathcal{X}_1 \oplus \mathcal{X}_2) \circ \mathcal{Y} \cong (\mathcal{X}_1 \circ \mathcal{Y}) \oplus (\mathcal{X}_2 \circ \mathcal{Y})$

Proof. Clear. $\square$

It is also distributive on the left with respect to product.

Lemma (4.5). $(\mathcal{X}_1 \otimes \mathcal{X}_2) \circ \mathcal{Y} \cong (\mathcal{X}_1 \circ \mathcal{Y}) \otimes (\mathcal{X}_2 \circ \mathcal{Y})$

Proof. By Lemma (4.4), it suffices to check this formula when $\mathcal{X}_1(j) = X_1$, $\mathcal{X}_2(k) = X_2$, $\mathcal{X}_1(i) = 0$, $i \neq j$ and $\mathcal{X}_1(i) = 0$, $i \neq k$. We have

$$\big((\mathcal{X}_1 \otimes \mathcal{X}_2) \circ \mathcal{Y}\big)(n) = \bigoplus_{q=0}^{n} \bigoplus_{\substack{\lambda \vdash n \\ \ell(\lambda)+q=j+k}} \mathrm{Ind}_{N(\mathbf{S}_\lambda)}^{\mathbf{S}_n}\Big(\mathrm{Ind}_{\mathbf{S}_j \times \mathbf{S}_k}^{\mathbf{S}_{j+k}}(X_1 \otimes X_2) \otimes \bigotimes_{1 \leq i \leq \ell(\lambda)} \mathcal{Y}(\lambda_i) \otimes \mathcal{Y}(0)^{\otimes q}\Big)^{\mathbf{S}_q}.$$

But we have

$$\bigoplus_{\substack{\lambda \vdash n \\ \ell(\lambda)+q=j+k}} \Big(\mathrm{Ind}_{\mathbf{S}_j \times \mathbf{S}_k}^{\mathbf{S}_{j+k}}(X_1 \otimes X_2) \otimes \bigotimes_{1 \leq i \leq \ell(\lambda)} \mathcal{Y}(\lambda_i) \otimes \mathcal{Y}(0)^{\otimes q}\Big)^{\mathbf{S}_q} \cong \bigoplus_{p=0}^{q} \bigoplus_{i=0}^{n}$$

$$\bigoplus_{\substack{\mu \vdash i \\ \ell(\mu)+p=j}} \Big(X_1 \otimes \bigotimes_{1 \leq i \leq \ell(\mu)} \mathcal{Y}(\lambda_i) \otimes \mathcal{Y}(0)^{\otimes p}\Big)^{\mathbf{S}_q} \otimes \bigoplus_{\substack{\lambda \vdash n-i \\ \ell(\lambda)+q-p=k}} \Big(X_2 \otimes \bigotimes_{1 \leq i \leq \ell(\lambda)} \mathcal{Y}(\lambda_i) \otimes \mathcal{Y}(0)^{\otimes q}\Big)^{\mathbf{S}_q},$$

from which the lemma follows. □

Lemma (4.6). *If $\mathcal{V}$ is a bounded $\mathbb{S}$-module in* Proj *and $\mathcal{X}$ is a bounded $\mathbb{S}$-module in $\mathcal{R}$,*

$$\mathrm{ch}(\mathcal{V}) \circ [\mathcal{X}] = [\mathcal{V} \circ \mathcal{X}].$$

Proof. By Lemma (4.4), we may assume that $\mathcal{V}$ is an irreducible $\mathbb{S}_n$-module V_λ. It remains to show that $\sigma_\lambda([\mathcal{X}]) = [V_\lambda \circ \mathcal{X}]$ for all partitions λ.

By Lemma (4.5), we see that for any partition μ with $\ell = \ell(\mu)$, we have

$$\big(\mathbb{1}_{\mu_1} \otimes \ldots \otimes \mathbb{1}_{\mu_\ell}\big) \circ \mathcal{X} \cong \big(\mathbb{1}_{\mu_1} \otimes \mathcal{X}\big) \otimes \ldots \otimes \big(\mathbb{1}_{\mu_\ell} \circ \mathcal{X}\big).$$

Taking the class in $K_0^{\mathbf{S}}(\mathcal{R})$ of both sides, we see that

$$\big[\big(\mathbb{1}_{\mu_1} \otimes \ldots \otimes \mathbb{1}_{\mu_\ell}\big) \circ \mathcal{X}\big] = \sigma_{\mu_1}([\mathcal{X}]) \ldots \sigma_{\mu_\ell}([\mathcal{X}]).$$

The irreducible representation V_λ is a linear combination of representations $\mathbb{1}_{\mu_1} \otimes \ldots \otimes \mathbb{1}_{\mu_\ell}$ with integral coefficients, and by Lemma (4.3), the Schur function s_λ is a linear combination of symmetric functions $h_{\mu_1} \otimes \ldots \otimes h_{\mu_\ell}$ with the same coefficients; the proof is completed by application of Lemma (4.4). □

Theorem (4.7). *The Grothendieck group $K_0^{\mathbf{S}}(\mathcal{R})$ of a Karoubian rring $\mathcal{R}$ over a field of characteristic zero is a λ-ring.*

Proof. If $f = \mathrm{ch}(\mathcal{V})$ and $g = \mathrm{ch}(\mathcal{W})$, where $\mathcal{V}$ and $\mathcal{W}$ are bounded $\mathbb{S}$-modules in Proj, and $x = [\mathcal{X}]$, where $\mathcal{X}$ is a bounded $\mathbb{S}$-module in $\mathcal{R}$, it follows from Lemma (4.6) that

$$f \circ \big(g \circ x\big) = \mathrm{ch}\big(\mathcal{V} \circ (\mathcal{W} \circ \mathcal{X})\big) = \mathrm{ch}\big((\mathcal{V} \circ \mathcal{W}) \circ \mathcal{X}\big) = \mathrm{ch}(\mathcal{V} \circ \mathcal{W}) \circ x.$$

Since ch is a morphism of λ-rings, we see that $\mathrm{ch}(\mathcal{V} \circ \mathcal{W}) = f \circ g$, and from which we obtain the formula (1.10) characterizing λ-rings in this case:

$$f \circ (g \circ x) = (f \circ g) \circ x.$$

It only remains to extend (1.10) to virtual elements $g = \mathrm{ch}(\mathcal{W}_0) - \mathrm{ch}(\mathcal{W}_1)$ and $x = [\mathcal{X}_0] - [\mathcal{X}_1]$. Both sides of (1.10) are polynomial functions of $g \in \Lambda$ and $x \in K_0^{\mathbf{S}}(\mathcal{R})$ and hence must coincide, since they are equal on a cone with non-empty interior. □

It follows that the Grothendieck group $K_0(\mathcal{R})$ is a λ-ring, namely the sub-λ-ring of $K_0^{\mathbf{S}}(\mathcal{R})$ consisting of virtual objects such that $X(n) = 0$, $n > 0$. The Peter-Weyl Theorem now has the following consequence.

Theorem (4.8). *If $\mathcal{R}$ is a Karoubian rring over a field of characteristic zero, there is an isomorphism $K_0^S(\mathcal{R}) \cong \Lambda \otimes K_0(\mathcal{R})$ of λ-rings.*

Proof. The Peter-Weyl Theorem gives isomorphisms of rings

$$\Lambda \otimes K_0(\mathcal{R}) \xleftarrow{ch \otimes 1} K_0^S(\mathsf{Proj}) \otimes K_0(\mathcal{R}) \xrightarrow{\boxtimes} K_0^S(\mathcal{R}).$$

The first of these arrows is an isomorphism of λ-rings by Lemma (4.3). As rings, both $K_0^S(\mathsf{Proj}) \otimes K_0(\mathcal{R})$ and $K_0^S(\mathcal{R})$ are generated by $K_0(\mathcal{R})$ and $[\mathbb{1}_n]$, $n \geq 1$, and $\boxtimes$ respects the σ-operations of these elements, proving that it is a map of λ-rings. □

5. The main result

If $\mathcal{R}$ is a Karoubian rring, denote by $\mathcal{R}[\mathbb{N}]$ the Karoubian ring of bounded sequences

$$(A^0, A^1, A^2, \cdots \mid A^n = 0 \text{ for } n \gg 0).$$

The sum on $\mathcal{R}[\mathbb{N}]$ is defined by $(A \oplus Y)^n = A^n \oplus Y^n$, while the product is defined by

$$(A \otimes B)^n = \bigoplus_{i+j=n} A^i \otimes B^j.$$

Definition (5.1). A **Künneth functor** with values in the Karoubian rring $\mathcal{R}$ is a rring functor E from the Cartesian rring Var of quasi-projective varieties and open embeddings to $\mathcal{R}[\mathbb{N}]$.

In other words, a functor $E : \mathsf{Var} \to \mathcal{R}[\mathbb{N}]$ is a Künneth functor if there are natural isomorphisms

$$E^i(X \coprod Y) \cong E^i(X) \oplus E^i(Y),$$
$$E^n(X \times Y) \cong \bigoplus_{n=i+j} E^i(X) \otimes E^j(Y).$$

If $E = \{E^n\}$ is a Künneth functor, denote by $e(X)$ the associated Euler characteristic

$$e(X) = \sum_{n=0}^{\infty} (-1)^n [E^n(X)]$$

in the Grothendieck group $K_0(\mathcal{R})$.

Definition (5.2). A **Serre functor** with values in $\mathcal{R}$ is a Künneth functor E such that for any closed sub-variety Z of X,

$$e(X) = e(X \setminus Z) + e(Z).$$

If E is a Serre functor and $X = X^0 \subset X^1 \subset X^2 \subset \ldots$ is a filtered quasi-projective variety such that $X^n = \emptyset$ for $n \gg 0$, we have

$$e(\mathrm{gr}\, X) \equiv \sum_n e(\mathrm{gr}^n X) = e(X). \tag{5.3}$$

Here are two examples of Serre functors:

i) The category of mixed Hodge structures over $\mathbb{C}$ is a rring, whose Grothendieck group may be identified with the polynomial ring $\mathbb{Z}[u, v]$ by means of the Serre polynomial (0.1). The functor $E^n(X)$ which takes a quasi-projective variety X to the mixed Hodge structure $(H_c^n(X, \mathbb{C}), F, W)$ over $\mathbb{C}$ is a Serre functor. The associated characteristic $e(X)$ may be identified with the Serre polynomial.

ii) Gillet and Soulé [8] have constructed a functor to the homotopy category of chain complexes of (pure effective rational) Chow motives; let $E^n(X)$ be the nth cohomology of this complex.

If $\mathcal{R}$ is a rring, let $T : \mathcal{R} \to [\![\mathbb{S}, \mathcal{R}]\!]$ be the rring functor with $T(X, n) = X^n$. (More precisely, $T(X, n)$ is defined by induction: $T(X, 0) = \mathbb{1}$, and $T(X, n) = T(X, n-1) \otimes X$.) The following result is a generalization of Macdonald's formula [14] for the Poincaré polynomial of the symmetric power $S^n X = X^n / \mathbb{S}_n$.

Proposition (5.4). *If X is a quasi-projective variety,*

$$e(T(X)) = \mathrm{Exp}\big(p_1 e(X)\big) \in \hat{K}_0^{\mathbf{S}}(\mathcal{R}).$$

Here, $e(T(X))$ denotes the class $n \mapsto e(T(X, n))$ in the Grothendieck group $\hat{K}_0^{\mathbf{S}}(\mathcal{R})$.

Proof. Since E is a rring-functor, $E \cdot T = T \cdot E$. By the Peter-Weyl Theorem,

$$E(T(X, n)) = T[E(X)](n) = \bigoplus_{\lambda \vdash n} V_\lambda \boxtimes \big(V_\lambda^* \otimes E(X)^{\otimes n}\big)^{\mathbf{S}_n}.$$

Descending to the Grothendieck group, we see that

$$e(T(X, n)) = \bigoplus_{\lambda \vdash n} s_\lambda \otimes \sigma_\lambda(e(X)) \in \Lambda_n \otimes K_0(\mathcal{R}) \subset K_0^{\mathbf{S}}(\mathcal{R}).$$

Summing over $n \geq 0$, and applying Cauchy's formula (1.6), we see that

$$e(T(X)) = \exp\Big(\sum_{k=1}^{\infty} \frac{p_k \otimes \psi_k e(X)}{k}\Big) \in \hat{K}_0^{\mathbf{S}}(\mathcal{R}).$$

The proposition now follows by the definition of $\mathrm{Exp}(\dots)$. □

Consider the following decreasing filtration on the $\mathbb{S}$-module $T(X)$, where X is a quasi-projective variety:

$$T^i(X)(n) = \{(z_1, \dots, z_n) \in X^n \mid \{z_1, \dots, z_n\} \text{ has cardinality at most } n - i\}.$$

Let $\mathrm{gr}^i T(X) = T^i(X) \setminus T^{i+1}(X)$ be the associated graded $\mathbb{S}$-module.

Lemma (5.5). *Let Z be the object of ${}^1[\![\mathbb{S}, \mathsf{Var}]\!]$*

$$Z(n) = \begin{cases} \mathbb{A}^0, & n > 0, \\ \emptyset, & n = 0. \end{cases}$$

Then $\mathrm{gr}\, T(X) = F(X) \circ Z$; in particular, $\mathrm{gr}^0 T(X) = F(X)$.

Proof. This lemma reflects the fact that an element of $\mathrm{gr}^i T(X, n)$ determines, and is determined by, a partition of the set $\{1, \dots, n\}$ into $n - i$ disjoint subsets, together a point in $F(X, n - i)$. □

We now arrive at the main theorem of this paper.

Theorem (5.6). *Let X be a quasi-projective variety over $\mathbb{C}$. If E is a Serre functor and V_λ is an irreducible representation of $\mathbb{S}_n$,*

$$e(F(X, n), V_\lambda) = \Phi_\lambda(e(X)).$$

Proof. If E is Serre functor, (5.3) and Lemma (5.5) show that

$$e(T(X)) = e(\operatorname{gr} T(X)) = e(F(X)) \circ e(Z).$$

To calculate $e(F(X))$, we invert the operation $-\circ e(Z)$ on $\hat{K}_0^{\mathbf{S}}(\mathcal{R})$. Indeed, $e(Z) = \operatorname{Exp}(p_1) - 1$ and by Lemma (2.2),

$$\begin{aligned} e(F(X)) &= e(F(X)) \circ \big(\operatorname{Exp}(p_1) - 1\big) \circ \Big(\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log(1 + p_n)\Big) \\ &= e(T(X)) \circ \Big(\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log(1 + p_n)\Big). \end{aligned}$$

By Proposition (5.4), this equals

$$\begin{aligned} \exp\Big(\sum_{k=1}^{\infty} \frac{p_k \cdot \psi_k e(X)}{k}\Big) \circ \Big(\sum_{\ell=1}^{\infty} \frac{\mu(\ell)}{\ell} \log(1 + p_\ell)\Big) &= \exp\Big(\sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} \frac{\mu(\ell)}{k\ell} \log(1 + p_{k\ell}) \psi_k e(X)\Big) \\ &= \exp\Big(\sum_{n=1}^{\infty} \sum_{d|n} \frac{\mu(n/d)}{n} \log(1 + p_n) \psi_d e(X)\Big), \end{aligned}$$

from which the theorem follows by extracting the coefficient of the Schur function s_λ on both sides. □

The concise formulation

$$e(F(X)) = \operatorname{Exp}(\operatorname{Log}(1 + p_1)\, e(X))$$

of this result makes the resemblance with the formula $e(T(X)) = \operatorname{Exp}(p_1\, e(X))$ clearer.

In the special cases $\lambda = (n)$ or $\lambda = (1^n)$, when Φ_λ is given by the explicit formula of Corollary (2.8), we obtain the following corollary.

Corollary (5.7). *If* $e(X) = \sum_{p,q} h_{pq} u^p v^q$ *is the Serre polynomial of* X, *then*

$$\begin{aligned} \sum_{n=0}^{\infty} t^n\, e(F(X,n)/\mathbb{S}_n) = \frac{\sigma_t(X)}{\sigma_{t^2}(X)} &= \prod_{p,q=0}^{\infty} \left(\frac{1 - t^2 u^p v^q}{1 - t u^p v^q}\right)^{h_{pq}}, \\ \sum_{n=0}^{\infty} t^n\, e(F(X,n), \varepsilon) = \sigma_{-t}(X)^{-1} &= \prod_{p,q=0}^{\infty} (1 + t u^p v^q)^{h_{pq}}. \end{aligned}$$

For example, if $X = \mathbb{C}$, $F(\mathbb{C}, n)/S\mathbb{S}_n$ is the classifying space $K(B_n, 1)$ of the braid group B_n on n strands. Our formula becomes in this case

$$\sum_{n=0}^{\infty} t^n\, e(F(\mathbb{C}, n)/\mathbb{S}_n) = \frac{1 - t^2 uv}{1 - tuv} = 1 + tL + t^2(L^2 - L) + t^3(L^3 - L^2) + \ldots,$$

reflecting the isomorphism of rational cohomology groups $H^\bullet(B_n, \mathbb{Q}) \cong H^\bullet(\mathbb{G}_m, \mathbb{Q})$ as mixed Hodge structures.

6. The Fulton-MacPherson compactification

Fulton and MacPherson [5] have introduced a sequence of functors $X \mapsto X[n]$ from Var to $[\mathbb{S}_n, \mathrm{Var}]$, with the following properties.

i) If X is projective, then so is $X[n]$.

ii) There is natural transformation of functors $F(X, n) \hookrightarrow X[n]$, which is an embedding.

iii) The complement $X[n] \setminus F(X)$ is a divisor with normal crossings.

In this section, we calculate the equivariant Serre polynomial $e(X[n])$. Denote by $FM(X)$ the functor $X \mapsto (n \mapsto X[n])$ from Var to $[\![\mathbb{S}, \mathrm{Var}]\!]$,

(6.1). **Trees and $\mathbb{S}$-modules.** Let $\Gamma(n)$, $n \geq 2$, be the set of isomorphism classes of labelled rooted trees with n leaves, such that each vertex has at least two branches. It is easily seen that $\Gamma(n)$ is finite: in fact, the generating function

$$x + \sum_{n=2}^{\infty} \frac{x^n |\Gamma(n)|}{n!} \tag{6.2}$$

is the inverse under composition of $x - x^2 - x^3 - x^4 - \dots$.

Given a tree $T \in \Gamma(n)$, denote by $\mathrm{Vert}(T)$ the set of vertices of T; given a vertex $v \in \mathrm{Vert}(T)$, denote by $n(v)$ the valence of v (its number of branches). Given a tree $T \in \Gamma(n)$ and an $\mathbb{S}$-module $\mathcal{V}$ in the rring $\mathcal{R}$, let $\mathcal{V}(T)$ be the object

$$\mathcal{V}(T) = \bigotimes_{v \in \mathrm{Vert}(T)} \mathcal{V}(n(v)), \tag{6.3}$$

and let $\mathbb{T}\mathcal{V}(n)$ be the $\mathbb{S}_n$-module

$$\mathbb{T}\mathcal{V}(n) = \bigoplus_{T \in \Gamma(n)} \mathcal{V}(T).$$

Thus, $\mathbb{T}$ is a functor from ${}^2[\![\mathbb{S}, \mathcal{R}]\!]$ to itself. (Recall that ${}^2[\![\mathbb{S}, \mathcal{R}]\!]$ is the full subcategory of $\mathbb{S}$-modules such that $X(0) = X(1) = 0$.)

A proof of the following formula for $\mathcal{R} = \mathrm{Proj}$ may be found in [7]; however, the same proof works in general. Observe that this theorem may be used to prove (6.2).

Theorem (6.4). *The elements*

$$f = h_1 - \sum_{n=2}^{\infty} [\mathcal{V}] \quad \textit{and} \quad g = h_1 + \sum_{n=2}^{\infty} [\mathbb{T}\mathcal{V}]$$

of $\check{K}_0^{\mathbb{S}}(\mathcal{R})$ *satisfy the formula* $f \circ g = g \circ f = h_1$.

(6.5). **The varieties $\overset{\circ}{P}_k(n)$.** The algebraic groups $\mathbb{C}^k$ and $\mathbb{G}_m$ act on the affine space $\mathbb{C}^k$ by translation and dilatation respectively; by functoriality, these actions extend to $F(\mathbb{C}^k, n)$. Denote by $G_k = \mathbb{C}^k \rtimes \mathbb{G}_m$ the semidirect product of these groups, and by $\overset{\circ}{P}_k(n)$, $n > 1$, the quotient of the configuration space $F(\mathbb{C}^k, n)$ by the free G_k-action. This action is $\mathbb{S}_n$-equivariant, and $\overset{\circ}{P}_k(n)$ is a smooth $\mathbb{S}_n$-variety of dimension $nk - k - 1$. For example, $\overset{\circ}{P}_k(2)$ is naturally isomorphic to the projective space $\mathbb{CP}^{k-1}$, with trivial $\mathbb{S}_2$-action.

Proposition (6.6).

$$e\big(\overset{\circ}{P}_k(n), \mathbb{S}_n\big) = \frac{e(F(\mathbb{C}^k, n), \mathbb{S}_n)}{e(\mathbb{C}^k)\, e(\mathbb{G}_m)} = \frac{e(F(\mathbb{C}^k, n), \mathbb{S}_n)}{L^k(L-1)}$$

Proof. We start with a lemma.

Lemma (6.7). *Let G be an algebraic group and P be a G-torsor with base $X = P/G$. If the projection $P \to X$ is locally trivial in the Zariski topology, $e(P) = e(G)\, e(X)$.*

Proof. We stratify X by locally closed subvarieties X_i of codimension i over which the torsor P is trivial. The strata are chosen inductively: X_{-1} is empty, while X_i is a Zariski-open subset of $X \setminus X_{i-1}$ over which P is trivial. The formula follows, since

$$e(P) = \sum_i e(P_i) = \sum_i e(G)\, e(X_i). \quad \square$$

The action of $\mathbb{C}^k$ on $F(\mathbb{C}^k, n)$ is not just locally, but globally, trivial: a global section is given by $(z_1, \ldots, z_n) \mapsto (z_1 - \bar{z}, \ldots, z_n - \bar{z})$, where $\bar{z} = \frac{1}{n}\sum_{i=1}^n z_i$. On the other hand, any free action of $\mathbb{G}_m$ on a variety is locally trivial in the Zariski topology: free actions with quotient X are classified by $H^1(X_{fl}, \mathbb{G}_m)$, locally trivial free actions with quotient X are classified by $H^1(X, \mathbb{G}_m)$, and these two groups are isomorphic by Hilbert's Theorem 90 (see Proposition XI.5.1 of Grothendieck [9]). $\square$

(6.8). **Stratification of $FM(X)$.** The $\mathbb{S}$-variety $\overset{\circ}{P}_k$ has a natural compactification to a smooth projective $\mathbb{S}$-variety P_k, which has a natural stratification. The strata are labelled by trees $T \in \Gamma(n)$, and the stratum associated to T is isomorphic to $\overset{\circ}{P}_k(T)$, in the notation of (6.3). It follows from Theorem (6.4) that $e(P_k)$ is the inverse of

$$\begin{aligned} h_1 - e\big(\overset{\circ}{P}_k\big) &= p_1 - \frac{\prod_{n=1}^\infty (1+p_n)^{\frac{1}{n}\sum_{d|n}\mu(n/d)L^{kd}} - 1 - L^k p_1}{L^k(L-1)} \\ &= \frac{L^{k+1}p_1 + 1 - \prod_{n=1}^\infty (1+p_n)^{\frac{1}{n}\sum_{d|n}\mu(n/d)L^{kd}}}{L^k(L-1)}. \end{aligned}$$

under plethysm.

The importance of the spaces $P^k(n)$ comes from the following result of Fulton and MacPherson.

Proposition (6.9). *The $\mathbb{S}$-module $FM(X)$ has a filtration such that*

$$\operatorname{gr} FM(X) \cong F(X) \circ P_k.$$

Since $X[n]$ is a projective Q-variety (it has singularities which are quotients of affine space by a finite group), $e(FM(X))(n)$ equals the $\mathbb{S}_n$-equivariant Hodge polynomial of $X[n]$. The above proposition shows that $e(FM(X)) = e(F(X)) \circ e(P_k)$, and leads to a practical algorithm for the calculation of the $\mathbb{S}_n$-equivariant Hodge numbers of $X[n]$.

On forgetting the action of the symmetric groups $\mathbb{S}_n$, we recover the formula of Fulton and Macpherson for the Poincaré polynomials of $FM(X, n)$, in a form stated by Manin [16]. On replacing h_n by $x^n/n!$, we obtain

$$1 + \sum_{n=1}^\infty x^n\, e(X[n]) = (1+x)^{e(X)} \circ \left(\frac{L^{k+1}x + 1 - (1+x)^{L^k}}{L^k(L-1)}\right)^{-1}$$

In this formula, we may take the limit $L \to 1$ using L'Hôpital's rule, obtaining a formula for the Euler characteristic of $FM(X, n)$:

$$1 + \sum_{n=1}^\infty x^n \chi(X[n]) = (1+x)^{\chi(X)} \circ \big((k+1)x - k(1+x)\log(1+x)\big)^{-1}.$$

The one dimensional case has special interest, since the spaces $\overset{\circ}{P}_1(n)$ and $P_1(n)$ are naturally isomorphic to the moduli spaces $\mathcal{M}_{0,n+1}$ and $\overline{\mathcal{M}}_{0,n+1}$; this isomorphism comes about because the translations and dilatations in one dimension generate the isotropy group of the point $\infty \in \mathbb{CP}^1$ with respect to the action of the group $PSL(2, \mathbb{C})$. This

identification means that the action of $\mathbb{S}_n$ on these spaces is the restriction of an action of $\mathbb{S}_{n+1}$. We have calculated the $\mathbb{S}_{n+1}$-equivariant Serre polynomials of these spaces in [6]; in a sequel to this paper, we calculate the $\mathbb{S}_n$-equivariant Serre polynomial of $\overline{\mathcal{M}}_{1,n}$.

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