Abelian Coverings of the Complex Projective Plane Branched Along Configurations of Real Lines

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Introduction

The basic topological invariants for classifying smooth complex projective surfaces are the Kodaira dimension κ, Chern numbers c_1^2 and c_2, and the first Betti number b_1. From these one can compute many other invariants, such as: the Euler characteristics, both topological and of the structure sheaf, the topological and geometric genus, and all the Betti numbers and Hodge numbers. An invariant depending on the algebraic structure of the surface is the Picard number, the rank of the Neron-Severi group, or subgroup of the second integral homology group generated by algebraic cycles.

Surfaces with $\kappa = -\infty, 0, 1$ have been thoroughly investigated and their Chern numbers and Betti numbers are well-understood [**B-P-V**]. For the large remaining class with $\kappa = 2$, surfaces of "general type," relatively little is known about which Chern numbers and Betti numbers occur, aside from the famous result due to Miyaoka [**My**] and Yau [**Y**], which states that for these surfaces

$$c_1^2 \leq 3c_2$$

and equality holds if and only if the surface is uniformized by the unit ball [**Y**].

In this paper we focus on surfaces introduced by Hirzebruch [**Hz**], which are minimal smooth models $\widehat{X}$ of certain abelian coverings X of $\mathbf{P}^2$ branched along configurations of lines $\mathcal{L}$. The branched coverings X are completions over $\mathbf{P}^2$ of the unbranched covering over $\mathbf{P}^2 - \mathcal{L}$ defined by the canonical map

$$\pi_1(\mathbf{P}^2 - \mathcal{L}) \to \mathrm{H}_1(\mathbf{P}^2 - \mathcal{L}; \mathbf{Z}/n\mathbf{Z})$$

for some integer $n \geq 2$. We will call $\widehat{X}$ the *Hirzebruch covering* associated to $\mathcal{L}$ and n. The coverings, see [**B-H-H**], are also referred to as Kummer coverings. When $\mathcal{L}$ contains triple or higher intersection points, X is singular, but $\widehat{X}$ itself can often be realized as a smooth abelian branched covering of a blowup of $\mathbf{P}^2$.

Hirzebruch's construction produces many examples of surfaces of general type, with the advantage that their Chern numbers are easy to calculate in terms of the simple combinatorics of the line configuration and the degree of the covering. In addition to these formulas, Hirzebruch also gives the Enriques-Kodaira classification of these surfaces. His results lead him to exhibit three examples of surfaces of general type satisfying $c_1^2 = 3c_2$, the extremal case of the Miyaoka-Yau inequality. As yet, however, there is no simple formula for the first Betti number or the Picard number of these surfaces.

The main result of this paper is an algorithm for computing the first Betti number of Hirzebruch coverings associated to configurations of real lines. We have developed a computer implementation of our algorithm and applied it to a large number of examples.

The study of the first Betti number for branched coverings goes back to work of Zariski for cyclic coverings [**Za1**], [**Za2**]. A recent extension of his work was achieved by Libgober [**Li1**]. Our basic approach is similar to that of Zariski and Libgober, in that we focus on the topology of the complement of the branch locus and the relation between the branched and unbranched parts.

In [**I**] Ishida outlines an algebraic method for computing the first Betti numbers, which works for any configuration of lines (not necessarily real). The method is described in a more general setting by Esnault in [**Es**]. Ishida develops an effective algorithm for the case when there are at most 12 triple intersections and none of higher order, and using this he explicitly computes b_1 for the three known Hirzebruch coverings satisfying $c_1^2 = 3c_2$. Our computation agrees with his on the example which is associated to a real line configuration.

Our algorithm has two main steps: the first is to find the first Betti number of the unbranched part of the covering and the second is to find the nullity of the intersection matrix of the curves above the branch locus. It can be shown (see Proposition I.6.3) that the difference between these two numbers equals b_1.

The first step in the algorithm uses classical methods which depend only on the topology of the embedding of the branch locus in $\mathbf{P}^2$ and the action of the group G of covering automorphisms. First, by constructing a fibration, computing its monodromy (see III.2, III.3) and using techniques developed by Zariski and van Kampen **[K]** (see also **[C]**), we find a presentation for the fundamental group of the complement of the branch locus in $\mathbf{P}^2$. Then using Fox calculus **[Fo1]** we construct a presentation matrix for the first homology group $\mathrm{H}_1(\widehat{X};\mathbf{Z})$ as a $\mathbf{Z}[G]$-module. Computing the rank of this group over $\mathbf{Z}$ is facilitated by a result of Libgober **[Li4]**.

The second step relies on new techniques which we develop in Chapter II. Given a smooth abelian branched covering $\rho : X \to Y$ and smooth curves C in the base space Y with normal crossings and certain other conditions, we give an intersection formula for the curves in the preimage of C. The formula requires some basic *lifting data* for the curves above each irreducible component of C. We show how to find such data for the branch loci of Hirzebruch coverings(III.4). Computing the rank of the resulting intersection matrix finishes the algorithm.

As an added bonus the rank of the intersection matrix gives a lower bound for another interesting invariant of smooth surfaces: the Picard number. This is the rank of the Neron-Severi group, i.e., the image of the divisors on $\widehat{X}$ in $H_2(\widehat{X};\mathbf{Z})$.

The Picard number has a natural upper bound, the Hodge number $h^{1,1}$, which can be computed from the first Betti number and the Chern numbers. Sometimes the two bounds agree, allowing us to find the Picard number exactly (see Chapter V for examples).

This paper is organised as follows. Chapter I contains background material on branched coverings. The key result of this paper is in Chapter II, which is concerned with intersection formulas for curves on smooth abelian covering surfaces. These are applied in Chapter III to give techniques for an algorithm to compute the first Betti number of Hirzebruch coverings associated to configurations of real lines. The actual steps of the algorithm are set down in Chapter IV. The final Chapter contains specific examples of Hirzebruch coverings. The numerical invariants were calculated using a computer program implementing the algorithm of Chapter IV.

It has recently been shown in [**Ho**], in answer to a question posed by Sarnak [**Sa**], that the first Betti number of Hirzebruch surfaces is "polynomial periodic". That is, if $\widehat{X}_n$ is the branched covering associated to

$$\pi_1(\mathbf{P}^2 - \mathcal{L}) \to \mathrm{H}_1(\mathbf{P}^2 - \mathcal{L}; \mathbf{Z}/n\mathbf{Z}),$$

then there exist polynomials $p_1, \ldots, p_N$ so that

$$b_1(\widehat{X}_n) = p_i(n) \qquad \text{whenever} \qquad n \equiv i (\operatorname{mod} N).$$

Put together with Zuo's results in [**Zu**], one sees that the polynomials p_i must all be of degree equal to one minus the maximum number of lines coming together. Furthermore, the polynomials only differ by at most a constant.

This suggests that the sequence of polynomials thus associated to $\mathcal{L}$ could be useful as isotopy invariants of the embedding of $\mathcal{L}$ in $\mathbf{P}^2$. In fact, the results mentioned above hold in slightly more generality than just for lines in $\mathbf{P}^2$. It seems possible that polynomial periodicity holds for arbitrary hypersurfaces on a smooth variety. The current problem is that the polynomials, even for the case of lines in $\mathbf{P}^2$ are difficult to compute.

This paper is based on my Ph.D. thesis at Brown University. I would like to thank Bill Fulton, who introduced me to the subject of branched coverings, my advisor Alan Landman, Anatoly Libgober, Curt McMullen and Dave Roberts for helpful comments and suggestions. Final editing of this paper and some new computations of examples were carried out at the Max-Planck-Institute in Bonn. During my stay there, this paper has benefited greatly from helpful conversations with Prof. F. Hirzebruch and others as well as from the excellent computer facilities.

Chapter I. Preliminaries

In this chapter we discuss some basic properties of branched coverings that are applied in the later chapters. The main objects we deal with in this thesis are abelian branched coverings of the complex projective plane and its blowups. Our emphasis is on the topology and combinatorics of these coverings. Because most related expositions on branched coverings are either algebraic and don't directly describe the basic topology, or are purely topological and don't deal specifically with branched coverings of complex varieties, some results which are essentially well known are restated and reproven here.

Section I.1 contains a definition of branched coverings of smooth varieties and an explanation of how to construct new branched coverings from old ones by blowing up the base space and pulling back. I.2 deals with basic topological properties of branched coverings, the unbranched part of the covering and fundamental groups. The most important result of this section is Proposition I.2.11: a homotopy lifting theorem for branched coverings. This result is applied in III.3 to find crucial combinatorial data about the preimage of the branch locus.

In I.3 we describe the stabilizer and inertia subgroups of the Galois group associated to subvarieties of the base space of regular coverings. These are the main tools for studying the geometry of the branched covering in terms of the geometry of the base space. In I.4 we show how to find generators for these subgroups. We give a criterion for the existence of singularities on the branched covering of a smooth surface in terms of the inertia and stabilizer subgroups in I.5. Finally, in I.6, we discuss some basic properties of the first Betti number and Picard number of a smooth surface. The key result, Proposition I.6.3, states that the difference between the first Betti number of the branched and unbranched parts of a covering equals the nullity of the intersection matrix of curves above the branch locus (assuming that the branch curves support an ample divisor.)

Most of the material in this section is well known and can be found in the literature. Possible exceptions are Proposition I.2.11 and Proposition I.6.3. The latter was communicated to me by Anatoly Libgober and Alan Landman. Some general sources on branched coverings of algebraic varieties are [**A**], [**Gr2**], [**Na**], [**Se**] and Chapter XIII and its appendix in [**Za4**]. We also use [**Mu**] and [**Ha**] for basic results from algebraic geometry. In describing the topology of branched coverings we use some of Fox's formulations [**Fo3**] and, for facts about topological unbranched coverings, we often refer to [**Ma**].

Before beginning, here is some basic terminology. In this thesis a *variety* is irreducible and complex projective; *curves* and *surfaces* are varieties of dimensions 1 and 2. *Hypersurfaces* are subvarieties of codimension 1. Varieties are given the *strong*, or *analytic*, topology.

I.1 Branched coverings of smooth varieties

This section gives the algebreo-geometric definition of branched coverings and associated definitions and results.

I.1.1 Definition. A *branched covering* $\rho : X \to Y$ is a finite surjective morphism between normal varieties. Let G be the group of isomorphisms $\alpha : X \to X$ so that $\rho(\alpha(x)) = \rho(x)$ for all x in X. G is called the group of *covering automorphisms* of the covering. If G acts transitively on all fibers of $\rho : X \to Y$, then the covering is called *Galois* or *regular*. In this case G is also referred to as the *Galois group* of the covering. An *abelian* covering is a branched covering which is Galois and has abelian Galois group. A branched covering $\rho : X \to Y$ is called *smooth* if X is smooth.

I.1.2 Remark. For any branched covering

$$\rho : X \to Y$$

there is a finite extension of function fields

$$\rho^* : \mathbb{C}(Y) \hookrightarrow \mathbb{C}(X)$$

given by $\rho^*(f) = f \circ \rho$ for $f \in \mathbb{C}(Y)$.

Conversely, given a smooth variety Y with function field $K = \mathbb{C}(Y)$ and a finite field extension

$$i : K \hookrightarrow L$$

there is a branched covering (unique up to an isomorphism commuting with the covering maps)

$$\rho : X \to Y$$

so that $\rho^* = i$.

The surface X is the normalization (see **[Ha]**, p. 23, for definition) of Y in L and ρ is its projection to Y.

I.1.3 Definition. Let $\rho : X \to Y$ be a branched covering and let $\sigma : \widehat{Y} \to Y$ be a birational morphism. This induces an isomorphism $\sigma^* : \mathbb{C}(Y) \to \mathbb{C}(\widehat{Y})$. Let $\widehat{\rho} : \widehat{X} \to \widehat{Y}$ be the branched covering associated to the field extension

$$\mathbb{C}(\widehat{Y}) \xrightarrow{(\sigma^{-1})^*} \mathbb{C}(Y) \xhookrightarrow{\rho^*} \mathbb{C}(X).$$

We call $\widehat{\rho} : \widehat{X} \to \widehat{Y}$ the *pullback branched covering* of the branched covering $\rho : X \to Y$ over $\sigma : \widehat{Y} \to Y$, since it is the minimal branched covering of $\widehat{Y}$ making the following diagram commute:

$$\begin{tikzcd} \widehat{X} \arrow[r, "\widehat{\sigma}"] \arrow[d, "\widehat{\rho}"'] & X \arrow[d, "\rho"] \\ \widehat{Y} \arrow[r, "\sigma"] & Y. \end{tikzcd}$$

I.2 The topology of branched coverings

In this section we give a topological definition of branched coverings and give some properties. We conclude by showing how the algebraic and topological definitions given in I.1 and this section are related.

Throughout this section assume that all topological spaces are locally path connected, semi-locally simply connected and Hausdorff and all maps between topological spaces are continuous.

We start with a topological definition of finite branched coverings following Fox [**Fo3**].

I.2.1 Definition. Let $\rho : X \to Y$ be a surjective map between topological spaces and suppose that the following properties hold:

(1) the topology on X is generated by connected components of preimages of open sets in Y;

(2) there is a dense open subset $Y^u \subset Y$ so that $X^u = \rho^{-1}(Y^u)$ is connected and dense in X and for $\rho^u = \rho|_{X^u}$

$$\rho^u : X^u \to Y^u$$

is a finite unbranched covering;

(3) the topology on Y has a basis of open sets whose intersections with Y^u are path connected (in other words, Y^u is *locally-connected in* Y);

(4) for any point $p \in Y$, there is a fundamental system of neighborhoods V of p so that each component of $\rho^{-1}(V)$ contains a single point in the fiber $\rho^{-1}(p)$;

(5) X^u is locally-connected in X.

Then $\rho : X \to Y$ is called a *finite topological branched covering* over Y. If Y^u is chosen to be maximal, then $B = Y - Y^u$ is called the *branch locus* of the covering and $\rho^u : X^u \to Y^u$ is called the *unbranched part* of the covering. A *topological covering automorphism* is a homeomorphism of X to itself preserving fibers of ρ.

Note that if $\rho : X \to Y$ is a finite topological branched covering U is an open subset of Y and $V \subset \rho^{-1}(U)$ is a connected component, then $\rho|_V : V \to U$ is also a finite topological branched covering.

I.2.2 LEMMA. *For any $\rho : X \to Y$ satisfying properties (1), (2), (3) and (4) of branched coverings, Y has a basis of open sets V so that, for any connected component U of $\rho^{-1}(V)$, $\rho(U) = V$ and the connected components of $\rho^{-1}(V)$ form a basis for the topology of X.*

Proof. By property (3) we can choose a basis of open sets V for Y so that $V \cap Y^u$ are path connected for all V. Property (1) implies that the connected components of $\rho^{-1}(V)$ form a basis for the topology of X.

Take any connected component U of $\rho^{-1}(V)$. Since, by property (2), X^u is dense in X, there is a point q_1 in $\rho(U \cap X^u)$. Let $p_1 = \rho(q_1)$. Take any p_2 in $V \cap Y^u$. Then there is a path γ from p_1 to p_2 in $V \cap Y^u$. By property (2) and the path lifting theorem for unbranched coverings γ lifts to any connected component of $\rho^{-1}(V \cap Y^u)$. Since $V \cap Y^u$ is contained in V, any connected component of $\rho^{-1}(V \cap Y^u)$ must be either contained in U or disjoint from U. Since $q_1 \in U$, there is a path lift of γ with endpoint q_1 which is contained in U. Therefore, there is a point q_2 in U so that $\rho(q_2) = p_2$. Thus, $\rho(U \cap X^u)$ maps onto $V \cap Y^u$. Since Y^u is dense in Y, $\rho(U)$ contains a dense subset of V.

To show that $\rho(U) = V$, take any point $p \in V$. For any neighborhood V_p of p in V, $\rho^{-1}(V_p)$ has a connected component U_p which intersects U. Therefore, U_p must be contained U. By property (4) U_p must contain at least one point in $\rho^{-1}(p)$. ∎

I.2.3 COROLLARY. *Any map $\rho : X \to Y$ satisfying properties (1), (2), (3) and (4) of branched coverings is open and proper.*

Proof. By property (1) and Lemma I.2.2 there is a basis of open sets V for Y so that the connected components U of $\rho^{-1}(V)$ form a basis of open sets for X and $\rho(U) = V$. Therefore, ρ is open.

By properties (2) and (4) Y has a basis of open sets V so that the number of connected components of $\rho^{-1}(V)$ is finite. To see that ρ is proper, take any compact subset F of Y. We need to show that $\rho^{-1}(F)$ is compact. By Lemma I.2.2 for any open covering of $\rho^{-1}(F)$ there is a refinement $\{U_\alpha\}$ so that for each α there is an open set V in Y so that U_α is a connected component of $\rho^{-1}(V)$ and $\rho(U_\alpha) = V$. Thus, we may assume that for some open covering $\{V_\beta\}$ of F, $\{U_\alpha\}$ consists of all the connected components of $\rho^{-1}(V_\beta)$ where V_β range over sets in this covering. The $\{V_\beta\}$ form an open covering of F so there is a finite subcovering. By taking the connected components of the preimages of these sets we get a finite subcovering of $\rho^{-1}(F)$. ∎

If we remove property (5) from Definition I.2.1, topological branched coverings have the following functorial property. This lemma will be used later in Proposition I.2.11, to prove the homotopy lifting theorem for branched coverings.

I.2.4 LEMMA. *Let $\rho : X \to Y$ be a continuous surjective map between topological spaces satisfying properties (1), (2), (3) and (4) of branched coverings, with branch locus B and let*

$$f : Z \to Y$$

be a continuous map from a space Z so that $f(Z)-B$ is dense and locally-connected in $f(Z)$. Let X_Z be the topological fiber product of Z and X over Y and ρ_Z the projection of X_Z on Z. Then ρ_Z satisfies properties (1), (2), (3) and (4) of branched coverings.

Proof. First, recall that

$$X_Z = \{(x,z) \in X \times Z | \rho(x) = f(z)\}.$$

Property (3) for ρ_Z follows from the hypothesis.

To prove (1), recall also that the topology on X_Z is the one induced by the product topology on $X \times Z$. Thus, given any point $(x,z) \in X_Z$ and neighborhood U', there is a smaller neighborhood $U \subset U'$ so that U is the intersection of $V \times W$ with X_Z, where V is a neighborhood of x in X and W is a neighborhood of z in Z. Since, by Corollary I.2.3, ρ is an open map, we can assume that $\rho(V)$ is open. Furthermore, since

$$(V \times W) \cap X_Z = (V \times W') \cap X_Z,$$

where W' is the largest subspace of W so that $f(W') \subset \rho(V)$, we can assume without loss of generality that $f(W) \subset \rho(V)$ and $\rho_Z(U) = W$.

Finally, since the topology on X is generated by connected components of preimages of open sets in Y, we can assume V is a connected component of $\rho^{-1}(\rho(V))$.

Let $V_1, V_2, \ldots, V_k$ be the connected components of $\rho^{-1}(\rho(V))$. Then $\rho_Z^{-1}(W)$ is contained in the disjoint union of intersections of $V_1 \times W, \ldots, V_k \times W$ with X_Z. Since U is connected and equals one of these sets, U must be a connected component of $\rho_Z^{-1}(W)$. This proves property (1) for ρ_Z.

By property (4) for ρ, by choosing V small enough we can assume each of the connected components $V_1, \ldots, V_k$ contains only one point in the fiber $\rho^{-1}(x)$. Thus, $\rho_Z^{-1}(W)$ is a disjoint union of open sets each containing a single point in the fiber $\rho^{-1}(z)$. This proves property (4) for ρ_Z.

To prove property (2) we need to show that ρ_Z restricted to $X_Z \cap (X^u \times Z)$ is an unbranched covering. For this it suffices to show that ρ_Z is a local homeomorphism. Let (x, z) be any point and U a neighborhood in $X_Z \cap (X^u \times Z)$. Assume without loss of generality that U is the intersection $X_Z \cap (V \times Z)$ where V is an open set in X and ρ is a homeomorphism when restricted to V. Then ρ_Z is a homeomorphism when restricted to U. ∎

In his paper [**Fo3**], Fox shows that unbranched coverings can be canonically completed to branched coverings.

I.2.5 THEOREM. *Given a finite unbranched covering*

$$\rho^u : X^u \to Y^u,$$

with X^u connected, and an imbedding $Y^u \hookrightarrow Y$ whose image is dense, there is a unique branched covering in the topological sense

$$\rho : X \to Y$$

whose unbranched part is $\rho^u : X^u \to Y^u$.

Note that without property (3) the uniqueness would not hold.

Let us recall some facts about topological unbranched coverings. Let Y^u be a connected, locally pathwise connected, semi-locally simply connected space. Then there is a bijection between unbranched coverings of Y^u and conjugacy classes of subgroups of $\pi_1(Y^u, y)$ which takes $\rho^u : X^u \to Y^u$ to the conjugacy class of the subgroup $\rho_*(\pi_1(X^u, x))$ in $\pi_1(Y^u, y)$ ([**Ma**], Theorem 10.2, p. 175). The covering is regular if and only if $\rho_*(\pi_1(X^u, x))$ is a normal subgroup of $\pi_1(Y^u, y)$ ([**Ma**], Lemma 8.1, p. 164.) In this case there is a canonical surjective map

$$\psi : \pi_1(Y^u, y) \to G,$$

where G is the group of covering automorphisms. Under this map a loop $\gamma \in \pi_1(Y^u, y)$ goes to the unique covering automorphism taking any point $x \in \rho^{-1}(y)$ to the endpoint of the lift of γ at x ([**Ma**], Theorem 7.2, p. 162.) The kernel of the map ψ equals $\rho^u_*(\pi_1(X^u, x))$ ([**Ma**], Corollary 7.4, p. 163.) It follows that abelian regular unbranched coverings lie in one-to-one correspondence with surjective maps

$$\phi : \mathrm{H}_1(Y^u, \mathbf{Z}) \to G$$

where G is an abelian group, since ψ must factor through the Hurewicz map

$$h : \pi_1(Y^u) \to \mathrm{H}_1(Y^u, \mathbf{Z})$$

taking loops to their homology classes, whose kernel is the commutator subgroup of $\pi_1(Y^u, y)$.

I.2.6 Definition. We call $\phi : \mathrm{H}_1(Y^u, \mathbf{Z}) \to G$ the *defining map* of the unbranched covering and canonically associated branched covering.

The next two lemmas hold generally for topological branched coverings.

I.2.7 LEMMA. *If $\rho : X \to Y$ is a topological branched covering with unbranched part $\rho^u : X^u \to Y^u$, then the natural map from the group of covering automorphisms of ρ to that of ρ^u, given by restriction, is an isomorphism onto.*

Proof. Let G be the group of topological configurations of ρ^u. Any $g \in G$ extends to a covering automorphism on X as follows. Let $p \in Y$ and $q \in \rho^{-1}(p)$. Let V be a neighborhood of p in Y so that the connected components of $\rho^{-1}(V)$ each contain a single point in $\rho^{-1}(p)$. Let W be the connected component of $\rho^{-1}(V)$ containing q whose intersection with X^u is connected. Let W_g be the connected component of $\rho^{-1}(V)$ containing $g(W \cap X^u)$ and define $g(q)$ to be the intersection of W_g with the fiber $\rho^{-1}(p)$.

By this definition, the extension of g is fiber preserving. To see that the extension is a homeomorphism, it suffices to show that g is an open map at each point $q \in X$. This follows from Corollary I.2.3. ∎

I.2.8 COROLLARY. *If the group G of covering automorphisms of a topological branched covering $\rho : X \to Y$ acts transitively on fibers in the unbranched part, then it acts transitively on all fibers.*

Proof. Let p be any point in Y. To see that G acts transitively on the fiber $\rho^{-1}(p)$, let V be a neighborhood of p so that the connected components of $\rho^{-1}(V)$ each contain a single point in the fiber $\rho^{-1}(p)$. Since G acts transitively on fibers in the unbranched part X^u of the covering, which is dense in X, G must act transitively on the connected components of $\rho^{-1}(V)$. Since each of these components contains a single point in the fiber $\rho^{-1}(p)$, G must also act transitively on the fiber $\rho^{-1}(p)$. ∎

We conclude this section with a result analogous to the homotopy lifting theorem for unbranched coverings.

I.2.9 Definition. Given a topological branched covering $\rho : X \to Y$ and a map $f : \Gamma \to Y$, a continuous map

$$h : [0,1] \times \Gamma \to Y,$$

such that (setting $h_t(\gamma) = h(t,\gamma)$) $h_0(\gamma) = f(\gamma)$ and $h_t(\gamma) \in Y - B$ for all $t > 0$ is called a *homotopy* of Γ off B.

I.2.10 Definition. Let $f : \Gamma \to Y$ be a continuous map and let

$$f' : \Gamma \to X$$

be any map so that $\rho(f'(\gamma)) = f(\gamma)$ for all γ in Γ. We call f' a *lifting map* for f.

Recall the following basic result from the theory of unbranched coverings. (See [**Ma**], Theorem 5.1, p. 156.) Let $f : \Gamma \to Y$ be a map between topological spaces (recall they must be locally connected) and suppose $\rho : X \to Y$ is a topological unbranched covering. If

$$f_*(\pi_1(\Gamma, *)) \subset \rho_*(\pi_1(X, *))$$

both considered as subgroups of $\pi_1(Y, *)$, then there is a lifting $f' : \Gamma \to X$. We generalize this to branched coverings in the following proposition.

I.2.11 PROPOSITION. *Let $f : \Gamma \to Y$ be a continuous map from any connected space Γ into Y and suppose there exists a homotopy*

$$h : [0,1] \times \Gamma \to Y$$

of Γ off B so that

$$(h_1)_*\pi_1(\Gamma, *) \subset (\rho^u)_*\pi_1(X^u, *)$$

as subgroups of $\pi_1(Y^u, *)$. Then there is a continuous lifting map

$$f' : \Gamma \to X$$

for f.

Before proving this we prove a lemma.

I.2.12 LEMMA. *Let Z be a connected space and $f : Z \to Y$ be any map so that $f(Z) - B$ is locally connected in $f(Z)$. Suppose there is a dense open connected subspace $U \subset Z$ which is locally connected in Z so that the restriction of f to U has a lift $f' : U \to X$. Then we can extend f' to a lifting on all of Z.*

Proof. Consider the topological fiber product $X_Z = X \times_Y Z$. Then by Lemma I.2.4 the projection $\rho_Z : X_Z \to Z$ satisfies properties (1),(2),(3) and (4) of branched coverings.

Let Z' be the closure of the preimage U' of the graph of f' in X_Z. We claim that ρ_Z restricted to Z' is a homeomorphism onto Z. Since, by Proposition I.2.3, ρ_Z is an proper mapping, it suffices to show that ρ_Z is a bijection from Z' to Z. Since ρ_Z sends closed sets to closed sets $\rho_Z(Z')$ contains the closure of U in Z, which is all of Z, so ρ_Z is onto. To see that it is one-to-one, take any $p \in Z$. By Lemma I.2.2, there is a connected open neighborhood V_p of p in Z so that any connected component of $\rho_Z^{-1}(V_p)$ maps onto V_p and contains a single point in the fiber $\rho_Z^{-1}(p)$. Suppose there are two points q_1 and q_2 in $\rho^{-1}(p) \cap Z'$. Then there are two distinct connected components W_1 and W_2 in $\rho_Z^{-1}(V_p)$ which intersect Z'. But, since U' is dense in Z', W_1 and W_2 must also intersect U'. This contradicts the fact that ρ_Z is one-to-one on U'.

Now, by composing the inverse of ρ_Z restricted to Z' with projection to X we obtain a lift of f on all of Z. ∎

Proof of Proposition I.2.11. We have

$$\begin{aligned} h_*\pi_1((0,1] \times \Gamma, *) &= (h_1)_*\pi_1(\Gamma, *) \\ &\subset (\rho^u)_*\pi_1(X^u, *), \end{aligned}$$

so there is a lifting map

$$h' : (0,1] \times \Gamma \to X^u$$

so that $\rho(h'(t,\gamma)) = h(t,\gamma)$ for $t \in (0,1]$ and $\gamma \in \Gamma$.

Let $Z = [0,1] \times \Gamma$, $U = (0,1] \times \Gamma$ and $f = h$. Then the rest follows from Lemma I.2.12. ∎

We now end this section by describing the relation between topological and algebraic branched coverings.

The following theorems were proven by Zariski in the 1930's.

I.2.13 THEOREM. *Let $\rho : X \to Y$ be a finite surjective morphism between normal varieties. Then, considered as a map between topological spaces, ρ is a topological branched covering.*

I.2.14 THEOREM. *The branch locus of a branched covering $\rho : X \to Y$ is either empty or a subvariety of Y of pure codimension one.*

The properties in Definition I.2.1 follow from the "fundamental openness theorem" (see for example [**Mu**], p. 43) and the unibranch property of normal surface, sometimes known as Zariski's main theorem. Zariski's paper on the "purity of the branch locus" [**Za3**] gives a proof of Theorem I.2.14.

Property (5) of topological branched coverings is analogous to the condition that branched coverings of varieties must be normal.

The following theorem is analogous to Lemma I.2.5.

I.2.15 THEOREM. *Let Y be a normal variety and B a finite union of proper subvariety of pure codimension one. Given a topological unbranched covering $\rho^u : X^u \to Y - B$, with X^u connected, there exists an irreducible normal variety X with a finite surjective morphism $\rho : X \to Y$ and a homeomorphism $s : X^u \to \rho^{-1}(Y - B)$ such that $\rho(x) = \rho^u(s(x))$ for all $x \in X^u$.*

This is a generalization of the Riemann-Enriques Existence Theorem [**En**], proved by Grauert and Remmert [**G-R**]. See Grothendieck's work [**Gr1**] for further generalizations. The statement given here is taken from Serre's introduction to his survey [**Se**]. Since normalizations are unique, there is only one branched covering $\rho : X \to Y$ over Y associated to an unbranched covering $\rho^u : X^u \to Y^u$, where Y^u is the complement of a finite union of subvarieties of codimension 1 in Y.

I.2.16 LEMMA. *Any topological covering automorphism of $\rho : X \to Y$ is an isomorphism from X to itself considered as a variety.*

Proof. Let σ be a topological covering automorphism. Since ρ is a local isomorphism on the unbranched part X^u, σ is an isomorphism from X^u to itself. The fact that σ extends to an isomorphism of X to itself follows from a weaker version of the Theorem I.2.15. ∎

I.3 Inertia and stabilizer subgroups

In order to translate from combinatorial data of branched coverings to geometric data about the covering space it is useful to study the actions of special subgroups of the group of covering automorphisms. Let $\rho : X \to Y$ be any branched covering with branch locus B and with group of covering automorphisms G.

I.3.1 Definition. For any subvariety W of X, the subgroup I_W of G defined by

$$I_W = \{g \in G \quad | \quad g(x) = x \quad \text{for all } x \in W\}$$

is called the *inertia subgroup* of W and the subgroup defined by

$$H_W = \{g \in G \quad | \quad g(x) \in W \quad \text{for all } x \in W\}$$

is called the *stabilizer subgroup* of W.

I.3.2 Remark. If the covering is regular, then the inertia subgroups (respectively, stabilizer subgroups) for different components of $\rho^{-1}(Z)$, where Z is a subvariety of Y, are conjugate. If the covering is also abelian, then conjugate subgroups are equal and we can define I_Z and H_Z for subvarieties of Y to be the inertia and stabilizer subgroup for any irreducible component of $\rho^{-1}(Z)$. In this case a subvariety Z of X is in the branch locus if and only if I_Z is nontrivial.

Hereafter, assume $\rho : X \to Y$ is abelian.

I.3.3 Lemma. *For any subvariety Z in Y, I_Z is the subgroup of G generated by elements of I_W for all irreducible components W of B containing V.*

Proof. Let S be the subgroup of G generated by I_W for all hypersurfaces W containing V, where $W \subset B$. Whenever Z is contained in W, any automorphism which fixes all points in $\rho^{-1}(W)$ must fix points in $\rho^{-1}(Z)$, so we have $I_W \subset I_Z$ and hence $S \subset I_Z$.

Conversely, suppose we take the quotient of the covering space X by S. The quotient covering

$$\overline{\rho} : X/S \to Y$$

is a branched covering and G/S equals its group of covering automorphisms. The new inertia subgroup $\overline{I_W}$ for W is the image of the original inertia subgroup I_W in G/S for any subvariety W of Y. Since $\overline{I_W}$ is trivial for all hypersurfaces W of Y with $V \subset W$, X/S is not branched over any hypersurface W containing Z. By Zariski's "purity of the branch locus," this implies that X/S is not branched over Z itself and therefore the image of I_Z in G/S is trivial. In other words, I_Z is contained in S. ▮

I.4 Generators for inertia and stabilizer subgroups

In this section we use some simple local topology to find special elements of the inertia and stabilizer subgroups of a curve C in the branch locus of an abelian branched covering over a smooth surface Y. These generate I_C and H_C when C is simply connected.

First, we study the more general case when Y can have any dimension.

I.4.1 Definition. Let B be a finite union of codimension-1 subvarieties of Y and let V be any irreducible component of B. For any smooth point p of B contained in V, let D be the unit complex disk and let

$$j : D \hookrightarrow Y$$

be an analytic embedding intersecting B transversally at p with $j(0) = p$. Note that for fixed p this is well defined up to homotopy. Let μ_p be the path defined by

$$\theta \mapsto j(e^{i\theta}), \qquad \text{for } 0 \leq \theta \leq 2\pi.$$

We will call μ_p the ***positively-oriented meridianal loop***, or just ***positive loop*** around V at p.

I.4.2 PROPOSITION. *Any two positive loops μ_p and μ_q around an irreducible component V of B at smooth points p and q of B on V are homologically equivalent in $Y - B$.*

Proof. For the case $p = q$ see Definition I.4.1. Assume $p \neq q$. Let Σ be the singular points of B and let γ be a path from p to q on $V - \Sigma$. (One exists since Σ is a proper subvariety of V, hence codimension one in an irreducible variety, so its complement is path connected.) Since $V - \Sigma$ is a smooth submanifold of B and γ is contained in $V - \Sigma$, γ has a tubular neighborhood $T(\gamma)$ in Y. That is, there is a (real disk bundle) $T(\gamma) \to \gamma$ and an embedding $T(\gamma) \hookrightarrow Y$ so that the zero section maps to γ, the rest lies in in $Y - B$ and the fibers over the endpoints p and q of γ equal the loops μ_p and μ_q. The boundary of the image of the S^1 bundle sitting inside the image of $T(\gamma)$ equals the difference between μ_p and μ_q. ∎

We will hereafter denote by μ_V the class in $\mathrm{H}_1(Y - B; \mathbf{Z})$ of a positively-oriented meridianal loop around V.

The following is a standard fact about topological unbranched covering (see, for example, [**Ma**], Proposition II.1, p. 177).

I.4.3 PROPOSITION. *Let $\rho : X \to Y$ be any topological unbranched covering, $V \subset Y$ a connected subset and W a connected component of $\rho^{-1}(V)$. Then the restriction of ρ_* to W is an unbranched covering map and for the inclusion map $i : V \hookrightarrow Y$ and any $w \in W$ we have*

$$\rho_*(\pi_1(W, w)) = i_*^{-1}(\rho_*(\pi_1(X, w))),$$

both considered as subgroups of $\pi_1(V, \rho(w))$.

I.4.4 COROLLARY. *If $V \subset Y$ is any hypersurface not contained in the branch locus B, then the stabilizer subgroup H_V equals*

$$\phi(i_* \mathrm{H}_1(V - B; \mathbf{Z}))$$

where i_* is induced by the inclusion $i : V - B \hookrightarrow Y - B$ and $\phi : \mathrm{H}_1(Y - B; \mathbf{Z}) \to G$ is the defining map for the covering.

Proof. Let $\psi : \pi_1(Y - B, v) \to G$ be the defining map for the covering (taking v to be in $V - B$). Then $\psi = \phi \circ h$, where h is the Hurewicz map.

Let W be any irreducible component of $\rho^{-1}(V)$. We need to show that the kernel of $\psi \circ i_*$ equals $\pi_1(W - \rho^{-1}(B), *)$. Since ψ is the defining map for $\rho : X \to Y$, the kernel of ψ equals $\rho_*(\pi_1(X - \rho^{-1}(B), w)$, where we may take the basepoint w to be in $W - \rho^{-1}(B)$ intersected with the fiber $\rho^{-1}(v)$. Thus, the kernel of $\phi \circ i_*$ equals $i_*^{-1}(\rho_*(\pi_1(X-\rho^{-1}(B), w)))$, which equals $\pi_1(W-\rho^{-1}(B), w)$, by Proposition I.4.3. ∎

I.4.5 PROPOSITION. *Given any irreducible component $V \subset B$, I_V is generated by $g_V = \phi(\mu_V)$.*

Proof. By Lemma I.3.3, if $p \in V - \overline{(B-V)}$, then $I_p = I_V$. For a small enough ball U centered at p, $\pi_1(U - B)$ is generated by a loop whose image under the Hurewicz map is homologically equivalent to μ_V. Thus, g_V generates the subgroup of G which stabilizes $U - B$. Since U can be taken to be arbitrarily small and G acts continuously, I_p must be generated by g_V. ∎

I.4.6 PROPOSITION. *For any point $p \in B$, I_p is generated by g_V for all irreducible components V in B passing through p.*

Proof. By Lemma I.3.3, I_p is generated by elements of I_V where V ranges over all irreducible components of B containing p. The rest follows from Proposition I.4.5. ∎

We now concentrate on the case that Y is a smooth surface. Let $C \subset Y$ be a curve not contained in the branch locus B. Let $p \in C \cap B$ and let $U \subset Y$ be a small ball around p in Y so that p is the only singular point on $U \cap (C \cup B)$. We will find special elements of H_C given as the images of the composition of maps

$$\mathrm{H}_1(U \cap C - p; \mathbf{Z}) \xrightarrow{j_*} \mathrm{H}_1(C - B; \mathbf{Z}) \xrightarrow{\phi \circ i_*} G,$$

where j_* is induced by the inclusion $j : U \cap C - p \to C - B$.

Consider $\partial U \cap C$, where ∂U, the boundary of U, is isomorphic to a 3-sphere S^3. Then $C \cap \partial U$ is a finite union of homeomorphic images of the circle S^1 oriented by the complex structure, so the inclusion of $C \cap \partial U$ in ∂U defines an oriented link L_C with components $K_1, \ldots, K_t$. These components lie in one-to-one correspondence with connected components of $(U-p)\cap C$. The closures $b_1,\ldots,b_t$ of these components are called the *branches* of C at p.

The following theorem can be found in [**Br**], Theorem 14, pp. 440-441.

I.4.7 THEOREM. *Let C and D be two analytic curves (not necessarily irreducible) defined in a complex disk U with origin p, and assume p is the only singular point of $C\cup D$. Let L_C and L_D be the intersections of C and D with ∂U thought of as oriented links on a 3-sphere. Then the intersection multiplicity $I_p(C,D)$ equals the linking number $\mathrm{lk}(L_C,L_D)$.*

I.4.8 COROLLARY. *The image of the composition of maps*

$$\mathrm{H}_1(U\cap(C-B);\mathbf{Z}) \xrightarrow{j_*} \mathrm{H}_1(C-B;\mathbf{Z}) \xrightarrow{i_*} \mathrm{H}_1(Y-B;\mathbf{Z}) \xrightarrow{\phi} G$$

is generated by elements of G of the form

$$\sum_{D\subset B} I_p(b,D)g_D,$$

where b ranges over branches of C at p and the sum is over curves D in B.

Proof. We have a commutative diagram

$$\begin{tikzcd}
\mathrm{H}_1(U\cap(C-B);\mathbf{Z}) \arrow[r, "j_*"] \arrow[d, "\alpha"'] & \mathrm{H}_1(C-B;\mathbf{Z}) \arrow[d, "i_*"'] \\
\mathrm{H}_1(U\cap(Y-B);\mathbf{Z}) \arrow[r, "\beta"] & \mathrm{H}_1(Y-B;\mathbf{Z})
\end{tikzcd}$$

where all maps are induced by inclusion. We will find the image of $\beta\circ\alpha$. Let L_B be the oriented link in ∂U given by $\partial U\cap B$. Then the pair

$$(\partial U\cap(C-B),\partial U\cap(Y-B))=(L_C,\partial U-L_B)$$

is a deformation retract of

$$(U\cap(C-B),U\cap(Y-B))$$

(see [Mi], Theorem 2.10, p. 18). Therefore, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{H}_1(L_C;\mathbf{Z}) & \longrightarrow & \mathrm{H}_1(\partial U - L_B;\mathbf{Z}) \\ \downarrow & & \downarrow \\ \mathrm{H}_1(U \cap (C-B);\mathbf{Z})) & \longrightarrow & \mathrm{H}_1(U \cap (Y-B);\mathbf{Z}) \end{array}$$

where all maps are induced by inclusions and vertical maps are isomorphisms. We will find the image of $\beta \circ \alpha$.

Let $K_1, \ldots, K_t$ be the oriented connected components of L_C. For each ℓ, the image of K_ℓ in $\mathrm{H}_1(\partial U - L_B;\mathbf{Z})$ equals, by definition of linking number (see [R], p. 132),

$$\sum_{D \subset B \cap U} \mathrm{lk}(K_\ell, L_D)\mu_D.$$

By Theorem I.4.7, $\beta \circ \alpha(K_\ell)$ equals

$$\sum_{D \subset B} I_p(b_\ell, D)\mu_D.$$

Applying the map ϕ we have

$$(\phi \circ \beta \circ \alpha)(K_i) = \sum_{D \subset B} I_p(b_\ell, D) g_D.$$

Since $K_1, \ldots, K_t$ generate $\mathrm{H}_1(L_C;\mathbf{Z})$, the elements described above generate their image under $\phi \circ \beta \circ \alpha$. ∎

I.4.9 PROPOSITION. *If C is a curve in Y not in the branch locus B, then H_C contains the elements of G of the form*

$$(*) \qquad \sum_{D \subset B} I_p(b, D) g_D$$

where p ranges over points in $C \cap B$ and b ranges over branches of C at p. If C is smooth and rational, then these elements generate H_C.

Proof. By Corollary I.4.8, the elements $(*)$ are in the image of

$$\phi(i_*(\mathrm{H}_1(C - B;\mathbf{Z})).$$

By Corollary I.4.4, they are elements of H_C. If C is smooth and rational then $\mathrm{H}_1(C - B; \mathbf{Z})$ is generated by the images of the maps

$$\mathrm{H}_1(U_p \cap (C - B); \mathbf{Z}) \to \mathrm{H}_1(C - B; \mathbf{Z})$$

induced by inclusions, where U_p are small balls around points $p \in C \cap B$. Therefore, the elements of the form $(*)$ generate H_C. ∎

I.4.10 PROPOSITION. *If C is a curve in B, then H_C contains g_C and elements in G of the form*

$$\sum_{D \subset \overline{(B-C)}} I_p(b, D) g_D,$$

where p ranges over points in $C \cap \overline{(B - C)}$ and b ranges over branches of C at p. If C is smooth and rational then these generate H_C.

Proof. Consider the covering

$$\begin{array}{c} X/I_C \\ \bar{p}\big\downarrow \\ Y \end{array}$$

obtained by taking the quotient of X by the action of I_C. The new branch locus $\overline{B}$ no longer contains C and, by Proposition I.4.9, the stabilizer subgroup $\overline{H_C}$ associated to C in the new covering contains elements in G of the form

$$\sum_{D \subset \overline{(B-C)}} I_p(b, D) \overline{g}_D,$$

where p ranges over points in $C \cap \overline{(B - C)}$, b ranges over branches of C at p and $\overline{g}_D$ is the image of g_D under the quotient map

$$G \to G/I_C.$$

These generate H_C if C is smooth and rational. Since $\overline{H_C}$ is the image of H_C under the quotient map and g_C generates I_C by Proposition I.4.5, the result follows. ∎

Finally, we describe the effect of pullbacks of branched coverings of Y over blowups of Y on the inertia and stabilizer subgroups.

I.4.11 PROPOSITION. *Let $\sigma : \widehat{Y} - Y$ be the blowup of Y at a point $p \in B$ and let $\widehat{\rho} : \widehat{X} \to \widehat{Y}$ be the pullback covering over $\widehat{Y}$ of $\rho : X \to Y$. Then the branch locus of the new covering $\widehat{\rho}$ consists of the proper transforms $\widehat{C}$ of curves C in B and possibly the exceptional set E_p. The inertia and stabilizer subgroups of each $\widehat{C}$ are the same as those for C. The inertia subgroup for E_p is generated by*

$$\sum m_p(C) g_C,$$

where $m_p(C)$ is the multiplicity of C at p and the sum is taken over all curves C in B containing p. The stabilizer subgroup is generated by $m_p(C)g_C$, where C ranges over all curves in B passing through p.

Proof. The coverings $\widehat{\rho} : \widehat{X} \to \widehat{Y}$ and $\rho : X \to Y$ agree on their unbranched part and hence so does the Galois group action.

Let C be any curve in Y, let $C_0 = C - \overline{B - C} - \mathrm{Sing}(C)$ and let $q \in C_0$ be chosen generically. Then, by Lemma I.3.3, $I_q = I_C$. Let $\widehat{q} \in \widehat{Y}$ be a point so that $\sigma(\widehat{q}) = q$. Let C_1 be the connected component of $\rho^{-1}(C_0)$ containing $\widehat{q}$. The restriction ρ_C of ρ to C_1 is a branched covering, since it is a finite morphism and it has Galois group G/I_C. Since q was chosen generically, ρ_C is unbranched near q. Therefore, the inertia subgroup of $\widehat{q}$ with respect to ρ_C is trivial and hence must be I_C with respect to ρ. Thus, the inertia subgroup $I_{\widehat{C}}$ equals the inertia subgroup I_C. It follows that only proper transforms of curves in B and possibly the exceptional curve, lie in the branch locus of $\widehat{\rho}$ by Remark I.3.2.

Let

$$\phi : \mathrm{H}_1(Y - B; \mathbf{Z}) \to G$$

be the defining map for ρ, let $\widehat{B}$ be the total transform of curves in B, and

$$\widehat{\phi} : \mathrm{H}_1(\widehat{Y} - \widehat{B}; \mathbf{Z}) \to G$$

be the defining map for $\widehat{\rho}$, where $\widehat{B}$ is the total transform of curves in B. Then $\widehat{\phi} = \phi \circ \sigma_*$, where

$$\sigma_* : \mathrm{H}_1(\widehat{Y} - \widehat{B}; \mathbf{Z}) \to \mathrm{H}_1(Y - B; \mathbf{Z})$$

is the isomorphism induced by σ.

By Corollary I.4.4, we know $H_{\widehat{C}}$ equals

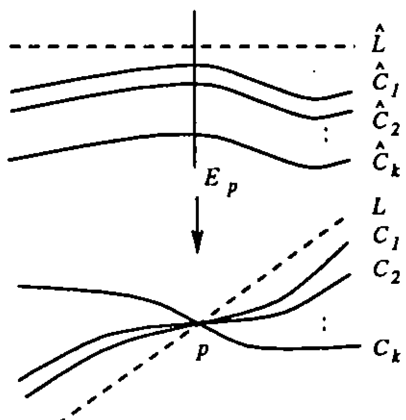
$$\widehat{\phi}(i_* \mathrm{H}_1(\widehat{C} - \widehat{B}; \mathbf{Z}))$$

where $i : \widehat{C} - \widehat{B} \to \widehat{Y} - \widehat{B}$ is the inclusion map. This equals

$$\phi(\sigma_* i_* \mathrm{H}_1(\widehat{C} - \widehat{B}; \mathbf{Z})) = \phi(j_* \mathrm{H}_1(C - B; \mathbf{Z})),$$

where $j : C - B \to Y - B$ is inclusion and the latter equals H_C, also by Corollary I.4.4. Therefore, H_C equals $H_{\widehat{C}}$.

We now find the stabilizer subgroup of the exceptional set E_p. Let $C_1, \ldots, C_t$ be curves in B passing through p. Let $\widehat{\gamma}$ be a loop on $\widehat{Y}$ around E_p, with image γ in Y and let $\gamma_1, \ldots, \gamma_t$ be loops on Y around $C_1, \ldots, C_t$, respectively. We can assume that γ lies on some line $\widehat{L}$ on $\widehat{Y}$ intersecting E_p at a general point. Projecting this line to Y, we obtain a new line L passing through p in general position with respect to the branch curves near p. By assumption γ lies on L.



Let U be a neighborhood of p isomorphic to a disk with center p. Assume U is small enough so that the intersections of branch curves $C_1, \ldots, C_t$ with the boundary ∂U are equal to $L_{p,C_1}, \ldots, L_{p,C_t}$ and $\partial U \cap L$ is homotopic to γ. Then in ∂U, γ is homotopic to

$$\sum_{s=1}^{t} \mathrm{lk}(L_{p,C_s}, \gamma)\gamma_s = \sum_{s=1}^{t} I_p(C_s, C)\gamma_s$$
$$= \sum_{s=1}^{t} m_p(C_s)\gamma_s.$$

The last equality comes from the fact that L is in general position with respect to the C_i. By Proposition I.4.5, it follows that I_{E_p} is generated by

$$\Sigma m_p(C) g_C,$$

where the sum is over curves C in B passing through p.

Applying Corollary I.4.10 to E_p, which is isomorphic to $\mathbf{P}^1$ and is simply connected, H_{E_p} is generated by

$$m_p(C) g_C,$$

where C ranges over curves in B passing through p. ∎

I.5 Criterion for the smoothness of coverings

In this section we give a criterion for an abelian covering $\rho : X \to Y$ over a smooth surface to be smooth in terms of conditions on the branch locus and the inertia subgroups associated to its irreducible components. Assume that the curves in the branch locus B are smooth and intersect in normal crossings. Given any branched covering one can construct one satisfying this hypothesis by taking a sequence of pullback coverings over blowups of the singularities in the branch locus.

I.5.1 PROPOSITION. *The covering surface X is smooth if and only if whenever two curves C and D in the branch locus intersect, the inertia subgroups I_C and I_D intersect only in the identity element.*

Proof. To study smoothness we need to look locally. Take any $p \in Y$. If p is not in the branch locus, then for any q in the fiber $\rho^{-1}(p)$, ρ is locally an analytic isomorphism near q. Since, in particular, Y is smooth at p, q must also be a smooth point of X.

We will now assume p is a point in B. Let U be a small ball around p isomorphic to a complex disk, so that, for any two distinct points q_1 and q_2 in $\rho^{-1}(p)$, the connected components V_1 and V_2 of $\rho^{-1}(U)$ containing q_1 and q_2 don't intersect. (See property (4) of Definition I.2.1.)

Suppose p lies on a single irreducible component C of B. By choosing U smaller if necessary, we can find complex coordinates x and y on U so that $U \cap B$ is given by the equation $x = 0$. For any q in $\rho^{-1}(p)$, let V_q be the connected component of $\rho^{-1}(U)$. Then the restriction of ρ to V_q is a branched covering over U branched along $U \cap B$.

The fundamental group of $U - B$ is isomorphic to $\mathbf{Z}$, so V_q must be a cyclic branched covering of U branched along B. By uniqueness of branched coverings, V_q must also be isomorphic to a complex disk and ρ restricted to V_q must be of the form

$$(x, y) \mapsto (x^k, y).$$

Suppose p is a point on the intersection of two curves C and D in B. We will show that any point q in $\rho^{-1}(p)$ is smooth if and only if $I_C \cap I_D$ contains only the identity element.

Again, take U small enough so that $U \cap B$ equals $U \cap (C \cup D)$. Choose complex coordinates x and y on U so that $U \cap C$ is given by the equation $x = 0$ and $U \cap D$ by $y = 0$.

Let V_q be the connected component of $\rho^{-1}(U)$ containing q. Since the restriction ρ_q of ρ to V_q is a branched covering (by the remark after Definition I.2.1) the isomorphism class of V_q is determined by the exact sequence

$$(*) \qquad 0 \to \pi_1(V_q - \rho^{-1}(B)) \xrightarrow{\rho_{q*}} \pi_1(U - B) \xrightarrow{\phi_q} G_q \to 0,$$

where G_q equals covering automorphisms defined on V_q. From Proposition I.4.3 it follows that one can consider G_q as the subgroup of G generated by g_C and g_D and, by Proposition I.3.3, it equals I_p.

Let r, s, t be nonnegative integers so that

$$rg_C + sg_D = 0 \qquad \text{and} \qquad tg_D = 0$$

generate the relations in G_q. Since G_q is finite, $r, t > 0$. We can also assume without loss of generality that $s < t$.

The numbers r and t are uniquely determined by the above and, if we assume also that s is minimal, then s is also determined. Note that $s = 0$ if and only if $I_C \cap I_D = (0)$.

Now q is a smooth point of X if and only if V_q is isomorphic to a complex disk. (Recall that by Definition I.2.1 the topology on X, in this case the complex topology, is generated by components of preimages of open sets in Y.) Thus, we need to classify all branched coverings of the complex disk to itself branched along $x = 0$ and $y = 0$.

All analytic maps from the disk to itself which are unbranched coverings over the complement of $x = 0$ and $y = 0$ can be put in the form

$$(x, y) \mapsto (x^a y^b, x^c y^d).$$

By a change of coordinates, one can write this as

$$(x, y) \mapsto (x^r, x^s y^t)$$

where $r, t > 0$ and $0 \leq s < t$. If we choose s to be minimal then r, s, t are uniquely determined. The map defines a branched covering (i.e. is finite) if and only if $s = 0$.

The unbranched part of this map is given by the same exact sequence as $(*)$. Thus, q is a smooth point of X if and only if $s = 0$. ∎

I.6 The first Betti number and the Picard number of a smooth surface

Let $\rho : X \to Y$ be a branched covering over a smooth surface Y and let $\sigma : \widehat{X} \to X$ be a desingularization. In this section we make a few remarks concerning two numerical invariants of $\widehat{X}$: the first Betti number and the Picard number. We show that if the branch locus supports an ample divisor, the problem of finding the first Betti number breaks up into two parts. One is finding the first Betti number of the unbranched part and the other is finding the nullity of the intersection matrix for curves in $\widehat{X}$ above the branch locus. The rank of this matrix gives a lower bound for the Picard number of $\widehat{X}$.

The first Betti number b_1 is the rank of the first homology group $\mathrm{H}_1(\widehat{X};\mathbf{Z})$ of $\widehat{X}$. It is not hard to see that the first Betti number (and, in fact, the fundamental group of a smooth variety) is invariant under blowing up. This follows by van Kampen's theorem and the fact that, topologically, blowing up consists of replacing a simply connected set with simply connected boundary in X (a ball around the point to be blown up) by another simply connected set with simply connected boundary (isomorphic to a 2-disk fiber bundle over $\mathbf{P}^1$).

I.6.1 Definition. Two divisors C and D on $\widehat{X}$ are said to be *numerically equivalent* if $C.H = D.H$ for all divisors H on $\widehat{X}$. The *Picard number* p is the rank of the group of divisors on $\widehat{X}$ modulo numerical equivalence, or, equivalently, the rank of the *Neron-Severi group* of $\widehat{X}$.

I.6.2 Remark. Any divisor determines a class in $\mathrm{H}_2(\widehat{X};\mathbf{Q})$, and intersections of divisors generalizes to intersections of homology 2-cycles. The intersection pairing on $\mathrm{H}_2(\widehat{X};\mathbf{Q})$ is nondegenerate by Poincaré duality, since $\widehat{X}$ is a compact 4-manifold. By the Hodge Index theorem, if a divisor D has the property that $D.H = 0$ for all divisors H on $\widehat{X}$, then considering D as an element of $\mathrm{H}_2(\widehat{X};\mathbf{Q})$, $D.Z = 0$ for any homology 2-cycle Z. Therefore, p can also be thought of as the rank of the image of the group of divisors on $\widehat{X}$ in $\mathrm{H}_2(\widehat{X};\mathbf{Q})$.

Since the first Betti number of the unbranched part of the covering $b_1(X^u)$ can be computed using essentially topological methods (which we describe in detail in Chapter III), our goal here is to find the difference $b_1(X^u) - b_1(\widehat{X})$. More generally we will show how the first Betti number of any smooth surface changes when one removes arbitrary unions of curves that support an ample divisor. The result applies to the difference between the first Betti numbers of branched and unbranched coverings as long as the branch locus supports an ample divisor, since then the preimage of the branch locus will also.

The following proposition was suggested to me by A. Libgober and A. Landman.

I.6.3. PROPOSITION. *If X is any smooth surface and $\mathcal{C}$ is a finite union of curves on X so that some linear combination defines an ample divisor on X, then*

$$b_1(X) = b_1(X - \mathcal{C}) - \mathrm{Null}(\mathcal{C}),$$

where $\mathrm{Null}(\mathcal{C})$ *is the nullity of the intersection matrix of* $\mathcal{C}$ *in* X.

Proof. Consider the exact homology sequence of the pair $(X, X - \mathcal{C})$

$$\begin{aligned} &\mathrm{H}_2(X) \to \mathrm{H}_2(X, X - \mathcal{C}) \\ \to \mathrm{H}_1(X - \mathcal{C}) \to &\mathrm{H}_1(X) \to \mathrm{H}_1(X, X - \mathcal{C}) \end{aligned}$$

with rational coefficients.

Intersection gives nondegenerate pairings

$$\mathrm{H}_k(\mathcal{C}) \times \mathrm{H}_{4-k}(X, X - \mathcal{C}) \to \mathbf{Q}$$

and

$$\mathrm{H}_k(X) \times \mathrm{H}_{4-k}(X) \to \mathbf{Q}$$

for $k = 0, 1, 2, 3, 4$.

In particular, the dual of $\mathrm{H}_1(X, X - \mathcal{C})$ is isomorphic to $\mathrm{H}_3(\mathcal{C})$ which is trivial since $\mathcal{C}$ is a 2-complex. So, $\mathrm{H}_1(X, X - \mathcal{C})$ is trivial. Therefore, the difference $b_1(X - \mathcal{C}) - b_1(X)$ equals the dimension of the cokernel of the map

$$r : \mathrm{H}_2(X) \to \mathrm{H}_2(X, X - \mathcal{C}),$$

or, equivalently, the nullity of the dual map

$$r^* : \mathrm{H}_2(X, X - \mathcal{C})^* \to \mathrm{H}_2(X)^*.$$

Let $i : \mathcal{C} \to X$ be the inclusion map. The intersection pairing also shows that, since $\mathrm{H}_2(\mathcal{C})$ is generated by cycles $[C]$ representing the irreducible components of $\mathcal{C}$, $\mathrm{H}_2(X, X - \mathcal{C})^*$ is generated by $[C]^*$, where

$$r^*[C]^*\alpha = [C]^*r(\alpha) = [C].r(\alpha) = i_*[C].\alpha$$

for all $\alpha \in H_2(X)$.

It follows that the kernel of r^* consists of $[D]^*$, where D is any linear combination of curves C in $\mathcal{C}$, such that

$$i_*[D].\alpha = 0,$$

for all $\alpha \in \mathrm{H}_2(X)$. In particular, for all curves C in $\mathcal{C}$,

$$i_*[D].i_*[C] = 0.$$

Thus, the rank of $\ker(r^*)$ is at most the nullity of the intersection matrix for curves in $\mathcal{C}$.

To see the equality, suppose D is a linear combination of curves in $\mathcal{C}$, and

$$i_*[D].i_*[C] = 0$$

for all curves C in $\mathcal{C}$. Then $i_*[D].i_*[D] = 0$ and, since $\mathcal{C}$ supports an ample divisor H, $i_*[D].H = 0$. By the Hodge index theorem this implies $i_*[D]$ is numerically equivalent to zero. Therefore, $i_*[D].\alpha = 0$ for all $\alpha \in \mathrm{H}_2(X)$ and $[D]^* \in \ker(r^*)$. ∎

Chapter II. Intersections of curves in abelian covering surfaces

In this chapter all branched coverings $\rho : X \to Y$ are smooth abelian branched coverings over a smooth surface Y. The goal here is to describe intersections of curves on X in terms of intersections in the base space Y and the action of the Galois group G. The results are applied later in Chapter III to find the intersection matrix for the preimage of the branch locus.

We use topological properties of branched coverings described in section I.2 and properties of the intertia and stabilizer subgroups defined in I.3. Because in our applications the curves we deal with are smooth and intersect in normal crossings and thus distinct curves lying above the same curve in the base space are disjoint (as we see in Lemma II.3.2), the problem of finding intersections of distinct curves reduces to counting points. The main difficulty in setting up the intersection formula is to find a language for relating the intersections of curves in the covering space to those in the base space.

Given a collection of curves $\mathcal{C}$ in the base space, our key result, Proposition II.3.1, gives intersection formulas for the curves in the preimage of $\mathcal{C}$ in terms of what we call *lifting data* for $\mathcal{C}$. Roughly, this has two parts. One is an enumeration of the curves above $\mathcal{C}$ and the other is some simple information determining which curves in $\rho^{-1}(C)$ and $\rho^{-1}(D)$ meet in the fiber of a point $p \in C \cap D$.

We set up the terminology in II.1. In II.2 we show how to apply graphs with certain properties imbedded in $\mathcal{C}$ to the problem of finding lifting data. If the graph lifts to the covering surface, then we show that finding the lifting data for $\mathcal{C}$ reduces to a local problem.

Finally, in II.3 we give intersection formulas for curves in $\rho^{-1}(\mathcal{C})$ in terms of lifting data, when $\mathcal{C}$ is a union of smooth curves so that $\mathcal{C} \cup B$ has normal crossings. When $\mathcal{C}$ is the branch locus of the covering, the nullity of the intersection matrix, which one can compute from the formulas, gives the difference between the first

Betti numbers of the branched and unbranched parts of the covering surface (see Proposition I.6.3) and the rank of the intersection matrix gives a lower bound for the Picard number of the surface (see Remark I.6.2).

II.1 Lifting data for curves in the base space

Let $\rho : X \to Y$ be an abelian branched covering with branch locus B and Galois group G. Let $\mathcal{C}$ be a finite union of curves in Y. For each $C \subset \mathcal{C}$, let H_C be the stabilizer subgroup of C. Then the curves in $\rho^{-1}(C)$ are a principal homogeneous space for G/H_C. Thus, choosing a fixed curve C' in $\rho^{-1}(C)$ determines a one-to-one correspondence between curves in $\rho^{-1}(C)$ and cosets G/H_C such that

$$\alpha C' \longleftrightarrow \alpha H_C, \qquad \text{for } \alpha \in G.$$

II.1.1 Definition. A choice of curves $C' \subset \rho^{-1}(C)$ for each curve C in $\mathcal{C}$ is called *a choice of liftings for $\mathcal{C}$*, or a *$\mathcal{C}$ lifting.*

Once we have liftings, we would like to know, given two curves C and D in $\mathcal{C}$, with $p \in C \cap D$, for which $\alpha, \beta \in G$ do the curves $\alpha(C')$ and $\beta(D')$ intersect in the fiber $\rho^{-1}(p)$.

II.1.2 Definition. Let S be the intersections on $\mathcal{C}$ and let $\mathcal{J}$ be the set of pairs (p, C) where $p \in S$, C is a curve in $\mathcal{C}$ and $p \in C$. Given a $\mathcal{C}$ lifting, let

$$\Psi : \mathcal{J} \to G$$

be a map so that, for each $p \in S$,

$$\Big(\bigcap_{p \in C \subset \mathcal{C}} \Psi(p, C)C' \Big) \cap \rho^{-1}(p) \neq \emptyset.$$

We call the map Ψ *lifting data* for the $\mathcal{C}$ lifting.

II.1.3 Remark. For any $\mathcal{C}$ lifting there exists lifting data Ψ, but it may be difficult to determine the map explicitly. One would like to find the simplest lifting data which can be associated to a choice of liftings of the curves. An interesting problem, which we have not been able to solve is whether there exists a $\mathcal{C}$ lifting so that the trivial map taking all pairs to the identity element is lifting data for this choice. We call a $\mathcal{C}$ lifting with this property a *good lifting.*

To conclude this section, we now explain how the lifting data transforms under blowups and pullback coverings in the special case that all pairs of curves in $\mathcal{C} \cup B$ intersect transversally. Let $\widehat{\sigma} : \widehat{Y} \to Y$ be a blowup of Y at some point p in $\mathcal{C}$. Let $\Psi : \mathcal{J} \to G$ be lifting data for a $\mathcal{C}$ lifting. We will show how to find lifting data for a choice of lifting for curves in the total transform $\widehat{\mathcal{C}}$ in the pull-back covering $\widehat{\rho} : \widehat{X} \to \widehat{Y}$.

Since σ is an isomorphism outside p, $\widehat{X}$ and X are isomorphic outside of the fibers above the exceptional set E_p and p. Therefore, there is a well-defined curve $\widehat{C}'$ corresponding to C' above the total transform $\widehat{C}$ of $C \in \mathcal{C}$.

By Proposition I.4.11 the stabilizer subgroup H_{E_p} is generated by

$$m_p(C)g_C,$$

where $C \subset B$ ranges over all curves passing through p. By assumption $m_p(C) = 1$ for all curves $C \subset B$ containing p. Therefore, $H_{E_p} = I_p$ by Proposition I.4.6. Therefore, there is one curve E'_p in $\widehat{X}$ mapping to the point p' in X, where p' is the point in $\rho^{-1}(p)$ associated to the lifting data and E'_p intersects $\Psi(p, C)\widehat{C}'$. If p lies on only one curve C in $\mathcal{C} \cup B$, let E'_p be any curve in $\widehat{X}$ mapping to p' which intersects $\Psi(p, C)\widehat{C}'$.

Let $\widehat{S}$ be the set of intersections on $\widehat{\mathcal{C}}$ and define

$$\widehat{\Psi} : \widehat{\mathcal{J}} \to G$$

so that $\widehat{\Psi}(q, \widehat{C})$ equals $\widehat{\Psi}(\sigma(q), C)$ for any $q \in \widehat{S} \cap C$ and $\widehat{\Psi}(q, E_p)$ equals the identity for all $q \in \widehat{S} \cap E_p$.

II.1.4 PROPOSITION. *The map $\widehat{\Psi}$ is lifting data for the $\widehat{\mathcal{C}}$ lifting.*

Proof. Take any $q \in \widehat{S}$. If $\sigma(q) \neq p$ then the fibers $\widehat{\rho}^{-1}(q)$ and $\rho^{-1}(\sigma(q))$ are canonically isomorphic and the isomorphism commutes with the action of G so the result follows. If $\sigma(q) = p$, then $q \in E_p \cap \widehat{C}$ for at most one curve $C \subset \mathcal{C} \cup B$, since, by assumption, all intersections of pairs of curves in $\mathcal{C} \cup B$ are transversal. Since E'_p intersects $\Psi(p, C)\widehat{C}'$ in at least one point, we are done. ∎

II.2 Intersection graphs

Let $\rho : X \to Y$ be any abelian branched covering with branch locus B and let $\mathcal{C}$ be a finite union of curves on Y. In this section we show how to find lifting data for $\mathcal{C}$ using graphs.

II.2.1 Definition. Let S be the set of intersections on $\mathcal{C}$, and let Γ be a graph with points in S as vertices and edges labelled C connecting vertices in $S \cap C$. Suppose $f : \Gamma \to Y$ is a continuous map sending vertices to their corresponding points in S and sending interiors of edges labelled C to paths on $C - (\overline{B - C})$. Suppose also that the subgraph Γ_C of Γ mapping to C under f is connected and nonempty. Then $f : \Gamma \to Y$ is called an *intersection graph* for $\mathcal{C}$.

II.2.2 Definition. Given an intersection graph $f : \Gamma \to Y$ for $\mathcal{C}$, a *lifting map* for f in X is a continuous map

$$f' : \Gamma \to X$$

so that $\rho(f'(\gamma)) = f(\gamma)$ for all $\gamma \in \Gamma$.

Note that given one lifting f' there are others given by $\sigma \circ f'$ where σ is any covering transformation.

II.2.3 Remark. If $f : \Gamma \to Y$ has a lifting map $f' : \Gamma \to X$ so that $f'(\Gamma_C)$ is contained in a single curve $C' \subset \rho^{-1}(C)$ for all curves C in $\mathcal{C}$, then we have a good lifting as described in Remark II.1.3.

Our aim now is to show that given a lifting map $f' : \Gamma \to X$ for an intersection graph $f : \Gamma \to Y$, we can find lifting data by a local study.

II.2.4 Definition. Let $f' : \Gamma \to X$ be a lifting map for an intersection graph $f : \Gamma \to Y$ for $\mathcal{C}$. Let $\mathcal{I}$ be the set of pairs of edges of Γ labelled by the same curve $C \subset \mathcal{C}$, meeting at a common vertex. Let

$$\psi : \mathcal{I} \to G$$

be a map so that for each $(e_1, e_2) \in \mathcal{I}$, there is a curve $C' \subset \rho^{-1}(C)$ such that $\psi(e_1, e_2)f'(e_1)$ and $f'(e_2)$ lie on C'. We call ψ the *shifting data* for $f' : \Gamma \to Y$.

The similarity of the notation with the lifting data associated to a $\mathcal{C}$ lifting will be explained in the next lemma. The problem of finding shifting data ψ is a local one in the following sense. If $(e_1, e_2) \in \mathcal{I}$ and e_1 and e_2 meet at a vertex corresponding to p, then $\psi(e_1, e_2)$ depends only on the combinatorics of the covering near the fiber $\rho^{-1}(p)$. Using the next lemma, we will show that the local information given by the shifting data leads to finding the global lifting data.

II.2.5 LEMMA. *For each curve C in $\mathcal{C}$, let C' be a choice of lifting of C in X so that for some edge e_C labelled C in Γ, $f'(e_C)$ is contained in C'. For any two curves C and D in $\mathcal{C}$ and $p \in C \cap D$, let*

$$e_1, e_2, \ldots, e_k$$
$$f_1, f_2, \ldots, f_\ell$$

be two sequences of edges labelled C and D, respectively, which are attached from end to end by common vertices, $e_C = e_1$, $e_D = f_1$, and the final endpoint of these strings of edges is a vertex associated to p. Define α_C and α_D in G by

$$\alpha_C = \psi(e_1, e_2)\psi(e_2, e_3)\ldots\psi(e_{k-1}, e_k)$$
$$\alpha_D = \psi(f_1, f_2)\psi(f_2, f_3)\ldots\psi(f_{\ell-1}, f_\ell).$$

Then $\alpha_C(C')$ and $\alpha_D(D')$ meet at a point in the fiber $\rho^{-1}(p)$.

II.2.6 Remark. It is easiest to visualize the curve $\alpha_C(C')$ as the curve obtained from C' by applying $\psi(e_1, e_2)$, then $\psi(e_2, e_3)$ successively in this order, although since G is abelian, the ordering doesn't matter.

Proof of Lemma II.2.5. Since $\psi(e_i, e_{i+1})f'(e_i)$ and $f'(e_{i+1})$ lie on the same irreducible component of $\rho^{-1}(C)$, for $i = 1, \ldots, k-1$,

$$\alpha_C(f'(e_C)) = \psi(e_1, e_2) \ldots \psi(e_{k-1}, e_k)(f'(e_C))$$

and $f'(e_k)$ lie on the same irreducible component $\alpha_C(C')$ of $\rho^{-1}(C)$. Similarly, $\alpha_D(f'(f_D))$ and $f'(f_\ell)$ both lie on $\alpha_D(D')$. Since, by definition, $f'(p) \in f'(f_\ell) \cap f'(e_k)$, $\alpha_C(C')$ and $\alpha_D(D')$ meet above p. (Note that the result is independent of the choice of sequences $e_2, \ldots, e_k$ and $f_2, \ldots, f_\ell$.) ∎

Thus, from a lifting $f' : \Gamma \to X$ and shifting data $\psi : \mathcal{I} \to G$, we can construct the lifting data for any $\mathcal{C}$ lifting satisfying the hypotheses of Lemma II.2.5 as follows. If C and D are two curves in $\mathcal{C}$ meeting at a point p, we can always find sequences of edges on Γ_C and Γ_D as in Lemma II.2.5, since Γ_C and Γ_D are connected. Then all we need to do is let $\Psi(p, C) = \alpha_C$ and $\Psi(p, D) = \alpha_D$ be as defined in Lemma II.2.5.

II.3 Intersection formulas for covering surfaces

Let $\rho : X \to Y$ be a smooth abelian branched covering of a smooth surface Y with branch locus B. Assume in addition that B is a finite union of smooth curves intersecting in normal crossings. (Recall the criterion for smoothness given in Proposition I.5.1.)

Let $\mathcal{C}$ be a finite union of smooth curves in Y so that the intersections in $\mathcal{C} \cup B$ are normal crossings. Suppose that for each curve $C \subset \mathcal{C}$, C' is a lifting, with lifting data

$$\Psi : \mathcal{J} \to G$$

as defined in Definition II.1.2. In this section we prove that, given such lifting data, we have the following intersection formulas.

II.3.1 Proposition (Intersection Formulas). *If C is any curve in $\mathcal{C}$ and α and β are in G, then*

$$(*) \qquad \alpha(C').\beta(C') = \frac{1}{|I_C|^2}|\alpha H_C \cap \beta H_C| C^2.$$

If C and D are distinct curves in $\mathcal{C}$ and α and β are in G, then

$$(**) \qquad \alpha(C').\beta(D') = \sum_{p \in C \cap D} \frac{1}{|I_C||I_D|} |\alpha\Psi(p,C)^{-1}H_C \cap \beta\Psi(p,D)^{-1}H_D|.$$

The proof requires a few lemmas. The first two concern intersections of curves in $\rho^{-1}(C)$ for a single curve C in $\mathcal{C}$.

II.3.2 LEMMA. *For any curve C in $\mathcal{C}$ the curves in $\rho^{-1}(C)$ are disjoint.*

Proof. Take any p in C. Since $\mathcal{C} \cup B$ contains only normal crossings, p lies in at most one curve $D \in B$ other than C and in this case $I_D \subset H_C$ by Proposition I.4.10. If there is such a curve D, then I_p is generated by the elements of I_C and I_D, otherwise just the elements of I_C. In either case, I_p is contained in H_C.

Now, suppose there are two curves C_1 and C_2 in $\rho^{-1}(C)$ intersecting at a point q in $\rho^{-1}(p)$. Let U be a neighborhood of p in Y so that each connected component of $\rho^{-1}(U)$ contains a distinct point in $\rho^{-1}(p)$ (see property (4) of branched coverings in Definition I.2.1). Let V_q be the connected component of $\rho^{-1}(U)$ containing q.

We will show C_1 and C_2 must be equal. For any point $p' \in C \cap U$, let $q_1 \in C_1$ and $q_2 \in C_2$ be points lying in $\rho^{-1}(p') \cap V_q$. Since G acts transitively, there is an automorphism $\alpha \in G$ so that $\alpha(q_1) = q_2$. Since α permutes the connected components of $\rho^{-1}(U)$ and q_1 and q_2 both lie in $V(q)$, it follows that $\alpha(V_q)$ equals V_q. The only point in $\rho^{-1}(p) \cap V_q$ is q, so $\alpha(q) = q$. But this implies that α is in I_p which is contained in H_C, so $\alpha(C_1) = C_1$ and $q_2 \in C_1$. This means that C_1 and C_2 intersect in $\rho^{-1}(p')$, but p' was chosen arbitrarily in U, so C_1 and C_2 intersect at all points in the open set $V_q \cap \rho^{-1}(C)$. Therefore, C_1 and C_2 must be the same curve. ∎

II.3.3 LEMMA. *If C' is any irreducible component of $\rho^{-1}(C)$, then the self intersection C'^2 equals*

$$\frac{|H_C|}{|I_C|^2}C^2.$$

Proof. Consider C as a divisor on Y and let ρ^*C be its pullback. Then by the general theory of intersections and pullbacks, $(\rho^*C)^2$ equals $|G|C^2$ (see, for example, [**Fu2**], Example 1.7.6, pp. 20 - 21). Each component of ρ^*C counts with multiplicity $|I_C|$. Furthermore, no pair of distinct components of $\rho^{-1}(C)$ meet by Lemma II.3.2, so we have

$$|G|C^2 = |I_C|^2 \sum_{C' \subset \rho^{-1}(C)} C'^2.$$

The number of irreducible components in $\rho^{-1}(C)$ is the index of H_C in G. Since the covering is Galois, all the components have the same self intersection. Therefore,

$$|G|C^2 = |I_C|^2 \frac{|G|}{|H_C|} C'^2,$$

for a given $C' \subset \rho^{-1}(C)$. Multiplying both sides of this equation by

$$\frac{|H_C|}{|I_C|^2|G|}$$

finishes the proof. ∎

If α and β are in G and C' is a curve in $\rho^{-1}(C)$, then, by Lemma II.3.2, $\alpha(C') \cap \beta(C')$ is nonempty only when they are the same curve. This only happens when αH_C equals βH_C, or equivalently, when the intersection $\alpha H_C \cap \beta H_C$ is nonempty. Thus, by Lemma II.3.3

$$\alpha(C').\beta(C') = \frac{1}{|I_C|^2} |\alpha H_C \cap \beta H_C| C^2,$$

and we have proven $(*)$.

Now assume that C and D are distinct curves in $\mathcal{C}$.

II.3.4 LEMMA. *If C' and D' are curves in $\rho^{-1}(C)$ and $\rho^{-1}(D)$ interesecting at q, then*

$$m_q(C', D') = 1,$$

i.e., the intersection is transversal.

Proof. As in the proof of Proposition I.5.1, since the covering X is smooth, the covering map ρ near q looks like

$$(x, y) \mapsto (x^r, y^t)$$

where r and t are integers greater than or equal to 0. The preimage of the branch locus is the union of $\{x = 0\}$, if $r > 0$ and $\{y = 0\}$, if $t > 0$ and the intersection, occurring if r and t are both positive, is transversal. ∎

II.3.5 COROLLARY. *If C' and D' are curves in $\rho^{-1}(C)$ and $\rho^{-1}(D)$, respectively, then $C'.D'$ equals the number of points at which C' and D' meet.*

II.3.6 LEMMA. *Two curves $\alpha(C')$ and $\beta(D')$ above C and D meet at a point in the fiber $\rho^{-1}(p)$ of a point $p \in C \cap D$ if and only if*

$$\alpha\Psi(p, C)^{-1} H_C \cap \beta\Psi(p, D)^{-1} H_D$$

is nonempty.

Proof. We know from the definition of Ψ that the two curves $\Psi(p, C)(C')$ and $\Psi(p, D)(D')$ intersect in at least one point in the fiber $\rho^{-1}(p)$.

For one direction, suppose γ is in the intersection

$$\alpha\Psi(p, C)^{-1} H_C \cap \beta\Psi(p, D)^{-1} H_D$$

then γH_C equals $\alpha\Psi(p, C)^{-1} H_C$ and γH_D equals $\beta\Psi(p, D)^{-1} H_D$, so

$$\begin{aligned}
\alpha C' \cap \beta D' &= \alpha\Psi(p, C)^{-1}\Psi(p, C)C' \cap \beta\Psi(p, D)^{-1}\Psi(p, D)D' \\
&= \gamma\Psi(p, C)C' \cap \gamma\Psi(p, D)D' \\
&= \gamma(\Psi(p, C)C' \cap \Psi(p, D)D').
\end{aligned}$$

Since γ is an automorphism and Ψ is lifting data, $\alpha C'$ and $\beta D'$ must intersect in a point in $\rho^{-1}(p)$.

Conversely, suppose that q_1 is a point in $\alpha C' \cap \beta D'$ so that $\rho(q_1) = p$. Let q_2 be a point in $\Psi(p, C)C' \cap \Psi(p, D)D'$ lying over p and let $\gamma \in G$ be an element taking q_2 to q_1. Then q_1 is in the intersection

$$\gamma\Psi(p, C)C' \cap \gamma\Psi(p, D)D'.$$

Therefore, $\gamma\Psi(p,C)C'$ and $\alpha C'$ intersect in q_1, and hence $\gamma\Psi(p,C)H_C$ and αH_C are equal cosets. Thus γ is contained in $\alpha\Psi(p,C)^{-1}H_C$ and, similarly, γ is contained in $\beta\Psi(p,C)^{-1}H_D$. Therefore,

$$\alpha\Psi(p,C)^{-1}H_C \cap \beta\Psi(p,D)^{-1}H_D$$

is nonempty. ∎

II.3.7 LEMMA. *Let $p \in C \cap D$ and let C' and D' be two curves in $\rho^{-1}(C)$ and $\rho^{-1}(D)$, respectively, so that C' and D' meet at a point above p. Then the number of points where C' and D' meet in the fiber $\rho^{-1}(p)$ equals*

$$\frac{|H_C \cap H_D|}{|I_C||I_D|}.$$

Proof. By Proposition I.5.1, I_C and I_D intersect only in the identity element, so $|I_C I_D|$ equals $|I_C||I_D|$. Also, since C and D intersect transversally, by Proposition I.4.10, I_C and I_D are contained in $H_C \cap H_D$. Thus, it suffices to show that $(H_C \cap H_D)/I_C I_D$ acts transitively and freely on the set $S = \rho^{-1}(p) \cap C' \cap D'$.

Since the covering is regular, G acts transitively on $\rho^{-1}(p)$. If α is in $H_C \cap H_D$ then $\alpha(C') = C'$ and $\alpha(D') = D'$, so $\alpha(C' \cap D') = C' \cap D'$. Thus, $H_C \cap H_D$ acts on S.

We have to show that the action is transitive. We know that for any q_1 and q_2 in S there is an element α of G so that $\alpha(q_1) = q_2$. Since the distinct curves in $\rho^{-1}(C)$ are disjoint, $\alpha(C') = C'$ and $\alpha(D') = D'$ imply $\alpha \in H_D$. Therefore, $\alpha \in H_C \cap H_D$.

Finally, we need to show that the kernel of the action is $I_C I_D$. We know from Lemma I.3.3 that $I_C I_D$ equals I_p. Therefore, $I_C I_D$ is the subgroup of G fixing each point in $\rho^{-1}(p)$. ∎

II.3.8 LEMMA. *If αH_C and βH_D intersect, then the number of elements in their intersection is the same as the number in $H_C \cap H_D$.*

Proof. Suppose γ is in $\alpha H_C \cap \beta H_D$. Then $\gamma H_C = \alpha H_C$ and $\gamma H_D = \beta H_D$, so $\gamma(H_C \cap H_D) = \alpha H_C \cap \beta H_D$. ∎

Now to prove (**) in Proposition II.3.1, we need only put together the above lemmas. By definition of Ψ, we know that, for $p \in C \cap D$,

$$\Psi(p, C)(C') \cap \Psi(p, D)(D')$$

is nonempty. By Lemma II.3.7, the number of elements in the intersection is

$$\frac{1}{|I_C I_D|}|H_C \cap H_D|.$$

By Lemma II.3.4, the intersections number of distinct curves in $\rho^{-1}(C)$ are, for any $\alpha, \beta \in G$, is given by

$$\alpha C'.\beta D' = \sum_{p \in C \cap D} |\alpha C' \cap \beta D' \cap \rho^{-1}(p)|.$$

By Lemmas II.3.6, II.3.7 and II.3.8, we have

$$|\alpha C' \cap \beta D' \cap \rho^{-1}(p)| = \frac{1}{|I_C||I_D|}|\alpha \Psi(p, C)^{-1} H_C \cap \beta \Psi(p, D)^{-1} H_D|.$$

Summing over all $p \in C \cap D$ gives the formula (**).

Chapter III. Hirzebruch covering surfaces

In this chapter we apply the previous results specifically to Hirzebruch surfaces and describe the techniques that lead to an effective algorithm for computing the first Betti number b_1 and lower bound for the Picard number p of Hirzebruch surfaces $\widehat{X}$ associated to configurations of real lines. The algorithm is given in Chapter IV.

We define Hirzebruch covering surfaces and give some properties following **[Hi]** in III.1. The surfaces are desingularizations of certain abelian branched coverings X of $\mathbf{P}^2$ whose branch locus $\mathcal{L}$ is a finite union. The surface X is desingularized (see Lemma III.1.2 and Remark III.1.3) by taking the pullback covering $\widehat{X}$ over the blowup of $\mathbf{P}^2$ at the triple and higher intersections on $\mathcal{L}$, with the exception of the case where $\mathcal{L}$ is two lines, in which case we take the blowup over the single intersection point. Thus, $\widehat{X}$ is a branched covering

$$\widehat{\rho} : \widehat{X} \to \widehat{\mathbf{P}}^2$$

branched along a subset of the total transform $\widehat{\mathcal{L}}$ of $\mathcal{L}$.

We describe the generators of the inertia and stabilizer subgroups for the branch locus of ρ and $\widehat{\rho}$. This is a useful part of the algorithm and also, together with Proposition I.5.1, leads to an easy proof that $\widehat{X}$ is smooth.

In order to compute b_1 the main steps are the following.

(1) Find a presentation for the fundamental group of $\pi_1(\mathbf{P}^2 - \mathcal{L})$.

(2) Find lifting data for a $\widehat{\mathcal{L}}$ lifting for $\widehat{\rho} : \widehat{X} \to \widehat{\mathbf{P}}^2$.

Using Fox calculus on the presentation for $\pi_1(\mathbf{P}^2 - \mathcal{L})$ and applying Libgober's result, we can compute the first Betti number b_1^u of the unbranched part of the covering. From the lifting data and generators for the stabilizer and inertia subgroups, we can find the intersection matrix I for the curves above the branch locus, using the formulas of Proposition II.3.1, and compute its nullity Null(I). By Proposition I.6.3, b_1 equals $b_1^u - \mathrm{Null}(I)$.

To find a presentation for $\pi_1(\mathbf{P}^2 - \mathcal{L})$, we use the technique often used by Moishezon [**Mo**] and Libgober [**Li2**], [**Li3**]. The idea is to project $\mathbf{P}^2 - \mathcal{L}$ to a general line H and compute the monodromy of the associated fibration as do Van Kampen [**V**] and Cheniot [**C**] and analyse the monodromy using braids. For configurations of real lines, the monodromy is easier to describe explicitly than in the general situation. We do this in III.2 and show how to find the presentation in III.3.

To find lifting data for the branch locus, we also study the local topology of real line configurations in $\mathbf{P}^2$. In III.4, we show how to find an intersection graph together with shifting data (see II.2) for the line configuration. Lemma II.2.5 shows how to convert this to lifting data.

III.1 Hirzebruch covering surfaces

The covering surfaces that we will deal with throughout the rest of this paper were defined in [**Hi**]. Here is an alternative definition using the language developed in section II.

III.1.1 Definition. Let $\mathcal{L}$ be a finite union of k lines in $\mathbf{P}^2$ and let $n \geq 2$ be an integer. Let

$$\rho : X \to \mathbf{P}^2$$

be the abelian branched covering determined by the defining map (see Definition I.2.6)

$$\phi : \mathrm{H}_1(\mathbf{P}^2 - \mathcal{L}; \mathbf{Z}) \to \mathrm{H}_1(\mathbf{P}^2 - \mathcal{L}; \mathbf{Z}/n\mathbf{Z}).$$

Let S be the points of intersection and let $T \subset S$ the points where at least three lines in $\mathcal{L}$ intersect. Define a surface $\widehat{\mathbf{P}}^2$ and a birational morphism $\sigma : \widehat{\mathbf{P}}^2 \to \mathbf{P}^2$ depending on $\mathcal{L}$ as follows.

(1) If $\mathcal{L}$ consists of two lines, let $\sigma : \widehat{\mathbf{P}}^2 \to \mathbf{P}^2$ be the blowup of $\mathbf{P}^2$ at the point of intersection.

(2) If T is empty and $k > 2$, let $\widehat{\mathbf{P}}^2$ equal $\mathbf{P}^2$ and let σ be the identity map.

(3) If T is nonempty, let $\sigma : \widehat{\mathbf{P}}^2 \to \mathbf{P}^2$ be the blowup of $\mathbf{P}^2$ at the points in T.

Let

$$\widehat{\rho} : \widehat{X} \to \widehat{\mathbf{P}}^2$$

be the pullback covering (see Definition I.1.7) of $\rho : X \to \mathbf{P}^2$. Then we call $\widehat{X}$ the *Hirzebruch covering* associated to $\mathcal{L}$ and n.

One particularly useful property of Hirzebruch coverings is the following.

III.1.2 LEMMA. *([**Hi**], p. 122) Hirzebruch coverings $\widehat{X}$ are smooth.*

A proof is sketched by Hirzebruch in [**Hi**]. We give a more detailed proof in Remark III.1.3 using the language developed in Chapter I. In the process we show how to find the generators of the stabilizer and inertia subgroups of the branch locus of ρ and $\widehat{\rho}$.

Recall from Definition I.4.1 that, if $\phi : \mathrm{H}_1(\mathbf{P}^2 - \mathcal{L}; \mathbf{Z}) \to \mathrm{H}_1(\mathbf{P}^2 - \mathcal{L}; \mathbf{Z}/n\mathbf{Z})$ is the defining map of the covering, to each line L in $\mathcal{L}$ there is a canonically associated element $\mu_L \in \mathrm{H}_1(\mathbf{P}^2 - \mathcal{L}; \mathbf{Z})$ which can be realized as a positively oriented meridianal loop around L.

LEMMA III.1.3. *If $\mathcal{L}$ is any finite union of k lines in $\mathbf{P}^2$, then $\mathrm{H}_1(\mathbf{P}^2 - \mathcal{L}; \mathbf{Z})$ is generated by μ_L for all $L \subset \mathcal{L}$ and has the only relation*

$$\sum_{L \subset \mathcal{L}} \mu_L = 0.$$

Proof. The Lefschetz hyperplane theorem states that for a general hyperplane H in $\mathbf{P}^2$ the map

$$\mathrm{H}_1(H - \mathcal{L}; \mathbf{Z}) \to \mathrm{H}_1(\mathbf{P}^2 - \mathcal{L}; \mathbf{Z})$$

is onto. By Proposition I.4.2, the μ_L can be represented by loops on $H - \mathcal{L}$. Therefore, the μ_L generate $\mathrm{H}_1(\mathbf{P}^2 - \mathcal{L}; \mathbf{Z})$, and satisfy the relation

$$\sum_{L \subset \mathcal{L}} \mu_L = 0.$$

We need to show that the above map is one-to-one. We can assume that $\mathcal{L}$ contains at least two lines (since if not, both domain and range are simply-connected, so the map must be an isomorphism.) It suffices then to show that any subset of the set of μ_L of order $k-1$ has no relations among its elements. But the Van Kampen method [K] for computing $\pi_1(\mathbf{P}^2 - \mathcal{L})$ (see Proposition III.3.3 for a more detailed explanation of this method) show that this group is generated by $k-1$ of the base-pointed loops μ_L and has relations given by the monodromy action of a generic pencil. Since the action is by conjugation, the relations are trivial after abelianization. ∎

For each line L in $\mathcal{L}$, let g_L be the image of μ_L under the map ϕ. Then the above proposition implies that the Galois group G is the abelian group generated by g_L, each having order n and the only relation among them is that the sum of the generators is 0.

As before, let S be the set of points where the lines in $\mathcal{L}$ meet. Define

$$g_p = \sum_{p \in L \subset \mathcal{L}} g_L.$$

III.1.4 PROPOSITION. *The inertia subgroup I_L associated to the line L is generated by the g_L, and the stabilizer subgroup H_L is generated by g_L and g_p where p ranges over points in $S \cap L$.*

Proof. This follows immediately from Propositions I.4.5 and I.4.10. ∎

Let T be the set of triple and higher order intersections on $\mathcal{L}$ and let $\sigma : \widehat{\mathbf{P}}^2 \to \mathbf{P}^2$ be the blowup at the points in T. Then the branch locus of $\widehat{\rho} : \widehat{X} \to \widehat{\mathbf{P}}^2$ is contained in $\widehat{\mathcal{L}} = \sigma^{-1}(\mathcal{L})$ (with equality except in the case where all lines pass through one point). The curves in $\widehat{\mathcal{L}}$ are the proper transforms $\widehat{L}$ of lines L in $\mathcal{L}$ and the exceptional sets E_p associated to points $p \in T$.

III.1.5 PROPOSITION. *Let $\widehat{L}$ be the proper transform of L. Then $I_{\widehat{L}}$ equals I_L and $H_{\widehat{L}}$ equals H_L. Let E_p be the exceptional set lying above $p \in T$. Then I_{E_p} is generated by g_p and H_{E_p} is generated by all g_L where $L \subset \mathcal{L}$ and $p \in \mathcal{L}$.*

Proof. This follows from Proposition I.4.11. ∎

III.1.6 Remark. Proposition III.1.3 and I.5.1 lead to a proof of Lemma III.1.3. Assume there are at least 3 lines in $\mathcal{L}$. Since the branch locus $\widehat{\mathcal{L}}$ of the covering $\widehat{\rho} : \widehat{X} \to \widehat{Y}$ is a union of smooth curves with normal crossings, we can apply Proposition I.5.1. Thus, we need only show that the inertia subgroups of two intersecting curves in $\widehat{\mathcal{L}}$ intersect in the identity.

Let $g_1, \ldots, g_k$ be the generators of G corresponding to the lines $L_1, \ldots, L_k$ in $\mathcal{L}$. Take any two intersecting curves C and D in $\widehat{\mathcal{L}}$. If C and D are the proper transforms $\widehat{L}_1$ and $\widehat{L}_2$, then $I_C = (g_{L_1})$ and $I_D = (g_{L_2})$. These intersect in the identity in G, since there is at least one more generator in G. If one of C and D is an exceptional curve, say $C = E_p$, and D is the proper transform $\widehat{L}$, then

$$I_C = (\sum_{p \in L' \subset \mathcal{L}} g_{L'})$$

and

$$I_D = (g_L).$$

In order for these to intersect nontrivially there must be a nontrivial relation among the $g_{L'}$ where L' ranges over lines in $\mathcal{L}$ passing through p. This can only happen if all the lines in $\mathcal{L}$ pass through p, but in this case $I_{E_p} = 0$. The case for $\mathcal{L}$ equal to two lines is the same as for the case where $\mathcal{L}$ consists of several lines all passing through one point. ∎

III.1.7 PROPOSITION. *Assume that not all lines in $\mathcal{L}$ pass through a single point p. For each line L in $\mathcal{L}$, let r_L be the number of points in $S \cap L$. Then the number of curves in $\widehat{\rho}^{-1}(\widehat{L})$ equals*

$$n^{k-r_L-1}.$$

For each point $p \in T$ let ℓ_p be the number of lines in $\mathcal{L}$ passing through p. Then the number of curves in $\widehat{\rho}^{-1}(E_p)$ equals

$$n^{k-\ell_p-1}.$$

Thus, the total number of curves in $\widehat{\rho}^{-1}(\widehat{\mathcal{L}})$ is a polynomial in n.

Proof. We know that G has order n^{k-1}.

In Proposition III.1.4, we saw that H_L is generated by g_L and g_p for all points $p \in S \cap L$. Since all lines in $\mathbf{P}^2$ intersect,

$$g_L + \sum_{p \in S \cap L} (g_p - g_L) = 0.$$

Since there is no other relation, H_L is generated freely by $g_p - g_L$ where p ranges over points in $S \cap L$. Since there are r_L of these, the order of the group H_L equals n^{r_L}. Therefore, the order of G/H_L equals n^{k-r_L-1}. By Proposition III.1.5, $H_{\widehat{L}}$ equals H_L so the number of curves in $\widehat{\rho}^{-1}(\widehat{L})$ equals n^{k-r_L-1}.

In Proposition III.1.5, we saw that H_{E_p} is generated by g_L, where L ranges over lines in $\mathcal{L}$ passing through p. Since not all lines in $\mathcal{L}$ pass through p, these generators have no relations. Therefore, the order of H_{E_p} equals n^{ℓ_p}, and the number of curves in $\widehat{\rho}^{-1}(E_p)$ equals $n^{k-\ell_p-1}$. ∎

III.2 Fibrations and monodromy

In this section we consider a finite union of lines $\mathcal{L}$ so that for some affine coordinates x, y, the intersection $\overline{\mathcal{L}}$ of $\mathcal{L}$ with $\mathbf{C}^2$ is a union of k lines defined by real equations in x, y and one line at infinity. (The constant k will be used in this and the next chapter as one less than the number of lines in $\mathcal{L}$.)

The choice of coordinates determines a projection $P_x : \mathbf{C}^2 - \overline{\mathcal{L}} \to \mathbf{C}$ onto the x-axis. Let $\overline{S}$ be the set of intersections of lines in $\overline{\mathcal{L}}$ and let $Q = P_x(\overline{S})$. If no fiber of P_x contains a line in $\overline{\mathcal{L}}$, then for points $q \in \mathbf{C}$ not in Q the fiber $P_x^{-1}(q)$ is canonically isomorphic to a copy of $\mathbf{C}$ minus k points. Therefore, P_x defines a fibration of $\mathbf{C}^2 - \overline{\mathcal{L}}$ over $\mathbf{C}$ with singular fibers above points in $P_x(\overline{S})$. The aim of this section is to study the monodromy of this fibration around singular fibers.

III.2.1 Conditions on the coordinates x, y. Rotate the coordinates x, y if necessary so that the following hold.

P1. Each L_α in $\overline{\mathcal{L}}$ is given by an equation of the form

$$y = m_\alpha x + b_\alpha,$$

where m_α and b_α are real.

P2. The projection P_x sends the set of all intersections $\overline{S}$ on $\overline{\mathcal{L}}$ to distinct (necessarily real) points Q in $\mathbf{C}$.

Note that the slopes m_α are not necessarily distinct.

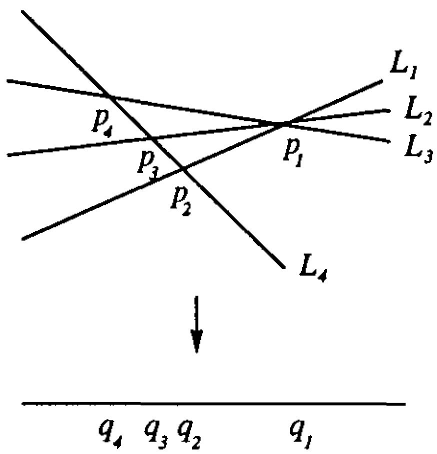
III.2.2 Definition. Given coordinates x, y satisfying the conditions in III.2.1 order the lines in $\overline{\mathcal{L}}$ so that

$$m_1 \geq m_2 \geq \cdots \geq m_k.$$

Order the points $p_1, \ldots, p_s$ in $\overline{S}$ so that if $q_1, \ldots, q_s$ are their images in Q, then

$$q_1 > q_2 > \cdots > q_s.$$

III.2.3 Example.



III.2.4 The fibration P_x. Let x, y be coordinates satisfying the conditions in III.2.1 and order the lines in $\overline{\mathcal{L}}$ and points of intersection $\overline{S}$ in $\overline{\mathcal{L}}$ as in Definition III.2.2. For any $q \in \mathbf{C} - Q$, the fiber $F_x = P_x^{-1}(q)$ equals a copy of $\mathbf{C}$, parameterized by y, minus k points T_q, where T_q is the set of $p \in \mathbf{C}$ so that (q, p) lies on $\mathcal{L}$. If $q \in \mathbf{R} - Q$, then the points $t_1, \ldots, t_k$ in T_q are real and can be ordered so that

$$t_1 > t_2 > \cdots > t_k.$$

III.2.5 Monodromy. Let $q_0 \in \mathbf{R}$ be a point so that $q_0 > q$ for all $q \in Q$. The *monodromy* of the fibration is the image of the natural map

$$(*) \qquad \pi_1(\mathbf{C} - Q, q_0) \to \mathrm{Mod}(F_{q_0})$$

where $\mathrm{Mod}(F_{q_0})$ is the *mapping class group*, or group of isotopy classes of homeomorphisms of F_{q_0} to itself which fix everything outside of a large disk in F_{q_0} containing T_{q_0}.

There is a canonical homomorphism

$$(**) \qquad \mathcal{B}_k \to \mathrm{Mod}(F_{q_0}),$$

where $\mathcal{B}_k$ is the braid group on k strands [**Mo**]. Let

$$\Sigma : \pi_1(\mathbf{C} - S, q_0) \to \mathcal{B}_k$$

be the map Σ which takes a loop $\gamma : [0,1] \to \mathbf{C} - S$ based at q_0 to the braid obtained by following $T_{\gamma(\theta)}$ as θ ranges between 0 and 1. Then the map $(*)$ is the composition of Σ and $(**)$.

III.2.6 Identification of fibers over real points. To explicitly find $\Sigma(\gamma)$ for paths $\gamma \in \pi_1(\mathbf{C} - Q, q_0)$, we use the fact that, whenever $q \in \mathbf{R}$, T_q is a set of real points, and hence has a canonical *local ordering* from largest to smallest. Note that this is different from the ordering on T_q induced by the global ordering of the lines $L_1, \ldots, L_k$. Thus, for all real points $q \in \mathbf{C}$ there is a homeomorphism of any fiber F_q to F_{q_0}, given by the local ordering, which is unique up to isotopy. Therefore, any path γ in $\mathbf{C} - Q$ with real endpoints defines an element in $\mathrm{Mod}(F_{q_0})$.

Explicitly, given any $q \in \mathbf{R} - \mathbf{Q}$ there is a unique isotopy class of maps

$$[\phi_q] : F_q \to F_{q_0}$$

with the following properties:

(1) the orderings of T_q and T_{q_0} are preserved;

(2) for any $\epsilon > 0$ there is a representative $\phi_q \in [\phi_q]$ so that the following diagram commutes

$$\begin{array}{ccc} F_q - \{|\operatorname{Im}(y)| < \epsilon\} & \xrightarrow{\phi_q} & F_x - \{|\operatorname{Im}(y)| < \epsilon\} \\ \downarrow & & \downarrow \\ \mathsf{C} - \{|\operatorname{Im}(y)| < \epsilon\} & \xrightarrow{\text{identity}} & \mathsf{C} - \{|\operatorname{Im}(y)| < \epsilon\}, \end{array}$$

where the vertical maps are the canonical identifications;

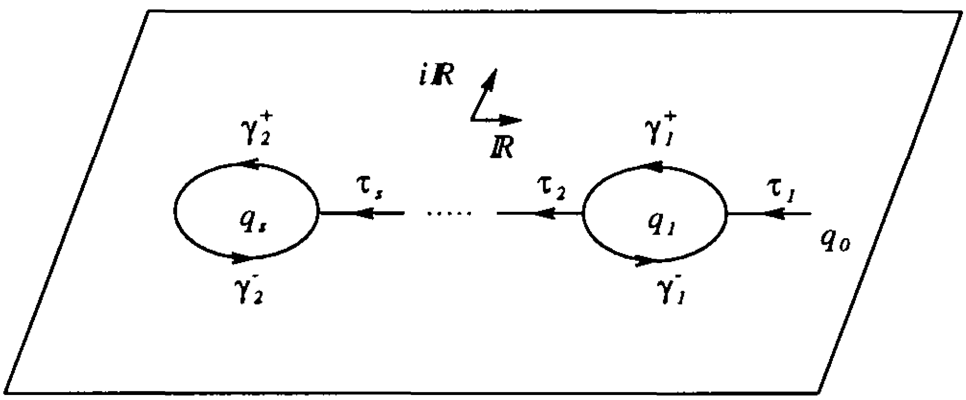
(3) any homeomorphism from F_q to F_{q_0} which has properties (1) and (2) is in the isotopy class $[\phi_q]$.

Using these maps, one can define a map from the set Pa of paths on $\mathsf{C} - Q$ whose endpoints are real to $\mathcal{B}_k$,

$$b : Pa \rightarrow \mathcal{B}_k$$

so that, for any path γ, $b(\gamma)$ is the braid obtained by following the points of T_q where q ranges over the image of γ. Then if γ_1 and γ_2 are paths so that the endpoint of γ_1 is the initial point of γ_2, then $b(\gamma_1\gamma_2) = b(\gamma_1)b(\gamma_2)$. Furthermore, the restriction of b to closed paths based at q_0 equals Σ.

III.2.7 Generators for $\pi_1(\mathsf{C} - Q, q_0)$. We construct generators as in the following diagram.



Assume without loss of generality that

$$|q_j - q_{j+1}| > 1$$

for all $j = 1, \ldots, s-1$. Let $\tau_1, \ldots, \tau_{s-1}$ be paths on $\mathbf{R} - Q$ defined by

$$\begin{aligned} \tau_1 : [0,1] &\to \mathbf{R} - Q \\ \theta &\mapsto q_0 + \theta(q_1 - q_0 + 1) \\ \tau_j : [0,1] &\to \mathbf{R} - Q, \qquad \text{for } j = 2, \ldots, s \\ \theta &\mapsto (q_{j-1} - 1) + \theta(q_j - q_{j-1} + 2) \end{aligned}$$

Let γ_j^+ and γ_j^- be paths on $\mathbf{C} - Q$ defined by

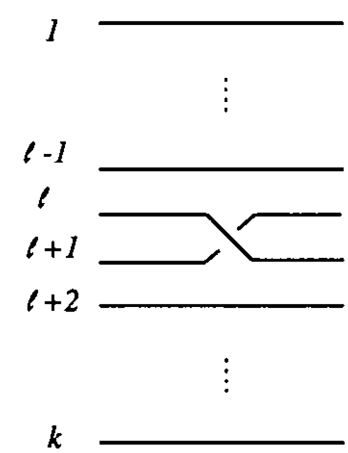
$$\begin{aligned} \gamma_j^+ : [0,1] &\to \mathbf{C} - Q \\ \theta &\mapsto q_j + e^{\pi i \theta} \\ \gamma_j^- : [0,1] &\to \mathbf{C} - Q \\ \theta &\mapsto q_j - e^{\pi i \theta} \end{aligned}$$

for $j = 1, \ldots, s$. Note that the endpoints of all paths defined above lie in $\mathbf{R} - Q$.

As can be seen by the previous diagram, the fundamental group $\pi_1(\mathbf{C} - Q, q_0)$ is generated by $\Gamma_1, \ldots, \Gamma_s$, where each Γ_j is defined by

$$\Gamma_j = \left(\prod_{r=1}^{j-1} \tau_r \gamma_r^+\right) \tau_j \gamma_j^+ \gamma_j^- \tau_j^{-1} \left(\prod_{r=1}^{j-1} \tau_r \gamma_r^+\right)^{-1}.$$

III.2.8 Generators for $\mathcal{B}_k$. Recall that $\mathcal{B}_k$ is generated by $\sigma_1, \ldots, \sigma_{k-1}$, where each σ_i is the braid



and has relations

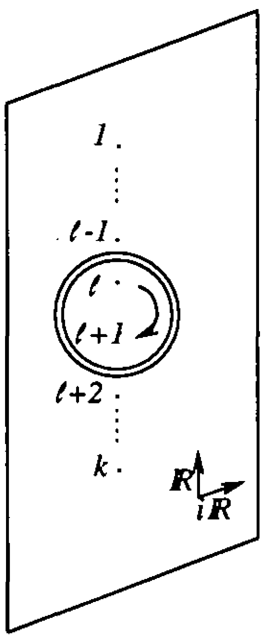
$$\sigma_i \sigma_j = \sigma_j \sigma_i$$

for $|i - j| \geq 2$ and

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

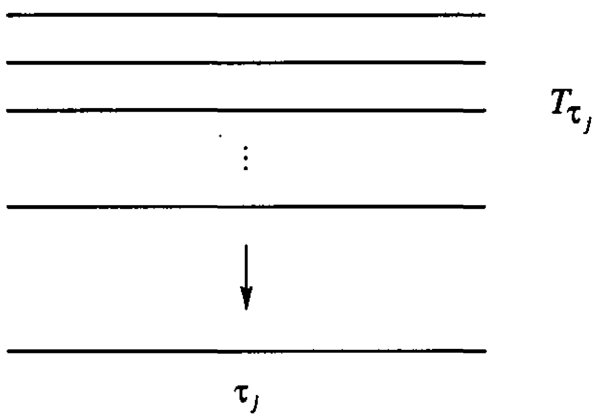
for $i = 1, \ldots, s-2$.

Recall that F_{q_0} equals $\mathbf{C}$ minus k ordered points lying on the real line. The braid σ_i corresponds to the element of $\mathrm{Mod}(F_{q_0})$ which can be represented by a homeomorphism which rotates a disk D, containing only the ith and $i+1$st point and centered between them, by 180 degrees and fixes all points outside of a disk $\overline{D}$ containing D.



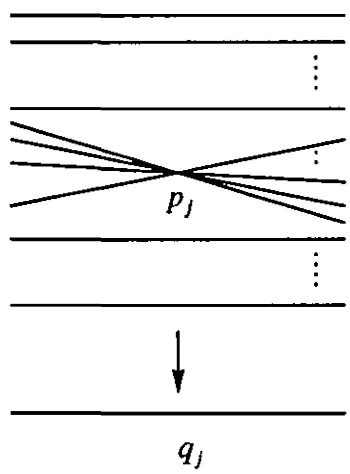
III.2.9 Generators for the Monodromy. To find generators for the image of Σ, we need only find $\Sigma(\Gamma_j)$ for $\Gamma_1, \ldots, \Gamma_s$ as defined in III.2.7. To do this we find $b(\tau_j)$, $b(\gamma_j^+)$ and $b(\gamma_j^-)$.

For each j, $\tau_j(\theta) \in \mathbf{R}$ for all θ and $\tau_j(\theta)$ doesn't pass through any points in Q, the points in $T_{\tau_j(\theta)}$ are real and their ordering is preserved as θ varies from 0 to 1. The local picture over the image of τ_j looks schematically as follows. Note that the lines are not really parallel as in the picture, but they might as well be, since they don't meet each other over this interval.



This is because a set of k points moving continuously on the real line cannot get permuted without coming into contact. Therefore, $b(\tau_j)$ is trivial for all j.

Now look at fibers over a path γ_j^+ or γ_j^- for any $j = 1, \ldots, s$. The real fibers of $P_x : \mathbf{R}^2 - \mathcal{L} \to \mathbf{R}$ over an interval containing q_j looks schematically like the following.



Consider the local ordering for the points $t_1, \ldots, t_k$ in T_q for q any real point to the right of q_j in this interval as in III.2.6. Let ℓ be the first index with respect to this local ordering so that L_ℓ passes through p_j. Translate coordinates so that

$$\overline{x} = x - q_j$$

$$\overline{y} = y - (b_\ell + m_\ell q_j)$$

where b_ℓ is the y-intercept and m_ℓ is the slope of L_ℓ with respect to x and y.

After the change of coordinates, and with respect to the local ordering, the lines in $\mathcal{L}$ are given by new equations

$$L_r : \overline{y} = m_r\overline{x} + c_r, \qquad \text{for } r = 1, \ldots, k,$$

where $c_r = (b_r - b_\ell) + (m_r - m_\ell)q_j$. For some $d \geq 2$

$$c_r = 0, \qquad \text{for all } r = \ell, \ldots, \ell + d - 1.$$

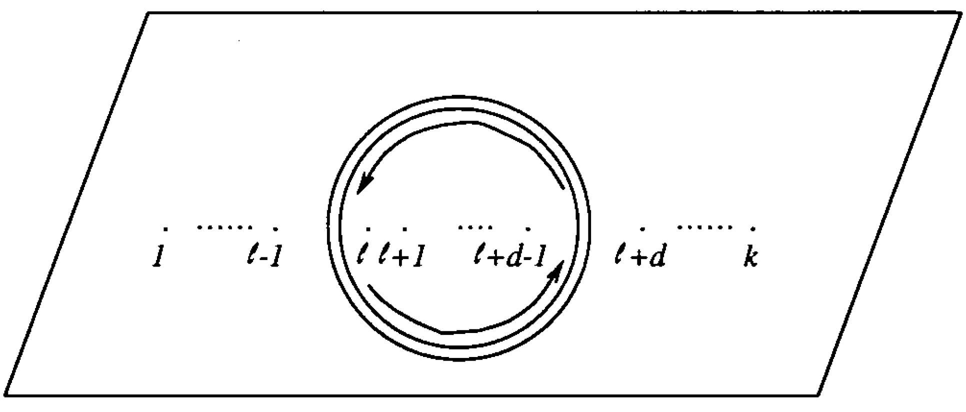
On $F_{\gamma_j^+(\theta)}$, we have

$$\begin{aligned} T_{\gamma_j^+(\theta)} = \{ & m_1 e^{\pi i\theta} + b_1 + m_1 q_j, \ldots, m_{\ell-1}e^{\pi i\theta} + b_{\ell-1} + m_{\ell-1}q_j, \\ & m_\ell e^{\pi i\theta}, \ldots, m_{\ell+d-1}e^{\pi i\theta}, \\ & m_{\ell+d}e^{\pi i\theta} + b_{\ell+d} + m_{\ell+d}q_j, \ldots, m_k e^{\pi i\theta} + b_k + m_k q_j\}. \end{aligned}$$

Similarly, on $F_{\gamma_j^-(\theta)}$, we have

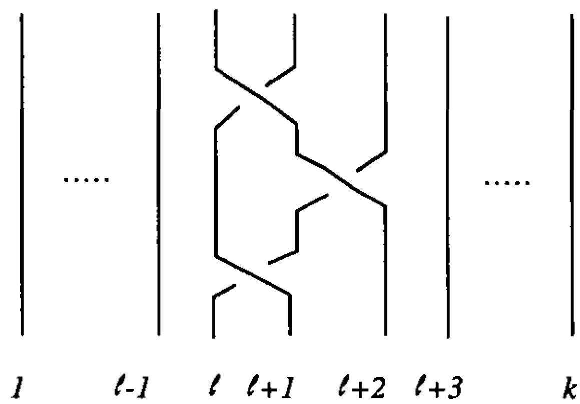
$$\begin{aligned} T_{\gamma_j^+(\theta)} = \{ & -m_1 e^{\pi i\theta} + b_1 + m_1 q_j, \ldots, -m_{\ell-1}e^{\pi i\theta} + b_{\ell-1} + m_{\ell-1}q_j, \\ & -m_\ell e^{\pi i\theta}, \ldots, -m_{\ell+d-1}e^{\pi i\theta}, \\ & -m_{\ell+d}e^{\pi i\theta} + b_{\ell+d} + m_{\ell+d}q_j, \ldots, -m_k e^{\pi i\theta} + b_k + m_k q_j\}. \end{aligned}$$

Thus, the element of $\mathrm{Mod}(F_{q_0})$ corresponding to γ^+ and γ^- rotates a disk containing $t_\ell, \ldots, t_{\ell+d-1}$ 180 degrees as in the following picture.



We will call this the *local monodromy around the fiber above* q_j.

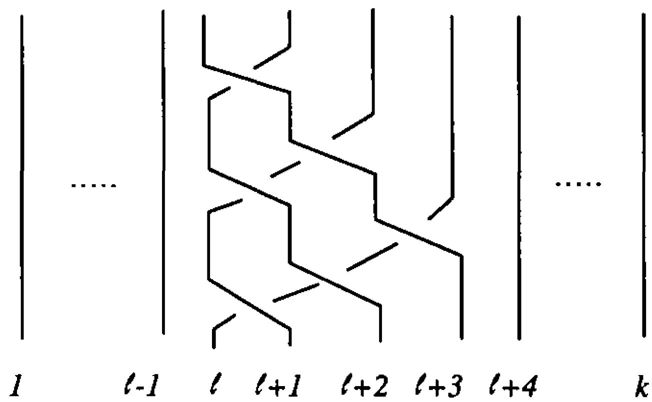
III.2.10 Examples. The corresponding braid for $d = 2$ is the generator element σ_ℓ. The corresponding braid for $d = 3$ is



which equals

$$\sigma_\ell \sigma_{\ell+1} \sigma_\ell.$$

For $d = 4$, the braid is



and equals

$$\sigma_\ell \sigma_{\ell+1} \sigma_{\ell+2} \sigma_\ell \sigma_{\ell+1} \sigma_\ell.$$

III.2.11 Definition. Let $\Sigma_{\ell,d}$ be the braid

$$\prod_{\alpha=d-1}^{1} \left(\prod_{\beta=1}^{\alpha} \sigma_{\ell+\beta-1}\right).$$

III.2.12 PROPOSITION. *If, by the local ordering at $\gamma_j^-(0) = \gamma_j^+(0)$, the lines indexed by $\ell, \ldots, \ell + d - 1$ come together, then the braids $b(\gamma_j^+)$ and $b(\gamma_j^-)$ equal $\Sigma_{\ell,d}$.*

Putting this local information together we have the following Proposition.

III.2.13 PROPOSITION. *The image of the monodromy*

$$\Sigma : \pi_1(\mathsf{C} - Q, q_0) \to \mathcal{B}_k,$$

is generated by

$$\Sigma(\Gamma_j) = [(\prod_{r=1}^{j-1} \Sigma_{\ell_r,d_r})\Sigma^2_{\ell_j,d_j}(\prod_{r=1}^{j-1} \Sigma_{\ell_r,d_r})^{-1}]$$

where $j = 1, \ldots, s$.

III.2.14 Example. Take the configuration in Example III.2.3. The monodromy is generated by

$$(\Sigma_{2,3})^2,$$
$$\Sigma_{2,3}(\Sigma_{1,2})^2\Sigma_{2,3}^{-1},$$
$$\Sigma_{2,3}\Sigma_{1,2}(\Sigma_{2,2})^2\Sigma_{1,2}^{-1}\Sigma_{2,3}^{-1},$$
$$\Sigma_{2,3}\Sigma_{1,2}\Sigma_{2,2}(\Sigma_{3,2})^2\Sigma_{2,2}^{-1}\Sigma_{1,2}^{-1}\Sigma_{2,3}^{-1}.$$

III.3 Fundamental group of the complement of real lines

In this section, we apply the results of III.2 to find the fundamental group of the complement of a configuration $\mathcal{L}$ of real lines in P^2.

Choose affine coordinates x, y in P^2, so that one of the lines in $\mathcal{L}$ is the line at infinity and satisfying the conditions P1 and P2 of III.2.1. Let $L_1, \ldots, L_k$ (we assume, for ease of notation, as in III.2, that the original number of lines was $k+1$) and $p_1, \ldots, p_s$ be the (globally) ordered lines in $\overline{\mathcal{L}}$ and points in $\overline{S}$ as in III.2.2.

III.3.1 Definition. Define a map

$$\mathcal{M} : \mathcal{B}_k \to \mathrm{Aut}(F_k),$$

where F_k is the free group on k generators $\mu_1, \ldots, \mu_k$ and $\mathrm{Aut}(F_k)$ is its group of automorphisms, by

$$\mathcal{M}(\sigma_i)(\mu_j) = \begin{cases} \mu_{j+1} & \text{if } i = j \\ \mu_{j+1}^{-1}\mu_j\mu_{j+1} & \text{if } i = j+1 \\ \mu_j & \text{otherwise.} \end{cases}$$

III.3.2 PROPOSITION. *The following is a presentation for $\pi_1(\mathbf{P}^2 - \mathcal{L})$:*

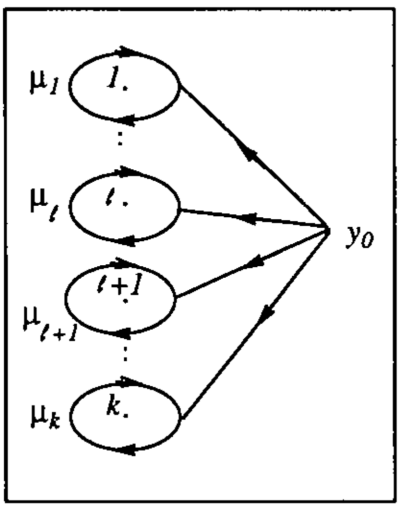
$$< \mu_1, \ldots, \mu_k : R_{\alpha,\beta}; \alpha = 1, \ldots, k; \beta = 1, \ldots, s >,$$

where the $\mu_1, \ldots, \mu_k$ correspond to positively oriented loops in $\mathbf{C}^2 - \overline{\mathcal{L}}$ around the lines $L_1, \ldots, L_k$,

$$R_{\alpha,\beta} = \mu_\alpha^{-1}\mathcal{M}(\Sigma_\beta)(\mu_\alpha)$$

and $\Sigma(\Gamma_\beta)$ is as described in Proposition III.2.13.

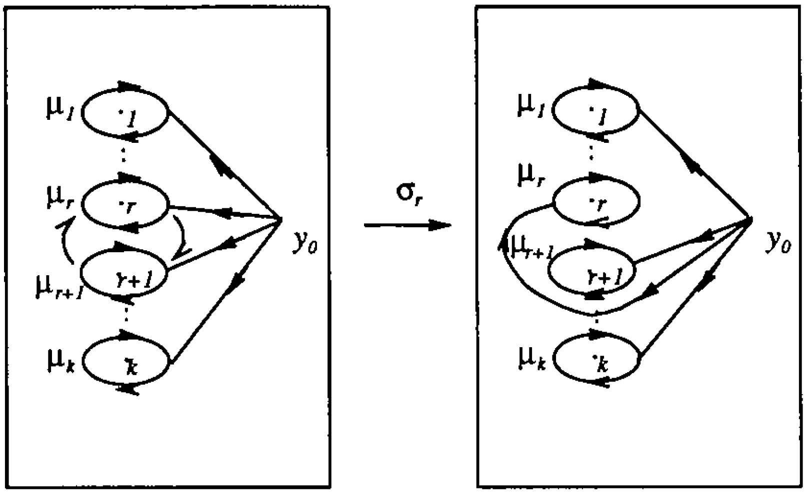
Proof. By a well-known result due to Zariski and van Kampen [K] (see also Cheniot's paper [C]), $\pi_1(\mathbf{C}^2 - \overline{\mathcal{L}}, p_0)$ (where p_0 is contained in F_{q_0}) is generated by loops $\mu_1, \ldots, \mu_k$ on the fiber F_{q_0} as given by the following picture



and has relations

$$\mu_i = \Sigma(\Gamma_j)^*(\mu_i).$$

To find $\Sigma(\Gamma_j)^*$ it suffices to find σ_r^* for each generator σ_r of $\mathcal{B}_k$. From the picture



we see that

$$\sigma_r^*(\mu_i) = \begin{cases} \mu_{r+1} & \text{if } i = r \\ \mu_{r+1}\mu_r\mu_{r+1}^{-1} & \text{if } i = r+1 \\ \mu_i & \text{otherwise} \end{cases}$$

Thus, $\mathcal{M}(\sigma_r) = \sigma_r^*$. ∎

III.4 Lifting data for curves above the branch locus

In this section, we use the methods described in section II.2 to define lifting data for curves in the branch locus $\widehat{\mathcal{L}}$ of $\widehat{\rho} : \widehat{X} \to \widehat{\mathbf{P}}^2$. Recall that this means that we define a map

$$\Psi : \widehat{\mathcal{J}} \to G$$

from the set of pairs

$$\widehat{\mathcal{J}} = \{(p, C) : p \in C, C \subset \widehat{\mathcal{L}}\}$$

to the Galois group $G = \mathrm{H}_1(\mathbf{P}^2 - \mathcal{L}; \mathbf{Z}/n\mathbf{Z})$ so that, for some choice of lifting C' for curves $C \subset \mathcal{L}$ and point $p' \in \rho^{-1}(p)$ for $p \in S$, we have $p' \in \Psi(p, C)C'$ for all $C \subset \mathcal{L}$ and $p \in \mathcal{C}$.

We first study the covering $\rho : X \to \mathbf{P}^2$ and then extend our findings to the pullback covering $\widehat{\rho} : \widehat{X} \to \widehat{\mathbf{P}}^2$.

III.4.1 More conditions on the coordinates x, y. Choose affine coordinates x, y satisfying the conditions P1 and P2 in III.2.1, and also assume the following.

P3. All intersections on $\mathcal{L}$ lie on the affine plane.

This condition implies that none of the lines in $\mathcal{L}$ is the line at infinity. Furthermore, the slopes $m_1, \ldots, m_k$ of the lines in $\mathcal{L}$ can be strictly ordered

$$m_1 > m_2 > \cdots > m_k.$$

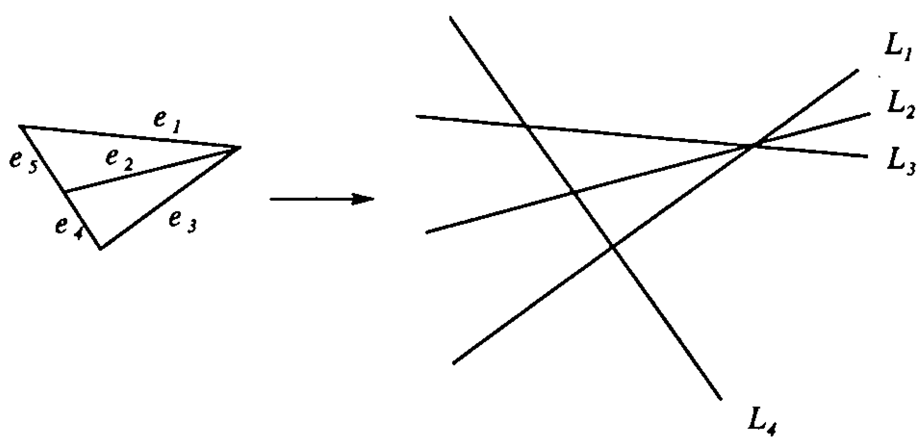
P4. All slopes are nonzero.

III.4.2 Intersection graph. Let Γ be the graph with vertices v corresponding to points of intersection S of $\mathcal{L}$, and with edges e labelled L given by the line segments lying between adjacent points of intersection on $L \cap \mathbf{R}^2$. Let

$$f : \Gamma \to \mathbf{P}^2$$

be the natural inclusion. Note that this graph satisfies the conditions of Definition II.2.2.

III.4.3 Example.



Here the edges are labelled as follows:

e_3	is labelled	L_1
e_2	"	L_2
e_1	"	L_3
e_4, e_5	are labelled	L_4

III.4.4 A lifting for $f : \Gamma \to \mathbf{P}^2$. We will use Proposition I.2.11 to show there is a lifting for $f : \Gamma \to \mathbf{P}^2$.

Note that the set

$$A = \{(x_0 + i\theta, y_0) : (x_0, y_0) \in \mathbf{R}^2, 0 < \theta \leq 1\},$$

which is homeomorphic to $\mathbf{R}^2 \times (0,1]$, is contained in $\mathbf{P}^2 - \mathcal{L}$. This is because all lines in $\mathcal{L}$ are given by equations of the form

$$y = mx + b$$

where $m \neq 0$ and $m, b \in \mathbf{R}$, so for $(x, y) \in A$ the imaginary part on the left side of the equation is 0 while on the right it is $m\theta > 0$.

Define

$$h : [0,1] \times \Gamma \to \mathbf{P}^2$$

by $h_\theta(\gamma) = f(\gamma) + (i\theta, 0)$. Then, for $\theta > 0$, $h_\theta(\gamma) \in A$, so

$$h((0,1] \times \Gamma) \subset A.$$

Since A is contractible and is contained in $\mathbf{P}^2 - \mathcal{L}$,

$$h_*(\pi_1((0,1] \times \Gamma))$$

is trivial. Therefore, by Proposition I.2.11, there is a lifting map

$$f' : \Gamma \to X$$

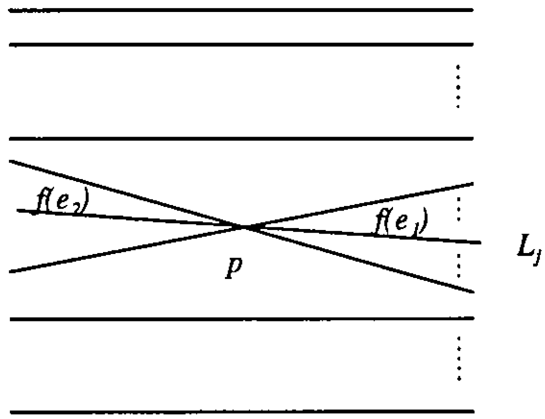
so that $\rho(f'(\gamma)) = f(\gamma)$.

III.4.5 Shifting data for the lifting. We now want to define a map

$$\psi : \mathcal{I} \to G$$

from the set $\mathcal{I}$ of pairs (e_1, e_2) labelled by the same line L and meeting at a common vertex v, to G, so that, if e_1 and e_2 are labelled L, $\psi(e_1, e_2)f'(e_1)$ and $f'(e_2)$ lie on the same curve in $\rho^{-1}(L)$.

We find ψ using the fibration and monodromy described in III.2. Suppose $f(e_1)$ and $f(e_2)$ lie on the line $L = L_j$ and $f(e_1)$ and $f(e_2)$ are joined at the point $p \in S$. Locally near p, $\mathcal{L} \cap \mathbf{R}^2$ looks like the following picture.



Let $P_x : \mathbf{C}^2 - \mathcal{L} \to \mathbf{C}$ be the projection $P_x(x, y) = x$ as in III.2.4 and assume, by a suitable change of coordinates if necessary, that $P_x(p) = 0$ and $P_x^{-1}([-1, 1]) \cap S = \{p\}$.

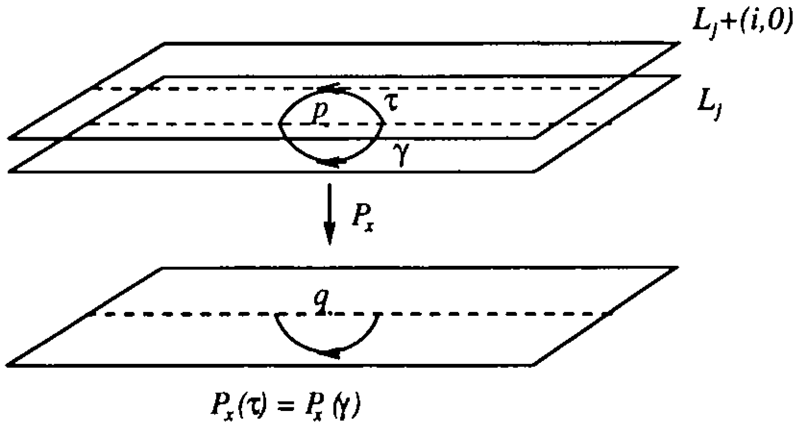
Define

$$\gamma : [0, 1] \to \mathbf{P}^2$$

so that $\gamma(\theta)$ equals the point $P_x^{-1}(e^{\pi i \theta}) \cap L$ and define

$$\tau : [0, 1] \to \mathbf{P}^2$$

so that $\tau(\theta)$ equals $(\sin(\pi\theta)i, 0) + P_x^{-1}(\cos(\pi\theta)) \cap L$. Note that $\gamma(0) = \tau(0)$, $\gamma(1) = \tau(1)$, the x coordinates of $\gamma(\theta)$ and $\tau(\theta)$ are equal, $\gamma(\theta) \subset L$ and $\tau(\theta) \subset A$ for all $0 < \theta < 1$.



Take the quotient covering $\overline{\rho} : \overline{X} \to \mathbf{P}^2$, where $\overline{X}$ is the surface obtained by modding X out by the action of I_{L_j}. Then the covering $\overline{\rho}$ is defined by the map

$$\overline{\phi} : \mathrm{H}_1(\mathbf{P}^2 - (\overline{\mathcal{L} - L_j}); \mathbf{Z}) \to \mathrm{H}_1(\mathbf{P}^2 - (\overline{\mathcal{L} - L_j}); \mathbf{Z}/n\mathbf{Z})$$

and the group of covering automorphisms of $\overline{\rho}$ is the group

$$\overline{G} = \mathrm{H}_1(\mathbf{P}^2 - (\overline{\mathcal{L} - L_j}); \mathbf{Z}/n\mathbf{Z}),$$

or G/I_{L_j}.

III.4.6 LEMMA. *Let $\psi(e_1, e_2)$ be any element of G so that the image of $\psi(e_1, e_2)$ in $\overline{G}$ equals $\overline{\phi}(\tau\gamma^{-1})$. Then $\psi(e_1, e_2)f'(e_1)$ and $f'(e_2)$ lie on the same curve in $\rho^{-1}(L_j)$.*

Proof. We have a composition of coverings

$$\begin{array}{ccc} X & \xrightarrow{\rho_j} & \overline{X} \\ & & \downarrow{\scriptstyle \overline{\rho}} \\ & & \mathbf{P}^2, \end{array}$$

where $\rho(x) = \overline{\rho}(\rho_j(x))$ for all $x \in X$ and I_j is the Galois group for ρ_j. Since I_{L_j} is the inertia subgroup for L_j in the composition covering ρ, $\overline{\rho}$ is one to one over $\rho^{-1}(L_j)$. Therefore, if $\overline{\psi}(e_1, e_2)$ is any element of G/I_j so that

(1) $\overline{\psi}(e_1, e_2)\rho_j(f'(e_1))$ and $\rho_j(f'(e_2))$ lie one the same curve in $\overline{\rho}^{-1}(L_j)$

(2) $\overline{\psi}(e_1, e_2)$ is the image of $\psi(e_1, e_2)$ in $\overline{G}$,

then $\psi(e_1, e_2)f'(e_1)$ and $f'(e_2)$ lie on the same curve in $\rho^{-1}(L_j)$.

Thus, we need to show that $\overline{\phi}(\tau\gamma^{-1})\rho_j(f'(e_1))$ and $\rho_j(f'(e_2))$ lie on the same curve in $\overline{\rho}^{-1}(L_j)$. The image of τ is contained in

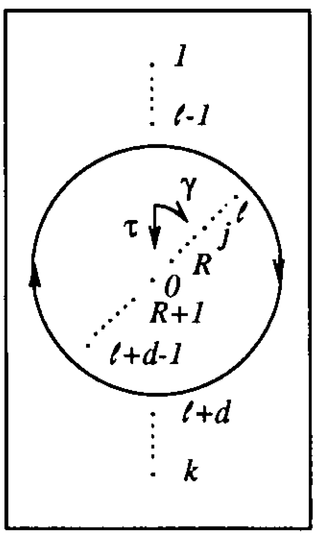
$$h([0, 1] \times \Gamma),$$

so any lift of τ with initial point in $\rho_j(f'(e_1))$ has endpoint on $\rho_j(f'(e_2))$. On the other hand, since γ doesn't pass through any points in S, the image of γ is contained in L_j, so any lift of γ^{-1} with initial point on $\rho_j(f'(e_2))$ has endpoint on the same curve in $\rho^{-1}(L_j)$ as the one containing $\rho_j(f'(e_2))$.

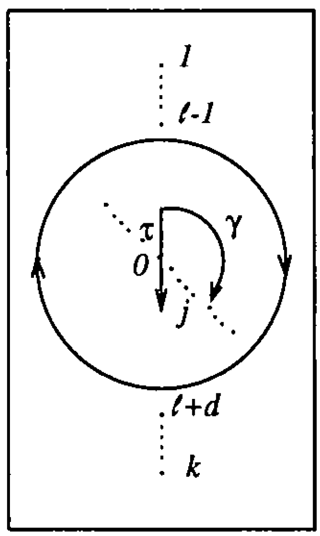
Therefore, the action of $\overline{\phi}(\tau\gamma^{-1})$ takes points on $\rho_j(f'(e_1))$ to points on the curve in $\overline{\rho}^{-1}(L_j)$ containing $\rho_j(f'(e_2))$. ∎

We now have left to find $\overline{\phi}(\tau\gamma^{-1})$. To do this we will look at the analysis in III.2.9 in more detail. Let $t_1, \ldots, t_k$ be the locally ordered points in T_1 with respect to the fiber $P_x^{-1}(1)$. We saw in III.2.9 that the local monodromy around the fiber over 0 fixes the locally ordered points numbered $1, \ldots, \ell - 1$ and $\ell + d, \ldots, k$, and rotates a disk containing the points numbered $\ell, \ldots, \ell + d - 1$ counterclockwise by 180 degrees. This corresponds to the braid $\Sigma_{d,\ell}$ as given in Definition III.2.11.

It is important to notice where the center of rotation of this disk is in relation to the points $\ell, \ldots, \ell + d - 1$. Let R be the first (global) index so that the line corresponding to t_R has positive slope and the line corresponding to t_{R+1} has negative slope. If R is between ℓ and $\ell + d - 1$ then the center of rotation of the disk occurs somewhere between R and $R+1$. Thus the fibers fibers $F_{P(\gamma^+(\theta))}$ and $F_{P(\gamma^-(\theta))}$ vary as in the following pictures. (Here τ and γ are drawn for the case that $\ell < j < R$.)



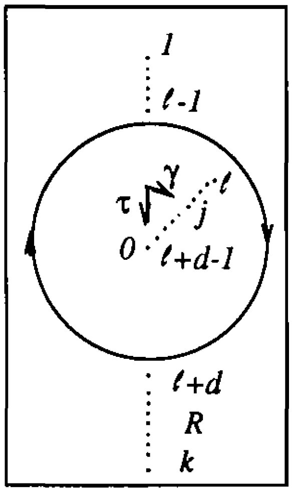
θ = 1/3



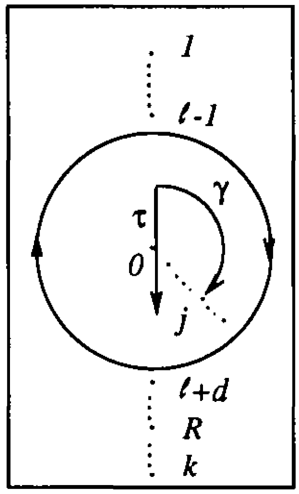
θ = 2/3

If R is not between ℓ and $\ell + d - 1$ then the center of rotation is either somewhere above ℓ (if $R < \ell$) or somewhere below $\ell + d - 1$ (if $R > \ell + d - 1$). For example, if

$R > \ell + d - 1$ we have the following picture.



$\theta = 1/3$



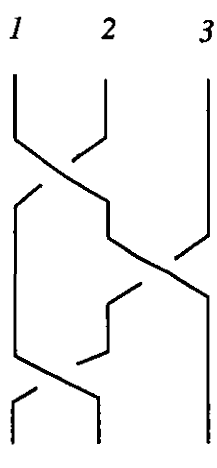
$\theta = 2/3$

Therefore,

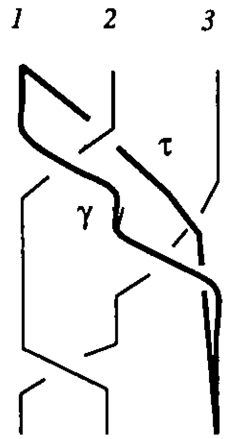
$$\overline{\phi}(\tau\gamma^{-1}) = \begin{cases} -\displaystyle\sum_{r=j+1}^{\min(R,\ell+d-1)} \overline{g}_{L_r} & \text{if } j < R \\ -\displaystyle\sum_{r=\max(R+1,\ell)}^{j-1} \overline{g}_{L_r} & \text{if } j > R \end{cases}$$

where $\overline{g}_L$ equals the image of g_L in $\overline{G}$.

III.4.7 Example. Assume $k = 3$, $d = 3$, $R = 2$ and $j = 1$. Then the braid associated to the monodromy looks like this:



and if we draw in the paths γ and τ in bold face, we have the following picture.



It is easy to see that $\tau\gamma^{-1} = \mu_2^{-1}$, so $\overline{\phi}(\tau) = -\overline{g}_2$.

We can put together the local information to obtain the following global result.

III.4.8 PROPOSITION. *For each $L \subset \mathcal{L}$, order the points $p_1, \ldots, p_{r_L} \in S \cap L$ so that*

$$P_x(p_1) > \cdots > P_x(p_{r_L}).$$

Let $e_1, \ldots, e_{r_L - 1}$ be the edges in Γ labelled L so that $P_x(e_i)$ is the interval between $P_x(p_i)$ and $P_x(p_{i+1})$. Let ϕ_i be any element of G mapping to $\overline{\phi}(\tau\gamma^{-1})$ in G/I_L as defined above for e_{i-1} and e_i. For each $p_j \in S \cap L$, let

$$\psi_{p_j} = \begin{cases} 0 & \text{if } j = 1 \\ \phi_1 \ldots \phi_{j-1} & \text{otherwise.} \end{cases}$$

Define

$$\Psi : \mathcal{J} \to G$$

so that for each L and $p \in S \cap L$

$$\psi(p, L) = \psi_p.$$

Then there exists a lifting L' of L for each $L \subset \mathcal{L}$ so that Ψ is lifting data for L'.

Proof. Define L' to be the lift of L containing the edge $f'(e_1)$. The rest follows from II.2.6. ∎

We are now ready to find lifting data for a $\widehat{\mathcal{L}}$ lifting in $\widehat{p} : \widehat{X} \to \widehat{Y}$.

III.4.9 PROPOSITION. *For each proper transform $\widehat{L} \in \widehat{\mathcal{L}}$ of a line L in $\mathcal{L}$, let $\widehat{L}'$ be the curve in $\widehat{\rho}^{-1}(\widehat{L})$ corresponding to L' under the birational map $\widehat{\sigma} : \widehat{X} \to X$. For each point $p \in T$, let E'_p be the curve in $\rho^{-1}(E_p)$ mapping to $f'(p)$ under $\widehat{\sigma}$. Let*

$$\widehat{\Psi} : \widehat{\mathcal{J}} \to G$$

be defined by $\widehat{\Psi}(q, \widehat{L}) = \Psi(\sigma(q), L)$ for all lines L in $\mathcal{L}$ and let $\widehat{\Psi}(E_p)$ be the identity element. Then $\widehat{\Psi}$ is lifting data for the liftings.

PROOF: Since $f'(p)$ is in $\Psi(p, L)L'$ for all $p \in S$ and $p \in L \subset \mathcal{L}$, the result follows from Proposition II.1.4. ∎

Chapter IV. Algorithm for Computing the First Betti Number

In this chapter we give an explicit algorithm for finding $b_1(\widehat{X})$. The algorithm breaks up into three parts:

INPUT. Create input for the algorithm. To do this we find a choice of coordinates satisfying certain criteria.

A. A point/line incidence matrix M for the line configuration $\mathcal{L}$ with respect to a choice of coordinates satisfying certain criteria.

B. An integer n so that the surface $\widehat{X}$ is determined by the canonical map

$$\pi_1(\mathbf{P}^2 - \mathcal{L}, *) \to \mathrm{H}_1(\mathbf{P}^2 - \mathcal{L}; \mathbf{Z}/n\mathbf{Z}).$$

C. The index R of the first line in $\mathcal{L}$ whose slope is negative.

STEP 1. Find the first Betti number b_1^u of the unbranched part X^u of $\rho : X \to \mathbf{P}^2$ using the following substeps.

A. Make a point/line incidence matrix $\overline{M}$ for lines $\mathbf{C}^2 - \overline{\mathcal{L}}$, where $\mathbf{C}^2$ denotes the affine plane given by $z \neq 0$ and $\overline{\mathcal{L}}$ is the intersection of $\mathcal{L}$ with $\mathbf{C}^2$.

B. Find a presentation for the fundamental group of $\mathbf{P}^2 - \mathcal{L}$.

C. Compute the Alexander matrix associated to the presentation.

D. Find b_1^u.

STEP 2. Find the nullity Null(I) of the intersection matrix for curves in $\widehat{\rho}^{-1}(\widehat{\mathcal{L}})$ using the following substeps.

A. Make a point/curve incidence matrix for curves in $\widehat{\mathcal{L}}$.

B. Make a shift matrix for $\widehat{\mathcal{L}}$.

C. Order the curves above $\widehat{\mathcal{L}}$ (using generators for the stabilizer subgroups).

D. Make an intersection matrix for curves above $\widehat{\mathcal{L}}$.

E. Find the nullity of the intersection matrix.

By Proposition I.6.3, the difference $b_1^u - \mathrm{Null}(I)$ equals $b_1(\widehat{X})$.

INPUT

The format of the input is important in making the later calculations from this input easier.

IV.1 Conditions on coordinates x, y. Recall the conditions on x, y given in III.2 and III.4. Properties P1 and P2 are needed to implement the algorithm for finding b_1^u described in III.4 and P1, P2, P3 and P4 are needed to implement the algorithm for finding the lifting data as described in III.4.

P1. Each L_α in $\overline{\mathcal{L}}$ is given by an equation of the form

$$y = m_\alpha x + b_\alpha,$$

where m_α and b_α are real.

P2. The projection P_x sends the set of all intersections S on $\mathcal{L} \cap \mathbf{C}^2$ to distinct (necessarily real) points Q in $\mathbf{C}$.

P3. All points in S lie on the affine plane.

P4. All slopes m_α are nonzero.

Add two more conditions.

P5. For some j_0, $p_j \in L_k$ for all $j \geq j_0$, and rotating the affine plane so that L_k becomes vertical doesn't change the ordering of the x-coordinates of points in $S - L_k$.

This property can always be achieved by changing coordinates if necessary so that the line $y = (m_k + \epsilon)x + b_k$ goes to infinity, where $\epsilon > 0$ is chosen small enough (this process would require changing the ordering of $L_1, \ldots, L_{k-1}$ and $p_1, \ldots, p_s$).

P6. $P_x(p) > 0$ for all $p \in S$.

By shifting x by a constant x_0 greater than $|P_x(p_s)|$ we can make sure property P6 holds without changing the previous conditions.

As a consequence of these conditions, we have orderings $L_1, \ldots, L_k$ of the lines in $\mathcal{L}$ so that the slopes are strictly decreasing:

$$m_1 > m_2 > \cdots > m_k,$$

and orderings $p_1, \ldots, p_s$ for points in the set of intersections S of $\mathcal{L}$ so that

$$P_x(p_1) > P_x(p_2) > \cdots > P_x(p_s).$$

Condition P6 implies that the y intercepts of the defining equations for $\mathcal{L}$ satisfy

$$b_1 < b_2 < \cdots < b_k.$$

We make the definition of a point/curve incidence correspondence for any collection of curves on a surface, since we will also use one for $\widehat{\mathcal{L}}$ in $\widehat{\mathbf{P}}^2$ in the algorithm. The definition also makes sense for curves lying on a quasi-projective surface, for example, $\mathbf{P}^2$ minus a line "at infinity."

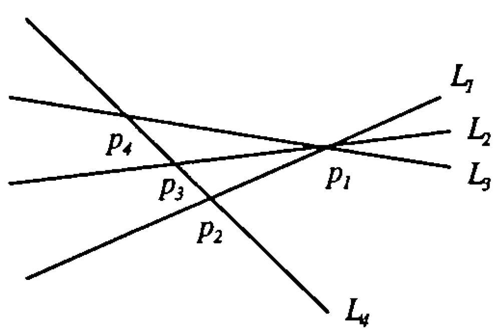
IV.2 Definition. Let C be a union of k curves on a quasi-projective surface Y with orderings $C_1, \ldots, C_k$ for the curves in C and $p_1, \ldots, p_s$ for the points of intersection S on C. The *point/curve incidence matrix* M for C with these orderings is the s by k matrix with entries

$$a_{i,j} = \begin{cases} 1 & \text{if } p_i \in C_j \\ 0 & \text{otherwise.} \end{cases}$$

In the special case where C is a configuration of lines in $\mathbf{P}^2$ we will also call this the *point/line incidence matrix.*

Let M be the point/line incidence matrix defined by the orderings of $\mathcal{L}$ and S determined by the coordinates x, y, as in Definition IV.2, let n be the order of the coefficient group, and let R be the last index so that L_R has positive slope with respect to the coordinate. The algorithm which we are about to describe takes as input M, n and R.

IV.3 Example. Recall the line configuration of III.2.3.



The point/line incidence matrix is

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

The last index R so that L_R has positive slope is 2

STEP 1: First Betti Number of Unbranched Part

Summary. To find the fundamental group we first send one line to infinity. This involves a change of coordinates which we need to show still satisfy properties P1 and P2 of III.3.1, and we get a new point/line incidence matrix. We then apply the methods described in III.2 and III.3 to find a presentation for $\pi_1(\mathbf{P}^2 - \mathcal{L})$. Applying Fox calculus, we obtain the Alexander matrix, a presentation matrix for $\mathrm{H}_1(X^u; \mathbf{Z})$ as a $\mathbf{Z}[G]$-module, where G is the Galois group of the covering.

A. Point/line incidence matrix for the affine part of $\mathbf{P}^2 - \mathcal{L}$

We first show that a new point/line incidence matrix $\overline{M}$ associated to a choice of coordinates where one line in $\mathcal{L}$ is sent to infinity can be obtained as follows.

IV.4 LEMMA. *The following change of coordinates leads to new coordinates satisfying the conditions of III.2.1.*

C1. Rotate the affine plane so that the equation for the line L_k becomes

$$x = 0;$$

C2. Apply the change of coordinates

$$[x : y : z] \mapsto [-z : y : x].$$

Let $\overline{M}$ be the matrix obtained from M by chopping off the rows with a 1 in the last column (i.e, the rows $j = j_0, \ldots, s$ as in property (P5) of the coordinates). Then $\overline{M}$ is the point/line incidence matrix for $L \cap \mathbf{C}^2$ with respect to this ordering.

Proof. Clearly, the two changes of coordinates preserve $\mathbf{RP}^2$, so the new coordinates still give real equations for $\mathcal{L}$ and thus satisfy P1.

To prove the rest of the lemma it suffices to show that the changes of coordinates preserve the ordering of the slopes of lines in $\mathcal{L}$ and of the x-coordinates of points in S.

Note that the ordering of the slopes of lines in $\mathcal{L}$ corresponds to the natural ordering (from largest to smallest) of the intersections of $\mathcal{L}$ with a vertical real line $x = \alpha$, where α is greater than the x-coordinate of any point in S. Any rotation of the affine plane preserves the ordering of these intersections for all lines which don't become vertical during the rotation.

Since m_k is the smallest slope of any line in $\mathcal{L}$, the rotation of C1 preserves the orderings of the slopes of the lines $L_1, \ldots, L_k$. By the same reasoning the ordering of the y-intercepts also do not change.

Herafter for this part of the algorithm we will replace $k-1$ by $\overline{k}$. Therefore, after C1 the new equations for $L_1, \ldots, L_{\overline{k}}$ are

$$L_i : y = \overline{m}_i x + \overline{b}_i, \qquad \text{for } i = 1, \ldots, \overline{k},$$

where

$$\overline{m}_1 > \overline{m}_2 > \cdots > \overline{m}_{\overline{k}}$$

and

$$(*) \qquad \overline{b}_1 < \overline{b}_2 < \cdots < \overline{b}_{\overline{k}}.$$

By property P5, the ordering of the x-coordinates of points in S is also perserved.

Clearly, if we follow with C2, L_k goes to the line at infinity. The affine equations for the lines $L_1, \ldots, L_{\overline{k}}$ become

$$L_i : y = -\overline{b}_i x + \overline{m}_i.$$

(The ordering of the y-intercepts reverses, but their ordering is not important for this part of the algorithm.) Thus, by $(*)$, the ordering of the slopes remains the same. Furthermore, if $x_1, \ldots, x_s$ are the ordered x-coordinates for points in S under the coordinate system obtained after Step (1), the new x coordinates are

$$-\frac{1}{x_1}, -\frac{1}{x_2}, \ldots, -\frac{1}{x_s},$$

so the ordering remains the same for points in S as well. ∎

B. Presentation of $\pi_1(\mathsf{P}^2 - \mathcal{L})$

We now use $\overline{M}$ to find a presentation for $\pi_1(\mathbf{C}^2 - \overline{\mathcal{L}})$. Let $\Sigma(\Gamma_j)$, $j = 1, \ldots, s$ be elements of $\mathcal{B}_k$ as in Proposition III.2.13, let $\mathcal{M}$ be the homomorphism

$$\mathcal{M} : \mathcal{B}_k \to \operatorname{Aut} F_{\overline{k}}$$

where F_k is the free group on generators $\mu_1, \ldots, \mu_{\overline{k}}$ as in Definition III.3.1, and let

$$R_{i,j} = \mu_i^{-1} \mathcal{M}(\Gamma_j)_*(\mu_i)$$

for $i = 1, \ldots, \overline{k}$ and $j = 1, \ldots, j_0 - 1$. Then as we saw by Proposition III.3.2,

$$< \mu_1, \ldots, \mu_{\overline{k}} : R_{i,j}, i = 1, \ldots, \overline{k}; j = 1, \ldots, j_s >$$

is a presentation for the fundamental group of $\mathsf{P}^2 - \mathcal{L}$.

To compute $R_{i,j}$ explicitly we use the following definition. For ease of notation we make this definition for an arbitrary point/line incidence matrix M corresponding to an ordering of k affine real lines $\mathcal{L}$ and intersection points S on $\mathcal{L}$, defined by coordinates x and y satisfying the conditions in Definition III.2.1.

IV.5 Definition. Define $\Sigma_1, \ldots, \Sigma_s$ in $\mathcal{B}_k$ as follows.

(1) Look at the first row of M. Let Σ_1 equal $\Sigma_{\ell,d}^2$, where ℓ is the first column of M containing a nonzero entry and d is the number of nonzero entries in M (they will be consecutive) and $\Sigma_{\ell,d}$ is as in Definition III.2.11. Let $\overline{\Sigma_1}$ equal $\Sigma_{\ell,d}$.

(2) Given the previous $\overline{\Sigma_r}$, let $\overline{\sigma_r}$ be the element of the symmetric group on $\overline{k}$ elements in the image of $\overline{\Sigma_r}$ under the natural map

$$\mathcal{B}_k \to \mathrm{Sym}_{\overline{k}}.$$

Define

$$\Sigma_r = \overline{\Sigma_r}(\Sigma_{\ell,d})^2\overline{\Sigma_r}^{-1},$$

where ℓ equals $\overline{\sigma_r}$ applied to the first column containing a nonzero entry in the current row and d is the number of nonzero entries in this row.

IV.6 PROPOSITION. *If we use the matrix $\overline{M}$, then the $\Sigma_1, \ldots, \Sigma_s$ defined in Definition IV.5 generate the monodromy of the fibration P_x on $\mathbf{C}^2 - \overline{\mathcal{L}}$.*

Proof. This follows from Proposition III.2.13. ∎

Now we can find $R_{i,j} = \mu_j^{-1}\mathcal{M}\Sigma_i(\mu_j)$ explicitly using Definition III.3.2.

C. Alexander Matrix

We compute the Alexander matrix of the presentation using Fox Calculus (see [**Fo1**]), which we summarize here.

IV.7 Definition. Given a free group H generated by $\mu_1, \ldots, \mu_{\overline{k}}$, *Fox derivatives* are maps $\frac{\partial}{\partial\mu_1}, \ldots, \frac{\partial}{\partial\mu_{k-1}}$ from H to the group ring $\mathbf{Z}[\widetilde{H}]$ of the abelianization $\widetilde{H}$ of H defined as follows.

$$\frac{\partial}{\partial\mu_i}(fg) = \frac{\partial}{\partial\mu_i}(f) + f\frac{\partial}{\partial\mu_i}(g);$$

$$\frac{\partial}{\partial\mu_i}(f) = 0 \qquad \text{if } f \text{ can be presented as a word not involving } \mu_i;$$

$$\frac{\partial}{\partial\mu_i}(\mu_i^n) = \begin{cases} 1 + \mu_i + \cdots + \mu_i^{n-1}, & \text{if } n > 0 \\ -\mu_i^{-1} - \mu_i^{-2} - \cdots - \mu_i^{n}, & \text{if } n < 0. \end{cases}$$

Let $\phi : \mathrm{H}_1(\mathbf{P}^2 - \mathcal{L}; \mathbf{Z}) \to G$ be the defining map for the covering. Then ϕ induces a map

$$\widetilde{\phi} : \mathrm{H}_1(\mathbf{P}^2 - \mathcal{L}; \mathbf{Z}) \to \mathbf{Z}[G],$$

given by composing ϕ with the inclusion of G in $\mathbf{Z}[G]$. Recall that $\mathrm{H}_1(\mathbf{P}^2 - \mathcal{L}; \mathbf{Z})$ is canonically isomorphic to the abelianization of $\pi_1(\mathbf{P}^2 - \mathcal{L}, *)$.

Let $< \mu_1, \ldots, \mu_k : R_1, \ldots, R_N >$ be a presentation for $\pi_1(\mathbf{P}^2 - \mathcal{L}, *)$. Let A be the matrix of Fox derivatives

$$\widetilde{\phi}(\frac{\partial R_i}{\partial\mu_j}).$$

The matrix A is called the *Alexander matrix* for the presentation.

The following proposition is a special case of Fox's result ([**Fo2**], (3.5), p. 411.)

IV.8 PROPOSITION. *The Alexander matrix A is a presentation matrix for the first homology group $\mathrm{H}_1(X^u, F; \mathbf{Z})$ considered as a $\mathbf{Z}[G]$-module, for F any fiber.*

D. Computing b_1^u

Let Ω^k be the set of k-tuples of nth roots of unity and, for each element $\omega = (\omega_1, \ldots, \omega_{\overline{k}})$ in $\Omega_{\overline{k}}$, let

$$\tau_\omega : \mathbf{Z}[G] \to \mathbf{Z}[\Omega^n]$$

be the $\mathbf{Z}$-module homomorphism defined by

$$\tau_\omega(g_i) = \omega_i,$$

where $g_i = \phi(\mu_i)$. Let A_ω be the matrix obtained from A by replacing each entry $a_{i,j}$ by $\tau_\omega(a_{i,j})$. Let r_ω be the rank of A_ω.

The following result is a consequence of [**Li4**], p. 2, Theorem 1 (see also [**Ho**], §2).

IV.9 THEOREM. *The first Betti number of X^u is given by*

$$b_1^u = \sum_{\omega \in \Omega^n} \overline{k} - 1 - r_\omega$$

This completes **STEP 1** of the algorithm.

STEP 2 : Intersection Matrix for Curves Above Branch Locus

Summary. As we saw in Chapter II, finding an intersection for curves in $\widehat{X}$ above $\widehat{\mathcal{L}}$ requires lifting data, i.e., a way to choose curves C' in the covering, one above each curve C in $\widehat{\mathcal{L}}$, together with the information of which group action makes two curves meet above a specified point. In Step 2, we make a point/curve incidence matrix $\widehat{M}$ for $\widehat{\mathcal{L}}$ and then replace each entry in $\widehat{M}$ by a group element, so that in the (p, C) entry we have the element $\Psi(p, C)$ with the property that

$$\Psi(p, C)C' \cap \Psi(p, D)D'$$

meet in the fiber $\rho^{-1}(p)$.

Using Propositions III.4.8 and III.4.9 we show how to define a map

$$\widehat{\Psi} : \widehat{\mathcal{J}} \to G,$$

using M and R, so that for some choice of lifting of the curves in $\widehat{\mathcal{L}}$ in the covering $\widehat{\rho} : \widehat{X} \to \widehat{\mathbf{P}}^2$, Ψ gives lifting data.

From this information, we use the formula in II.3.1 to find the intersection matrix I of the curves in $\widehat{\rho}^{-1}(\widehat{\mathcal{L}})$. Then, by Proposition I.6.3,

$$b_1 = b_1^u - \mathrm{Null}(I).$$

A. Point/curve incidence matrix for $\widehat{\mathcal{L}}$

We begin by ordering the curves in $\widehat{\mathcal{L}}$ and the points $\widehat{S}$ of intersection on $\widehat{\mathcal{L}}$.

The curves in $\widehat{\mathcal{L}}$ are proper transforms $\widehat{L}_1, \dots, \widehat{L}_k$ of lines $\mathcal{L}$ and exceptional curves E_q for points $q \in T$. Order the points $q_1, \dots, q_t$ in T so that each $q_i = p_r$, where p_r is the ith point in the sequence $p_1, \dots, p_s$ through which more than two lines pass.

Order the curves in $\mathcal{L}$ as follows:

$$\widehat{L}_1, \dots, \widehat{L}_k, E_{q_1}, \dots, E_{q_t}.$$

For each point $p_r \in S$ with only two lines L_{j_1} and L_{j_2} passing through p_r, set $d_r = 1$. There is a single corresponding point $\widehat{p}_r$ in $\widehat{L}_{j_1} \cap \widehat{L}_{j_2}$.

For each point $p_r \in S$ with $p_r = q_u$ for some q_u in T, there are d_r distinct points $\widehat{p}_{r,1}, \dots, \widehat{p}_{r,r_{d_r}}$ in $\widehat{\sigma}^{-1}(p_r) = E_{q_u}$ so that the proper transforms $\widehat{L}_{j_1}, \dots, \widehat{L}_{j_{d_r}}$ intersect E_{q_u}.

We thus have an ordering for the points in $\widehat{S}$:

$$\widehat{p}_{1,1}, \dots, \widehat{p}_{1,d_1}, \widehat{p}_{2,1}, \dots, \widehat{p}_{2,d_2}, \dots, \widehat{p}_{s,1}, \dots, \widehat{p}_{s,d_s}.$$

Define $\widehat{M_1}, \dots, \widehat{M_s}$ to be the matrices defined as follows.

(1) If row r of M has only two columns j_1 and j_2 with entry equal to 1 then let $\widehat{M_r}$ be the $1 \times (k+t)$ matrix with a 1 in the j_1 and j_2 columns and zero elsewhere.

(2) If row r of M has columns $j_1, \dots, j_d$ with entries equal to 1, with $d > 2$, then let $\widehat{M}_r$ be the $d \times (k+t)$ matrix with a 1 in the ℓ, j_ℓ and $\ell, k+u$ place for $\ell = 1, \dots, d$, if r is the uth row of M containing more than two entries equal to 1.

Let $\widehat{M}$ be the matrix obtained by concatenating these matrices.

The following proposition can be checked easily from the definitions.

IV.10 PROPOSITION. *The matrix $\widehat{M}$ defined as above is the point/curve incidence matrix for the curves in $\widehat{\mathcal{L}}$ and the points of intersection $\widehat{S}$ in $\widehat{\mathcal{L}}$ ordered as above.*

B. Shift matrix for $\widehat{\mathcal{L}}$

To find the shift matrix for $\widehat{\mathcal{L}}$ we begin by finding one for $\mathcal{L}$.

IV.11 Definition. Let $a_{i,j}$, $i = 1, \dots, s$; $j = 1, \dots, k$ be the entries of M. Define the shift matrix $\mathrm{Sh}(\mathcal{L})$ with entries $b_{i,j}$ inductively on i as follows.

(1) Row 1: $b_{1,j} = 0$ for all $j = 1, \dots, k$.

(2) Row i:

$$b_{i,j} = \begin{cases} b_{i-1,j} & \text{if } a_{i,j} = 0 \text{ or } j = R \text{ or } j = R+1; \\ b_{i-1,j} - \displaystyle\sum_{\alpha=j+1}^{R} g_\alpha a_{i,\alpha} & \text{if } a_{i,j} = 1 \text{ and } j < R; \\ b_{i-1,j} - \displaystyle\sum_{\alpha=R+1}^{j-1} g_\alpha a_{i,\alpha} & \text{if } a_{i,j} = 1 \text{ and } j > R. \end{cases}$$

By Proposition III.4.8, there is a choice of lifting L'_j for each L_j in $\mathcal{L}$ so that

$$\Psi : \mathcal{J} \to G$$

$$(p_i, L_j) \mapsto b_{i,j}$$

are lifting data.

Now we are ready to find the shift matrix for $\widehat{\mathcal{L}}$.

IV.12 Definition. For $r = 1, \dots, s$, let $\widehat{\mathrm{Sh}}_r$ be the matrix defined as follows

(1) If row r of M has only two columns j_1 and j_2 with entries equal to 1, then let $\widehat{\mathrm{Sh}}_r$ be the $1 \times (k+t)$ matrix with entries b_{r,j_1} in the j_1 column, b_{r,j_2} in the j_2 column and zeros elsewhere.

(2) If row r of M has columns $j_1, \dots, j_d$ with entries equal to 1 and $d > 2$, then let $\widehat{\mathrm{Sh}}_r$ be the $d \times (k+t)$ matrix with entries b_{r,j_ℓ} in the ℓ, j_ℓ place for $\ell = 1, \dots, d$ and zeros elsewhere.

Let $\mathrm{Sh}(\widehat{\mathcal{L}})$ be the matrix given by concatenating $\widehat{\mathrm{Sh}}_1, \dots, \widehat{\mathrm{Sh}}_s$. Since $\mathrm{Sh}(\widehat{\mathcal{L}})$ has the same dimensions as $\widehat{M}$, there is a well-defined bijection from $\widehat{\mathcal{J}}$ to entries of $\mathrm{Sh}(\widehat{\mathcal{L}})$, which we can think of as a correspondence between pairs (p, C) in $\widehat{\mathcal{J}}$ to integer pairs (i_p, j_C), where i_p is the row corresponding to the point p and j_C is the column corresponding to the curve C in $\widehat{M}$.

By Proposition III.4.9, there is a choice of lifting C' for each curve C in $\widehat{\mathcal{L}}$ in the covering

$$\widehat{\rho} : \widehat{X} \to \widehat{\mathbf{P}}^2$$

so that

$$\Psi : \widehat{\mathcal{J}} \to G$$

$$(p, C) \mapsto b_{i_p, j_C}$$

is lifting data for the C'.

C. Ordering curves above $\widehat{\mathcal{L}}$

To find the intersection matrix for the curves in $\widehat{\rho}^{-1}(\widehat{\mathcal{L}})$ explicitly, we need to be able to order the curves in $\widehat{\rho}^{-1}(\widehat{\mathcal{L}})$ and find their intersection numbers.

Recall that the curves in $\rho^{-1}(C)$ for any curve C in $\widehat{\mathcal{L}}$ are in one to one correspondence with cosets of the stabilizer subgroup H_C associated to C. Thus our goal now is to find the stabilizer subgroups explicitly.

IV.13 PROPOSITION. *Consider G as the quotient of the free abelian $\mathbf{Z}/n\mathbf{Z}$-module A_k of rank k with basis $g_1, \dots, g_k$ by the submodule generated by $g_1 + g_2 + \cdots + g_k$. For each curve $C \subset \widehat{\mathcal{L}}$ we have:*

(1) *if $C = \widehat{L}_i$, for $i = 1, \dots, k$, then H_C is the submodule of G generated by the relation-free elements*

$$\sum_{p \in L_j} g_j$$

for all $p \in L_i \cap S$;

(2) if $C = E_{q_u}$, for $u = 1, \dots, t$, then H_C is the submodule of G generated by the relation-free elements $g_{j_1}, \dots, g_{j_d}$, where $L_{j_1}, \dots, L_{j_d}$ are the lines in $\mathcal{L}$ passing through q_u.

Proof. This is a restatement of Propositions III.1.4 and III.1.5. ∎

IV.14 Corollary. *The number of curves in $\widehat{\rho}^{-1}(C)$ for $C \subset \widehat{\mathcal{L}}$ is*

(1) n^{k-r-1} *if $C = \widehat{L}_j$ for some $j = 1, \dots, k$ and r is the number of entries in the jth column of M equal to 1;*

(2) n^{k-d-1} *if $C = E_{q_u}$ for some $u = 1, \dots, t$ and, for i such that $p_i = q_u$, d is the number entries in the ith row of M equal to 1.*

IV.15 Corollary. *With notation as in IV.14, the quotient G/H_C for $C \subset \widehat{\mathcal{L}}$ is a free $\mathbf{Z}/n\mathbf{Z}$-module of rank*

(1) $R_C = k - r - 1$ *if $C = \widehat{L}_j$,*

(2) $R_C = k - d - 1$ *if $C = E_{q_u}$.*

Furthermore, we can choose bases for these quotients as follows.

(1) *If $C = \widehat{L}_j$, G/H_C is freely generated by the images of elements of the form*

$$g_{i_1} - g_{i_2},$$

where L_{i_1} and L_{i_2} pass through a point $p \in L_j$ and i_2 is the largest index ($< j$) of a line in $\mathcal{L}$ passing through p;

(2) *If $C = E_{q_u}$, G/H_C is freely generated by the images of elements of the form*

$$g_{i_1} - g_{i_2},$$

where L_{i_1} and L_{i_2} don't pass through q_u and i_2 is the largest index of any line in $\mathcal{L}$ not passing through q_u.

Proof. The first part of the corollary follows trivially from Corollary IV.7. To show that the elements described above generate the quotient modules, we first check that the ranks are correct. For case (2) this is obvious. For case (1) assume there are r points in $S \cap \mathcal{L}$ and there are a total of d_i lines through the ith point in $S \cap L_j$ for each $i = 1, \ldots, r$. By definition we have $d_i - 2$ generators for each $p_i \in S \cap L_j$. Since all lines intersect in $\mathbf{P}^2$,

$$\sum_{i=1}^{r}(d_i - 1) = k - 1.$$

Therefore,

$$\sum_{i=1}^{r}(d_i - 2) = k - r - 1.$$

Let $\overline{G}$ be the quotient of G by the subgroup generated by the generators described above. Since the generators described are independent in G and H_C is clearly contained in the kernel of the map

$$G \to \overline{G},$$

$\overline{G}$ must be isomorphic to G/H_C. ∎

Now we order the curves in $\rho^{-1}(C)$ by ordering the elements of G/H_C in lexicographic order with respect to the choice of basis given in Corollary IV.15.

D. Intersection matrix for curves above $\widehat{\mathcal{L}}$

From Theorem III.2.1, to find the intersection number of curves lying above C and D we need to find the number of elements in

$$\alpha H_C \cap \beta H_D$$

explicitly for C and D in $\widehat{\mathcal{L}}$ and $\alpha, \beta \in G$.

Let τ_C be the quotient map

$$\tau_C : G \to G/H_C$$

for each $C \subset \widehat{\mathcal{L}}$. The number of elements in $\alpha H_C \cap \beta H_D$ equals the number of simultaneous solutions to

$$\tau_C(g) = \overline{\alpha}, \quad \tau_D(g) = \overline{\beta},$$

where $\overline{\alpha}$ and $\overline{\beta}$ are the images of α and β in G/H_C and G/H_D.

It is easiest to find the number of simultaneous solutions by writing the maps τ_C in matrix form. Recall that G is isomorphic to the quotient of A_k by the submodule I generated by $\sum_{i=1}^{k} x_i$. For each $C \subset \widehat{\mathcal{L}}$ define the matrices T_C as follows

(1) If $C = \widehat{L}_j$, for each row i in M with $d \geq 3$ column entries equal to 1, let $j_1, \ldots, j_{d-1}$ be the indices of these columns excluding j. Let $T_{C,i}$ be the $(d-2) \times k$ matrix with a -1 in all the j_{d-1} column entries, a 1 in the ℓ, j_ℓ entries, and zeros elsewhere. Concatenate the $T_{C,i}$ in the order of increasing i to get T_C.

(2) If $C = E_{p_r}$, where p_r corresponds to the ith row of M, let $j_1, \ldots, j_k$ be the columns with entries 0 in row i. Let T_C be the $(k-d-1) \times k$ matrix with entries -1 in the entire $j_{k-\ell}$ column, 1 in the ℓ, j_ℓ entries, and zeros elsewhere.

IV.16 PROPOSITION. *For each C, T_C is a matrix which represents a surjective module homomorphism*

$$T_C : A_k \to G/H_C$$

so that T_C is the composition of the quotient maps $A_k \to A_k/I = G$ and $G \to G/H_C$.

Proof. One observes that the rows of T_C correspond to the generators found in Corollary IV.15. ∎

IV.17 Blocks of the intersection matrix. We construct the intersection matrix I in blocks $I_{C,D}$ corresponding to how the curves above C and curves above D intersect, for C and D in $\widehat{\mathcal{L}}$.

Each curve in $\widehat{\mathcal{L}}$ corresponds to a column of the point/curve incidence matrix $\widehat{M}$. Let c, d denote the two columns corresponding to the curves C and D. Let R_C and R_D be as in Corollary IV.14. Define $M_{c,d}$ to be the $n^{R_c} \times n^{R_d}$ matrix with entries as follows.

(1) If $c = d \leq k$, then $M_{c,d}$ has entries

$$\frac{1}{n^3}(\text{rank}(T_C))(1 - R_C)$$

on the diagonal and zeros elsewhere.

(2) If $c = d > k$, then $I_{C,D}$ has entries

$$\frac{1}{n^3}(\text{rank}(T_C))(-1)$$

on the diagonal and zero elsewhere.

(3) If $c \neq d$ and $\widehat{M}$ has a row whose c and d columns don't both have entry 1, then $M_{c,d}$ is the zero matrix.

(4) If $c \neq d$ and $\widehat{M}$ has a row whose c and d columns have entry 1, let $\overline{\alpha}$ and $\overline{\beta}$ run through elements of G/H_C and G/H_D ordered lexicographically, and let $I_{C,D}$ be the matrix with entries

$$\frac{1}{n^3}(\text{number of solutions to } T_C X = \overline{\alpha},\, T_D X = \overline{\beta})$$

in the $\overline{\alpha}, \overline{\beta}$ place.

The $I_{C,D}$ defined in IV.17 is the intersection matrix for curves in $\widehat{\rho}^{-1}(C)$ and $\widehat{\rho}^{-1}(D)$. After concatenation we get the intersection matrix I for all curves in $\widehat{\rho}^{-1}(\widehat{\mathcal{L}})$.

E. Computing the nullity of the matrix I

We compute the nullity of I using basic integer row reduction. A problem with the algorithm is that the size of the matrix I grows as a polynomial in n, so it quickly becomes too large for a computer to handle.

Putting together the results of **STEP 1** and **STEP 2** gives the first Betti number of $\widehat{X}$.

Chapter V. Examples

In this chapter we summarize the classification of Hirzebruch coverings, following [**Hi**], and calculate geometric invariants which can be obtained from the algorithms and formulas described in this paper. These are the Betti numbers, Chern numbers, bounds on the Picard number, irregularity, algebraic and geometric genera.

The types of surfaces which occur as Hirzebruch coverings are ruled, elliptic, K3 and general type. We review Hirzebruch's classification and properties in V.1. We also prove in this section, following Ishida's analysis in [**I**], that the branched coverings X are complete intersections when $\mathcal{L}$ is a union of lines not all passing through a single point.

If the branch locus is a configuration of $k > 3$ lines in general position then the covering is simply connected and of general type (see V.2). There are specific kinds of line configurations which give rise to surfaces which are birationally equivalent to a product of curves (see V.3) or fibrations (see V.4). Examples of K3 surfaces and elliptic surfaces (see V.5) occur when we consider coverings with $n = 2$. The largest class of Hirzebruch coverings are general type (see V.6). In V.7 we give of a list of some computer output.

V.1 Classification of Hirzebruch coverings

We give here properties of Hirzebruch coverings and formulas for the Chern numbers and other geometric invariants. The results are essentially contained in [**Hi**], but we use some different notation here to make our computer calculations simpler.

As we saw in Lemma III.1.2 Hirzebruch coverings are smooth. In most cases they are also minimal and one can speak of their Kodaira-Enriques classification.

V.1.1 THEOREM. *([**Hi**], p. 127) The Hirzebruch surface $\widehat{X}$ is a minimal surface except in the following cases:*

(1) *T contains a single point p and all but at most 2 line in $\mathcal{L}$ passes through p;*

(2) *T contains two points p and q, all lines in $\mathcal{L}$ pass through p or q, and there is one lines in $\mathcal{L}$ containing both p and q.*

For $n \geq 3$, the canonical dimension of $\widehat{X}$ is 2, that is, $\widehat{X}$ is a surface of general type. For $n = 2$ and $k = 6$, $\widehat{X}$ is a K3 surface, and for $n = 2$ and $k \geq 7$, $\widehat{X}$ has canonical dimension greater than or equal to 0, and it is elliptic or of general type.

We will deal with the exceptional cases in V.3 and V.4.

A useful aspect of Hirzebruch coverings is the ease with which their Chern numbers can be calculated.

Let t be the number of points in T and let s be the number of points in S. For each $p \in S$, let ℓ_p be the number of lines in $\mathcal{L}$ passing through p and for each line $L \subset \mathcal{L}$, let r_L be the number of points in $S \cap L$.

V.1.2 THEOREM. *([**Hi**], pp. 123-125) If not all lines pass through a single point, then we have*

$$K_{\widehat{X}} = \widehat{\rho}^*(\sigma^* K_{\mathbf{P}^2} + \sum_{p\in T}(1 + \frac{n-1}{n}(1-\ell_p))E_p + \sum_{L\subset\mathcal{L}} \frac{n-1}{n}\sigma^* L),$$

where E_p is the exceptional curve lying above the point p. This implies that

$$\begin{aligned} c_1^2 &\stackrel{\text{def}}{=} K_{\widehat{X}}^2 \\ &= n^{k-1}((-3 + k(\frac{n-1}{n}))^2 - \sum_{p\in T}(1 + \frac{n-1}{n}(1-\ell_p))^2). \end{aligned}$$

The second Chern number equals

$$\begin{aligned} c_2 &\stackrel{\text{def}}{=} \chi_{\text{top}}(\widehat{X}) \\ &= n^{k-1}(3 - s - \sum_{L\subset\mathcal{L}}(2-r_L)) + n^{k-2}(\sum_{p\in T}(2-\ell_p) + \sum_{L\subset\mathcal{L}}(2-r_L)) \\ &\qquad + n^{k-3}(s - t + \sum_{p\in T}\ell_p). \end{aligned}$$

If all lines pass through a single point p, then

$$K_{\widehat{X}} = \widehat{\rho}^*(\sigma^* K_{\mathbf{P}^2} + E_p + \sum \frac{n-1}{n}\sigma^* L)$$

and hence

$$c_1^2 = n^{k-1}[(-3 + \frac{n-1}{n}k)^2 - 1].$$

The second Chern number equals

$$c_2 = 2(2-k)n^{k-1} + 2kn^{k-2}.$$

V.1.3 Remark. Noether's formula gives the Euler number of the structure sheaf

$$\chi(\mathcal{O}_{\widehat{X}}) = \frac{1}{12}(c_1^2 + c_2).$$

Thus, once one finds the first Betti number b_1 of $\widehat{X}$, we have the following additional invariants:

(1) From the Hodge decomposition of $\mathrm{H}_1(\widehat{X}; \mathbf{C})$ and Poincaré duality

$$\begin{aligned} b_1 &= h^{1,0} + h^{0,1} \\ &= 2q, \end{aligned}$$

and we get the formula

$$q = \frac{1}{2}b_1$$

for the *irregularity* q of $\widehat{X}$;

(2) From Poincaré duality we get all the *Betti numbers*:

$$b_0 = b_4 = 1$$

and

$$b_3 = b_1,$$

and, since $c_2 = \chi_{\text{top}}(\widehat{X}) = b_0 - b_1 + b_2 - b_3 + b_4$, we have

$$b_2 = c_2 - 2 + 2b_1;$$

(3) From the decomposition $\chi(\mathcal{O}_{\widehat{X}}) = 1 - h^{1,0} + h^{2,0}$ we obtain the *arithmetic genus*

$$\begin{aligned} p_a &\stackrel{\text{def}}{=} \chi(\mathcal{O}_{\widehat{X}}) - 1 \\ &= \frac{1}{12}(c_1^2 + c_2) - 1 \end{aligned}$$

and the *geometric genus*

$$
\begin{aligned}
p_g &\stackrel{\text{def}}{=} h^{2,0} \\
&= p_a + q;
\end{aligned}
$$

(4) From the Hodge decomposition of $\mathrm{H}_2(\widehat{X};\mathbf{C})$ and Poincaré duality

$$
\begin{aligned}
b_2 &= h^{2,0} + h^{1,1} + h^{0,2} \\
&= 2p_g + h^{1,1},
\end{aligned}
$$

and solving for $h^{1,1}$ we obtain

$$h^{1,1} = \frac{5}{6}c_2 - \frac{1}{6}c_1^2 + b_1;$$

(5) Hodge theory tells us that the Picard number p is bounded

$$p \leq h^{1,1}.$$

Another interesting property of Hirzebruch surfaces is that X, the covering before desingularization, is a complete intersection when the lines in $\mathcal{L}$ don't all meet at a single point. The proof follows from the analysis of Ishida [I] which leads him to a different method for finding the first Betti number.

V.1.4 PROPOSITION. *If $\mathcal{L}$ is a configuration of $k \geq 3$ lines not all meeting at a single point and $n > 1$, then the branched covering X of $\mathbf{P}^2$ defined by the map*

$$\pi_1(\mathbf{P}^2 - \mathcal{L}) \to \mathrm{H}_1(\mathbf{P}^2 - \mathcal{L}; \mathbf{Z}/n\mathbf{Z})$$

is a complete intersection. If no $k-1$ of the k lines pass through a single point, then X is the complete intersection of smooth hypersurfaces.

Proof. Define $\ell : \mathbf{P}^2 \to \mathbf{P}^{k-1}$ by

$$(W_1, W_2, W_3) = (\ell_1(W_1, W_2, W_3), \ldots, \ell_k(W_1, W_2, W_3))$$

where W_1, W_2, W_3 are homogeneous coordinates for $\mathbf{P}^2$ and $\ell_1, \dots, \ell_k$ are homogeneous equations for the lines in $\mathcal{L}$. The map ℓ is induced by a linear map $m : \mathbf{C}^3 \to \mathbf{C}^k$. Since not all lines in $\mathcal{L}$ pass through one point, this map has nonzero kernel, so ℓ defines an immersion $\mathbf{P}^2 \to \mathbf{P}^{k-1}$, i.e., ℓ is an isomorphism of varieties from $\mathbf{P}^2$ to $\ell(\mathbf{P}^2)$.

Consider the perpendicular space P to $m(\mathbf{C}^3)$. This is a $(k-3)$-dimensional linear subspace of $\mathbf{C}^k$. Let

$$(a_{i,1}, \dots, a_{i,k}) \qquad i = 1, \dots, k-3,$$

be a basis for P. Let $h_1, \dots, h_{k-3}$ be linear homogeneous equations on $\mathbf{P}^{k-1}$ defined by

$$h_i = \sum_{j=1}^{k} a_{i,j} Y_j \qquad i = 1, \dots, k-3,$$

where $Y_1, \dots, Y_k$ are the homogeneous coordinates for $\mathbf{P}^{k-1}$. Then $\ell(\mathbf{P}^2)$ is the complete intersection of the hyperplanes defined by the equations $h_i = 0$.

Let c_n be the morphism defined by

$$c_n : \mathbf{P}^{k-1} \to \mathbf{P}^{k-1}$$
$$[X_1 : \cdots : X_k] \mapsto [X_1^n : \cdots : X_k^n].$$

Then c_n is a branched covering branched along the coordinate axes

$$\bigcup_{i=1}^{k} \{Y_i = 0\}$$

defined by the natural map

$$\pi_1(\mathbf{P}^{k-1} - \bigcup_{i=1}^{k} \{Y_i = 0\}) \to \mathrm{H}_1(\mathbf{P}^{k-1} - \bigcup_{i=1}^{k} \{Y_i = 0\}; \mathbf{Z}/n\mathbf{Z}).$$

Let $X = c_n^{-1}(\ell(\mathbf{P}^2))$. Since ℓ induces an isomorphism on the fundamental groups of $\mathbf{P}^2 - \mathcal{L}$ and $\mathbf{P}^{k-1} - \bigcup_{i=1}^{k} \{Y_i = 0\}$, by Corollary I.4.4 the stabilizer subgroup of $\ell(\mathbf{P}^2)$ is the whole Galois group. Therefore, X is irreducible. Since c_n is a finite morphism, its restriction to X is a finite morphism. Furthermore, by Proposition I.4.3, its restriction to $\rho^{-1}(\ell(\mathsf{P}^2 - \mathcal{L}))$ is an unbranched covering defined by the natural map

$$\pi_1(\ell(\mathsf{P}^2 - \mathcal{L})) \to \mathrm{H}_1(\ell(\mathsf{P}^2 - \mathcal{L}); \mathbf{Z}/n\mathbf{Z}).$$

Therefore, to show that X is the Hirzebruch surface associated to $\mathcal{L}$ and n it suffices to show that X is normal. We will show this and the statement of the proposition by showing that X is a global complete intersection with singularities in codimension 2 (see **[Ha]** p. 188.)

Consider the equations

$$f_i \stackrel{\mathrm{def}}{=} h_i(X_1^n, \ldots, X_k^n) = 0 \qquad i = 1, \ldots, k-3.$$

At least set theoretically, X is the intersection of these hypersurfaces. To show that X is normal and equal to the complete intersection, it suffices to show that the Jacobian matrix for the set

$$\{f_1, \ldots, f_{k-3}\}$$

has rank $k-3$ for all but a finite number of points in X.

The Jacobian matrix has entries $na_{i,j}X_i^{n-1}$. We claim that if p is a point in X so that not more than two of its coordinates are zero (i.e., a point p so that $c_n(p)$ lies in the image of at most two lines in $\mathcal{L}$) the Jacobian matrix at p has rank $k-3$. Define M_{j_1,j_2} to be the matrix obtained from $[a_{i,j}]$ by setting the j_1 and j_2 columns equal to 0. Since $n \geq 2$, the matrix $[na_{i,j}X_i^{n-1}]$ at a point p lying on the image of $L_{j_1} \cup L_{j_2}$, but not on the image of any other lines in $\mathcal{L}$, has the same rank as M_{j_1,j_2}.

We need to show that M_{j_1,j_2} has rank $k-3$. Suppose there was a linear relation among the rows of M_{j_1,j_2}. Then, since $[a_{i,j}]$ has full rank, this would imply that there is an element of P where only the j_1 and j_2 entries are nonzero, i.e., an element giving a linear relation between $m(\ell_1)$ and $m(\ell_2)$. This implies that ℓ_1 and ℓ_2 are linearly dependent, which means that the lines L_1 and L_2 must be equal. Thus, M_{j_1,j_2} must have rank $k-3$.

This shows that the scheme defined by the functions $f_1, \ldots, f_{k-3}$ is reduced and can only have singularities at points above the image of triple and higher intersections on $\mathcal{L}$. Therefore, it must be a normal complete intersection.

Suppose no $k-1$ of the lines meet in a single point. If $y_1, \ldots, y_k$ are the coordinates for $\mathbf{C}^k$, P does not lie in any coordinate hyperplane $y_j = 0$. If it did, then this would imply that any triple of lines other than L_j are dependent vectors in the dual space $\check{\mathbf{C}}^3$ to $\mathbf{C}^3$. That is any triple of lines not containing L_j intersect in a single point. This implies that all $k-1$ lines in $\mathcal{L}$ other than L_j intersect in a single point contradicting the hypothesis. Therefore, P has a basis

$$(a_{i,1}, \ldots, a_{i,k}) \qquad i = 1, \ldots, k-3,$$

where none of the $a_{i,j}$ are 0. In this case, it is easy to see that the hypersurfaces defined by $f_1, \ldots, f_{k-3}$ are smooth. ∎

V.2. Lines in general position

Assume $\mathcal{L}$ contains only double points, i.e., T is empty. Then there is no need for blowups and pullbacks, and the Hirzebruch surface $\widehat{X}$ associated to $\mathcal{L}$ and any positive integer n equals X. By Proposition III.2.6, we know that X is the complete intersection of smooth hypersurfaces in $\mathbf{P}^{k-1}$. Since X is smooth, these surfaces must be in general position. Thus, we can embed $\mathbf{P}^{k-1}$ into $\mathbf{P}^N$ for appropriate N so that $\widehat{X}$ is the complete intersection of the image of $\mathbf{P}^k$ and hyperplanes in general position. The Lefschetz hyperplane theorem states that, for any variety Y of dimension greater than or equal to 3 and generic hyperplane H in the ambient projective space, the map

$$\pi_1(Y \cap H) \to \pi_1(Y)$$

induced by inclusion is an isomorphism. Therefore, since $\mathbf{P}^{k-1}$ is simply connected, so is X.

Here is a proof of the following weaker statement, which illustrates the techniques of the general algorithm of this thesis.

V.2.1 PROPOSITION. *The first Betti number of X is 0.*

Proof. Zariski's conjecture on the fundamental group of the complement of nodal curves, proven by Deligne [**D**] and Fulton [**Fu1**], implies that

$$\pi_1(\mathbf{P}^2 - \mathcal{L})$$

is abelian. (In fact, the theorem asserts that the fundamental group is abelian for any union of curves with only normal crossings.)

Consider the commutative diagram

$$\begin{array}{ccc} \pi_1(X^u, *) & \xrightarrow{h} & \mathrm{H}_1(X^u; \mathbf{Z}) \\ {\scriptstyle \rho_*^u}\downarrow & & {\scriptstyle \rho_*^u}\downarrow \\ \pi_1(\mathbf{P}^2 - \mathcal{L}, *) & \xrightarrow{h} & \mathrm{H}_1(\mathbf{P}^2 - \mathcal{L}; \mathbf{Z}) \end{array}$$

where the horizonal maps h are the Hurewicz homomorphisms taking loops to their homology classes.

Then, since $\pi_1(\mathbf{P}^2 - \mathcal{L})$ is abelian and $\rho_*^u : \pi_1(X^u) \to \pi_1(\mathbf{P}^2 - \mathcal{L})$ is injective, we have

$$\rho_*^u : \mathrm{H}_1(X^u; \mathbf{Z}) \to \mathrm{H}_1(\mathbf{P}^2 - \mathcal{L}; \mathbf{Z})$$

is injective. The cokernel of ρ_*^u is isomorphic to the cokernel of the map

$$\rho_*^u : \pi_1(X^u) \to \pi_1(\mathbf{P}^2 - \mathcal{L}),$$

and is isomorphic to G, a finite group. Therefore, the image of ρ_*^u has finite index in $\mathrm{H}_1(\mathbf{P}^2 - \mathcal{L}; \mathbf{Z})$. Thus, the rank of $\mathrm{H}_1(X^u; \mathbf{Z})$ equals that of $\mathrm{H}_1(\mathbf{P}^2 - \mathcal{L}; \mathbf{Z})$ or $k-1$. Therefore, $b_1^u = k - 1$. Now consider the intersection matrix for $\rho^{-1}(\widehat{L})$ in X.

Any line $L \subset \mathcal{L}$ intersects all other lines in $\mathcal{L}$ in distinct points, so $H_L = G$. Therefore, $L' = \rho^{-1}(L)$ is irreducible for all $L \subset \mathcal{L}$.

If L_1 and L_2 are two distinct lines in L, then by Proposition II.3.1

$$\begin{aligned} L_1' L_2' &= \frac{1}{n^2} |H_{L_1} \cap H_{L_2}| \\ &= \frac{1}{n^2} |G| \\ &= n^{k-3} \end{aligned}$$

If L is any line in $\mathcal{L}$, then

$$L'.L' = \frac{1}{n^2}|G| = n^{k-3}.$$

Therefore, the intersection matrix for $\rho^{-1}(\mathcal{L})$ equals n^{k-3} times the intersection matrix $I(\mathcal{L})$ for $\mathcal{L}$, and hence the rank and nullity of the matrices are the same and equal 1 and $k-1$, respectively.

Putting this together with b_1^u, we get

$$\begin{aligned} b_1 &= b_1^u - \mathrm{Null}(I(\rho^{-1}(\mathcal{L}))) \\ &= k-1-(k-1) \\ &= 0. \end{aligned}$$

∎

The Chern numbers for X associated to lines in general position go as follows.

$$c_1^2 = n^{k-1}(-3+k(\frac{n-1}{n}))^2.$$
$$c_2 = n^{k-1}(3 - \frac{k(k-1)}{2} - k(3-k)) + n^{k-2}k(3-k) + n^{k-3}\frac{k(k-1)}{2}.$$

V.2.2 EXAMPLE: If $k=3$, then $X = \mathbf{P}^2$ and

$$\rho : \mathbf{P}^2 \to \mathbf{P}^2$$

is given by

$$\rho([X:Y:Z]) = [X^n : Y^n : Z^n]$$

for some choice of coordinates $[X:Y:Z]$.

By the above formulas we get the Chern numbers:

$$\begin{aligned} c_1^2 &= n^2(-3+3\frac{n-1}{n})^2 \\ &= (-3n+3(n-1))^2 \\ &= 9; \end{aligned}$$

and

$$c_2 = n^2(3-3)+0+1(3) = 3$$

as expected.

V.3. Fibrations

In this section we deal with exceptional case (1) of Theorem V.1.1. (See also [**Hi**], p.131.)

Before we begin, we do a calculation which will be useful in both this and the next section.

V.3.1 LEMMA. *Let $\mathcal{P}$ be k points in $\mathbf{P}^1$ and let C be the branched covering over $\mathbf{P}^1$ defined by*

$$\pi_1(\mathbf{P}^1 - P) \to \mathrm{H}_1(\mathbf{P}^1 - P; \mathbf{Z}/n\mathbf{Z}).$$

Then C has topological Euler characteristic

$$\chi_{\mathrm{top}}(C) = (2-k)n^{k-1} + kn^{k-2}$$

and genus

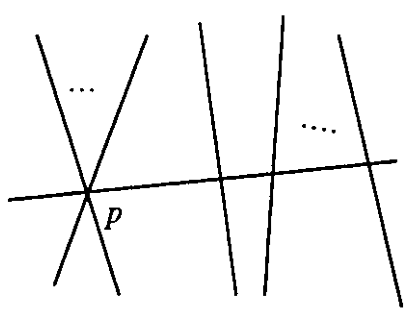
$$g(C) = \frac{1}{2}((k-2)n^{k-1} - kn^{k-2} + 2).$$

Proof. It suffices to show the formula for χ_{top}, since $g(C) = \frac{1}{2}(2 - \chi_{\mathrm{top}})$. Let $C^u = \rho^{-1}(\mathbf{P}^1 - P)$. By a general property of unbranched coverings, since n^{k-1} is the degree of the covering and $2-k$ is the topological Euler characteristic of $\mathbf{P}^1 - P$, we have

$$\chi_{\mathrm{top}}(C^u) = n^{k-1}(2-k).$$

The completion C is obtained by adding n^{k-2} points above each of the k points in P. The claim follows. ∎

Case (1) in Theorem V.1.1 generalizes to the case where T contains only one point p, as in the following diagram (note that not all intersections are drawn here.)



As usual let $\widehat{\mathbf{P}}^2$ be the blowup of $\mathbf{P}^2$ at p. There is a natural $\mathbf{P}^1$ fibration $f : \widehat{\mathbf{P}}^2 \to E_p$ given by projecting along the proper transforms of lines through p. Let ℓ_p be the number of lines through p.

Consider the composition of maps $\widehat{X} \xrightarrow{\widehat{\rho}} \widehat{\mathbf{P}}^2 \xrightarrow{f} E_p$. By Stein factorization, there is a curve C so that this composition factors as $\widehat{X} \xrightarrow{f'} C \to E_p$, where $\widehat{X} \to C$ has connected fibers and $C \to E_p$ is a finite surjective morphism (i.e., a branched covering.) It follows that $\widehat{X}$ can be described as a fibration over the curve C.

To find C explicitly, look at the commutative diagram

$$\begin{tikzcd} C' \arrow[r, "i'"] \arrow[d] & \widehat{X} \arrow[r, "f'"] \arrow[d, "\widehat{\rho}"'] & C \arrow[d] \\ E_p \arrow[r, "i"] & \widehat{\mathbf{P}}^2 \arrow[r, "f"] & E_p \end{tikzcd}$$

where C' is a connected component of $\widehat{\rho}^{-1}(E_p)$ and the maps $i : E_p \to \widehat{\mathbf{P}}^2$ and $i' : C' \to \widehat{X}$ are inclusions. Note that C' irreducible by Lemma II.3.2. Since $f \circ i : E_p \to E_p$ is an isomorphism and $f' \circ i' : C' \to C$ is a one-to-one and onto map preserving fiber, C' and C are isomorphic and C is the branched covering of $\mathbf{P}^1$ branched along ℓ_p points.

The general fibers of this fibration are branched along $k - \ell_p + 1$ points.

From the above discussion, Proposition I.4.3 and Lemma V.3.1, we have the following proposition.

V.3.2 PROPOSITION. *The surface $\widehat{X}$ is a fibration over the curve C with genus*

$$g(C) = \frac{1}{2}((\ell_p - 2)n^{\ell_p - 1} - \ell_p n^{\ell_p - 2} + 2)$$

and with fibers F of genus

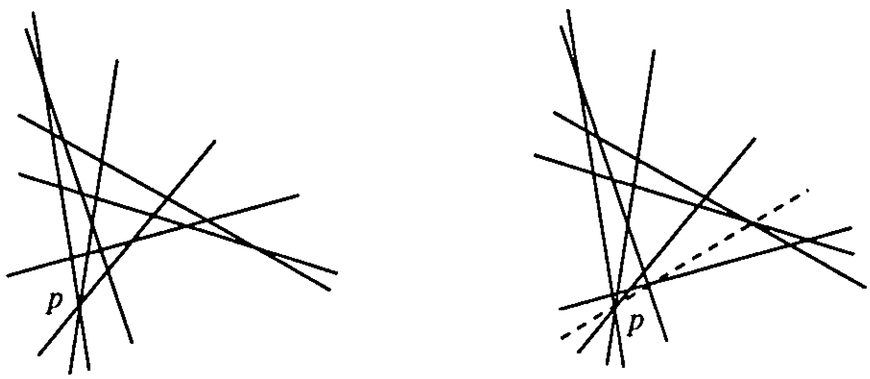
$$g(F) = \begin{cases} \frac{1}{2}((k - \ell_p - 1)n^{k-\ell_p} - (k - \ell_p + 1)n^{k-\ell_p-1} + 2) & \text{if } \ell_p \leq k - 2 \\ & 0 \text{otherwise} \end{cases}$$

When $\ell_p = k, k-1$, $\widehat{X}$ is a ruled surface for all $n \geq 2$. When $\ell_p = k-2$, $\widehat{X}$ is ruled surface for $n = 2$, an elliptic surface for $n = 3$ and a surface of general type for $n \geq 3$. When $\ell_p \leq k-3$, $\widehat{X}$ is of general type for all $n \geq 2$.

In this case, if $n = 2$ one gets a ruled surface, if $n = 3$ one gets an elliptic surface and for $n > 3$ one gets a surface of general type. For ℓ_p not equal to $k, k-1$ or $k-2$, and $n \geq 2$, $\widehat{X}$ is of general type.

V.3.3 REMARK: The special fibers of the fibration depend on the positions of the double points.

Consider, for example, the following configurations drawn on the "real part" of $\mathbf{P}^2$.



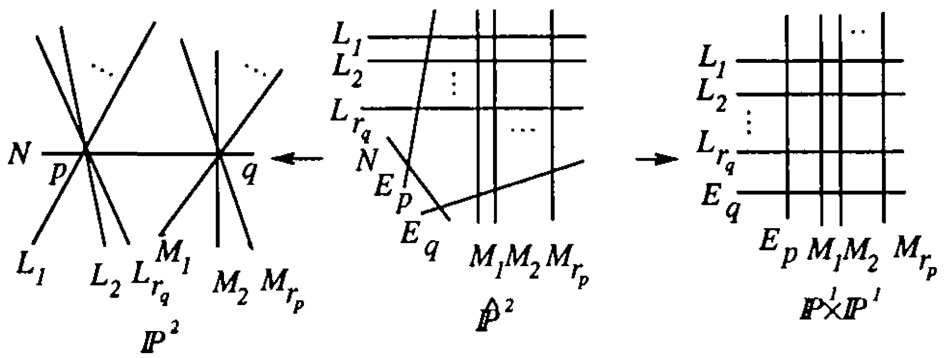
Assume, say, that $n = 2$. The coverings corresponding to these configurations are naturally fibrations over $\mathbf{P}^1$ with general fibers of genus 5. For the left configuration, all special fibers have genus 1, but for the right configuration, there are special fibers lying over the dotted line with genus 0.

V.4 Coverings birational to the product of two curves

We now deal with exceptional case (2) of Theorem V.1.3. Let p and q be the two points in T and let ℓ_p and ℓ_q be the number of lines through p and q. Then $\widehat{\mathbf{P}}^2$ is obtained from $\mathbf{P}^2$ by blowing up the points p and q. As is well known, if one blows down the proper transform of the line $N \subset \widehat{L}$ passing through p and q one obtains $\mathbf{P}^1 \times \mathbf{P}^1$. We have

$$\mathbf{P}^2 \leftarrow \widehat{\mathbf{P}}^2 \rightarrow \mathbf{P}^1 \times \mathbf{P}^1$$

where the arrows are birational morphisms. The preimage $\widehat{L}$ of $\mathcal{L}$ in $\widehat{\mathbf{P}}^2$ equals the union of the proper transforms of the lines in $\mathcal{L}$ and two exceptional divisors E_p and E_q. The image of $\widehat{L}$ in $\mathbf{P}^1 \times \mathbf{P}^1$ equals $\ell_p + 1$ lines in one ruling and $\ell_q + 1$ lines in the other ruling. The maps are isomorphisms on the complement $\mathcal{U}$ of these sets.



Note that in the above diagram, the notation for curves and their proper transforms are the same, as no confusion should arise.

Let Z be the completion over $\mathbf{P}^1 \times \mathbf{P}^1$ of the unbranched covering of $\mathcal{U}$ given by restricting ρ. Then Z is birationally equivalent to $\widehat{X}$ and agrees over $\mathcal{U}$. We will show that Z is a product of curves, which can be given as branched coverings of the components of $\mathbf{P}^1 \times \mathbf{P}^1$.

Let $L_1, \ldots, L_{r_p}$ (respectively, $M_1, \ldots, M_{r_p}$) be the lines in $\mathcal{L}$, other than N, going through p (respectively q). Let $\gamma_1, \ldots, \gamma_{r_p}$ be the meridianal loops around $L_1, \ldots, L_{r_p}$, let $\mu_1, \ldots, \mu_{r_q}$ be the loops around $M_1, \ldots, M_{r_q}$ and let τ be the loop around N.

Recall that $G = \mathrm{H}_1(\mathbf{P}^2 - \mathcal{L}; \mathbf{Z}/n\mathbf{Z})$ is generated by

$$\gamma_1, \ldots, \gamma_{r_p}, \mu_1, \ldots, \mu_{r_q}, \tau$$

and they have the relation

$$\sum_{i=1}^{r_p} \gamma_i + \sum_{i=1}^{r_q} \mu_i + \tau = 0.$$

Thus, $\gamma_1, \ldots, \gamma_{r_p}$ and $\mu_1, \ldots, \mu_{r_q}$ generate subgroups G_p and G_q, with the property that $G = G_p \times G_q$.

Now, $\mathcal{U}$ is isomorphic to

$$\mathbf{P}^1 - \{r_p + 1\text{points}\} \times \mathbf{P}^1 - \{r_q + 1 \text{ points}\}$$

and the defining map $\pi_1(\mathcal{U}) \to G$ can be seen as the product of the maps

$$\pi_1(\mathbf{P}^1 - \{r_p + 1\text{points}\}) \to G_p$$

and

$$\pi_1(\mathbf{P}^1 - \{r_q + 1 \text{ points}\}) \to G_q.$$

We thus have proved the following. (See also [**Hi**], p. 131.)

V.4.1 *Proposition. Suppose T contains two points p and q, all lines in $\mathcal{L}$ pass through p or q and there is a line in $\mathcal{L}$ passing through both p and q. Then $\widehat{X}$ is birationally equivalent to the product of curves C and D, where*

$$g(C) = \frac{1}{2}((r_p - 1)n^{r_p} - (r_p + 1)n^{r_p - 1} + 2)$$

and

$$g(D) = \frac{1}{2}((r_q - 1)n^{r_q} - (r_q + 1)n^{r_q - 1} + 2)$$

In particular, if $r_p = 2$ (or $r_q = 2$ and $n = 2$ then $\widehat{X}$ is ruled, if $n = 3$ then $\widehat{X}$ is elliptic and if $n > 3$ then $\widehat{X}$ is of general type. If $r_p = 3$ or $r_q = 3$, then $\widehat{X}$ is elliptic for $n = 2$, general type for $n \geq 3$. If both r_p and r_q are greater than or equal to r, then $\widehat{X}$ is of general type.

V.5 K3 surfaces and elliptic surfaces

Assume $\mathcal{L}$ does not fall under one of the exceptional cases of Theorem V.1.1. Recall the equation for $K_{\widehat{X}}$ from Theorem V.1.2. Replacing n by 2, we have, for H a general line on $\mathbf{P}^2$,

$$\begin{aligned} K_{\widehat{X}} &= \widehat{\rho}^*(\sigma^* K_{\mathbf{P}^2} + \sum_{p \in T} \frac{1}{2}(3 - \ell_p)E_p + \sum_{L \subset \mathcal{L}} \frac{1}{2}\sigma^* L) \\ &= \widehat{\rho}^*(-3\sigma^* H + \sum_{p \in T} \frac{1}{2}(3 - \ell_p)E_p + \frac{k}{2}\sigma^* H) \\ &= \widehat{\rho}^*((-3 + \frac{k}{2})\sigma^* H + \sum_{p \in T} \frac{1}{2}(3 - \ell_p)E_p). \end{aligned}$$

Here equality means linear equivalence as divisors.

Thus, if $k = 6$ and $\mathcal{L}$ has no quadruple or higher order points, then the canonical divisor $K_{\widehat{X}}$ is trivial and $\widehat{X}$ is a K3 surface. If $k \geq 7$ then $K_{\widehat{X}}$ is effective (one does not subtract off more E_p's than one adds with the $\sigma^* H$'s.)

From Theorem V.1.2, the formula for c_1^2 when $n = 2$ is

$$c_1^2 = 2^{k-1}((-3 + \frac{k}{2})^2 - \sum_{p \in T}(3 - \ell_p)^2).$$

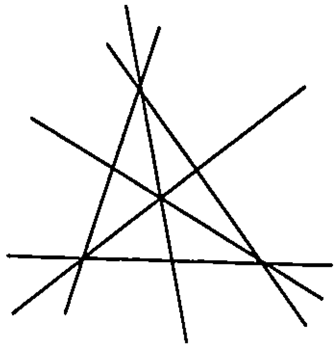
Thus, for example, if $k = 7$, then $\widehat{X}$ is an elliptic surface if and only if $\mathcal{L}$ has one quadruple point and the rest are double or triple points.

V.5 Calculations of invariants

We end with a list of output from computer aided calculations implementing the algorithm described in this paper. The invariants which we focus on are the Betti numbers b_1, b_2, the Chern numbers c_1^2, c_2 and bounds on the Picard number given by the rank of the intersection matrix for curves above the branch locus and the Hodge number $h^{1,1}$. (Note that given c_2 and b_1, one can calculate b_2 directly.)

Although surfaces with Kodaira dimension less than two have been studied in detail and their Betti numbers as well as Chern numbers are understood. This is not true for surfaces of general type. We have seen that most of the examples arising as Hirzebruch coverings are of general type. For example, if $n \geq 3$, $k \geq 7$ and T has at least three points, this is the case.

According to the Miyaoka-Yau inequality, we have $c_1^2 \leq 3c_2$ for minimal surfaces of general type, with equality occuring when the surface is uniformized by the complex ball. An example of a Hirzebruch covering surface whose Chern numbers satisfy the equality, with branch locus defined by real equations, occurs when we have the folowing configuration, with $n = 5$.

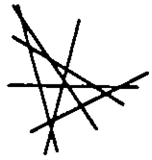


The above configuration is also interesting because when $n = 2$ one gets a K3 surface with Picard number equal to 20, which equals $h^{1,1}$. When $n = 3$ one gets a surface of general for which the Picard number is also equal to $h^{1,1}$.

In the following we list computer calculations for configurations of 6 and 7 lines as well as two more examples. The existence of the first (a configuration with 9 lines) can be proven using Pappus' Theorem. The second, is the set of lines through 5 points in general position.

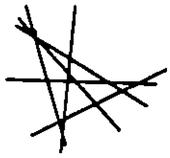
Configurations of 6 lines

1



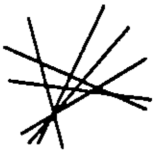
$n = 2$: K3, $b_1 = 0$, $b_2 = 22$, $c_1^2 = 0$, $c_2 = 24$, $1 \le p \le 20$
$n = 3$: general, $b_1 = 0$, $b_2 = 403$, $c_1^2 = 243$, $c_2 = 405$, $1 \le p \le 297$
$n = 5$: general, $b_1 = 0$, $b_2 = 9373$, $c_1^2 = 10125$, $c_2 = 9375$, $1 \le p \le 6125$

2



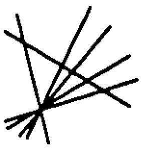
$n = 2$: K3, $b_1 = 0$, $b_2 = 22$, $c_1^2 = 0$, $c_2 = 24$, $5 \le p \le 20$
$n = 3$: general, $b_1 = 2$, $b_2 = 326$, $c_1^2 = 216$, $c_2 = 324$, $10 \le p \le 236$
$n = 5$: general, $b_1 = 12$, $b_2 = 7522$, $c_1^2 = 9000$, $c_2 = 7500$, $26 \le p \le 4762$

3



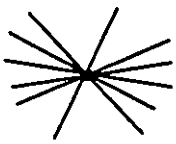
$n = 2$: ruled*, $b_1 = 2$, $b_2 = 10$, $c_1^2 = -8$, $c_2 = 8$, $3 \le p \le 10$
$n = 3$: elliptic, $b_1 = 20$, $b_2 = 148$, $c_1^2 = 0$, $c_2 = 108$, $4 \le p \le 110$
$n = 5$: general, $b_1 = 152$, $b_2 = 3802$, $c_1^2 = 4000$, $c_2 = 3500$, $6 \le p \le 2402$

4



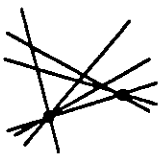
$n = 2$: ruled, $b_1 = 10$, $b_2 = 2$, $c_1^2 = -32$, $c_2 = -16$, $2 \le p \le 2$
$n = 3$: ruled, $b_1 = 110$, $b_2 = 2$, $c_1^2 = -432$, $c_2 = -216$, $2 \le p \le 2$

5



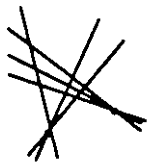
$n = 2$: ruled, $b_1 = 34$, $b_2 = 2$, $c_1^2 = -32$, $c_2 = -64$, $2 \le p \le 6$
$n = 3$: ruled, $b_1 = 488$, $b_2 = 2$, $c_1^2 = 0$, $c_2 = -972$, $2 \le p \le 128$

6



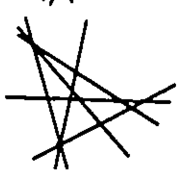
$n = 2$: K3*, $b_1 = 2$, $b_2 = 10$, $c_1^2 = -8$, $c_2 = 8$, $10 \le p \le 10$
$n = 3$: elliptic*, $b_1 = 22$, $b_2 = 69$, $c_1^2 = -27$, $c_2 = 27$, $29 \le p \le 49$

7



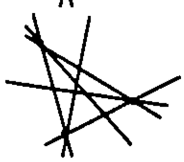
$n = 2$: K3, $b_1 = 0$, $b_2 = 22$, $c_1^2 = 0$, $c_2 = 24$, $9 \le p \le 20$
$n = 3$: general, $b_1 = 4$, $b_2 = 249$, $c_1^2 = 189$, $c_2 = 243$, $19 \le p \le 175$
$n = 5$: general, $b_1 = 24$, $b_2 = 5671$, $c_1^2 = 7875$, $c_2 = 5625$, $51 \le p \le 3399$

8



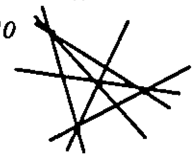
$n = 2$: K3, $b_1 = 0$, $b_2 = 22$, $c_1^2 = 0$, $c_2 = 24$, $10 \le p \le 20$
$n = 3$: general, $b_1 = 4$, $b_2 = 249$, $c_1^2 = 189$, $c_2 = 243$, $23 \le p \le 175$
$n = 5$: general, $b_1 = 24$, $b_2 = 5671$, $c_1^2 = 7875$, $c_2 = 5625$, $67 \le p \le 3399$

9



$n = 2$: K3, $b_1 = 0$, $b_2 = 22$, $c_1^2 = 0$, $c_2 = 24$, $16 \le p \le 20$
$n = 3$: general, $b_1 = 6$, $b_2 = 172$, $c_1^2 = 162$, $c_2 = 162$, $40 \le p \le 114$
$n = 5$: general, $b_1 = 36$, $b_2 = 3820$, $c_1^2 = 6750$, $c_2 = 3750$, $124 \le p \le 2036$

10

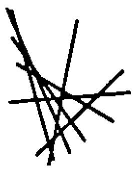


$n = 2$: K3, $b_1 = 0$, $b_2 = 22$, $c_1^2 = 0$, $c_2 = 24$, $20 \le p \le 20$
$n = 3$: general, $b_1 = 10$, $b_2 = 99$, $c_1^2 = 135$, $c_2 = 81$, $55 \le p \le 55$
$n = 5$: general, $b_1 = 60$, $b_2 = 1993$, $c_1^2 = 5625$, $c_2 = 1875$, $185 \le p \le 685$

* not minimal model

Configurations of 7 lines

1

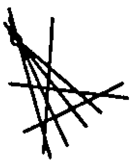


$n = 2$: general, $b_1 = 0$, $b_2 = 78$, $c_1^2 = 16$, $c_2 = 80$, $1 \leq p \leq 64$
$n = 3$: general, $b_1 = 0$, $b_2 = 2185$, $c_1^2 = 2025$, $c_2 = 2187$, $1 \leq p \leq 1485$

2

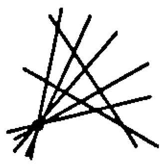
$n = 2$: general, $b_1 = 0$, $b_2 = 78$, $c_1^2 = 16$, $c_2 = 80$, $9 \leq p \leq 64$
$n = 3$: general, $b_1 = 2$, $b_2 = 1946$, $c_1^2 = 1944$, $c_2 = 1944$, $28 \leq p \leq 1298$

3



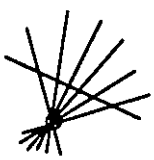
$n = 2$: elliptic, $b_1 = 2$, $b_2 = 50$, $c_1^2 = 0$, $c_2 = 48$, $5 \leq p \leq 42$
$n = 3$: general, $b_1 = 20$, $b_2 = 1334$, $c_1^2 = 1296$, $c_2 = 1296$, $10 \leq p \leq 884$

4



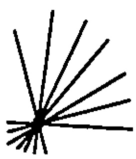
$n = 2$: ruled*, $b_1 = 10$, $b_2 = 18$, $c_1^2 = -48$, $c_2 = 0$, $3 \leq p \leq 18$
$n = 3$: elliptic, $b_1 = 110$, $b_2 = 542$, $c_1^2 = 0$, $c_2 = 324$, $4 \leq p \leq 380$

5



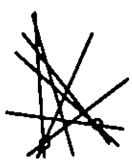
$n = 2$: ruled, $b_1 = 34$, $b_2 = 2$, $c_1^2 = -128$, $c_2 = -64$, $2 \leq p \leq 2$
$n = 3$: ruled, $b_1 = 488$, $b_2 = 2$, $c_1^2 = -1944$, $c_2 = -972$, $2 \leq p \leq 2$

6



$n = 2$: ruled, $b_1 = 98$, $b_2 = 2$, $c_1^2 = -48$, $c_2 = -192$, $2 \leq p \leq 18$

7

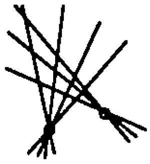


$n = 2$: general, $b_1 = 0$, $b_2 = 78$, $c_1^2 = 16$, $c_2 = 80$, $17 \leq p \leq 64$
$n = 3$: general, $b_1 = 4$, $b_2 = 1707$, $c_1^2 = 1863$, $c_2 = 1701$, $55 \leq p \leq 1111$

8

$n = 2$: general, $b_1 = 0$, $b_2 = 78$, $c_1^2 = 16$, $c_2 = 80$, $18 \leq p \leq 64$
$n = 3$: general, $b_1 = 4$, $b_2 = 1707$, $c_1^2 = 1863$, $c_2 = 1701$, $59 \leq p \leq 1111$

9

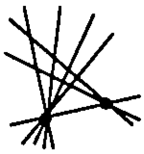


$n = 2$: elliptic, $b_1 = 2$, $b_2 = 50$, $c_1^2 = 0$, $c_2 = 48$, $13 \leq p \leq 42$
$n = 3$: general, $b_1 = 22$, $b_2 = 1095$, $c_1^2 = 1215$, $c_2 = 1053$, $37 \leq p \leq 697$

10

$n = 2$: elliptic, $b_1 = 2$, $b_2 = 50$, $c_1^2 = 0$, $c_2 = 48$, $16 \leq p \leq 42$
$n = 3$: general, $b_1 = 22$, $b_2 = 1095$, $c_1^2 = 1215$, $c_2 = 1053$, $53 \leq p \leq 697$

11

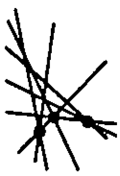


$n = 2$: ruled*, $b_1 = 10$, $b_2 = 18$, $c_1^2 = -48$, $c_2 = 0$, $18 \leq p \leq 18$
$n = 3$: elliptic*, $b_1 = 112$, $b_2 = 303$, $c_1^2 = -81$, $c_2 = 81$, $83 \leq p \leq 193$

12

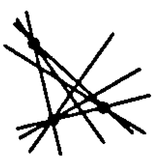
$n = 2$: elliptic*, $b_1 = 4$, $b_2 = 22$, $c_1^2 = -16$, $c_2 = 16$, $18 \leq p \leq 20$
$n = 3$: general*, $b_1 = 40$, $b_2 = 483$, $c_1^2 = 567$, $c_2 = 405$, $83 \leq p \leq 283$

13



$n = 2$: general, $b_1 = 0$, $b_2 = 78$, $c_1^2 = 16$, $c_2 = 80$, $27 \leq p \leq 64$
$n = 3$: general, $b_1 = 6$, $b_2 = 1468$, $c_1^2 = 1782$, $c_2 = 1458$, $90 \leq p \leq 924$

14

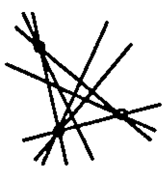


$n = 2$: general, $b_1 = 0$, $b_2 = 78$, $c_1^2 = 16$, $c_2 = 80$, $28 \leq p \leq 64$
$n = 3$: general, $b_1 = 6$, $b_2 = 1468$, $c_1^2 = 1782$, $c_2 = 1458$, $94 \leq p \leq 924$

15

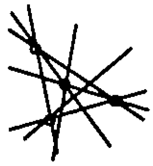
$n = 2$: general, $b_1 = 0$, $b_2 = 78$, $c_1^2 = 16$, $c_2 = 80$, $29 \leq p \leq 64$
$n = 3$: general, $b_1 = 6$, $b_2 = 1468$, $c_1^2 = 1782$, $c_2 = 1458$, $102 \leq p \leq 924$

16



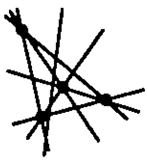
$n = 2$: elliptic, $b_1 = 2$, $b_2 = 50$, $c_1^2 = 0$, $c_2 = 48$, $28 \leq p \leq 42$
$n = 3$: general, $b_1 = 24$, $b_2 = 856$, $c_1^2 = 1134$, $c_2 = 810$, $100 \leq p \leq 510$

17



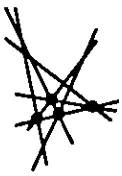
$n = 2$: general, $b_1 = 0$, $b_2 = 62$, $c_1^2 = 16$, $c_2 = 64$, $38 \leq p \leq 50$
$n = 3$: general, $b_1 = 8$, $b_2 = 905$, $c_1^2 = 1701$, $c_2 = 891$, $129 \leq p \leq 467$

18



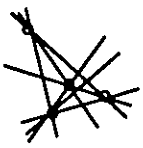
$n = 2$: general, $b_1 = 0$, $b_2 = 78$, $c_1^2 = 16$, $c_2 = 80$, $36 \leq p \leq 64$
$n = 3$: general, $b_1 = 10$, $b_2 = 1233$, $c_1^2 = 1701$, $c_2 = 1215$, $127 \leq p \leq 739$

19



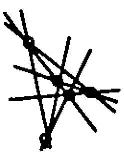
$n = 2$: general, $b_1 = 0$, $b_2 = 78$, $c_1^2 = 16$, $c_2 = 80$, $40 \leq p \leq 64$
$n = 3$: general, $b_1 = 8$, $b_2 = 1229$, $c_1^2 = 1701$, $c_2 = 1215$, $141 \leq p \leq 737$

20



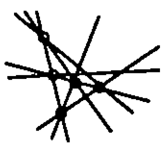
$n = 2$: elliptic, $b_1 = 2$, $b_2 = 50$, $c_1^2 = 0$, $c_2 = 48$, $38 \leq p \leq 42$
$n = 3$: general, $b_1 = 28$, $b_2 = 621$, $c_1^2 = 1053$, $c_2 = 567$, $145 \leq p \leq 325$

21



$n = 2$: general, $b_1 = 0$, $b_2 = 78$, $c_1^2 = 16$, $c_2 = 80$, $51 \leq p \leq 64$
$n = 3$: general, $b_1 = 10$, $b_2 = 990$, $c_1^2 = 1620$, $c_2 = 972$, $180 \leq p \leq 550$

22



$n = 2$: general, $b_1 = 0$, $b_2 = 78$, $c_1^2 = 16$, $c_2 = 80$, $50 \leq p \leq 64$
$n = 3$: general, $b_1 = 12$, $b_2 = 994$, $c_1^2 = 1620$, $c_2 = 972$, $186 \leq p \leq 552$

23

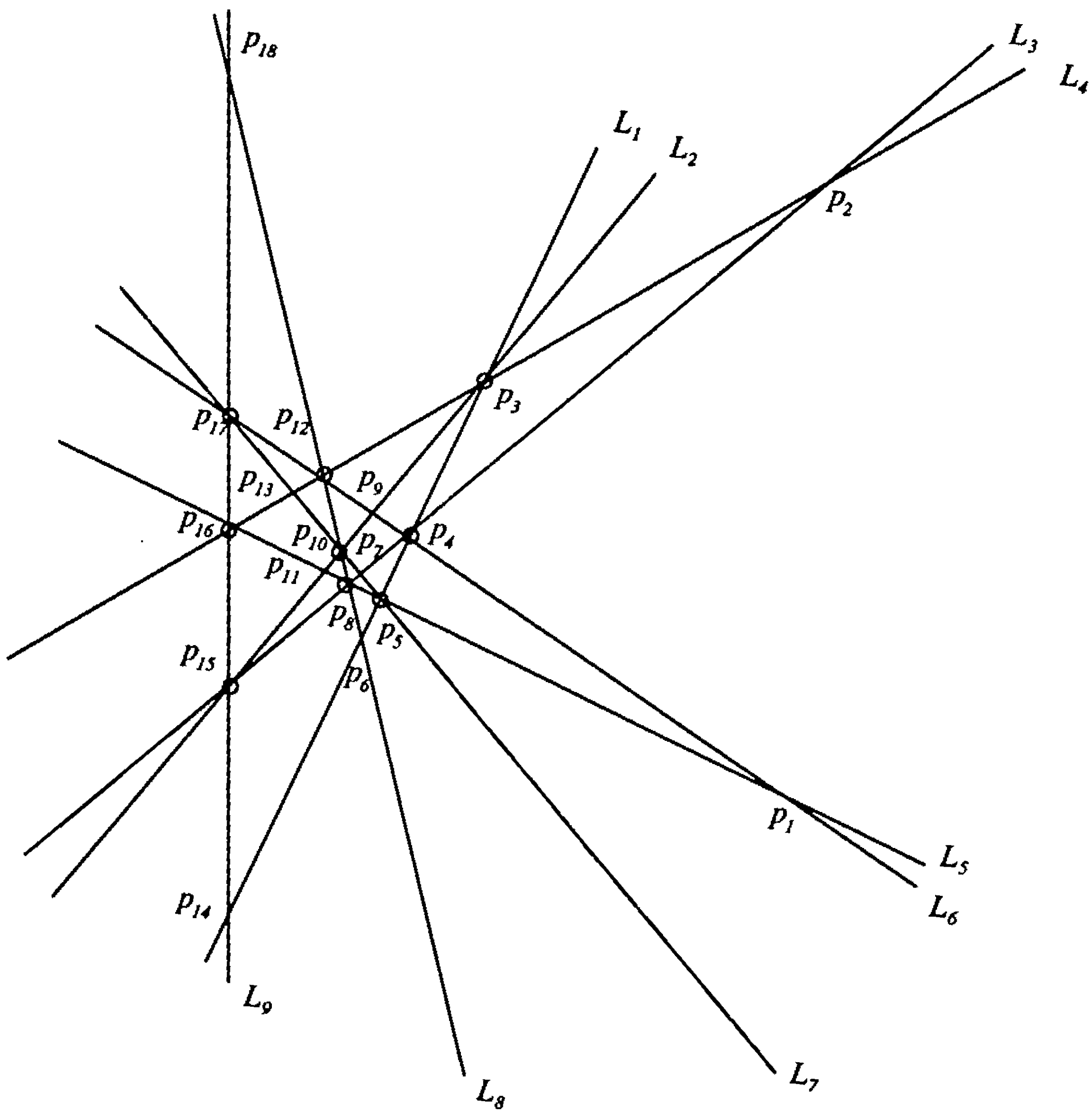
$n = 2$: general, $b_1 = 2$, $b_2 = 82$, $c_1^2 = 16$, $c_2 = 80$, $62 \leq p \leq 66$
$n = 3$: general, $b_1 = 18$, $b_2 = 763$, $c_1^2 = 1539$, $c_2 = 729$, $237 \leq p \leq 369$

* not a minimal surface

INPUT

DATA DETERMINING THE SURFACE:

Real line configuration in $\mathbf{P}^2$:



Order of cyclic group: 2.
Index of center of rotation: 4.

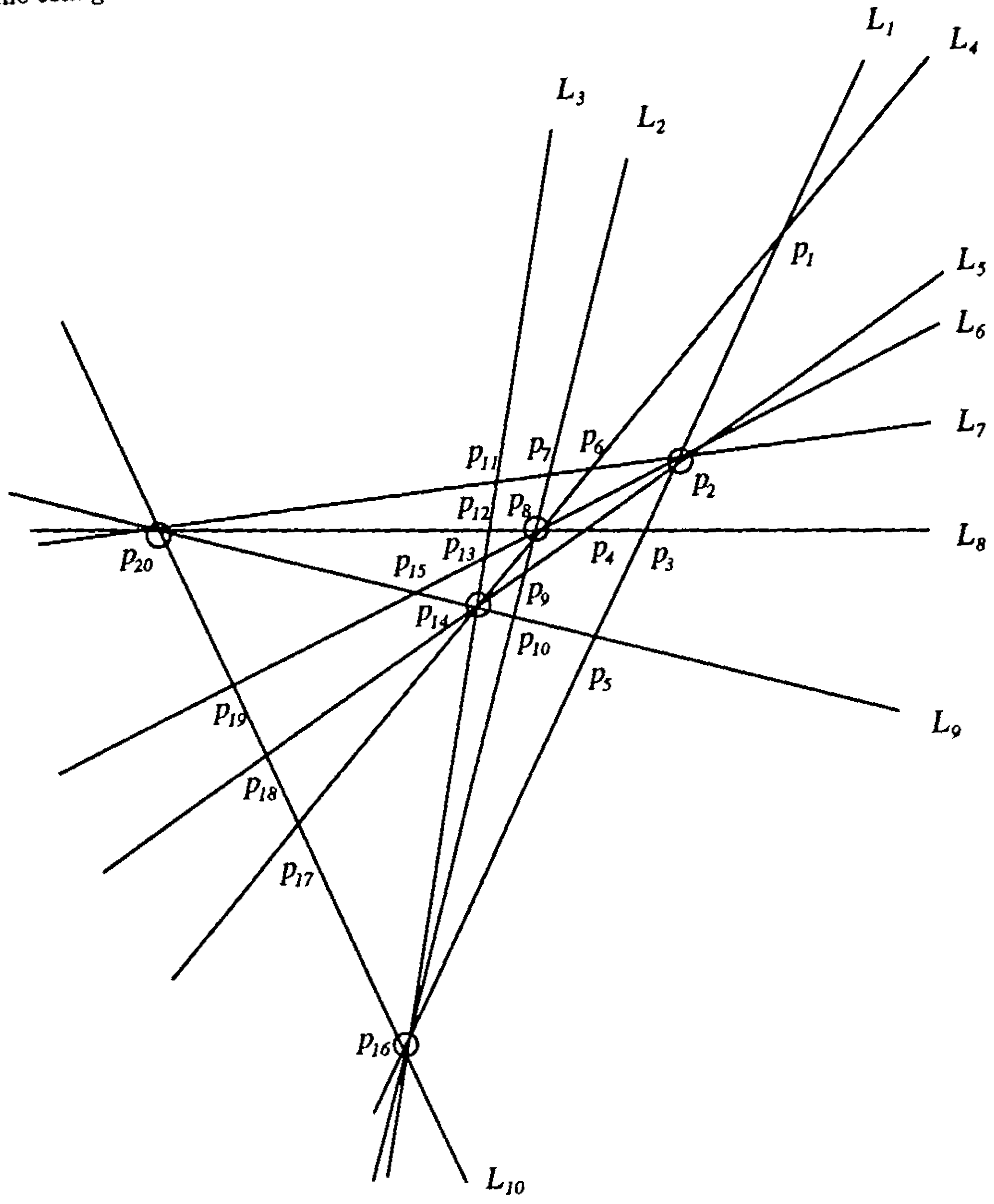
Point line incidence correspondence:

	$L1$	$L2$	$L3$	$L4$	$L5$	$L6$	$L7$	$L8$	$L9$
p_1	0	0	0	0	1	1	0	0	0
p_2	0	0	1	1	0	0	0	0	0
p_3	1	1	0	1	0	0	0	0	0
p_4	1	0	1	0	0	1	0	0	0
p_5	1	0	0	0	1	0	1	0	0
p_6	1	0	0	0	0	0	0	1	0
p_7	0	0	1	0	0	0	1	0	0
p_8	0	0	1	0	1	0	0	1	0
p_9	0	1	0	0	0	1	0	0	0
p_{10}	0	1	0	0	0	0	1	1	0
p_{11}	0	1	0	0	1	0	0	0	0
p_{12}	0	0	0	1	0	1	0	1	0
p_{13}	0	0	0	1	0	0	1	0	0
p_{14}	1	0	0	0	0	0	0	0	1
p_{15}	0	1	1	0	0	0	0	0	1
p_{16}	0	0	0	1	1	0	0	0	1
p_{17}	0	0	0	0	0	1	1	0	1
p_{18}	0	0	0	0	0	0	0	1	1

OUTPUT

First Betti Number of Unbranched Part: $b_1^u = 38$.
First Betti Number: $b_1 = 0$.
Second Betti Number: $b_2 = 766$.
Chern Numbers: $c_1^2 = 576$, $c_2 = 768$.
Bounds on the Picard number: rank $= 322 \leq p \leq$ h11 $= 544$.
Euler number of $O_{\widehat{X}}$: $\chi(O_{\widehat{X}}) = 112$.
Irregularity: q = 0.
Arithmetic genus: p_a= 111.
Geometric genus: p_g= 111.

INPUT
DATA DETERMINING THE SURFACE:
Real line configuration in $\mathbf{P}^2$:



Order of cyclic group: 2.
Index of center of rotation: 7.

Point line incidence correspondence:

	L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}
p_1	0	0	1	1	0	0	0	0	0	0
p_2	0	0	1	0	1	1	1	0	0	0
p_3	0	0	1	0	0	0	0	1	0	0
p_4	0	0	0	0	1	0	0	1	0	0
p_5	0	0	1	0	0	0	0	0	1	0
p_6	0	0	0	1	0	0	1	0	0	0
p_7	0	1	0	0	0	0	1	0	0	0
p_8	0	1	0	1	0	1	0	1	0	0
p_9	0	1	0	0	1	0	0	0	0	0
p_{10}	0	1	0	0	0	0	0	0	1	0
p_{11}	1	0	0	0	0	0	1	0	0	0
p_{12}	1	0	0	0	0	0	0	1	0	0
p_{13}	1	0	0	0	0	1	0	0	0	0
p_{14}	1	0	0	1	1	0	0	0	1	0
p_{15}	0	0	0	0	0	1	0	0	1	0
p_{16}	1	1	1	0	0	0	0	0	0	1
p_{17}	0	0	0	1	0	0	0	0	0	1
p_{18}	0	0	0	0	1	0	0	0	0	1
p_{19}	0	0	0	0	0	1	0	0	0	1
p_{20}	0	0	0	0	0	0	1	1	1	1

OUTPUT

First Betti Number of Unbranched Part: $b_1^u = 94$.
First Betti Number: $b_1 = 10$.
Second Betti Number: $b_2 = 914$.
Chern Numbers: $c_1^2 = 1408$, $c_2 = 896$.
Bounds on the Picard number: $236 \leq p \leq 522$.
Euler number of $O_{\widehat{X}}$: $\chi(O_{\widehat{X}}) = 192$.
Irregularity: q = 5.
Arithmetic genus: $p_a = 191$.
Geometric genus: $p_g = 196$.

References

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[**B-P-V**] W. Barth, C. Peters, A. Van de Ven, "Compact Complex Surfaces," Springer-Verlag, Berlin, 1984.

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[**Br**] E. Brieskorn, H. Knörrer, "Plane Algebraic Curves," Birkhäuser-Verlag, Boston, 1986.

[**C**] D. Cheniot, *Une démonstration du théorème de Zariski sur les sections hyperplanes d'une hypersurface projective et du théorème de Van Kampen sur le groupe fondamental du complémentaire d'une courbe projective plane*, Comp. Math. **27** (1973), 141–158.

[**D**] P. Deligne, *Le groupe fondamental du complément d'une courbe plane n'ayant que des points double ordinaires est abélien*, Séminaire Bourbaki 32e année no. 543 (1979).

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[**Es**] H. Esnault, E. Viehweg, *Revêtements cycliques*, Proceedings of the CIME conference at Varenna, Threefolds (1981).

[**Fo1**] R. Fox, *Free Differential Calculus I*, Annals of Math. **57** (1953), 547–560.

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