CHOW GROUPS
and
BOREL-MOORE SCHEMES

by

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Introduction

This paper is a continuation of Rosselló-Xambó [1987]. Our interest in the computation of Chow groups stems mainly from Enumerative Geometry. This branch of Geometry is concerned with the formulation of principles and development of methods that suffice to solve questions that ask how many geometric figures of some kind satisfy some suitable number of conditions, the construction of the figures not being required (see Schubert [1879]). Usually Chow groups enter the scene, in the process of solving an enumerative problem, in steps 4 and 5 of the "program" below.

Step 1: Parametrization. First we need to parametrize the figures we are interested in by the points of some variety, say S. So, for example, irreducible conics in $\mathbf{P}^3$ may be parametrized by an open set U of $\mathbf{P}(S^2(E^*))$, where E is the rank 3 tautological bundle over $\mathbf{P}^3$.

Step 2: Interpretation of conditions as cycles. Next we interpret conditions imposed on the figures as cycles, or families of cycles, on S. Often conditions are determined by some geometric relation of our figures to some other kind of geometric entity (räumliche Bedingungen, in Schubert' s terminology). Such relations give rise to algebraic (often rational) families of cycles. For example, the conics in $\mathbf{P}^3$ meeting a line form an irreducible hypersurface of the open set U described in Step 1, and when the line moves we get a rational family of hypersurfaces of U. A condition that expresses a geometric restriction, or special configuration, on the internal structure of our figures (absolute conditions) gives rise to a distinguished cycle on S. For example, if we take tetrads of distinct colinear points, we may consider the condition that the points form an harmonic tetrad.

Step 3: Interpretation of the enumerative problem as the computation of the degree of a 0-cycle obtained intersecting the cycles corresponding to the conditions given in the problem. Here sometimes the difficulty lies in proving that the locus of solutions has indeed dimension 0, as for instance in the problem of finding the number of $(n-2)$-dimensional linear spaces in $\mathbf{P}^n$ that are $(2n-2)$-secant of a given curve C, provided this number is finite.

Step 4: Algebraization via compactification. Take the closure of the intersecting cycles in some suitable complete variety S' containing S and in such a way that *the degree N of the product of the rational classes of the closed conditions is the number we are looking for.*

Step 5: Computation. To compute N effectively usually involves writing one of the conditions as linear combinations of degeneration classes, that is, classes of components of $S'-S$ (boundary components), which has the efect of reducing the problem to computations on the degeneration varieties. For this reduction one must know how the conditions involved in the definition of N restrict to the boundary components. Sometimes we can compute the numbers using a variety S' which is a 'partial compactification' of S, that is, obtained adding only a few boundary components to S.

Steps 1 and 2 are usually solved by current general techniques, while Step 3 is often achieved using transversality results (for instance Kleiman's transverslity of generic translates, or, more generally, Casas [1987] and Speiser [1988]) that guarantee that relational conditions meet properly under rather general circumstances.

Step 4 is often difficult. Here the points in $S'-S$ may be interpreted as some sort of figures obtained degenerating the figures in S, so that usually the problem is to understand what possible degenerations our figures of type S may undergo. In general, the more complicated the problem we want to solve, the deeper the knowledge of the degenerations we need.

Let us remark in passing that a simple way of producing degenerations is by means of the homolography process, that is, a family of homologies in projective space whose modulus goes to 0 (or to ∞). In this way one can obtain, for example, *all* possible degenerations of the cuspidal cubics (Miret-Xambó [1987]).

In step 5 it is needed to know:

(a) How the Chow groups behave under the diverse opperations that we perform to arrive at S' starting from S (for instance under closure or blow up (or down));

(b) What information can we get from the knowledge of structural properties of a given variety (for instance from the existence of distinguished closed filtrations, or from the existence of distinguished fibrations over other varieties), and

(c) In which ways the Chow groups of a given variety are interrelated (existence of some form of duality and so on).

In what follows we will focus on some general statements that solve a number of questions in the directions pointed at by (a), (b) and (c), which improve, to some extend, existing results, and also on some new applications of them. We refer to Miret-Xambó [1987, 88] for the application of the program above to plane cuspidal cubics, and also for details about the program itself, particularly concerning the use of partial compactifications.

Contents.

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1. Borel–Moore schemes

1.1. Notations. Let $\mathbf{k}$ be an algebraically closed field of characteristic $p \geq 0$. Here the term *scheme* will mean an algebraic $\mathbf{k}$–scheme of finite type which admits a closed imbedding into a regular $\mathbf{k}$–scheme of finite type and *variety* will mean an irreducible, reduced scheme. If X is a scheme, $A_k(X)$ will denote the k–dimensional Chow group of X. R will always denote a noetherian domain wich is flat over $\mathbf{Z}$, the ring of integers. The field of fractions of R will be denoted $K(R)$. Finally we shall often write $A_k(X,R)$ instead of $A_k(X) \otimes R$.

1.2. Homology. We shall let $H_*(\ ,R)$ denote a homology theory for schemes, that is, a family of functors from schemes to R–modules, *covariant for proper maps* and *contravariant for open embeddings*, satisfying properties (a)–(e) below.

(a) If X is a scheme of dimension d, then $H_i(X,R) = 0$ for $i < 0$ and $i > 2d$.

(b) Let X be a scheme, Y a closed subscheme and $U = X - Y$. Let $i: Y \hookrightarrow X$ and $u: U \hookrightarrow X$ be the inclusion maps. Then there exists a functorial long exact sequence

$$\ldots \to H_{k+1}(U,R) \xrightarrow{\partial} H_k(Y,R) \xrightarrow{i_*} H_k(X,R) \xrightarrow{u^*} H_k(U,R) \to \ldots$$

(c) For any finite disjoint union of schemes $\bigsqcup X_i$ and for all k

$$H_k(\bigsqcup_i X_i, R) = \bigoplus_i H_k(X_i, R).$$

(d) For all schemes X and all integers k there exists a map

$$cl_X^k : A_k(X) \longrightarrow H_{2k}(X,R)$$

which commutes with push-forward by proper morphisms and with restrictions to open subschemes. After tensoring with R, it induces a map from $A_k(X,R)$ to $H_{2k}(X,R)$, which we shall denote $cl^k_{X,R}$, or simply cl^k_X if there is no danger of confusion.

A Borel–Moore scheme with respect to R, or simply an *R-Borel-Moore scheme* ($R-BM$ for short), will be a scheme X for which cl_X is an isomorphism, by which we mean that for all k (i) $cl^k_{X,R}$ is an isomorphism and (ii) $H_{2k+1}(X,R) = 0$ (cf. Fulton [1984], Ex. 19.1.11, and Rosselló-Xambó [1987], Sect. 2).

(e) If X is an R–Borel-Moore scheme, then any projective bundle over X is an R–Borel-Moore scheme.

If $p = 0$ the *Borel-Moore homology* satisfies (a)–(e) with respect to $R = \mathbf{Z}$ (see Iversen [1986], Ch. IX, X; see also Fulton [1984], Ch. 19, and Verdier [1976]). If $p > 0$ there exists a homology theory that satisfies (a)–(e) with respect to $R = \mathbf{Z}_\ell$, ℓ any prime number different from p (see Laumon [1976]). Henceforth a Borel–Moore scheme will be a Borel–Moore scheme with respect to $\mathbf{Z}$ if $p = 0$ and a Borel–Moore scheme with respect to $\mathbf{Z}_\ell$, for all $\ell \neq p$, if $p > 0$.

1.3. Lemma. *Let $i: Y \hookrightarrow X$ a closed embedding and assume that Y is an R–BM scheme. Let $U = X - Y$ and $u: U \hookrightarrow X$. Then the following two conditions are equivalent:*

i) *U is an R–Borel-Moore scheme.*
ii) *X is an R–Borel-Moore scheme and $i_* : A_*(Y,R) \to A_*(X,R)$ is injective.*

Proof: This Lemma follows from the definitions and a little chasing over the diagram

$$\begin{array}{ccccccccccc}
0 \to H_{2k+1}(X,R) \to H_{2k+1}(U,R) & \xrightarrow{\partial} & H_{2k}(Y,R) & \xrightarrow{i_*} & H_{2k}(X,R) & \xrightarrow{u^*} & H_{2k}(U,R) & \to 0 \\
 & & \uparrow{\scriptstyle cl^k_Y} & & \uparrow{\scriptstyle cl^k_X} & & \uparrow{\scriptstyle cl^k_U} & \\
0 \to \quad \mathrm{Ker}(i_*) & \to & A_k(Y,R) & \xrightarrow{i_*} & A_k(X,R) & \xrightarrow{u^*} & A_k(U,R) & \to 0
\end{array}$$

where the top row is exact by **1.2** (b) and the hypothesis, and the bottom row is exact by Fulton [1984], Prop. 1.8.1. □

1.4. Corollary. *With the same notations as in* **1.3**, *the following two conditions are equivalent:*

i) *U is a Borel–Moore scheme.*
ii) *X is a Borel–Moore scheme and $i_*: A_*(Y) \to A_*(X)$ is injective.*

Proof: Apply **1.3** to $R = \mathbf{Z}$, when $p = 0$, and to $R = \mathbf{Z}_\ell$ when $p > 0$. In the latter case notice that if U is a Borel–Moore scheme then $(\operatorname{Ker} i_*) \otimes \mathbf{Z}_\ell = 0$, for all $\ell \neq p$, which implies that $\operatorname{Ker} i_* = 0$. □

1.5. As a corollary of **1.3**, it is also easy to prove that any vector bundle over an R–Borel–Moore scheme is an R–Borel–Moore scheme (see Rosselló-Xambó [1987], Lemma in section 2).

1.6. Remark. The conditions (i) and (ii) in the definition of Borel–Moore schemes are independent. Indeed, on one hand $\mathbf{A}^n - \{0\}$, $n > 0$, is a scheme such that cl^k is an isomorphism for all k but with $H_1 = \mathbf{Z}$, as it can be easily seen arguing as in the proof of **1.3**. On the other hand, $K3$ surfaces have vanishing odd homology and for them cl^1 is not an isomorphism (see Barth et al. [1984], VIII, 3).

2. Filtrations and fibrations

In this Section some statements for BM schemes in Rosselló-Xambó [1987] and Rosselló [1988] are recast to include R-BM schemes. Since the proofs are similar they are mostly omited.

2.1. An *R-Borel-Moore filtration* of a scheme X is a sequence of closed subschemes X_i of X,

$$(*) \qquad \emptyset = X_{-1} \subset X_0 \subset X_1 \subset \ldots \subset X_{n-1} \subset X_n = X,$$

such that all schemes $Z_i = X_i - X_{i-1}$ are R–Borel–Moore schemes. When all differences Z_i are Borel–Moore schemes (i. e., $\mathbf{Z}$–Borel–Moore schemes in the characteristic 0 case, and $\mathbf{Z}_\ell$–Borel–Moore schemes in the characteristic $p \neq 0$ case, for all $\ell \neq p$) we will call (*) a *Borel–Moore filtration.* An R–Borel–Moore filtration will be called *R-free* if $A_*(Z_i, R)$ is a free R–module, for all i. Instead of $\mathbf{Z}$-free and $\mathbf{Z}_\ell$-free, as the case may be, we will say free for short.

2.2. Proposition (cf. Roselló-Xambó [1987], Th. 1)

a) If a scheme X admits an R–Borel–Moore filtration then X is an R–Borel–Moore scheme.

b) If the R–Borel–Moore filtration of X is R–free, then $A_k(X, R)$ is a free R-module, for all k. Moreover, the classes of the closures in X of representative cycles of R-free bases for $A_k(Z_i, R)$, $0 \leq i \leq n$, form an R-free basis for $A_k(X, R)$.

c) In particular, if X admits a free Borel–Moore filtration, then X is a Borel–Moore scheme, $A_k(X)$ is a free group for all k, and the union of the classes of the closures in X of representative cycles of free bases for $A_k(Z_i)$, $0 \leq i \leq n$, form a free basis for $A_k(X)$.

2.3. Folklore theorem. As a corollary of **Proposition 2.2 (c)** and **1.5**, we get that if a scheme X admits a cellular decomposition in the sense of Fulton [1984], Ex. 1.9.1 (i. e., a closed filtration such that each Z_i is isomorphic to finite disjoint union of affine spaces (the *cells* of the decomposition), then X is a Borel-Moore scheme and $A_k(X)$ is freely generated by the classes of the closures of the k–dimensional cells (cf. Rosselló-Xambó [1987], Corollary in Sect. 1; see also Fulton [1984], Ex. 19.1.11 (b)).

2.4. Theorem (cf. Rosselló-Xambó [1987], Th. 2)

a) Let X be a scheme which admits a cellular decomposition and let $f\colon X' \longrightarrow X$ a morphism which is trivial over any cell in the cellular decomposition of X, with fiber a fixed scheme F. Then for all k there exists an epimorphism

$$g\colon \bigoplus_{r+s=k} A_r(X) \otimes A_s(F) \longrightarrow A_k(X').$$

a′) If the fiber F in (a) is an R–Borel–Moore scheme, then X' is also a R–Borel–Moore scheme and $g \otimes K(R)$ is an isomorphism.

a″) If the fiber F in (a) is an R–Borel–Moore scheme and $A_k(F, R)$ is a free R–module, then X' is also an R–Borel–Moore scheme and $g \otimes R$ is an isomorphism.

b) With the notations and assumptions of (a), if X admits an R–Borel–Moore filtration and F admits a cellular decomposition, then X' is an R–Borel–Moore scheme and there exists an epimorphism g as in (a) such that $g \otimes K(R)$ is an isomorphism.

b′) If the R–Borel–Moore filtration of X is R-free, then $g \otimes R$ is an isomorphism.

Statements (a), (a′) and (a″) are proved in Rosselló-Xambó [1987], Sect. 4, for the Borel–Moore case. The extension here, as well as the proof of (b) and (b′), can be done in a similar way.

To end this Section, we recall an application of **Theorem 2.4** which will be used repeatedly in **Section 4**. It is worked out in Rosselló [1988], Th. 2.2, but for convenience of the reader we include a sketch of its proof. The notations for Schubert cycles on $\mathrm{Gr}(k,n)$ (the Grassmann variety of k-planes in $\mathbf{P}^n$) appearing in its statement and proof are those of Fulton [1984], §14.7.

2.5. Theorem. *Given a non-singular variety $\mathcal{W}$ and a morphism*

$$p\colon \mathcal{W} \longrightarrow Gr(k,n),$$

$(k < n)$, assume that p is trivial over all open Schubert cells (with respect to a fixed complete flag) of $Gr(k,n)$, with fiber a fixed scheme W, and that moreover the following properties are satisfied:

i) W is a non-singular complete Borel–Moore variety.

ii) $A_*(W)$ is a finitely generated free group.

iii) For all m, the intersection product induces an isomorphism between $A^m(W)$ and $A_m(W)^*$ (the dual of $A_m(W)$).

Then:

a) $\mathcal{W}$ is a Borel–Moore scheme of dimension $(k+1)(n-k)+dim(W)$;

b) All Chow groups $A_m(\mathcal{W})$ are finitely generated free groups of ranks

$$b_m(\mathcal{W}) = \sum_{i=0}^{m} b_i(W) b_{m-i}(Gr(k,n));$$

c) The intersection product induces an isomorphism $A^m(\mathcal{W}) \cong A_m(\mathcal{W})^*$;

d) Suppose given a general linear space $L_k \hookrightarrow \mathbf{P}^n$ of dimension k and a subset

$$\mathcal{Z}_l = \{\mathcal{Z}_{l,j}\}_j$$

of $A_l(\mathcal{W})$, for all l, such that its restriction to the fiber $p^{-1}(\{L_k\}) \cong W$ is a free basis of $A_{l-(k+1)(n-k)}(W)$. Then the set

$$\mathcal{B}_m = \left\{p^*(a_0, a_1, \ldots, a_k) \cdot \mathcal{Z}_{((k+1)(n-k)+m-\dim(a_0,\ldots,a_k)),j}\right\}_{\substack{(a_0,a_1,\ldots,a_k),\ j \\ 0 \le a_0 < a_1 \ldots < a_k \le n}}$$

in $A_m(\mathcal{W})$ is a free basis of this group, for all m.

Proof: Statements (a) and (b) are direct consequences of **Theorem 2.4.** We shall prove now (c) and (d) simultaneously.

First notice that (b) implies $b_m(\mathcal{W}) = b_{\dim \mathcal{W} - m}(\mathcal{W})$ and card $\mathcal{B}_m = b_m(\mathcal{W})$.

Applying the properties of the intersection product of Schubert cycles (Fulton [1984], §14.7, Griffiths-Harris [1978], pp. 197-198), the commutativity of the intersection product and its compatibility with flat pull-backs and Gysin-homomorphisms (Fulton [1984], Th. 6.2.(b) and Th. 6.4), for any two cycles Z_1 and Z_2 on $\mathcal{W}$ we have the following:

1) Given two partitions $(a_0, a_1, \dots, a_k)$, $(b_0, b_1, \dots, b_k) \in \mathbf{Z}^{k+1}$ (with $0 \leq a_0 < \dots < a_k \leq n$ and $0 \leq b_0 < \dots < b_k \leq n$) such that there exists some i with $b_{k-i} + a_i < n$, then

$$(p^*(a_0, a_1, \dots, a_k) \cdot Z_1) \cdot (p^*(b_0, b_1, \dots, b_k) \cdot Z_2) = 0$$

2) Given any partition $(a_0, a_1, \dots, a_k) \in \mathbf{Z}^{k+1}$ (with $0 \leq a_0 < \dots < a_k \leq n$), then

$$(p^*(a_0, a_1, \dots, a_k) \cdot Z_1) \cdot (p^*(n - a_k, n - 1 - a_{k-1}, \dots, n - k - a_0) \cdot Z_2) = i^*(Z_1) \cdot i^*(Z_2)$$

where i is the regular embedding $i \colon W \cong f^{-1}(\{H_k\}) \hookrightarrow \mathcal{W}$.

From (1) and (2) we infer that the intersection products of elements of $\mathcal{B}_m$ and $\mathcal{B}_{\dim \mathcal{W} - m}$, in a suitable order, give a square "triangular" matrix

$$\begin{pmatrix} M_1 & * & * & \dots & * \\ 0 & M_2 & * & \dots & * \\ 0 & 0 & M_3 & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \end{pmatrix}$$

where the boxes on the diagonal ate matrices whose entries are the intersection products of pairs of cycles belonging to bases for two Chow groups of W of complementary dimensions, so that (by (iii)) each M_i is unimodular. We conclude that the whole matrix is unimodular, which implies (c) and (d) (cf. Lemma in Elencwajg-Le Barz [1983]). □

3. Chow groups of blow ups and other modifications

3.1. Notations. Let X be a scheme and let Y be a closed subscheme of X such that the embedding $i \colon Y \hookrightarrow X$ is regular of codimension $r = e + 1 \geq 2$. Let

$$\begin{array}{ccc} \widetilde{Y} & \xrightarrow{j} & \widetilde{X} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[i]{} & X. \end{array}$$

denote the fibre square corresponding to the blow-up of X along Y. If $\mathbf{N}$ denotes the normal bundle to Y in X, recall that $\widetilde{Y} = \mathbf{P}(\mathbf{N})$. We will write $h = c_1(\mathcal{O}_{\widetilde{Y}}(1))$ and $\mathbf{E} = g^*\mathbf{N}/\mathcal{O}_{\widetilde{Y}}(-1)$.

In next theorem, the case when X is a smooth variety and Y a smooth subvariety of codimension 2 is contained in Samuel [1958], §3, and with the same hypothesis but arbitrary codimension in Jouanolou [1977], 9.10.2 and Beauville [1977], Prop. 0.1.3. The arguments in our proof, although simpler, are rooted in those older proofs and use the level of generality afforded by today intersection theory.

3.2. Theorem.

a) *For all k, there exists a split exact sequence*

$$0 \to \bigoplus_{i=1}^{e} A_{k-i}(Y) \xrightarrow{\varphi} A_k(\widetilde{X}) \xrightarrow{f_*} A_k(X) \to 0$$

where

$$\varphi(y_1, \ldots, y_e) = j_* \sum_{i=1}^{e} h^{e-i} g^*(y_i)$$

for all $y_i \in A_{k-i}(Y)$, $i = 1, \ldots, e$. Moreover, a splitting of the sequence is given by the map $f^*: A_k(X) \longrightarrow A_k(\widetilde{X})$.

b) *In particular, for all k there exists an isomorphism*

$$\Phi\colon A_k(X) \oplus A_{k-1}(Y) \oplus \ldots \oplus A_{k-e}(Y) \longrightarrow A_k(\widetilde{X})$$

given by the formula

$$\Phi(x, y_1, \ldots, y_e) = f^*(x) + \varphi(y_1, \ldots, y_e).$$

c) *The inverse of Φ is the map Ψ given by the relation*

$$\Psi(\tilde{x}) = (f_*(\tilde{x}), -g_*j^*(\tilde{x}), -g_*c_1(\mathbf{E})j^*(\tilde{x}), \ldots, -g_*c_{e-1}(\mathbf{E})j^*(\tilde{x})).$$

Proof: Consider the sequence

$$0 \to A_k(\widetilde{Y}) \xrightarrow{\alpha} A_k(Y) \oplus A_k(\widetilde{X}) \xrightarrow{\beta} A_k(X) \to 0,$$

where

$$\alpha(\tilde{y}) = (g_*(\tilde{y}), -j_*(\tilde{y})), \quad \beta(y, \tilde{x}) = i_*(y) + f_*(\tilde{x}).$$

Then, except for the 0 on the left, it is exact (Fulton [1984], Ex. 1.8.1). But α is also injective, by Fulton [1984], Prop. 6.7 (c). Moreover, if we let

$$\gamma\colon A_k(X) \longrightarrow A_k(Y) \oplus A_k(\widetilde{X})$$

be the map defined by

$$\gamma(x) = (0, f^*(x)),$$

then

$$\beta\gamma = f_* f^* = id,$$

by Fulton [1984], Prop. 6.7 (b). So our sequence is split. By the snake lemma applied to the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A_k(\widetilde{Y}) \arrow[r, "\alpha"] \arrow[d, "g_*"] & A_k(Y) \oplus A_k(\widetilde{X}) \arrow[r, "\beta"] \arrow[d, "pr_1"] & A_k(X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_k(Y) \arrow[r, "id"] & A_k(Y) \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

we obtain that

$$0 \to \operatorname{Ker} g_* \xrightarrow{j_*} A_k(\widetilde{X}) \xrightarrow{f_*} A_k(X) \to 0$$

is exact and split, hence

$$A_k(X) \oplus \operatorname{Ker} g_* \simeq A_k(\widetilde{X}),$$

via the map $(x, \widetilde{y}) \mapsto f^*(x) + j_*(\widetilde{y})$.

Now

$$A_k(\widetilde{Y}) \simeq A_k(Y) \oplus A_{k-1}(Y) \oplus \ldots \oplus A_{k-e}(Y),$$

by the map $(y_0, y_1, \cdots, y_e) \mapsto h^e g^*(y_0) + \ldots + g^*(y_e)$ and under this isomorphism, $g_*(y_0, y_1, \cdots, y_e) = y_0$. Hence $\operatorname{Ker} g_* \simeq A_{k-1}(Y) \oplus \ldots \oplus A_e(Y)$. From this it follows that Φ is indeed an isomorphism. Checking that Ψ is the inverse of Φ follows easily from the relation

$$g_*(c_k(\mathbf{E}) h^j g^*(y)) = y \text{ if } k + j = e, \ 0 \text{ otherwise}$$

(which in turn is equivalent to the relation $c(\mathbf{N})s(\mathbf{N}) = 1$, c and s the total Chern and Segre classes of $\mathbf{N}$) and the self-intersection formula for j, $j^* j_* = -h$. □

3.3. Corollary. *For all k, g_* induces an isomorphism between $(\operatorname{Ker} j_*)_k$ and $(\operatorname{Ker} i_*)_k$. The inverse homomorphism is given by the map*

$$\theta(y) = \sum_{j=0}^{e} h^{e-j} g^*(c_j(\mathbf{N}) y).$$

Proof: From the definition of $\mathbf{E}$ and the *key formula* (Fulton [1984], Prop. 6.7 (a)) we easily obtain that, for all $y \in A_k(Y)$,

$$j_*(h^e g^*(y)) = f^* i_*(y) - j_* \sum_{j=1}^{e} h^{e-j} g^*(c_j(\mathbf{N}) y).$$

Aplying this formula, we get

$$\begin{aligned} j_*(\sum_{j=0}^{e} h^{e-j} g^*(y_j)) &= f^* i_*(y_0) + j_* \sum_{j=1}^{e} h^{e-j} g^*(y_i - c_j(\mathrm{N})y_0) \\ &= \Phi(i_*(y_0), y_1 - c_1(\mathrm{N})y_0, \ldots, y_e - c_e(\mathrm{N})y_0) \end{aligned}$$

for all $y_j \in A_{k-j}(Y)$, $j = 0, \ldots, e$. Since Φ is an isomorphism, we deduce

$$j_*(\sum_{j=0}^{e} h^{e-j} g^*(y_j)) = 0 \text{ if and only if } \begin{cases} i_*(y_0) = 0 \\ y_i = c_i(\mathrm{N})y_0 \quad i = 1, \ldots, e. \end{cases}$$

Finally, since $g_*(\sum_{j=0}^{e} h^{e-j} g^*(y_j)) = y_0$, we get the desired isomorphism. □

3.4. Proposition. *Suppose that Y is an R–Borel–Moore scheme. Then X is an R–Borel–Moore scheme if and only if $\widetilde{X}$ is an R–Borel–Moore scheme.*

Proof: We shall prove that if X is an R–Borel–Moore scheme then $\widetilde{X}$ is also an R–Borel–Moore scheme. The proof of the inverse implication is very similar, so we leave it to the reader.

Let $U = X - Y$ and let $u: U \hookrightarrow X$ be the inclusion. Reasoning over the commutative diagram

$$\begin{array}{ccccccccc} 0 \to H_{2k+1}(U,R) & \xrightarrow{\partial_k} & H_{2k}(Y,R) & \xrightarrow{i_*} & H_{2k}(X,R) & \xrightarrow{u^*} & H_{2k}(U,R) \to 0 \\ & & \uparrow{\scriptstyle cl_Y^k} & & \uparrow{\scriptstyle cl_X^k} & & \uparrow{\scriptstyle cl_U^k} \\ 0 \to (\mathrm{Ker}\, i_*)_k & \to & A_k(Y,R) & \xrightarrow{i_*} & A_k(X,R) & \xrightarrow{u^*} & A_k(U,R) \to 0 \end{array}$$

leads easily to the following conclusions:

(i) ∂_k is injective;

(ii) cl_Y^k induces an isomorphism between $(\mathrm{Ker}\, i_*)_k$ and $\partial_k H_{2k+1}(U,R)$.

(iii) cl_U^k is an isomorphism;

Now set $\widetilde{u}: \widetilde{U} = \widetilde{X} - \widetilde{Y} \hookrightarrow \widetilde{X}$ and let $p: \widetilde{U} \longrightarrow U$ be the restriction of f to $\widetilde{U}$. Since p is an isomorphism, from (iii) we get that $cl_{\widetilde{U}}^k$ is also an isomorphism. Finally, recall that, since Y is an R–Borel–Moore scheme and $\widetilde{Y}$ is a projective bundle over Y, then $\widetilde{Y}$ is also an R–Borel–Moore scheme (**1.2** (e)). Let's consider now the following commutative diagram:

$$(*) \quad \begin{array}{ccccccccc} 0 \to H_{2k+1}(\widetilde{X},R) \to H_{2k+1}(\widetilde{U},R) & \xrightarrow{\partial'_k} & H_{2k}(\widetilde{Y},R) & \xrightarrow{j_*} & H_{2k}(\widetilde{X},R) & \xrightarrow{\widetilde{u}^*} & H_{2k}(\widetilde{U},R) \to 0 \\ & & \uparrow{\scriptstyle cl_{\widetilde{Y}}^k} & & \uparrow{\scriptstyle cl_{\widetilde{X}}^k} & & \uparrow{\scriptstyle cl_{\widetilde{U}}^k} \\ 0 \to (\mathrm{Ker}\, j_*)_k & \to & A_k(\widetilde{Y},R) & \xrightarrow{j_*} & A_k(\widetilde{X},R) & \xrightarrow{\widetilde{u}^*} & A_k(\widetilde{U},R) \to 0 \end{array}$$

From the functoriality of the long exact sequence in **1.2** (b), the injectivity of ∂_k and the fact that p_* is an isomorphism, we easily get that

(*iv*) $H_{2k+1}(\widetilde{X}, R) = 0$.

From **3.3** and (*ii*) it is not difficult to deduce that

(*v*) $cl^k_{\widetilde{Y}}$ induces an isomorphism between $(\operatorname{Ker} j_*)_k$ and $\partial'_k H_{2k+1}(\widetilde{U}, R)$.

Finally it is easy to see that (*v*) implies

(*vi*) $cl^k_{\widetilde{X}}$ is an isomorphism.

Now (*iv*) and (*vi*), and the fact that k is an arbitrary integer, tell us that $\widetilde{X}$ is an R–Borel–Moore scheme. □

3.5. Let's suppose now that Y is a closed subscheme of X (no longer regularly embedded into X), and let

$$\begin{array}{ccc} Y' & \xrightarrow{j} & X' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[i]{} & X. \end{array}$$

be a proper modification of X along Y (i. e., a fiber square where f is a proper map such that it induces an isomorphism between $U' = X' - Y'$ and $U = X - Y$). Assume that g is a projective bundle of rank e.

3.6. Theorem. *In the conditions of* **3.5**, *assume that* Y *and* U *are* R*–Borel–Moore schemes. Then*

a) X *and* X' *are* R*–Borel–Moore schemes and the maps* $i_* : A_*(Y, R) \to A_*(X, R)$, $j_* : A_*(Y', R) \to A_*(X', R)$ *are monomorphisms.*

b) For all k, *there exists an exact sequence*

$$0 \longrightarrow \bigoplus_{i=1}^{e} A_{k-i}(Y, R) \xrightarrow{\varphi} A_k(X', R) \xrightarrow{f_*} A_k(X, R) \longrightarrow 0$$

where

$$\varphi(y_1, \ldots, y_e) = j_* \sum_{i=1}^{e} h^{e-i} g^*(y_i),$$

for all $y_i \in A_{k-i}(Y, R)$, $i = 1, \ldots, e$ *(with* $h = c_1(\mathcal{L})$, $\mathcal{L}$ *the tautological line bundle on* Y'*).*

Proof: Part (a) is straighforward from the hypothesis and **1.3**.

As far as part (b) goes, when we apply Fulton [1984], Ex. 1.8.1, to the fibre square in **3.5** we get the exact sequence:

$$A_k(Y') \xrightarrow{\alpha} A_k(Y) \oplus A_k(X') \xrightarrow{\beta} A_k(X) \to 0$$

where $\alpha(y') = (g_*(y'), j_*(y'))$ and $\beta(y, x') = i_*(y) - f_*(x')$ for all $y' \in A_k(Y')$, $y \in A_k(Y)$ and $x' \in A_k(X')$.

Tensoring this exact sequence by R and noticing that, by (a), $j_* : A_*(Y', R) \to A_*(X', R)$ is injective, we obtain that the sequence

$$0 \to A_k(Y', R) \xrightarrow{\alpha} A_k(Y, R) \oplus A_k(X', R) \xrightarrow{\beta} A_k(X, R) \to 0$$

is exact for all k.

From this exact sequence one easily deduces that the sequence

$$(*) \qquad 0 \to (\operatorname{Ker} g_*)_k \otimes R \xrightarrow{j_*} A_k(X', R) \xrightarrow{f_*} A_k(X, R) \to 0$$

is also exact, for all k.

Finally, the exact sequence in the statement comes from (*) and the isomorphism

$$\Phi : \bigoplus_{i=1}^{e} A_{k-i}(Y) \longrightarrow (\operatorname{Ker} g_*)_k$$

given by

$$\Phi(y_1, \dots, y_e) = \sum_{i=1}^{e} h^{e-i} q'^*(y_i)$$

(Fulton [1984], Prop. 3.1 (a) and Th. 3.3 (b)). □

In particular, taking $R = \mathbf{Z}$ or $\mathbf{Z}_\ell$, as the case may be, we have:

3.7. Corollary. *In the conditions of* **3.5**, *assume that Y and U are Borel–Moore schemes. Then*

a) *The schemes X and X' are Borel–Moore schemes and the maps $i_* : A_*(Y) \to A_*(X)$, $j_* : A_*(Y') \to A_*(X')$ are monomorphisms.*

b) *For all k, there exists an exact sequence*

$$0 \longrightarrow \bigoplus_{i=1}^{e} A_{k-i}(Y) \xrightarrow{\varphi} A_k(X') \xrightarrow{f_*} A_k(X) \longrightarrow 0$$

where

$$\varphi(y_1,\ldots,y_e) = j_* \sum_{i=1}^{e} h^{e-i} g^*(y_i),$$

for all $y_i \in A_{k-i}(Y)$, $i = 1,\ldots,e$ *(with* $h = c_1(\mathcal{L})$, $\mathcal{L}$ *the tautological line bundle on* Y'*).*

3.8. Remark. We don't know whether this exact sequence is, in general, split.

4. $\widetilde{\mathrm{Cop}^k}\mathbf{P}^n$ revisited and the Chow groups of $\mathrm{Cop}^k\mathbf{P}^n$

4.1. Let $\mathrm{Al}^k\mathbf{P}^n$ ($n \geq 2$, $k \geq 3$) denote the closed subscheme of $\mathrm{Hilb}^k\mathbf{P}^n$ whose points are the collinear k–tuples (i. e., k–tuples contained in lines), and let

$$\mathrm{Axis}\colon \mathrm{Al}^k\mathbf{P}^n \longrightarrow \mathrm{Gr}\,(1,n)$$

denote the map sending any collinear k–tuple to the unique line containing it. It is known that Axis is a projective bundle of rank k (Le Barz [1987], Prop. 1 and Rém. 2). Then, applying **1.2** (e), we get that $\mathrm{Al}^k\mathbf{P}^n$ is a Borel–Moore variety, and we can compute its Chow groups using **2.5**.

4.2. Now let $\mathrm{Cop}^k\mathbf{P}^n$ ($n \geq 2, k \geq 3$) denote the closed subscheme of $\mathrm{Hilb}^k\mathbf{P}^n$ parametrizing the coplanary k–tuples (i. e., k–tuples contained in planes). Notice that there does not exist a regular map from $\mathrm{Cop}^k\mathbf{P}^n$ to $\mathrm{Gr}(2,n)$ analogous to Axis in **4.1**, for collinear k-tuples are contained in infinitely many planes.

When $n = 2$, $\mathrm{Cop}^k\mathbf{P}^2 = \mathrm{Hilb}^k\mathbf{P}^2$ is known to be a non-singular Borel–Moore variety of dimension $2k$, for all k (Fogarty [1968], Ellingsrud-Strømme [1987]), and its Chow groups have been proved to be free and their ranks have been computed in Ellingsrud-Strømme [1987], Th. 1.1, applying the results in Bialynicki-Birula [1973], [1976] on cellular decompositions defined by a torus action.

When $k = 3$, $\mathrm{Cop}^3\mathbf{P}^n = \mathrm{Hilb}^3\mathbf{P}^n$ is also known to be a non-singular Borel–Moore variety of dimension $3n$, for all n, and its Chow groups have been computed in Rosselló [1986], Th. 2, using **2.5** and **3.2**, and in Rosselló-Xambó [1987] §5, using **Theorem 2.4** (b) and Bialynicki-Birula's theorems.

On the other hand, it is not difficult to check that, when $n \geq 3$ and $k \geq 4$, $\mathrm{Cop}^k\mathbf{P}^n$ is a singular variety (for instance, along the collinear k–tuples supported by a single point). In Rosselló [1986] the Chow groups of $\widetilde{\mathrm{Cop}^k}\mathbf{P}^n$, a natural desingularization of $\mathrm{Cop}^k\mathbf{P}^n$ (see below), were studied. Here we will show that both $\widetilde{\mathrm{Cop}^k}\mathbf{P}^n$ and $\mathrm{Cop}^k\mathbf{P}^n$ are Borel-Moore schemes and will determine the Chow groups of the latter (see **4.7**).

4.3. To do this, let's introduce the following auxiliar incidence variety (see Elencwajg-Le Barz [1983], Rosselló [1987]):

$$\widetilde{\mathrm{Cop}^k\mathbf{P}^n} = \left\{(\theta,\Pi) \in \mathrm{Hilb}^k\mathbf{P}^n \times \mathrm{Gr}\,(2,n) \mid \quad \theta \subset \Pi\right\}.$$

Let p and q the restrictions to $\widetilde{\mathrm{Cop}^k\mathbf{P}^n}$ of the natural projections from $\mathrm{Hilb}^k\mathbf{P}^n \times \mathrm{Gr}\,(2,n)$,

$$p(\theta,\Pi) = \Pi \quad \text{and} \quad q(\theta,\Pi) = \theta.$$

4.4. Lemma (Rosselló [1986], Lems. 1, 2).*The morphism* p: $\widetilde{\mathrm{Cop}^k\mathbf{P}^n} \to \mathrm{Gr}(2,n)$ *is trivial over any Schubert cell on* $\mathrm{Gr}(2,n)$, *with fiber* $\mathrm{Hilb}^k\mathbf{P}^2$.

4.5. Theorem (cf. Rosselló [1986], Th. 1).*For all* $k \geq 2$, $n \geq 3$

a) $\widetilde{\mathrm{Cop}^k\mathbf{P}^n}$ *is a non-singular projective Borel–Moore variety of dimension* $3n+2(k-3)$.

b) For all m, *the Chow group* $A_m(\widetilde{\mathrm{Cop}^k\mathbf{P}^n})$ *is free of rank*

$$b_m(\widetilde{\mathrm{Cop}^k\mathbf{P}^n}) = \sum_{i=0}^{m} b_i(\mathrm{Hilb}^k\mathbf{P}^2)b_{m-i}(\mathrm{Gr}(2,n))$$

c) For all m, *the intersection product induces an isomorphism between* $A^m(\widetilde{\mathrm{Cop}^k\mathbf{P}^n})$ *and* $A_m(\widetilde{\mathrm{Cop}^k\mathbf{P}^n})^*$.

Proof: Just apply **2.5** to p. □

4.6. Now let us consider the following fiber square:

$$(*) \qquad \begin{array}{ccc} \widetilde{\mathrm{Al}^k\mathbf{P}^n} & \xrightarrow{j} & \widetilde{\mathrm{Cop}^k\mathbf{P}^n} \\ {\scriptstyle q'}\downarrow & & \downarrow{\scriptstyle q} \\ \mathrm{Al}^k\mathbf{P}^n & \xrightarrow[i]{} & \mathrm{Cop}^k\mathbf{P}^n \end{array}$$

It is very easy to check that q induces an isomorphism between $\widetilde{\mathrm{Cop}^k\mathbf{P}^n} - \widetilde{\mathrm{Al}^k\mathbf{P}^n}$ and $\mathrm{Cop}^k\mathbf{P}^n - \mathrm{Al}^k\mathbf{P}^n$, and that q' is a projective bundle of rank $n-2$. In particular, the codimension of $\widetilde{\mathrm{Al}^k\mathbf{P}^n}$ in $\widetilde{\mathrm{Cop}^k\mathbf{P}^n}$ is $k-2$. Thus, when $k=3$ it is not difficult to see that $\widetilde{\mathrm{Cop}^3\mathbf{P}^n}$ is the blow-up of $\mathrm{Cop}^3\mathbf{P}^n = \mathrm{Hilb}^3\mathbf{P}^n$ along $\mathrm{Al}^3\mathbf{P}^n$ (Elencwajg-Le Barz [1983]). This fact allowed us to apply **3.2** in order to compute the Chow groups of $\mathrm{Hilb}^3\mathbf{P}^n$ from those of $\widetilde{\mathrm{Cop}^3\mathbf{P}^n}$ (Rosselló [1986], Th. 2). On the other hand, when $k > 3$, (*) is no longer a blowing-up fiber square, because in this case $\widetilde{\mathrm{Al}^k\mathbf{P}^n}$ is not a divisor on $\widetilde{\mathrm{Cop}^k\mathbf{P}^n}$. Then, in order to compute the Chow groups of $\mathrm{Cop}^k\mathbf{P}^n$, **3.2** no longer applies, but fortunately **3.6** does.

4.7. Proposition. *With the notations of* **4.6**, *for all* $k \geq 3$, $n \geq 3$*:*

a) The map $i_*: A_m(\mathrm{Al}^k\mathbf{P}^n) \longrightarrow A_m(\mathrm{Cop}^k\mathbf{P}^n)$ *is injective, for all* m.

b) $\mathrm{Cop}^k\mathbf{P}^n$ *is a Borel–Moore variety of dimension* $3n + 2(k-3)$*;*

c) For all m*, there is an exact sequence*

$$0 \to \bigoplus_{i=1}^{n-2} A_{m-i}(\mathrm{Al}^k\mathbf{P}^n) \xrightarrow{\varphi} A_m(\widetilde{\mathrm{Cop}^k\mathbf{P}^n}) \xrightarrow{q_*} A_m(\mathrm{Cop}^k\mathbf{P}^n) \to 0$$

where φ *is defined as*

$$\varphi(y_1, \ldots, y_e) = j_* \sum_{i=1}^{e} h^{e-i} q'^*(y_i),$$

for all $y_i \in A_{m-i}(\mathrm{Al}^k\mathbf{P}^n)$, $i = 1, \ldots, e$ *(with* $h = c_1(\mathcal{L})$, $\mathcal{L}$ *the tautological line bundle over* $\widetilde{Al^k\mathbf{P}^n}$*).*

Proof: Since $\widetilde{Al^k\mathbf{P}^n}$ (which is a projective bundle over a projective bundle over $\mathrm{Gr}(1,n)$) and $\widetilde{\mathrm{Cop}^k\mathbf{P}^n}$ are Borel–Moore schemes, in order to apply **3.6** we only need to prove that $\widetilde{\mathrm{Cop}^k\mathbf{P}^n} - \widetilde{Al^k\mathbf{P}^n}$ is also a Borel–Moore scheme, or equivalently, that

$$j_*: A_m(\widetilde{Al^k\mathbf{P}^n}) \longrightarrow A_m(\widetilde{\mathrm{Cop}^k\mathbf{P}^n})$$

is a monomorphism for all m. But, since both $A_*(\widetilde{Al^k\mathbf{P}^n})$ and $A_*(\widetilde{\mathrm{Cop}^k\mathbf{P}^n})$ are finitely generated free modules, in fact we only need to prove that, for all m,

$$j_* \otimes \mathbf{Q}: A_m(\widetilde{Al^k\mathbf{P}^n}, \mathbf{Q}) \longrightarrow A_m(\widetilde{\mathrm{Cop}^k\mathbf{P}^n}, \mathbf{Q})$$

is a monomorphism.

To do this, we first prove the following

4.8. Claim. *For all* m*, the Gysin morphism*

$$j^* \otimes \mathbf{Q}: A_m(\widetilde{\mathrm{Cop}^k\mathbf{P}^n}, \mathbf{Q}) \longrightarrow A_{m-k+2}(\widetilde{Al^k\mathbf{P}^n}, \mathbf{Q})$$

is an epimorphism.

Proof: Fix a complete linear flag L on $\mathbf{P}^n$,

$$\emptyset = L_{-1} \subset L_0 \subset \ldots \subset L_n = \mathbf{P}^n,$$

a complete linear flag L'

$$\emptyset = L'_{-1} \subset L'_0 \subset \ldots \subset L'_n = \mathbf{P}^n$$

in general position with respect to L (i. e., such that for all i and j, $\dim L_i \cap L'_j = i+j-n$) and k general hyperplanes $H_1, \ldots, H_k$.

By **2.5**, a basis for $A_l(\mathrm{Al}^k\mathbf{P}^n)$, for all l, is given by the cycles

$$\left\{ [\{\theta \in \mathrm{Al}^k\mathbf{P}^n \mid \mathrm{Axis}(\theta) \cap L_{a_0} \neq \emptyset,\ \mathrm{Axis}(\theta) \subset L_{a_1}, \theta \cap H_i \neq \emptyset, \text{ for } i = 1, \ldots, r\}] \right\}_{\substack{0 \leq a_0 < a_1 \leq n,\ 0 \leq r \leq k \\ l = a_0 + a_1 + k - r - 1}}$$

Given a set of 4 positive integers (a_0, a_1, r, j), with $0 \leq a_0 < a_1' \leq n$, $0 \leq r \leq k$ and $0 \leq j \leq n-2$, we define the cycle on $\widetilde{\mathrm{Al}^k\mathbf{P}^n}$

$$Z_{a_0,a_1,r,j} = [\{(\theta, \Pi) \in \widetilde{\mathrm{Al}^k\mathbf{P}^n} \mid \mathrm{Axis}(\theta) \cap L_{a_0} \neq \emptyset,\ \mathrm{Axis}(\theta) \subset L_{a_1}, \Pi \cap L'_{n-2-j} \neq \emptyset,\ \theta \cap H_i \neq \emptyset, \text{ for } i = 1, \ldots, r\}]$$

Applying **2.5** to q' we get that the cycles

$$\left\{ Z_{a_0,a_1,r,j} \right\}_{\substack{0 \leq a_0 < a_1 \leq n,\ 0 \leq r \leq k,\ 0 \leq j \leq n-2, \\ a_0 + a_1 + k - r + n - 3 - j = m}}$$

give a basis of $A_m(\widetilde{\mathrm{Al}^k\mathbf{P}^n})$, for all m.

Now let us define, for all (a_0, a_1, r, j) as before, the cycle

$$Z'_{a_0,a_1,r,j} = [\{(\theta, \Pi) \in \widetilde{\mathrm{Cop}^k\mathbf{P}^n} \mid \theta \cap H_i \neq \emptyset, \text{ for } i = 1, \ldots, r, \ \theta \text{ contains a 2–tuple collinear with } L_{a_0}, \ \theta \cap L_{a_1} \neq \emptyset,\ \Pi \cap L'_{n-2-j} \neq \emptyset\}]$$

on $\widetilde{\mathrm{Cop}^k\mathbf{P}^n}$. It's straighforward to check that

$$Z'_{a_0,a_1,r,j} \in A_{a_0+a_1-1+n-r-j+2(k-2)}(\widetilde{\mathrm{Cop}^k\mathbf{P}^n})$$

and that $j^*(Z'_{a_0,a_1,r,j})$ is a multiple of $Z_{a_0,a_1,r,j}$. We have then found inverse images by j^* of multiples of the elements of a basis for $A_m(\widetilde{\mathrm{Al}^k\mathbf{P}^n})$, thus $j^* \otimes \mathbf{Q}$ is an epimorphism. ∎

In order to simplify the notations, from now on we will write j_* (resp. j^*) instead of $j_* \otimes \mathbf{Q}$ (resp. $j^* \otimes \mathbf{Q}$). We want to prove that

$$j_* : A_m\left(\widetilde{\mathrm{Al}^k}\mathbf{P}^n, \mathbf{Q}\right) \longrightarrow A_m\left(\widetilde{\mathrm{Cop}^k}\mathbf{P}^n, \mathbf{Q}\right)$$

is injective for all m.

On one side, since $\widetilde{\mathrm{Al}^k}\mathbf{P}^n$ is a projective bundle over a projective bundle over $\mathrm{Gr}(1,n)$, and the intersection product induces an isomorphism between $A^k(\mathrm{Gr}(1,n))$ and $A_k(\mathrm{Gr}(1,n))^*$ for all k (Fulton [1984] §14.6), it is not difficult to see that the same assertion holds for $\widetilde{\mathrm{Al}^k}\mathbf{P}^n$ (apply, for example, Roberts [1987], Prop. 7.1, or even **2.5**).

Given $\mathbf{Q}$–bases $\{y_1, \ldots, y_{b_m}\}$ and $\{y'_1, \ldots, y'_{b_m}\}$ of $A_m\left(\widetilde{\mathrm{Al}^k}\mathbf{P}^n, \mathbf{Q}\right)$ and $A^m\left(\widetilde{\mathrm{Al}^k}\mathbf{P}^n, \mathbf{Q}\right)$, respectively, take elements $\{x_1, \ldots, x_{b_m}\}$ in $A^m\left(\widetilde{\mathrm{Cop}^k}\mathbf{P}^n, \mathbf{Q}\right)$ such that $j^*(x_i) = y'_i$ for all $i = 1, \ldots, b_m$. Then we have

$$\int_{\widetilde{\mathrm{Al}^k}\mathbf{P}^n} y_i \cdot y'_j = \int_{\widetilde{\mathrm{Cop}^k}\mathbf{P}^n} j_*(y_i \cdot y'_j) = \int_{\widetilde{\mathrm{Cop}^k}\mathbf{P}^n} j_*(y_i \cdot j^*(x_j)) = \int_{\widetilde{\mathrm{Cop}^k}\mathbf{P}^n} x_j \cdot j_*(y_i)$$

for all $1 \leq i, j \leq b_m$. Since the intersection product on $\widetilde{\mathrm{Al}^k}\mathbf{P}^n$ is unimodular, it turns out that

$$\det\left(\{x_j \cdot j_*(y_i)\}_{i,j}\right) = \det\left(\{y_i \cdot y'_j\}_{i,j}\right) \neq 0,$$

and from this it is straighforward to deduce that j_* is a monomorphism. □

5. The Chow groups of Le Barz's "complete 3–tuples" of $\mathbf{P}^n$.

5.1. The goal of this Section is to compute the Chow groups of the variety parametrizing ordered triangles in $\mathbf{P}^n$, which was introduced, for a general scheme, in Le Barz [1986]. The relation of this variety to W^*, the triangle variety of Semple [1954] (see also Roberts-Speiser [1984], [1986] and [1987], and Roberts [1987]) is very similar to the relation of $\mathrm{Hilb}^3\mathbf{P}^n$ to $\mathrm{Hilb}^3\mathbf{P}^2$, so that we shall argue, in order to compute the Chow groups of the "complete 3-tuples", much as we did in **4.6**.

First we recall some facts on W^*.

5.2. Definition. *Let W^* denote the closure in $(\mathbf{P}^2)^3 \times ((\mathbf{P}^2)^*)^3 \times Gr(2,5)$ of the locally closed subscheme*

$$W^o = \{(x_1, x_2, x_3, l_1, l_2, l_3, \Sigma) \mid \forall i \neq j \; x_i \neq x_j, \; x_i \in l_j;$$
$$\Sigma \textit{ is the pencil of all conics containing } x_1, x_2 \textit{ and } x_3\}$$

5.3. Theorem

a) (Semple [1954]) W^* *is a non-singular projective variety of dimension* 6.

b) (Collino-Fulton [1987]) W^* *admits a cellular descomposition, so it is a Borel–Moore scheme.*

c) (Roberts-Speiser [1987]) *The Chow groups of* W^* *are free, with ranks given by the following table:*

i	0	1	2	3	4	5	6
$b_i(W^*)$	1	7	17	22	17	7	1

d) (Roberts-Speiser [1987]) *For all* k, *the intersection product induces an isomorphism between* $A^k(W^*)$ *and* $(A_k(W^*))^*$.

5.4. Definition (Le Barz [1986]).*For all* $n \geq 1$ *let*

$$\begin{aligned}\widehat{H^3}(\mathbf{P}^n) = \{(p_1,p_2,p_3,d_{1,2},d_{1,3},d_{2,3},t) \in (\mathbf{P}^n)^3 \times (\mathrm{Hilb}^2\mathbf{P}^n)^3 \times \mathrm{Hilb}^3\mathbf{P}^n \mid \\ \forall i \neq j \ p_i,p_j \in d_{i,j} \subset t,\ d_{i,j} - p_i = p_j \text{ and } d_{i,j} - p_j = p_i; \\ t - d_{i,j} = p_k \ \forall i,j,k \text{ different}\}\end{aligned}$$

which will be called the variety of complete 3–tuples of $\mathbf{P}^n$.

5.5. Theorem (Le Barz, loc. cit.)

a) For all n, $\widehat{H^3}(\mathbf{P}^n)$ *is a non-singular projective variety of dimension* $3n$.

b) When $n = 2$, *the map*

$$\varphi\colon \widehat{H^3}(\mathbf{P}^2) \longrightarrow W^*$$

defined as

$$\varphi(p_1,p_2,p_3,d_{1,2},d_{1,3},d_{2,3},t) = (p_1,p_2,p_3,l_1,l_2,l_3,\Sigma_t)$$

(where, for i,j,k *different,* $l_i = \mathrm{Axis}(d_{j,k})$ *and* Σ_t *is the family of all conics containing* t*) is an isomorphism.*

Our goal is to prove the following result:

5.6. Proposition. *For all* $n \geq 3$:

a) $\widehat{H^3}(\mathbf{P}^n)$ *is a Borel–Moore scheme.*

b) For all k, *the Chow group* $A_k(\widehat{H^3}(\mathbf{P}^n))$ *is free of rank*

$$b_k(\widehat{H^3}(\mathbf{P}^n)) = \sum_{i=0}^{k} b_i(\mathrm{Gr}(2,n))b_{k-i}(W^*) - \sum_{i=1}^{n-2}(\sum_{j=0}^{k-i} b_j(\mathrm{Gr}(1,n))b_{k-i-j}((\mathbf{P}^1)^3)).$$

c) For all k, the intersection product induces an isomorphism between $A^k(\widehat{H^3}(\mathbf{P}^n))$ and $(A_k(\widehat{H^3}(\mathbf{P}^n)))^*$

Proof:

5.7. Lemma. *For all $n \geq 3$, let's define the following auxiliar incidence variety:*

$$\widetilde{H^3}(\mathbf{P}^n) = \{(p_1, p_2, p_3, d_{1,2}, d_{1,3}, d_{2,3}, t, \Pi) \in (\mathbf{P}^n)^3 \times (\mathrm{Hilb}^2\mathbf{P}^n)^3 \times \mathrm{Hilb}^3\mathbf{P}^n \times \mathrm{Gr}(2,n) \mid (p_1, p_2, p_3, d_{1,2}, d_{1,3}, d_{2,3}, t) \in \widehat{H^3}(\mathbf{P}^n) \text{ and } t \subset \Pi\}$$

Then:

i) $\widetilde{H^3}(\mathbf{P}^n)$ is a non-singular projective Borel–Moore variety of dimension $3n$.

ii) For all k, $A_k(W_n^*)$ is free of rank

$$b_k(\widetilde{H^3}(\mathbf{P}^n)) = \sum_{i=0}^{k} b_i(\mathrm{Gr}(2,n)) b_{k-i}(W^*).$$

iii) For all k, the intersection product induces an isomorphism between $A^k(\widetilde{H^3}(\mathbf{P}^n))$ and $(A_k(\widetilde{H^3}(\mathbf{P}^n)))^*$

Proof: Just notice that the natural projection $p\colon \widetilde{H^3}(\mathbf{P}^n) \longrightarrow \mathrm{Gr}(2,n)$ is trivial over any Schubert cell on $\mathrm{Gr}(2,n)$, with fiber $\widehat{H^3}(\mathbf{P}^2) \cong W^*$ (this assertion can be verified arguing as it is done in Rosselló [1986], Lem. 1, 2, for the map $\widetilde{\mathrm{Cop}^k\mathbf{P}^n} \longrightarrow \mathrm{Gr}(2,n)$). Then, by **5.3**, we can apply **2.5.** ∎

5.8. Remark. It is not difficult to prove that the isomorphism $\widehat{H^3}(\mathbf{P}^2) \longrightarrow W^*$ can be "globalized", giving an isomorphism between $\widetilde{H^3}(\mathbf{P}^n)$ and the variety W_n^* introduced in Rosselló-Xambó [1987], Rem. 5. Notice that (i) and (ii) were stated there for W_n^*.

5.9. Lemma. *For all $n \geq 2$, let $\widehat{AH^3}(\mathbf{P}^n)$ denote the closed subvariety of $\widehat{H^3}(\mathbf{P}^n)$ parametrizing all complete collinear 3–tuples, i. e.*

$$\widehat{AH^3}(\mathbf{P}^n) = \{(p_1, p_2, p_3, d_{1,2}, d_{1,3}, d_{2,3}, t) \in \widehat{H^3}(\mathbf{P}^n) \mid t \in \mathrm{Al}^3\mathbf{P}^n\}.$$

i) $\widehat{AH^3}(\mathbf{P}^n)$ is a non-singular projective Borel–Moore variety of dimension $2n+1$.

ii) For all k, the Chow group $A_k(\widehat{AH^3}(\mathbf{P}^n))$ is free of rank:

$$b_k(\widehat{AH^3}(\mathbf{P}^n)) = \sum_{i=1}^{k} b_i(\mathrm{Gr}(1,n)) b_{k-i}((\mathbf{P}^1)^3).$$

iii) For all k, the intersection product induces an isomorphism between $A^k(\widehat{AH^3}(\mathbf{P}^n))$ *and* $A_k(\widehat{AH^3}(\mathbf{P}^n))^*$.

Proof: Let's consider the morphism

$$\gamma\colon \widehat{AH^3}(\mathbf{P}^n) \longrightarrow \mathrm{Gr}(1,n)$$

defined as

$$\gamma((p_1,p_2,p_3,d_{1,2},d_{1,3},d_{2,3},t)) = \mathrm{Axis}(t).$$

Arguing again as in Rosselló [1986] we see that γ is trivial over any Schubert cell on $\mathrm{Gr}(1,n)$, with fiber $\widehat{H^3}(\mathbf{P}^1)$. On the other hand, since any 3–tuple on $\mathbf{P}^1$ is determined by its support and its multiplicity on each point in its support, it is very easy to check that the natural projection

$$\begin{array}{ccc} \widehat{H^3}(\mathbf{P}^1) & \longrightarrow & (\mathbf{P}^1)^3 \\ (p_1,p_2,p_3,d_{1,2},d_{1,3},d_{2,3},t) & \longrightarrow & (p_1,p_2,p_3) \end{array}$$

is an isomorphism. We can then apply **2.5.** ▮

Now it is straighforward to check that the natural projection

$$\pi\colon \widetilde{H^3}(\mathbf{P}^n) \longrightarrow \widehat{H^3}(\mathbf{P}^n)$$

sending $(p_1,p_2,p_3,d_{1,2},d_{1,3},d_{2,3},t,\Pi)$ to $(p_1,p_2,p_3,d_{1,2},d_{1,3},d_{2,3},t)$ is isomorphic to the blow-up of $\widehat{H^3}(\mathbf{P}^n)$ along $\widehat{AH^3}(\mathbf{P}^n)$. Then, from **3.4** we deduce that $\widehat{H^3}(\mathbf{P}^n)$ is a Borel–Moore scheme; from **3.2** we get that all its Chow groups are free with ranks

$$b_k(\widehat{H^3}(\mathbf{P}^n)) = b_k(W_n^*) - \sum_{i=1}^{n-2} b_{k-i}(\widehat{AH^3}(\mathbf{P}^n)) =$$

$$= \sum_{i=0}^{k} b_i(\mathrm{Gr}(2,n))b_{k-i}(W^*) - \sum_{i=1}^{n-2}(\sum_{j=0}^{k-i} b_j(\mathrm{Gr}(1,n))b_{k-i-j}((\mathbf{P}^1)^3)).$$

And from Roberts [1987], Th. 7.5, it turns out that, for all k, the intersection product induces an isomorphism between $A^k(\widehat{H^3}(\mathbf{P}^n))$ and $A_k(\widehat{H^3}(\mathbf{P}^n))^*$. □

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