ON THE COMPLETIONS OF THE SPACES OF MATRICS ON AN OPEN MANIFOLD II.

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Abstract

Given a set X we construct a metric ρ on the set $\mathcal{S}(X)$ of semi-metrics on X. We prove that ρ is complete and that a variety of interesting subsets of $\mathcal{S}(X)$ are closed, giving rise to complete metric spaces of semi-metrics. In the second part we generalize this to a result about finite separating families of semi-metrics. In the third part of the paper we apply the results from the first part by constructing canonical metrics on spaces of riemannian metrics on an open manifold, which metricize some of the uniform structures defined in [3]. Finally we construct some spaces of riemannian metrics, which are related to the remaining uniform structures from [3].

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1 Some General Semi-Metric Space Theory.

By a semi-metric on a set X we will in the following mean a symmetric map $d : X \times X \mapsto [0, \infty)$, which vanishes on the diagonal and satisfies the triangle inequality. A semi-metric d induces a uniform structure $\mathcal{U}_d$ with a neigbourhood basis given by the sets

$$U_\delta = \{(x, x') \in X \times X \mid d(x, x') < \delta\}$$

The uniform structure $\mathcal{U}_d$ again induces a topology τ_d on X, with a basis given by the balls

$$B_{\varepsilon,d}(x_0) = \{x \in X \mid d(x, x_0) < \varepsilon\}$$

for $\varepsilon > 0$ and $x_0 \in X$. The topological space (X, τ_d) is a Hausdorff space if and only if d is a metric. See [8] for details about uniform structures.

Let in the following X be any set, and let $\mathcal{S}(X)$ denote the set of semi-metrics on X and $\mathcal{M}_{all}(X)$ the set of metrics on X. Since at least the discrete metric (given by $d(x, y) = 1$ for $x \neq y$ and $d(x, x) = 0$ for all $x, y \in X$) is a metric on X and the trivial semi-metric (given by $d(x, y) = 0$ for all x, y) is a semi-metric, $\mathcal{M}_{all}(X)$ is a nonempty subset of $\mathcal{S}(X)$, which is proper if X has at least two elements.

In the following we will use the convention, that the supremum taken over the empty set is 0. This causes no problem with respect to the properties of the supremum as long as we only take supremums over sets of non-negative real numbers.

Using the inequality [7, Lemma 1]

$$\frac{|a-b|}{a+b} \leq \frac{|a-c|}{a+c} + \frac{|c-b|}{c+b} \tag{1}$$

for $a > 0$, $b > 0$ and $c > 0$, it is not difficult to see, that for $\rho : \mathcal{S}(X) \times \mathcal{S}(X) \mapsto \mathbb{R}$ given by

$$\rho(d_1, d_2) = \sup_{x,y \in X, d_1(x,y)+d_2(x,y) \neq 0} \frac{|d_1(x,y) - d_2(x,y)|}{d_1(x,y) + d_2(x,y)}, \tag{2}$$

that $(\mathcal{S}(X), \rho)$ is a metric space.

From the definition it immediately follows, that ρ respects the cone structure of $\mathcal{S}(X)$ in the way, that for $d_1, d_2, d_3 \in \mathcal{S}(X)$,

$$\rho(\alpha d_1, \alpha d_2) = \rho(d_1, d_2) \quad ; \alpha > 0$$

and

$$\rho(d_1 + d_3, d_2 + d_3) \leq \rho(d_1, d_2)$$

Lemma 1.1 *The metric space $(\mathcal{S}(X), \rho)$ is complete. Further, for each semi-metric d_0, the set*

$$\{d \in \mathcal{S}(X) \mid d(x,y) = 0 \Leftrightarrow d_0(x,y) = 0\} \tag{3}$$

is closed. In particular, the space $(\mathcal{M}_{all}(X), \rho)$ is a complete metric space.

Proof: First notice, that if for some $x, y \in X$, $d_1(x,y) = 0$ but $d_2(x,y) \neq 0$, then

$$\frac{|d_1(x,y) - d_2(x,y)|}{d_1(x,y) + d_2(x,y)} = 1$$

Thus every Cauchy sequence is from some point contained in a set of the form (3) and cannot converge towards any point in any other such set. This proves the second statement of the lemma. The third follows from the first and the second in combination. It remains to prove the first.

Let $\{d_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $(\mathcal{S}(X), \rho)$. Then for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$, such that for $n, m > N$, we have for all $x, y \in X$ with $d_N(x,y) > 0$, that

$$\frac{|d_n(x,y) - d_m(x,y)|}{d_n(x,y) + d_m(x,y)} < \varepsilon \tag{4}$$

If we for a moment fix $0 < \varepsilon < 1$ and n and let m vary, we see, that $d_m(x,y)$ must be bounded from above in order not to get a contradiction against (4). Thus there exists a constant C such that

$$|d_n(x,y) - d_m(x,y)| < C \frac{|d_n(x,y) - d_m(x,y)|}{d_n(x,y) + d_m(x,y)}$$

It follows, that the sequence $\{d_n(x,y)\}_{n=1}^{\infty}$ is a Cauchy sequence, and by the completeness of $\mathbb{R}$, we may define

$$d(x,y) = \begin{cases} \lim_{n\to\infty} d_n(x,y) & ; d_N(x,y) > 0 \\ 0 & ; d_N(x,y) = 0 \end{cases}$$

Symmetry and the triangle inequality are trivially fullfilled. Thus d is a semi-metric on X. Using the full statement of (4) it is easy to check, that $\rho(d_n, d) \to 0$. This proves the lemma. □

Lemma 1.2 *Let τ be a topology on X. Then the set $\mathcal{S}_\tau(X) \subseteq \mathcal{S}(X)$ of semi-metrics on X, which induce the topology τ on X is closed.*

Proof: Assume that d_1 and d_2 are semi-metrics on X, which induce different topologies on X. After swapping d_1 and d_2, if necessary, there exists a subset $U \subset X$, which is open with respect to d_1, but not with respect to d_2. That means, that there exists a point $x_0 \in U$ and $\varepsilon_0 > 0$ such that $B_{\varepsilon_0, d_1}(x_0) \subseteq U$, but for no $\varepsilon > 0$, $B_{\varepsilon, d_2}(x_0) \subseteq U$. Thus there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of points in X with

$$d_1(x_n, x_0) \geq \varepsilon_0$$

and

$$d_2(x_n, x_0) \overset{n\to\infty}{\longrightarrow} 0$$

It follows, that $\rho(d_1, d_2) = 1$. Thus any Cauchy sequence of semi-metrics induce the topology of its limit point from some point. This proves the lemma. □

Let d_1 be a semi-metric on X. The set of semi-metrics quasi isometric to d_1 is given by

$$\mathcal{S}_{QI,d_1}(X) =$$

$$\{d_2 \in \mathcal{S}(X) | \exists k_1, k_2, K_1, K_2 \in \mathbb{R}_+ : \forall x, y \in X : d_2(x,y) \leq k_1 d_1(x,y) + K_1 \leq k_2 d_2(x,y) + K_2\} \quad (5)$$

Lemma 1.3 *The subset $\mathcal{S}_{QI,d_1}(X)$ of $(\mathcal{S}, \rho)$ is closed.*

Proof: Assume d_2 does not satisfy the first inequality of (5) for any pair k_1, K_1. Fix $K_1 \geq 0$. Then there exists a sequence $\{(x_n, y_n)\}_{n=1}^{\infty} \subseteq X \times X$ such that

$$d_2(x_n, y_n) > n d_1(x_n, y_n) + K_1$$

Since the function $a \mapsto \frac{a-b}{a+b}$ is increasing for $a > 0$, $b > 0$, we may estimate

$$\frac{|d_2(x_n,y_n) - d_1(x_n,y_n)|}{d_2(x_n,y_n) + d_1(x_n,y_n)} = \frac{d_2(x_n,y_n) - d_1(x_n,y_n)}{d_2(x_n,y_n) + d_1(x_n,y_n)} > \frac{(n-1)d_1(x_n,y_n) + K_1}{(n+1)d_1(x_n,y_n) + K_1}$$

Since

$$\left| 1 - \frac{(n-1)d_1(x_n,y_n) + K_1}{(n+1)d_1(x_n,y_n) + K_1} \right| = \frac{2d_1(x_n,y_n)}{(n+1)d_1(x_n,y_n) + K_1} \leq \frac{2}{n+1}$$

it follows, that $\rho(d_1, d_2) = 1$. In a similar way we prove, that $\rho(d_1, d_2) = 1$ if the second inequality of (5) is not satisfied for any k_2, K_2. From that the lemma easily follows. □

We say, that two semi-metrics d_1 and d_2 are metrically equivalent if there exist constants $0 < c \leq C < \infty$ such that $c d_2(x,y) \leq d_1(x,y) \leq C d_2(x,y)$ for all $x, y \in X$.

Corollary 1.4 *The subset $S_{met,d_1}(X) \subseteq \mathcal{S}(X)$ of semi-metrics metrically equivalent to d_1 is closed.*

Proof: Like the proof of Lemma 1.3 with $K_1 = K_2 = 0$, $k_1 = \frac{1}{c}$ and $k_2 = \frac{C}{c}$. □

Another closedness result concerning quasi isometries can be obtained as follows: Let $\mathcal{S}_{QI,d_1,k_1,k_2,K_1,K_2}(X)$ be the set of metrics d_2 quasi isometric to d_1 with constants k_1, k_2, K_1 and K_2. Then $\mathcal{S}_{QI,d_1,k_1,k_2,K_1,K_2}(X)$ is a closed subset of $(\mathcal{S}(X), \rho)$. The proof follows since convergence with respect to ρ implies pointwise convergence.

Lemma 1.5 *The subset $\mathcal{S}_{cplt}(X) \subseteq \mathcal{S}(X)$ making X into a complete uniform space is closed.*

Proof: It suffices to prove, that if d_1 and d_2 are semi-metrics on X such that (X, d_1) is complete but (X, d_2) is not, then $\rho(d_1, d_2) = 1$. If d_1 and d_2 do not induce the same topology on X, we allready have, that $\rho(d_1, d_2) = 1$. Thus it suffices to consider the case, where d_1 and d_2 induce the same topology. In this case, there exists a non-convergent Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ with respect to d_2. Since d_1 is complete and induces the same topology as d_2, $\{x_n\}_{n=1}^{\infty}$ is not a Cauchy sequence with respect to d_1. It follows, that there exists some $\varepsilon > 0$ and a sequence

$$n_1 < m_1 < n_2 < m_2 < \dots$$

such that for all $k \in \mathbb{N}$, $d_1(x_{n_k}, x_{m_k}) > \varepsilon$ and $d_2(x_{n_k}, x_{m_k}) \overset{k\to\infty}{\longrightarrow} 0$. Thus $\rho(d_1, d_2) = 1$. This proves the lemma. □

Notice, that since $\mathcal{M}_{all}(X) \subseteq \mathcal{S}(X)$ is closed, all the completeness results above still hold, if we restrict attention to metrics on X.

2 Separating Families of Semi-metrics.

Often a metric d on a metric space X is constructed from a family of semi-metrics. We will restrict attention to metrics, which occur as a sum of finitely many semi-metrics

$$d(x, y) = \sum_{i=1}^{m} d_i(x, y)$$

Examples of such metrics are the Sobolev space norms on spaces of functions on $\mathbb{R}^n$ and the C^m-norm on the space of C^m-functions on a closed riemannian manifold. Some other examples are given in Section 3.

Our point of view in this section will be, that we want to metricize the space of separating families of semi-metrics giving a certain Hausdorff topology on a set X. In the abstract setup, this gives more information than if we just metricized the space of metrics giving the same topology on X.

Let in the following $S'_m(X)$ be the space

$$S'_m(X) = \{(d_1, \dots, d_m) \subseteq \mathcal{S}(X)^m \mid d_1 + \dots + d_m \text{ is a metric on } X\}$$

Then we may construct a metric ρ^m on $S'_m(X)$ given by

$$\rho^m((d_{11}, \dots, d_{1m}), (d_{21}, \dots, d_{2m})) = \sum_{i=1}^{m} \rho(d_{1i}, d_{2i})$$

Lemma 2.1 *The space $(S'_m(X), \rho^m)$ is a complete metric space.*

Proof: A finite product of complete metric spaces is complete. Thus it suffices to show, that the set of m-tubles of semi-metrics $(d_1, \ldots, d_m)$, for which $d_1 + \ldots + d_m$ is a metric is closed. But this follows by continuity of the map $\sum : \mathcal{S}(X)^m \mapsto \mathcal{S}(X)$, given by $\sum(d_1, \ldots, d_m) = d_1 + \ldots + d_m$, and closedness of $\mathcal{M}_{all}$. □

Remark: In the applications in Section 3, the results are *very* sensitive to the way of decomposing a metric in a sum of semi-metrics. Slightly different choices than the ones presented in Section 3 give rise to completely absurd topologies. See Example 3.3.

3 Canonical Metrics on Spaces of Riemannian Metrics.

In this section X will be a smooth paracompact manifold of dimension $n > 0$. In [3] a sequence of uniform structures is defined on the space $\mathcal{M} = \mathcal{M}(X)$ of complete smooth riemannian metrics on X. Like in [7] we will not repeat the construction from [3], but only recall, that for each $m \in \mathbb{N}$ a basis for the uniform structure is given by the sets

$$V^m_\delta = \{(g, g') \in \mathcal{M} \times \mathcal{M} \mid \sup_{x \in X} |g - g'|_{g',x} < \infty \text{ and}$$

$$^{b,m}|g - g'|_g = \sup_{x \in X} |g - g'|_{g,x} + \sum_{j=0}^{m-1} \sup_{x \in X} \left|(\nabla^g)^j (\nabla^g - \nabla^{g'})\right|_{g,x} < \delta\}$$

for $\delta > 0$. Let in the following ${}^b_m\mathcal{M}$ denote the topological space $(\mathcal{M}, \tau_m)$, where τ_m is the topology induced on $\mathcal{M}$ by the m'th uniform structure, and let ${}^{b,m}\mathcal{M}$ denote the completion of ${}^b_m\mathcal{M}$. In [7] it is proved, that ${}^{b,m}\mathcal{M}$ consists of m times continuously differentiable riemannian metrics on X. In particular, ${}^{b,m}|h - l|_g$ is well defined (in $\mathbb{R}_+ \cup \{\infty\}$) for any $g, h, l \in {}^{b,m}\mathcal{M}$.

Notice, that the neigbourhood basis has a countable subbasis. Thus any convergent net has a convergent subsequence, and any Cauchy net has a Cauchy subsequence. Consequently we will be able to work with sequences rather than nets. The Cauchy sequences of ${}^b_m\mathcal{M}$ can be described by

$$\{g_\nu\}_{\nu \in \mathbf{N}} \text{ is a Cauchy sequence iff}$$

$$\forall \delta > 0 : \exists \nu_0 > 0 : \{x_\nu\}_{\nu > \nu_0} \times \{x_\nu\}_{\nu > \nu_0} \subseteq V^m_\delta$$

The connected components of ${}^{b,m}\mathcal{M}$ are described in [3]. For any $g_0 \in {}^{b,m}\mathcal{M}$, the connected component containing g_0 is given by

$$\text{comp}^{b,m}(g_0) = \{g' \in {}^{b,m}\mathcal{M} \mid \sup_{x \in X} |g_0 - g'|_{g',x} < \infty \text{ and } {}^{b,m}|g_0 - g'|_{g_0,x} < \infty\}$$

There is another similar approach, which gives the same uniform structures. For E be a bundle in the tensor algebra generated by TX and T^*X and ϕ a smooth section in E, let

$$^{b,m}\|\phi\|_g := \sum_{i=0}^{m} \sup_{x \in X} |(\nabla^g)^i \phi|_{g,x}$$

and let

$$W_\delta := \{(g,h) \in \mathcal{M} \mid \sup_{x\in X} |g-h|_{h_x} < \infty \text{ and } {}^{b,m}\|g-h\|_g < \delta\}$$

Then also $\{W_\delta\}_{\delta>0}$ is a basis for a uniform structure on ${}^{b,m}\mathcal{M}$. We postpone the proof of this until we have proved some relations between the systems $\{V_\delta\}$ and $\{W_\delta\}$. By [3, Remark 2.11], the topology induced by $\{W_\delta\}$ is the same as the one induced by $\{V_\delta\}$. In fact, more is true:

Lemma 3.1 *For each m there exists polynomials R_m and R'_m with non-negative coefficients and vanishing constant terms such that for all g, h with ${}^{b,m}\|g-h\| < \infty$ or ${}^{b,m}|g-h| < \infty$, we have*

$${}^{b,m}|g-h|_g \leq R_m({}^{b,m}\|g-h\|_g)$$

and

$${}^{b,m}\|g-h\|_g \leq R'_m({}^{b,m}|g-h|_g)$$

Proof: For $m = 0$, this is obvious since ${}^{b,0}|\cdot|_g = {}^{b,0}\|\cdot\|_g$. In the following all estimates will be pointwise, but the variable $x \in X$ will be left out. For $m = 1$ we get

$$\nabla^g(g-h) = -\nabla^g h = (\nabla^h - \nabla^g)h$$

This gives

$$|\nabla^g(g-h)| \leq |\nabla^g - \nabla^h|_g |h|_g \leq |\nabla^g - \nabla^h|_g(|g-h|_g + 1)$$

Thus $R'_1(x) = 2x + x^2$ satisfies the hypothesis. For $m > 1$,

$$|(\nabla^g)^m(g-h)|_g = |(\nabla^g)^{m-1}(\nabla^g - \nabla^h)h|_g$$

$$= \left|\sum_{j=0}^{m-1}\binom{m-1}{j}\left\{(\nabla^g)^j(\nabla^g - \nabla^h)\right\}\left\{(\nabla^g)^{m-1-j}h\right\}\right|_g$$

$$\leq \left|(\nabla^g)^{m-1}(\nabla^g - \nabla^h)\right|_g(|g-h|_g + 1) + \sum_{j=0}^{m-2}\binom{m-1}{j}\left|(\nabla^g)^j(\nabla^g - \nabla^h)\right|_g\left|(\nabla^g)^{m-1-j}h\right|_g$$

By induction, R'_m exists for all m. The existence of R is more tricky. First recall the identity

$$g((\nabla^g_X - \nabla^h_X)Y, Z) = \frac{1}{2}\Big((\nabla^h_X(g-h))(Y,Z) + (\nabla^h_Y(g-h))(X,Z) - (\nabla^h_Z(g-h))(X,Y)\Big) \quad (6)$$

holding for $X, Y, Z \in C^\infty(TM)$. The vector-field free version of (6) is the identity

$$g \circ ((\nabla^g - \nabla^h) \otimes 1) = \tau(\nabla^h(g-h))$$

where, for any vector-bundle E supplied with a connection ∇^E, we define the section $\tau \in C^\infty(End(End(TM \otimes TM \otimes TM, E)))$ by

$$(\tau A)(X,Y,Z) = \frac{1}{2}\{A(X,Y,Z) + A(Y,X,Z) - A(Z,X,Y)\}$$

Then $\nabla\tau = 0$ for any connection ∇ induced from a Levi-Civita connection on TM and ∇^E in the normal way. By induction we then get the identity

$$g \circ \Big((\nabla^g)^m(\nabla^g - \nabla^h) \otimes 1\Big) = \tau((\nabla^g)^m \nabla^h(g-h))$$

$$= \tau((\nabla^g)^{m+1}(g-h)) - \tau((\nabla^g)^m(\nabla^g - \nabla^h)(g-h))$$

The existence of R_m follows from this, induction and the identity

$$\Big|(\nabla^g)^m(\nabla^g - \nabla^h)\Big|_g = \Big|g \circ \Big((\nabla^g)^m(\nabla^h - \nabla^g) \otimes 1\Big)\Big|_g$$

This proves the lemma. □

We are now in a position, where it is easy to prove, that $\{W_\delta\}$ constitutes a basis for a uniform structure. It will then follow from Lemma 3.1, that this uniform structure is equivalent to the one induced by $\{V_\delta\}$. By [8], what we have to prove is

U1 Every W_δ contains the diagonal.

U2 For each $W_\delta, W_{\delta'}$, there exists δ'', such that $W_{\delta''} \subseteq W_\delta \cap W_{\delta'}$.

U3 For each δ there exists δ' such that $W_{\delta'}^2 \subseteq W_\delta$.

U4 For each δ there exists δ' such that $W_{\delta'}^{-1} \subseteq W_\delta$.

U1 and U2 are trivial. We prove U3. Given δ, choose ε such that $V_\varepsilon \subseteq W_\delta$. Since $\{V_\delta\}$ is the basis of a uniform structure, there exists ε', such that $V_{\varepsilon'}^2 \subseteq V_\varepsilon$. Now choose δ' such that $W_{\delta'} \subseteq V_{\varepsilon'}$. Then δ' satisfies U3. The proof of $U4$ is similar.

Let $j, l \in \mathrm{comp}^{b,m}(g_0)$. From the inequality

$$^{b,m}\|j - l\|_{g_0} \leq {}^{b,m}\|j - g_0\|_{g_0} + {}^{b,m}\|g_0 - l\|_{g_0} < \infty$$

it follows, that $^{b,m}\| \cdot - \cdot \|_g$ is a metric on $\mathrm{comp}^{b,m}(g_0)$ for any $g \in \mathrm{comp}^{b,m}(g_0)$. Thus we get a map

$$\Phi^m : \mathrm{comp}^{b,m}(g_0) \mapsto \mathcal{M}_{all}(\mathrm{comp}^{b,m}(g_0)),$$

which to $g \in \mathrm{comp}^{b,m}(g_0)$ assigns $^{b,m}\| \cdot - \cdot \|_g \in \mathcal{M}_{all}(\mathrm{comp}^{b,m}(g_0))$. Write for $g, h \in \mathrm{comp}^{b,m}(g_0)$

$$\rho^m(g,h) := \sum_{i=0}^{m} \rho(\Phi^i(g), \Phi^i(h))$$

Then clearly ρ^m is a semi-metric on $\mathrm{comp}^{b,m}(g_0)$. The main result, which we will go for, is the following:

Theorem 3.2 *ρ^m is a metric on comp$^{b,m}(g_0)$, which gives the same uniform structure on comp$^{b,m}(g_0)$ as the one inherited from $^{b,m}\mathcal{M}$.* □

Example 3.3 *Let* $\tilde{\Phi}^m : comp^{b,m}(g_0) \mapsto \mathcal{M}_{all}(comp^{b,m}(g_0))$ *be the map, which to* g *assigns the semi-norm* ${}^{b,0}\|(\nabla^g)^m(\cdot - \cdot)\|_g$. *Another semi-metric* $\tilde{\rho}^m$ *is then given by*

$$\tilde{\rho}^m(g,h) := \sum_{i=0}^{m} \rho(\tilde{\Phi}^i(g), \tilde{\Phi}^i(h))$$

It is not difficult to see, that a necessary condition for g *and* h *to be in the same connected component with respect to* $\tilde{\rho}$ *is, that* h *is parallel with respect to* ∇^g *and* g *is parallel with respect to* ∇^h. *In particular* $\tilde{\rho}$ *does not give the same topology as the uniform structure on* ${}^{b,m}\mathcal{M}$. *A closer look shows, that* $\tilde{\rho}$ *induces a strictly stronger topology than* ρ *and the uniform structure.*

If we replace ρ^m *by the semi-metric* $\rho(\Phi^0(g), \Phi^0(h)) + \rho(\Phi^m(g), \Phi^m(h))$ *we get an equivalent uniform structure. This will follow, since the proof of Theorem 3.2 goes through for this semi-metric also.* □

The proof of Theorem 3.2 will be separated in a series of lemmas.

Lemma 3.4 *Let* $a, b > 0$ *and assume, that for some* $0 < \delta < 1$,

$$\frac{|a-b|}{a+b} < \delta$$

Then

$$\frac{|a-b|}{a} < \frac{2\delta}{1-\delta}$$

Proof: By assumption

$$-\delta < \frac{a-b}{a+b} < \delta$$

so

$$-\delta(a+b) < a-b < \delta(a+b)$$

and thus

$$(1-\delta)\frac{b}{a} - \delta < 1 < (1+\delta)\frac{b}{a} + \delta.$$

This gives the inequality

$$\frac{1-\delta}{1+\delta} < \frac{b}{a} < \frac{1+\delta}{1-\delta},$$

from which it follows

$$\frac{|a-b|}{a} = (1 + \frac{1+\delta}{1-\delta})\frac{|a-b|}{a + \frac{1+\delta}{1-\delta}a} < (1 + \frac{1+\delta}{1-\delta})\frac{|a-b|}{a+b} = \frac{2}{1-\delta}\frac{|a-b|}{a+b} < \frac{2\delta}{1-\delta}$$

□

Recall, that for each $m \in \mathbb{N}$, a basis for the uniform structure associated to the semi-metric ρ^m is given by the sets

$$U_\delta^m = \{(g,g') \in {}^{b,0}\mathcal{M} \times {}^{b,0}\mathcal{M} \mid \rho^m(g,g') < \delta\}$$

Lemma 3.5 *Theorem 3.2 is true for $m = 0$.*

Proof: Let in the following $T_x^\times X = T_xX \setminus \{0\}$. We estimate

$$\rho^0(g,h) = \sup_{j\neq l\in \mathrm{comp}^{b,0}(g_0)} \frac{\left|\sup_{x\in X;Y,Z\in T_x^\times X}\frac{|(j_x-l_x)(Y,Z)|}{g_x(Y,Y)^{\frac12}g_x(Z,Z)^{\frac12}} - \sup_{x\in X;Y,Z\in T_x^\times X}\frac{|(j_x-l_x)(Y,Z)|}{h_x(Y,Y)^{\frac12}h_x(Z,Z)^{\frac12}}\right|}{\sup_{x\in X;Y,Z\in T_x^\times X}\frac{|(j_x-l_x)(Y,Z)|}{g_x(Y,Y)^{\frac12}g_x(Z,Z)^{\frac12}} + \sup_{x\in X;Y,Z\in T_x^\times X}\frac{|(j_x-l_x)(Y,Z)|}{h_x(Y,Y)^{\frac12}h_x(Z,Z)^{\frac12}}}$$

$$\leq \sup_{j\neq l\in \mathrm{comp}^{b,0}(g_0)} \frac{\sup_{x\in X;Y,Z\in T_x^\times X}\left|\frac{|(j_x-l_x)(Y,Z)|}{g_x(Y,Y)^{\frac12}g_x(Z,Z)^{\frac12}} - \frac{|(j_x-l_x)(Y,Z)|}{h_x(Y,Y)^{\frac12}h_x(Z,Z)^{\frac12}}\right|}{\sup_{x\in X;Y,Z\in T_x^\times X}\left(\frac{|(j_x-l_x)(Y,Z)|}{g_x(Y,Y)^{\frac12}g_x(Z,Z)^{\frac12}} + \frac{|(j_x-l_x)(Y,Z)|}{h_x(Y,Y)^{\frac12}h_x(Z,Z)^{\frac12}}\right)}$$

$$\leq \sup_{j\neq l\in \mathrm{comp}^{b,0}(g_0)}\ \sup_{x\in X;Y,Z\in T_x^\times X;(j-l)_x(Y,Z)\neq 0} \frac{\left|\frac{|(j_x-l_x)(Y,Z)|}{g_x(Y,Y)^{\frac12}g_x(Z,Z)^{\frac12}} - \frac{|(j_x-l_x)(Y,Z)|}{h_x(Y,Y)^{\frac12}h_x(Z,Z)^{\frac12}}\right|}{\left(\frac{|(j_x-l_x)(Y,Z)|}{g_x(Y,Y)^{\frac12}g_x(Z,Z)^{\frac12}} + \frac{|(j_x-l_x)(Y,Z)|}{h_x(Y,Y)^{\frac12}h_x(Z,Z)^{\frac12}}\right)}$$

$$= \sup_{j\neq l\in \mathrm{comp}^{b,0}(g_0)}\ \sup_{x\in X;Y,Z\in T_x^\times X}\left|\frac{(h_x(Y,Y)^{\frac12}h_x(Z,Z)^{\frac12} - g_x(Y,Y)^{\frac12}g_x(Z,Z)^{\frac12})}{(h_x(Y,Y)^{\frac12}h_x(Z,Z)^{\frac12} + g_x(Y,Y)^{\frac12}g_x(Z,Z)^{\frac12})}\right|$$

$$\leq \sup_{x\in X;Y\in T_x^\times X}\left|\frac{h_x(Y,Y)^{\frac12} - g_x(Y,Y)^{\frac12}}{h_x(Y,Y)^{\frac12} + g_x(Y,Y)^{\frac12}}\right| + \sup_{x\in X;Z\in T_x^\times X}\left|\frac{h_x(Z,Z)^{\frac12} - g_x(Z,Z)^{\frac12}}{h_x(Z,Z)^{\frac12} + g_x(Z,Z)^{\frac12}}\right|$$

Now, for $a > 0$, $b > 0$, the estimate

$$\frac{|a^{\frac12} - b^{\frac12}|}{a^{\frac12} + b^{\frac12}} = \frac{|(a^{\frac12} - b^{\frac12})(a^{\frac12} + b^{\frac12})|}{(a^{\frac12} + b^{\frac12})^2} \leq \frac{|a-b|}{a+b}$$

gives, that we may estimate further

$$\leq 2 \sup_{x\in X;Y\in T_x^\times X}\frac{|h_x(Y,Y) - g_x(Y,Y)|}{h_x(Y,Y) + g_x(Y,Y)} \leq 2\,{}^{b,0}|g-h|_g$$

This proves, that $V^0_{\frac{\delta}{2}} \subseteq U^0_\delta$ for all $\delta > 0$. The lemma now follows, if we can prove, that if $g, h \in \mathrm{comp}^{b,0}(g_0)$ with $\rho^0(g,h) < \delta$ then for each $x_0 \in X$ and $Y_0, Z_0 \in T^\times_{x_0}X$

$$\frac{|g_{x_0}(Y_0,Z_0) - h_{x_0}(Y_0,Z_0)|}{g_{x_0}(Y_0,Y_0)^{\frac12}g_{x_0}(Z_0,Z_0)^{\frac12}} < \frac{2\delta}{1-\delta} \tag{7}$$

First we show, that it is enough to consider the case $Y_0 = Z_0$. By homogeneity it is enough to consider the case, where $g(Y_0,Y_0) = g(Z_0,Z_0) = 1$. Let in the following $S^g_xX \subset T_xX$ denote the unit sphere of T_xX with respect to g. Then we have to estimate

$$\sup_{(Y_0,Z_0)\in S^g_xX\times S^g_xX} |(g_{x_0} - h_{x_0})(Y_0,Z_0)|$$

Since $g_{x_0} - h_{x_0}$ is a symmetric bilinear form, we know from spectral theory, that the maximum is taken for $Y_0 = Z_0$ an eigenvector of the g-symmetric operator A_{x_0} given by

$$g_{x_0}(A_{x_0}Y, Z) = (g_{x_0} - h_{x_0})(Y,Z)$$

Thus the supremum of the left side of (7) in Y_0, Z_0 is allways taken for $Y_0 = Z_0$.

That $\rho^0(g,h) < \delta$ implies, that for every $j, l \in \mathrm{comp}^{b,0}(g_0)$ we have

$$\frac{\left|\sup_{x\in X;Y\in T_x^\times X} \frac{|(j-l)_x(Y,Y)|}{g_x(Y,Y)} - \sup_{x\in X;Y\in T_x^\times X} \frac{|(j-l)_x(Y,Y)|}{h_x(Y,Y)}\right|}{\sup_{x\in X;Y\in T_x^\times X} \frac{|(j-l)_x(Y,Y)|}{g_x(Y,Y)} + \sup_{x\in X;Y\in T_x^\times X} \frac{|(j-l)_x(Y,Y)|}{h_x(Y,Y)}} < \delta$$

We will occasionally consider g_x and h_x as maps from T_xX to T_x^*X given by

$$g_x(X)(Y) = g_x(X,Y) \quad ; h_x(X)(Y) = h_x(X,Y)$$

Then $h((h^{-1}g)_x(X), Y) = g(X,Y)$ and

$$g((h^{-1}g)(X), Y) = h((h^{-1}g)(X), (h^{-1}g)(Y)) = g(X, (h^{-1}g)(Y))$$

Thus $h^{-1}g$ is symmetric and positive with respect to both g and h. It follows, that its square root is well defined, independent of the metric and symmetric with respect to both metrics.

Let U be a neigbourhood of x_0 diffeomorphic to a ball in $\mathbb{R}^n$, and let $\psi \in C_0^\infty(X)$ be a function with $0 \le \psi \le 1$, which is identical to 1 on a neigbourhood of x_0 and has support in U. Extend Y_0 to a non-vanishing continuous section Y in TU. Set for $x \in U$

$$\varphi(x) := \psi(x) \min\left\{\frac{g_{x_0}(Y_0,Y_0)}{g_x(Y,Y)}, \frac{g_{x_0}((h^{-1}g)_{x_0}(Y_0),Y_0)}{g_x((h^{-1}g)_x(Y),Y)}\right\}$$

and extend φ to X by 0. Let $l \in \mathrm{comp}^{b,0}(g_0)$ be arbitrary and let j be given by

$$j_x(Z,W) = l_x(Z,W) + \varphi(x) g_x(Y,Z) g_x(Y,W)$$

Since j is a compact pertubation of l, it is easy to see, that $j \in \mathrm{comp}^{b,0}(g_0)$. Further

$$j_x - l_x = \varphi(x) g_x(Y,\cdot) g_x(Y,\cdot)$$

Thus

$$\frac{|(j_x - l_x)(Z,Z)|}{g_x(Z,Z)} = \frac{\varphi(x) g_x(Y,Z) g_x(Y,Z)}{g_x(Z,Z)} \tag{8}$$

Clearly, for each x, the supremum of (8) in Z is taken for $Z = Y$. Thus

$$\sup_{x\in X;Z\in T_x^\times X} \frac{\varphi(x) g_x(Y,Z) g_x(Y,Z)}{g_x(Z,Z)} = \sup_{x\in X} |\varphi(x) g_x(Y,Y)| = g_{x_0}(Y_0,Y_0)$$

The supremum with respect to Z of

$$\frac{|\varphi(x) g_x(Y,Z) g_x(Y,Z)|}{h_x(Z,Z)} \tag{9}$$

is slightly more complicated to compute. Let $A : T_xX \mapsto T_xX$ be the symmetric linear operator given by

$$g_x(Y,Z) g_x(Y,W) = h_x(AZ,W)$$

more explicitly

$$AZ = g_x(Y, Z)(h^{-1}g)_x(Y)$$

Then the numerically biggest eigenvalue of A is given by $g_x(Y, (h^{-1}g)_x(Y))$ (to the eigenvector $(h^{-1}g)_x(Y)$, the only other eigenvalue is 0). It follows, that the supremum of (9) is given by

$$\sup_{x \in X} |\varphi(x) g_x(Y, (h^{-1}g)_x(Y))|$$

By construction of φ, this supremum is given by

$$|g_{x_0}(Y_0, (h^{-1}g)_{x_0}(Y_0))|$$

Putting everything together, we get the inequality

$$\frac{||g_{x_0}(Y_0, Y_0)| - |g_{x_0}(Y_0, (h^{-1}g)_{x_0}(Y_0))||}{|g_{x_0}(Y_0, Y_0)| + |g_{x_0}(Y_0, (h^{-1}g)_{x_0}(Y_0))|} < \delta \tag{10}$$

Using that Y_0 was arbitrary, we may replace Y_0 by $(g^{-1}h)^{\frac{1}{2}}(Y_0)$ in (10). This renders

$$\frac{|h_{x_0}(Y_0, Y_0) - g_{x_0}(Y_0, Y_0)|}{h_{x_0}(Y_0, Y_0) + g_{x_0}(Y_0, Y_0)} < \delta$$

By Lemma 3.4 this implies

$$\frac{|h_{x_0}(Y_0, Y_0) - g_{x_0}(Y_0, Y_0)|}{g_{x_0}(Y_0, Y_0)} < \frac{2\delta}{1 - \delta}$$

This again implies (7), and the proof is complete. □

Before we proceed with higher m, we will again have to prove some lemmas.

Lemma 3.6 *Let $g, h, j, l \in {}^{b,m}\mathcal{M}$. For all $p > 0$ the tensor*

$$\left\{((\nabla^g)^p - (\nabla^h)^p)(j - l)\right\}$$

is a polynomial without constant term in the **tensors** *(extended to the full tensor algebra in the usual way) $(\nabla^g)^i(\nabla^g - \nabla^h)$ and $(\nabla^g)^k(j - l)$, $i, k = 1, \ldots, p-1$. It is linear in the vector $((j - l), \ldots, (\nabla^g)^{p-1}(j - l))$, and for fixed j, l, the constant term vanishes.*

Proof: By Leibnitz rule it suffices to prove, that the operator $(\nabla^h)^p$ can be written as a finite sum of products with p factors of the **operators** ∇^g and $\nabla^g - \nabla^h$, and that the coefficient to $(\nabla^g)^p$ is equal to one. For $p = 1$ this is trivial. We proceed by induction. Assume the hypothesis holds for $p - 1$. Then

$$(\nabla^h)^p = (\nabla^h)^{p-1}(\nabla^g) - (\nabla^h)^{p-1}(\nabla^g - \nabla^h)$$

is easily seen to satisfy the hypothesis. □

Corollary 3.7 *For each $p > 0$ there exists a polynomial P_p with non-negative coefficients and vanishing constant coefficient of degree at most p in the variables $|(\nabla^g)^i(\nabla^g - \nabla^h)|_{g,x}$ and $|(\nabla^g)^k(j-l)|_{g,x}$ such that*

$$|((\nabla^g)^p - (\nabla^h)^p)(j-l)|_{g,x} \leq P_p\left(\{|(\nabla^g)^i(\nabla^g - \nabla^h)|_{g,x}\}_{i=1}^{p-1}, \{|(\nabla^g)^k(j-l)|_{g,x}\}_{k=1}^{p-1}\right)$$

P_p is linear in the vector $(|(j-l)|_{g,x}, \ldots, |(\nabla^g)^{p-1}(j-l)|_{g,x})$ and for fixed j, l, the constant term vanishes. □

Lemma 3.8 *For $s \geq 0$ and $a > -\frac{s}{2}, b > -\frac{s}{2}$ and $c > -\frac{s}{2}$, the inequality*

$$\frac{|a-b|}{s+a+b} \leq \frac{|a-c|}{s+a+c} + \frac{|c-b|}{s+c+b}$$

holds.

Proof: By substitution of (a, b, c) by $(a + \frac{s}{2}, b + \frac{s}{2}, c + \frac{s}{2})$ in (1). □

Lemma 3.9 *For $m > 0$, $a_1, \ldots, a_m > 0$ and $b_1, \ldots, b_m > 0$, there exists a polynomial S_m of the variables $\frac{|a_i - b_i|}{b_i}$ with vanishing constant coefficient such that*

$$\left|1 - \frac{a_1 \cdots a_m}{b_1 \cdots b_m}\right| \leq S_m(\frac{|a_1 - b_1|}{b_1}, \ldots, \frac{|a_m - b_m|}{b_m})$$

Proof: For $m = 1$ this is trivial. For $m > 1$ we proceed by induction

$$\begin{aligned}
\left|1 - \frac{a_1 \ldots a_m}{b_1 \ldots b_m}\right| &= \frac{|b_1 \ldots b_m - a_1 \ldots a_m|}{b_1 \ldots b_m} \\
&\leq \frac{|b_1 \ldots b_m - b_1 \ldots b_{m-1} a_m|}{b_1 \ldots b_m} + \frac{|b_1 \ldots b_{m-1} a_m - a_1 \ldots a_m|}{b_1 \ldots b_m} \\
&\leq \frac{|a_m - b_m|}{b_m} + \left(\frac{|b_m - a_m|}{b_m} + 1\right) S_{m-1}(\frac{|b_1 - a_1|}{b_1}, \ldots, \frac{|b_{m-1} - a_{m-1}|}{b_{m-1}})
\end{aligned}$$

This proves the lemma. □

We are ready for the next serious step:

Lemma 3.10 *For $m \geq 0$ the identity map $comp^{b,m}(g_0) \mapsto (comp^{b,m}(g_0), \rho^m)$ is uniformly continuous.*

Proof: We prove this by induction. For $m = 0$ this is proved in Lemma 3.5. By induction it suffices to estimate

$$\rho(\Phi^m(g), \Phi^m(h)) = \sup_{j,l \in \mathrm{comp}^{b,m}(g_0)} \frac{\left|\sum_{i=0}^m \left\{|(\nabla^g)^i(j-l)|_g - |(\nabla^h)^i(j-l)|_h\right\}\right|}{\sum_{i=0}^m \left\{|(\nabla^g)^i(j-l)|_g + |(\nabla^h)^i(j-l)|_h\right\}}$$

$$\leq \sup_{j,l \in \mathrm{comp}^{b,m}(g_0)} \frac{\left|\sum_{i=0}^{m-1} \left\{|(\nabla^g)^i(j-l)|_g - |(\nabla^h)^i(j-l)|_h\right\}\right|}{\sum_{i=0}^m \left\{|(\nabla^g)^i(j-l)|_g + |(\nabla^h)^i(j-l)|_h\right\}}$$

$$+ \sup_{j,l \in \text{comp}^{b,m}(g_0)} \frac{\left|\left\{|(\nabla^g)^m(j-l)|_g - |(\nabla^h)^m(j-l)|_h\right\}\right|}{\sum_{i=0}^m \left\{|(\nabla^g)^i(j-l)|_g + |(\nabla^h)^i(j-l)|_h\right\}}$$

The first term can be estimated uniformly by induction. The second one can be estimated from above by the supremum over $j, l \in \text{comp}^{b,m}(g_0)$, $x \in X$ and $Y_1, \ldots, Y_{m+2} \in T_x^\times X$ such that

$$\sum_{i=0}^m \frac{|(\nabla^g)^i(j-l)_x(Y_1,\ldots,Y_{i+2})|}{\prod_{k=1}^{i+2}|Y_k|_g} + \frac{|(\nabla^h)^i(j-l)_x(Y_1,\ldots,Y_{i+2})|}{\prod_{k=1}^{i+2}|Y_k|_h} > 0$$

of

$$\frac{\left|\frac{|(\nabla^g)^m(j-l)_x(Y_1,\ldots,Y_{m+2})|}{\prod_{k=1}^{m+2}|Y_k|_g} - \frac{|(\nabla^h)^m(j-l)_x(Y_1,\ldots,Y_{m+2})|}{\prod_{k=1}^{m+2}|Y_k|_h}\right|}{\sum_{i=0}^m \frac{|(\nabla^g)^i(j-l)_x(Y_1,\ldots,Y_{i+2})|}{\prod_{k=1}^{i+2}|Y_k|_g} + \frac{|(\nabla^h)^i(j-l)_x(Y_1,\ldots,Y_{i+2})|}{\prod_{k=1}^{i+2}|Y_k|_h}} =$$

$$\frac{\left||(\nabla^g)^m(j-l)_x(Y_1,\ldots,Y_{m+2})|\prod_{k=1}^{m+2}|Y_k|_h - |(\nabla^h)^m(j-l)_x(Y_1,\ldots,Y_{m+2})|\prod_{k=1}^{m+2}|Y_k|_g\right|}{\sum_{i=0}^m |(\nabla^g)^i(j-l)_x(Y_1,\ldots,Y_{i+2})|\left(\prod_{k=1}^{i+2}|Y_k|_h\right)\left(\prod_{k=i+3}^{m+2}|Y_k|_g\right) + |(\nabla^h)^i(j-l)_x(Y_1,\ldots,Y_{i+2})|\left(\prod_{k=1}^{i+2}|Y_k|_g\right)\left(\prod_{k=i+3}^{m+2}|Y_k|_h\right)}$$

By Lemma 3.8 followed by the removal of some terms in the denominators and a reduction, this can be estimated by

$$\frac{\left||(\nabla^g)^m(j-l)_x(Y_1,\ldots,Y_{m+2})| - |(\nabla^h)^m(j-l)_x(Y_1,\ldots,Y_{m+2})|\right|}{\sum_{i=0}^m |(\nabla^g)^i(j-l)_x(Y_1,\ldots,Y_{i+2})|\left(\prod_{k=i+3}^{m+2}|Y_k|_g\right)} + \frac{\left|\prod_{k=1}^{m+2}|Y_k|_h - \prod_{k=1}^{m+2}|Y_k|_g\right|}{\prod_{k=1}^{m+2}|Y_k|_g}$$

The second term can be estimated by Lemma 3.9 and the inequality $\frac{|a-b|}{a} \le \frac{|a^2-b^2|}{a^2}$. The first term can be estimated by

$$\begin{aligned} &\frac{|((\nabla^g)^m - (\nabla^h)^m)(j-l)_x|_g}{{}^{b,m}\|j-l\|_g} \\ \le\ &\frac{P_m(\{|(\nabla^g)^i(\nabla^g - \nabla^h)_x|_g\}, \{|(\nabla^g)^k(j-l)|\})}{{}^{b,m}\|j-l\|_g} \\ \le\ &P_m(\{|(\nabla^g)^i(\nabla^g - \nabla^h)_x|_g\}, (1,1,\ldots,1)) \end{aligned}$$

By Corollary 3.7 this is a uniform estimate in terms of ${}^{b,m}|g-h|_g$. This proves the lemma. □

The last step in the proof of Theorem 3.2 is surprisingly easy, when one has first seen the trick. First notice, that if $g \in \text{comp}^{b,m}(g_0) = \text{comp}^{b,m}(g)$, then $\alpha g \in \text{comp}^{b,m}(g_0)$ for $0 < \alpha < \infty$. This follows since $|g|_g = 1$ and $(\nabla^g)^i g = 0$ for all $i > 0$.

Lemma 3.11 *The map $(comp^{b,m}(g_0), \rho^m) \mapsto comp^{b,m}(g_0)$ is uniformly continuous for $m \ge 0$.*

Proof: Since $\text{comp}^{b,m}(g_0)$ is dense in $\text{comp}^{b,m'}(g_0)$ for $m' < m$, we may apply induction. For $m = 0$ this has already been proved. We proceed for $m \ge 1$. Assume $\rho^m(g,h) < \delta < 1$. By inserting $j = 2g$ and $l = g$ in the definition of ρ and using $\nabla^g g = 0$, we get

$$\frac{|{}^{b,0}\|g\|_g - {}^{b,m}\|g\|_h|}{{}^{b,0}\|g\|_g + {}^{b,m}\|g\|_h} < \delta$$

so, by Lemma 3.4, and since ${}^{b,0}\|g\|_g = 1$, we get

$$|\,{}^{b,0}\|g\|_g - {}^{b,m}\|g\|_h| < \frac{2\delta}{1-\delta}$$

This again implies

$$|({}^{b,0}\|g\|_g - {}^{b,0}\|g\|_h) - ({}^{b,m}\|g\|_h - {}^{b,0}\|g\|_h)| < \frac{2\delta}{1-\delta}$$

Now, $|{}^{b,0}\|g\|_g - {}^{b,0}\|g\|_h| \leq {}^{b,0}\|g-h\|_h$. Thus

$$\sum_{i=1}^{m} {}^{b,0}\|(\nabla^h)^i(g-h)\|_h = |\,{}^{b,m}\|g\|_h - {}^{b,0}\|g\|_h| < \frac{2\delta}{1-\delta} + {}^{b,0}\|g-h\|_h$$

By the case $m = 0$, this is a uniform estimate. The lemma follows by adding ${}^{b,0}\|g-h\|_h$ on both sides. □

4 More Canonical Metrics.

Constructions similar to the ones above can also be carried out for the spaces $\mathcal{M}^{p,r}$ defined in [3, p. 268]. Let $\mathcal{M}(I, B_k)$ be the space of smooth complete metrics g on X, such that the injectivity radius of (X, g) has a lower bound, and such that the derivatives, $|(\nabla^g)^j R^g|$, of the curvature tensor are bounded for $j \leq k$. Let for $g, g' \in \mathcal{M}(I, B_k)$, $p \in [1, \infty)$ and $r \in \mathbb{N}$:

$$|g - g'|_{g,p,r} = \left(\int_X (|g-g'|^p_{g,x} + \sum_{i=0}^{r-1} |(\nabla^g)^i(\nabla^{g'} - \nabla^g)|^p_{g,x}) dvol_g(x)\right)^{\frac{1}{p}}$$

For $k \geq r > \frac{n}{p} + 1$, a metrizable uniform structure on $\mathcal{M}(I, B_k)$ is given by the following neigbourhood basis:

$$V_\delta = \{(g, g') \in \mathcal{M}(I, B_k) \times \mathcal{M}(I, B_k) \mid g \text{ and } g' \text{ are quasi isometric and } |g - g'|_{g,p,r} < \delta\}$$

Let $\mathcal{M}^p_r(I, B_k)$ be the space $\mathcal{M}(I, B_k)$ supplied with the uniform structure given above, and let $\mathcal{M}^{p,r}(I, B_k)$ be the completion of $\mathcal{M}^p_r(I, B_k)$. By [7, Lemma 2.4], $\mathcal{M}^{p,r} := \mathcal{M}^{p,r}(I, B_k)$ consists of C^1 riemannian metrics. By [3], the connected component containing $g \in \mathcal{M}^{p,r}$ is given by

$$V_\infty(g) = \{g' \in \mathcal{M}^{p,r} \mid g \text{ and } g' \text{ are quasi isometric and } |g - g'|_{g,p,r} < \infty\}$$

As in the beginning of Section 3, we will like to define an alternative neigbourhood basis for the uniform structure on $\mathcal{M}^{p,r}(I, B_k)$. Define for ϕ a smooth section in a bundle E in the tensor algebra generated by TX and T^*X:

$$\|\phi\|_{g,p,r} := \left(\int_X \left(\sum_{i=0}^{r} |(\nabla^g)^i \phi|^p_{g,x}\right) dvol_g(x)\right)^{\frac{1}{p}}$$

and let $\Omega^{p,r}(E)$ be the completion of the space

$$\{\phi \in C^\infty(X,E) \mid \|\phi\|_{g,p,r} < \infty\}$$

supplied with the Banach-space norm $\|\cdot\|_{g,p,r}$. It follows like in the text after Lemma 3.1, that the sets

$$U_\delta = \{(g,g') \in \mathcal{M}^{p,r} \mid g \text{ and } g' \text{ are quasi isometric and } \|g-g'\|_{g,p,r} < \delta\}$$

give an alternative neigbourhood basis for the uniform structure on $\mathcal{M}^{p,r}$. The substitute for Lemma 3.1 is the following lemma:

Lemma 4.1 *For $k \geq r > \frac{n}{p}$ there exists polynomials Q_r and Q'_r with vanishing constant terms, such that for all $g,h \in \mathcal{M}^{p,r}$*

$$|g-h|_{g,p,r} \leq Q_r(\|g-h\|_{g,p,r})$$

and

$$\|g-h\|_{g,p,r} \leq Q'_r(|g-h|_{g,p,r})$$

Proof: First we recall the module structure theorem for manifolds and vector bundles with bounded geometry. We will here present it for tensor products. From that it will immediately follow for bounded parallel products, i.e. products of the form

$$E_1 \times E_2 \longmapsto E_1 \otimes E_2 \stackrel{*}{\longmapsto} E_3$$

where E_1, E_2, E_3 are Hermitian vector-bundles supplied with Hermitian connections and $* \in C^\infty(End(E_1 \otimes E_2, E_3))$ is bounded and parallel. All the products, which we make use of here, are bounded and parallel with respect to the connections applied. The module structure theorem asserts:

Theorem 4.2 *Let X be a riemannian manifold with bounded geometry of order k and let $E_i \mapsto M$, $i = 1,2$, be Hermitian vector-bundles with compatible connections ∇^i with bounded curvature of order k. Assume, that $p, p_1, p_2 \in [1,\infty)$, $p \geq \frac{p_1 p_2}{p_1+p_2}$, $r_1, r_2, r < k$, $r < \min\{r_1, r_2\}$; $r_1 - \frac{n}{p_1} \geq r - \frac{n}{p}$, $r_2 - \frac{n}{p_2} \geq r - \frac{n}{p}$ and $r_1 - \frac{n}{p_1} + r_2 - \frac{n}{p_2} > (r - \frac{n}{p})$. Then the imbedding $\Omega^{p_1,r_1}(E,\nabla^1) \otimes \Omega^{p_2,r_2}(E_2,\nabla^2) \mapsto \Omega^{p,r}(E_1 \otimes E_2, \nabla^1 \otimes \nabla^2)$ exists and is bounded in the sense, that there exists a constant $C < \infty$ such that for $f_1 \in \Omega^{p_1,r_1}(E_1)$ and $f_2 \in \Omega^{p_2,r_2}(E_2)$,*

$$\|f_1 \otimes f_2\|_{p,r} \leq C \|f_1\|_{p_1,r_1} \|f_2\|_{p_2,r_2}$$

□

We will only be interested in the case, where $p_1 = p_2 = p$, $r_1 + r_2 > \frac{n}{p}$ and $r = 0$. In this case, we get is a continuous imbedding $\Omega^{p,r_1}(E,\nabla^1) \otimes \Omega^{p,r_2}(E_2,\nabla^2) \mapsto \Omega^{p,0}(E_1 \otimes E_2, \nabla^1 \otimes \nabla^2)$.

Using the computations from the proof of Lemma 3.1, we now estimate

$$\|(\nabla^g)^m(g-h)\|_{g,p,0} \leq K\Big(\quad |g-h|_{g,p,m}({}^{b,0}|g-h|+1) +$$

$$\sum_{j=0}^{m-2} \binom{m-1}{j} \|(\nabla^g)^j(\nabla^g - \nabla^h)\|_{g,p,r-j}\|(\nabla^g)^{m-1-j}(g-h)\|_{g,p,j}\Big)$$

From that, the Sobolev imbedding theorem, which gives ${}^{b,0}|g-h| \leq C\|g-h\|_{g,p,r}$, and induction in m it follows, that there exist polynomials $Q_{m,r}$ such that

$$|g-h|_{g,p,m} \leq Q_{m,r}(\|g-h\|_{g,p,r})$$

The first part of the lemma now follows with $Q_r = Q_{r,r}$. The second part is similar, using the last estimates in the proof of Lemma 3.1. □

Let in the following $\Phi^i = \Phi^{p,i} : \mathrm{comp}_{p,r}(g_0) \mapsto \mathcal{M}_{all}(\mathrm{comp}_{p,r}(g_0))$ be the map, which to a metric g assigns $\| \cdot - \cdot \|_{g,p,i}$. Define

$$\rho^r(g,h) = \rho^{p,r}(g,h) := \rho(\Phi^{p,0}(g), \Phi^{p,0}(h)) + \rho(\Phi^{p,r}(g), \Phi^{p,r}(h))$$

Then ρ^r is a semi-metric on $\mathrm{comp}_{p,r}(g_0)$.

Lemma 4.3 *Let $k \geq r > \frac{n}{p}$. For each $i \leq r$, there exists a polynomial $T_i = T_{k,r,n,p,i}$ with non-negative coefficients and vanishing constant coefficient in the variable $\|g-h\|_{g,p,r}$, such that for all $j, l, g, h \in comp_{p,r}(g_0)$*

$$\left(\int_X |((\nabla^g)^i - (\nabla^h)^i)(j-l)|^p_{g,x} dvol_g(x)\right)^{\frac{1}{p}} \leq T_i(\|g-h\|_{g,p,r})\|j-l\|_{g,p,i}$$

Proof: By the proof of Lemma 3.6 we have, that $((\nabla^g)^i - (\nabla^h)^i)(j-l)$ is a linear combination of products of the form

$$\{(\nabla^g)^{t_1}(\nabla^g - \nabla^h)\} \cdots \{(\nabla^g)^{t_d}(\nabla^g - \nabla^h)\}\{(\nabla^g)^s(j-l)\}$$

Following the proof of Lemma 3.6, it is not difficult to see, that $t_1 + \ldots + t_d + s \leq i-1$. The case $d = 0$ does not occur. For $d > 0$ we estimate by the module structure theorem

$$\|\{(\nabla^g)^{t_1}(\nabla^g - \nabla^h)\} \cdots \{(\nabla^g)^{t_d}(\nabla^g - \nabla^h)\}\{(\nabla^g)^s(j-l)\}\|_{g,p,0} \leq$$

$$\|\{(\nabla^g)^{t_1}(\nabla^g - \nabla^h)\}\|_{g,p,r-t_1}\|\{(\nabla^g)^{t_2}(\nabla^g - \nabla^h)\} \cdots \{(\nabla^g)^{t_d}(\nabla^g - \nabla^h)\}\{(\nabla^g)^s(j-l)\}\|_{g,p,t_1}$$

If $d = 1$ this gives what we want. For $d > 1$, the result follows by the multinomial formula and induction in d. □

Lemma 4.4 *For each p there exists a continuous function $v_{n,p} : [0,\infty)^2 \mapsto [0,\infty)$, which vanishes in 0, such that for $r > \frac{n}{p}$, and $g, h \in comp_{p,r}(g_0)$,*

$$\sup_{x\in X}\left(\left|1 - \left(\frac{dvol_g}{dvol_h}(x)\right)^{\frac{1}{p}}\right|\right) \leq v_{n,p}(\|g-h\|_{g,p,r}, \|g-h\|_{h,p,r})$$

Proof: By using the formula $dvol_g(x) = \frac{dx}{\sqrt{\det(g_x)}}$ in local coordinates, we conclude

$$\frac{dvol_g}{dvol_h}(x) = \sqrt{\det(g^{-1}h)}$$

where here $g : TM \mapsto T^*M$ and $h : TM \mapsto T^*M$. It follows from the inequality

$$\|(g^{-1}h)^{-1}\|_h^{-\frac{n}{2}} \le \sqrt{\det(g^{-1}h)} \le \|g^{-1}h\|_g^{\frac{n}{2}}$$

that

$$\|g\|_{h,x}^{\frac{-n}{2p}} \le \left(\frac{dvol_g}{dvol_h}(x)\right)^{\frac{1}{p}} \le \|h\|_g^{\frac{n}{2p}}.$$

This again gives

$$\left|1 - \left(\frac{dvol_g}{dvol_h}(x)\right)^{\frac{1}{p}}\right| \le \max\left\{\left|1 - |1 + {}^{b,0}|g-h|_h|^{\frac{-n}{2p}}\right|, \left|1 - |1 + {}^{b,0}|g-h|_g|^{\frac{n}{2p}}\right|\right\}$$

The lemma follows from this and the Sobolev imbedding theorem. □

Lemma 4.5 *Assume $g, h, j, l \in comp_{p,r}(g_0)$ and $r > \frac{n}{p} + 1$. Then, for $i \le r$*

$$\left(\int_X \left||(\nabla^g)^i(j-l)|_{g,x} - |(\nabla^g)^i(j-l)|_{h,x}\right|^p dvol_g(x)\right)^{\frac{1}{p}}$$

$$\le S_{i+2}({}^{b,0}|g-h|_h)\left(\int_X |(\nabla^g)^i(j-l)|_{g,x}^p dvol_g(x)\right)^{\frac{1}{p}}$$

where S_{i+2} is the polynomial from Lemma 3.9.

Proof: By the pointwise estimate

$$\left||(\nabla^g)^i(j-l)|_{g,x} - |(\nabla^g)^i(j-l)|_{h,x}\right| \le$$

$$\sup_{Y_1,\dots,Y_{i+2}\in T_x^\times X}\left|\frac{(\nabla^g)^i(j-l)_x(Y_1,\dots,Y_{i+2})}{|Y_1|_g\cdots|Y_{i+2}|_g} - \frac{(\nabla^g)^i(j-l)_x(Y_1,\dots,Y_{i+2})}{|Y_1|_h\cdots|Y_{i+2}|_h}\right| \le$$

$$\sup_{Y_1,\dots,Y_{i+2}\in T_x^\times X}\left|1 - \frac{|Y_1|_g\cdots|Y_{i+2}|_g}{|Y_1|_h\cdots|Y_{i+2}|_h}\right||(\nabla^g)^i(j-l)|_{g,x} \le S_{i+2}({}^{b,0}|g-h|_h)|(\nabla^g)^i(j-l)|_{g,x}$$

and Lemma 3.9. □

Lemma 4.6 *For $k \ge r > \frac{n}{p} + 1$, the map $comp_{p,r}(g_0) \mapsto (comp_{p,r}(g_0), \rho^r)$ is uniformly continuous.*

Proof: We estimate for $0 \leq i \leq r$

$$\left|\left(\int_X |(\nabla^g)^i(j-l)|^p_{g,x} dvol_g(x)\right)^{\frac{1}{p}} - \left(\int_X |(\nabla^h)^i(j-l)|^p_{h,x} dvol_h(x)\right)^{\frac{1}{p}}\right| \leq$$

$$\left|\left(\int_X |(\nabla^g)^i(j-l)|^p_{g,x} dvol_g(x)\right)^{\frac{1}{p}} - \left(\int_X |(\nabla^h)^i(j-l)|^p_{g,x} dvol_g(x)\right)^{\frac{1}{p}}\right| +$$

$$\left|\left(\int_X |(\nabla^h)^i(j-l)|^p_{g,x} dvol_g(x)\right)^{\frac{1}{p}} - \left(\int_X |(\nabla^h)^i(j-l)|^p_{h,x} dvol_g(x)\right)^{\frac{1}{p}}\right| +$$

$$\left|\left(\int_X |(\nabla^h)^i(j-l)|^p_{h,x} dvol_g(x)\right)^{\frac{1}{p}} - \left(\int_X |(\nabla^h)^i(j-l)|^p_{h,x} dvol_h(x)\right)^{\frac{1}{p}}\right| \leq$$

$$\left(\int_X \left||(\nabla^g)^i(j-l)|_{g,x} - |(\nabla^h)^i(j-l)|_{g,x}\right|^p dvol_g(x)\right)^{\frac{1}{p}} +$$

$$\left(\int_X \left||(\nabla^h)^i(j-l)|_{g,x} - |(\nabla^h)^i(j-l)|_{h,x}\right|^p dvol_g(x)\right)^{\frac{1}{p}} +$$

$$\left|\left(\int_X \left(|(\nabla^h)^i(j-l)|_{h,x}\left(\frac{dvol_g}{dvol_h}(x)\right)^{\frac{1}{p}}\right)^p dvol_h(x)\right)^{\frac{1}{p}} - \left(\int_X |(\nabla^h)^i(j-l)|^p_{h,x} dvol_h(x)\right)^{\frac{1}{p}}\right| \leq$$

$$T_i(\|g-h\|_{g,p,r})\|j-l\|_{g,p,i} + S_{i+2}({}^{b,0}|g-h|_g, \ldots, {}^{b,0}|g-h|_g)\|j-l\|_{g,p,i}$$
$$+ v_{n,p}(\|g-h\|_{g,p,r}, \|g-h\|_{h,p,r})\|j-l\|_{h,p,i}$$

For $i = 0, \ldots, r$ the above estimate gives, that $\rho(\Phi^i(g), \Phi^i(h))$ can be estimated uniformly. This proves the lemma. □

It is at the time being not known to the author, under which conditions the map $(\text{comp}_{p,r}(g_0), \rho^r) \mapsto \text{comp}_{p,r}(g_0)$ is continuous - not to talk about uniformly continuous. We will therefore have to satisfy ourselves with some partial results and the proof, that it is indeed continuous in the case, where X is a closed manifold.

A partial converse of Lemma 3.9 is, that if for some $q \geq 1$ we have, that $|1 - a^q| < \varepsilon < 1$. Then $|1 - a| < \varepsilon$.

Lemma 4.7 *Let $\delta > 0$ be such, that $S_n(\delta) < 1$. Further, let g and h be measurable riemannian metrics on X and let $x \in X$. Assume that for all $Y \in T_x^\times X$ and some $0 < \alpha \leq 1$, that*

$$\left|1 - \frac{g(Y,Y)}{h(Y,Y)}\left(\frac{dvol_h}{dvol_g}(x)\right)^\alpha\right| < \delta \tag{11}$$

then

$$\left|1 - \left(\frac{dvol_h}{dvol_g}(x)\right)^\alpha\right| < S_n(\delta)$$

Proof: First recall, that $\frac{dvol_h}{dvol_g}(x) = \sqrt{\det(h^{-1}g)_x}$. Next notice, that (11) can be rewritten

$$\left|1 - \frac{h_x((h^{-1}g)_x Y, Y)}{h_x(Y,Y)} \left(\det(h^{-1}g)_x\right)^{\frac{\alpha}{2}}\right| < \delta$$

Let $Y_1, \ldots, Y_n$ be a basis of eigenvectors for $(h^{-1}g)_x$. Then by Lemma 3.9

$$\left|1 - \det((h^{-1}g)_x)^{\frac{n\alpha+2}{2}}\right| = \left|1 - \left(\prod_{i=1}^{n} \frac{g_x(Y_i, Y_i)}{h_x(Y_i, Y_i)} \left(\det((h^{-1}g)_x)\right)^{\frac{\alpha}{2}}\right)\right| < S_n(\delta)$$

The lemma now follows by the remark above it. □

Lemma 4.8 *Let p, k, r be given with $k \geq r > \frac{n}{p} + 1$. There exists some $\delta_0 > 0$ such that if $g, h \in comp_{p,r}(g_0)$ satisfy, that for all $j \neq l \in comp_{p,r}(g_0)$ and some $0 < \delta < \delta_0$, that*

$$\frac{\left|\left(\int_X |j-l|^p_{g,x} dvol_g(x)\right)^{\frac{1}{p}} - \left(\int_X |j-l|^p_{h,x} dvol_h(x)\right)^{\frac{1}{p}}\right|}{\left(\int_X |j-l|^p_{g,x} dvol_g(x)\right)^{\frac{1}{p}} + \left(\int_X |j-l|^p_{h,x} dvol_h(x)\right)^{\frac{1}{p}}} \leq \delta \tag{12}$$

Then

$${}^{b,0}\|g-h\|_g \leq \frac{2(\delta + S_n(\frac{2\delta}{1-\delta}))}{1 - (\delta + S_n(\frac{2\delta}{1-\delta}))}$$

Proof: First notice, that g and h are continuous by the Sobolev embedding theorem. Next notice, that (12) may be completed with respect to $j - l$ in such a way, that we may work with bounded measurable test-sections of compact support in the bundle of symmetric tensors in $T^*X \otimes T^*X$. Given any $j, l \in \text{comp}_{p,r}(g_0)$, define

$$\varphi_{m,y} = \chi_{B_{\frac{1}{m},g}(y)} vol_g(B_{\frac{1}{m},g}(y))^{-\frac{1}{p}}(j-l)$$

Then, for every y such that $(j-l)_y \neq 0$

$$\delta \geq \lim_{m\to\infty} \frac{\left|\left(\int_X |\varphi_{m,y}|^p_{g,x} dvol_g(x)\right)^{\frac{1}{p}} - \left(\int_X |\varphi_{m,y}|^p_{h,x} dvol_h(x)\right)^{\frac{1}{p}}\right|}{\left(\int_X |\varphi_{m,y}|^p_{g,x} dvol_g(x)\right)^{\frac{1}{p}} + \left(\int_X |\varphi_{m,y}|^p_{h,x} dvol_h(x)\right)^{\frac{1}{p}}}$$

$$= \frac{\left||j-l|_{g,y} - |j-l|_{h,y}\left(\frac{dvol_h}{dvol_g}(y)\right)^{\frac{1}{p}}\right|}{|j-l|_{g,y} + |j-l|_{h,y}\left(\frac{dvol_h}{dvol_g}(y)\right)^{\frac{1}{p}}}$$

Like in the proof of Lemma 3.5, for each $Y \in T_y^{\times}X$, we may choose j, l such that

$$|j-l|_{g,y} = h_y(Y,Y) \quad ; |j-l|_{h,y} = g_y(Y,Y)$$

This gives

$$\frac{\left|h_y(Y,Y) - g_y(Y,Y)\left(\frac{dvol_h}{dvol_g}(y)\right)^{\frac{1}{p}}\right|}{h_y(Y,Y) - g_y(Y,Y)\left(\frac{dvol_h}{dvol_g}(y)\right)^{\frac{1}{p}}} \leq \delta$$

By Lemma 3.4 and a reduction

$$\left|1-\frac{g_y(Y,Y)}{h_y(Y,Y)}\left(\frac{dvol_h}{dvol_g}(y)\right)^{\frac{1}{p}}\right|\le\frac{2\delta}{1-\delta}$$

By Lemma 4.7 this implies

$$\left|1-\left(\frac{dvol_h}{dvol_g}(y)\right)^{\frac{1}{p}}\right|\le S_n(\frac{2\delta}{1-\delta}) \tag{13}$$

We may now estimate

$$\delta\ge\frac{\left|\left(\int|j-l|^p_{g,x}dvol_g(x)\right)^{\frac{1}{p}}-\left(\int|j-l|^p_{h,x}dvol_h(x)\right)^{\frac{1}{p}}\right|}{\left(\int|j-l|^p_{g,x}dvol_g(x)\right)^{\frac{1}{p}}+\left(\int|j-l|^p_{h,x}dvol_h(x)\right)^{\frac{1}{p}}}\ge$$

$$\left|\frac{\left|\left(\int|j-l|^p_{g,x}dvol_g(x)\right)^{\frac{1}{p}}-\left(\int|j-l|^p_{h,x}dvol_g(x)\right)^{\frac{1}{p}}\right|}{\left(\int|j-l|^p_{g,x}dvol_g(x)\right)^{\frac{1}{p}}+\left(\int|j-l|^p_{h,x}dvol_g(x)\right)^{\frac{1}{p}}}-\right.$$
$$\left.\frac{\left|\left(\int|j-l|^p_{h,x}dvol_g(x)\right)^{\frac{1}{p}}-\left(\int|j-l|^p_{h,x}dvol_h(x)\right)^{\frac{1}{p}}\right|}{\left(\int|j-l|^p_{h,x}dvol_g(x)\right)^{\frac{1}{p}}+\left(\int|j-l|^p_{h,x}dvol_h(x)\right)^{\frac{1}{p}}}\right| \tag{14}$$

The last term inside the norm in (14) can be estimated

$$\frac{\left|\left(\int|j-l|^p_{h,x}dvol_g(x)\right)^{\frac{1}{p}}-\left(\int|j-l|^p_{h,x}dvol_h(x)\right)^{\frac{1}{p}}\right|}{\left(\int|j-l|^p_{h,x}dvol_g(x)\right)^{\frac{1}{p}}+\left(\int|j-l|^p_{h,x}dvol_h(x)\right)^{\frac{1}{p}}}\le$$

$$\frac{\left(\int|j-l|^p_{h,x}\left|1-\left(\frac{dvol_h}{dvol_g}(x)\right)^{\frac{1}{p}}\right|^p dvol_g(x)\right)^{\frac{1}{p}}}{\left(\int|j-l|^p_{h,x}dvol_g(x)\right)^{\frac{1}{p}}+\left(\int|j-l|^p_{h,x}dvol_h(x)\right)^{\frac{1}{p}}}\le S_n(\frac{2\delta}{1-\delta}) \tag{15}$$

Consequently, by (13) and (15),

$$\frac{\left|\left(\int|j-l|^p_{g,x}dvol_g(x)\right)^{\frac{1}{p}}-\left(\int|j-l|^p_{h,x}dvol_g(x)\right)^{\frac{1}{p}}\right|}{\left(\int|j-l|^p_{g,x}dvol_g(x)\right)^{\frac{1}{p}}+\left(\int|j-l|^p_{h,x}dvol_g(x)\right)^{\frac{1}{p}}}\le\delta+S_n(\frac{2\delta}{1-\delta}) \tag{16}$$

Proceeding like in the start of this proof with (16) we obtain, that for all $y\in X$ and $Y\in T^*_yX$,

$$\frac{|h_y(Y,Y)-g_y(Y,Y)|}{h_y(Y,Y)+g_y(Y,Y)}\le\delta+S_n(\frac{2\delta}{1-\delta})$$

The lemma follows by applying Lemma 3.4 once more. □

Corollary 4.9 *The inclusion* $(comp_{p,r}(g_0), \rho^r) \mapsto {}^{b,0}\mathcal{M}$ *is uniformly continuous. Further, if* X *is closed, the map* $(comp_{p,r}(g_0), \rho^r) \mapsto comp_{p,0}(g_0)$ *is continuous.* □

Corollary 4.10 ρ^r *is a metric on* $comp_{p,r}(g_0)$. □

Corollary 4.11 *If* X *is closed, the inclusion* $(comp_{p,r}(g_0), \rho^r) \mapsto comp_{p,0}(g_0)$ *is continuous.* □

Corollary 4.12 *The completion of* $(comp_{p,r}(g_0), \rho^r)$ *consists of continuous riemannian metrics metrically equivalent to any riemannian metric in* $comp_{p,r}(g_0)$. □

Proposition 4.13 *If* X *is closed, the identity* $(comp_{p,r}(g_0), \rho^r) \mapsto comp_{p,r}(g_0)$ *is continuous. Further, the space* $(comp_{p,r}(g_0), \rho^r)$ *is complete.*

Proof: By Corollary 4.11, the inclusion $(\mathrm{comp}_{p,r}(g_0), \rho^r) \mapsto (\mathrm{comp}_{p,0}(g_0), \rho^0)$ is continuous. Since X is closed, all multiples of any metric in $\mathrm{comp}_{p,r}(g_0)$ are contained in $\mathrm{comp}_{p,r}(g_0)$. In particular, setting $j = 2g$, $l = g$, the condition $\rho^r(g,h) < \delta < 1$ together with the equality $\nabla^g g = 0$ implies

$$\frac{|||g||_{g,p,0} - ||g||_{h,p,0}|}{||g||_{g,p,0} + ||g||_{h,p,0}} < \delta$$

$$\frac{|||g||_{g,p,0} - ||g||_{h,p,r}|}{||g||_{g,p,0} + ||g||_{h,p,r}} < \delta$$

By Lemma 3.4 this gives

$$|||g||_{g,p,0} - ||g||_{h,p,0}| \leq vol_g(X)^{\frac{1}{p}} \frac{2\delta}{1-\delta}$$

$$|||g||_{g,p,0} - ||g||_{h,p,r}| \leq vol_g(X)^{\frac{1}{p}} \frac{2\delta}{1-\delta}$$

Proceeding like in Lemma 3.11 we get

$$\sum_{i=1}^{m} ||(\nabla^h)^i (g-h)||_{h,p,0} = \sum_{i=1}^{m} ||(\nabla^h)^i g||_{h,p,0} \leq vol_g(X)^{\frac{1}{p}} \frac{2\delta}{1-\delta} + |||g||_{g,p,0} - ||g||_{h,p,0}| \qquad (17)$$

This proves the continuity. Now, let $\{g_i\}_{i=1}^{\infty}$ be a Cauchy sequence in $(\mathrm{comp}_{p,r}(g_0), \rho^r)$. By Lemma 4.8, $\{g_i\}_{i=1}^{\infty}$ is a Cauchy sequence in ${}^{b,0}\mathcal{M}$. This implies, that the volume $vol_{g_i}(X)$ is bounded. (17) now gives, that $\{g_i\}_{i=1}^{\infty}$ is a Cauchy sequence in $\mathrm{comp}_{p,r}(g_0)$, so that the limit $g = \lim_{i\to\infty} g_i$ exists in $\mathrm{comp}_{p,r}(g_0)$. That it also exists in $(\mathrm{comp}_{p,r}(g_0), \rho^r)$ follows from Lemma 4.6. □

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