24. MATHEMATISCHE ARBEITSTAGUNG

1983

Sonderforschungsbereich 40
Theoretische Mathematik
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für Mathematik
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Mathematisches Institut der Universität Bonn Wegelerstr. 10, Bonn 1

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Max-Planck-Institut für Mathematik Gottfried-Claren-Str. 26, Bonn 3

Programm der Mathematischen Arbeitstagung 1983 (I)

Donnerstag, den 16.6.:

17.00 - 18.00 Uhr: M.F. Atiyah: Instantons, monopoles and rational maps

Freitag, den 17.6.:

10.15 - 11.15 Uhr: G. Faltings: The conjectures of Tate and Mordell I

12.00 - 13.00 Uhr: B. Gross: Heights and L-series I (On the conjecture of Birch and Swinnerton-Dyer)

17.00 - 18.00 Uhr: S.W. Donaldson: Stable holomorphic bundles and curvature

Samstag, den 18.6.:

10.00 - 10.15 Uhr: Festlegung der nächsten Vorträge

10.15 - 11.15 Uhr: C. Procesi: The solution of the Scholtky problem (Characterization of Jacobian varieties)

12.00 - 13.00 Uhr: D. Quillen: Cyclic homology and Hochschild-homology

17.00 - 18.00 Uhr: N. Hingston: Equivariant Morse theory

Die Vorträge finden alle im "Großen Hörsaal" (Wegelerstr. 10) statt.

Erfrischungspausen mit Tee: Freitag und Samstag, 11.30-12.00 Uhr vor dem Großen Hörsaal, nachmittags ab 15.30 Uhr im Diskussionsraum Beringstraße 1.

Post liegt während der Teepausen aus.

Tischtennis im Keller des Hauses Beringstr. 4.

Den Tagungsbeitrag bitte an Frau Gerber (SFB-Büro, Beringstr. 4) bezahlen.

Alle Teilnehmer mögen sich bitte in die Teilnehmerlisten eintragen. Teilnehmerlisten und andere Informationen liegen vor dem Diskussiosraum Beringstr.1 aus.

Alle Tagungsteilnehmer mit ihren Damen oder Herren sind herzlich zum Empfang des Rektors eingeladen. Zeit: Donnerstag, den 16.6., 20.00 Uhr. Ort: Festsaal der Universität, Hauptgebäude; Eingang von der Straße "Am Hof" durch das Tor gegenüber Buchhandlung Röhrscheid.

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Programm der Mathematischen Arbeitstagung 1983 (II)

Sonntag, den 19.6.:

10.15 - 11.15 Uhr: D. Zagier: Heights and L-series II (and applications to the class number problem)

12.00 - 13.00 Uhr: D. McDuff: The Arnold conjecture on symplectic fixed points (after Conley and Zehnder)

17.00 - 18.00 Uhr: G. Faltings: The conjectures of Tate and Mordell II (Moduli spaces of abelian varieties)

Montag, den 20.6.:

9.00 Uhr: Ganztägiger Schiffsausflug nach Kobern-Gondorf. Abfahrt pünktlich um 9.00 Uhr mit Motorschiff "Carmen Silva" am Alten Zoll.

Dienstag, den 21.6.:

10.15 - 11.15 Uhr: J. Milnor: Monotonicity for the entropy of quadratic maps

Der Vortrag Dienstag, 10.15 Uhr, findet im "Kleinen Hörsaal" statt; alle andern Vorträge sind im "Großen Hörsaal" (Wegelerstraße 10).

Erfrischungspausen mit Tee: Sonntag und Dienstag vormittags von 11.30-12.00 Uhr vor dem Großen Hörsaal, Sonntag Nachmittag ab 15.30 Uhr im Diskussionsraum Beringstraße 1.

Die Post liegt während der Teepausen aus.

Tischtennis im Keller des Hauses Beringstr. 4.

Den Tagungsbeitrag bitte an Frau Gerber bezahlen (SFB-Büro, Beringstr. 4).

Alle Tagungsteilnehmer mögen sich bitte in die Teilnehmerlisten eintragen. Teilnehmerlisten und Informationen liegen vor dem Großen Hörsaal bzw. dem Diskussionsraum Beringstr. 1 aus.

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Programm der Mathematischen Arbeitstagung 1983 (III)

Dienstag, den 21.6.:

12.00 - 13.00 Uhr: E. Friedlander: On the conjectures of Lichtenbaum and Quillen (after Suslin and others)

15.00 - 16.00 Uhr: informal lecture outside the main program
D. Quillen: Arithmetic surfaces and analytic torsion

17.00 - 18.00 Uhr: B. Moonen: Polar multiplicities and curvature integrals

Mittwoch, den 22.6.:

10.15 - 11.15 Uhr: D. Eisenbud: Special divisors on curves and Kodaira dimension of the moduli space (mostly after Mumford and Harris and Gieseker)

12.00 - 13.00 Uhr: F. Kirwan: Cohomology of quotients in algebraic and symplecti geometry

15.00 - 16.00 Uhr: lecture outside the main program
W. Tutschke: Generalizations of the Cauchy-Kowalewski and Holmgren theorems to the case of generalized analytic functio

17.00 - 18.00 Uhr: G. Wüstholz: Group varieties and transcendence

The above lectures are in the "Großer Hörsaal" (Wegelerstraße 10).

Extra activity on Thursday, June 23, 10.15-11.00 and 11.15-12.00, Kleiner Hörsaal Wegelerstraße 10: M.F. Atiyah (special lecture in a course of F. Hirzebruch) "Convex Polyhedra and Algebraic Geometry".

E. Friedlander will talk in the Oberseminar (Thursday, 3.00-4.00 p.m. at the Max-Planck-Institut, Gottfried-Claren-Str. 26) on "Rational cohomology of algebraic groups". There will be tea afterwards in the MPI.

On Friday, June 24, from 4.30-5.30 p.m. D. Gromoll will talk in the Colloquium (Kleiner Hörsaal, Wegelerstraße 10) on "Riemannian spaces with positive Ricci curvature".

Die Referenten werden gebeten, ihre Kurzfassungen bis Dienstag Nachmittag (17.00 Herrn Kohnen zu geben, da wir den Tagungsbericht allen Teilnehmern noch vor ihrer Abreise aushändigen möchten.

Erfrischungspausen mit Tee: Dienstag und Mittwoch vormittags von 11.30-12.00 Uhr vor dem Großen Hörsaal, Wegelerstr. 10, nachmittags ab 15.30 Uhr im Diskussionsra Beringstraße 1.

Die Post liegt während der Teepausen aus. Den Tagungsbeitrag bitte an Frau Gerbe (SFB-Büro, Beringstraße 4) bezahlen. Programme und Informationen liegen vor dem Diskussionsraum, Beringstr. 1, aus.

Titel: Instantons, Monopoles and Rational Maps

Autor: M. F. ATIYAH

Adresse: Mathematical Institute, Oxford.

I will discuss three variational problems and outline a programme to relate them in various ways.

<u>Rational Maps</u> from S^2 to S^2 are the absolute minima of the Energy functional $E(f) = \frac{1}{2}\int |df|^2$. More generally consider rational maps from S^2 to Kähler manifolds, more particularly the homogeneous manifolds of compact Lie groups G arising as orbits $G\lambda$ of points λ in the Lie algebra (or its dual). If we normalize such maps by requiring them to preserve base points we will denote the moduli space of rational maps $f: S^2 \to G\lambda$ with given homology class ν by $Rat(G, \lambda, \nu)$. Note that ν can be viewed as an element of the integral lattice in the Lie algebra t of the maximal torus of G.

<u>Instantons</u> for a group G are connections over R^4 for which the curvature F is self-dual. They are minima of the Yang-Mills functional $\frac{1}{2}\int |F|^2$ for a G-bundle over $S^4 = R^4 \cup \infty$ (by conformal

compactification). For simple G we have a topological invariant $n \in \pi_3(G) = Z$. Identifying connections which differ by a bundle automorphism trivial at ∞ we get a moduli space denoted by $Inst(G, n)$. This is of interest to physicists and has been much studied.

<u>Loop Groups</u> It is now well known that the loop space ΩG has a natural Kähler structure analogous to that of the finite-dimensional homogeneous spaces. It is therefore natural to look at rational maps $S^2 \to \Omega G$. The moduli space of (based) rational maps in a given homology class $n \in H_2(\Omega G) = Z$ is denoted by $Rat(\Omega G, n)$. This is related to the instanton moduli space and it should be possible to prove the following

"Theorem 1" $Inst(G, n) = Rat(\Omega G, n)$.

<u>Magnetic Monopoles</u> are connections on R^3 with a Higgs field Φ (section of adjoint bundle) satisfying the Bogomolny equations $\nabla\Phi = *F$ and with asymptotic conditions implying that at ∞ $\Phi_\infty : S^2 \to G)$ lies in a fixed adjoint orbit

For monopoles with the homology class $[\phi_\infty] = \nu$ the moduli space (used bundle automorphisms trivial at a base point) is denoted by $\mathrm{Mon}(G, \lambda, \nu)$.

By considering the fixed points of the natural S^1-action on both sides of theorem 1 one should be able to deduce:

"Theorem 2" $\mathrm{Mon}(G, \lambda, \nu) = \mathrm{Rat}(G, \lambda, \nu)$

In fact this deduction needs one further important ingredient: S^1-invariant instantons can naturally be interpreted as monopoles on the hyperbolic 3-space H^3 because of the conformal equivalence

$$R^4 - R^2 \sim S^1 \times H^3.$$

However by rescaling and letting the curvature of H^3 tend to 0 it should be possible to identify the moduli space of monopoles on Hyperbolic and Euclidean space.

The "proof" of Theorem 1 will depend in an essential way on ideas of S. Donaldson (cf. his talk in this Arbeitstagung). The connection with hyperbolic space has been noted independently by A. Chakrabarti.

REFERENCES

M. F. Atiyah, Geometry of Yang-Mills Fields, Fermi Lectures, Pisa

A. Chakrabarti, Phys. Rev. D vol 25, 3282-3298

Autor : G. Faltings

Adresse : Bergische Universität-
Gesamthochschule Wuppertal

THE CONJECTURES OF TATE AND MORDELL

The proof of these conjectures over number-fields is a translation of the function-field case. The main technical ingredient is the computation of the height of a point in the moduli-space of abelian varieties, in terms of the corresponding abelian variety over a number field.

More precisely, let A/K be an abelian variety over a number field K (with integers R), which has semistable reduction. Extend A to a semiabelian

$$p : A \longrightarrow \operatorname{Spec}(R)$$

with zero-section

$$s : \operatorname{Spec}(R) \longrightarrow A ,$$

and let

$$\omega_{A/R} = s^*(\Omega^{top}_{A/R})$$

$\omega_{A/R}$ is a line-bundle over $\operatorname{Spec}(R)$ and has norms at the infinite places of K (square-integration) and thus has a degree. Let

$$h(A) = \frac{1}{[K:\mathbb{Q}]} \cdot \deg(\omega_{A/R}) .$$

Then there exist only finitely many principally polarized A/K's as above, with bounded $h(A)$.

The next step is to look at the change of $h(A)$ under an isogeny

$$\phi : A_1 \to A_2 .$$

If $G = \mathrm{Ker}(\phi)$, then

$$h(A_2)-h(A_1) = \frac{1}{2}\log(\deg(\phi)) - \frac{1}{[K:\mathbb{Q}]}\log(\# s^*\Omega^1_{G/R})$$

Especially $\exp(2[K:\mathbb{Q}](h(A_2)-h(A_1)))$ is a rational number. We first apply this to the case that

$$G = \cup G_n \subseteq A[l^\infty]$$

is an l-divisible subgroup. Then:

<u>Lemma:</u> $h(A/G_n) = h(A) \quad \forall n$

<u>Sketch of proof:</u>

Reduce to $K=\mathbb{Q}$ (restriction of scalars) To G corresponds a Galois-stable sublattice

$$W \subsetneq T_l(A) \text{ (Tate-Modul)}$$

of rank $h=\mathrm{height}(G)$.

If

$$d = \dim(G) ,$$

$$h(A_n)-h(A) = n\cdot\log(l)\left(\frac{h}{2}-d\right) ,$$

and

$$\pi = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$$

operates on

$$\Lambda^h W \subseteq \Lambda^h(T_l(A)$$

via $\chi_0^{\,d}$, where χ_0 denotes the cyclotomic character.
By the Weil-conjectures $d = \frac{h}{2}$.

From this lemma one derives by wellknown methods (due to Tate and Zarhin) the Tate conjecture:

a) $T_l(A) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ is a semisimple π-module

b) $\mathrm{End}_K(A) \otimes_{\mathbb{Z}} \mathbb{Z}_l \cong \mathrm{End}_\pi(T_l(A))$.

The next step is to show that up to isogeny there are only finitely many A/K's with given set of bad reduction. We must bound the number of isomorphism classes of the $T_l(A) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$, which follows from Čebotarev and the fact that the trace of a Frobenius on $T_l(A) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ is bounded by the Weil-conjectures.

Finally we show that there are only finitely many isomorphism classes of principally polarized A/K's with given set of bad reduction. The main point here ist that for l big enough $h(A)$ does not change under an isogeny of l-power order. The method of proof is similar as in the Tate conjecture:

Last not least by Torelly there are only finitely many classes of curves over K with given set of bad reduction.

Parshin has shown how to derive Mordell form this:

If $X_{/K}$ is a curve of genus $g \geq 2$ and $x \in X(K)$, one constructs a covering

$$Y(x) \to X$$

ramified only in x. The result above can be applied to $Y(x)$ and leads to finiteness of $X(K)$.

Titel: Stable holomorphic bundles and Curvature.

Autor: S. K. Donaldson

Adresse: The Mathematical Institute, 24-29 St. Giles, Oxford, England

On any Riemannian 4-manifold one can study the self dual and anti self dual connections (particular solutions of the Yang-Mills equations). Take the manifold to be a projective algebraic surface $X \subseteq \mathbb{CP}^n$ with a Kahler metric ω dual in cohomology to the hyperplane section class. Then the anti self dual connections are related to the holomorphic vector bundles over X via the decomposition of the 2-forms.

$$\Omega^2_X = \Omega^2_+ \oplus \Omega^2_- = [\Omega^{2,0} \oplus \Omega^{0,2} \oplus \langle\omega\rangle] \oplus P^{1,1}$$

: an anti-self dual connection has curvature of type (1,1) and so defines a holomorphic structure. Conversely given a holomorphic bundle one can seek a hermitian metric on the bundle such that the induced connection is anti self dual: in terms of local trivialisations the equations becomes:

$$\hat{F}_H = \sum g^{\alpha\bar{\beta}} F_{\alpha\bar{\beta}} = \Delta H + \underset{\text{terms}}{\text{non-linear}} = 0$$

so is a non-linear variant of the Laplace equation.

<u>Thm.</u> A holomorphic bundle $\mathcal{E}$ has an (irreducible, projectively) anti self dual connection if and only if it is <u>stable</u> and in that case the connection is unique.

The concept of stability of vector bundles was introduced in algebraic geometry in order to get a good moduli space. The result has a potential generalisation to bundles over arbitrary Kahler manifolds; in particular for algebraic curves the corresponding fact is a theorem of Narasimhan-Seshadri – a bundle over a curve is stable iff it has a projectively <u>flat</u> connection.

Using ~~by~~ theorem of Uhlenbeck on weak compactness of spaces of connections with bounded curvature it suffices to prove that if $\mathcal{E}$ is a semistable bundle (eg of degree 0) then the inf of $\|\hat{F}\|$ over connections on $\mathcal{E}$ is zero. To find such a good sequence of connections one can uses a gradient flow for metrics on $\mathcal{E}$:

$$\frac{\partial H}{\partial t} = -2i\, H \hat{F}$$

A solution to this "heat equation" exist for all positive time.

We introduce a functional M_X on the metrics on $\mathcal{E}$ such that:

$$\frac{\partial}{\partial t} M_X(H_t) = -2\|\hat{F}\|^2$$

It suffices then to prove that $M_X(H_t)$ is bounded below if $\mathcal{E}$ is semi-stable. This is done by using a theorem of Mehta and Ramanathan to pick a curve, without loss, $C \in |H|$ in X such that $\mathcal{E}|_C$ is semi-stable. There is a similar functional M_C on connections on $\mathcal{E}|_C$ which one knows is bounded below by the Narasimhan-Seshadri theorem, and a "residue formula":

$$M_X(H_t) = M_C(H_t) + \int_X \rho \cdot \mathrm{Tr}(F_{H_t}^2)$$

where ρ is a function such that $\omega = C + i\bar{\partial}\partial\rho$ as currents. The functionals M_X, M_C and much of the rest of the proof have a natural interpretation in terms of the symplectic geometry of the space of connections and the associated "determinant line bundle".

Ref.
M.F. Atiyah and R. Bott: The Yang-Mills equations on Riemann Surfaces.
S. Kobayashi: Curvature and stability of vector bundles (Proc. J.A.S) 1982.
D. Quillen: Lecture at 1982 Arbeitstagung.
K. Uhlenbeck: Connections with L^p bounds on curvature (Commun. Math Phys. 1982)

Titel: A CHARACTERIZATION OF RIEMANN MATRICES (ACCORDING TO ARBARELLO - DE CONCINI) 1

Autor: PROCESI CLAUDIO

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Given a compact Riemann surface of genus g and a symplectic basis $x_1, \dots, x_g, y_1, \dots, y_g$ of $H_2(S, Z)$ one has the normalized differentials of first kind $\omega_1, \dots, \omega_g$ with $\int_{x_i} \omega_j = \delta^i_j$. The period matrix $\tau = (\tau_{ij})$ is $\tau_{ij} = \int_{y_i} \omega_j$. τ is in Siegel space S_g and changes under basis change by a transformation $(A\tau + B)(C\tau + D)^{-1}$, $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g, Z)$

Classical problem of Riemann; characterize $\mathcal{M}_g$; $= \{\tau$ in S_g/Sp coming from Riemann surfaces$\}$.

Various approaches have been taken starting from the classical approach of Schottky using Prym differentials to the recent approach of Novikov coming from the Kadomtsev - Petviashvili non linear equation $3/4\, u_{yy} = \frac{\partial}{\partial x}\left[u_t - \frac{1}{4}(6uu_x + u_{xxx})\right]$.

Both approaches furnish relations among Θ null werte defining some subvariety of $S_g / Sp(2g, Z)$ in each case it has been proved (Van Geemen 1983) (Dubrovin 1981) that $\mathcal{M}_g$ is a

component of the corresponding variety.
Introduce the Θ functions:

$$\theta[n](\tau, z) = \sum_{m \in \mathbb{Z}^g} \exp(2\pi i [\tfrac{1}{2}\, {}^t(m+n)\tau(m+n)$$

$$+ {}^t(m+n) z]) \quad , \quad n \in \tfrac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g .$$

Introduce for $\tau \in S_g$ the abelian variety $X_\tau = \mathbb{C}^g/(1, \tau)$ then $\theta[n](\tau, z)$ map X_τ in $\mathbb{P}^{2^g-1}$ with image the Kummer variety K_τ.

The proposed solution will come by comparing the equations coming from the Novikov conjecture with a geometric property of the Kummer variety associated to a curve.

The KP equation has solutions of a special type according to Krichever: S a Riemann surface of genus g, $Q \in C$ a point z a local coordinate in Q, $\Omega_1, \Omega_2, \Omega_3$ differentials of second kind with polar divisors $d\frac{1}{z}$, $d\frac{1}{z^2}$, $d\frac{1}{z^3}$ in Q and $\int_{X_i} \Omega_j = 0$ then fixing a non special divisor $D = \sum P_i$ one can find a Baker-Akhiezer function $\psi(x,y,t:z)$ with essential singularity at Q of type

$$e^{\frac{1}{z}x + \frac{1}{z^2}y + \frac{1}{z^3}t}\left(1 + \xi_1 z + \xi_2 z^2 + \cdots\right)$$

and poles at D. Such function is unique one has then that $u = -2\frac{\partial \xi_1}{\partial x}$ is a solution of KP.

Explicitly

$$\psi(x,y,t;P) = \exp\left(x\int_{P_0}^{P}\Omega_1 + y\int_{P_0}^{P}\Omega_2 + t\int_{P_0}^{P}\Omega_3\right)$$

$$\times \frac{\theta\left(A(P) - A(D) + xD_1 + yD_2 + tD_3 - k\right)}{\theta\left(A(P) - A(D) - k\right)}$$

where $A : S \to Jac(S)$ is Abel's map, K Rieman's constant, D_1, D_2, D_3 the vectors of g periods of $\Omega_1, \Omega_2, \Omega_3$.

Then $u = 2\frac{\partial^2}{\partial x^2}\log\theta\left(xD_1 + yD_2 + tD_3 + z_0\right) + c$.

Extracting the information from KP one has the relation

*) $D_1^4 \hat\theta[n] - D_1 D_3 \hat\theta[n] + \frac{3}{4} D_2^2 \hat\theta[n] + d \cdot \hat\theta[n] = 0$

where $\hat\theta[n](\tau, z) = \theta[n](2\tau, z)$

and D_i is thought as a (constant) differential operator $D_i = \sum_j D_{ij} \frac{\partial}{\partial z_j}$ and after differentiating one sets $z = 0$.

Now from work of Welters based on Gunning's work one can interpret the relation *) in totally different language of trisecants of K_τ.

If $\psi : X_\tau \to K_\tau \to \mathbb{P}^{2^g - 1}$ we say that $\zeta+a, \zeta+b, \zeta+c$ is a trisecant if $\psi(\zeta+a)\ \psi(\zeta+b)\ \psi(\zeta+c)$ are on a line

if $X_\tau = \mathrm{Jac}(C)$ one has

<u>Theorem</u> (Fay-Mumford) If $a, b, c \in C$

$$\tfrac{1}{2}(C - a - b - c) = \{\zeta \mid \zeta+a\ \ \zeta+b\ \ \zeta+c \text{ is trisecant}\}.$$

<u>Gunning</u>: Given any X_τ and $Y = \{a, b, c\} \in X_\tau$ set $V_Y = \{\zeta \in X_\tau \mid \zeta + Y \text{ is trisecant}\}$.

then $V_Y = 2\hat{V}_Y$ contains $\{-a-b, -a-c, -b-c\}$ and if it has dimension > 0 in one of this point it contains an irreducible curve V through these points smooth at them.

Furthermore the endomorphism
$\alpha_V : X_\tau \to X_\tau$ given by $\alpha_V(x) = \sum (\Theta_x - \Theta) \cdot V$
is the identity at the previous points.

Welters has an infinitesimal variation of this and fixes $Y = \operatorname{Spec} \mathbb{C}[\varepsilon]/(\varepsilon^3) \subset (X_\tau, 0)$

$\left(x_i = D_{1i}\varepsilon + D_{2i}\varepsilon^2 \right)$, then

$$\hat{V}_Y = \{ z \mid \operatorname{rk} \mid \theta''(z), D_1\theta''(z) \ \left(\tfrac{1}{2}D_1^2 + D_2\right)\theta''(z) \mid \leq 2 \} \quad \left(\theta''(z) = \hat{\theta}[n](\tau, 2z) = \theta[n](2\tau, 2z) \right)$$

D_1, D_2 operators as before.

Then setting $V_Y = 2\hat{V}_Y$ one can see that V_Y contains the scheme Y and Novikov's equation consists in assuming that $V_Y \supset \operatorname{Spec} \mathbb{C}[\varepsilon]/(\varepsilon^4)$ a further approximation to the curve.

Now the approach of Arbarello - De Concini is

1) Impose that V_Y contains high order approximations of the curve

2) Show that after a sufficiently high approximation is valid then V_Y is a curve V in O.

3) Repeat the same procedure at near by points so to ensure that

$\alpha_V = 1$ on V.

Once all these steps are done for which one needs a suitable notion of degree of the varieties in study one can finally write complete explicit equations for M_g these are:

$$\sum_{i+r=\ell} R_i^{(v)} f_{n,r}(z,\tau)\Big|_{z=0} = 0 \qquad \ell + v \leq M+1$$

where $n = n_1, n_2, n_3 \in \left(\frac{1}{2} \mathbb{Z}^g/\mathbb{Z}^g\right)^3$

$$f_{n,r} = \sum_{\substack{\rho+\lambda-3=r \\ \lambda \geq 2 \\ \rho \geq 1}} \det \left| \vec{\theta}^n(z),\ \rho D_\rho \vec{\theta}^n(z),\ \frac{1}{2} \sum_{v+\mu=\lambda} v\mu D_v D_\mu + \binom{\lambda}{2} D_\lambda \vec{\theta}^n(z) \right|$$

$$\vec{\theta}^n = \begin{pmatrix} \theta^{n_1} \\ \theta^{n_2} \\ \theta^{n_3} \end{pmatrix}, \quad M = (6^g g!)^2 (4g-4)(g+1)$$

$$\sum_{i=1}^{\infty} R_i^{(v)} t^i = \Delta_v\left(D_1(t), 2D_2(t), \ldots, 2^{v-1} D_v(t)\right)$$

$$\Delta_v(\ ,\ ,\ ) = \sum_{h_1+2h_2+\cdots+vh_v=v} \frac{1}{h_1! \cdots h_v!} D_1^{h_1} \cdots D_v^{h_v}$$

$$D_h(t) = \sum_{\mu=h}^{\infty} \binom{\mu}{h} D_\mu t^{\mu-h}.$$

$$D_1 \neq 0$$

Titel: Cyclic homology and Hochschild homology.

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This is a report on an interesting new development in homological algebra, the theory of cyclic homology for an associative algebra. Most of the theory is due to A. Connes [1, 2, 3].

Let A be an associative algebra (with 1) over a field $k \supset \mathbb{Q}$. The Hochschild homology $H_n(A,A)$ is the homology of the Hochschild complex

$$\xrightarrow{\ b\ } A\otimes A\otimes A \xrightarrow{\ b\ } A\otimes A \xrightarrow{\ b\ } A$$

where $b(a_0,\dots,a_n) = \sum_{i=0}^{n-1}(-1)^i(a_0,\dots,a_ia_{i+1},\dots,a_n) + (-1)^n(a_na_0,a_1,\dots,a_{n-1})$.

Let the cyclic group of order n act on $A^{\otimes n}$ by

$$t(a_1,\dots,a_n) = (-1)^{n-1}(a_n,a_1,\dots,a_{n-1})$$

and form the quotient $A^{\otimes n}/(1-t)$ of this action. The <u>cyclic homology</u> is defined to be the homology of the quotient complex so obtained:

$$HC_n(A) = H_n\left\{ A^{\otimes(*+1)}/(1-t),\ b \right\}.$$

One has [4, 5, 6]:

<u>Thm.</u> $HC_{n-1}(A)$ is isomorphic to the primitive part of the homology of the Lie algebra $gl(A) = \bigcup gl_n(A)$ of matrices over A with coefficients k.

In many cases the Hochschild homology can be calculated by the tools of homological algebra. The cyclic homology can then be found using the following [2, [redacted] 5,6]

Thm: There is a long exact sequence

$$\cdots \longrightarrow H_n(A,A) \longrightarrow HC_n(A) \xrightarrow{S} HC_{n-2}(A) \longrightarrow H_{n-1}(A,A) \longrightarrow \cdots$$

The Hochschild homology is known when A is a commutative algebra which is smooth over k:

$$H_n(A,A) = \Lambda^n_A \Omega^1_A = \Omega^n_A .$$

Hence one can use the exact sequence to obtain:

Thm: When A is smooth over k

$$HC_n(A) = \Omega^n_A / d\Omega^{n-1}_A \oplus H^{n-2}_{DR}(A) \oplus H^{n-4}_{DR}(A) \oplus \cdots$$

where $H^*_{DR}(A)$ is the algebraic de Rham cohomology of A.

Cor: For n large $HC_n(A) = H^{even}_{DR}(A)$ or $H^{odd}_{DR}(A)$ according to the parity of n.

Therefore cyclic homology provides a generalization of de Rham cohomology to non-commutative rings.

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Titel: Equivariant Morse theory

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We disuss applications of the equivariant Morse theory, as used by Atiyah-Bott, to the existence of closed geodesics. Let M be a compact, simply connected Riemannian manifold.

Theorem 1. Suppose M has the rational homotopy type of S^n or P^n ($P^n = \mathbb{C}P^n$, $\mathbb{H}P^n$ or CaP^2). If all closed geodesics on M are hyperbolic, then there are infinitely many.

This theorem is of interest because of the theorem (Moser, Klingenberg-Takens): For a generic metric with a nonhyperbolic geodesic there are infinitely many. Together these theorems give infinitely many closed geodesics for a generic metric on a manifold with one of these rational homotopy types.

Theorem 2. Let M_s be S^n or P^n with the standard metric. Suppose all geodesic loops on M have length $\geq 2\pi$, and that there is a homotopy equivalence $M_s \to M$ so that the images of circles have length $< 4\pi$. Then M has $\geq g(\lambda, n)$ simple closed geodesics of length $< 4\pi$. Generically there will be $\lambda(\lambda+1)\, n(n+1)/4$.

Here $g(\lambda, n)$ is the cuplength of the space of unparameterized great circles on M_s.

Anosov and Ballmann-Thorbergsson-Ziller have given proofs of versions of this theorem for S^n. Rademacher has shown that $g(\lambda, n)$ can be replaced by the cuplength $g_1(\lambda, n)$ of the space of parameterized great circles mod the orientation reversing involution, and that $g_1 > g$ for most projective spaces.

Theorem 3. Let $A: S^2 \to S^2$ be an orientation preserving diffeomorphism of finite order > 2. For a generic A-invariant metric on S^2 there are infinitely many A-invariant geodesics.

Here A-invariant means that A takes the geodesic to itself, preserving orientation. Note that in the simplest example, that of a rotation on S^2 with the standard metric, the only A-invariant geodesic is the equator.

Equivariant Morse theory seems in many ways to be natural for the study of closed geodesics. We mention some remaining questions in closed geodesics.

(1) In theorem 1 we get the growth estimate $\liminf N(\ell)\log\ell/\ell > 0$ for the number $N(\ell)$ of closed geodesics of length $\leq \ell$. Ballmann, Thorbergsson and Ziller have remarked that one can get the same estimate from the Moser argument. Much better estimates should be possible.

(2) Using the results for S^n and P^n together with the theorems of Gromoll-Meyer and Vigué-Pourrier-Sullivan, one would have generic existence of infinitely many closed geodesics for any simply connected Riemannian manifold <u>if</u> one could prove the analog of the Vigué-Pourrier-Sullivan theorem for $\mathbb{Z}_p$ coefficients.

(3) Manifolds with infinite fundamental group. Bangert and the author have proved the existence of infinitely many closed geodesics on a manifold with $\pi_1 = \mathbb{Z}$. Manifolds with infinite fundamental group for which the existence of infinitely many closed geodesics remains unproved are those for which π_1 has no abelian subgroup of finite index and so that there exist $\alpha_1, \ldots, \alpha_N$ in π_1 so that every $\beta \in \pi_1$ is conjugate to α_i^j for some i, j.

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Titel: Heights and L-Series I, II (with applications to the Birch-Swinnerton-Dyer conjecture and the class number problem)

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The result we will present is most interesting in connection with the Birch - Swinnerton-Dyer conjecture, which we now recall.

An elliptic curve E over $\mathbb{Q}$ can be given by an equation of the form $y^2 = f_3(x)$, where $f_3(x) \in \mathbb{Z}[x]$ is a cubic polynomial with distinct roots (and hence discriminant $\Delta \neq 0$). To E are associated two basic invariants:

1) $E(\mathbb{Q}) = \{(x,y) \in \mathbb{Q}^2 \mid y^2 = f_3(x)\} \cup \{\infty\}$, the set of rational points on E,

2) $L(E/\mathbb{Q},s) = \sum_{n=1}^{\infty} a(n)\, n^{-s}$, the L-series of E; here $a(n)$ is a multiplicative function (i.e. $a(p_1^{\nu_1} \ldots a_t^{\nu_t}) = a(p_1^{\nu_1}) \ldots a(p_t^{\nu_t})$) with

$$a(p) = p - \#\{x,y \in \mathbb{Z}/p\mathbb{Z} \mid y^2 \equiv f_3(x) \pmod p\} = -\sum_{x \,(\mathrm{mod}\, p)} \left(\frac{f_3(x)}{p}\right),$$

$$a(p^{\nu+1}) = a(p)a(p^{\nu}) - pa(p^{\nu-1})$$

for $p \nmid \Delta$.

The following is known about these:

1) $E(\mathbb{Q})$ is an abelian group, the group law being such that three points in $E(\mathbb{Q})$ sum to zero iff they are colinear (Diophantus, Poincaré). This group is finitely generated, i.e. $E(\mathbb{Q}) \simeq \mathbb{Z}^r \oplus T$ for some $r \in \mathbb{Z}_{\geq 0}$ and finite T (Mordell, 1922). Given E, the group T can be determined algorithmically whereas for r we can in general get only lower bounds by trial-and-error and upper bounds by "descent" (Fermat). The group T is always isomorphic to $\mathbb{Z}/n\mathbb{Z}$ with $n \leq 12$, $n \neq 11$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2n\mathbb{Z}$ with $n \leq 4$ (Mazur, 1977). The number r, which in most examples is 0 or 1, can be at least as large as 12 (Mestre, 1982). On $E(\mathbb{Q}) \otimes \mathbb{R}$ there is a canonical positive definite bilinear pairing $\langle\, ,\, \rangle$, the height, characterized by the fact that $\langle P,P\rangle -$ log (denominator of x-coordinate of P) is bounded for $P \in E(\mathbb{Q})$ (Mordell, Néron, Tate).

2) The coefficients $a(p)$ satisfy $|a(p)| < 2\sqrt{p}$ (Hasse, 1933), so $L(E,s)$ converges absolutely for $\mathrm{Re}(s) > 3/2$. Nothing else is known in general. However, there is a large class of curves, usually called Weil curves, for which we know that $L(E,s)$ is an entire function of s and satisfies the functional equation

$$\left(\frac{N}{2\pi}\right)^{s} \Gamma(s)\, L(E,s) = \varepsilon \left(\frac{N}{2\pi}\right)^{2-s} \Gamma(2-s)\, L(E,2-s)$$

with $\varepsilon = \pm 1$, N being a positive integer (the conductor of E) containing only primes dividing Δ. There is an algorithm to determine whether a given elliptic curve E is a Weil curve, and this has been done for hundreds of examples. The Taniyama-Weil conjecture says that every elliptic curve over $\mathbb{Q}$ is a Weil curve. The defining property of Weil curves is that $f(z) := \sum_{n=1}^{\infty} a(n)\, e^{2\pi i n z}$ $(z \in H)$ is a modular form (and then automatically a cusp form and an eigenform for all Hecke operators) of weight 2 for $\Gamma_0(N) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}) \mid c \equiv 0 \pmod N\}$, i.e. the differential form $\omega = f(z)\,dz$ on H is invariant under $\Gamma_0(N)$. Then the set $\Lambda = \{\int_\gamma \omega \mid \gamma \in H_1(X_0(N))\}$ (where $X_0(N) = H/\Gamma_0(N) \cup \{\text{cusps}\}$) is a lattice in $\mathbb{C}$ with $\mathbb{C}/\Lambda$ isomorphic to $E(\mathbb{C})$, and $z \longrightarrow \int_z^\infty \omega$ gives a map $\pi: X_0(N) \to E$ of finite degree and defined over $\mathbb{Q}$.

No general way to relate $E(\mathbb{Q})$ and $L(E,s)$ is known. However, the Birch-Swinnerton-Dyer conjecture says: Let $\rho = \operatorname{ord}_{s=1} L(E,s)$ and $c = \lim_{s\to 1} L(E,s)/(s-1)^\rho$ (this makes sense if E is a Weil curve), while $E(\mathbb{Q}) = \mathbb{Z}P_1 \oplus \ldots \oplus \mathbb{Z}P_r \oplus T$ with T finite. Then $\rho = r$ and $c = \alpha\,\Omega \det(\langle P_i, P_j\rangle)$, where the "period" Ω is a positive real number (belonging to the lattice Λ) and α an explicitly given rational number. Up to elementary rational factors, α is $|\text{Ш}|/|T|^2$, where Ш is the Tate-Shafarevich group of E (definition omitted). However, Ш is not known to be finite for a single E.

A special case of the result we prove gives some support to this conjecture.

THEOREM 1. Let E be a Weil curve with conductor N and $\varepsilon = -1$ (thus ρ is odd, so the BSD conjecture predicts that $E(\mathbb{Q})$ is infinite). For each negative discriminant D congruent to a square modulo $4N$, there is a point $P_D \in E(\mathbb{Q})$ whose height is related to the L-series of E by

$$(*) \qquad L'(E,1)\, L(E^{(D)},1) = \frac{\pi^2 \omega}{\sqrt{|D|}} \langle P_D, P_D\rangle .$$

Here $E^{(D)}$ is the "twisted" curve $Dy^2 = f_3(x)$ and $\omega \in \mathbb{R}_{>0}$ a certain period (roughly, the product of those of E and $E^{(D)}$).

By a result of Waldspurger, there are always (infinitely many) D with $L(E^{(D)},1) \neq 0$. Hence, even without knowing the definition of P_D, we can give three corollaries:

Corollary 1. $\rho = 1 \Rightarrow r \geq 1$, i.e. if $L(E,1) = 0$ and $L'(E,1) \neq 0$ then $E(\mathbb{Q})$ is infinite.

Indeed, under this hypothesis the left-hand side of (*) is non-zero, so $P_D \in E(\mathbb{Q})$ is not a torsion point.

Corollary 2. $L'(E,1) \geq 0$.

This supports both the BSD conjecture and the Riemann hypothesis for $L(E,s)$

Corollary 3. If $\rho = r = 1$, then the formula for c in the BSD conjecture is true up to a positive rational constant.

Indeed, the height $\langle P_D, P_D \rangle$ is a square integer multiple of the 1×1 height pairing determinant in the BSD conjecture, and $L(E^{(D)},1)$ is known to be a rational multiple of the corresponding period. Note that in this corollary the mysterious group Ш has dropped out; if the BSD conjecture is true, then (*) implies a relation between the order of Ш and the index of P_D in $E(\mathbb{Q})$.

The points P_D are defined as follows. Given a discriminant $D<0$, the class number $h(D)$ is defined as the number of $SL_2(\mathbb{Z})$-inequivalent primitive positive definite binary quadratic forms of discriminant D, or equivalently the number of points z in a fundamental domain for the action of $SL_2(\mathbb{Z})$ on H satisfying a quadratic equation $az^2+bz+c=0$ with $(a,b,c)=1$, $b^2-4ac=D$. In a fundamental domain for $\Gamma_0(N)$ there will be $[SL_2(\mathbb{Z}):\Gamma_0(N)]h(D)$ such points, but if D is as in the theorem and we fix an integer r with $r^2 \equiv D \pmod{4N}$, then exactly $h(D)$ of them will satisfy $a \equiv 0 \pmod{N}$, $b \equiv r \pmod{2N}$. Call these $z_1,\ldots,z_h \in H/\Gamma_0(N)$; then $P_D = \pi(z_1)+\ldots+\pi(z_h) \in E$. The points z_i are defined only over the ring class field of discriminant D, but they are conjugate over $\mathbb{Q}$, so $P_D \in E(\mathbb{Q})$. The points z_i are called Heegner points (though the definition here is due to Birch) and have the following modular interpretation: $X_0(N)$ parametrizes pairs (E,C) consisting of an elliptic curve E and a cyclic subgroup $C \subset E$ of order N, or equivalently pairs of elliptic curves E and E' $(=E/C)$ together with a cyclic isogeny $E \longrightarrow E'$ of degree N. A Heegner point corresponds to E and E' having complex multiplication by the same order, i.e. $\mathrm{End}(E) \simeq \mathrm{End}(E') \simeq \mathcal{O} = \{(x+y\sqrt{D})/2 \mid x,y \in \mathbb{Z},\ x \equiv Dy \pmod 2\}$; then $E \cong \mathbb{C}/\mathfrak{a}$, $E' \cong \mathbb{C}/\mathfrak{a}\mathfrak{n}^{-1}$ for some $\mathcal{O}$-ideal $\mathfrak{a}$ and some primitive $\mathcal{O}$-ideal $\mathfrak{n}$ of norm N, the class of $\mathfrak{a}$ corresponding to the index $i=1,\ldots,h(D)$ of z_i and the choice of $\mathfrak{n}$ to the choice of $r \pmod{2N}$ with $r^2 \equiv D \pmod{4N}$.

In some cases, we can show directly that $P_D = 0$. In particular, if $N=37$ and $D=-139$ (with $h(D)=3$), then we can show that $(z_1)+(z_2)+(z_3)-3(\infty)$ is zero on the Jacobian of $X_0(N)$ (and hence a fortiori on any quotient of it). This is because N is the norm of a principal ideal in $\mathcal{O}$ (namely $37 = (9+139)/4 = N(\lambda)$, $\lambda = (3+\sqrt{D})/2$), so that the function $u(z) =$

$\sqrt[12]{\Delta(z)/\Delta(37z)}$ ($\Delta(z)$ = classical discriminant function) takes on the same value λ at each z_i (if the ideal $\mathfrak{n}$ with norm N were non-principal, its values at these three points would be conjugate numbers in a ring class field). Also, $u(z) = q^{-3} + \ldots$ at infinity ($q = e^{2\pi iz}$). Therefore the function $u(z) - \lambda$ has divisor $(z_1)+(z_2)+(z_3)-3(\infty)$. The Jacobian of $X_0(37)$ is isogenous to $E_1 \times E_2$, where E_1 is given by $y^2 = x^3+4x^2-48x+80$ and satisfies $L(E_1,1) \neq 0$; applying Theorem 1 to the twist $E_1^{(-139)}$ gives $L'(E^{(-139)},1) = 0$ and hence:

Corollary 4. *The L-series of the elliptic curve* $-139y^2 = x^3+4x^2-48x+80$ *has a triple zero at* $s = 1$.

The curve in question has rank 3, so this agrees with the BSD conjecture.

* * *

Corollary 4 is of interest because it can be combined with a beautiful result of Goldfeld (Journées Arithmétiques de Caen 1976, Astérisque 41-42) to yield:

THEOREM 2. *There is an effectively computable constant* c *such that*

$$h(D) > c\,(\log |D|)^{1 - \sqrt{1323/\log\log |D|}}$$

for all discriminants $D < 0$.

Indeed, if $(\frac{D}{37}) = +1$ this is trivial, for then 37 is the norm of a prime ideal $\mathfrak{p}$ in the order $\mathcal{O}$ of discriminant D and $\mathfrak{p}^{h(D)}$ is a principal ideal $(\frac{x+y\sqrt{D}}{2})$ with $y \neq 0$, giving the much better estimate $37^{h(D)} = \frac{x^2+y^2|D|}{4} > |D|/4$. If $(\frac{D}{37}) = 0$ or -1, then the twist $E_0^{(D)}$, where E_0 is the curve in Corollary 4, has a sign $\varepsilon = -1$ in its functional equation, so $L(E_0,s)\cdot L(E_0^{(D)},s)$ has at least a quadruple zero at $s = 1$, and the desired estimate follows from Goldfeld's theorem.

Theorem 2 gives an effective solution to a problem posed by Gauss in the Disquisitiones. Gauss conjectured that the last D with $h(D)=1$ is -163, and more generally that $h(D) \to \infty$ as $D \to -\infty$. The latter fact was proved by Heilbronn in 1934, and Siegel (1935) even showed that $h(D) > c_\alpha |D|^\alpha$ for any $\alpha < \frac{1}{2}$ and some $c_\alpha > 0$, but neither result was effective. The case $h = 1$ was finally resolved by Heegner in the early 50's and by Baker and Stark in the late 60's, and the latter authors also settled the case $h = 2$ in 1971, but until now the best effective lower bound known was $h(D) \geq 3$ ($|D| > 427$). Of course, the constants in Theorem 2 as it now stands are too poor to settle even the next open case $h = 3$ (which conjecturally implies $|D| \le 907$) explicitly: the bound obtained would be of the order of $10^{10^{600}}$.

* * *

The identity (*) in Theorem 1 should be thought of as a limit formula of the same sort as Dirichlet's ("solution of Pell's equation by circular functions," 1837) and Kronecker's ("solution of Pell's equation by elliptic functions," 1863); and indeed, there are some philosophical reasons, based on ideas of Drinfel'd, to imagine that these are the only three explicit limit formulas of this kind over number fields. The proof of Theorem 1 involves computing both sides independently, the left by a variant of "Rankin's method" in the theory of modular forms and the right by a computation of local heights of Heegner points at all finite and infinite places. Both computations lead to formulas containing many terms, and at the end of the calculation these match up term for term. Amusingly, the computation of the local heights are of interest even for $N = 1$, when the global height on the Jacobian is 0 and Theorem 1 is empty (because $X_0(N)$ has genus 0), because these local heights, which must sum to zero, do not vanish individually. What one obtains is a result on the prime factorization of special values of the classical modular invariant

$$j(z) = e^{-2\pi iz} + 744 + 196884\, e^{2\pi iz} + \ldots$$

which gives the isomorphism between $X_0(1)$ and $\mathbb{P}^1$. By the theory of complex multiplication, one knows that $j(z)$ is an algebraic integer of degree $h(D)$ over $\mathbb{Q}$ at a point $z \in \mathfrak{H}$ satisfying a primitive quadratic equation of discriminant D. The result we obtain gives the complete prime factorization of the norm of this number (and in fact of the number itself in the algebraic number field in which it lies); in particular:

THEOREM 3. <u>Any prime</u> p <u>dividing</u> $N(j(z))$ <u>is</u> $\le \frac{3|D|}{4}$. <u>More precisely, if</u> $3 \nmid D$ <u>then</u> p <u>occurs if and only if</u> $p \mid \frac{3|D|-x^2}{4}$ <u>for some integer</u> $x < \sqrt{3|D|}$ <u>and</u> p <u>is the only prime with</u> $(\frac{D}{p}) \ne 1$ <u>which divides</u> $\frac{3|D|-x^2}{4}$ <u>to an odd power</u>.

For example, $e^{\pi\sqrt{163}} = 262537412640768743.99999999999924\ldots$, so $j(\frac{1+\sqrt{-163}}{2}) = -262537412640768000 = -2^{18}\cdot 3^3\cdot 5^3\cdot 23^3\cdot 29^3$, and the numbers $(3\cdot 163-x^2)/4$ are $2\cdot\underline{61}$, $2^3\cdot 3\cdot 5$, $2^2\cdot\underline{29}$, $2\cdot 5\cdot 11$, $2\cdot 3\cdot 17$, $2^2\cdot\underline{23}$, $2^4\cdot 5$, $2\cdot 3\cdot 11$, $\underline{2}\cdot 5^2$, $\underline{2}^5$ and $2^2\cdot\underline{3}$, where the underlined primes are those satisfying the conditions of the theorem (note that $(\frac{-163}{p}) = -1$ for all $p < 41$). There is a similar result for differences $j(z_1)-j(z_2)$ with z_1 and z_2 of (possibly different) discriminants D_1 and D_2 and $(3|D|-x^2)/4$ replaced by $(D_1D_2-x^2)/4$ in the theorem. For example, $j(\frac{1+\sqrt{-7}}{2}) = -3375$ and $j(\frac{1+\sqrt{-7}}{2}) - j(\frac{1+\sqrt{-163}}{2}) = 3^8\cdot 5^3\cdot 7\cdot 13\cdot 17\cdot 31\cdot 103\cdot 229\cdot 283$, where all primes occurring divide $\frac{7\cdot 163-x^2}{4}$ for some x and we again have formulas for the exact powers to which they occur.

Titel: Arnold's conjecture on fixed points of symplectic mappings (after Conley and Zehnder)

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Let M^{2n} be a compact, smooth manifold of dimension $2n$. A symplectic form ω on ~~a smooth manifold~~ M^{2n} is a closed, non-degenerate ($\omega^n \neq 0$), smooth 2-form. Let $\mathcal{D} = \mathcal{D}iff_{sym} M$ be the group of diffeomorphisms of M which preserve ω, and let $\mathcal{D}_0 = \mathcal{D}iff_{sym,0} M$ be its identity component. These groups are locally connected by differentiable arcs, and so the universal cover $\tilde{\mathcal{D}}_0$ of $\mathcal{D}_0$ consists of pairs $(g, \{g_t\})$, where $g \in \mathcal{D}_0$ and $\{g_t\}$ is a homotopy ~~path~~ class of smooth paths in $\mathcal{D}_0$ with $g_0 = id.$ and $g_1 = g$. Define $\tilde{C} : \tilde{\mathcal{D}}_0 \longrightarrow H^1(M; \mathbb{R})$ by $\tilde{C}(g, \{g_t\}) = \left[\int_0^1 \dot{g}_t \lrcorner \omega \, dt\right]$. (Here $\dot{g}_t$ is the vector field whose value at x is $\frac{\partial g_t}{\partial t}(y)$ where $x = g_t(y)$. Since $\mathcal{L}_{\dot{g}_t} \omega = \dot{g}_t \lrcorner d\omega + d(\dot{g}_t \lrcorner \omega) = 0$, the 1-form $\dot{g}_t \lrcorner \omega$ is closed.) The map $\tilde{C}$ is a surjective homomorphism. Quotienting out by $\pi_1 \mathcal{D}_0$ and its image $\Gamma = \tilde{C}(\pi_1 \mathcal{D}_0)$, one obtains the <u>Calabi homomorphism</u> $C : \mathcal{D}_0 \longrightarrow H^1(M; \mathbb{R})/\Gamma$. (For all this see [2] or [3].)

<u>Example</u> 1. When $M = T^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ with the standard symplectic form ω induced from $dx_1 \wedge dx_2 + \dots + dx_{2n-1} \wedge dx_{2n}$, $\pi_1 \mathcal{D}_0$ is generated by loops of rotations and the subgroup Γ is $H^1(T^{2n}; \mathbb{Z})$. If F is a fundamental domain for T^{2n} in $\mathbb{R}^{2n}$, one may interpret $C(g) \in \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ to be $cg(\bar{g}F) - cg(F)$, where $\bar{g} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ lifts g and

where $cg(A)$ denotes the center of gravity of A. Thus C is C^0-continuous in this case.

We say that a smooth map $g: M \to M$ has *enough fixed points* if the number of its fixed points is:

(a) $\geqslant 1 + CL(M)$, where the cup length $CL(M)$ of M is the maximum length ℓ of a non-zero cup product $\alpha_1 \cup \dots \cup \alpha_\ell$ in $H^*(M; \mathbb{R})$, and

(b) $\geqslant SB(M) =$ sum of Betti numbers of M, if all the fixed points x of g are known to be non-degenerate (ie 1 is not an eigenvalue of dg_x).

Example 2 Let ϕ_t be a 1-parameter subgroup of $\mathcal{D}_0$. Then ϕ_t is contained in $\ker C$ iff the 1-form $\dot{\phi}_t \lrcorner \omega$ (which is independent of t) is exact and hence of the form dH for some (Hamiltonian) function H on M. Since H has enough critical points, the flow ϕ_t ~~has~~ will then have enough fixed points.

Arnold's Conjecture [1, Appendix 9] *Every symplectic diffeomorphism* g *which is in the kernel of the Calabi homomorphism has enough fixed points.*

Thm 3 (Conley-Zehnder [4]) *This conjecture is true if* (M, ω) *is the torus* T^{2n} *with its standard symplectic form.*

Thm 4 (Weinstein [8]). This conjecture holds for any (M,ω) provided that g is sufficiently C^0-close to the identity.

Note. If g is assumed C^1-close to the identity, it is easy to prove the conjecture: see [1, App 9] and [7].

Cor to Thm 3 When $n>1$, the group $\mathcal{D}iff_{sym} T^{2n}$, where T^{2n} has the standard symplectic form ω, is not C^0-dense in the group $\mathcal{D}iff_{vol} T^{2n}$ of diffeomorphisms of T^{2n} which preserve the volume form ω^n.

Pf. It is clear from example 1 that the Calabi homomorphism C extends to $\mathcal{D}iff_{vol,0} T^{2n}$. (See also [2].) But there are elements in $\ker C \cap \mathcal{D}iff_{vol,0} T^{2n}$ which have no fixed points. For example, one can take the flow generated by an non-vanishing vector field ξ such that $\xi \lrcorner \omega^n$ is exact. These exist by Gromov's method of convex integration.

Because there are no natural groups between $\mathcal{D}iff_{sym} M$ and $\mathcal{D}iff_{vol} M$, one would then conjecture that $\mathcal{D}iff_{sym} M$ is C^0-closed in $\mathcal{D}iff_{vol} M$. Eliashberg has announced in [6] a proof of this, together with many other related results, including the full conjecture of Arnold.

A generalization of Theorem 3 to intersections of Lagrangians is given in [3]. This article also contains a proof of Theorem 3, as well as many interesting remarks and conjectures.

Finally, related results for contact structures in dimension 3 are proved by Bennequin. See his thesis (Ecole Normale Sup., Paris 1982) and [5].

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Titel: Conjectures of Tate and Mordell II

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The purpose of this talk is to provide the input for part I, namely that one can define a good hight-function on A_g, the moduli-space of principally polarized abelian varieties of dimension g, by

$$h(A) = \deg(\omega_A).$$

(A is a semiabelian variety over the integers of a number-field)

The main reason why this is true is the fact that ω of the universal abelian variety over A_g is ample, but there are some technical difficulties due to the fact that so far there is no modular compactification of A_g over Spec(Z):

a) A_g is only a coarse moduli space

b) A_g is not proper over Spec(Z)

c) If we take a proper compactification of A_g, there is no ample line-bundle on it which has a modular interpretation.

Difficulty a) can be ignored, and for b) one can use the compactifications over the complex numbers constructed by Baily-Borel, Mumford, Namikawa....., and the fact that the metrics on the natural bundles on A_g have only logarithmic singularities at the boundary.

For c) one writes the universal abelian variety over A as the quotient of a jacobian, and uses the theory of the moduli-spaces of curves.

Titel: The monotonicity theorem for real quadratic maps

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Define the "discriminant" of a quadratic mapping $x \mapsto f(x) = Ax^2 + Bx + C$, $A \neq 0$, from the real (or complex) line to itself to be the number $\Delta = \Delta_f = B^2 - 4AC - 2B$.

Two such maps f and g are <u>linearly conjugate</u> ($f = \ell \circ g \circ \ell^{-1}$ with ℓ linear) if and only if they have the same discriminant. It is easy to check that f maps some finite interval I into itself $\Longleftrightarrow$ $-1 \leq \Delta \leq 8$.

Numerical evidence early suggested three conjectures:

1. (Obviously true?). For a dense open set of values of Δ in $[-1, 8]$, f has an <u>attractive</u> periodic orbit. That is, some iterate

$f^n = \underbrace{f \circ \cdots \circ f}_{n}$ has a fixed point x_0 at which the derivative Df^n satisfies $|Df^n x_0| <$

2. (Apparently harder). As Δ increases, the dynamical behavior of f becomes more complex.

3. (Hardest.) For a set of values of Δ of positive Lebesgue measure, there is no attractive periodic orbit.

This last conjecture was the first to be proved, by Jakobson. (See also Carleson and Benedicks, and an important generalization to many-parameter families of complex rational maps by Rees.)

The second conjecture was proved by Hubbard, Douady, Sullivan, Thurston and myself, and will be described below. The first conjecture has not yet been proved.

Here is a precise statement.

<u>Monotonicity Theorem</u> If $\Delta_f < \Delta_g$, then for any iterate f^n, the number $|\mathrm{fix}\, f^n|$ of fixed points satisfies

$$|\mathrm{fix}\, f^n| \leq |\mathrm{fix}\, g^n|,$$

and the number $|\mathrm{crit}\, f^n|$ of critical points satisfies

$$|\mathrm{crit}\, f^n| \leq |\mathrm{crit}\, g^n|.$$

Misiurewicz and Szlenk have shown that the <u>topological entropy</u> $h(f)$ can be computed as

$$\lim_{n\to\infty} \frac{1}{n} \log |\mathrm{crit}\, f^n|.$$

Thus it follows as a corollary that

$$h(f) \leq h(g).$$

Although the monotonicity theorem involves only real variables, all known proofs make use of complex variable methods to prove the following key lemma. Call a periodic orbit

$$x_0 \mapsto x_1 \mapsto \cdots \mapsto x_n = x_0$$

<u>superattractive</u> if it contains the critical point, so that $Df^n(x_0) = 0$.

<u>Lemma</u> (Hubbard, Douady, Sullivan). If two quadratic maps f and g both have superattractive orbits of period n, <u>which are ordered in the same way</u>, then they are linearly conjugate.

Several proofs have been given. I will describe one due to Thurston, based on Teichmüller theory.

Given this Lemma, together with the Milnor-Thurston theory of "kneading invariants", and Thurston's <u>Intermediate Value Theorem</u> for kneading invariants, the proof is quite straightforward.

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Titel: On Conjectures of Lichtenbaum and Quillen

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These conjectures concern the computation of algebraic K-groups of certain arithmetic and algebro-geometric rings. To motivate K-theory, we recalled a theorem of D. Quillen concerning the relationship of K-groups to chow groups of an algebraic variety [9] and a theorem of A.S. Mekuriev - A.A. Suslin relating K-theory (K_2/nK_2) to Brauer groups [7].

We consider K-theory mod-n of a commutative ring A, $K_*(A, \mathbb{Z}/n)$, which fits in the long exact sequence

$$\cdots \to K_i(A) \xrightarrow{n} K_i(A) \to K_i(A, \mathbb{Z}/n) \xrightarrow{\partial} K_{i-1}(A) \to$$

With little evidence other than Quillen's computation of the K-theory of finite fields, the following program has been proposed:

I. For k an alg. closed field of char $= p \geq 0$, prove

$$K_i(k, \mathbb{Z}/n) = \begin{cases} \mathbb{Z}/n & i \text{ even} \\ 0 & i \text{ odd} \end{cases} \quad \text{for } (n,p) = 1$$

II. For A the local ring of germs of algebraic functions at a point on a variety (with respect to the etale topology), prove $K_*(A, \mathbb{Z}/n) \xrightarrow{\sim} K_*(k, \mathbb{Z}/n)$ where $A \to k$ is the residue map.

III. Prove the existence of a local-to-global (for the etale topology) spectral sequence for K-theory mod-n.

IV. Construct a computable topological theory isomorphic to algebraic K-theory mod-n for "nice" rings A, one that satisfies I.) $\to$ III.)

For char(k) > 0, I.) follows from Quillen's computation of $K_*(\bar{\mathbb{F}}_p)$ and the following

<u>Theorem</u> (Suslin [10]). Let $F \to k$ be an extension of algebraically closed fields. Then $K_*(F, \mathbb{Z}/n) \to K_*(k, \mathbb{Z}/n)$ is an isomorphism.

II.) is implied by the following theorem announced by O. Gabber, and proved independently in a special case by H. Gillet and R. Thomason [4]

<u>Theorem</u> (Gabber [3]) Let A be a hensel local ring with residue field k and assume $1/n \in A$. Then $K_*(A, \mathbb{Z}/n) \to K_*(k, \mathbb{Z}/n)$ is an isomorphism.

Using this last theorem, Suslin completed the proof of I.) by proving the following.

<u>Theorem</u> (Suslin [11]) Let G^δ denote a Lie group viewed as a discrete group. Then

$$BGL(\mathbb{R})^\delta \to BGL(\mathbb{R}), \quad BGL(\mathbb{C})^\delta \to BGL(\mathbb{C})$$

determine isomorphisms in $\mathbb{Z}/n$ homology. In particular, $K_i(\mathbb{C}, \mathbb{Z}/n) = \begin{cases} \mathbb{Z}/n & i \text{ even} \\ 0 & i \text{ odd}. \end{cases}$

Using sophisticated homological algebra techiques, R. Thomason has proved the following weak form of III.)

<u>Theorem</u> (Thomason [12]) Let $\mathcal{K}_*(A)$ denote algebraic K-theory of A mod-n with "the Bott element inverted" Then for sufficiently nice $\mathbb{Z}[1/n]$-algebras A, there exists a spectral sequence

$$E_2^{p,-q} = H^p_{et}(\operatorname{Spec} A, \mu_n^{\otimes -q/2}) \Rightarrow \mathcal{K}_{-(p+q)}(A)$$

For IV.) W. Dwyer and I have constructed a theory depending only on the etale topology of A, $K^{et}_*(A, \mathbb{Z}/n)$, and have proved the following

<u>Theorem</u> (Dwyer-Friedlander [1]). Let ℓ be an odd prime. There exists a natural transformation of $\mathbb{Z}[1/\ell]$-algebras

$$\varphi : K_*(A, \mathbb{Z}/\ell^\nu) \to K^{et}_*(A, \mathbb{Z}/\ell^\nu) \qquad \text{any } \nu > 0$$

satisfying

a.) φ is an isomorphism for finite fields and algebraically closed fields

b.) φ is surjective for fields of mod-ℓ etale cohomological dimension ≤ 3

c.) The Lichtenbaum Conjecture for global fields at the prime ℓ is equivalent to the conjecture that φ is an isomorphism for A = ring of integers in a global field (with $1/\ell$ adjoined) and all $\nu > 0$.

d.) There is a local-global spectral sequence

$$E_2^{p,-q} = H^p_{et}(\operatorname{Spec} A, \mu_{\ell^\nu}^{\otimes -q/2}) \Rightarrow K^{et}_{-(p+q)}(A, \mathbb{Z}/\ell^\nu)$$

In view of stability theorems relating $H_i(BGL_m(k))$ with $H_i(BGL(k))$, the orthogonal and symplectic analogues of I.) proved by M. Karoubi [5], and stability theorems for these groups, one might conjecture

<u>Conjecture</u> For any algebraically closed field k, any reductive complex Lie group G with associated algebraic group G_k over k, there is a natural isomorphism.

$$H^*(BG_k(k), \mathbb{Z}/n) \simeq H^*(BG, \mathbb{Z}/n) \qquad 1/n \in k$$

where $G_k(k)$ is the discrete group of rational points of G_k.

Recently, G. Mislin and I proved the following

<u>Theorem</u> (Friedlander-Mislin[2])

a.) The above conjecture is true for $k = \overline{\mathbb{F}}_p$

b.) For any k, there is a natural split injection

$$H^*(BG_k(k), \mathbb{Z}/n) \hookrightarrow H^*(BG, \mathbb{Z}/n)$$

c.) This map is an isomorphism $\Leftrightarrow$ for any $x \in H^i(BG_k(k), \mathbb{Z}/n)$, $\exists$ a finite subgroup $\pi \subset G_k(k)$ such that x restricts non-trivially in $H^i(B\pi, \mathbb{Z}/n)$.

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Titel: Polar multiplicities and curvature integrals

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A more appropriate (but longer) title would be "Polar multiplicities, curvature integrals, projective invariants and deformations to the tangent cone" since we use a (long known) 1-parameter deformation deforming the inclusion $\{x\} \hookrightarrow X$ of a singularity $(X,x) \subset (\mathbb{C}^N, 0)$ to the inclusion $\{x\} \hookrightarrow C(X,x)$ in the tangent cone*) to relate invariants of the singularity to classical projective invariants of the projectivized tangent cone (hopefully) and to calculate the limits of certain curvature integrals.

(Remark This deformation is a special case of the "deformation to the normal cone" which BAUM-FULTON-MACPHERSON used in their proof of RIEMANN-ROCH and which is useful in other contexts, e.g. in defining algebraic intersections, see [1], [3]).

The invariants in question are the local polar multiplicities $m_k(X,x)$, $0 \le k \le d-1$, for a reduced pure d-dimensional singularity, defined by LÊ DŨNG TRÁNG and TEISSIER, see [5], [9]; these seem to be very useful and interesting invariants since they may be used to define the CHERN-MACPHERSON-SCHWARTZ-classes for singular varieties ([6], [5], [7]; and to characterize WHITNEY-conditions numerically ([9], [7]).

*) given by a flat map $p: Y \to \mathbb{C}$ s.th. $p^{-1}(t) = X_t \cong X$ for $t \neq 0$ and $p^{-1}(0) = X_0 = C(X,x)$.

The theorem referred to in the title is

Theorem Let $(X,x) \hookrightarrow (\mathbb{C}^N, 0)$ be the representative of a germ of a reduced, equidimensional complex space of dimension d, ϕ be the standard KÄHLER-2-form on $\mathbb{C}^N$, and let for $0 \le k \le d$ c_k denote the k-th CHERN forms of the tangentbundle TX_{reg} with respect to the metric on X_{reg} induced by ϕ *). Then, for $0 \le k \le d-1$:

1) The integrals $\int_{X \cap B_r} c_k \wedge \phi^{d-k}$ exist, where B_r is the ball of small enough radius r in $\mathbb{C}^N$ around 0 and the integrals are to be interpreted as improper ones over $X_{reg} \cap B_r$

2) $\lim_{r \to 0} \frac{(-1)^k}{r^{2d-2k}} \int_{X \cap B_r} c_k \wedge \phi^{d-k}$ exists and equals $\pi^{d-k} \cdot m_k(X,x)$

Remarks 1. The case $k = 0$:

$$\lim_{r \to 0} \frac{1}{r^{2d}} \int_{X \cap B_r} \phi^d = \pi^d \cdot \text{multiplicity of } x \text{ on } X$$

is due to THIE and DRAPER. [10], [2]

2. The case $k = d$ is also true, since both expressions are zero.

Sketch of proof The proof hinges on the following facts:

*) X_{reg} denotes the regular locus of X

a) In the blowup-diagram

$$\begin{tikzcd} \mathcal{O}(-1) = \{(z,\ell) \in \mathbb{C}^N \times \mathbb{P}^{N-1} \mid z \in \ell\} \arrow[r, "q"] \arrow[d, "e"'] & \mathbb{P}^{N-1} \\ \mathbb{C}^N & \end{tikzcd}$$

(with q, e the projections) of $0 \in \mathbb{C}^N$, one has, on $\mathcal{O}(-1)$ - zerosection $= \mathbb{C}^N - \{0\}$:

$$\phi = dd^c \|z\|^2 \qquad \begin{pmatrix} d = \bar{\partial} + \partial \\ d^c = \frac{i}{4}(\bar{\partial} - \partial) \end{pmatrix}$$
$$\omega = dd^c \log \|z\|^2$$

where $\omega = \pi \cdot$ FUBINI-STUDY-form of $\mathbb{P}^{N-1}$; thus on $S_r = \partial B_r$:

$$\frac{1}{r^2}\phi = d\eta = \omega$$

b) In the basic blowup-diagram of [5]

$$\begin{tikzcd} \mathbb{P}^{N-1} \times \mathbb{C}^N \times \mathrm{Grass}_d \supset \mathfrak{X} \arrow[r, "\hat{e}"] \arrow[d, "\tilde{\nu}"'] \arrow[dr, "\xi"] & \hat{X} \subset \mathbb{C}^N \times \mathrm{Grass}_d(\mathbb{C}^N) \arrow[d, "\nu"] & \hat{T}X \arrow[l] \\ \mathcal{O}(-1) \supset \tilde{X} \arrow[r, "e"'] & X \subset \mathbb{C}^N & \end{tikzcd}$$

$\tilde{X} \subset \mathbb{P}^{N-1} \times \mathbb{C}^N$

we have, using a) and STOKES:

$$*) \qquad \frac{1}{r^{2d-2k}} \int_{X \cap B_r} c_k \wedge \phi^{d-k} = \int_{\mathfrak{X} \cap \xi^{-1}(S_r)} c_k \wedge \eta \wedge \omega^{d-k-1}$$

A first conclusion is (where $B_{r,s} = \{|z| \mid r \leq |z| \leq s\}$)

$$\frac{1}{s^{2d-2k}} \int_{X \cap B_s} c_k \wedge \phi^{d-k} - \frac{1}{r^{2d-2k}} \int_{X \cap B_r} c_k \wedge \phi^{d-k} \underset{\text{STOKES again}}{=} \int_{\mathcal{X} \cap \tilde{g}^{-1}(B_{r,s})} c_k \wedge \omega^{d-k}$$

Since the CHERN forms have sign properties in the holomorphic case, one knows

$$(-1)^k \int_{\mathcal{X} \cap \tilde{g}^{-1}(B_{r,s})} c_k \wedge \omega^{d-k} \geq 0$$

so that $\frac{(-1)^k}{r^{2d-2k}} \int_{X \cap B_r} c_k \wedge \phi^{d-k}$ is monotonically increasing, and $\lim_{r \to 0} \frac{1}{r^{2d-2k}} \int_{X \cap B_r} c_k \wedge \phi^{d-k}$ exists.

c) There is a "good description" of $m_k(X,x)$ (which is both convenient geometrically and algebraically):

$$m_k(X,x) = (-1)^k \int_{Y} c_k \wedge \omega_{st}^{d-k-1}$$

where $Y = \varphi^{-1}(0)$, the exceptional divisor, and $\omega_{st} = \omega/\pi$; this is the main result (5.1.1.) of [5]

d) Finally, there is the <u>continuity of the fibre integral</u> due to STOLL (see [ ]); applying it to the relative NASH transform of the deformation $p: Y \to \mathbb{C}$ of X to the tangent cone one may conclude, putting $X_t = \bar{p}^{-1}(t)$

$$\lim_{r\to 0} \frac{1}{r^{2d-2k}} \int_{X\cap B_r} c_k \wedge \phi^{d-k} = \lim_{\substack{r\to 0\\ t\to 0}} \frac{1}{r^{2d-2k}} \int_{X_t\cap B_r} c_k \wedge \phi^{d-k}$$

$$= \lim_{r\to 0} \frac{1}{r^{2d-2k}} \int_{C(X,x)} c_k \wedge \phi^{d-k}$$

By $*$), and FUBINI $\frac{(-1)^k}{r^{2d-2k}} \int_{C(X,x)} c_k \wedge \phi^{d-k} = (-1)^k \pi \int_{Y} c_k \wedge \omega^{d-k-1}$

(modulo technicalities)

$$= (-1)^k \pi^{d-k} \int_{Y} c_k \wedge \omega_{st}^{d-k-1}$$

$$= \pi^{d-k} \cdot m(X,x) \text{ (by c)}.$$

QED

Applications 1) Derivation of GRIFFITH's formula for isolated hypersurface singularities ([4], (5.22)

2) $m_k(X,x)$ can be computed on the tangent cone:

$$m_k(X,x) = \sum_{j=0}^{k} (-1)^j \binom{d-j}{k-j} \int_Z c_j(Z) \wedge \omega_{st}^{d-j-1}, \quad Z = \mathbb{P}C(X,x)$$

Further developments Modulo technicalities, the result of LÊ and TEISSIER c) says $m_k(X,x)$ = k-th class of $\mathbb{P}C(X,x)$ in the sense of [8]. Now the polarvarieties $P_k(X,x)$ of [5], defining the $m_k(X,x)$, should be thought of as defining classes in suitable local homology groups $H_*^{\{x\}}(X,x)$. There are specialization homomorphisms, constructed by FULTON [1] (appendix) and VERDIER [11], $\sigma_t : H_*^{\{x_t\}}(X_t) \to H_*^{\{x\}}(C(X,x)) = H_*(\mathcal{O}(-1), \mathcal{O}(-1)\text{ - zerosect.})$

Combining with the THOM homomorphism $H_*(\mathcal{O}(-1), \mathcal{O}(-1)\text{ - zerosect.}) \xrightarrow[\cap[\mathbb{P}C(X,x)]]{} H_{*-1}(\mathbb{P}C(X,x))$, the following hopefully holds:

Theorem Under this homomorphism, the classes $[P_k(X,x)]$ specialize to the polar classes of $\mathbb{P}C(X,x)$ in the sense of [8].

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Titel: The irrationality of the moduli space of Curves after Harris and Mumford
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Let $\mathcal{M}_g$ be the moduli space of complex curves of genus g. $\mathcal{M}_1$ is of course $\mathbb{A}^1$, and it was observed by Severi that $\mathcal{M}_g$ can be covered by a rational variety for $g \leq 10$ (Sernesi has now proved this also for $g=12$). Severi conjectured, and even outlined a proof, that this is true for all g. However:

<u>Theorem</u> ([H-M] and [H]): For large g, $\mathcal{M}_g$ is of general type; in particular, there is not even a rational curve through the general point of $\mathcal{M}_g$.

Specifically, the theorem holds for g odd and ≥ 25; and for g even and ≥ 40. Further, nearly completed work by Harris and myself seems likely to show that the same is true for $g \geq 28$, and perhaps even for $g \geq 24$. [H-M] also proves that $\overline{M}_{23}$ has an effective pluricanonical divisor, but Harris conjectures that for $g < 23$, $\overline{M}_g$ has none.

The proof of the Theorem has two major parts. The first is to show that pluricanonical divisors on the regular locus of $\overline{M}_g$ all extend to a desingularization of $\overline{M}_g$. Since $\overline{M}_g$ is locally the quotient of a ~~sy~~ smooth space, the Reid-Tai criterion can be used, but unfortunately it does not apply in a simple way in all cases, and considerable computation is required.

One computes, using the Grothendieck-Riemann-Roch Theorem, that the canonical class K is given by

$$K = 13\lambda - 2\Delta_0 - \tfrac{3}{2}\Delta_1 - 2\Delta_2 - 2\Delta_3 - \dots,$$

where

λ is the "Hodge" bundle and

$\Delta_0, \Delta_1, \dots, \Delta_{[g/2]}$ are the components of $\overline{M}_g - M_g$.

From the theory of theta-functions on the moduli space of abelian varieties, one knows that λ is ample on M_g, so it is enough to show that for large g there is an effective divisor D and integers $m, n, r \geq 1$ with

$$mK = n\lambda + rD + (\text{effective}).$$

This is clearly possible for D of the form

$$D = a\lambda - \sum_{i=0}^{[g/2]} b_i \Delta_i$$

as long as

$$a/b_i < 13/2 \quad \text{for } i \neq 1$$

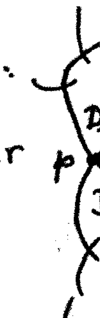
$$a/b_1 < 13/(3/2) \quad .$$

The second main step is to produce such a divisor. The theory of linear series on a reducible curve is used in an essential way, in the form of admissible coverings developed by Beauville and Knudsen. I will describe a different approach, from my work with Harris, that leads in some cases to simplification:

Let C be a reduced connected curve whose ~~ea~~ intersection graph is a tree:

If C is irreducible, then a g^r_d on C is just a line bundle $\mathcal{L}$ on C together with an $r+1$-dimensional vectorspace of sections $V \subset H^0(C; \mathcal{L})$. In general, if C has components D_i, we define a <u>limit g^r_d</u> on C to be a g^r_d on each D_i:

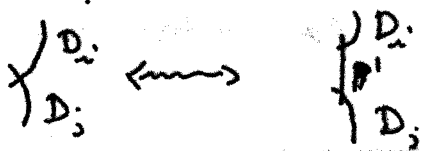
$$V_i \subset H^0(D_i, \mathcal{L}_i)$$



such that whenever a D_i meets another component D_j in a point, say p, and $a_0 < \ldots < a_r$ are the degrees with which sections in V_i vanish at p, while $b_0 < \ldots < b_r$ are the degrees with which sections of V_j vanish at p, then

$$a_i + b_{r-i} = d.$$

We say that two curves are equivalent if we can obtain one from the another by operations of the type



.

Returning to the Theorem, we may now describe the necessary divisors $D \subset \mathcal{M}_g$ or $\overline{\mathcal{M}}_g$. For many values

$r, d \geq 0$ with $\rho = g-(r+1)(g-d+r) \geq -1$, the set of smooth curves C <u>not</u> satisfying: "Petri's Condition" – The scheme $G^r_d(C)$ of g^r_d's on C is smooth of dimension ρ — is a divisor $AP^r_d \subset \mathcal{M}_g$ (the "anti-Petri divisor".) (See also [G] and [E-H]).

If $g = 2k-1$ is odd, then the divisor $D \subset \overline{\mathcal{M}}_g$ necessary for the proof of the Theorem is the closure of AP^1_k, the set of curves posessing a g^1_k (or, in the reducible case, such that some equivalent curve posseses a limit g^1_k).

If $g = 2k-2$ is even, the situation is somewhat more complicated. Harris [H] makes use of the divisor in $\overline{M}_g$ which is the closure of the divisor of curves posessing a g^1_k with a ramification point of order $2k-g+1$. However, it seems likely that the closure of AP^1_k leads to a better result, proving the Theorem for $k \geq 28$, as mentioned above. For $g = 26$, it seems that one should work with AP^2_9 while for $g = 24$ with AP^4_{23}.

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Titel: THE COHOMOLOGY OF QUOTIENTS IN ALGEBRAIC AND SYMPLECTIC GEOMETRY

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The aim of this talk is to describe a general procedure for calculating the cohomology (at least the Betti numbers) of the quotient varieties associated to complex reductive group actions in algebraic geometry. Such varieties are interesting in particular because of their relevance to moduli problems.

There are two approaches to the problem leading to the same results. One is purely algebraic while the other uses ideas of equivariant Morse theory and symplectic geometry. I shall try to explain the second approach here.

Let $X \subseteq \mathbb{P}_n(\mathbb{C})$ be a nonsingular complex projective variety; and let G be an algebraic group acting linearly on X. We must assume that G is reductive, i.e. that it is the complexification of a maximal compact subgroup K. For simplicity suppose $G \subseteq GL(n+1)$ and $K \subseteq U(n+1)$.

Let $A(X)$ be the graded algebra of homogeneous polynomials on X. Then the algebro-geometric "quotient" of X by G is defined as the projective variety corresponding to the invariant subalgebra $A(X)^G$ of $A(X)$ (see [M]). The inclusion $A(X)^G \hookrightarrow A(X)$ induces a rational map

$$\phi : X \dashrightarrow X/\!/G.$$

We can define open subsets $X^s_{(0)} \subseteq X^{ss} \subseteq X$ (the stable and semistable points respectively) such that

(i) ϕ induces a G-invariant surjective morphism

$$\phi : X^{ss} \longrightarrow X/\!/G$$

and (ii) every fibre of ϕ which meets X^s is a single G-orbit, (see [M] again)

Main aim: to calculate $H^*(X/\!/G; \mathbb{Q})$ in the good cases when $X^{ss} = X^s_{(0)}$. (With some more work, information can be obtained about the general case). In such cases topologically we have

$$X/\!/G = X^{ss}/G.$$

Recall that if Y is a G-space then its equivariant cohomology is $H^*_G(Y) := H^*(Y \times_G EG)$ where $EG \to BG$ is the universal G-bundle. If G acts with only finite stabilisers then $H^*_G(Y; \mathbb{Q}) = H^*(Y/G; \mathbb{Q})$.

So $$\boxed{X^{ss} = X^s_{(0)} \implies H^*(X/\!/G; \mathbb{Q}) = H^*_G(X^{ss}; \mathbb{Q})}$$

We shall obtain a formula for $H^*_G(X^{ss}; \mathbb{Q})$ in all cases.

IDEA: define a K-invariant "Morse" function f on X and show that

① $X^{ss} = X^{min}$, the minimal Morse stratum for f;

② the equivariant Morse inequalities are in fact equalities, i.e. $P^G_t(X) = P^G_t(X^{ss}) + \sum_{\text{other strata } S_\beta} t^{\operatorname{codim} S_\beta} P^G_t(S_\beta)$;

(3) $P_t^G(X) = P_t(X)\, P_t(BG)$; and

(4) For each unstable stratum S_β, there is a nonsingular $X_\beta \subsetneqq X$ invariant under a reductive subgroup $G_\beta \subseteq G$ such that $P_t^G(S_\beta) = P_t^{G_\beta}(X_\beta^{ss})$.

(The idea for this is borrowed from [A&B])

N.B. Here P_t denotes the Poincaré series and P_t^G the equivariant Poincaré series. Since $G \sim^{hty} K$ we have $P_t^G = P_t^K$.

This gives

Theorem. The equivariant Betti numbers of X^{ss} are given by the inductive formula

$$P_t^G(X^{ss}) = P_t(X)\, P_t(BG) - \sum_\beta t^{\operatorname{codim} S_\beta} P_t^{G_\beta}(X_\beta^{ss})$$

Moreover the Morse stratification $\{S_\beta \mid \beta \in \mathcal{B}\}$ of f is "computable" for example

— there is a combinatorial description of $\mathcal{B}$ in terms of the weights of the representation of G on $\mathbb{C}^{n+1}$,

— each X_β is the intersection of X with a linear subvariety of $\mathbb{P}_n$, etc.

Hence we can derive an explicit formula for $P_t^G(X^{ss})$ (which in good cases ~~gives~~ equals $P_t(X/\!/G)$).

The function f comes from symplectic geometry. The Kähler form ω on X gives X a symplectic structure preserved by K. There is a <u>moment map</u>

$$\mu : X \to \mathfrak{k}^* \text{ the dual Lie algebra of } K$$

for this action (for the definition see e.g. [G&S])
Then

<u>Defn</u> For $x \in X$, let $f(x) = \|\mu(x)\|^2$

(We can take any convex K-invariant function of μ here instead of $\|\mu\|^2$).

<u>Problem</u>. f is not a nondegenerate Morse function (in the sense of Bott). However f is "minimally degenerate" which is enough to ensure that the Morse theory still works.

This method applies more generally to any compact group action on a compact symplectic manifold X for which a moment map exists. The "symplectic quotient" is then defined as $\mu^{-1}(0)/K$. In the algebraic case this quotient coincides with $X/\!/G$ (cf. ① above).

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Titel: Group varieties and transcendence
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1. Historical remarks

The first result on group varieties and transcendence was proved in 1873 by Ch. Hermite who proved that $e \notin \overline{\mathbb{Q}}$. This result was generalized in 1882 by F. Lindemann. He proved the remarkable result that for complex $x \neq 0$

$$\{x, e^x\} \not\subset \overline{\mathbb{Q}}.$$

This theorem can also be interpretated as a result on arithmetical properties of the exponential map of the complex Lie-group $\mathbb{C}^\times = \mathbb{G}_m(\mathbb{C})$, the complex valued points of the multiplicative group $\mathbb{G}_m$.

The next interesting result in this field was proved in 1932 by C.L. Siegel, namely that the periods of elliptic integrals of the first kind on an elliptic curve E defined over $\overline{\mathbb{Q}}$ are not all algebraic. Shortly after, in 1936, Th. Schneider proved that these periods are either zero or transcendental. More generally he proved that for complex numbers x with $\wp(x) \neq \infty$ one has

$$\{x, \wp(x)\} \not\subset \overline{\mathbb{Q}}.$$

Here $\wp(x)$ denotes the Weierstraß elliptic function and we assume that g_2, g_3 are algebraic. Also this result can be interpretated as a result on the arithmetic properties of the exponential map of the elliptic curve.

Then Cartier asked the following problem: given a commutative algebraic group G defined over $\overline{\mathbb{Q}}$ with tangent space $T(G)$ at the neutral element of G and exponential map

$$T(G) \xrightarrow{\exp} G.$$

Is it true that for an algebraic tangent vector $0 \neq x \in T(G)(\overline{\mathbb{Q}})$ the image $\exp_G(x)$ is transcendental, i.e. not in $G(\overline{\mathbb{Q}})$?

This problem was solved by S. Lang.

2. Baker's Theorem

In 1967 A. Baker proved the following

result. Let $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in \overline{\mathbb{Q}}$, $\log \alpha_1, \ldots, \log \alpha_n$ $\mathbb{Q}$-linearly independant and $\beta_1, \ldots, \beta_n$ not all zero.

<u>Theorem (Baker)</u>. $\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n \notin \overline{\mathbb{Q}}$.

Again this result can be interpretated as a result on group varieties and their transcendence properties

3. A general result

In order to formulate our result let as before be G a commutative algebraic group defined over $\overline{\mathbb{Q}}$ of dimension n with tangent space $T(G)$ and exponential map

$$T(G) \xrightarrow{\exp} G.$$

Let A be an analytic subgroup of G of dimension $< n$ defined over $\overline{\mathbb{Q}}$. Then one can ask the question to describe the set of $\overline{\mathbb{Q}}$-rational points $A(\overline{\mathbb{Q}})$ of A. Obviously one has $A(\overline{\mathbb{Q}}) \neq (0)$ if there exist an algebraic subgroup $H \neq (0)$ with $H \subseteq A$. But we also have the converse.

<u>Theorem 1.</u> Suppose that $T(A) \cap \exp_G^{-1}(G(\overline{\mathbb{Q}}))$ is different from (0). Then there exists an algebraic subgroup H of G with

(i) $H \subseteq A$

(ii) $\dim H > 0$.

<u>Example 1.</u> One can easily deduce from this result Baker's Theorem stated

above.

Example 2. Let A be a simple abelian variety of dimension g defined over $\overline{\mathbb{Q}}$. Let $u_1, \dots, u_n \in T(A)$ be tangent vectors such that $\exp_A(u_i) \in A(\overline{\mathbb{Q}})$, $1 \leq i \leq n$. Suppose that $u_1, \dots, u_n$ are linearly independent over $(\operatorname{End} A) \otimes \mathbb{Q}$. Then if we write $u_i = (u_{i,1}, \dots, u_{i,g})$ for $1 \leq i \leq n$ and a suitable basis for $T(A)$ we have

Theorem 2. The ng complex numbers $u_{1,1}, \dots, u_{n,g}$ are $\overline{\mathbb{Q}}$-linearly independent.

Remark. This theorem implies a conjecture of Waldschmidt.

4. Periods

Consider the elliptic curve

$$y^2 = 4x^3 - 4x .$$

Then the fundamental periods are given by

$$\omega = 2\int_1^\infty \frac{dx}{\sqrt{4x^3-4x}} = \frac{\Gamma(1/4)^2}{2\sqrt{2\pi}} \notin \overline{\mathbb{Q}}$$

$$\omega' = i\omega .$$

So special values of special functions appear as periods.

We consider the general case of a smooth projective variety X and a non-exact closed meromorphic 1-form ξ on X defined both over $\overline{\mathbb{Q}}$. Let D be the polar divisor of ξ and $Y = X - |D|$.

Then let C be a closed path in Y.

<u>Theorem 3</u>. $\int_C \xi$ is either zero or transcendental.

Titel: Generalization of the Cauchy-Kowalewski and Holmgren theorems to the case of generalized analytic functions

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1. Statement of the problem

We regard differential equations of type

$$(*) \qquad \frac{\partial u}{\partial t} = Lu .$$

The function u, for which we look, depends on the time t and on a spacelike variable x, where x is a point in the R^n. The differential operator L is of first order. If x denotes a variable in the plane, then we will write z instead of x. If the sought function is complex-valued, then we will denote it by w.

A special case of the differential equation (*) is the hans Lewy differential equation [4]

$$(1) \qquad \frac{\partial w}{\partial t} = - \frac{\partial w}{\partial \bar{z}} \cdot \frac{i}{z} + f(z,t),$$

which is (for z≠o) a linear differential equation without singularities. Hans Lewy had proved, that there are infinitely differentiable functions f = f(z,t), such that the differential equation (1) does not possess any solution.

On the other hand the Cauchy-Kowalewski theorem shows, that (*) is solvable, if the coefficients of L are holomorphic. In this case, moreover, it is possible to prescribe the initial values

$$(**) \qquad u(.,o) = u_o$$

of the sought solution. However we must assume, that the initial function u_o is holomorphic.

The reason for the solvability of the initial value problem (*), (**) in the holomorphic case is the following one:

If Φ is holomorphic in the bounded domain G and continuous in the closure, then the maximum norm of the derivative in a compact subset K with distance δ from the boundary of G may be estimated by

(2)

On the other hand we know, that generalized analytic functions possess many common properties with holomorphic functions in the classical sense. For instance, an estimate analogous to (2) is valid for generalized analytic functions. Using such an analogous estimate we will generalize the Cauchy-Kowalewski theorem to the case of generalized analytic functions. This means, especially, that we can replace holomorphic initial functions by generalized analytic ones.

2. Generalized analytic functions (see I.N.Vekua [14])

Generalized analytic functions are solutions of differential equations of type

$$(3) \qquad \frac{\partial w}{\partial \bar{z}} = A(z)w + B(z)\bar{w} ,$$

where A(z) and B(z) are given coefficients.The differential equation (3) is the canonical form of a uniformly elliptic lin. system for two unknown, real-valued functions in the plane.

The theory of generalized analytic functions is based on the useof the T_G- and Π_G-operators:

$$(T_G h)[z] = -\frac{1}{\pi} \iint_G \frac{h(\zeta)}{\zeta - z}\, d\xi\, d\eta, \quad \zeta = \xi + i\eta ,$$

$$(\Pi_G h)[z] = -\frac{1}{\pi} \iint_G \frac{h(\zeta)}{(\zeta - z)^2}\, d\xi\, d\eta .$$

The basic properties of this two integral operators are:

$$\frac{\partial}{\partial \bar{z}} T_G h = h, \quad \frac{\partial}{\partial z} T_G h = \Pi_G h.$$

Using Weyl's lemma for any solution w=w(z) of (3) the function

$$\Phi = w - T_G(Aw+Bw)$$

is holomorphic. Thus we can apply properties of holomorphic functions in order to prove analogous properties of generalized analytic functions (cf. [3, 7, 14]).

3. Associated differential operators

An operator with holomorphic coefficients transforms the space of holomorhic functions into itself.

A simple example of an operator, to which the classical Cauchy-Kowalewski theory is not applicable, is

$$Lw = \frac{\partial w}{\partial z} + \bar{w} .$$

This operator does not transform the space of holomorphic functions into itself. The space of generalized analytic functions defined by

$$lw = \frac{\partial w}{\partial \bar{z}} + w - \bar{w} = o$$

is transforms into itself by this operator.

If L is a given differential operator. Then we look for a second operator l, such that $l(Lw) = o$ for all solutions w of $lw = o$. The pair L, l is called associated.

In the paper [9] for given differential operators L with partial complex derivatives associated pairs had been calculated. The calculation can be reduced to the inhomogeneous Cauchy-Riemann system with an additional linear algebraic side condition. Such a problem is overdetermined. In this way we get the following result:

If the coefficients of L fulfill a complex compatibility condition, then there exists an associated operator l.

4. The Cauchy-Kowalewski theorem with generalized analytic functions as initial functions

Let L be a given differential operator (with partial complex derivatives) possessing an associated operator l. Then the following theorem holds (see [8]):

Theorem: If the initial function w_o fulfills the side condition

(4) $\qquad lw = o,$

then the initial value problem

$$(5) \qquad \frac{\partial w}{\partial t} \equiv Lw, \; w(.,o) = w_o$$

possesses a solution, that fulfill the side condition (4) for every t. The solution may be constructed by the method of successive approximations.

The proof is based on the fact, that the initial value problem (*), (**) is equivalent to the integro-differential equation

$$u(t) = u_o + \int_{\tau=0}^{t} (Lu)[\tau]\, d\tau$$

(see F. Treves [6]).

5. The Holmgren theorem for generalized analytic functions

In the holomorphic case the Holmgren theorem says, that every C^1-solution of the corresponding initial value problem must be holomorphic for every t. In the case of generalized analytic functions an analogous statementis the following one:

If the initial function w_o satisfy the side condition (4), then every solution of the initial value problem (5) must satisfy the side condition (4) for every t, too. Thus the solution of (5) is unique.

6. Further generalizations

a) Initial functions belonging to L_p , $p > 2$, are considered in [5] (in [8] the initial functions are assumed to be Hölder-continuous).
b) Initial value problems with pseudoanalytic initial functions (in the sense of L. Bers [1]) are solved in [13] .
c) In [1o] the initial functions are potential vectors in R^3.
d) Monogenic functions in R^n (see F.Brackx, R. Delanghe, and F. Sommen [2]) as initial functions are considered in [12].

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Hint. The book [3] written by an international team of authors will be appear at the end of this year. It discusses the meaning of complex analysis for the mathematics as a whole (its bibliography contains more than 1ooo titles).