Special Normal Form of a hyperbolic CR manifold in $\mathbb{C}^4$ [*†]

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Abstract

In this paper we give a special normal form for a non-semiquadratic hyperbolic CR-manifold M of codimension 2 in $\mathbb{C}^4$, i.e., a construction of coordinates where the equation of M satisfies certain conditions. The coordinates are determined up to a linear coordinate change.

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1 Introduction

In 1907 Poincaré [Poi07] posed the problem of local biholomorphic equivalence of two real-analytic hypersurfaces in $\mathbb{C}^2$. It is interesting to consider this problem for real hypersurfaces in $\mathbb{C}^N$ or, more generally for real-analytic CR-submanifolds in $\mathbb{C}^N$. One approach to this problem, that is due to Chern and Moser [CM74], is the construction of normal forms of the hypersurfaces, i.e., special coordinate systems with the property that certain monomials in the equation of the hypersurface, considered as a power series, vanish. This construction is universal for all hypersurfaces with a given signature of the Levi-form. The normal form condition determines the coordinate system up to a rather complicated action of the finite-dimensional isotropy group of the quadric of maximal osculation. This group is covered by $\mathrm{SU}(n-p, p+1)$, where $n = N-1$ and p is the signature of the Levi-form. If both hypersurfaces are given in normal form the equivalence problem reduces to the (still complicated) problem to find a transformation out of a finite-dimensional variety that maps one hypersurface to the other.

Webster introduced in 1978 [Web78] a *special normal form* that takes into account higher order terms of a given hypersurface. By imposing additional requirements he reduces the freedom in the choice of normal coordinates to a subgroup of $\mathrm{GL}(n, \mathbb{C})$ that acts by *linear* transformations.

In this paper we construct a special normal form for hyperbolic CR-manifolds of codimension 2 in $\mathbb{C}^4$. By hyperbolic CR-manifold we mean a CR-submanifold $M \subset \mathbb{C}^4$ that can be written in the form

$$\begin{aligned} v_1 &= |z_1|^2 + N_1(z, \bar{z}, u) \\ v_2 &= |z_2|^2 + N_2(z, \bar{z}, u), \end{aligned} \tag{1}$$

where z_1, z_2, w_1, w_2 are coordinates in $\mathbb{C}^4$ and $w_j = u_j + iv_j$ and N_j does not contain any terms of order less than 3. Hyperbolic CR-manifolds in C^4 can also be characterised as the strictly pseudo-convex CR-manifolds of codimension 2 in $\mathbb{C}^4$. For such manifolds Loboda constructed a normal form that is analogous to Chern-Moser's normal form for hypersurfaces. Loboda's normal form conditions can be described as follows:

Set

$$N_j(z, \bar{z}, u) = N_j^1(z_j, \bar{z}_j, u_j) + N_j^2(z, \bar{z}, u),$$

where $N_j^1(z_j, \bar{z}_j, u_j)$ consists of all monomials that depend only on $z_j, \bar{z}_j, u_j$. Then

$$N_j^1 = 2\,\mathrm{Re}(h_{42}^j(u) z_j^4 \bar{z}_j^2) + \sum_{\substack{k,l \geq 2 \\ k+l \geq 7}} h_{kl}^j(u) z_j^k \bar{z}_j^l,$$

where $h_{kl}^j(u)$ are real-analytic functions of u. Notice that these are exactly Chern-Moser's conditions for the normal form of a hypersurface in $\mathbb{C}^2$.

For the terms N_2^j the following holds

$$\begin{aligned}
(N_2^j)_{k0} &\equiv 0, \quad \text{for } k \geq 1 & (N_2^j)_{11} &\equiv 0 \\
\frac{\partial}{\partial \bar{z}_j}(N_2^j)_{k1} &\equiv 0, \quad \text{for } k \geq 2 & \frac{\partial^2}{\partial z_1 \partial z_2}(N_2^j)_{21} &\equiv 0 \\
\frac{\partial^4}{\partial z_1 \partial z_2 \partial \bar{z}_1 \partial \bar{z}_2}(N_2^j)_{22} &\equiv 0
\end{aligned}$$

By $\mathcal{N}_0(M)$ we denote the family of biholomorphic, origin preserving transformations that map M (in normal form) into M' that has also normal form. If $\Phi \in \mathcal{N}_0(M): \quad z' = f(z,w)$, $w' = g(z,w)$ then

$$L_C: \ z' = Cz, \qquad w' = \rho w \tag{2}$$

with

$$C = \left(\frac{\partial f}{\partial z}\right)\bigg|_0, \quad \rho = \left(\frac{\partial g}{\partial w}\right)\bigg|_0$$

also belongs to $\mathcal{N}_0(M)$. Moreover, C and ρ must then have the form

$$C = \begin{pmatrix} \lambda_1 e^{i\psi_1} & 0 \\ 0 & \lambda_2 e^{i\psi_2} \end{pmatrix}, \quad \rho = \begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix},$$

or,

$$C = \begin{pmatrix} 0 & \lambda_1 e^{i\psi_1} \\ \lambda_2 e^{i\psi_2} & 0 \end{pmatrix}, \quad \rho = \begin{pmatrix} 0 & \lambda_1^2 \\ \lambda_2^2 & 0 \end{pmatrix},$$

where $\lambda_j > 0$, $\psi_j \in \mathbb{R}$. The group of all (C,ρ)- transformations will be denoted by $\mathrm{GL}(Q)$.

Thus, any $\Psi \in \mathcal{N}_0(M)$ can be decomposed into $\Psi = L_C \circ \Psi_{\mathrm{id}}$ with $\Psi_{\mathrm{id}} \in \mathcal{N}_{0,\mathrm{id}}(M)$, where

$$\mathcal{N}_{0,\mathrm{id}}(M) = \left\{ \Phi \in \mathcal{N}_0(M) : \left. \left(\frac{\partial f}{\partial z} \right) \right|_0 = \mathrm{id} \right\}.$$

It follows from Loboda's theorem that any $\Phi \in \mathcal{N}_{0,\mathrm{id}}(M)$ is uniquely determined by the data

$$a_j := \left. \frac{\partial f_j}{\partial w_j} \right|_0, \quad r_j := \frac{1}{2} \left. \frac{\partial^2 g_j}{(\partial w_j)^2} \right|_0.$$

If M is the hyperbolic quadric

$$M : \ v_1 = |z_1|^2, \quad v_2 = |z_2|^2$$

then for any possible initial data there exists an automorphism, hence any normalisation is automatically an automorphism.

For arbitrary M the right hand side of equation (1) can be expanded into a series of homogeneous polynomials where the weighted degree of each monomial is the sum of the degree with respect to z and $\bar{z}$ and twice the degree with respect to u. If M is non-quadric and given in normal form then there is a smallest weight γ such that the homogeneous component N_γ with weight γ of $N = (N_1, N_2)$ is different from 0. We will refer to this N_γ as to the first non-quadratic term. It can be easily verified that any normalisation $\Phi \in \mathcal{N}_0(M)$ is equal to the quadric automorphism with the same initial data up to terms of weighted degree $\geq \gamma$ (For more elaborate explication see [Sch95]).

Among the non-quadratic hyperbolic manifolds there is a special type of manifolds that are called semi-quadratic manifolds and should be separately dealt with. A semi-quadratic manifolds is given by the equations

$$\begin{aligned} v_1 &= |z_1|^2 \\ v_2 &= |z_2|^2 + N^2(z_2, \bar{z}_2, u_2), \end{aligned}$$

where z_1, z_2, w_1, w_2 are normal coordinates, i.e., they are a Cartesian product of a sphere in $\mathbb{C}^2$ and a non-spherical hypersurface in $\mathbb{C}^2$ that is given in Chern-Moser's normal form.

The construction of a special normal form reduces then to Chern-Moser-Webster's special normal form for hypersurfaces. A coordinate change from one special normal form to another is the a fractional linear sphere automorphism in the variables z_1, w_1 and a linear transformation in the variables z_2, w_2.

Now, we may restrict ourselves to the case of non-semiquadratic manifolds. The construction and the condition of the special normal form for a given non-semiquadratic manifold M depends on the concrete form of the first non-quadratic, resp. non-semiquadratic, term. We will consider 4 different cases of non-semiquadratic manifolds that appeared in [Sch95] where it was proved that for such manifolds there exist normal coordinates where all automorphisms are linear. This is, of course, necessary for a construction of special normal forms with the property that passing to another special normal coordinates is provided by linear transformations because any automorphism is particularly a transformation to other special normal coordinates.

In Section 2 we recall the rough scheme of the proof of the following theorem from [Sch95]:

Theorem 1 *Let M be a non-semiquadratic, real-analytic CR-manifold in $\mathbb{C}^4$. Then any isotropic automorphism $\Phi = (f, g)$, considered as a normalisation, is uniquely determined by the parameter*

$$C = \left.\frac{\partial f}{\partial z}\right|_0$$

Section 3 is devoted to the description of the special normal form and to the proof of the existence and uniqueness (up to linear coordinate changes) special coordinates.

We remark that a special normal form for *non-umbilic* hyperbolic CR manifolds of codimension 2 in $\mathbb{C}^4$ was constructed by Loboda in [Lob90].

2 Sketch of the proof of Theorem 1

The proof of Theorem 1 is very long and very technical. Here we will only sketch the main ideas.

Step 1. One has to show that any automorphism with $C = \mathrm{id}$ is trivial. Any such automorphism can be embedded in some one-parametric subgroup Φ_t. From

$$\operatorname{Im} g_t(z,w) - \langle f_t(z,w), f_t(z,w)\rangle - N(f_t, \bar{f}_t, \operatorname{Re} g_t)\big|_{\operatorname{Im} w = \langle z,z\rangle + N} \equiv 0 \qquad (3)$$

one obtains that the derivative with respect to t for $t=0$ must vanish. Denote this function by $P(z,\bar{z},u)$.

Step 2. The function $P(z,\bar{z},u)$ can be expressed by

$$P(z,\bar{z},u) = 2\operatorname{Re}\chi(v - \langle z,z\rangle - N(z,\bar{z},u))\big|_{\operatorname{Im} w = \langle z,z\rangle + N},$$

where χ is a vector field that is decomposed into a series of homogeneous components with respect to the weight

$$\chi = \chi_1 + \chi_2 + \chi_{\gamma-1} + \cdots$$

Here, one has to take into account that the basis vectors $\frac{\partial}{\partial z_j}$, resp., $\frac{\partial}{\partial w_j}$ have the weight -1, resp., -2. Then the index indicates the weight of the component. The vector fields χ_1 and χ_2 do not depend on M and have the explicit form

$$\begin{aligned}
\chi_1 &= \sum_{j=1,2} \left(a_j w_j + 2i\bar{a}_j (z_j)^2\right) \frac{\partial}{\partial z_j} + 2i\bar{a}_j z_j w_j \frac{\partial}{\partial w_j} \\
\chi_2 &= \sum_{j=1,2} r_j z_j w_j \frac{\partial}{\partial z_j} + r_j (w_j)^2 \frac{\partial}{\partial w_j}.
\end{aligned}$$

Step 3. Besides of the weighted degree, there is another characteristic of a monomial that plays an important rôle in the proof. This characteristic is called *j-defect* (for $j=1,2$) and is defined as the difference of the degree in z_j and in $\bar{z}_j$.

Now, suppose that $a_j \neq 0$. Without loss of generality one can then assume that $a_1 = i$. One has to show that then equation (3) cannot be satisfied. This implies that there are no isotropic automorphisms Φ of M with $\Phi \in \mathrm{Aut}_{0,\mathrm{id}}$. We distinguish four different cases:

1. M has the form

$$\begin{aligned}
v_1 &= |z_1|^2 + N^1(z,\bar{z},u) \\
v_2 &= |z_2|^2 + N^2(z,\bar{z},u),
\end{aligned} \qquad (4)$$

where $N^1_{\gamma_1,\delta_1} \not\equiv 0$ is the term of maximal 1-defect δ_1+1 of the first non-vanishing term of N^1. Then the term in P^1 of weight γ_1 and 1-defect δ_1 takes the form

$$P^1_{\gamma_1+1,\delta_1+1} = \chi_{(1)}\big|_{v_1=|z_1|^2} N^1_{\gamma_1,\delta_1} - 2z_1 N^1_{\gamma_1,\delta_1} + \chi_{\gamma_1-1,\delta_1+1}\big|_{v_1=|z_1|^2}(|z_1|^2 - v_1),$$

where

$$\chi_{(j)} = i\bar{a}_j\left(2(z_j)^2\frac{\partial}{\partial z_j} + 2(u_j+iv_j)z_j\frac{\partial}{\partial w_j} - i(u_j-iv_j)\frac{\partial}{\partial \bar{z}_j}\right)$$

is the part of the vector field χ_1 that increases the j-defect by 1.

Due to (3), this expression must vanish. For the first case we suppose that the sum the terms including N^1 is different from 0. Then they must be compensated by the last term that includes $\chi_{\gamma_1-1,\delta_1+1}$. This turns out to be impossible because then the last term has non-normal components that cannot occur in the first two terms. Hence, $a_1 \neq 0$ implies that the polynomials $P^1_{\gamma_1+1,\delta_1+1}$ cannot vanish, i.e., for a_1 running over $\mathbb{C}$ they form a one-dimensional subspace in the space of polynomials with weight γ_1+1 and j-defect δ_1+1.

2. For the second case we assume that M has the form

$$\begin{aligned} v_1 &= |z_1|^2 \\ v_2 &= |z_2|^2 + N^2(z,\bar{z},u), \end{aligned} \tag{5}$$

where $N^2_{\gamma_2,\delta_2} \not\equiv 0$ is the first non-vanishing term with maximal 2-defect, $N^2_{\gamma_1,\delta_1}$ is the first term that depends on one of the variables $z_1, \bar{z}_1, u_1$ with weight γ_1 and maximal 1-defect δ_1. Then $a_1 \neq 0$ implies $P^2_{\gamma_1+1,\delta_1+1} \neq 0$ (even without the assumption $N^1 \equiv 0$). Furthermore, the second case shall be characterised by $a_2 \neq 0$ implies $P^2_{\gamma_2+1,\delta_2+1} \neq 0$, as above. In this case the polynomials $P^j_{\gamma_j+1,\delta_j+1}$ form one-dimensional subspaces in the spaces of polynomials with weight γ_j+1 and j-defect δ_j+1 for a_j running over $\mathbb{C}$.

3. For the third case M has the form (4) but $a_1 \neq 0$ does not imply that $P^1_{\gamma_1+1} \neq 0$. Then one obtains that $N^1_{\gamma_1}$ must have the special form

$$v_1 = |z_1|^2 + (u_1 + i|z_1|^2)(u_1 - i|z_1|^2)\Pi_2(z_2, \bar{z}_2, u_2) + \cdots,$$

where Π_2 is not constant and depends only on the indicated variables. Then it follows from $a_2 \neq 0$ that $P^1_{\gamma_1+1} \neq 0$. There are two possibilities for N^2: either there is a first component $N^2_{\gamma_2}$ that depends on one of the variables $z_1, \bar{z}_1, u_1$ and $a_1 \neq 0$ implies $P^2_{\gamma_2+1} \neq 0$, or N^2 depends only on $z_2, \bar{z}_2, u_2$. The first alternative characterises the third case. Then one finds out that a_j running over $\mathbb{C}$ produces one-dimensional subspaces of $P^l_{\gamma_j+1,\delta_j+1}$ $(j \neq l)$ in the space of polynomials with weight $\gamma_j + 1$ and j-defect $\delta_j + 1$.

4. The second alternative from above constitutes the fourth case. Then there exists a number k with $2 \leq k \leq \gamma$ such that $P^1_{\gamma_1+k} \neq 0$. We will call M exceptional of order k.

Step 4. If $r_j \neq 0$ then $P_{\gamma+2} \neq 0$.

3 The special normal form

Consider the vector fields

$$\chi_{(j)} = i\bar{a}_j \left(2(z_j)^2 \frac{\partial}{\partial z_j} + 2(u_j + iv_j) z_j \frac{\partial}{\partial w_j} - i(u_j - iv_j) \frac{\partial}{\partial \bar{z}_j} \right) \qquad (6)$$

$$\chi_2^j = r_j w_j z_j \frac{\partial}{\partial z_j} + r_j (w_j)^2 \frac{\partial}{\partial w_j} \qquad (7)$$

$$\chi_\gamma = f_{\gamma+1}(z,w) \frac{\partial}{\partial z} + g_{\gamma+2}(z,w) \frac{\partial}{\partial w}, \qquad \gamma \geq 3, \qquad (8)$$

with $a_j \in \mathbb{C}$, $r_j \in \mathbb{R}$. Notice that $\chi_{(j)}$ increases the j-defect of a monomial by 1 and that

$$\begin{aligned} 2\,\mathrm{Re}\,\chi_1 &= \chi_{(1)} + \chi_{(2)} + \overline{\chi_{(1)}} + \overline{\chi_{(2)}} \\ \chi_2 &= \chi_2^1 + \chi_2^2. \end{aligned}$$

By χ_{γ,δ_j} resp. $\chi_{\gamma,d}$ we will denote the component of j-defect δ_j resp. of *total defect* $d = \delta_1 + \delta_2$ in χ_γ.

We distinguish now four types of manifolds in dependence on the first non-quadratic terms in the expansion of the equation in normal form. Since the first order term of any renormalisation is is a linear mapping L_C as described above, the dependence of the first non-quadratic terms in the equation of M depends in a simple way from the chosen normal coordinates: only the coefficients at the monomials change. Therefore, the definitions below are correct.

These four types of manifolds are according to the four cases in Step 3 of the proof above.

1. Both $N^j \not\equiv 0$ and have not the special form indicated in case 3. Let $N^j_{\gamma_j}$ be the first non-quadratic terms. Collect in $N^j_{\gamma_j}$ the terms of maximal total defect d_j, then within this polynomial the terms of maximal j-defect δ_j and then the terms of maximal weight ω_j with respect to $(z_j, \bar{z}_j, u_j)$. Finally, choose in the resulting expression a polynomial of the form

$$\hat{N}^j_{\gamma_j,d_j} = \tilde{N}^j_{\omega_j,\delta_j}(z_j, \bar{z}_j, u_j)\Pi^j,$$

where Π^j is a monomial in $z_l, \bar{z}_l, u_l$ $(l \neq j)$ of minimal degree with respect to $z_l, \bar{z}_l$. Consider space

$$\mathcal{P}^j = \left\{ P^j_{\gamma_j+1,\delta_j+1} = \chi_{(j)}\hat{N}^j_{\gamma_j,\delta_j} - 2z_j\hat{N}^j_{\gamma_j,\delta_j} + \chi_{\gamma_j-1,\delta_j+1}(|z_j|^2 - v_j)\big|_{v_j=|z_j|^2} \right\}, \tag{9}$$

where $\chi_{\gamma_j-1,\delta_j+1}$ is uniquely determined for given a_j by the condition that P^j consists of normal terms. The arguments from [Sch95] (cf. Section 2) applied to normalisations (instead of automorphisms) show that these spaces are 1-dimensional complex spaces for a_j running over $\mathbb{C}$. We can make use of these arguments because the action of automorphisms and normalisations on the first non-quadratic terms are the same.

Analogously, we define the 1-dimensional real spaces

$$\mathcal{Q}^j = \left\{ Q^j_{\gamma_j+2,\delta_j} = \chi^j_2 \hat{N}^j_{\gamma_j,d_j}\Big|_{v_j=|z_j|^2} \right\}. \tag{10}$$

2. One of the $N^j \equiv 0$, say $N^1 \equiv 0$. Since M is not semi-quadratic, N^2 must then depend on one of the $z_1, \bar{z}_1, u_1$. We define $\hat{N}^2$ in the same way as above. $\hat{N}^1$ will be defined analogously, but starting from $N^2_{\gamma_1,\delta_1}$ the term in N^2 of minimal weight γ_1 and maximal 1-defect δ_1 that depends on one of the variables $z_1, \bar{z}_1, u_1$. Now, we define by (9), resp. (10), the 1-dimensional complex spaces $\mathcal{P}^j$, resp. the 1-dimensional real spaces $\mathcal{Q}^j$.

3. One of the N^j, has the special form from case 3. and, therefore, must depend on one of the variables $z_l, \bar{z}_l, u_l$ $(l \neq j)$, and N^l depends on $z_j, \bar{z}_j, u_j$. $\hat{N}^j$ is then derived from $N^l_{\gamma_j,\delta_j}$, $(l \neq j)$, the component of minimal weight and maximal j-defect that contains one of the variables $z_j, \bar{z}_j, u_j$.

   The spaces $\mathcal{P}^j$ and $\mathcal{Q}^j$ will also be defined by (9) and (10). According to [Sch95], $\mathcal{P}^j$ are complex 1-dimensional for a_j running over $\mathbb{C}$ and the $\mathcal{Q}^j$ are real 1-dimensional for r_j running over $\mathbb{R}$.

4. One of the N^j, say N^1 has the special form from case 3., and N^2 does not depend on $z_1, \bar{z}_1, u_1$. Then there exist an integer k such that the spaces generated by $P^1_{\gamma_1+k,1}$, resp., $P^1\gamma_1 + 1, \delta_2 + 1$ for a_1, resp., a_2 in $\mathbb{C}$ are one-dimensional. In this case $\hat{N}^2$, $\mathcal{P}^2$ and $\mathcal{Q}^j$ are defined as for manifolds of type 3. $\hat{N}^1$ will be derived from $N^1_{\gamma_1,\delta_1}$, $(\gamma_1 = \gamma_2 + k - 1)$, the term of minimal weight and maximal 1-defect that produces a a one-dimensional space of polynomials $P_{\gamma_2+k,1}$.

   Set

$$\begin{aligned}\mathcal{P}^1 = \Big\{ P^1_{\gamma_1+k,\delta_1+1} = \chi_{(j)}\hat{N}^1_{\gamma_j,\delta_j} &- 2z_1\hat{N}^1_{\gamma_1+k-1,\delta_1} \\ &+ \chi_{\gamma_1+k-2,\delta_1+1}\big(|z_1|^2 - v_1\big)\big|_{v_1=|z_1|^2} \Big\}.\end{aligned}$$

In the spaces of polynomials with given weight we introduce an inner product such that the monomials form an orthonormal basis. We will use this inner product in order to choose complementary subspaces.

Now we are ready to give the definition of the s.n.f.

Definition 1 *A non-semiquadratic CR-manifold M has s.n.f. if for some choice of $\hat{N}^j$ the following conditions are satisfied:*

If M is of type 1, then

$$N^j_{\gamma_j+1,\delta_j+1} \in \mathcal{P}^{j\perp}, \text{ and } N^j_{\gamma_j+2,\delta_j} \in \mathcal{Q}^{j\perp}.$$

If M is of type 2, then

$$N^2_{\gamma_j+1,\delta_j+1} \in \mathcal{P}^{j\perp}, \text{ and } N^2_{\gamma_j+2,\delta_j} \in \mathcal{Q}^{j\perp}.$$

For M of type 3

$$N^l_{\gamma_j+1,\delta_j+1} \in \mathcal{P}^{j\perp}, \text{ and } N^l_{\gamma_j+2,\delta_j} \in \mathcal{Q}^{j\perp}, \qquad (l \neq j),$$

and for M of exceptional type

$$N^1_{\gamma_1+k,\delta_1+1} \in \mathcal{P}^{1\perp}, \qquad N^1_{\gamma_1+1,\delta_2+1} \in \mathcal{P}^{2\perp}, \text{ and } N^j_{\gamma_j+2,\delta_j} \in \mathcal{Q}^{j\perp}.$$

By definition M has also s.n.f. if its equation can be obtained from one of the equations above by interchanging z_1 with z_2 and w_1 with w_2.

Theorem 2 *If M is a hyperbolic non-semiquadratic CR-manifold in $\mathbb{C}^4$ of codimension 2 then there exist coordinates such that the equation of M has special normal form (s.n.f.). The coordinates where M has s.n.f. are determined up to a linear transformation of the form (2).*

Proof. Let M be given in normal form in sense of Loboda and let $\Phi_a = (f_a, g_a)$ be a normalisation of M with parameters $(C = \mathrm{id}, a = (a_1, a_2), r = 0)$. The expansion of this normalisation into weighted homogeneous components differs from a quadric automorphism only starting from term of weight γ in f_a and of weight $\gamma + 1$ in g_a. The spaces $\mathcal{P}^j$ were chosen so that for suitable a_1, a_2 the $\mathcal{P}^j$-component of the corresponding term in the new equation can be made equal to zero.

After performing a normalisation with parameters $(C = \mathrm{id}, a = 0, r = (r_1, r_2))$ with suitable r_j the condition that involves the $\mathcal{Q}^j$ will be also satisfied.

Now we turn to the proof of the uniqueness statement. By inspection one easily verifies that any linear mapping of the form (2) preserves the s.n.f.

Now, let Φ be a local coordinate change such that the new equation has also s.n.f. Then Φ is a renormalisation and its linear part is equal to some L_C. The mapping $L_C^{-1} \circ \Phi$ is also a coordinate change that takes M to s.n.f. and its linear part is the identity. Such renormalisations act on the first non-quadratic terms like quadric automorphisms and, therefore, would produce non-normal terms for one of the a_j, r_j being different from 0. This shows that $a_j = r_j = 0$. But, according to Loboda's theorem, then $L_C^{-1} \circ \Phi = \mathrm{id}$, i.e., $\Phi = L_C$. $\square$

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