26. MATHEMATISCHE ARBEITSTAGUNG

1986

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str.26
5300 Bonn 3

Mathematisches Institut
der Universität Bonn
Wegelerstr. 10
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MPI 86-26

Inhalt

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Programm der Mathematischen Arbeitstagung 1986 (I)

Freitag, den 13.6.1986

16.00 - 17.00 Uhr	M.F. ATIYAH: The logarithm of the Dedekind η-function

Samstag, den 14.6.1986

10.00 - 11.00 Uhr	CH. SOULÉ: Higher dimensional Arakelov theory
11.45 - 12.45 Uhr	M. KRECK: 7-dimensional Einstein manifolds with $SU(3) \times SU(2) \times U(1)$ - symmetry
17.00 - 18.00 Uhr	K. FUKAYA: Collapsing and eigenvalues

Sonntag, den 15.6.1986

9.45 - 10.00 Uhr	Festlegung der nächsten Vorträge
10.00 - 11.00 Uhr	A. FLOER: Holomorphic curves and fixed points of symplectic maps
11.45 - 12.45 Uhr	P. KRONHEIMER: Gravitational instantons and Kleinian singularities
17.00 - 18.00 Uhr	G. FALTINGS: Hodge-Tate structures

Die Vorträge finden alle im "*Großen Hörsaal*", Wegelerstraße 10, statt.

Erfrischungspausen mit Tee: Samstag und Sonntag 11.00 - 11.30 Uhr vor dem Großen Hörsaal und ab 15.00 Uhr im Max-Planck-Institut.

Post, die nicht ans MPI adressiert ist, liegt während der Teepausen vor dem Großen Hörsaal aus. Alle Teilnehmer mögen sich bitte in die *Teilnehmerlisten* eintragen Teilnehmerlisten und *Informationen* liegen vor dem Großen Hörsaal aus.

Für Diskussionen stehen das *MPI*, der *Diskussionsraum* Beringstraße 1 und der *Sitzungssaal* (Raum 4) Beringstraße 4 zur Verfügung. *Tischtennis* im Keller des Hauses Beringstraße 4.

Den *Tagungsbeitrag* bitte an *Frau Karge* bezahlen (Samstag und Sonntag 13.00 - 17.00 Uhr im Empfang des MPI).

Alle Tagungsteilnehmer mit ihren Damen oder Herren sind herzlich zum *Empfang des Rektors* eingeladen. Zeit: Freitag, 13.6.1986, 20.00 Uhr. Ort: Festsaal der Universität, Hauptgebäude; Eingang von der Straße "Am Hof" durch das Tor gegenüber Buchhandlung Röhrscheid.

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Programm der Mathematischen Arbeitstagung 1986 (II)

Montag, den 16.6.1986

10.30 - 11.30 Uhr	M. KERVAIRE: Jones's knot polynomials
13.00 Uhr	Schiffsausflug nach Bad Hönningen. Abfahrt pünktlich um 13.00 Uhr mit Motorschiff "Carmen Silva" am Alten Zoll. Rückkehr ca. 19.30 Uhr.

Dienstag, den 17.6.1986

10.00 - 10.15 Uhr	Festlegung der nächsten Vorträge
10.15 - 11.15 Uhr	K. RIBET: Modular forms, ℓ-adic representations and Fermat's last theorem
11.45 - 12.45 Uhr	M.F. ATIYAH: Donaldson invariants for 4-manifolds
15.30 - 16.30 Uhr (informal talk)	F.A. BOGOMOLOV: Rationality problems, stable cohomology
17.00 - 18.00 Uhr	U. PINKALL: New minimal surfaces in S^3

Mittwoch, den 18.6.1986

10.15 - 11.15 Uhr	H. KNÖRRER: Fermi curves and density of states

Die Vorträge finden alle im *"Großen Hörsaal"*, Wegelerstr. 10, statt.

Erfrischungspausen mit Tee: Dienstag 11.00 - 11.30 Uhr und 16.30 - 17.00 Uhr vor dem Großen Hörsaal.

Post, die nicht ans MPI adressiert ist, liegt während der Teepausen vor dem Großen Hörsaal aus.

Alle Tagungsteilnehmer mögen sich bitte in die *Teilnehmerlisten* eintragen. *Teilnehmerlisten* und *Informationen* liegen vor dem Großen Hörsaal aus.

Den *Tagungsbeitrag* bitte an *Frau Karge* bezahlen (Dienstag von 13.00 - 17.00 Uhr im Empfang des MPI).

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Programm der Mathematischen Arbeitstagung 1986 (III)

Mittwoch, den 18.6.1986

16.30 - 17.30 Uhr	G. WÜSTHOLZ: Baker's method and effectivity
17.45 - 18.45 Uhr	R. LEE: Finite group actions on the complex projective plane

Donnerstag, den 19.6.1986

10.15 - 11.15 Uhr	J. BRÜNING: L^2-index theorems for regular singular operators
12.00 - 13.00 Uhr	H. ESNAULT: Characteristic classes of flat bundles
17.00 - 18.00 Uhr	F. HIRZEBRUCH: 3-folds with $c_1 = 0$

Freitag, den 20.6.1986

12.30 - 13.30 Uhr	M. REID: Progress in 3-folds (Sondervortrag)
16.30 - 17.30 Uhr	H. KURKE: Modulräume von Vektor-Bündeln (Kolloquiumsvortrag)

Der *Sondervortrag* findet im Seminarraum des MPI für Mathematik statt. Der *Kolloquiumsvortrag* findet im "*Kleinen Hörsaal*", Wegelerstr. 10 statt. Alle *anderen Vorträge* finden im "*Großen Hörsaal*" statt.

Post, die nicht ans MPI adressiert ist, lieg während der Teepausen aus.

Erfrischungspausen mit Tee: Mittwoch, 16.00-16.30 vor dem Großen Hörsaal. Donnerstag 11.15 - 11.45 vor dem Großen Hörsaal, ab 15.00 Uhr im MPI. Freitag, 16.00 - 16.30 Uhr vor dem Großen Hörsaal.

Informationen liegen vor dem Großen Hörsaal aus.

Den *Tagungsbeitrag* bitte an *Frau Karge* bezahlen (Mittwoch und Donnerstag von 13.00 - 17.00 in Raum 20-4 des MPI).

Titel: THE LOGARITHM OF THE DEDEKIND η- FUNCTION

Autor: M. F. ATIYAH

Adresse: MATHEMATICAL INSTITUTE, OXFORD.

The Dedekind η-function is defined by

$$\eta(z) = e^{\frac{\pi i z}{12}} \prod_{n=1}^{\infty} \left(1 - e^{2\pi i n z}\right), \quad z \in H$$

It is a non-vanishing holomorphic function in the upper half-plane and η^{24} is a modular form of weight 12 for $SL(2,Z)$. The problem of investigating the action of $SL(2,Z)$ on $\log \eta(z)$ was studied by Dedekind and explicitly solved in terms of Dedekind sums. Since then this question has been studied by many people including Rademacher, W. Meyer, Hirzebruch, Zagier, Atiyah-Singer-Patodi, Donnelly. Very recently it also links with ideas from physics due to Witten and investigated mathematically by Quillen, Bismut and Freed.

Here I try to present a coherent approach on geometrical lines to these old ideas.

We consider only hyperbolic elements of $SL(2,\mathbb{Z})$: elliptic and parabolic elements are rather special. If $A \in SL(2,\mathbb{Z})$ has real fixed points α, β, then $\omega_A = \frac{du}{u}$ with $u = (z-\alpha)/z-\beta$ is an A-invariant differential and $F(z) = \eta(z)^{24} dz^6 / \omega_A^6$ is an A-invariant function. Hence we can define an invariant $\varphi(A) \in \mathbb{Z}$ by

$$F(Az) - F(z) = 2\pi i \, \varphi(A).$$

Now let X be a compact Riemann surface with boundary consisting of a number of circles S_i. Let $\alpha : \pi_1(X) \to SL(2,\mathbb{Z})$ be a homomorphism, and E_α the associated local coefficient system on X. Then $H^1(X, \partial X; E_\alpha)$ has a <u>quadratic form</u> and hence a signature, denoted by $\mathrm{sign}(\alpha)$. Assume for simplicity that the conjugacy class A_i associated by α to each S_i is hyperbolic. Then one has

$$\mathrm{sign}(\alpha) = 2 \sum_i \varphi(A_i) \qquad \text{[W. Meyer]}$$

This formula can be proved by applying the index theorem for elliptic operators to $(X, \partial X)$. The boundary contribution is zero because each A_i, being hyperbolic, is conjugate to A_i^{-1}.

Moreover this proof extends to the case when $SL(2,\mathbb{Z})$ is replaced by any discrete group Γ of $SL(2,\mathbb{R})$ with the property that $H/\Gamma = C$, with finitely many branch points.

We can also construct a 4-manifold T_α fibred over X, with fibre the 2-torus, and boundary Y the union of 3-manifolds $Y(A_i)$, each fibred over the circle. The index theorem for (X,Y) gives a formula

$$\operatorname{sign}(\alpha) = \operatorname{sign}(T_\alpha) = \frac{1}{3}\int_{T_\alpha} p_1 - \sum_i L_i$$

when we use a certain natural metric near Y.

The L_i are the values for $s=0$ of the spectral L-function associated to a differential operator on $Y(A_i)$. By a rescaling argument, involving replacing each torus by its quotient by points of period n, and letting $n \to \infty$ one finds the integral term vanishes. Comparison with Meyer's formula then shows that $L_i = -2\Phi(A_i)$.

<u>Note</u> By direct analysis one also can identify L_i with the value at $s=0$ of classical L-functions of the quadratic field defined by the eigenvalues of A_i.

Computation of $\varphi(A)$

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $c > 1$, then there is an elementary formula for $\varphi(A)$ in terms of $a+d = \mathrm{Tr}\,A$.

In general, for $c > 1$, let

$$B = \begin{pmatrix} a & bc \\ 1 & d \end{pmatrix}$$

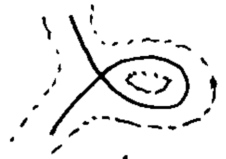
Then $Y(B) \to Y(A)$ is a c-fold covering, and one can investigate $\varphi(A)$ by looking at $\varphi(B) - c\,\varphi(A)$, the deviation from multiplicativity.

For the spectral L-invariant this problem in general has been much studied and can be solved if we can find a 4-manifold W with $\partial W = Y(B)$ and extend the action of the covering group to W with fixed points.

The answer involves in particular the fixed-point contributions in the G-signature theorem, where each fixed point contributes terms like $\cot \alpha/2 \cdot \cot \beta/2$, where α, β are rotation angles.

It is these quantities which are related to Dedekind sums (as noted by Hirzebruch + Zagier).

In our case one constructs W easily by taking a nodal rational curve with normal bundle having degree $-\operatorname{tr} A$. Then W is a neighbourhood of the zero section, and the node is the unique fixed point.



Remarks .1) This W is a special case of the Hirzebruch resolution of cusp singularities for Hilbert modular surfaces.

2) Actually $Y(B) \to Y(A)$ is not a Galois covering, so the argument needs to be modified slightly.

The relation with work of Witten etc, arises from the fact that

$$\eta(z)^2 = \det{}'(\bar\partial_z)$$

where $\bar\partial_z$ is the $\bar\partial$-operator of the elliptic curve defined by $z \in H$ and $\det'$ indicates a regularized determinant (excluding the 0-eigenvalue, coming from constants). Our invariant $\Phi(A)$ represents a variation in phase of $\det'(\bar\partial_z)$ along a path in H. The relation with the L-function is a special case of a theorem of Bismut & Freed.

REF. H. RADEMACHER, Math. Zeitschr. 63 (1956) 445-463.

Titel: Higher dimensional Arakelov Theory

Autor: Soulé, Christophe.

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In a joint work with H. Gillet, we tried to develop an algebraic
geometry of hermitian vector bundles which would extend to
higher dimensions the work of Arakelov, Szpiro and Faltings on the
intersection theory over arithmetic surfaces. At the Arbeitstagung of 1984,
Manin asked whether Faltings Riemann-Roch theorem for hermitian line bundles
over arithmetic surfaces was a special case of an arithmetic Riemann-Roch
-Grothendieck theorem. We can phrase such a statement.

1. <u>Intersection theory</u>:

1.1. Define an <u>arithmetic variety</u> to be a regular scheme which is quasi-projective
and flat over $\mathbb{Z}$ (notice however that what we shall say about such varieties
remains valid, mutatis mutandis, for smooth complex analytic manifolds).
Given an arithmetic variety X and $p \geq 0$ an integer, a codimension p <u>cycle</u>
on X is a finite sum $\sum_\alpha n_\alpha Z_\alpha$, where $n_\alpha \in \mathbb{Z}$ and Z_α is an integral codimension
p closed subscheme in X. Let δ_Z be the current on the complex variety
$X(\mathbb{C})$ of complex points of X which sends a form η to $\sum_\alpha n_\alpha \int_{Z_\alpha(\mathbb{C})} \eta$.
A real current g of type $(p-1, p-1)$ is said to be a <u>Green current</u>
for Z when the following equation holds

$$(1.1.1.) \qquad \frac{1}{\pi i}\partial\bar\partial g = \delta_Z - \omega(Z,g),$$

where $\omega(Z,g)$ is a smooth (p,p) form over $X(\mathbb{C})$. Let us define $\hat{Z}^p(X)$ to be
the group of pairs (Z,g), where Z is a codimension p cycle and g a Green
current for Z. Given a closed integral subscheme Y of codimension $p-1$ in X,
and a nonzero rational function f on Y, the pair $(\mathrm{div}(f), \log|f|) = \widehat{\mathrm{div}}(f)$
is an element of $\hat{Z}^p(X)$. Here $\mathrm{div}(f)$ is the divisor of f and $(\log|f|)(\eta)$ is
the integral of $\log|f|\,\eta$ on the smooth part of $Y(\mathbb{C})$. The <u>arithmetic Chow group</u>
$\widehat{CH}^p(X)$ is the quotient of $\hat{Z}^p(X)$ by the subgroup generated by these $\widehat{\mathrm{div}}(f)$'s

and by pairs of type $(0, \partial u + \bar\partial v)$.

Remark: to recover Arakelov's original definition take $p = 1$, $\dim X = 2$ and assume that $\omega(Z, g)$ is harmonic with respect to a fixed Kähler metric on $X(\mathbb{C})$.

1.2. Any morphism $f : X \to Y$ between arithmetic varieties induces a pull-back map $f^* : \widehat{CH}^p(Y)_{\mathbb{Q}} \longrightarrow \widehat{CH}^p(X)_{\mathbb{Q}}$, $p \geq 0$, where $A_{\mathbb{Q}} = A \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$.

A proper map $f : X \to Y$ which is surjective and generically smooth induces a direct image $f_* : \widehat{CH}^p(X) \longrightarrow \widehat{CH}^{p+\delta}(Y)$, $\delta = \dim(Y) - \dim(X)$.

We can also define a intersection cup-product

$$\widehat{CH}^p(X)_{\mathbb{Q}} \otimes \widehat{CH}^q(X)_{\mathbb{Q}} \xrightarrow{\ \cup\ } \widehat{CH}^{p+q}(X)_{\mathbb{Q}}.$$

It is given by the formula

(1.2.1.) $$(Z, g) \cup (Z', g') = (Z \cap Z', g\delta_{Z'} + \omega(Z, g) g').$$

In this formula, to make sense of $g\,\delta_{Z'}$ one chooses g to be smooth outside $Z(\mathbb{C})$ and to have logarithmic growth along $Z(\mathbb{C})$. The Stokes formula shows that this product is commutative.

1.3. To make sense of $Z \cap Z'$ in the formula (1.2.1.) (in the absence of a moving lemma over X) we use the isomorphism $CH^p_Y(X)_{\mathbb{Q}} \simeq K_0^Y(X)^{(p)}$. Here X can be any regular noetherian scheme of finite dimension, $CH^p_Y(X)$ is the Chow of codimension p cycles on X with support in Y and $K_0^Y(X)^{(p)}$ is the subspace of the K-theory of X with support in Y (tensored with $\mathbb{Q}$) where all Adams operations ψ^k act by multiplication by k^p. The tensor product of modules gives a multiplicative structure on $K_0^Y(X)$ and $\psi^k(xy) = \psi^k(x)\psi^k(y)$, therefore we get a cup-product

$$CH^p_Y(X)_{\mathbb{Q}} \otimes CH^q_Y(X)_{\mathbb{Q}} \longrightarrow CH^{p+q}_Y(X)_{\mathbb{Q}}.$$

1.4. This approach to intersection theory using Adams operations led us to the following result (conjectured by Serre, and proved by another method by P. Roberts).

Theorem: Let A be a regular noetherian local ring, M and N two finitely generated A-modules such that the length $\ell(M \underset{A}{\otimes} N)$ of their tensor product is finite. Assume that $\operatorname{codim} \operatorname{Supp}(M) + \operatorname{codim} \operatorname{Supp}(N) > \dim(A)$.

Then $\sum_{i\geq 0} (-1)^i \, \ell\,(\mathrm{Tor}_i^A(M,N)) = 0$.

2. Hermitian vector bundles:

2.1. Let (L,h) be the pair of a line bundle L on the arithmetic variety X and an hermitian metric h on $L(\mathbb{C})$, the holomorphic line bundle over $X(\mathbb{C})$ attached to L. Given any rational section s of L, the pair $(\mathrm{div}(s), \log\|s\|)$ lies in $\hat{Z}^1(X)$ ($\|\cdot\|$ is the norm attached to h) and its class $\hat{c}_1(L,h)$ in $\widehat{CH}^1(X)$ does not depend on the choice of s. This class $\hat{c}_1(L,h)$ determines (L,h) up to algebraic isomorphisms which preserve the metrics.

2.2. Given a pair (E,h) of a vector bundle E on X and an hermitian metric h on $E(\mathbb{C})$, we can define a Chern character $\widehat{ch}(E,h)$ in $\widehat{CH}(X)_{\mathbb{Q}} = \bigoplus_{p\geq 0} \widehat{CH}^p(X)_{\mathbb{Q}}$. This Chern character commutes with inverse image, it is additive under orthogonal direct sums, multiplicative under tensor products and satisfies the equality $\widehat{ch}(L,h) = \exp \hat{c}_1(L,h)$ for any hermitian line bundle (L,h). Let $A^{pp}(X)$ be the space of real (p,p) forms and $\omega : \widehat{CH}^p(X) \to A^{pp}(X)$ the map which sends (Z,g) to the form $\omega(Z,g)$ defined by (1.1.1.). One gets that $\omega(\widehat{ch}(E,h))$ is $\mathrm{tr}\left(\exp\left(-\frac{1}{2\pi i}\nabla^2\right)\right)$, where ∇ is the unique connection on $E(\mathbb{C})$ which is unitary and compatible with the complex structure.

2.3. Let

$$\mathcal{E} : \quad 0 \to S \to E \to Q \to 0$$

be an exact sequence of vector bundles over X. Let h', h, h'' be arbitrary metrics on $S(\mathbb{C})$, $E(\mathbb{C})$, $Q(\mathbb{C})$ respectively. Bott and Chern defined a secondary class $\widetilde{ch}(\mathcal{E})$ in $\mathbf{A}(X) = \bigoplus_{p\geq 0} A^{pp}(X) / (\mathrm{Im}\,\partial + \mathrm{Im}\,\bar\partial)$ such that

$$\frac{1}{\pi i}\partial\bar\partial\, \widetilde{ch}(\mathcal{E}) = ch(S,h') + ch(Q,h'') - ch(E,h).$$

In $\widehat{CH}(X)_{\mathbb{Q}}$ the following equality holds:

$$\widehat{ch}(E,h) - \widehat{ch}(S,h') - \widehat{ch}(Q,h'') = (0, \widetilde{ch}(\mathcal{E})).$$

Define a group $\widehat{K}_0(X)$ as generated by triples $(E,h;\eta)$ where (E,h) is a hermitian vector bundle on X and $\eta \in \tilde{A}(X)$, with relations

$$(S,h';\eta') + (Q,h'';\eta'') = (E,h;\eta'+\eta''-\widetilde{ch}(\mathcal{E})),$$

for any exact sequence $\mathcal{E}$ and forms η', η'' in $\tilde{A}(X)$. The map $\widehat{ch}$ extends to a map $\widehat{ch} : \widehat{K}_0(X)_{\mathbb{Q}} \to \widehat{CH}(X)_{\mathbb{Q}}$ sending $(E,h;\eta)$ to $\widehat{ch}(E,h)+(0,\eta)$. When X is projective this is an isomorphism.

3. Direct image of hermitian vector bundles:

3.1. Let X and Y be arithmetic varieties, and $f : X \times Y \to X$ the first projection. We assume that Y is projective and equipped with a Kähler form ω_Y. We want to define a direct image morphism $f_! : \widehat{K}_0(X\times Y) \to \widehat{K}_0(X)$. Let (E,h) be a hermitian vector bundle on X. Assume that $R^k f_* E = 0$ for all $k>0$. The direct image sheaf $f_* E$ then defines a vector bundle on X. Let $f_* h$ be the L^2-metric on $f_* E$ (obtained by integration on the fibers of f). We shall define an analytic torsion form $T \in \tilde{A}(X)$ such that

$$(3.1.1.) \quad \frac{1}{\pi i}\partial\bar{\partial} T = ch(f_*E, f_*h) - f_*(ch(E,h)\,Td(Y,\omega_Y)),$$

and $f_!(E,h) \in \widehat{K}_0(X)$ will be the class of $(f_*E, f_*h; T)$.

3.2. In degree zero T is the Ray-Singer analytic torsion, as proposed by Quillen. Namely, fix a point $x \in X(\mathbb{C})$ and let Δ^q be the Laplace operator on $(0,q)$ forms of $Y(\mathbb{C}) = Y(\mathbb{C}) \times \{x\}$ with coefficients in E. Let $\zeta_q(s) = \mathrm{Tr}(\Delta^q)^{-s}$ be the zeta function of Δ^q (it is well defined when $\mathrm{Re}(s)$ is big enough and extends meromorphically to the whole complex plane). Denote by $\gamma = -\Gamma'(1)$ the Euler constant. Then

$$T^0(x) = \sum_{q\geq 0} (-1)^q\, q\, (\zeta_q'(0) - \gamma\, \zeta_q(0)) - \gamma \dim_{\mathbb{C}} (f_*E)_x .$$

3.3. The complete definition of $T \in \tilde{A}(X)$ is inspired by the work of Quillen and Bismut-Freed. Let D^q be the (infinite dimensional) C^∞ bundle on X whose sections on U are sections on $f^{-1}(U)$ of $E \otimes A^{0q}_Y$. Let $D^{-1} = f_*E$,

$j : D^{-1} \to D^0$ the natural inclusion, $D_+ = \bigoplus_{q \text{ even}} D^q$ and $D_- = \bigoplus_{q \text{ odd}} D^q$. For any $q \geq -1$, D^q is equipped with a L^2 metric. Let $p : X \times \mathbb{C}^* \to X$ be the first projection, $u : p^* D_+ \to p^* D_-$ the map $z\bar\partial_Y + \bar z \bar\partial_Y^* + \bar z j^*$ (where $z \in \mathbb{C}^*$ and $*$ means taking the adjoint), and $\nabla_\pm$ the connection on $D_\pm$ obtained by lifting horizontally the ~~vectors~~ tangent vectors from $X \times \mathbb{C}^*$ to $X \times Y \times \mathbb{C}^*$. We endow the graded bundle $p^*D_+ \oplus p^*D_-$ with the superconnection

$$\tilde\nabla = \left(\begin{array}{c|c} \nabla_+ & iu^* \\ \hline iu & \nabla_- \end{array}\right).$$

Let $\omega(z)$ be the supertrace $\mathrm{tr}_s \exp(\tilde\nabla^2)$.

For any real number $r > 0$ the integral

$$I(r) = \int_{|z|^2 \geq r} \omega(z) \log|z|$$

converges and defines a form on X. When r tends to zero, $I(r)$ has an asymptotic development with finitely many divergences of type r^{-j-1} and $r^{-j}\log(r)$, $j \geq 0$. Let $I(0)$ be the finite part of $I(r)$ as r goes to zero and $T = I(0)'$, where, given α in $A^{pp}(X)$, we set $\alpha' = \left(\frac{-1}{2\pi i}\right)^{p+1} \alpha$.

3.4. Using Bismut's local index theorem for families one can prove (3.1.1.). Furthermore $f_!$ extends to a morphism $f_! : \widehat{K}_0(X \times Y) \to \widehat{K}_0(X)$ such that $\omega(\widehat{ch}(f_! \alpha)) = f_*(\omega(ch\, \alpha)\, Td(Y, \omega_Y))$. We expect that a Riemann-Roch-Grothendieck theorem with values in $\widehat{CH}(X)_{\mathbb{Q}}$ holds, which would involve the different notions we introduced. Cases of it were obtained by Deligne and Belavin-Knižnik.

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Titel: 7-dimensional Einsteins Manifolds with SU(3) x SU(2) x U(1)-symmetry.

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It is an interesting open question which smooth manifolds admit Einstein metrics and how many. Except in dimension 4 where Hitchin proved that for an Einstein manifold $|\mathrm{sign}\,(M)| \leq 2/_3\,\chi(M)$ no obstructions are known. On the other hand only for rather few classes of manifolds one has a positive answer for the existence of an Einstein metric (for instance Calabi-Yau manifolds). In the recent years some families with positive scalar curvature were constructed by M. Wang and W. Ziller [M.Z.], mainly homogeneous spaces but also some others. They asked for a classification of the diffeomorphism type of these families which is of some importance for the question how many Einstein metrics these manifolds have.

Independently some of these families where also introduced by theoretical physists. Especially E. Witten [W] in his study of 11-dimensional Kaluza-Klein theory considered 7-dimensional closed manifolds M with $G = SU(3) \times SU(2) \times U(1)$-symmetry. By this we mean that the group of isometries (with respect to some Riemannian metric on M) is G and that G acts transitively implying M is a homogeneous space.

Although there are some serious difficulties with this version of Kaluza-Klein theory (the physisists changed their viewpoint in the meantime from 11 to 10 dimensions) one should analyse the differential topology of these manifolds occuring in physics and differential geometry.

More precisely the manifolds introduced by Witten are homogeneous spaces $SU(3) \times SU(2) \times U(1)/_{SU(2) \times U(1) \times U(1)}$ for the different embeddings of subgroups. In this note we restrict ourself to the simply connected case (in general the fundamental group is finite cyclic). From the point of view of differential topology or of algebraic geometry the following description of Witten is more appropriate : The manifolds are total spaces $M_{p,q}$ of S^1-bundles over $\mathbb{C}P^2 \times \mathbb{C}P^1$ where the first Chern class of this bundle is $qx + py$ for x and y generators of $H^2(\mathbb{C}P^2)$ and $H^2(\mathbb{C}P^1)$. To guarantee that the symmetry group is not bigger than G one has to exclude the cases where p or q is zero and $M_{p,q}$ is 1-connected iff $(p,q) = 1$.

The cohomological structure of $M_{p,q}$ is a follows:

$H^2(M_{p,q}) \cong \mathbb{Z}\alpha$, $H^3(M_{p,q}) = \{0\}$, $H^4(M_{p,q}) \cong \mathbb{Z}_{q^2}\cdot\alpha^2$.

The linking form is given by $L(\alpha^2, \alpha^2) = p^{-3} \bmod q^2 \in \mathbb{Z}_{q^2}$. The relevant characteristic classes are $w_2(M_{p,q}) = p \bmod 2$ and $p_1(M_{p,q}) = 3\, p^2\alpha^2 \bmod q^2$.

Theorem: Let q be prime to 3. a) The diffeomorphism type of a Spin-manifold M of the form $M_{p,q}$ is determined by two invariants:

$|H^4(M;\mathbb{Z})|$ and

$S(M) = S(M,g) := \frac{1}{8}\eta(B,0)+14(\dim \mathrm{Ker} D + \eta(D,0) - \frac{1}{2^5}\int_M p_1(g)\wedge h \in \mathbb{Q}/_{28\mathbb{Z}}$ where g is some metric on M, B and D are the signature and Dirac operator, η is the invariant of Atiyah-Patodi-Singer [A.P.S.],

$p_1(g)$ is the Pontajagin form and h is a 3-form s.t. $dh = p_1(g)$.

b) The homeomorphism type is determined by $|H^4(M;\mathbb{Z})|$ and $\bar{S}(M) = [S(M)] \in \mathbb{Q}/\mathbb{Z}$.

c) The values of these invariants in terms of p and q are:

$|H^4(M_{p,q})| = q^2$

$$S(M_{p,q}) = \frac{3p(q^2+3)(q^2-1)}{2^5\, q^2} \in \mathbb{Q}/_{28\mathbb{Z}}$$

We also have a classification if 3 divides q and in the non-Spin case but the formulation is more complicated.

Corollary: If p is even and $3 \nmid q$ then $M_{p,q}$ is diffeomorphic to $M_{p',q'} \Leftrightarrow q = \pm q'$ and

	$16 \nmid q^2-1$	$16 \mid q^2-1$
$q \equiv 0,3,4 \bmod 7$	$p' \equiv p \bmod 28\, q^2$	$p' \equiv p \bmod 14\, q^2$
$q \equiv 1,2,5,6 \bmod 7$	$p' \equiv p \bmod 4\, q^2$	$p' \equiv p \bmod 2\, q^2$

$M_{p,q}$ is homeomorphic to $M_{p',q'} \Longleftrightarrow q=\pm q'$ and $p'\equiv p \mod 2\, q^2$. The statement of this Corollary holds also if 3 divides q (This result was misstated in the talk).

Since the middle of the 60th the Hsiang brothers asked at various occasions [H.H.] whether homeomorphic homogeneous spaces are diffeomorhic. Our result shows that this in general not true:

Corollary: There exist homeomorphic homogeneous spaces which are not diffeomorphic.

Remark: As $H^3(M_{p,q}) = \{o\}$, general smoothing theory [K.S.] implies that the maximal number of smooth structures on such a 7-manifold is 28 and if two such manifolds M and M' are homeomorphic there exists a homotopy sphere Σ s.t. $M \# \Sigma$ is diffeomorphic to M'. Our classification result in the non-Spin case implies that $M_{19,56}$ has 28 different smooth structures which are all homogeneous spaces G/H with different embeddings of H into G.

Corollary: There exist Einstein manifolds (with positive scalar curvature) which have an exotic structure which is again Einstein (with positive scalar curvature)

Corollary: For all $M_{p,q}$ ($(p,q)=1$) the moduli space of Einstein metrics has infinitely many components.

This follows immediately from our Theorem and the results of [W.Z.] where this result is contained for the case q = 1.

Remark (M.F. Atiyah): The simplest manifolds with exotic smooth structure, the 7-spheres, have two natural descriptions: they are total spaces of S^3-bundles over S^4 (Milnor) and links of isolated singularities (Brieskorn). No direct proof is known that those constructions give the same homotopy spheres. If $p < o$ and $q < o$ the manifolds $M_{p,q}$ unify both aspects: They are total spaces of S^1-bundles over $\mathbb{C}P^2 \times \mathbb{C}P^1$ or if we project to $\mathbb{C}P^2$ of Lens-space bundles over $\mathbb{C}P^2$. And they are links of isolated singularities: Consider the associated disk-bundle $W_{p,q}$ and blow the zero-section down to obtain an isolated singularity with link $M_{p,q}$.

The proof of the Theorem is based on the modified surgery theory of [K]. In our situation this theory deals with the classification of $M_{p,q}$'s together with a map $M \longrightarrow \mathbb{C}P^\infty$ inducing an isomorphism on H^2

Up to sign and homotopy such a map is unique. If $F: W \to \mathbb{C}P^\infty$ is a bordism between M and M' there is an obstruction $\theta(W)$ in a monoid $\ell_0(\{e\})$ for transforming W into an h-cobordism. The monoid $\ell_0(\{e\})$ is in general very complicated but if we assume that M and M' have same linking form and Pontrjagin class it is shown in [K] that $\theta(W)$ is completely determined by relative characteristic numbers of W. In the case where $W = W_{p,q} - W_{p',q'}$ is the union of the corresponding disk-bundles these numbers can easily be computed and are non-zero. The diffeomorphism classification is derived from this information by computing $\Omega_8^{Spin}(\mathbb{C}P^\infty)$ (using Riemann-Roch theorems) which determines the possible variations of the characteristic numbers of W for different bordisms.

The formula for $S(M_{p,q})$ is proved with the Atiyah-Patodi-Singer Index theorem for the Signature and Dirac operator in the case of compact manifolds with boundary.

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Titel: Collapsing and eigenvalues

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We shall study the eigenvalues $0 = \lambda_0(M,g) \lneq \lambda_1(M,g) \leq \cdots$ of the Laplace operator on functions of Riemanian manifold (M,g). We shall discuss so called the singular perturbation problem, which is stated as follows.

「When a sequence of Riemannian manifolds (M_i, g_i) converges to a space X, caluculate the limit $\lim_{i\to\infty} \lambda_k(M_i, g_i)$ in terms of X」

Our purpose is to develope the systematic study and clearify what controlls the limit of eigenvalues. Then, we begin with defining the convergence of spaces.

Definition 1 (Gromov).

- Let X, Y be compact metric spaces, and $\varphi: X \to Y$ be a (not necessarily continuous) map. We say that φ is an ε-Hausdorff approximation if
  1. the ε-neighborhood of $\varphi(X)$ is Y,
  2. for $x, y \in X$
  $$|d(x,y) - d(\varphi(x), \varphi(y))| < \varepsilon.$$
- We define the Hausdorff distance $d_H(X, Y)$ by

$$d_H(X,Y) = \inf\{\varepsilon \mid \text{there exist } \varepsilon\text{-Hausdorff approximations from } X \text{ to } Y \text{ and from } Y \text{ to } X\}$$

But this distance is not suffice for our purpose.

Example 2

Put $M = S^1 \times S^1$, $g = g_\varepsilon(c) = dt^2 \oplus \varepsilon^2 c(t)^2 ds^2$. Here c ; $S^1 \to \mathbb{R}^+$ is a C^∞-function. $(S^1\times S^1, g_\varepsilon(c))$ convergens to S^1 with respect to the Hausdorff distance, hence the limit is independent of c. But the eigenvalues $\lambda_k(S^1\times S^1, g_\varepsilon(c))$ converges to the eigenvalues of the operator

$$P_c f = -\frac{d^2 f}{dt^2} - \left(\frac{dc}{dt}\Big/c\right)\frac{df}{dt}.$$

The limit $\lim_{\varepsilon\to 0}\lambda_k(S^1\times S^1, g_\varepsilon(c))$ does depend on c.

Thus $\lim_{\varepsilon\to 0}\lambda_k(S^1\times S^1, g_\varepsilon(c))$ can not be determined by the limit of $(S^1\times S^1, g_\varepsilon(c))$ with respect to the Hausdorff distance. (We say M_i collapses to X if M_i converges to X and if $\dim X \lneqq \dim M_i$.)

Hence we need a finer topology.

Definition 3

- Put $\mathfrak{M}\mathfrak{M} = \{(X,\mu) \mid X : \text{compact metric space}, \ \mu : \text{Borel measure on } X \text{ with } \mu(X) = 1\}$
- If M is a Riemannian manifold we put $\mu_M = \Omega_M/\mathrm{Vol}(M)$, where Ω_M is the volume element of M, and regard M as the element (M,μ_M) of $\mathfrak{M}\mathfrak{M}$

- For $(X_i,\mu_i), (X,\mu) \in \mathcal{MM}$, we say that (X_i,μ_i) converges to (X,μ) with respect to the <u>measured Hausdorff topology</u> if there exists ε_i and $\varphi_i : X_i \to X$ such that
  ① $\lim_{i\to\infty} \varepsilon_i = 0$
  ② φ_i is an ε_i-Hausdorff approximation
  ③ $(\varphi_i)_* \mu_i$ converges to μ with respect to the weak topology that is $\int f\circ\varphi_i \, d\mu_i$ converges to $\int f d\mu$ for every continuous function f on X.

<u>Theorem 1</u> <u>If a sequence of Riemannian manifolds (M_i, g_i) converges to $(X,\mu) \in \mathcal{MM}$ with respect to the measured Hausdorff topology</u>
<u>then</u>
$$\lim_{i\to\infty} \lambda_k(M_i, g_i) \leqq \lambda_k(X,\mu).$$

(We can define $\lambda_k(X,\mu)$ for arbitrary $(X,\mu)\in\mathcal{MM}$. We omit the definition of general case, and give it later a special case.)

<u>Conjecture</u> $=$ in Theorem 1 holds, if $\mathrm{Ricc}(M_i, g_i) \geqq$ const.

<u>Theorem 2</u> <u>We assume furthermore</u>

$$\underline{|\text{sectional curvature of } (M_i, g_i)| \leq \text{const}.}$$

<u>Then</u> $\underline{\lim_{i\to\infty} \lambda_k(M_i, g_i) = \lambda_k(X,\mu).}$

<u>Definition 4</u> We assume that $X = X_{reg} \cup X_{sing}$, X_{reg} is a Riemanian manifold, $\dim X_{sing} \lneqq \dim X$, $\mu(X_{sing}) = 0$. For a C^1-function f on X we put

$$D(f,f) = \int \langle df, df \rangle_{X_{reg}} d\mu .$$

And, let $\lambda_k(X,\mu)$ be the k-th eigenfunction of D on $L^2(X,\mu)$. In case when $\mu = \chi \lrcorner \Omega_{X_{reg}}$ for a C^1-function χ, $\lambda_k(X,\mu)$ is the k-th eigenvalue of the operator

$$P(f) = \Delta_{X_{reg}} f - \langle df, d\chi \rangle / \chi .$$

These assumptions are almost satisfied under the assumption of Theorem 2.

<u>Theorem 3</u> (Fukaya; Gromov) <u>Let (X,μ) be as in Theorem 2.</u>

① <u>For $p \in X$ there exists a neighborhood U of p which is isometric to $(\mathbb{R}^m, g)/T$. Here g is a T invariant metric of $C^{1+\alpha}$ class and T is a compact subgroup of $O(m)$ whose connected component is a torus.</u>

② <u>There exists a continuous function χ on X such that</u>

$$\mu = \chi \cdot \Omega_{X_{reg}} .$$

Example 2 satisfies the assumption of Theorem 2. We shall give other examples.

Examples 5 ∘ Let (S^3, g_{can}) be the standard sphere with standard metric. $SO(4)$ acts on S^3 by isometries. Take a maximal torus T^2 of $SO(4)$. Put $\mathbb{R}_1 = (1,\pi)\mathbb{R}/\mathbb{Z}^2 \subseteq \mathbb{R}^2/\mathbb{Z}^2 = T^2$ and

$$g_\varepsilon(V,V) = \begin{cases} \varepsilon\, g_{can}(V,V) & \text{if } V \text{ is tangent to an } \mathbb{R}\text{-orbit.} \\ g_{can}(V,V) & \text{if } V \text{ is perpendicular to an } \mathbb{R}\text{-orbit.} \end{cases}$$

Then (S^3, g_ε) conveges to $(([0,\frac{\pi}{2}], dt^2), \cos t \sin t\, dt)$ when ε tends to 0. In this case, the operator P in definition 4 is

$$Pf = -\frac{d^2f}{dt^2} - (\cot t - \tan t)\frac{df}{dt}.$$

P has a regular singularity on boundary.

∘ More generaly, suppose that M is a Riemannian manifold on which a torus T^k acts by isometries. Assume that, for each $p \in M$, the isotropy group $\{g \in T^k \mid g(p) = p\}$ does not coincide with T^k. Then there is a sequence of metrics g_ε such that (M, g_ε) converges to $(M/T^k, \chi \cdot \Omega_{M/T^k})$. Here $\chi(p)$ is the k-dimensional volume of the orbit corresponding to p. In this case, the operator P also has a regular singularity at the singular point.

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Titel: Holomorphic curves and fixed points of symplectic maps

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Let P be a smooth compact manifold with a symplectic, i.e. closed and nondegenerate, 2-form ω. Then for every (time dependent) function H_t: $P \to \mathbb{R}$, one defines the (time dependent) Hamiltonian vector field X_t by

$$\omega(X_t, \cdot) = dH_t \quad .$$

If we integrate such a family of vector fields, we obtain a family of diffeomorphisms ϕ_t of P, which preserve the symplectic form. The set of all diffeomorphisms obtained in such a way by any function H_t is called the set $\mathcal{D}_e$ of exact deformations of P.

We are interested in the fixed points of exact deformations. Clearly, the number of fixed points of $\phi \in \mathcal{D}_e$ satisfies the same estimates as the number of zeroes of a smooth vector field on P. This follows from the Lefschetz fixed point theorem, since each $\phi \in \mathcal{D}_e$ is homotopic to the identity. It has been conjectured by V.I. Arnold that due to the additional restrictions on $\mathcal{D}_e$, the number of fixed points of an exact deformation even satisfies estimates similar to those for the critical points of a smooth function on P. We have the following result in this direction:

Theorem: Let (P,ω) be a compact symplectic manifold and let ϕ be an exact deformation. If all fixed points of ϕ are nondegenerate, then there number is greater than or equal to the sum of the Z_2 - Betti numbers of P.

More precisely, we show that if C denotes the free Z_2-module over the set of fixed points of ϕ, then there exists a homomorphism $\delta : C \to C$ satisfying $\delta\delta = 0$ and

$$H^{*}(P, Z_2) = \ker \hat{\delta} / \operatorname{im} \hat{\delta} .$$

If ϕ is a deformation induced by a time independent function H, then this follows immediately from Morse theory, see f.e. M . In the general case, there is still a variational approatch, however one has to pass to the infinite dimensional space

$$\Omega(\phi) = \{ z \subset C^{\infty}([o,1] ,P) \mid z(1) = \phi(z(o)) \} .$$

There is a function $\mathcal{a} : \Omega \to \mathbb{R}$ (sometimes defined only locally), which is called the symplectic action functional and which satisfies

$$d\mathcal{a}(z)\,\xi = \int_0^1 \omega(\xi, \dot{z}) .$$

The (nondegenerate) critical points correspond to the (nondegenerate) fixed points of $\mathcal{a}$. Since standard variational methods do not apply in this case (for reasons of lack of Palais-Smale compactness and definiteness), we use the following method: For suitable almost complex structures J on P , maps $u : \mathbb{R} \to \Omega$, $u(s)(t) = u(s,t)$ which satisfy

$$u = \frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} = 0$$

can be interpreted as trajectories of the "gradient flow" of $\mathcal{a}$. ~~Hence we are interested~~

On the other hand, they can be considered as holomorphic maps from $\mathbb{R} \times [0,1] \subset \mathbb{C}$ into (P,J). It turns out that if such a holomorphic map has finite area, then it represents a trajectory connecting two critical points. Moreover, there exists a "relative Morse index" $\mu(x,y)$ for critical points x and y so that the set of trajectories connecting x and y is a smooth manifold of dimension $\mu(x,y)$. In particular, if $\mu(x,y) = 1$, then there are is a discrete and, in fact, finite set of one dimensional trajectories. If we use these numbers as the matrix elements of the coboundary operator δ, then by establishing existence and continuation properties for holomorphic curves one can show that δ has the properties stated above. This proves the theorem.

[M] : Milnor, Lectures on the H-cobordism theorem

[W] : Witten, Supersymmetry and Morse theory

[FU] Freed/Uhlenbeck: Instantons and 4-dimensional topology

Titel: Gravitational instantons and Kleinian Singularities

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A hyperKähler manifold is a Riemannian manifold (M, g) equipped with three integrable, Kähler complex structures I, J, K satisfying the relations of the quaternion algebra. Such a manifold is a symplectic manifold with respect to each of the three Kähler forms $\omega_1, \omega_2, \omega_3$ corresponding to I, J, K.

Let Γ be a finite subgroup of $SU(2)$ and $q : X \to \mathbb{C}^2/\Gamma$ the minimal resolution of the Kleinian singularity $\mathbb{C}^2/\Gamma$. Forgetting its complex structure, we regard X as a 4-manifold.

THEOREM. Let three classes $\alpha_1, \alpha_2, \alpha_3 \in H^2(X;\mathbb{R})$ be given which satisfy the condition that for all $\Sigma \in H_2(X;\mathbb{Z})$ with $\Sigma \cdot \Sigma = -2$ there exist $i \in \{1,2,3\}$ with $\alpha_i(\Sigma) \neq 0$. Then there exists on X a hyperkähler structure (g, I, J, K) with $[\omega_i] = \alpha_i$, $i = 1,2,3$, and such that g is asymptotically an isometry to the Euclidean metric on $\mathbb{C}^2/\Gamma$. //

When Γ is the cyclic group of order 2, the space X is the cotangent bundle $T^*\mathbb{C}P^1$ of the projective line, and the metric g whose existence is asserted here is the Eguchi-Hanson metric. The larger cyclic groups give the multi-Eguchi-Hanson metrics. On the basis of the twistor description of the cyclic case, the theorem above was conjectured by Hitchin [2].

The proof of the theorem is an application of the hyperkähler quotient construction of Hitchin and

Autor: KRONHEIMER Seite: 4

Roček. As a corollary one obtains an indepent proof of a theorem due to Brieskorn: every deformation of a Kleinian singularity admits a simultaneous resolution [1].

[1] E. Brieskorn, Singular elements of semi-simple algebraic groups, Actes Congrès Intern. Math, 1970, 2, 279-284.

[2] N.J. Hitchin, Polygons and gravitons, Math. Proc. Camb. Phil. Soc., 1979, 85, 455-476

[3] P.B. Kronheimer, Instantons gravitationelles et singularités de Klein, (to appear in Contes Rendues, 1986).

Titel: P-adic Hodge-Theory

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Let K denote a p-adic field (i.e. $K \supseteq \mathbb{Q}_p$ a finite extension). If X/K is a smooth proper variety, we have an isomorphism of $Gal(\bar{K}/K)$-modules:

$$H^n_{et}(X \otimes_K \bar{K}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \hat{\bar{K}} \cong \bigoplus_{a+b=n} H^a(X, \Omega^b_X) \otimes_K \hat{\bar{K}}(-a)$$

("$(-a)$" = Tate-twist) (short form: $H^*_{et}(X) = H^*_{Hodge}(X)$)

The isomorphism is functorial, and respects the usual structures (Cup-product, Künneth, Chern-classes, cycle-classes).

As a corollary we obtain algebraic proofs for the degeneration of the Hodge-spectral-sequence, and of the symmetry of Hodge-numbers for projective varieties (use hard Lefschetz). This suffices for many applications. A variante (due to Fontaine and Messing) gives relations between étale and chrystalline cohomology. This is stronger than our result, but works only under more restrictive assumptions (good reduction, "$e=1$", e.t.c.) Their proof uses the syntomic topology.

Ideas of proof:

a) Almost unramified extensions (Tate):
Let V = integers of K, V_∞ = normalisation of V in $K(\mu_{p^\infty})$ $\Rightarrow$ $\bar{K}/K$ is almost unramified, that is:
If L/K is a finite extension, $L_n = L(\mu_{p^n}) \supseteq K_n = K(\mu_{p^n})$, then the discriminant $\delta(L_n/K_n)$ converges to 0 for $n \to \infty$.
Hence for many purposes it suffices to consider the wellknown field K_∞ instead of $\bar{K}$.

b) Let R = étale extension of $V[T_1^{\pm 1}, \ldots, T_d^{\pm 1}]$,
R_∞ = normalisation of R in extension generated by μ_{p^∞} and $T_i^{p^{-\infty}}$

$\tilde{R}$ = normalisation of R in extension generated by $\{\sqrt[p^\infty]{u} \mid u \in R^*\}$

$\bar{R}$ = maximal extension of R unramified in Char 0.

Then $R \subseteq R_\infty \subseteq \tilde{R} \subseteq \bar{R}$, and $\tilde{R}/R_\infty$ is almost unramified. ($\bar{R}/R_\infty$ is only almost unramified in codim ≤ 2)
We obtain an extension

$$0 \to \Omega^1_{V_\infty/V} \otimes \tilde{R} \to \Omega^1_{\tilde{R}/R} \to \Omega^1_{\tilde{R}/RV_\infty} \to 0$$

or

$$0 \to \tilde{R}[\tfrac{1}{p}]/\tfrac{1}{\xi}\tilde{R}(1) \to \Omega^1_{\tilde{R}/R} \to \Omega^1_{R/V} \otimes (\tilde{R}[\tfrac{1}{p}]/\tilde{R}) \to 0$$

(using $\Omega^1_{V_\infty/V} \cong K_\infty/\rho^{-1}V_\infty(1)$, for some $\rho \in V_\infty$ ($\rho \neq 0$),
via $V_\infty \otimes \mu_{p^\infty} \to \Omega^1_{V_\infty/V}$

$$\alpha \otimes \varepsilon \mapsto \alpha\, d\log \varepsilon$$

This gives a map

$$\Omega^1_{R/V} \to H^1(\Delta, \hat{\bar{R}}(1)) \quad (\Delta = Gal(\bar{R}/R\bar{V})),$$

and the relation between étale cohomology and Hodge-cohomology.

c) Let X/V be smooth and projective. Use affine hypercoverings of X by Spec(R)'s (R as in b)) to construct a cohomology theory $\mathcal{H}^*(X)$ (globalisation of $H^*(\Delta, \hat{\bar{R}})$) with natural transformations

$$H^*_{ét}(X) \to \mathcal{H}^*(X) \leftarrow H^*_{Hodge}(X).$$

$\mathcal{H}^*(X)$ has a cup-product, and a trace-map

$$\mathcal{H}^{2\dim(X)}(X) \to \hat{\bar{R}}(-\dim(X)).$$

d) Use that both $H^*_{ét}(X)$ and $H^*_{Hodge}(X)$ are good cohomology-theories, to derive that the pairing

$$H^*_{ét}(X) \times H^{2\dim(X)-*}_{Hodge}(X) \to \mathcal{H}^{2\dim(X)}(X) \to \hat{\bar{R}}(-d)$$

is induced from a natural isomorphism

$$H^*_{ét} \cong H^*_{Hodge}$$

e) For bad reduction things are more complicated:

i) In the construction of $\mathcal{H}^*$, we have to replace $\hat{\bar{R}}$ by a system of coefficients $\underline{\Phi}$.

ii) We have to use rigid hypercoverings

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J.M. Fontaine: Formes differentielles.....
Invent. math. 65 (1982), 379-409

G. Faltings: Hodge-Tate structures and modular forms
submitted (1985) to Mathem. Annalen

Titel: JONES' LINK POLYNOMIALS

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The Hecke algebras H_n (associated with the symmetric groups S_n) generated over a complex rational function field on 2 variables $K = \mathbb{C}(q, z)$ by $T_1, T_2, \dots, T_{n-1}$ with relations

$$T_i T_j = T_j T_i \quad \text{for } |i-j| \geq 2,$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$$

$$T_i^2 = (q-1) T_i + q$$

turn out to have compatible trace functions

$$\begin{tikzcd} H_n \arrow[r, hook] \arrow[dr, "\mathrm{Tr}"'] & H_{n+1} \arrow[d, "\mathrm{Tr}"] \\ & \mathbb{C}(q,z) \end{tikzcd}$$

satisfying $\mathrm{Tr}(1) = 1$ and

$$\mathrm{Tr}(a T_n b) = z\, \mathrm{Tr}(ab)$$

for $a, b \in H_n$.

This fundamental observation of V. Jones provides a braid invariant $V : B_n \to K(\sqrt{\frac{q}{zw}})$

$$V_\alpha(q,z) = \left(\frac{1}{z}\right)^{\frac{n+e(\alpha)-1}{2}} \left(\frac{q}{w}\right)^{\frac{n-e(\alpha)-1}{2}} . \operatorname{Tr}(\rho(\alpha)),$$

where $\rho : B_n \to H_n$ is the obvious representation of the braid group

$B_n = \langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| \geq 2,\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \rangle$

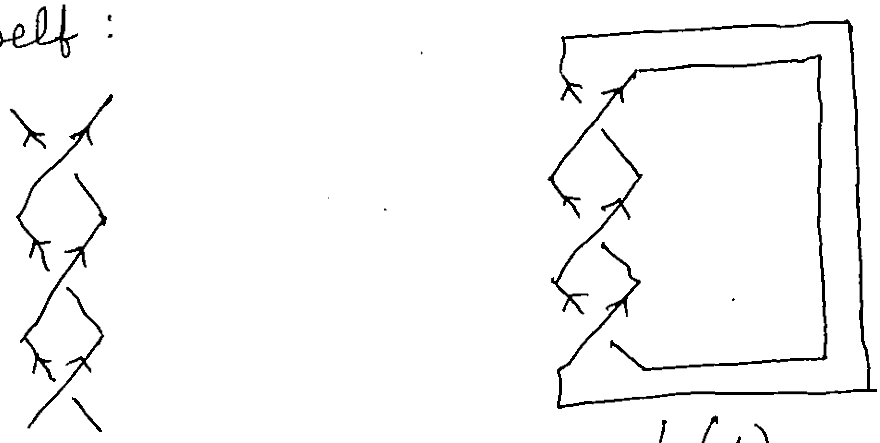
given by $\rho(\sigma_i) = T_i$, and where, for $\alpha \in B_n$,

$e(\alpha) = \sum_{i=1}^{k} e_i$ = exponent sum of $\alpha = \sigma_{i_1}^{e_1} \cdot \ldots \cdot \sigma_{i_k}^{e_k}$

and $w = 1 - q + z$.

[observe that in H_n, one has $T_i^{-1} = \frac{1}{q}(1-q+T_i)$]

It is a quite remarkable fact that $V_\alpha(q,z)$ depends only on the oriented link $L(\alpha)$ obtained by "closing" the braid α, rather than on α itself:

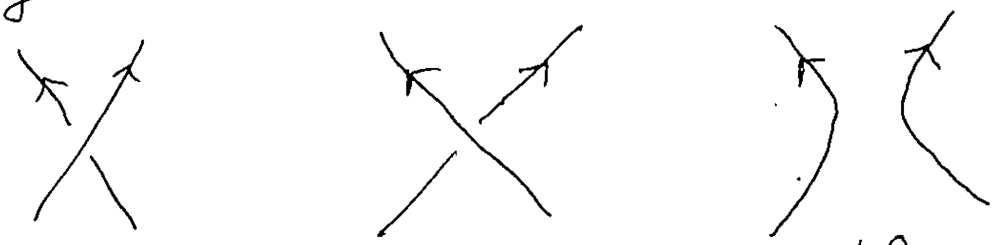


α $L(\alpha)$

"closing" a braid α

Since every oriented link L can be obtained as some $L(\alpha)$, with $\alpha \in B_n$ for some n (Alexander), $V_\alpha(q, z)$ is thus an invariant of oriented links.

Suppose now that L^+, L^- and L^0 are 3 links with diagrams which are identical except in one crossing, where they look like this:



L^+ L^- L^0

A link invariant $P(L)$ with values in some commutative ring A is said to be skein invariant if it satisfies $P(\text{unknot}) = 1$ and

$$a_+ P(L^+) + a_- P(L^-) + a_0 P(L^0) = 0$$

for some invertible elements $a_+, a_-, a_0 \in A$.

Setting $\ell = i\left(\frac{z}{w}\right)^{1/2}$, $m = i(q^{1/2} - q^{-1/2})$ the invariant V_L satisfies

$$\ell V_{L^+} + \ell^{-1} V_{L^-} + m V_{L^0} = 0$$

Thus V_L is really a Laurent polynomial in the variables ℓ and m with integral coefficients. (Use induction on the link "complexity" and the skein invariance. Thus, here $A = \mathbb{Z}[\ell, \ell^{-1}, m, m^{-1}]$.)

In fact, it is the universal skein invariant. In particular it specializes to the Alexander polynomial (as normalized by J. Conway) under $\ell \mapsto i$, $m \mapsto -i(t^{1/2} - t^{-1/2})$ and to the 1-variable Jones' polynomial under $\ell \mapsto it$, $m \mapsto -i\cdot(t^{1/2} - t^{-1/2})$ (Bull. AMS Vol. 12, 1985, p. 239.)

There are applications of the use of the polynomial to the calculation of the braid number of a link L (the smallest n such that $L = L(\alpha)$ with $\alpha \in B_n$), by H. Morton and

V. Jones. Another striking application is the solution of a long standing conjecture of P. G. Tait (1877): the ~~minimal~~ crossing number of a~~n~~ simple alternating link projection is a topological invariant ~~knot is given by a minimal alternating projection~~. (L. Kauffman and K. Murasugi.)

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Titel:

Fermat's Last Theorem, Modular Forms,...

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The talk concerned recent efforts to show that "Fermat's Last Theorem" is a consequence of general conjectures in various branches of arithmetic. One starts with an odd prime r and a non-trivial co-prime solution (a,b,c) to the Fermat equation

$$a^r + b^r = c^r.$$

Using it, one writes down the elliptic curve E defined by the affine equation

$$y^2 = x(x-a^r)(x-c^r).$$

Then E is a semi-stable elliptic curve over $\mathbf{Q}$, whose conductor N is the product of all primes p which divide abc. Moreover, the discriminant Δ associated with the above affine equation is the product of a certain power of 2 (namely: 64) with an r^{th} power (namely $(abc)^{2r}$). Here r is a prime number which is not ridiculously small (e.g., we will use that $r \geq 23$, which follows from the work of Kummer). The problem is to show that E does not exist, i.e., to derive a contradiction.

For this, one writes down the 2-dimensional $\mathbf{F}_r$ vector space

$$V = E[r],$$

defined to be the kernel of multiplication of r on E. We view it initially as a representation of the group $G = \mathrm{Gal}_{\mathbf{Q}}$. It is irreducible by a theorem of Mazur, since r is not small. Also, it is unramified at all primes p prime to Nr, simply because E has good reduction at all primes outside N.

The hypothesis that Δ is an r^{th} power at all $p|N$ implies further that V is unramified at all primes $p|N$ other than 2 and r. Moreover, if $r|N$, then the hypothesis on Δ implies an analogous (though more subtle) fact about the behavior of V at r. The end result is that V may be viewed as the G-representation attached to a finite flat commutative group scheme $\mathbf{V}$ over $\mathbf{Z}[1/2]$, of type (r,r), which is "semistable at 2". Most people seem to feel that there should be no such group schemes except the trivial ones: products of copies of $\mathbf{Z}/r\mathbf{Z}$ and $\boldsymbol{\mu}_r$. In particular, one can hope that the methods of affine group schemes (à al Fontaine) will someday show that $\mathbf{V}$ must be reducible; if so, we will get a contradiction and thereby prove Fermat's Last Theorem.

I should have mentioned at the beginning of this summary that the idea of writing down E, given a solution to the Fermat equation, is due to G. Frey [1]. Frey's philosophy, I believe, is it may be too hard to prove outright that E does not exist. However, it may be easier to show, at least, that the existence of E is incompatible with the standard conjecture, due to Taniyama and Shimura, that all elliptic curves over **Q** are modular.

Suppose, then, that E is modular. This means that there is a weight-2 newform

$$f = \Sigma\, a_n q^n$$

with integral coefficients, on $\Gamma_0(N)$, whose L-function coincides with that of E. In particular, the representation

$$\rho\colon G \to \mathrm{Aut}(V)$$

defined by $V = E[r]$ is isomorphic to the mod r representation of G which is attached to f. We can say, informally, that ρ is "modular of level N (and weight 2)."

In general, now, suppose that

$$\sigma\colon G \to GL(2,\mathbf{F}_{r^\nu})$$

is modular of level N and that σ is unramified at a prime number p|N. Then there are strong reasons to believe that σ comes from a newform of level N/p (as well as from a newform of level N). Analogously, if r|N and if σ has appropriate local properties at r, one should be able to deduce that σ is modular of level N/r.

It may be possible to obtain results of this nature in the near future. Assume that they are at our disposal, and apply them to ρ. We obtain, by induction, that ρ is modular of level 2. The desired incompatibility then follows from the observation that there are no non-zero cusp forms on $\Gamma_0(2)$!

Precise conjectures about the possibility of replacing N by N/p have been given by Serre in [2] (and in [3]). Serre takes a somewhat different viewpoint from the one above. Namely, he starts with a representation

$$\sigma: G \to GL(2, \mathbf{F}_{r^\nu})$$

whose determinant is an odd character of G (it takes complex conjugations to -1). He conjectures that σ is automatically modular, specifying a level N and a weight $k \geq 2$ for σ. (The modular form giving σ will be on $\Gamma_1(N)$, rather than $\Gamma_0(N)$, if $\det(\sigma)$ is ramified outside r.) Serre's general recipe gives $N = 2$ and $k = 2$ in the special case $\sigma = \rho$, giving again the incompatibility.

There is yet another way of trying to obtain an incompatibility: by giving an upper bound for Δ in terms of N. Thanks are due to S. Lang for pointing this out after my talk. He has written an appendix to this résumé which summarizes his remarks.

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1. G. Frey. Links between stable elliptic curves and certain diophantine equations. Annales Universitatis Saraviensis 1 (1986)

2. J-P. Serre. Letter to J-F. Mestre, August 13, 1985. To appear in: Current Trends in Arithmetical Algebraic Geometry. (Contemporary Mathematics, AMS)

3. J-P. Serre. Letter to J-M. Fontaine, May 6, 1986

Titel: Appendix to Ken Ribet's talk

Autor: Serge LANG

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In his talk Ribet explained the connection of Fermat's Last Theorem with adic representations and finite group schemes via Frey's elliptic curve. I want here to mention another approach, also given in Frey's paper. Associated with a Fermat point

$$a^n+b^n = c^n$$

with positive relatively prime integers (a,b,c) we have the discriminant

$$\Delta = (abc)^{2n} = g_2^3 - 27g_3^2.$$

A bound for n then follows immediately from.

<u>SZPIRO's conjecture. There exists a number</u> $c > 0$ <u>such that if</u> N <u>is the conductor, then</u> $|\Delta| \leqq N^c$.

This conjecture can be formulated independently of elliptic curves:

<u>There exists a number</u> $c > 0$ <u>such that for all integers</u> u, v <u>relatively prime we have</u>

$$|u^3-v^2| \leqq \prod_p p^c$$

<u>where the product is taken over all primes dividing</u> u^3-v^2.

There is a similar statement if instead of u^3-v^2 one considers Au^3+Bv^2, or Au^m+Bv^n, with u, v relatively prime, or divisible only by a bounded power of primes. The constant c then depends on such data. Such a conjecture is reminiscent of the Hall conjecture, which gives a lower bound $|u^3-v^2| \gg u^{1/2-\varepsilon}$ if $u^3-v^2 \neq 0$.

Szpiro's conjecture can then be viewed as a conjecture purely in number theory, free of elliptic or modular interpretations, only as a diophantine inequality. If one decides to give it a modular interpretation a la Taniyama-Shimura, then it is equivalent to the following statement concerning the degree of a Taniyama-Shimura representation of the elliptic curve:

There exists a number $c > 0$ such that if $TS: X_0(N) \longrightarrow E$ is the Taniyama-Shimura representation of an elliptic curve of conductor N by the modular curve of level N, then $\deg TS \leqq N^c$.

On the other hand, one cannot dismiss the possibility of proving Szpiro's conjecture by other means, for instance via diophantine inequalities. In fact, Vojta's conjectured height inequality, motivated by Nevanlinna theory, also implies a bound on the degrees of Fermat curves as follows. A very special case of Vojta's inequality, for curves, can be formulated as follows:

Let X be a curve of genus $\geqq 2$, over the rationals. Let $d(P)$ be the log of the absolute discriminant of an algebraic point P on X. Let h_K be the absolute height with respect to the canonical class K. Then for all algebraic points P of bounded degree on X one has

$$h_K(P) \leqq c.d(P) + O(1)$$

where the constant c and $O(1)$ depend only on X and the degree.

Vojta also has a higher dimensional version, with a divisor D, but the above suffices for Fermat. Indeed, let X be the curve defined by $x^4+y^4=z^4$, of genus 3. To each Fermat point (a,b,c) we associate the point $P = (a^{n/4} : b^{n/4} : c^{n/4})$ on X. Then

$$h_K(P) = \frac{n}{4} \log \max(a,b,c) \ll \log(abc) + O(1),$$

which gives the desired bound on n.

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Y. TANIYAMA, Problems, Tokyo-Nikko Number Theory Conference, 1955

G. FREY, Links between stable elliptic curves and certain diophantine equations, Annles Universitatis Saraviensis Vol. 1 No. 1, 1986

P. VOJTA, Diophantine approximation and value distribution theory, to appear, Springer Lecture Notes

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Remark added after these two pages were typed: Szpiro's conjecture was actually proved by Parshin ca. 1973 for surfaces over the complex numbers, and more recently by Szpiro's in characteristic p (the "function field case") with a low constant c, using the methods of intersection theory on surfaces, that is diophantine (in)equalities.

Titel: DONALDSON INVARIANTS FOR 4-MANIFOLDS

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In recent years S. K. Donaldson has introduced new methods (use of instantons) into four-dimensional geometry and obtained many spectacular results. Some of these have been reported on at earlier Arbeitstagungs. In this lecture I will present briefly the most recent results of Donaldson, based on lectures given at Oxford but as yet unpublished.

The main result is that, from the differentiable view-point, algebraic surfaces are essentially indecomposable. The precise result is as follows:

THEOREM. Let X be a simply-connected algebraic surface and assume that, differentiably, X is the connected sum of 2 4-manifolds X_1, X_2. Then X_1 or X_2 has negative definite quadratic form on H^2.

COROLLARY Let $X_d \subset P_3$ be an algebraic surface of degree d, with d odd and ≥ 5. Then X_d is homeomorphic but not diffeomorphic to a connected sum of copies of P_2 and $\bar{P}_2$ (ie P_2 with opposite orientation)

The Theorem is proved using new invariants introduced by Donaldson. These invariants are defined for simply-connected smooth 4-manifolds X with b_2^+ odd and ≥ 3 (here $b_2^+ = \dim H_+^2$, where H_+^2 is a maximal positive definite subspace of H^2). The invariants are integer polynomials Φ_k on $H_2(X, Z)$. They are defined for large k ($k > k_0(b_2^+)$) and have degree $d(k) = 4k - 3(p+1)$ where $b_2^+ = 2p+1$. They depend on an orientation of X and also on an orientation of H_2^+.

The Donaldson polynomials have two key properties:

(1) If $X = X_1 \# X_2$ with $b_2^+(X_i) > 0$ for $i = 1, 2$ then all $\Phi_k = 0$.

(2) If X is algebraic then for large k all Φ_k are non-zero: more precisely if C is a hyperplane section $\Phi_k(C) > 0$.

The Theorem is an immediate consequence of these two properties of the Φ_k.

CONSTRUCTION OF THE Φ_k

Fix an $SU(2)$-bundle P_k over X with $c_2=k$, and consider connections A on P_k. Let F be the curvature of A, then the *Yang-Mills* functional is $\|F\|^2$. This depends on a choice of Riemannian metric g on X. A k-instanton on X is a connection A which gives $\|F\|^2$ its absolute minimum $8\pi^2 k$: it is a solution of the equations $*F=-F$. Let $\mathcal{G} = \mathrm{Aut}\, P_k$ be the group of gauge transformations, and let $\mathcal{A}$ be the space of all A. Then the *instanton moduli space* $M_k(X,g)$ is a subspace of the infinite-dimensional space $\mathcal{A}/\mathcal{G}$. For generic g and large k one proves:

1) M_k is a manifold of dimension $2d(k)$

2) M_k can be (abstractly) compactified by adding a boundary of codimension ≥ 2

3) M_k inherits a natural orientation from the orientation of $H^2_+(X)$.

From 1) – 3) it follows that the homology class $[M_k(X,g)] \in H_{2d(k)}(\mathcal{A}/\mathcal{G}, \mathbb{Z})$ is defined. One then proves it is independent of the metric g.

On the other hand one can define by standard topological methods a homomorphism

$$\mu : H_2(X, \mathbb{Z}) \longrightarrow H^2(\mathcal{A}/\mathcal{G}, \mathbb{Z}).$$

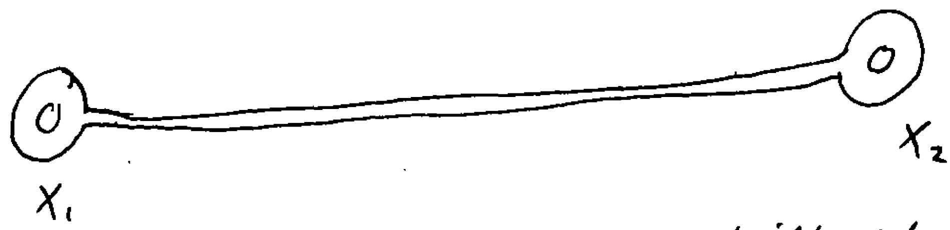
Finally one defines

$$\Phi_k(\alpha) = \mu(\alpha)^{d(k)}[M_k] \quad , \alpha \in H_2(X, \mathbb{Z}).$$

It remains to discuss the key properties of Φ_k. First if X is algebraic and one takes g to be a Kähler metric, then Donaldson [1] has shown that M_k can be identified with the moduli space of stable algebraic vector bundles ($\text{rank} = 2$, $c_1 = 0$, $c_2 = k$). Moreover $\Phi_k(C)$ turns out to be the degree of M_k (or its Kähler volume) and so is positive.

Actually care is needed here because a Kähler metric need not be "generic". But this problem can be solved by taking large enough k.

If $X = X_1 \# X_2$ then one chooses a metric as indicated in the picture



k-instantons on X are then built up out of j-instantons on X_1 and $(k-j)$-instantons on X_2 together with a 'glueing'. This is enough to prove that all $\Phi_k = 0$: essentially the cohomology classes to be evaluated are independent of the glueing and so the 'integration' does not involve all variables.

[1] S.K Donaldson: Proc. Lond. Math. Soc. 50 (1985), 1-26.

[2] S.K. Donaldson: J. Diff.-Geometry 18 (1983), 279-315

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Rationality problems, stable cohomology

An algebraic variety X of dimension n is defined to be rational if there exists a dense Zariski open set U of X which is isomorphic to an open set of $\mathbb{C}^n$; equivalently, the rational function field $\mathbb{C}(X)$ is a purely transcendental extension of $\mathbb{C}$.

Here we are concerned with quotients of the form $X = V/G$ where V is a vector space with a linear action of a group G; recent progress has been made in two directions: (1) new examples where rationality has been proved (esp. results of Katsylo and Shepherd-Barron). For example, Theorem. If V is a representation of $G = SL_2$ then V/G rational. The key case here was the proof of the rationality of the moduli space of hyperelliptic curves, where the group is actually PGL_2. (2) Counter-examples, esp. that of D.Saltman, and a new understanding of obstructions to rationality.

§1. Stable rationality.

This section runs through the results of [1]. For the duration of §1, G is a connected, simply-connected linear algebraic group; we usually consider 'generically faithful' representations V, satisfying the condition

(f) for general $x \in V$, $\mathrm{stab}(x) = (1)$.

A weakening of the rationality condition is the following:
Definition. X is stably rational if $X \times \mathbb{C}^m$ is rational for some m.

The advantage of this notion is that it depends only on G:
Proposition. Let V_1, V_2 be two representations of G satisfying (f); then V_1/G is stably rational if and only if V_2/G is.

The idea of the proof is that if we consider the diagonal action of G on $V_1 \times V_2 = W$ then W/G is a Zariski fibre bundle over a dense open set of V_1/G with fibre V_2, so that V_1/G is stably birationally equivalent to W/G, hence to V_2/G.

More generally, the same kind of argument shows that if $V_n \to V_{n-1} \to \dots \to V_1 = V$ is a G-tower of varieties such that each step is generically a vector bundle then V_n/G is stably birationally equivalent to V/G.

Theorem. Suppose that G has no E_8 factor (that is, $G \not\cong G' \oplus E_8$); then V/G is stably rational for any representation V satisfying (f).

The statement reduces using arguments of the above type to considering only a simple group G; this case can be reduced further by the following notion:
Definition. Let H be a subgroup of G; a subvariety $S \subset V$ is a (G, H)-section if

(1) $\overline{GS} = V$; and (2) for $x \in S$, $gx \in S \Leftrightarrow g \in H$.

The point is that $V/G = S/H$, so that the problem reduces to a smaller group.

It turns out that for some groups the problem reduces directly to a quotient of the form L/H with L a linear space and H a finite group, and that the quotient L/H can be shown to be rational by direct computations. In the remaining exceptional cases a reduction of the problem can be obtained by considering small-dimensional representations.

§2. Computing the Brauer invariant.

For finite groups G, the quotient $X = V/G$ will not usually be stably rational; counter-examples with G of order p^9 were given by D. Saltman [2]. Here we are mainly interested in discussing the obstruction to rationality used, namely the torsion subgroup of $H^3(X, \mathbb{Z})$.

Let V be a representation of G and write V^L for the free locus, and X for a smooth compactification of the quotient V^L/G. Now set $Br_v(G) := H^3(X, \mathbb{Z})_{tors}$ (the subscript v stands for 'unramified at every valuation').

<u>Theorem</u>. $Br_v(G) = \{\gamma \in H^2(G, \mathbb{Q}/\mathbb{Z}) \mid \gamma|_A = 0 \ \forall$ Abelian subgp. $A \subset G\}$.

Recall that $Br(\mathbb{C}(X))$ is the group of $\mathbb{P}^n$-fibre bundles over open sets of X modulo those of the form $\mathbb{P}(E)$ for E a vector bundle; a $\mathbb{P}^n$-bundle over V^L/G corresponds to an extension $\mathbb{Z}_m \to G_\gamma \to G$ and a faithful representation W of G_γ such that the cyclic subgroup $\mathbb{Z}_m$ acts by multiplication by a character: the bundle is $(V \times \mathbb{P}(W))/G$; now a $\mathbb{P}^n$-bundle over an open set of X given by $\gamma \in H^3(X, \mathbb{Z})$ is a Brauer element if and only if the bundle extends over the general point of every divisor of X. Using the relation between divisors and discrete valuation rings A_v of $\mathbb{C}(X)$, the condition that γ must satisfy is $\gamma \in \bigcap_v Br(A_v)$; now if for given v we write G_v for the local Galois group, we have

$$\gamma \in Br(A_v) \Leftrightarrow \gamma|_{G_v} \text{ comes from } G_v/(\text{centre}).$$

It is not hard to get from this to the condition in the theorem.

Incidentally, the same kind of argument shows that $Br_v(G)$ is well-defined for linear algebraic groups (and is zero if G is connected). In fact if G acts on any variety V, we can compute in a similar way the contribution of $H^2(G, \mathbb{Q}/\mathbb{Z})$ in $Br(V/G)$.

Before discussing the examples, we make some remarks. First of all, computation show that the invariant $Br_v(G)$ tends to be zero for simple groups. Next,

<u>Lemma 1</u>. If $\gamma \in Br_v G$ is an element of order p then $\gamma \in Br(S_p)$, where S_p is the Sylow p-subgroup.

Thus for simple examples we can stick to p-groups.

<u>Lemma 2</u>. If G is a p-group of order $< p^6$ then $Br_v(G) = 0$; also there exists a group of order p^6 (generated by 3 elements) for which $Br_v(G) \neq 0$.

We now turn to examples of Saltman type. Suppose that G is a central extension of G^{ab} of the form

$$C \to G \to G^{ab}$$

where C and G^{ab} are elementary p-groups. In this case C and G^{ab} can be viewed as vector spaces over the field $\mathbb{F}_p$, and the invariant $Br_v(G)$ turns into a nice problem of finite geometry over $\mathbb{F}_p$. Taking commutators in the exact sequence for G defines a linear map

$$c : \Lambda^2 G^{ab} \to C,$$

and we write S for the kernel of c. Then write $S_\wedge \subset S$ for the vector space generated by the elementary tensors $x \wedge y$ with $x, y \in G^{ab}$. Then

<u>Proposition</u>. $Br_v(G) = (S/S_\wedge)^*$.

This is an easy calculation from the above Theorem: elements of $Br_v(G)$ are lifted from G^{ab}, and are trivial on $x \wedge y$.

Now consider the case when G^{ab} is 4-dimensional, so that $\Lambda^2 G^{ab}$ is a 6-dimensional vector space. The elementary tensors form a quadric Q in the corresponding projective space $\mathbb{P}^5$; then by the proposition, $Br_V(G) \neq 0$ if the linear subspace $\mathbb{P}(S)$ is not the linear span of its intersection with Q. There are various cases, of which the first 3 are geometric:

(1) $\mathbb{P}(S)$ = point not on Q; this gives the Saltman example with $|G| = p^9$;

(2) $\mathbb{P}(S)$ = line tangent to Q; this gives $|G| = p^8$;

(3) $\mathbb{P}(S)$ = plane touching Q along a line, giving $|G| = p^7$;

then two cases which depend on the field $\mathbb{F}_p$ being non-closed:

(4) a line not meeting Q;

(5) a plane intersecting Q in one point only.

The example considered here can be considerably generalised, see for example [3].

§3. Stable cohomology.

The purpose of this section is to point out that the invariant $Br_V(X)$ we have been considering is a particular case of a more general construction which might be of use in other contexts as an obstruction to rationality.

We define the stable cohomology groups of a variety X to be the limit in the cohomology of $\mathbb{C}(X)$ of $H^i(X, F)$; more precisely, define

$$H^i_S(X, F) = H^i(X, F)/K,$$

where

$$K = \sum \ker\{H^i(X, F) \to H^i(X - \cup D_i, F)\},$$

here D_i are prime divisors of X and the sum takes place over all finite union of D_i. This is the part of the cohomology of X which survives on passing to an arbitrarily small Zariski open.

Remark. We are grateful to Atiyah for pointing out the similarity with the construction of [4].

The importance of the construction is the following result.

Lemma. If X is a smooth projective variety and F is a locally constant sheaf, then $H^i_S(X, F)$ is a birational invariant of X, and in particular if X is rational then $H^i_S(X, F) = 0$ for all $i > 0$.

For $i = 2$, the invariant $H^i_S(X, \mathbb{Q}/\mathbb{Z})$ coincides with the above invariant $Br_V(X)$.

In the spirit of §2 we can define stable cohomology of a group G to be

$$H^i_S(G, F) := \mathrm{Im}\{H^i(G, F) \to H^i_S(X, F)\}$$

where X is V^L/G for any representation of G satisfying (f) of §1. This definition turns out not to depend on the representation V. The group $H^i_S(G, F)$ when $F = \mathbb{Z}$ or $\mathbb{Z}_\ell$ contains an invariant part which comes from a smooth projective completion of V^L/G. The quotient spaces V^L/G carry a collection of stable cohomology clases with coefficients in finite sheaves which are universal.

Further details can be found in §§1-2 and §4 of [5], (although the reader should be warned that §3 of this paper contains essential gaps and errors which invalidate the proof of the main theorem).

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I am grateful to Dr. Miles Reid for help with preparation of these lecture notes.

Titel: New minimal surfaces in S^3

Autor: U. Pinkall

Adresse: MPI für Mathematik

Here we explain a new method for the construction of compact embedded minimal surfaces in the 3-sphere. This method was elaborated recently by H. Karcher, I. Sterling and the speaker.

In 1970 H.B. Lawson constructed compact minimal surfaces in S^3 by starting with a certain geodesic quadrilateral Q and spanning in a Plateau solution M. Rotating M by $180°$ around one of its boundary edges one obtains (using the reflection principle) an analytic continuation of M. If the quadrilateral is suitably chosen, the copies of M generated by repeated reflections around edges piece together to form a compact embedded surface of genus g.

All surfaces constructed in this way divide S^3 into two congruent components Ω_1, Ω_2: The $180°$-rotations used in the construction interchange these components. In general Lawson proved that every compact embedded minimal surface in S^3 divides S^3 into two diffeomorphic components. Moreover there was a conjecture (Yau's problem list N° 98) that these two components always have the same volume.

Looking for a counterexample to this conjecture we had to construct a compact embedded minimal surface in S^3 that does not contain any great circle (otherwise the reflection principle would imply that Ω_1 and Ω_2 are congruent).

We obtained 7 such surfaces with the following method: Start with a regular tesselation of S^3

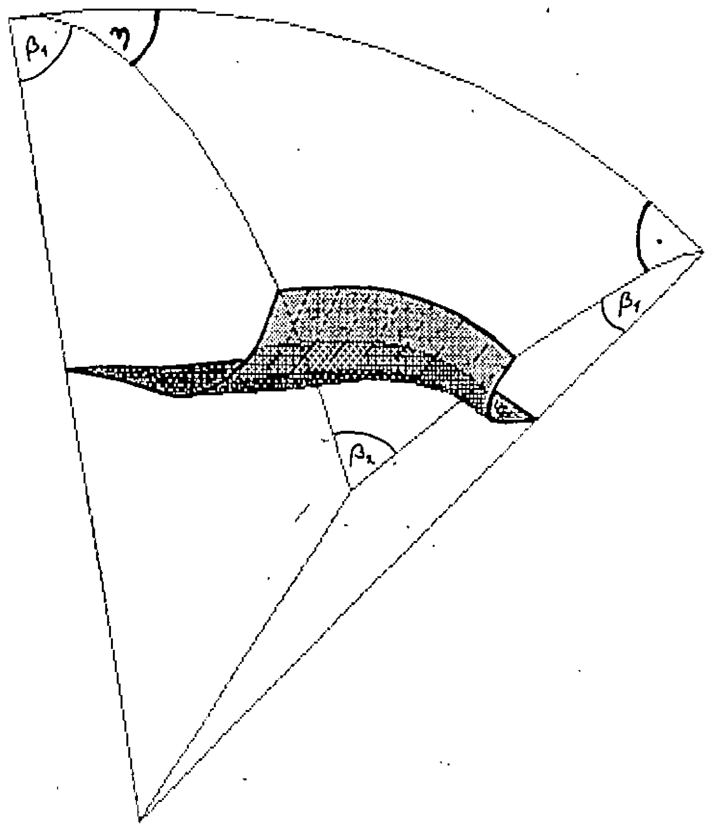


Figure 1 ("patch")

into polyhedral cells, each having the symmetry of a platonic solid in $\mathbb{R}^3$. We also allow the possibility that the cells have dihedral angle 180°, so that there are only two cells.

Dividing a cell by its planes of symmetry one obtains as a fundamental region for the group of symmetries a tetrahedron T. To construct the desired closed surface we first find a minimal surface with boundary, called a "patch", which intersects orthogonally all plane-faces of T (this is actually the hard part, see the remarks below). Figure 1 shows such a patch in stereographic projection. From the patch we obtain a certain piece of the whole

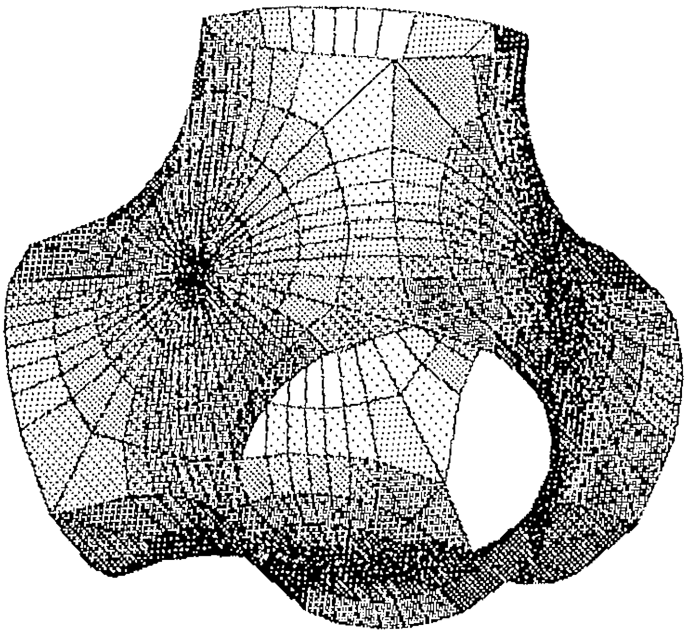


Figure 2 ("bone")

surface, called a "bone", by repeatedly reflecting patches through those plane-faces of T which are not contained in faces of the cell. Figure 2 shows a bone inside a tetrahedral cell with dihedral angles 120°.

Finally, we build the complete surface using reflections through faces of the cells. Figure 3 shows the result for the case of a tesselation of S^3 into two "cubes" with edge-angles 180°.

Figure 4 shows 1/2 of a surface that stays within some narrow neighborhood of an equatorial 2-sphere in S^3. Clearly this surface is a counter-example to the conjecture mentioned above.

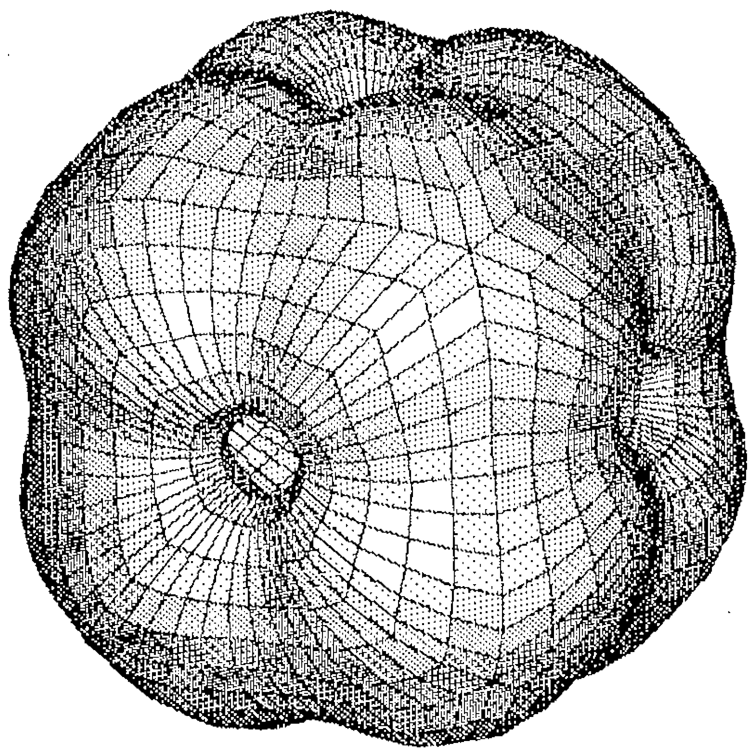
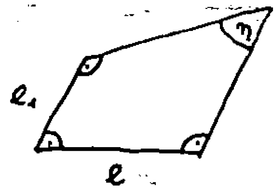


Figure 3 (genus 5)

We now roughly sketch the construction of the patch.



Consider quadrilaterals Q in S^3 with three right angles and one angle equal to η (defined in Figure 1). This leaves us with two free parameters l_1, l_2. Let ${}^Q\bar{M}$ be the "conjugate" minimal surface corresponding to a Plateau solution ${}^QM$ with boundary Q. Then ${}^Q\bar{M}$ looks almost like the required patch, except that the angles β_1, β_2 defined in Figure 1 have to be adjusted correctly by choosing l_1, l_2 suitably. To prove that this is possible requires detailed geometric estimates about the Plateau solution ${}^QM$. These estimates are obtained by comparing M with certain ruled minimal surfaces and making extensive use of the maximum principle.

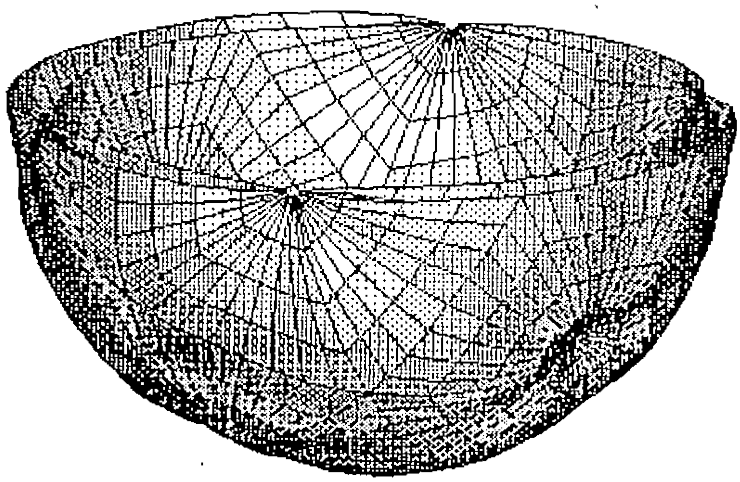


Figure 4 (half a surface of genus 11)

Titel: Fermi curves and density of states

Autor: Horst Knörrer

Adresse: Mathematisches Institut, Universität Bonn.

joined work with D. Gieseker and E. Trubowitz.

In the "almost - free electron approximation" for the behavior of an electron in a metal or crystal one assumes that all the forces acting on the electron can be subsumed in a periodic potential (cf. [1]). The quantum - description of the electron then leads to the following problem:

Let $V : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function which is periodic with respect to a lattice $\mathbb{Z}\cdot\gamma_1 \oplus \ldots \oplus \mathbb{Z}\gamma_n$ $(n = 2, 3)$. One looks for eigenfunctions ψ of $-\Delta + V$ for which there are complex number $\xi_1, \ldots, \xi_n$ of absolute value 1 such that $\psi(x+\gamma_i) = \xi_i \cdot \psi(x)$ for $x \in \mathbb{R}^n$ ("Bloch's theorem" [1], ch. 8). The numbers $\frac{1}{2\pi i} \ln \xi_i$ are called quasi - momenta of the electron.

Many macroscopic properties of the metal depend only on the sets

$$F_\lambda := \{ (\xi_1, \ldots, \xi_n) \in (\mathbb{C}^*)^n \,/\, |\xi_i| = 1, \text{ there is an eigenfunction } \psi \text{ of } (-\Delta + V) \text{ to the eigenvalue } \lambda \text{ with } \psi(x+\gamma_i) = \xi_i \cdot \psi(x) \},$$

which are called Fermi-curves resp. -surfaces for $n=2$ resp. $n=3$. Another physically important quantity is the density of states $\rho_V(\lambda)$ which is defined as the expectation value of finding an electron of energy λ. One observes that this quantity can be described as follows:

Let $B = B(V) = \{ (\xi,\lambda) \in (\mathbb{C}^*)^n \times \mathbb{R} \mid |\xi_i| = 1,\ \xi \in F_\lambda \}$, and $\pi : B \longrightarrow \mathbb{R}$ the projection $(\xi,\lambda) \mapsto \lambda$. ~~If~~ Then

$$\rho_V(\lambda) = \int_{F_\lambda} \omega_\lambda$$

where ω is the relative differential form on B characterized by $\omega \wedge \pi^*(d\lambda) = \frac{1}{(2\pi i)^n} \frac{d\xi_1}{\xi_1} \wedge \dots \wedge \frac{d\xi_n}{\xi_n}$.

For $n=1$, the relation between the potential V, the variety B and the density of states is quite well understood (cf. [2]). In the case $n=2$ we obtain the following results, if one replaces the spectral problem described above by a difference approximation:

Theorem :

(i) If for two potentials V, V' the functions $\varrho_V(\lambda)$ and $\varrho_{V'}(\lambda)$ coincide in the neighbourhood of some point λ_0 of the spectrum, and if the analytic continuation of this function has only ordinary logarithmic singularities, then $B(V) = B(V')$.
(For generic potentials V the density of states function $\varrho_V(\lambda)$ has the property mentioned above)

(ii) There is a Zariski-open dense subset $\mathcal{U}$ of the space $\mathcal{V}$ of all potentials such that for all $V \in \mathcal{U}$, $V' \in \mathcal{V}$ with $B(V) = B(V')$ ~~there is a~~ one has

$$V'(x) = V(x+T) \quad \text{for some } T$$

The proof of the theorem uses the natural complexification of the spectral problem described above, which leads to a proper family $\pi_{\mathbb{C}} : \overline{B}(V) \longrightarrow \mathbb{P}_1(\mathbb{C})$ of "complex Fermi-curves".
(ii) is based on the fact that the surface $\overline{B}(V)$ has irregularity zero and hence there are no isospectral deformations of V.
In the proof of (i) one uses monodromy in the family $\pi : \overline{B}(V) \longrightarrow \mathbb{P}_1(\mathbb{C})$ to recover from $\varrho_V(\lambda)$ the periods of ω_λ over any element in $H_1(F_\lambda, \mathbb{Z})$. Using the

"theorem of the fixed part" ([3], 4.1) one concludes that $\pi : \overline{B}(V) \longrightarrow \mathbb{P}_1(\mathbb{C})$ and $\pi' : \overline{B}(V') \longrightarrow \mathbb{P}_1(\mathbb{C})$ define isomorphic families of Hodge structures over the complement of a finite set in $\mathbb{P}_1(\mathbb{C})$. The isomorphism between $\overline{B}(V)$ and $\overline{B}(V')$ is then constructed with help of Torelli's theorem.

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Titel: Baker's Method and Effectivity

Autor: G. Wüstholz

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In the talk I gave a report on effectivity questions in the context of Siegel's theorem on integral points on curves.

Let K be a number field, $C/_K$ a smooth proper curve, $D/_K$ a very ample divisor with support $|D|$. The divisor D defines an embedding $\varphi_D : C \longrightarrow \mathbb{P}^N$ so that $D = \varphi(C) \cap \{X_0 = 0\}$. Then we denote by C_D the affine curve $\mathcal{C} \setminus |D|$. If $X_0, \dots, X_N$ are the projective coordinates and if we put

$$x_i = \frac{X_i}{X_0} \quad (i = 1, \dots, N)$$

then a point $P \in C_D(K)$ is called integral (with respect to D) if

$$x_i(P) \in \mathcal{O}_K \qquad (i = 1, \dots, N)$$

where $\mathcal{O}_K$ is the ring of integers in K. We denote the set of integral points on C_D by $C_D(\mathcal{O}_K)$.

Theorem (Siegel, 1929). *If the genus g of C is ≥ 1, then $\# C_D(\mathcal{O}_K) < \infty$.*

Remark. Siegel actually deals also with curves of genus zero but we disregarded this case in our lecture.

The proof of Siegel's theorem has two main ingredients, a diophantine approximation result which is in its modern version Roth's theorem. The second main ingredient is the theorem of Mordell-Weil, that the group of rational points $A(K)$ on an abelian variety defined over a number field K is finitely generated. Both results are completely non-effective and it is an extremely difficult problem to make them results effective.

There are nowadays two approaches to effectivity. One is through Baker's theory on linear forms in logarithms in algebraic numbers and Baker was able to prove bounds for the seize of integral points for special classes of curves.

The second approach is by linear forms in abelian logarithms and was suggested by S. Lang. To describe it we have to introduce the Weil-height of a point $P \in \mathbb{P}^N(K)$. Write

$$P = (X_0(P), \dots, X_N(P))$$

and define the (logarithmic) (Weil-) height $h(P)$ as

$$h(P) := \sum_v \max_i (\log \|X_i(P)\|_v)$$

where v runs over all valuations of K and $\| \ \|_v$ is a suitable normalized absolute value. This can be written as

$$h(P) = h_\infty(P) + h_f(P)$$

where h_∞ is the sum over all infinite places and h_f the sum over all finite places in the representation of $h(P)$ above. Then P is integral on $\ddot{C}_D$ if $h(P) = h_\infty(P)$ and one shows that for $P \in C_D(\mathcal{O}_K)$ there exists a $Q \in |D|$ such that the (logarithmic) distance $d(P, Q)$ satisfies:

$$d(P, Q) \leq -c_0 h(P)$$

where $c_0 > 0$ is an effective constant. So one is lead via the Jacobian of C to the following

Conjecture. Let $A/_K$ be an abelian variety over K, $P \in A(K)$ and $O \in A(K)$ the zero section. Then there exists an effectively computable constant $c_1 > 0$ not depending on P such that the distance $d(P,O)$ satisfies

$$d(P,O) \geqq -c_1 \log h(P).$$

By the box principle one can show that for a suitable small c_1' the reversed inequality with c_1 replaced by c_1' has infinitely many solutions.

Proposition. The conjecture implies the effective Siegel's theorem (ES). In other words: there exists an effectively computable constant $c_2 > 0$ such that for points $P \in C_D(\mathcal{O}_K)$ we have $h(P) \leqq c_2$.

<u>Proof</u>. If $P \in C_D(\mathcal{O}_K)$ is integral then we have seen that this leads to an inequality of the type

$$d(P,O) \leqq -c_3\, h(P)$$

where O is the zero section on the Jacobian $J(C)$ of C. By the Conjecture

$$d(P,O) \geqq -c_1 \log h(P).$$

Hence we get a bound for $h(P)$, namely

$$h(P) \leqq c_4$$

where c_4 depends effectively on c_3 and c_1 and is therefore effective.

In the talk we discussed then what is known about the conjecture. One result is the following result.

Theorem. There exists an effectively computable constant $c > 0$ depending on a set of generators for $A(K)$ such that

$$d(P, O) \geq -c\,(\log h(P))^{rg+2+\varepsilon}$$

where r is the rank of $A(K)$, $g = \dim A$ and $\varepsilon > 0$ arbitrary.

The proof of this theorem uses a vast generalisation of Baker's theorem on linear forms in logarithms and recent results on multiplicity estimates on group varieties.

There is another approach to an ES by using Coverings and Selmer groups. However there is still one link missing so that there does not exist yet a general version of effective Siegel ES.

Titel: Finite Group Actions on $\mathbb{P}^2(\mathbb{C})$

Autor: Ronnie Lee

Adresse: Mathematics Department, Yale University
New Haven, C.T. 06520, U.S.A.

We start with the question: which finite groups operate as symmetries of $\mathbb{P}^2(\mathbb{C})$? Any finite subgroup of $PGL_3(\mathbb{C})$ operates as a group of collineations and these give the linear models. The list of such groups is relatively short but contains for example abelian groups of rank ≤ 2 and the simple groups A_5, A_6 and $PSL_2(\mathbb{F}_7)$. It turns out that these are the only groups which can operate topologically on $\mathbb{P}^2(\mathbb{C})$ with reasonable behavior near the singular set. An action is called "locally linear" if each singular point has an invariant neighborhood which is equivariantly homeomorphic to a neighborhood of 0 in a real representation space.

<u>Theorem</u> Let G be a finite group with

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a locally linear action on $\mathbb{P}^2(\mathbb{C})$. If the induced action on cohomology is trivial, then G is isomorphic to a subgroup of $\mathbb{P}GL_3(\mathbb{C})$.

The proof of the theorem are seperated into two parts: the case of solvable groups and the case of simple groups. In the first case, we need a theorem of Bredon on the fixed point set of the action of cyclic prime order group on $\mathbb{P}^n(\mathbb{C})$. As for the actions of simple groups, we need the classification theory of finite simple groups of 2-rank at most 2.

(Joint work with Ian Hambleton).

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Titel: Finite group actions on $\mathbb{P}^2(\mathbb{C})$.

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Titel: L^2 - INDEX THEOREMS FOR REGULAR - SINGULAR OPERATORS

Autor: J. BRÜNING

Adresse: INSTITUT FÜR MATHEMATIK DER UNIVERSITÄT AUGSBURG, MEMMINGER STR. 6, 8900 AUGSBURG

This is a report on joint work with R. Seeley. In [CH] Cheeger began to extend the spectral theory of the Laplace operator on smooth manifolds to manifolds with conic singularities, that is Riemannian manifolds M such that there is an open subset U with the following properties: (1) $M \setminus U$ is a smooth compact manifold with boundary, (2) U is isometric to the metric cone $C_{(0,x_1)}N$ which is by definition the product $(0,x_1) \times N$ with metric $dx^2 + x^2 ds_N^2$ for some compact Riemannian manifold N. Using separation of variables and the Hankel transform Cheeger proved formulae for the L^2 Euler-characteristic $\hat{\chi}(M)$ and the L^2-signature $\hat{\sigma}(M)$ of M, involving the integrals of characteristic forms known from the smooth case plus singular corrections. The latter are spectral invariants of N, and coincide for $\hat{\sigma}(M)$ with the η-invariant introduced by [A+P+S]. We were interested in the common structure of these theorems as index theorems, and we were also looking for a proof that avoids separation of variables as far as possible, hoping to extend the results to more general cases. Any form ω on $C_{(0,x_1)}N$ can be written as $\omega = \omega_1(x) + \omega_2(x) \wedge dx$ with $\omega_i(x) \in C^\infty((0,x_1), \Omega(N))$, $\Omega(N) = \bigoplus_{j \geq 0} \Omega^j(N)$ the space of smooth forms on N. Then one computes that the geometric operators on $C_{(0,x_1)}N$ have the form

$$D = \partial_x + x^{-1} S_0 \quad \text{acting on } C^\infty((0,x_1), H_1)$$

for a suitable space H_1. E.g. if $D = D_{GB} = d + \delta : \Omega^{ev}(M) \to \Omega^{odd}(M)$ is the Gauss-Bonnet operator we have $H_1 = \Omega(N)$, and for $\omega(x) = \begin{pmatrix} \omega_0(x) \\ \vdots \\ \omega_n(x) \end{pmatrix}$

$$S_0\, \omega(x) = \begin{pmatrix} c_0 & \delta_N & & 0 \\ d_N & \ddots & \ddots & \\ & \ddots & \ddots & \delta_N \\ 0 & & d_N & c_n \end{pmatrix} \begin{pmatrix} \omega_0(x) \\ \vdots \\ \omega_n(x) \end{pmatrix}, \quad c_i = (-1)^i (i - n/2), \quad n = \dim N.$$

The construction of a parametrix for these operators near $x=0$ amounts, via eigenspace decomposition, to the integration of first order ode's with regular singularities. This leads us to the definition of (first order) <u>regular singular operators</u>: by this we mean a first order elliptic differential operator $D : C^\infty(E) \to C^\infty(F)$ with hermitian vector bundles E, F over M such that on some "singular" set U with property (1) above $D \mid C_0^\infty(E|U)$ is unitarily equivalent to an operator valued ode

$$\partial_x + x^{-1}(S_0 + S_1(x)) \text{ acting on } C_0^\infty((0,x_1), C^\infty(G)).$$

Here G is another hermitian vector bundle over a compact Riemannian manifold N, S_0 is a self-adjoint first order elliptic differential operator, and $S_1(x)$ is a family of first order operators depending smoothly on $x \in (0, x_1)$. In order that $x^{-1}S_1$ can be treated as a perturbation we require

(3) $\{-\tfrac{1}{2}, \tfrac{1}{2}\} \notin \operatorname{spec} S_0$, (4) $\|(|S_0|+1)^{-1} S_1(x)\| + \|S_1(x)(|S_0|+1)^{-1}\|$ $= o(x)$ as $x \to 0$.

This is enough to establish the Fredholmness of such operators.

<u>Theorem 1</u> Each closed extension of D in $L^2(E)$ is Fredholm. The closed extensions correspond to the subspaces of the finite-dimensional space $W := \mathcal{D}(D_{max}) / \mathcal{D}(D_{min})$, and if D_V denotes the extension associated with $V \subset W$ then

$$\operatorname{ind} D_V = \operatorname{ind} D_{min} + \dim V.$$

Moreover, $W = \{0\}$ if $\overline{W} := \bigoplus_{|s|<1/2} \ker(S_0 - s) = \{0\}$, and if (4) is strengthened to $O(x)$ then $W \simeq \overline{W}$.

The index formula can now be computed by the heat equation method. Considering separately contributions from the interior and from the singular part we obtain from the interior the regularized integral of the "index form" ω_D where $\omega_D(p,p)$ denotes the constant term in the pointwise asymptotic expansion of

$$\operatorname{tr}_{E \boxtimes E} e^{-tD^*D}(p,p) - \operatorname{tr}_{F \boxtimes F} e^{-tDD^*}(p,p), \quad t \searrow 0.$$

In dealing with the singular contribution we may deform S_1 to have $S_1(x) \equiv 0$ near $x = 0$. Then we have to determine the heat kernel asymptotics for the operators $-\partial_x^2 + x^{-2}(S_0^2 \pm S_0)$ plus certain boundary conditions at 0. Treating the resolvent first we note that expanding the resolvent trace,

$$\operatorname{tr}_{L^2(\mathbb{R}_+, L^2(G))} \varphi\left(-\partial_x^2 + x^{-2}(S_0^2 \pm S_0) + z^2\right)^{-m},$$

as $z \to \infty$ amounts to the expansion of $\int_0^\infty \sigma(x, xz)\, dx$ where the "symbol" $\sigma(x,\xi)$ allows an expansion in ξ as $\xi \to \infty$ and satisfies an integrability condition. We prove a singular asymptotics lemma [B+S] which gives explicit coefficients in terms of σ. Carrying out the calculations (using Bessel functions only here) we obtain the contribution to the constant term expressed by the zetafunctions of $|S_0 \pm \frac{1}{2}|$. Their difference can be written in terms of the η-function

$$\eta_{S_0}(z) = \sum_{\substack{s \in \operatorname{spec} S_0 \\ s \neq 0}} \operatorname{sign} s\, |s|^{-z}.$$

<u>Theorem 2</u>

$$\operatorname{ind} D_{\min} = \lim_{\varepsilon \to 0} \int_{x > \varepsilon} \omega_D - \frac{1}{2}\left(\eta_{S_0} + \dim \ker S_0\right)$$

$$- \sum_{-\frac{1}{2} < s < 0} \dim \ker (S_0 - s) + \sum_{k \geq 1} \alpha_k \operatorname{Res} \eta_{S_0}(2k).$$

With a particular choice of V in Thm. 1 we thus obtain Cheeger's formulae, written in a uniform way as index theorems, except for the term $R(S_0) := \sum_{k \geq 1} \alpha_k \operatorname{Res} \eta_{S_0}(2k)$. Here the α_k are universal nonzero numbers. For most geometric operators it is known that η_{S_0} is regular in $\operatorname{Re} z > -1/2$, and it would follow if we could replace in the above theorem S_0 by βS_0 for β ranging through a positive interval. The original treatment of the η-function in [A+P+S] leads one to consider also complete noncompact manifolds with ends which are warped products.

In the setting above U is isometric to $(x_0, \infty) \times N$ for some $x_0 > 0$ with metric $dx^2 + f(x)^2 ds_N^2$, f smooth and positive. Then the geometric operators have the form $D = \partial_x + f(x)^{-1} S_0 + f'(x) f(x)^{-1} S_1$ where e.g. for D_{GB}

$$(5) \quad S_0 = \begin{pmatrix} 0 & \delta_N & & 0 \\ d_N & \ddots & \ddots & \\ & \ddots & & \delta_N \\ 0 & & d_N & 0 \end{pmatrix}, \quad S_1 = \begin{pmatrix} c_0 & & 0 \\ & \ddots & \\ 0 & & c_n \end{pmatrix}.$$

But U is conformally equivalent to a metric cone $C_{(0,x_1)} N$ if $\int_{x_0}^{\infty} \frac{dx}{f(x)} = \infty$. This leads to a unitary equivalence

$$g D g \simeq -\partial_x + x^{-1} (S_0 + S_1(x))$$

where the "weight function" is $g^2(x) = f(x) e^{\int_{x_0}^{x} \frac{du}{f(u)}}$ and $S_1(x)$ satisfies (4) if $\lim_{x \to \infty} f'(x) = 0$. Now these conditions remain valid if we replace f by ρf and S_0 by ρS_0 for some $\rho > 0$ hence we may assume that (3) is satisfied, too. The closed extensions of gDg are Fredholm by thm. 1 and it is natural to compare their indices with

$$L^2 \operatorname{ind} D := \dim \ker D \cap L^2(E) - \dim \ker D^* \cap L^2(F).$$

If Q denotes the orthogonal projection onto $\ker S_0 = \bigoplus H^j(N)$ in (5) then Q reduces the bounded operator $S_1 \cdot {}^{j \geq 0}$ We write

$$Q S_1 Q = \bigoplus_{t \in \mathbb{R}} t\, Q_t .$$

<u>Theorem 3</u>

$$L^2 \operatorname{ind} D = \lim_{R \to \infty} \int_{x < R} \omega_D - \frac{1}{2} (\eta_{S_0} + \dim \ker S_0)$$

$$+ h_0 + h_1 ,$$

where $h_0 = \sum_{\substack{t \\ f^{-t} \in L^2}} \dim Q_t$, $\quad h_1 \leq \sum_{\substack{t \\ f^{-t}, f^t \notin L^2}} \dim Q_t$.

This Thm. generalizes various L^2 index theorems e.g. the one given in [A+P+S] for cylinders. h_1 seems to be difficult to compute in general; Gauss-Bonnet for surfaces shows that it may be $\neq 0$. An abstract version of the above scaling argument enables us to show that $R(S_0) = 0$ under natural abstract assumptions on the operator S_0, reducing the index formula in Thm. 2 to the results of Cheeger if applied to the proper setting.

<u>Theorem 4</u> Let S_0 be a self-adjoint operator in some Hilbert space H such that $(|S_0|+1)^{-p}$ is trace class for some $p > 0$ and the zeta functions of the operators $|S_0 \pm 1/2|$ are meromorphic in $\mathbb{C}$ with only simple poles. Then the η-function η_{S_0} of S_0 is also meromorphic in $\mathbb{C}$ with at most simple poles, and the residues at the points $2k$, $k = 0, 1, 2, \dots$, vanish.

<u>Remark</u> It seems likely that the assumption on the simplicity of the poles can be removed.

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Titel: Characteristic classes of flat bundles.

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On a complex projective variety X we consider a flat bundle E . This means that E has a holomorphic integrable connection ∇ . By a modified splitting principle we defined classes $c_p(E,\nabla)$ in $H^{2p-1}(X,\mathbb{C}/\mathbb{Z}(p))$ (with $\mathbb{Z}(p) = (2i\pi)^p\cdot\mathbb{Z}$) which map to the Chern classes $c_p^D(E)$ in the Deligne cohomology (and thereby to the Chern classes $c_p^{top}(E)$ in $H^{2p}(X,\mathbb{Z}(p))$. They are functorial and additive in the sense that if $0 \to G \to E \to F \to 0$ is a flat exact sequence (i.e : $\nabla G \subset \Omega^1 \otimes G$) then $c(E,\nabla) = c(G,\nabla)\cdot c(F,\nabla)$ (see [E]).

Let us say a few words about the construction.

Write $\mathbb{C}/\mathbb{Z}(p)[-1] = \mathbb{Z}(p) \to \Omega_X^{\cdot}$ ($\Omega_X^{\cdot}$ is the De Rham complex). This maps to $\mathbb{Z}(p)_D = \mathbb{Z}(p) \to \underline{O}_X \to \dots \to \Omega_X^{p-1}$. One knows that $H^2(X,\mathbb{Z}(1) \to \underline{O}_X)$ is identified with the group of isomorphism classes of rank one bundles. P.Deligne [B] remarked that the group $H^2(X,\mathbb{Z}(1) \to \underline{O}_X \to \Omega_X^1)$ is identified with the group of isomorphism classes (E,∇) of rank one bundles E with holomorphic connection ∇ .
As $H^2(X,\mathbb{Z}(1) \to \Omega_X^{\cdot})$ is embedded in $H^2(X,\mathbb{Z}(1) \to \underline{O}_X \to \Omega_X^{\cdot})$, one sees that ∇ is integrable if and only if (E,∇) lies in $H^2(X,\mathbb{Z}(1) \to \Omega_X^{\cdot})$.
Define the product

$$(*)\quad (\mathbb{Z}(p) \to \Omega_X^{\cdot}) \times (\mathbb{Z}(q) \to \Omega_X^{\cdot}) \to (\mathbb{Z}(p+q) \to \Omega_X^{\cdot}) \quad \text{by}$$

$$x \quad , \quad x' \quad \longmapsto \begin{cases} x\cdot x' & \text{if degree } x = 0 \\ 0 & \text{otherwise.} \end{cases}$$

If a higher rank bundle (E,∇) has a flat filtration $E_{k-1} \subset E_k$ (i.e $\nabla E_k \subset \Omega_X^1 \otimes E_k$) with rank one quotients $(L_k = E_k/E_{k-1},\nabla)$, one defines $c_p(E,\nabla)$ as the p-th symmetric product of the classes (L_k,∇) in $H^2(X,\mathbb{Z}(1)\longrightarrow \Omega^{\cdot}_X)$. In particular, as (*) factorizes over the product $\mathbb{Z}(p) \times (\mathbb{Z}(q)\longrightarrow \Omega^{\cdot}_X) \longrightarrow (\mathbb{Z}(p+q)\longrightarrow \Omega^{\cdot}_X)$, $(x,x')\longrightarrow x\cdot x'$, $c_p(E,\nabla)$ is torsion for $p\geq 2$ in this case.

In general consider the canonical filtration $E_{k-1}\subset E_k$ of f^*E on the flag bundle $f : P\longrightarrow X$. This is not a flat filtration. Therefore the question is to find a substitute for flatness. One shows that there is a morphism of complexes $\tau : \Omega^{\cdot}_P \longrightarrow \Omega^{\cdot}_\tau$, with $Rf_*\Omega^{\cdot}_\tau = \Omega^{\cdot}_X$, such that the integrable τ-connection $\tau f^*\nabla$ respects the filtration E_k. This defines classes $(L_k,\tau f^*\nabla)$ in $H^2(P,\mathbb{Z}(1)\longrightarrow \Omega^{\cdot}_\tau)$, and taking formally the same product as (*), classes $c_p(f^*E,f^*\nabla)$ in $H^{2p}(P,\mathbb{Z}(1)\longrightarrow \Omega^{\cdot}_\tau)$. The point is that this "τ-cohomology" is not a free module over $H^{\cdot}(X,\mathbb{Z}(.)\longrightarrow \Omega^{\cdot}_X)$, and one can not apply the standard Hirzebruch-Grothendieck formalism. It is easy to see that $c_p(f^*E,f^*\nabla) = f^{-1}c_p(E,\nabla)$ for a well defined class $c_p(E,\nabla)$ in $H^{2p}(X,\mathbb{Z}(p)\longrightarrow \Omega^{\cdot}_X)$ which maps to $c_p^D(E)$. Functoriality and additivity (and therefore equivalence of this τ-construction with the direct construction on X in case of a flat filtration) are a bit more tricky.

J. Cheeger and J. Simons [C.S] constructed in a differential geometric framework, when X is a C^∞ manifold, classes $\hat{c}_p(E)\in H^{2p-1}(X,\mathbb{R}/\mathbb{Z})$, which - according to S. Bloch - map to $c_p^D(E)$. M. Karoubi [K] defined also classes $\check{c}_p(E) \in H^{2p-1}(X,\mathbb{C}/\mathbb{Z}(p))$ with K-theory and cyclic homology. We did not study the relationship between $\hat{c}_p(E)$, $\check{c}_p(E)$ and $c_p(E,\nabla)$.

Our construction can be performed without supplementary difficulties if ∇ is <u>an integrable holomorphic connection with logarithmic poles along a normal crossing divisor Y</u>. One obtaines classes $c_{p,Y}(E,\nabla)$ in $H^{2p}(X,\mathbb{Z}(p) \longrightarrow Rj_*\mathbb{C})$ (where $j: X-Y \longrightarrow X$ is the open embedding) which map to $c_p^{top}(E)$ and to the image of $c_p^D(E)$ in

$H^{2p}(X,\mathbb{Z}(p) \longrightarrow \underline{O}_X \longrightarrow \dots \longrightarrow \Omega_X^{p-1}\langle Y\rangle)$. They are again functorial and additive for flat exact sequences.

The Atiyah class of a vector bundle with a connection ∇ with logarithmic singularities along Y is the image in $H^1(X,\Omega_X^1 \otimes \underline{\mathrm{End}}(E))$ of the residue of ∇. One obtaines the De Rham - Chern classes of E in terms of ∇. In general, integrable logarithmic connections can replace non integrable C^∞ connections in many situations arising in algebraic geometry (see [E.V]).

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Titel: *3-folds with $c_1 = 0$*

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Suppose that we have a Kähler manifold X of dimension three with $c_1 = 0$. Then X has the following invariants: $K \equiv 0$, $h^{0,0} = 1 = h^{0,3}$, $h^{0,1} = 0 = h^{0,2}$. By duality, the only variable Hodge numbers are $h^{1,1}$ and $h^{2,1}$. From general theory on Hodge numbers the Euler number $e(X)$ of X is $2h^{1,1} - 2h^{2,1}$. The most obvious examples are complete intersections of k smooth hypersurfaces in $\mathbb{P}^{3+k}(\mathbb{C})$ in general position:

a) A quintic in $\mathbb{P}^4$ with Euler number -200.

b) The intersection of a quartic and a quadric resp. two cubics in $\mathbb{P}^5$ with Euler number -176 resp. -144.

c) The intersections of a cubic and two quadrics in $\mathbb{P}^6$ with Euler number -144.

d) The intersection of four quadrics in $\mathbb{P}^7$; its Euler number is -128.

Other important examples are double (resp. triple) coverings of $\mathbb{P}^3(\mathbb{C})$, branched along smooth octic (resp. sextic) surfaces; the Euler numbers of these threefolds are equal to -296 (resp. -204).

Physicists studying superstring theory are interested in 3-dim. Kähler manifolds which have $c_1 = 0$ and absolute value of the Euler number as small as possible, but not equal to zero. In order to get such manifolds, they take the examples given above, look for some groups acting freely on the manifolds, and divide by these group-actions. Another method for getting new examples with different Euler numbers will be described now: we introduce singularities and then resolve them in different ways. Let us consider the double covering of $\mathbb{P}^3(\mathbb{C})$, branched along an octic surface which is allowed to have singularities. Assume first, that these singularities are of type $g(u,v,z) = 0$, where g is a homogeneous polynomial defining a smooth curve of degree 4. Then in local affine coordinates this threefold singularity is given, for example, by

$$(*) \qquad w^2 + u^4 + v^4 + z^4 = 0 .$$

Blowing up the singular point of the branch divisor in $\mathbb{P}^3$, the exceptional divisor D is isomorphic to $\mathbb{P}^2$. The proper transform $\tilde{B}$ of the branching surface B cuts out a curve of degree 4 of this exceptional divisor. The singular point p of the threefold is then resolved into a double cover of $\mathbb{P}^2$, branched along the curve of degree 4. This is a del-Pezzo-surface, which is isomorphic to $\mathbb{P}^2$ blown up in seven points. So the Euler number of the surface, which replaces that singular point of the threefold is 10. The second Betti number of the Milnor fiber of the singularity (*) is 27. When the singular point is taken out of the threefold, the Euler number changes in the same manner as if the Milnor fiber is taken out of a smooth model. So in our example the Euler number decreases by $1 - 27$, that means: it increases by 26. Gluing in the del-Pezzo-surface enlarges the Euler number again by 10. So with every singularity of the described type the Euler number of $\tilde{X}$, which is the double covering of the blown up $\mathbb{P}^3$, branched along $\tilde{B}$, increases by 36. The canonical class of $\tilde{X}$ vanishes because of the special type of the singularities.

Now let us construct an octic surface with 8 singularities of type (4,4,4). The Euler number of $\tilde{X}$ is then equal to -8. In homogeneous coordinates $X_0,\dots,X_3$ consider the quartic $X_1^4 + X_2^4 + X_3^4 = 0$ with one singularity. Now use coordinates $T_0,\dots,T_3$ with

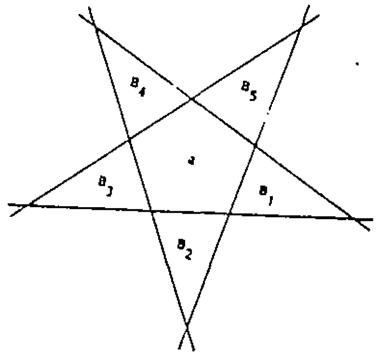
$$X_1 - X_0 = T_1^2 , \; X_2 - X_0 = T_2^2 , \; X_3 - X_0 = T_3^2 , \; X_0 = T_0^2 .$$

The inverse image of the quartic $\sum_{i=1}^{3} x_i^4 = 0$ is an octic with 8 singularities of the type we require.

Similar examples are given by triple coverings of $\mathbb{P}^3(\mathbb{C})$, branched along sextic surfaces with singularities $g(u,v,z) = 0$, g homogeneous of degree 3.

Assume now that the singularities of the branching octic are ordinary nodes. Then the double cover has ordinary nodes $\sum_{i=1}^{4} u_i^2 = 0$. This can be written as $\phi_1\phi_2 = \phi_3\phi_4$. The local meromorphic function $\frac{\phi_1}{\phi_3} = \frac{\phi_4}{\phi_2}$ has a point of indeterminancy at the critical point $\phi_1 = \phi_2 = \phi_3 = \phi_4 = 0$. The graph of this meromorphic function is smooth and contains a $\mathbb{P}^1$ at that critical point; the singularity is replaced by a set of codimension two. Therefore this "small" resolution does not influence the canonical class. The meromorphic function $\frac{\phi_1}{\phi_4} = \frac{\phi_3}{\phi_2}$ gives us a different small resolution. The Euler number increases by 2 with every small resolution. Now a different problem comes into the game: It is uncertain whether or not the small resolutions $\hat{X}$ are still Kähler. This depends on the number of nodes and their special position. In general the manifolds $\hat{X}$ are only Moišeson: the transcendence degree of the function field is 3. The results of Moišeson tell us that a manifold is projective algebraic if and only if it is Moišeson and Kähler. So in our examples the properties "Kähler" and projective algebraic are equivalent. If we take, for example, the Čmutov octic $\sum_{i=1}^{3} T_8(x_i) - 1 = 0$ with 108 nodes as branching surface - T_8 is the Čebyšev polynomial of degree 8 - then no small resolution is Kähler. But if we take the Čmutov octic $\sum_{i=1}^{3} T_8(x_i) + 1 = 0$ with 144 nodes, Werner proved that some $\hat{X}$ are projective algebraic. These are Kähler manifolds with trivial canonical bundle and Euler number $-296 + 2 \cdot 144 = -8$.

A similar example is given by the quintic Čmutov hypersurface $\sum_{i=1}^{4} T_5(x_i) = 0$ in $\mathbb{P}^4(\mathbb{C})$ with 96 nodes. The small resolutions of that singular variety have Euler number $-200 + 2 \cdot 96 = -8$, again some are projective algebraic.



Now let $f(x,y) = 0$ be the quintic curve in the affine plane, which is given by the product of the five lines of a regular pentagon. As a function of two real variables, f has relative extrema in the center a of the pentagon and in one point b_i of every triangle B_i. So both partial derivatives of f vanish at these six points and at the ten intersection points of the five lines. By symmetry $f(b_i) = f(b_j)$ for all i and j. Consider the threefold given in four affine coordinates in $\mathbb{P}^4(\mathbb{C})$ by the equation $f(u,v) - f(z,w) = 0$. This threefold has 126 nodes, the Euler number of a small resolution is given by $-200 + 2 \cdot 126 = +52$. It is an open question, whether some of the small resolutions are projective algebraic or not.

The quintic $\sum_{i=0}^{4} x_i^5 - 5 \prod_{i=0}^{4} x_i = 0$ in $\mathbb{P}^4(\mathbb{C})$ - given by Schoen - has 125 nodes. They are $(\xi_5^{a_0}, \dots, \xi_5^{a_4})$ with a primitive 5-th root of unity ξ_5 and $\sum_{i=0}^{4} a_i \equiv 0(5)$, It can be proved that some of the small resolutions are projective algebraic with Euler number $+50$.

Furthermore Chad Schoen wrote me that the fibre product of a rational surface with itself gives rise to examples of nodal threefolds with $c_1 = 0$. Special small resolutions of various examples lead to projective threefolds with $c_1 = 0$ and Euler number of small absolute value. Other examples have large positive Euler number.

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