Geometric Realization of Conformal Field
Theory on Riemann Surfaces

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Abstract

Conformal field theory on a family of Riemann surfaces is formulated.
We derive equations of motion of vacua which are parametrized by
moduli of Riemann surfaces and show that these vacua are
characterized uniquely by these equations. Our theory has a deep
connection with Sato's theory of KP equations.

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§0 Introduction

Recently it has been recognized that the conformal field theory (CFT) on Riemann surfaces of arbitrary genus plays an essential role to understand the profound mechanism of the string theory [F.S.][Fr.]. Among others very important insights have been brought by a formulation of bosonization rule [D.J.K.M.] [A-G.B.M.N.V.][E.O.] and an observation that the Virasoro (energy-momentum tensor) operator deforms the moduli of Riemann surfaces [E.O.1][B.M.S.].

One approach to the CFT on Riemann surfaces is based on the path-integral method initiated by Polyakov [P.]. This approach can be regarded as a geometric one which is recently developed into the algebro-geometric level [B.K.].

Another approach to the CFT is an algebraic one based on the representation theory of the Virasoro algebra, and was initiated by Belavin Polyakov and Zamolodchikov [B.P.Z.]. This approach has an essential connection with solvable models of statistical mechanics and Kac-Moody Lie algebras.

One of the aims of this paper is to unify these two approaches by constructing a CFT on a family of Riemann surfaces in an operator formalism. Another aim is to establish a solid mathematical basis for a class of CFT on Riemann surfaces.

The main ingredient of our theory is M. Sato's theory of KP equations [Sa.][S.S.]. Originally his theory was developed to solve a problem of soliton equations, but here we show that his theory actually covers the CFT on Riemann surfaces. Here two notions, the universal Grasmann manifold (UGM) and the τ-function, play

the most essential role.

Theory of KP equations was reconstructed by Date, Jimbo, Kashiwara and Miwa in an operator formalism [D.J.K.M.]. Our theory is deeply interrelated with their formulation and is in a sense an extended version to a completed Fock space; in order to treat theta functions we need completion. It should be compared with the treatment of Segal and Wilson [S.W.] which treats the same content in the Hilbert space formalism.

The relation between the theory of KP equations and Riemann surfaces was formulated by Krichever as the theory of the Baker-Akhiezer function [Kr.][Mum.2], which plays an important role in this paper. This relation to Riemann surfaces was studied further by Mulase and Shiota in connection with the Schottky problem [Mul.][Sh.].

Fundamental operators in our theory are the free fermions $\psi(z)$, $\bar{\psi}(z)$, the current $J(z)$ and the energy-momentum tensor $T(z)$, acting on the Fock space $\mathcal{F}$. They are provided, a priori, without reference to Riemann surfaces. Informations of Riemann surfaces are carried by vacuum states $[X] \in P(\mathcal{F})$ (the projective Fock space). In this respect our theory is based on the interaction picture in physicist's terminology. The space $\mathcal{E}$ (resp. $\mathcal{E}$ which we call Weierstrass system (§ 2)) of all geometric data sets is a dressed moduli space of Riemann surfaces (resp. Riemann surfaces and line bundles) The physical vacuum $[X] \in P(\mathcal{F})$ moves with a parametrization of data $X \in \mathcal{E}$ (or $\mathcal{E}$). An important fact is that these spaces are infinitesimally homogeneous spaces on which infinite dimensional Lie algebras $\mathfrak{G} = \mathbb{C}((z^{-1}))\frac{d}{dz}$ and $\tilde{\mathfrak{G}} = \mathfrak{G} \oplus \mathbb{C}((z^{-1}))$ act respectively.

Our main view point is to interprete the map $\mathcal{B}$(resp. $\mathfrak{X}$) $\longrightarrow \mathbb{P}(\mathcal{F})$, $X \longmapsto [X]$ as a period map of the moduli space $\mathcal{B}$(resp. $\mathfrak{X}$) and is to investigate the deformation of moduli generated by operators $\ell \in \mathfrak{G}$(resp. $\widetilde{\mathfrak{G}}$). An important fact is that the view as a period map matches very well with the method of field theory. For example the fundamental operators $T(z)$ and $J(z)$ are just the deformation generators mentioned above. From physical point of view this deformation equation can be considered as an equation of motion of the vacua. On the other hand it can be regarded as the Gauss-Manin connection from an algebro-geometric point of view.

We have tried to make this article self-contained as much as possible, since no detailed reference on Sato's theory is available in western language. This paper is organized as follows:

§1 and §2 are devoted to geometrical setup for a description of the period map. In §1 we give a description of UGM (universal Grassmann manifold) and its Plücker embedding into projective Fock space $\mathbb{P}(\mathcal{F})$, essentially following M.Sato. In §2 we construct the dressed moduli space of curves $\mathcal{B}$ (following [B.M.S.]) and its generalization $\mathfrak{X}$ including the Picard variety, which are infinite dimensional complex manifolds.

Using the theory of the abelian functions, we can define a period map from $\mathcal{B}$(resp. $\mathfrak{X}$) to UGM and derive Torelli-type theorems (2.28),(2.29) as a main result of §2. The action of the modular group on $\mathcal{B}$(resp. $\mathfrak{X}$) is also important.

§3 is devoted to a preparation of algebraic setup. In terms of fermion operator $\psi(z)$, $\overline{\psi}(z)$ acting on the Fock space $\mathcal{F}$, we construct a Fock space representation of a central extension of $\mathfrak{G}$

(or $\widetilde{\mathfrak{G}}$), where the Virasoro algebra and the current algebra appear.

In §4 we provide the bosonization $B:\mathcal{F} \longrightarrow \mathcal{H}$ (bosonized Fock space) and show that the image of UGM in $\mathcal{H}$ can be characterized by a conjugate pair of wave functions and the Hirota equations for τ-functions. We essentially follow [D.J.K.M.] but reformulate the theory in an appropriate form for field theory.

The contents of §1-§4 can be summarized in the following diagram:

$$
\begin{array}{ccccccccccc}
\mathcal{B} & \longrightarrow & \mathcal{B}(\mathcal{F}) & \longrightarrow & \widetilde{P}(\mathfrak{L}) & \longrightarrow & \widetilde{\mathrm{UGM}} & \longrightarrow & \mathcal{F}^{\times} & \overset{\sim}{\longrightarrow} & \mathcal{H}^{\times} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{B} & \longrightarrow & \mathcal{B}(\mathcal{F}) & \longrightarrow & P(\mathfrak{L}) & \longrightarrow & \mathrm{UGM} & \longrightarrow & P(\mathcal{F}) & \overset{\sim}{\longrightarrow} & P(\mathcal{H}).
\end{array}
$$

$\Delta_{1/2}:$ (to the left of the bottom row)

The space $\mathcal{B}(\mathcal{F})$ is, briefly to say, the deformation of $\mathcal{B}$ in the direction to the Jacobian varieties with a gauge fixing (see §2,F)). The symbols with " $\sim$ " ($\widetilde{\mathcal{B}}$ etc.) stand for the induced $\mathbb{C}^{\times}$-bundles.

$$
\begin{array}{ccc}
 & \overset{\tau}{\nearrow} & \mathcal{H}^{\times} \\
 & & \downarrow \\
\mathcal{B}_{g}(\mathcal{F}) & \underset{\Delta_{1/2}}{\longrightarrow} & P(\mathcal{H})
\end{array}
$$

In §5 we construct a well-behaved lifting which we call the τ-function of the period map $\Delta_{1/2}$ with a method based on the Krichever's theory concerning Baker-Akhiezer function (i.e. a wave function associated to the curve). This τ-function can be explicitly written down in terms of classical Riemann's theta function and the Jacobian embedding of Riemann surfaces:

$$
\tau(\mathfrak{t},X_{\mathbb{c}}) = e^{\frac{1}{2}q(\mathfrak{t})}\theta(I(\mathfrak{t})+c|\Omega).
$$

This explicit form of the τ-function has been already given by several authors [I.M.O.][A-G.G.R.] [V.].

In §6 the explicit forms of actions of fundamental operators on the τ-function and N-point functions are obtained by using the concrete expression of the τ-function. It is interesting to note that we can obtain the addition formulae of θ-functions associated with the Jacobian of a curve systematically, which are closely related with the Schottky problem in algebraic geometry ([Fay],[Mum.3],[v.G.] ,[Mul.],[Sh.]).

In §7 the fundamental differential equations satisfied by the τ-function will be derived. This provides the Gauss-Manin connection of this period map. The main result of this paper can be stated as follows: The τ-function is characterized by the differential linear equations with infinite degrees of freedom:

$$[\theta(\ell) + a(\ell, X_c)]\tau(t, X_c) = \Phi_B(\ell)\tau(t, X_c), \quad {}^\forall \ell \in \mathfrak{G}$$

$$\sum_{i=1}^{g} \frac{1}{2\pi\sqrt{-1}} \int_{\beta_i} d\varphi \frac{\partial}{\partial c^i} \tau(t, X_c) = \Phi_B(\varphi)\tau(t, X_c), \quad {}^\forall \varphi(z) \in \mathfrak{X}(X)$$

and together with the automorphy:

$$\tau(t, X_{c+\Omega a+b}) = \exp\left(-2\pi\sqrt{-1}\left(\frac{1}{2}{}^t a\Omega a + {}^t a(I(t)+c)\right)\right)\tau(t, X_c).$$

Here X_c lies in the moduli space $\mathfrak{E}(\mathfrak{F})$ of Riemann surfaces (§2, F)) and $t = (t_1, t_2, \ldots)$ is the parameter of the bosonic Fock space $\mathcal{H}$ (§4, A)). For $\ell \in \mathfrak{G}$, $\theta(\ell)$ stands for its action on X_c as a vector field on $\mathfrak{E}(\mathfrak{F})$ (§2, D)) and $\Phi_B(\ell)$ acts on t as a quantized operator on $\mathcal{H}$ ((3.20),(4.4)). For the notations in the second type of equations (concerning gauge transformations) see Appendix. It is a remarkable fact that the τ-function chracterized by the above *linear* equations satisfies Hirota's *nonlinear* equations.

In Appendix we collect the notations and the formulae in the theory of abelian functions on Riemann surfaces which we have used

for explicit descriptions of our results in this article.

One of the main characteristics of our theory is that we have constructed the whole theory to keep the complex analyticity throughout the formulation. Another characteristic is that any infinite dimensional manifold like $\mathcal{G}$ and $\mathcal{K}$ is treated as a projective limit of a finite dimensional manifold. The latter treatment is compatible with the completion of the Fock space.

Moreover the structure of our theory has an intimate similarity with recent theory of arithmetic surfaces due to Arakelov and Faltings [Ar.][Fal.].

Some of the results in §5 and §6 have already been given by Ishibahsi, Matsuo and Ooguri [I.M.O.] and Alvarez-Gaume, Gomez and Reina [A-G.G.R.]. These sections are, however, reformulated with the help of the preparations of §1- §4 within the framework of the theory of infinite dimensional complex analytic manifolds. Our main contribution in this paper is to clear up the geometric setting of the whole theory and to provide a system of differential equations which characterizes the τ-function· uniquely.

After this work has been completed a very interesting paper by Krichever and Novikov [K.N.] came to our attention, which seems to have a close relation with our paper.

§1. UGM and its Plücker embedding.

A)Universal Grassmann manifold (UGM)

(1.0) The theory of UGM was created and developed by M. Sato and plays an essential role in his theory of KP equations ([Sato], [S.S]). Thanks to the additional structure of filtration one can obtain an almost complete analogy to the theory of usual Grassmann manifolds for finite dimensional spaces [S.N.]. We recall here in §1 the most elementary part of the theory (or the part which is the same as usual finite dimensional cases). The deeper part where we need an essential modification including the theory of τ-functions will be treated in §3 and §4.

(1.1) Let V be a linear space (over $\mathbb{C}$) equipped with a filtration $\{F^m V\}_{m \in \mathbb{Z}}$ satisfying the following conditions:

1) $\{F^m V\}_{m \in \mathbb{Z}}$ is decreasing, $\cdots \supset F^m V \supset F^{m+1} V \supset \cdots$;

2) $\underset{m \in \mathbb{Z}}{\cup} F^m V = V$, and $\underset{m \in \mathbb{Z}}{\cap} F^m V = \{0\}$;

3) $\dim_{\mathbb{C}} F^m V / F^{m+1} V = 1$;

4) the topology induced from the filtration (i.e. $\{F^m V\}_{m \in \mathbb{Z}}$ form a system of neighbourhoods of 0) is complete.

Such V is actually unique up to isomorphisms.

(1.2) For concrete calculation we choose a (topological) basis of V as follows. For later use we employ $\mathbb{Z}h = \mathbb{Z} + (1/2) = \{n+(1/2); n \in \mathbb{Z}\}$ as an index set and choose e^μ in $F^{\mu-(1/2)} V - F^{\mu+(1/2)} V$ for each $\mu \in \mathbb{Z}h$. Then every element v in V can be expressed as

$$v = \sum_{\substack{-\infty < < \mu < \infty \\ \mu \in \mathbb{Z}h}} v_\mu e^\mu, \qquad v_\mu \in \mathbb{C}$$

and

$$F^m V = \{v = \sum_{m \leq \mu} v_\mu e^\mu\}.$$

Example(1.3). $V = \hat{K}\sqrt{d\xi}$ ($K = \mathbb{C}((\xi))$ is the field of formal Laurent series). The filtration comes from the non-archimedian valuation $\nu: V \to \bar{Z} = Z \cup \{\infty\}$ with $\nu(\xi^n\sqrt{d\xi}) = n$. We use a basis $\{e^{m+(1/2)} = \xi^m\sqrt{d\xi}\}_{m \in Z}$. For later use we introduce another indeterminate z with the relation $z\xi = 1$. Namely z is a coordinate whose value tends to ∞ at the reference point. When we use this coordinate in $V = \mathbb{C}((z^{-1}))\sqrt{dz}$ the valuation is given by $\nu(z^m\sqrt{dz}) = \nu(\xi^{-m-1}\sqrt{d\xi}) = -m-1$.

(1.4) Let $\bar{V} = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ be the topological dual space (namely the space of *continuous* $\mathbb{C}$-linear functions on V with discrete topology on $\mathbb{C}$). Then $\bar{V}$ has a canonical dual structure of a filtered space as

$$F_m\bar{V} = \mathrm{Ker}(\mathrm{Hom}(V, \mathbb{C}) \to \mathrm{Hom}(F^m V, \mathbb{C})).$$

If we set $F^m\bar{V} = F_{-m}\bar{V}$, and $F_m V = F^{-m}V$, then $(\bar{V}, \{F^m\bar{V}\})$ satisfies the same properties as in (1.1) and V and $\bar{V}$ are dual to each other. In particular

$$(\mid) : \bar{V} \times V \longrightarrow \mathbb{C}$$
$$\cup \qquad\qquad \cup$$
$$(\bar{v}, v) \longmapsto (\bar{v}\mid v) = \bar{v}(v)$$

is a complete dual pairing of topological vector spaces.

For a basis $\{e^\mu\}_{\mu \in Zh}$ of V the dual basis $\{\bar{e}^\mu\}_{\mu \in Zh}$ of $\bar{V}$ can be defined by the relations:

$$(\bar{e}^\mu \mid e^\nu) = \delta^{\mu+\nu, 0}.$$

Then $\bar{e}^{m+(1/2)} \in F^m\bar{V} - F^{m+1}\bar{V}$. Every element $\bar{v} \in F^m\bar{V} \subset \bar{V}$ can be expressed as

$$\bar{v} = \sum_{m \leq \mu} \bar{v}_\mu \bar{e}^\mu, \qquad \bar{v}_\mu \in \mathbb{C}.$$

(1.5) For later use we introduce another notation:

$$e_\mu = e^{-\mu}, \quad \bar{e}_\mu = \bar{e}^{-\mu}, \quad \mu \in \mathbb{Z}h.$$

Then the pairing on $\bar{V} \times V$ becomes the natural one:

$$(\bar{e}_\mu | e^\nu) = \delta_\mu{}^\nu, \quad (\bar{e}^\mu | e_\nu) = \delta^\mu{}_\nu.$$

Definition (1.6). A subset M of $\mathbb{Z}h$ is called a *Maya-diagram* if both $M \cap \mathbb{Z}h_{>0}$ and $M^c \cap \mathbb{Z}h_{<0}$ are finite sets where $\mathbb{Z}h_{>0} = \{\mu \in \mathbb{Z}h; \mu > 0\}$ etc. and M^c is the complement of M in $\mathbb{Z}h$. The integer $\chi(M) = \#(M \cap \mathbb{Z}h_{>0}) - \#(M^c \cap \mathbb{Z}h_{<0})$ is called the *charge* or the *Euler-characteristic* of M.

We denote by M (resp. M_p) the set of all Maya-diagrams (resp. Maya-diagrams of charge p).

Definition (1.7). For $M \in M_p$ we express it with an increasing function $\mu : \mathbb{Z}h_{<p} = \{\mu \in \mathbb{Z}h; \mu < p\} \to \mathbb{Z}h$ as

$$M = \{ \cdots, \mu(p-(3/2)), \mu(p-(1/2)) \}.$$

This uniquely defined μ is called the *characteristic function* of M. Note that $\mu(\nu) = \nu$ for $\nu \ll \infty$. Therefore the set
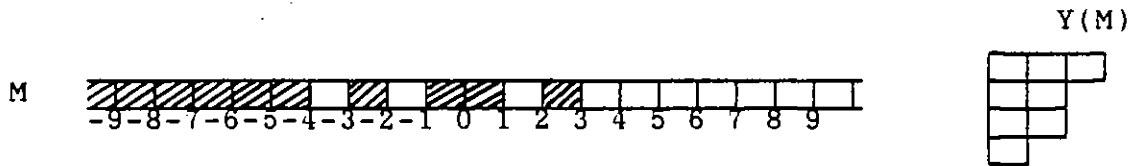
$$\{\mu(\nu) - \nu; \nu \in \mathbb{Z}h, \mu(\nu) - \nu > 0\}$$

is finite and defines a Young diagram Y(M). We say that Y(M) is *associated with* M. The number of boxes

$$d(M) = \sum_\nu (\mu(\nu) - \nu)$$

is called the *degree* of M.

M

Y(M)

$$\chi = 0, \quad d = 8$$

Denote by Y (resp. Y_d) the set of all Young diagrams (resp.

Young diagrams of degree d). Note that $\#\mathfrak{Y}_d = p(d)$, the number of partitions of d.

Lemma (1.8). We have a canonical bijection

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\ \sim\ } & \mathbb{Z} \times \mathfrak{Y} \\ \cup\!\!| & & \cup\!\!| \\ M & \longmapsto & (\chi(M),\ Y(M)). \end{array}$$

Definition (1.9). We fix $p \in \mathbb{Z}$. The *universal Grassmann manifold* (of charge p) UGM^p is the set of closed subspaces U of V s.t. i) the kernel and cokernel of the natural map $f : U \to V/F^0 V$ is of finite dimension; ii) dim Ker(f) - dim Coker(f) = p. These U's in UGM^p are called semi-infinite subspaces (of charge p). Set $UGM = \bigcup_{p \in \mathbb{Z}} UGM^p$.

Note that, by definition, the topology on U, which is induced by V, is discrete.

(1.10) By considering the image of $F^{-m}V \cap U$ in $V/F^m V$, $m \in \mathbb{N}$, one can introduce on UGM^p a canonical structure of a scheme as a projective limit of schemes of finite type (Grassmann manifolds for finite dimension), but we will not make it precise here in order to avoid unnecessary complication. Specialists on abstract algebraic geometry could easily fill up logical gaps. For non-specialists it is enough to know that UGM^p admits a 'good' structure of an infinite dimensional complex manifold as a 'limit' of finite dimensional manifolds.

(1.11) Let U be a closed subspace in V. If we consider the induced filtration $\{F^m U = F^m V \cap U\}$ on U, then dim $F^m U/F^{m+1} U \leq 1$. If we set

$$M(U) = \{m+(1/2) \in \mathbb{Z}h; \dim F^m U/F^{m+1} U = 1\},$$

then $U \in UGM^p$ if and only if $M(U)$ is a Maya-diagram of charge p.

We can then define the Young diagram $Y(U) = Y(M(U))$ associated with U, and the degree of U, $d(U) = d(M(U))$.

B) Tangent space of UGM.

Proposition (1.12)([S.N.]). For $U \in UGM$ we have a canonical isomorphism

$$T_U UGM \simeq Hom_{\mathbb{C}}(U, V/U)$$

(here $T_U UGM$ denotes the 'holomorphic' tangent space of UGM at U).

Definition (1.13). An *endomorphism* φ on V is a continuous linear map from V to V such that there exists an integer $n \in \mathbb{Z}$ with $\varphi(F^m V) \subset F^{m+n}(V)$ for all $m \in \mathbb{Z}$. Denote by $F^n End(V)$ the set of such endomorphisms and put $End(V) = \cup F^n End(V)$.

$End(V)$ is thus a filtered $\mathbb{C}$-algebra (i.e. if $\varphi_1 \in F^m$, $\varphi_2 \in F^n$, then $\varphi_1 \circ \varphi_2 \in F^{m+n}$) whose filtration topology is complete.

For each $\varphi \in End(V)$ define $\theta_U(\varphi) \in T_U UGM$ by $U \xrightarrow{i} V \xrightarrow{\varphi} V \longrightarrow V/U$.

Proposition (1.14). The linear map $\theta : End(V) \longrightarrow H^0(UGM, \theta) = $ {global holomorphic vector fields on UGM} is an anti-homomorphism of Lie algebras, i.e. $[\theta(\varphi_1), \theta(\varphi_2)] = \theta([\varphi_2, \varphi_1])$.

C) Frame bundle on UGM

Definition (1.15). Let $U \in UGM^p$. A *frame* ξ of U is a basis of U

$$\xi = \{ \cdots, \xi^{p-(3/2)}, \xi^{p-(1/2)} \}$$

such that for $\exists \mu_0 \in \mathbb{Z}h$ and $\forall \mu \leq \mu_0$, $\xi^\mu \in F^{\mu-(1/2)}U - F^{\mu+(1/2)}U$ and $\xi^\mu \equiv e^\mu \mod F^{\mu+(1/2)}U$. (Since the topology of U is discrete, all vectors in U can be expressed as a linear combination of a *finite* number of vectors in ξ.) If we express ξ with $\{e^\mu\}$, then we have

$$\xi^r = e^\nu \xi_\nu{}^r \text{ (with Einstein's notation) } \nu \in \mathbb{Z}h, r \in \mathbb{Z}h_{<p}$$

or in a form of matrices

$$\xi_\nu{}^r = \begin{vmatrix} \cdot \cdot & & & \\ & 1 & & \\ & 1 & 1 & \\ * & & & \cdot \\ & & & * \\ \hline * & & & \\ * & & & * \end{vmatrix} \begin{matrix} \nu \\ \vdots \\ \vdots \\ \vdots \\ p-1/2 \\ p+1/2 \end{matrix}$$

$$r \to \cdots\cdots p-3/2 \quad p-1/2$$

We denote by FUGM^p the set of all frames of elements in UGM^p. The natural map $\pi : \mathrm{FUGM} \longrightarrow \mathrm{UGM}$ is holomorphic.

Definition (1.16). Let $\mathrm{GL}(\mathbb{Z}h_{\langle p})$ be a group of semi-finite matrices of the form

$$A = (a_r{}^s)_{r,s \in \mathbb{Z}h_{\langle p}} = \begin{vmatrix} \cdot \cdot & & & 0 \\ & 1 & & \\ & 1 & 1 & \\ & & & \\ \hline * & & & \tilde{A} \\ & & & \end{vmatrix} \begin{matrix} r \\ \vdots \\ \vdots \\ \vdots \\ p-3/2 \\ p-1/2 \end{matrix}$$

$$s \to \cdots\cdots\cdots\cdots\cdots p-1/2$$

$$\det \tilde{A} \neq 0$$

For an element $A \in \mathrm{GL}(\mathbb{Z}h_{\langle p})$ as above we define $\det(A) = \det(\tilde{A})$, which is easily seen to be well-defined. We define

$$\mathrm{SL}(\mathbb{Z}h_{\langle p}) = \{A \in \mathrm{GL}(\mathbb{Z}h_{\langle p}) : \det A = 1\}.$$

The group $\mathrm{GL}(\mathbb{Z}h_{\langle p})$ operates on FUGM^p from the right as follows;

$$(\xi A)^r = \xi^s a_s{}^r \quad A = (a_s{}^r) \in \mathrm{GL}(\mathbb{Z}h_{\langle p}), \quad \xi = (\xi^r)_{r \in \mathbb{Z}h_{\langle p}} \in \mathrm{FUGM}^p.$$

Proposition (1.17)

1) $\pi : \mathrm{FUGM}^p \longrightarrow \mathrm{UGM}^p$ is a principal $\mathrm{GL}(\mathbb{Z}h_{\langle p})$-bundle,

2) $\widetilde{\mathrm{UGM}}^p \equiv \mathrm{FUGM}^p/\mathrm{SL}(\mathbb{Z}h_{\langle p}) \longrightarrow \mathrm{UGM}^p$ is a principal $\mathbb{C}^*$-bundle.

Definition (1.18). Let $\mathrm{GL}^f(\mathbb{Z}h)$ be the group of invertible linear transformations $g : g(e^\mu) = e^\nu g_\nu{}^\mu$ of V such that $g_\nu{}^\mu = \delta_\nu{}^\mu$ except for a

finite number of μ, ν's:

$$\begin{vmatrix} \ddots & & & & \\ & 1_1 & & & 0 \\ & & \ddots & & \\ & & & \boxed{*} & \\ & 0 & & & \ddots \\ & & & & 1_1 \\ & & & & & \ddots \end{vmatrix}$$

Also $P_- = \{1+\varphi; \varphi \in F^1 End(V)\}$ is a group since any element $1+\varphi \in P_-$ is invertible: $(1+\varphi)^{-1} = \sum_{n=0}^{\infty} (-1)^n \varphi^n$. An element $g \in P_-$ is represented in the following matrix form;

$$(g_\nu{}^\mu) = \begin{vmatrix} \ddots & & & & \\ & 1_1 & & & 0 \\ & & 1_1 & & \\ & & & 1_1 & \\ & & & & \ddots \\ & * & & & \ddots \end{vmatrix}$$

We denote by $GL_f(\mathbb{Z}h)$ the linear transformation group on V generated by $GL^f(\mathbb{Z}h)$ and P_-. Any element g in $GL_f(\mathbb{Z}h)$ can be expressed as

$$(g_\nu{}^\mu) = \begin{vmatrix} \ddots & & & & \\ & 1_1 & & 0 & \\ & * & \boxed{*} & \\ & & & & 1_1 \\ & * & * & & \ddots \end{vmatrix}$$

Proposition (1.19).

1) The group $GL_f(\mathbb{Z}h)$ acts on $FUGM^p$ from the left transitively.

2) The action of $GL_f(\mathbb{Z}h)$ is commutative with the action of $GL(\mathbb{Z}h_{<p})$:

$$(g\xi)A = g(\xi A), \text{ for } {}^\forall g \in GL_f(\mathbb{Z}h), \; {}^\forall A \in GL(\mathbb{Z}h_{<p}), \; {}^\forall \xi \in FUGM^p.$$

3) $GL_f(\mathbb{Z}h)$ acts on $\widetilde{UGM}^p$ and UGM^p.

Proposition (1.20) (Bruhat decomposition) ([S.N.]).

1) For ${}^\forall p \in \mathbb{Z}$ and ${}^\forall Y \in \mathscr{Y}$, $UGM^{p,Y} = \{U \in UGM^p; Y(U) = Y\}$ is a locally closed subspace of UGM^p, and P_- acts on each $UGM^{p,Y}$ transitively.

2) $UGM^p = \underset{Y \in \mathcal{Y}}{\amalg} UGM^{p,Y}$;disjoint union (Bruhat decomposition).

3) $UGM^{p,\phi}$ is an open dense subset of UGM^p whose complement is of codimension 1 in UGM^p. We call this subspace $UGM^{p,\phi}$ the *big cell* (in UGM^p). We denote $UGM^{0,\phi}$ simply by UGM^ϕ.

We denote $F^1 End(V)$ by $\mathfrak{B}_-$ which is the Lie algebra of P_-. Note that in $\mathfrak{B}_-$ the Lie antihomomorphism θ in (1.14) lifts to

$$\mathfrak{T} : \mathfrak{B}_- \longrightarrow H^0(\widetilde{UGM}, \theta) \quad , \quad \varphi \longmapsto \mathfrak{T}(\varphi)$$

In general it is necessary to "subtract an infinity" when we lift the action of $\varphi \in End(V)-\mathfrak{B}_-$ to that on $\widetilde{UGM}$. This procedure will be treated in §3.

D) Fock space (Semi-infinite form)

(1.22) For each Maya-diagram $M = \{ \cdots \mu(p-(3/2)) \; \mu(p-(1/2)) \}$ we denote by e^M or $|M\rangle$ the symbol

$$e^{\mu(p-(1/2))} {}_\wedge \; e^{\mu(p-(3/2))} {}_\wedge \; \cdots \; .$$

We consider also finite permutations of $\mu(v)$'s, under which e^M is multiplied by ± 1 as usual.

Definition (1.23). The *Fock space* (of charge p) is a direct product

$$\mathcal{F}_p = \underset{M \in \mathcal{M}_p}{\Pi} \mathbb{C}e^M$$

and the whole Fock space is

$$\mathcal{F} = \underset{p}{\oplus} \mathcal{F}_p.$$

We define two fundamental operators on $\mathcal{F}$, the *charge operator* and the *mass operator* as

$$J_0(\alpha) = p\alpha \quad \text{if } \alpha \in \mathcal{F}_p,$$
$$M(\alpha) = d\alpha \quad \text{if } \alpha = e^M \text{ with } d(M) = d.$$

They commute each other ($[J_0, M] = 0$). If we set

$$\mathcal{F}_p(d) = \{\alpha \in \mathcal{F}; J_0(\alpha) = p\alpha, \ M(\alpha) = d\alpha\},$$

we obtain

$$\mathcal{F} = \bigoplus_p \prod_d \mathcal{F}_p(d).$$

Note that

$$\mathcal{F}_p(d) = \bigoplus_{\chi(M)=p, d(M)=d} \mathbb{C}e^M,$$

and it is of dimension $p(d)$ (cf.(1.7)).

We define moreover the *energy operator* as

$$L_0 = \frac{1}{2}J_0^2 + M.$$

On $\mathcal{F}_p(d)$, L_0 is a scalar operator; $L_0 = (\frac{1}{2}p^2 + d)\mathrm{id}$.

We introduce a complete Hausdorff linear topology on $\mathcal{F}_p$ by the filtration $F^d(\mathcal{F}_p) = \prod_{d' \geq d} \mathcal{F}_p(d')$; $\cdots \supseteq F^d(\mathcal{F}_p) \supseteq F^{d+1}(\mathcal{F}_p) \supseteq \cdots$.

(1.24) For a Maya-diagram M as above we define the (dual) symbol $\bar{e}_M$ by

$$\bar{e}_M = \cdots \wedge \bar{e}_{\mu(p-(3/2))} \wedge \bar{e}_{\mu(p-(1/2))}$$

which is also denoted by $\langle M|$.

Then we define the *dual* Fock space as a *direct sum* with *discrete* topology:

$$\bar{\mathcal{F}} = \bigoplus_{M \in \mathcal{M}_p} \mathbb{C}\bar{e}_M.$$

We can call it 'dual' since we have a continuous bi-linear map

$$\langle \ | \ \rangle : \bar{\mathcal{F}} \times \mathcal{F} \longrightarrow \mathbb{C}$$
$$\cup\!\!\!| \qquad\qquad \cup\!\!\!|$$
$$(\bar{e}_M, \ e^N) \longmapsto \langle M|N\rangle = \delta_M^{\ N},$$

by which $\bar{\mathcal{F}}$ and $\mathcal{F}$ are topologically dual to each other (with discrete topology on $\mathbb{C}$).

Any element α in $\mathcal{F}$ can be written as

$$\alpha = \sum_M \langle \bar{e}_M | \alpha \rangle e^M = \sum_M \alpha_M \, e^M$$

(1.25) We set

$$|p\rangle = e^{p-(1/2)} \wedge e^{p-(3/2)} \wedge \cdots ,$$

$$\langle p| = \cdots \wedge \bar{e}_{p-(3/2)} \wedge \bar{e}_{p-(1/2)}$$
$$= \cdots \wedge \bar{e}^{-p+(3/2)} \wedge \bar{e}^{-p+(1/2)} .$$

Then $\mathcal{F}_p(0) = \mathbb{C}|p\rangle$.

Proposition (1.26). The group $GL_f(\mathbb{Z}h)$ acts (topologically) on $\mathcal{F}$ by

$$G = G[g]: \quad \mathcal{F} \quad \longrightarrow \quad \mathcal{F}$$
$$\Cup \qquad\qquad\qquad \Cup$$

$$e^{\mu(p-(1/2))} \wedge e^{\mu(p-(3/2))} \wedge \cdots$$

$$\longmapsto g(e^{\mu(p-(1/2))}) \wedge g(e^{\mu(p-(3/2))}) \wedge \cdots$$

for $g \in GL_f(\mathbb{Z}h)$ where $g(e^*) \wedge \cdots$ is understood according to the rules of usual exterior algebra.

Proof. For each $M \in M_p$

$$G(e^M) = \sum_{N \in M_p} G_N^M \, e^N$$

where $G_N^M = \langle \bar{e}_N | G(e^M) \rangle = \det((a_{\nu(\alpha)}{}^{\mu(\beta)})_{\alpha < p, \beta < p})$
with $M = \{ \cdots \mu(p-(3/2)) \; \mu(p-(1/2)) \}$, $N = \{ \cdots \nu(p-(3/2)) \; \nu(p-(1/2)) \}$.
Since g belongs $GL_f(\mathbb{Z}h)$, the determinant of the infinite matrix is well defined. ∎

Corollary (1.27). The group $GL_f(\mathbb{Z}h)$ acts also on $\bar{\mathcal{F}}$ as $\bar{G} = \bar{G}[g] \in$ End($\bar{\mathcal{F}}$) with the condition that

$$\langle \bar{G}(\bar{\alpha}) | G(\alpha') \rangle = \langle \alpha | \alpha' \rangle$$

for $\forall \bar{\alpha} \in \bar{\mathcal{F}}$, $\alpha' \in \mathcal{F}$.

Remark (1.28). One can write $\bar{G}(\bar{\alpha})$ explicitly as above by using the matrix representation of g^{-1}.

E) Plücker embedding

(1.29) We consider $\xi = \{\cdots, \xi^{p-(3/2)}, \xi^{p-(1/2)}\} \in \mathrm{FUGM}^p$. Then we see that

$$\wedge\xi = \xi^{p-(1/2)} \wedge \xi^{p-(3/2)} \wedge \cdots \in \mathcal{F}_p^\times = \mathcal{F}_p - \{0\}.$$

Hence we obtain a holomorphic map

$$\mathrm{FUGM}^p \longrightarrow \mathcal{F}_p .$$

Proposition (1.30). For $\xi \in \mathrm{FUGM}^p$ and $A \in \mathrm{GL}(\mathbb{Z}h_{<p})$ we have

$$\wedge(\xi A) = \det A \cdot (\wedge\xi).$$

Corollary (1.31). We have a commutative diagram of holomorphic maps

$$\begin{array}{ccc}
\widetilde{\mathrm{UGM}}^p & \xrightarrow{\ \mathbf{P}\ } & \mathcal{F}_p^\times \\
p\downarrow & & \downarrow \\
\mathrm{UGM}^p & \xrightarrow{\ P\ } & \mathbb{P}(\mathcal{F}_p)
\end{array}$$

with which the action of $\mathrm{GL}_f(\mathbb{Z}h)$ is compatible, where $\mathbb{P}(\mathcal{F}_p)$ is the projective space $\mathcal{F}_p^\times/\mathbb{C}^*$ associated with $\mathcal{F}_p$.

Definition (1.32). The map $P:\mathrm{UGM}^p \longrightarrow \mathbb{P}(\mathcal{F}_p)$ is called the Plücker embedding of UGM^p in the Fock space.

Theorem (1.33) ([S.S.] [S.N.]). 1) P is injective and dp_U: $T_U \mathrm{UGM}^p \longrightarrow T_{P(U)}\mathbb{P}(\mathcal{F}_p)$ is also injective for $^\forall U \in \mathrm{UGM}^p$.

2) The image of P is a closed submanifold in $\mathbb{P}(\mathcal{F}_p)$ defined by the Plücker relations.

§2. Moduli of Riemann Surfaces and Line Bundles

(2.0) In this section we prepare geometric framework of our theory. The key idea, due to [B.M.S.], is to introduce an apparently redundant parameter, a (formal) local coordinate, which leads us to work on infinite dimensional manifolds such as the moduli space of locally framed Riemann surfaces. Generalizing [B.M.S], we construct moduli spaces to include Picard varieties, in order to realize current operators in our geometric settings. The theory then becomes more natural by this generalization. This gigantic moduli space, which we call the *Weierstrass system*, can be embedded into UGM, and is later identified with the space of physical vacua.

A) Field of formal Laurent series

(2.1) Here we fix a number of notations concerning universal coordinates.

$$\hat{R} = \mathbb{C}((\varsigma)) = \{ \sum_{n\gg-\infty} a_n \varsigma^n \; ; \; a_n \in \mathbb{C} \} : \text{the field of formal}$$

Laurent series,

$$\hat{\mathcal{O}} = \mathbb{C}[[\varsigma]] = \{ \sum_{n=0}^{\infty} a_n \varsigma^n \} : \text{the ring of formal power series,}$$

$$\hat{m} = \varsigma\hat{\mathcal{O}} = \{ \sum_{n=1}^{\infty} a_n \varsigma^n \} : \text{the unique maximal ideal of } \hat{\mathcal{O}}.$$

(2.2) In $\hat{R}$ we define a decreasing filtration satisfying the properties in (1.1) by

$$F^n\hat{R} = \{ \sum_{k=n}^{\infty} a_k \varsigma^k \}.$$

This filtration is the one induced from the unique (non-archimedian) valuation $\nu : \hat{R} \to \bar{Z} = Z \cup \{\infty\}$ with $\nu(\varsigma^n) = n$ as

$$F^n\hat{R} = \nu^{-1}([n, \infty]).$$

We have moreover

$$F^0 \hat{R} = \hat{0}, \quad F^1 \hat{R} = \hat{m}, \quad F^n \hat{R} = \hat{m}^n \ (n > 0).$$

(2.3) Let $G_0 = \text{Aut}_{\mathbb{C}}(\hat{0})$ be the group of automorphisms of $\hat{0}$ as $\mathbb{C}$-algebra, which we call *coordinate transformations*. We have a natural bijection:

$$\begin{array}{ccc} G_0 & \xrightarrow{\ \sim\ } & \hat{m} - \hat{m}^2 \\ \text{\rotatebox{90}{$\in$}} & & \text{\rotatebox{90}{$\in$}} \\ \varphi & \longmapsto & \varphi(\zeta) = b_1 \zeta + b_2 \zeta^2 + \cdots, \quad b_1 \neq 0. \end{array}$$

Let G_n $(n \geq 1)$ be the subgroup of G_0 consisting of φ's with $\varphi(\zeta) = \zeta + b_{n+1} \zeta^{n+1} + \cdots$.

(2.4) Consider $\hat{0}^* = \hat{0} - \hat{m} = \{\sigma = a_0 + a_1 \zeta + \cdots; a_0 \neq 0\}$ which acts on $\hat{0}$ by multiplication. We call the elements of $\hat{0}^*$ *gauge transformations*. Denote by $\hat{0}_n^*$ the subgroup of elements σ with $\sigma = 1 + a_n \zeta^n + \cdots$.

We define a semidirect product group $\tilde{G}_0 = G_0 \times \hat{0}^*$ by commuting relations $\varphi \circ \sigma = \varphi(\sigma) \circ \varphi$ for $\varphi \in G_0$, $\sigma \in \hat{0}^*$.

Then $\tilde{G}_0$ acts on $\hat{0}$ as $\mathbb{C}$-linear endomorphisms by $(\varphi, \sigma)(f) = \varphi(\sigma \cdot f)$ Note that (φ, σ) is uniquely determined by its values at 1 and ζ. We set moreover $\tilde{G}_n = G_n \times \hat{0}_n^*$. Then $\tilde{G}_1$ is a normal subgroup of $\tilde{G}_0$.

(2.5) Let $\mathfrak{G} = \text{Der}(K) = \hat{R} \frac{d}{d\zeta}$ be the Lie-algebra of derivations on $\hat{R}$. Define its filtration by

$$F^n \mathfrak{G} = \mathfrak{G}_n = \{l \in \mathfrak{G} : F^m \hat{R} \longrightarrow F^{m+n} \hat{R}\} = \hat{0} \zeta^{n+1} \frac{d}{d\zeta}$$

$$\cdots \supseteq \mathfrak{G}_n \supseteq \mathfrak{G}_{n+1} \supseteq \cdots.$$

Proposition(2.6)

1) $\qquad [\mathfrak{G}_n, \mathfrak{G}_m] \subseteq \mathfrak{G}_{n+m}$,

2) $\qquad \dim \mathfrak{G}_n / \mathfrak{G}_{n+1} = 1$,

3) $\qquad \mathfrak{G}_n$ $(n \geq -1)$ is a subalgebra of $\mathfrak{G}$.

Note that $\mathfrak{G}_n = \text{Lie } G_n$ for $n \geq 0$ but $\mathfrak{G}_{-1}$ does not correspond to any

holomorphic Lie group. We denote $\mathfrak{G}_0 = \mathfrak{B}$, $\mathfrak{G}_1 = \mathfrak{B}_-$ in the following.

(2.7) Let $\mathfrak{T} = \mathfrak{D}^1(K) = \hat{K}\frac{d}{d\xi} + \hat{K}$ be the Lie algebra of differential operators of order ≤ 1 on $\hat{K}$. Then it satisfies the following commutation relation for $\ell_i \frac{d}{i d\xi} + s_i \in \mathfrak{G}$ ($i = 1,2$)

$$[\ell_1 \frac{d}{d\xi} + s_1, \ \ell_2 \frac{d}{d\xi} + s_2]$$

$$= \left(\ell_1 \frac{d\ell_2}{d\xi} - \ell_2 \frac{d\ell_1}{d\xi}\right)\frac{d}{d\xi} + \left(\ell_1 \frac{ds_2}{d\xi} - \ell_2 \frac{ds_1}{d\xi}\right).$$

Define its filtration by

$$F^n\mathfrak{T} = \mathfrak{T}_n = \{D \in \mathfrak{T}: F^m\hat{K} \longrightarrow F^{m+n}\hat{K}\} = \hat{\mathfrak{O}}\, \xi^{n+1}\frac{d}{d\xi} \oplus \hat{\mathfrak{O}}\, \xi^n$$

$$\cdots \supseteq \mathfrak{T}_n \supseteq \mathfrak{T}_{n+1} \supseteq \cdots$$

Proposition (2.8).

1) $\quad [\mathfrak{T}_n, \mathfrak{T}_m] \subseteq \mathfrak{T}_{n+m}$

2) $\quad \mathfrak{T}_n$ ($n \geq 0$) is a subalgebra of $\mathfrak{T}$

Note that $\mathfrak{T}_n = \mathrm{Lie}\ \mathfrak{T}_n$ ($n \geq 0$). In the following, we denote $\mathfrak{T}_0 = \mathfrak{B} \oplus \mathfrak{A}$ and $\mathfrak{T}_1 = \mathfrak{T}_- = \mathfrak{B}_- \oplus \mathfrak{A}_-$ where $\mathfrak{A} = \mathrm{Lie}\ \mathfrak{O}^* = \mathfrak{O}$ and $\mathfrak{A}_- = \mathrm{Lie}\ \mathfrak{O}_1^* = \hat{\mathfrak{m}}$.

Remark (2.9) The algebra $\mathfrak{T}_0$ acts on $\mathfrak{O}$ as

$$(\ell\frac{d}{d\xi} + s)\ (f) = \ell\frac{df}{d\xi} + sf.$$

Proposition (2.10). The exponential map $\exp : \mathfrak{T}_0 \to \mathfrak{T}_0$ sending $D \in \mathfrak{T}_0$ to $e^D \in \mathfrak{T}_0$ is well-defined and surjective, and it induces a bijection $\exp : \mathfrak{T}_1 \xrightarrow{\sim} \mathfrak{T}_1$.

Remark (2.11). Later we use mainly a coordinate $z = \xi^{-1}$ (i.e. $K = \mathbb{C}((z^{-1}))$) to adapt our notations to the convention in physics. Since $z\frac{d}{dz} = -\xi\frac{d}{d\xi}$, the feature of $\mathfrak{G}$ does not change much:

$$\mathfrak{G}_n = \mathbb{C}[[z^{-1}]]z^{-n+1}\frac{d}{dz}$$

$$\mathfrak{T}_n = \mathbb{C}[[z^{-1}]]z^{-n+1}\frac{d}{dz} \oplus \mathbb{C}[[z^{-1}]]\, z^{-n}$$

B)Riemann surfaces and their Picard varieties

(2.12) We recall some elementary notions on Riemann surfaces. For details we refer the reader to [Si.], [F.K.].

(2.13) Let R be a Riemann surface of genus $g \geq 0$ and $K = K(R)$ the field of meromorphic functions on R. Note that K determines R uniquely up to isomorphisms.

Let Q be a point on R. Denote by $\mathcal{O}_Q$ the ring of meromorphic functions on R which are holomorphic at Q. $\mathfrak{m}_Q$ is the unique maximal ideal of $\mathcal{O}_Q$ consisting of meromorphic fuctions vanishing at Q.

There is a valuation ν_Q at Q in K s.t. $\nu_Q(f)$ = the order of zero at Q = - the order of pole at Q , which defines a filtration $\{F_Q^n K\}$ in K. This ν_Q determines a point Q with the relation $\mathcal{O}_Q = \{f \in K; \nu_Q(f) \geq 0\}$. We further define $\hat{R}_Q = \varprojlim_n K/\mathfrak{m}_Q^{n+1}\mathcal{O}_Q$ (the completion of K with respect to $\mathfrak{m}_Q$-adic topology).

Definition (2.14). We denote by $A(Q) = A(R, Q)$ the space $H^0(R, \mathcal{O}(*Q)) = \{f \in K; \text{holomorphic except at } Q\}$.

This is a subring in K s.t. $A(Q) \cap \mathcal{O}_Q = \mathbb{C}1$. It has a filtration $\{F^n A(Q)\}$ induced from $\{F_Q^n K\}$ in K.

(2.15) More generally let $\mathscr{L}$ be (a sheaf of germs of holomorphic sections of) a line bundle on R, $\mathscr{L}_Q$ the vector space of meromorphic sections of $\mathscr{L}$ which are holomorphic at Q. Note that $\mathscr{L}_Q$ is an $\mathcal{O}_Q$-module ,free of rank 1.

We denote by $H^0(\mathscr{L}(*Q))$ the space of meromorphic sections of $\mathscr{L}$ on R which are holomorphic except at Q. It has a filtration:

$$F^m H^0(\mathscr{L}(*Q)) = \{f \in H^0(\mathscr{L}(*Q)): \text{order of zero at } Q \geq m\}, m \in \mathbb{Z}.$$

In particular we set:

$$\bar{B}(R, Q) = H^0(R, \Theta(*Q)) = \{\text{meromorphic vector fields,}$$

holomorphic outside Q},

$$B(R,Q) = \{v \in \bar{B}(R,Q); v \text{ has a pole at } Q\},$$

where θ is the sheaf of germs of holomorphic vector fields on R. Note that $B(R,Q) = \bar{B}(R,Q)$ in case the genus of $R > 1$.

(2.16) The set $P = Pic(R)$ of holomorphic line bundles on R has a natural structure of an abelian Lie group. Its connected components are parametrized by the degree d (i.e. the first Chern class) or the Euler-characteristic χ:

$$P = \coprod_{d \in \mathbb{Z}} P^{(d)}$$

$$= \coprod_{\chi \in \mathbb{Z}} P^{\chi}, \quad \chi = d + 1 - g.$$

The connected component $P^{(0)}$ of 0 (i.e. the trivial line bundle) is an abelian variety $J = J(R)$ called the *Jacobian variety* of R. If we choose a line bundle $\mathcal{L}_1 \in P^{(1)}$, we have an isomorphism of groups $J(R) \times \mathbb{Z} \xrightarrow{\sim} P$ sending $(\mathcal{L}, d)$ to $\mathcal{L} \otimes \mathcal{L}_1^{\otimes d}$ (with the convension: $\mathcal{L}_1^{\otimes(-1)} = \mathcal{L}_1^{\vee}$ where $\mathcal{L}_1^{\vee}$ denotes the dual of $\mathcal{L}$).

P has a canonical involution (*charge conjugation*) $*: P^{\chi} \to P^{-\chi}$ sending $\mathcal{L}$ to $\Omega \otimes \mathcal{L}^{\smile}$ where $\Omega (= \theta^{\vee})$ is the cotangent bundle.

C) Formal uniformization

(2.17) Let R be a Riemann surface. Denote by $\hat{R}$ the set of isomorphisms (as $\mathbb{C}$-algebras) $u: \hat{\mathcal{O}}_Q \xrightarrow{\sim} \hat{\mathcal{O}} = \mathbb{C}[[\xi]]$ (2.1) with $Q \in R$ where $\hat{\mathcal{O}}_Q$ denotes the completion $\varprojlim_{n} \mathcal{O}_Q/\mathfrak{m}_Q^{n+1}$ of $\mathcal{O}_Q$. A specification of u is equivalent to give a valuation-preserving $\mathbb{C}$-algebra homomorphism $\mathcal{O}_Q \to \hat{\mathcal{O}}$. If moreover the image of $u: \mathcal{O}_Q \to \hat{\mathcal{O}}$ is contained in the subring of convergent power series $\mathcal{O} \subset \hat{\mathcal{O}}$, then the specification of u is nothing but to choose a holomorphic coordinate at Q. Since K is the quotient field of $\mathcal{O}_Q$, $u: \mathcal{O}_Q \to \hat{\mathcal{O}}$ induces an

injective ℂ-algebra homomorphism u: K ⟶K which preserves the valuations and further induces an isomorphism u: $\hat{K}_Q$⟶$\hat{K}$. Such u is called a *formal uniformization* at Q.

There is a natural surjective map $\tilde{\omega}$: $\hat{R}$ → R sending u to Q. This $\hat{R}$ has a canonical structure of an infinite dimensional complex manifold as a projective limit of finite dimensional ones : $R^{(n)} = \{u_n : \mathcal{O}_Q/\mathfrak{m}_Q^{n+1} \xrightarrow{\sim} \mathcal{O}/\hat{\mathfrak{m}}^{n+1}\}$.

On R the group G_0 = $\text{Aut}_{\mathbb{C}}(\mathcal{O})$ acts (from the left) as u → $\varphi \cdot u$ for φ ∈ G_0, u ∈ R. With this action R becomes a (holomorphic) principal fibre space over R with G_0 as fibres. For n ≥ 1, $R^{(n)}$ ⟶ R becomes a principal fibre space over R with $G^{(n)}$ = G_0/G_n as fibres, and $\hat{R}$ = $\varprojlim_n R^{(n)}$, G_0 = $\varprojlim_n G^{(n)}$.

(2.18) Let (R, (α,β)) be a (homologically) marked Riemann surface with a canonical basis $\{\alpha_1, \cdots, \alpha_g, \beta_1, \cdots, \beta_g\}$ of $H_1(R, \mathbb{Z})$ (i.e. $(\alpha_i, \beta_j) = \delta_{ij}$, $(\alpha_i, \alpha_j) = (\beta_i, \beta_j) = 0$). (R,$(\alpha,\beta)$) and (R',$(\alpha',\beta')$) are isomorphic if there exist an isomorphism f: R ⟶R' s.t. $f_*(\alpha_i) = \alpha_i'$, $f_*(\beta_i) = \beta_i'$ (f_*: $H_1(R)$ ⟶$H_1(R')$). Denote the set of isomorphism classes of (R,(α,β)) by $\mathcal{T}_g$. It admits a canonical structre of a complex manifold of dimension 0 (g=0), 1 (g=1), 3g - 3 (g≥2). We call this $\mathcal{T}_g$,the Torelli space.

On $\mathcal{T}_g$ the modular group M (=Sp(2g,$\mathbb{Z}$)) acts properly discontinuously (as base changes in $H_1(R,\mathbb{Z})$) and the quotient space $\mathcal{T}_g/M$ is isomorphic to the coarse moduli space $\mathcal{M}_g$ of Riemann surfaces of genus g (i.e. the set of isomorphism classes). Over $\mathcal{T}_g$ there is a universal family of Riemann surfaces $\pi: \mathcal{C}_g$ → $\mathcal{T}_g$ where $\mathcal{C}_g = \underset{([R],(\alpha\beta)) \in \mathcal{T}_g}{U} R$. See Fig.1.

What we need here essentially is that the tangent space $T_{[R]}\mathcal{T}_g$ of $\mathcal{T}_g$ at $[R]$ is canonically isomorphic to $H^1(R, \theta_R)$ via the Kodaira-Spencer map [Ko.]. Then for any $Q \in R$ the tangent space of $\mathcal{B}_g$ at Q is canonically isomorphic to

$$T_Q\mathcal{B}_g \xrightarrow{\sim} H^1(R, \mathfrak{m}_Q\theta_R) = H^1(R, \theta_R(-Q)), \quad (g \geq 2),$$

$$T_Q\mathcal{B}_g/T_Q(\mathcal{B}_g/\mathcal{T}_g) \xrightarrow{\sim} H^1(R, \mathfrak{m}_Q\theta_R) = H^1(R, \theta_R) \quad (g = 0, 1)$$

where $T_Q(\mathcal{B}_g/\mathcal{T}_g)$ is the tangent space at Q of the fibre of $\pi: \mathcal{B}_g \longrightarrow \mathcal{T}_g$.

For the universal family of Riemann surfaces $\pi: \mathcal{B}_g \to \mathcal{T}_g$ the union of $\hat{R}$; $\mathcal{\hat{B}}_g = \bigcup\limits_{([R],(\alpha\beta))\in\mathcal{T}_g} \hat{R}$ (resp. $R^{(n)}$; $\mathcal{B}_g^{(n)} = \bigcup\limits_{([R],(\alpha,\beta))\in\mathcal{T}_g} R^{(n)}$)

constructed above admits a natural structure of a holomorphic family

$$\tilde{\omega} : \mathcal{\hat{B}}_g \longrightarrow \mathcal{B}_g \quad (\text{resp. } \tilde{\omega}_n: \mathcal{B}_g^{(n)} \to \mathcal{B}_g)$$

which are again fibre bundles with structure group G_0 (resp. $G^{(n)}$). For convenience we set $\mathcal{B}_g^{(0)} = \mathcal{\hat{B}}_g$. Then we have

$$T_{(Q,u_n)}(\mathcal{B}_g^{(n)}/\text{Aut}^0(R)) \xrightarrow{\sim} H^1(R, \theta_R((-n-1)Q)) \quad (n \geq 0).$$

For each data $X = (R,(\alpha,\beta),Q,u:\mathcal{\hat{B}}_Q \to \mathcal{\hat{B}}) \in \mathcal{\hat{B}}_g$, we define a $\mathbb{C}$-subalgebra of $\hat{R}$, $A(X)$, by $uA(R,Q)$ and Lie subalgebras $B(X)$ and $\bar{B}(X)$ of $\text{Der}(\hat{K}) = \hat{K}\frac{d}{d\xi}$ by $uB(R,Q)$, $u\bar{B}(R,Q)$ respectively (cf.(2.14),(2.15)).

Proposition (2.19) [B.M.S.]. 1) We have canonical isomorphisms:

$$H^1(R, \theta_R) \xleftarrow{\sim} \text{Der}(\hat{K})/(\mathcal{O}\frac{d}{d\xi} + \bar{B}(X))$$

$$H^1(R, \theta_R((-n-1)Q)) \xleftarrow{\sim} \text{Der}(\hat{K})/(\mathcal{O}\ \xi^{n+1}\frac{d}{d\xi} + \bar{B}(X))$$

$$\text{at } X = (R,(\alpha\beta),Q,u) \in \mathcal{\hat{B}}_g \ (n \geq 0).$$

2) In the limit $(n \to \infty)$ we have

$$T_X\mathcal{\hat{B}}_g \xrightarrow{\sim} \text{Der}(\hat{K})/B(X)$$

which induces a commutative diagram of short exact sequences:

$$0 \longrightarrow T_X\mathcal{\hat{B}}_g/\mathcal{B}_g \longrightarrow T_X\ \mathcal{\hat{B}}_g \longrightarrow \tilde{\omega}^*T_Q\mathcal{B}_g \longrightarrow 0$$

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$$0 \longrightarrow \mathfrak{G}_0 = \mathcal{O}\xi\frac{d}{d\xi} \longrightarrow \frac{\mathrm{Der}(\hat{R})}{B(X)} \longrightarrow \frac{\mathrm{Der}(\hat{R})}{\mathcal{O}\xi\frac{d}{d\xi} + B(X)} \longrightarrow 0$$

where $T\hat{\mathcal{B}}_g/\mathcal{B}_g$ is the set of tangent vectors along the fibre of $\widetilde{\omega}$.

3) The above isomorphism defines a Lie anti-homomorphism

$$\theta \; : \; \mathfrak{G} = \mathrm{Der}(\hat{R}) \longrightarrow H^0(\hat{\mathcal{B}}_g, \; \theta_{\hat{\mathcal{B}}_g})$$

whose restriction to $\mathfrak{G}_0$ coincides with the one induced from the action of G_0

$$\theta \; : \; \mathfrak{G}_0 = \mathcal{O}\xi\frac{d}{d\xi} \longrightarrow H^0(\hat{\mathcal{B}}_g, \; \theta_{\hat{\mathcal{B}}_g}).$$

4) For $^\forall X \in \hat{\mathcal{B}}_g$ the following diagram is commutative:

$$\begin{array}{ccc}
\theta_X \; : \; \mathfrak{G}_{-1} & \longrightarrow & T_X(\hat{\mathcal{B}}_g/\mathcal{T}_g) \\
\cup & & \cup \\
\theta_X \; : \; \mathfrak{G}_0 & \longrightarrow & T_X(\hat{\mathcal{B}}_g/\mathcal{B}_g)
\end{array}$$

Remark (2.20) For $^\forall l \in \mathfrak{G}$, $\theta(l)$ is a holomorphic vector field on $\hat{\mathcal{B}}$, and for $^\forall X \in \hat{\mathcal{B}}_g$, $T_X\hat{\mathcal{B}}_g$ is spanned by $\{\theta_X(l), \; l \in \mathfrak{G}\}$. In this sense we say that the triple $(\hat{\mathcal{B}}_g, \; \mathfrak{G}, \; \theta)$ is an infinitesimally homogeneous space ([B.R.]).

D) Formal trivialization

(2.21) We still consider the family of Riemann surfaces in (2.18). The union of Picard varieties of fibres forms a holomorphic family of abelian Lie groups:

$$p \; : \; \mathcal{P}_g = \bigcup_{[R]\in\mathcal{T}_g} P(R) \longrightarrow \mathcal{T}_g.$$

It decomposes into connected components as

$$\mathcal{P}_g = \coprod_{d\in\mathbb{Z}} \mathcal{P}_g^{(d)} = \coprod_{\chi\in\mathbb{Z}} \mathcal{P}_g^\chi \qquad (\text{cf. } (2.16)).$$

The fibre product $\mathcal{X}_g = \hat{\mathcal{B}}_g \times_{\mathcal{T}_g} \mathcal{P}_g$ is the set of triples $\{(R,(\alpha,\beta),Q, \mathcal{L})$ with $Q \in R, \; \mathcal{L} \in P(R)\}$ and by projection we have a commutative diagram

$$\begin{array}{ccc}
\mathfrak{X}_g & \longrightarrow & \mathcal{B}_g \\
\downarrow & & \downarrow \pi \\
\mathcal{P}_g & \overset{p}{\longrightarrow} & \mathcal{T}_g .
\end{array}$$

Definition (2.22). 1) For each $\mathcal{L} \in \text{Pic}(R)$ with $Q \in R$ we define
$\hat{\mathcal{L}}_Q = \varprojlim_n \mathcal{L}_Q / m_Q^{n+1} \mathcal{L}_Q$, which is left $\hat{\mathcal{O}}_Q$-module free of rank 1.

2) The space $\hat{\mathfrak{X}}_g$ is a set of quintuples $\{(X, L) = (R,(\alpha,\beta),Q,\mathcal{L},u,t)$; $Q \in R$, $\mathcal{L} \in P(R)$, $u:\hat{\mathcal{O}}_Q \xrightarrow{\sim} \hat{\mathcal{O}}$, $t:\hat{\mathcal{L}}_Q \xrightarrow{\sim} \hat{\mathcal{O}}\}$ where u is a formal uniformization (2.18) and t is a u-linear isomorphism of modules (i.e. for $f \in \hat{\mathcal{O}}_Q$ and $s \in \hat{\mathcal{L}}_Q$, $t(fs) = u(f)t(s)$) called a *formal trivialization* or *formal gauge fixing* of $\mathcal{L}$ at Q. This t is determined by the (formal) section $\hat{s} = t^{-1}(1)$ which is a (formal) generator of $\mathcal{L}$ at Q.

For a given data $(R,Q,\mathcal{L})$ with a Riemann surface R, $Q \in R$ and $\mathcal{L} \in P(R)$ we denote by $K_{\mathcal{L}}$ the $\mathbb{C}$-vector space of meromorphic sections of $\mathcal{L}$ which is a free $K = K(R)$-module of rank 1. We further define $\mathcal{D}_K^1(K_{\mathcal{L}}) = \{\mathcal{D}; K_{\mathcal{L}} \longrightarrow K_{\mathcal{L}}, \mathbb{C}$-linear map s.t. $\exists \bar{D} \in \text{Der}(K)$, for $\forall f \in K$, $\forall \varphi \in K_{\mathcal{L}}$; $D(f\varphi) = \bar{D}(f)\varphi + fD(\varphi)\}$. We then obtain the following exact sequence:

$$0 \longrightarrow K \xrightarrow{\ i\ } \mathcal{D}_K^1(K_{\mathcal{L}}) \xrightarrow{\ \pi\ } \text{Der}(K) \longrightarrow 0 .$$

For $g \in K$, $i(g) \in \mathcal{D}_K^1(K_{\mathcal{L}})$ is defined by $i(g)(\varphi) = g\varphi$ for $\forall \varphi \in K_{\mathcal{L}}$, and for $D \in \mathcal{D}_K^1(K_{\mathcal{L}})$, $\pi(D) = \bar{D}$.

We next define the Lie subalgebras of $\mathcal{D}_K^1(K_{\mathcal{L}})$ by
$\bar{B}(R,Q,\mathcal{L}) \equiv \{D \in \mathcal{D}_K^1(K_{\mathcal{L}}) | D: H^0(R,\mathcal{L}(*Q)) \longrightarrow H^0(R,\mathcal{L}(*Q))\}$,
$B(R,Q,\mathcal{L}) = \pi^{-1}(B(R,Q))$.

Proposition (2.23). With the notations above we obtain the following commutative diagram of the short exact sequence:

$$
\begin{array}{ccccccc}
& & 0 & & 0 & & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow & A(R,Q) & \longrightarrow & B(R,Q,\mathcal{L}) & \longrightarrow & B(R,Q) & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & K & \longrightarrow & \mathcal{D}^1_K(K_{\mathcal{L}}) & \longrightarrow & \mathrm{Der}(K) & \longrightarrow 0
\end{array}
$$

We further define $\hat{K}_{\mathcal{L},Q} \equiv \varprojlim\limits_{n} K_{\mathcal{L}}/m_Q^{n+1}\mathcal{L}_Q$, which is a free

$\hat{R}_Q$-module of rank 1. The following relations hold naturally:

$$\hat{K}_{\mathcal{L},Q} = K \underset{\mathcal{O}_Q}{\otimes} \hat{\mathcal{L}}_Q = \hat{R}_Q \underset{\mathcal{O}_Q}{\otimes} \mathcal{L}_Q.$$

We have natural projections $\rho : \hat{\mathfrak{T}}_g \longrightarrow \mathfrak{T}_g$, $\hat{\mathfrak{T}}_g \longrightarrow \hat{\mathcal{B}}_g$ which form a
commutative diagram of holomorphic maps

$$
\begin{array}{ccc}
\hat{\mathfrak{T}}_g & \longrightarrow & \hat{\mathcal{B}}_g \\
\rho \downarrow & & \downarrow \tilde{\omega} \\
\mathfrak{T}_g & \longrightarrow & \mathcal{B}_g \\
\downarrow & & \downarrow \pi \\
\mathcal{P}_g & \overset{p}{\longrightarrow} & \mathcal{T}_g
\end{array}
$$

Like $\hat{\mathcal{B}}_g$ this $\hat{\mathfrak{T}}_g$ also is a projective limit of finite
dimensional complex manifolds. $\hat{\mathfrak{T}}_g$ is a disjoint union of connected
components $\hat{\mathfrak{T}}_g^X$ each of which lies over $\mathcal{P}_g^X$ respectively.

On $\hat{\mathfrak{T}}_g$ the group $\tilde{G}_0$ (2.6) acts (from the left) as $(u, t) \to (\varphi \cdot u,$
$\varphi(\sigma \cdot t))$ for $\varphi \in G_0$, $\sigma \in \hat{\mathcal{O}}^*$. With this action $\hat{\mathfrak{T}}_g$ becomes a (holomorphic
principal homogeneous space over $\mathfrak{T}_g$ with $\tilde{G}_0$ as fibres.

Definition (2.24). Denote by $P(\hat{\mathfrak{T}}_g)$ the projective space
associated with $\hat{\mathfrak{T}}_g$ on which $\mathbb{C}^*$ acts by $\mathbb{C}^* \subset \hat{\mathcal{O}}^*$. This is equivalent to
consider the formal trivialization up to constants. We can take the
quotient since $\mathbb{C}^*$ is contained in the centre of $\tilde{G}_0$. This space is what
we are mainly concerned with.

For each data $(X,L) = (R,(\alpha,\beta),Q,\mathcal{L},u,t) \in \hat{\mathfrak{T}}_g$, u-linear ($u : \hat{\mathcal{O}}_Q \longrightarrow \hat{\mathcal{O}}$

isomorphism t: $\mathcal{L}_Q \longrightarrow \mathcal{O}_Q$ induces u-linear (u: $\hat{K}_Q \xrightarrow{\sim} \hat{K}$) isomorphism t: $\hat{K}_{\mathcal{L},Q} \longrightarrow \hat{K}$. Then each element $D \in \mathcal{D}^1_K(K_{\mathcal{L}})$ defines $t(D) \in \mathcal{D}^1(K)$. We then define a Lie subalgebra of $\mathcal{D}^1(K)$ by $B(X,L) = \{t(D); D \in B(R,Q,\mathcal{L})\}$. From Proposition (2.23) we obtain the following exact sequence:

$$0 \longrightarrow A(X) \longrightarrow B(X,L) \longrightarrow B(X) \longrightarrow 0 .$$

For $(X,L) \in \hat{\mathcal{R}}_g$ we denote the corresponding element in $P(\hat{\mathcal{R}}_g)$ by $((X,L))$.

Then in the same way as (2.19) we have the following.

Proposition (2.25). 1) There is an isomorphism

$$T_{((X,L))}P(\hat{\mathcal{R}}_g) \xrightarrow{\sim} \mathcal{D}^1(\hat{K})/B(X,L)$$

which induces a commutative diagram of short exact sequences

$$0 \longrightarrow T_{((X,L))}P(\hat{\mathcal{R}}_g)/\mathcal{I}_g \longrightarrow T_{((X,L))}P(\hat{\mathcal{R}}_g) \longrightarrow \rho^* T_{(Q,\mathcal{L})}\mathcal{I}_g \longrightarrow 0$$

$$\downarrow \qquad\qquad\qquad \downarrow \qquad\qquad\qquad \downarrow$$

$$0 \longrightarrow \mathcal{B} \oplus \mathcal{U}_- \longrightarrow \frac{\mathcal{D}^1(K)}{B(X,L)} \longrightarrow \frac{\mathcal{D}^1(\hat{K})}{B(X,L) + \mathcal{B} + \mathcal{U}_-} \longrightarrow 0$$

2) The above isomorphism defines a Lie anti-homomorphism

$$\theta : \mathcal{T} = \mathcal{D}^1(\hat{K}) \longrightarrow H^0(P(\hat{\mathcal{R}}_g), \theta_{P(\hat{\mathcal{R}}_g)})$$

whose restriction to $\mathcal{T}_0 = \mathcal{B} + \mathcal{U}$ coincides with that induced from the action of $\mathcal{T}_0$ on $\hat{\mathcal{R}}_g$:

$$\theta : \mathcal{T}_0 \longrightarrow H^0(P(\hat{\mathcal{R}}_g), \theta_{P(\hat{\mathcal{R}}_g)}) .$$

(Again $(P(\hat{\mathcal{R}}_g), \mathcal{T}_0, \theta)$ is an infinitesimal homogeneous space in the sense of Remark (2.20)).

E) Embedding of Weierstrass system into UGM.

(2.26) Recall that $M = Sp(2g, Z)$ acts on the Torelli space $\mathcal{T}_g$ (2.18). It is easy to see from the construction that the action of M lifts to that on $\hat{\mathcal{R}}_g$ etc. and is compatible with the diagram in (2.23).

We can take the quotient $\hat{\mathfrak{X}}_g = P(\hat{\mathfrak{X}}_g)/M$. This moduli space $\hat{\mathfrak{X}}_g$ of framed and gauged Riemann surfaces is called the *Weierstrass system* (of genus g).

(2.27) We define a map

$$\Gamma : P(\mathfrak{X}) \longrightarrow UGM(K)$$

sending $(X,L) = (R,(\alpha,\beta),Q,\mathcal{L},u,t)$ to $t(H^0(R,\mathcal{L}(*Q))) \subset \hat{K}$, which is seen to be well-defined and holomorphic. In other words the image is the set of Laurent series expansions of meromorphic sections of $\mathcal{L}$ holomorphic outside Q by means of the chosen (formal) coordinates and gauge. Γ can be regarded as *the period map* of $P(\hat{\mathfrak{X}}_g)$. Note that the period domain UGM does not depend on the genus g.

We have the main theorem in this section, which corresponds to Torelli's theorem for the Weierstrass system.

Theorem (2.28).
1) Γ maps $P(\hat{\mathfrak{X}}^X)$ into UGM^X.
2) The value of Γ does not change under the action of M, hence we have a holomorphic map

$$\bar{\Gamma} : \hat{\mathfrak{X}}_g = P(\hat{\mathfrak{X}}_g)/M \longrightarrow UGM(\hat{K}).$$

3) $\bar{\Gamma}$ is injective for $g > 1$ (for $g \leq 1$, replace M by M × Aut(R)). Namely $U = t(H^0(R, \mathcal{L}(*Q))) \subset \hat{K}$ determines the data (X,L) up to isomorphisms and scalar.

Proof. Only the last statement 3) is non-trivial. We have however

$$K = K(R) = \{f/g \in K; f, g \in U\}$$

and

$$A(R,Q) = \{h \in K; hU \subset U\}.$$

Hence R is determined by K and Q by A(R,Q) since we have

$$\mathcal{O}_Q = \{f/g;\ f,g \in A(R,Q),\ \nu_Q(f) \geq \nu_Q(g)\}$$

$$(\nu_Q(f) = \max\ \{n;\ f \in F^n\})$$

On the other hand, as $R - \{Q\}$ is affine, sections in $H^0(\mathcal{L}(*Q))$

generates $\mathcal{L}_{Q'}$, $Q' \neq Q$ and $\mathcal{L}_Q$ is characterized as

$$\{f \in H^0(\mathcal{L}(*Q)) \underset{A(R,Q)}{\otimes} K\ ;\ \nu_Q(f) \geq 0\}$$

(considered as $\mathcal{L} \subset \hat{K}_{\mathcal{L}}$ (constant sheaf)). Since A and U are already

contained in $\hat{K}$, the formal uniformization and the local

trivialization are also uniquely determined (up to scalar). ∎

Theorem (2.29). For each pair $(X,L) \in \hat{\mathfrak{A}}_g$ and $((X,L)) \in P(\hat{\mathfrak{A}}_g)$,

1) the following diagram is commutative;

$$
\begin{array}{ccc}
\mathcal{U} & \xrightarrow{\hspace{2cm}} & \text{End}(\hat{K}) \\
\theta \downarrow & & \downarrow \theta \\
T_{((X,L))}P(\hat{\mathfrak{A}}_g) & \xrightarrow{\ \Gamma_*\ } & T_{U(X,L)}\text{UGM}\ ;
\end{array}
$$

2) for each $\mathcal{U} \ni D$, the holomorphic vector field $\theta(D)$ on $\hat{\mathfrak{A}}_g$ is

modular invariant,

$$\gamma_*\theta_{(X,L)}(D) = \theta_{\gamma(X,L)}(D) \qquad \text{for } \gamma \in M;$$

3) (the local Torelli theorem) the map

$$\Gamma_*;\ T_{((X,L))}P(\hat{\mathfrak{A}}) \longrightarrow T_{U(X,L)}\text{UGM}$$

is injective if $g > 1$ (in general $\text{Ker}\ \Gamma_* = \bar{B}(X,L)/B(X,L)$).

Proof. 1) and 2) are trivial from the definition of θ. We prove

3). From Proposition (2.25) we have a natural

identification: $T_{((X,L))}P(\hat{\mathfrak{A}}) \cong \mathfrak{D}^1(K)/B(X,L)$. For a given

$D \in \mathfrak{D}^1(\hat{K})$ the element

$$\Gamma_*(D) \in T_{U(X,L)}\text{UGM} \cong \text{Hom}_{\mathbb{C}}(U(X,L),\hat{K}/U(X,L))$$

is represented by the map

$$\Gamma_*(D):\ U(X,L) \longrightarrow \hat{K} \xrightarrow{\ D\ } \hat{K} \longrightarrow \hat{K}/U(X,L).$$

If we assume $\Gamma_*(D) = 0$, then we recognize that $D: H^0(R,\mathcal{L}(*Q)) \longrightarrow$ $H^0(R,\mathcal{L}(*Q))$ since $U(X,L) = t H^0(R,\mathcal{L}(*Q))$ where we have omitted the symbol t. On the other hand $K = K(R)$ is generated by $\{\psi/\varphi;\ \psi,\varphi \in H^0(R,\mathcal{L}(*Q)), \varphi \neq 0\}$ as a $\mathbb{C}$-algebra. Considering $\bar{D} = \pi(D) \in \mathrm{Der}(K)$, we obtain:

$$\bar{D}(\psi/\varphi) = \frac{(D\psi)\varphi - \psi D(\varphi)}{\varphi^2} \qquad \text{for } \varphi,\psi \in H^0(R,\mathcal{L}(*Q)).$$

Since $D(\varphi), D(\psi) \in H^0(R,\mathcal{L}(*Q))$, $\bar{D}$ belongs to $\mathrm{Der}(K)$. On the other hand D preserves $H^0(R,\mathcal{L}(*Q))$ and $K_{\mathcal{L}} = K \underset{A(R,Q)}{\otimes} H^0(R,\mathcal{L}(*Q))$, hence we have $D: K_{\mathcal{L}} \longrightarrow K_{\mathcal{L}}$, in other words $D \in \mathcal{D}^1_K(K_{\mathcal{L}})$. From the condition $D: H^0(R,\mathcal{L}(*Q)) \longrightarrow H^0(R,\mathcal{L}(*Q))$ we reach the result $D \in B(X)$. ∎

F) Sections of spin j.

(2.30) For $j \in \mathbb{Z}$ we have a section $\sigma_j : \mathcal{T}_g \longrightarrow \mathcal{P}_g$ sending $[R]$ to $([R], \Omega_R^{\otimes j})$, where Ω_R is the canonical sheaf (or the sheaf of holomorphic forms on R). Note that the degree of $\Omega_R^{\otimes j}$ is equal to $2(g-1)j$, hence the Euler-characteristic is $(2j-1)(g-1)$.

This section lifts to $\partial_j : \mathcal{B}_g \to \mathcal{R}_g$ (called of *spin* j) since $u: \mathcal{O}_Q \to \mathcal{O}$ induces $(du)^j : \Omega_{R,Q}^{\otimes j} \to \mathcal{O}$ sending $f(du^{-1}(\zeta))^j$ to $u(f)$.

(2.31) Now we want to define such section for $j \in \frac{1}{2}\mathbb{Z}$. Using the theory of Riemann constant for each $(R, (\alpha,\beta)) \in \mathcal{T}_g$, we can canonically associate a line bundle $\mathcal{L}_\Delta(R,(\alpha,\beta)) = \mathcal{L}_\Delta$ s.t. $\mathcal{L}_\Delta^2 = \Omega_R$ [B.M.][A-G.N.V.]. Here we define $\sigma_j ; \mathcal{T}_g \longrightarrow \mathcal{P}_g^{(2j-1)(g-1)}$, as $\sigma_j(R,(\alpha,\beta)) \equiv (R,(\alpha,\beta),\mathcal{L}_\Delta^{2j})$. We define the following formal triviarization $\sqrt{du}\ ;\ \mathcal{L}_{\Delta,Q} \overset{\sim}{\longrightarrow} \mathcal{O}_Q$ with $(\sqrt{du})^2 = du$: $(\mathcal{L}_{\Delta,Q})^2 = \Omega_Q \overset{\sim}{\longrightarrow} \mathcal{O}$, which is determined up to constants ± 1.

For $j \in \frac{1}{2}\mathbb{Z}$ with an identification $\Omega_R^j = \mathcal{L}_\Delta^{2j}$ we define a mapping $\partial_j ; \mathcal{B}_g \longrightarrow \mathbb{P}(\mathcal{R}_g^{(2j-1)(g-1)})$, by

$$\hat{\sigma}_j((R,(\alpha,\beta),Q,u)) \equiv (R,(\alpha,\beta),Q,u,z_\Delta^{2j},(du)^j)$$

where $(\sqrt{du})^{2j}$; $\mathfrak{Z}_{\Delta,Q}^{2j} \cong (\mathfrak{Z}_{\Delta,Q})^{\otimes 2j} \xrightarrow{\sim} \mathfrak{O}$.

Then we have a commutative diagram

$$\hat{\sigma}_j : \mathscr{E}_g \longrightarrow P(\mathfrak{L}_g^{(2j-1)(g-1)})$$
$$\downarrow \qquad \downarrow$$
$$\sigma_j : \mathscr{T}_g \longrightarrow \mathscr{P}_g^{(2j-1)(g-1)}$$

We define the following composite holomorphic map

$$\Gamma \cdot \hat{\sigma}_j : \mathscr{E}_g \longrightarrow UGM^{(2j-1)(g-1)}$$

which we call the *modular embedding of spin j*.

Proposition (2.32). For $j \in \frac{1}{2}\mathbb{Z}$ and a section

$$s_j : Der(\hat{R}) \longrightarrow \mathfrak{D}^1(\hat{R})$$
$$\Cup \qquad \qquad \Cup$$
$$f(\zeta)\frac{d}{d\zeta} \longmapsto f(\zeta)\frac{d}{d\zeta} + j\frac{df}{d\zeta}(\zeta),$$

we have a commutative diagram

$$\begin{array}{ccc}
Der(\hat{R}) & \xrightarrow{\quad\theta\quad} & T_u\mathscr{E} \\
s_j\downarrow & & \downarrow(\hat{\sigma}_j)_* \\
\mathfrak{D}^1(\hat{R}) & \xrightarrow{\quad\theta\quad} & T_{((u,t_u))}P(\mathfrak{X})
\end{array}$$

where horizontal arrows are defined in (2.19), (2.25).

We are mainly interested in the case $j = \frac{1}{2}$. For simplicity we denote $\hat{\sigma}_{1/2}$; $\mathscr{E}_g \longrightarrow P(\mathfrak{X}^0)$ by $\hat{\sigma}$.

(2.33) By using the period matrix Ω, the jacobian variety J of R can be expressed as the quotient of $\mathbb{C}^g$ by a lattice $L = (1_g, \Omega)\mathbb{Z}^{2g}$ of rank $2g$ through the identification of $\mathscr{J} = Hom(H^0(R,\Omega_R), \mathbb{C})$ with $\mathbb{C}^g$. For an element $c = {}^t(c_1, \ldots, c_g) \in \mathbb{C}^g$, the corresponding element $c \in \mathscr{J}$ satisfies $c(\omega_i) = c_i$. Choosing a point Q

$\in R$, we have an embedding $I_Q : R \longrightarrow J$, $P \longmapsto (\int_Q^P \omega^i)$ mod L.

(2.34) For $c \in \mathcal{J}$ $(= \mathbb{C}^g)$ denote by $\mathcal{L}_c$ the line bundle of degree 0 on R corresponding to c mod L in J. The section of $\mathcal{L}_c$ can be identified with the space of multiplicative functions on R

(*) $\{f : \tilde{R} \to \mathbb{C} ; f(\tilde{p} + {}^t m\alpha + {}^t n\beta) = \exp(2\pi\sqrt{-1}\,{}^t nc) f(\tilde{p}), \; m,n \in \mathbb{Z}^g\}$

where $\tilde{R}$ is the maximal abelian covering of R. Moreover a multiplicative function on $\tilde{R}$ can be considered as a multivalued meromorphic function on R. This implies that for $Q \in R$ and $c \in \mathcal{J}$ we have chosen a local trivialization $s_c : (\mathcal{L}_c)_Q \simeq \mathcal{O}_Q$ via this identification.

Denote by $\mathcal{F}_g$ $(= (\pi_* \Omega_{\mathcal{G}/\mathcal{J}})^\vee)$ the family of universal coverings of $J(R)$'s, which is a vector bundle on $\mathcal{J}_g$ of rank g. The dual basis of $\{\omega_i\}$ gives the trivialization of vector bundle $\pi : \mathcal{F}_g \longrightarrow \mathcal{J}_g$;

$$\begin{array}{ccc}
\mathcal{F}_g & \longrightarrow & \mathcal{J}_g \times \mathbb{C}^g \\
& \pi \searrow \quad \swarrow \pi_1 & \\
& \mathcal{J}_g &
\end{array}$$

We set

$$\mathcal{O}_g(\mathcal{F}) \equiv \mathcal{O}_g \times_{\mathcal{J}_g} \mathcal{F} = \text{a set of data } (R, (\alpha, \beta), Q, u, \mathcal{L}_c)$$

and define a map $\tilde{\sigma}_{1/2} : \mathcal{O}_g(\mathcal{F}) \longrightarrow P(\mathcal{L}^{(0)})$ by

$$\tilde{\sigma} = \tilde{\sigma}_{1/2} : (R, (\alpha, \beta), Q, u, \mathcal{L}_c) \longrightarrow (R, Q, \mathcal{L}_\Delta \otimes \mathcal{L}_c, u, \pm\sqrt{du} \otimes (u \cdot s_c))$$

with s_c defined above.

Proposition (2.35). 1) The following diagram is commutative

$$\begin{array}{ccc}
\mathcal{O}_g & \xrightarrow{\;\;\partial\;\;} & P(\mathcal{L}^0) \\
& \text{zero} \searrow \quad \swarrow \tilde{\sigma} & \\
\text{section} & & \\
& \mathcal{O}_g(\mathcal{F}) &
\end{array}$$

2) The following action of vector fields are compatible:

$$\begin{array}{ccccc}
\hat{\theta}_g & \longrightarrow & \hat{\theta}_g(\mathcal{T}) & \longrightarrow & P(\hat{\mathcal{X}}^0) \\
\theta \uparrow & \nearrow & \theta \downarrow & & \theta \uparrow \\
\mathfrak{S} & \xrightarrow{\quad S_{1/2} \quad} & & & \mathcal{D}^1(\mathcal{R})
\end{array}$$

where $\theta(\mathfrak{S})$ acts on $\hat{\theta}_g(\mathcal{T}) = \hat{\theta}_g \underset{\mathcal{T}_g}{\times} \mathcal{T}$ to its first component only.

(2.36) Define the actions of Z^{2g} and the modular group M on $\theta(\mathcal{T})$ as follows:

1) $Z^{2g} \ni (m,n)$ acts on $\theta_g(\mathcal{T}) \longrightarrow \theta_g$ as a fibrewise transformation by $c \longrightarrow c + \Omega m + n$.

2) The modular group M acts on $\theta_g(\mathcal{T})$ as

$$\gamma(X,c) = (\gamma(X), {}^t(C\Omega+D)^{-1}c) \text{ for } \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M.$$

Proposition (2.37). With the above notations, the map $\tilde{\sigma}_{1/2}$: $\theta_g(\mathcal{T}) \longrightarrow P(\hat{\mathcal{X}}_g^0)$ is invariant under the action of $\tilde{M}_\Delta = M_\Delta \ltimes Z^{2g}$, where M_Δ $(= M \cap \Gamma(2,4)) \subset M$ is the subgroup of transformations which preserv the Riemann constant Δ (cf.§5,C)).

§3 Fermion Fock Space and the Second Quantization

(3.0) The main object of this section is to extend and lift the infinitesimal action of $\text{End}(V)$ on UGM (as holomorphic vector fields on it) to the whole diagram

$$
\begin{array}{ccc}
\text{UGM} & \longrightarrow & \mathcal{F}^{\times} \\
\downarrow & & \downarrow \\
\text{UGM} & \xrightarrow{\quad P \quad} & \mathbb{P}(\mathcal{F})
\end{array} \quad .
$$

where P is the Plücker embedding (1.31).

Here we encounter the difficulties of divergences coming essentially from the infinity of the dimension of UGM. We use a procedure of regularization (a technique of normal ordering, cf.C)). In fact what acts on $\mathcal{F}$ is not $\text{End}(V)$ itself but its non-trivial extension by a 1-dimensional center. This phenomenon is what physicists call anomaly. The Virasoro algebra and the current algebra appear as fundamental subalgebras of this extension (cf. D)).

A) Fermion Operators

Definition (3.1). i) The associative algebra with 1 generated by ψ_μ, $\bar{\psi}_\mu$, $\mu \in \mathbb{Z}h$, with the relations:

$$[\psi_\mu, \psi_\nu]_+ = [\bar{\psi}_\mu, \bar{\psi}_\nu]_+ = 0, \quad [\psi_\mu, \bar{\psi}_\nu]_+ = \delta_{\mu+\nu,0}$$
$$\text{for } \forall \mu, \nu \in \mathbb{Z}h, \text{ where } [A,B]_+ = AB + BA;$$

(i.e. the Clifford algebra) is called the *fermion operator algebra*, which we denote by $\mathcal{A}$.

We define actions of ψ_μ, $\bar{\psi}_\mu$ $(= \bar{\psi}^{-\mu})$ on $\mathcal{F}$ and $\bar{\mathcal{F}}$ as follows:

$$\psi_\mu = \frac{\partial}{\partial e^\mu} \quad , \quad \bar{\psi}^\mu = e^\mu \wedge \quad \text{(left action on } \mathcal{F}),$$

$$\psi_\mu = \wedge \bar{e}_\mu \quad , \quad \bar{\psi}^\mu = \frac{\partial}{\partial \bar{e}_\mu} \quad \text{(right action on } \bar{\mathcal{F}}).$$

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Then the above anti-commutation relations hold, i.e. $\mathcal{A}$ acts on $\mathcal{F}$ and $\bar{\mathcal{F}}$.

We denote by W (resp $\bar{W}$) the vector space generated by $\{\psi_\mu\}$ (resp. $\{\bar{\psi}_\mu\}$) $(\mu \in \mathbb{Z}h)$ and set $\tilde{V} = W \oplus \bar{W}$.

Proposition(3.2). These operators satisfy the following relations:

1) for $^\forall \langle u| \in \bar{\mathcal{F}}$, $|v\rangle \in \mathcal{F}$, $\mu \in \mathbb{Z}h$,

$$\langle u| (\psi_\mu |v\rangle) = (\langle u|\psi_\mu)|v\rangle$$
$$\langle u| (\bar{\psi}_\mu |v\rangle) = (\langle u|\bar{\psi}_\mu)|v\rangle \; ;$$

2) for $^\forall p \in \mathbb{Z}$,

$$\psi_\mu |p\rangle = 0 \quad (\mu > p), \quad \langle p|\psi_\mu = 0 \quad (\mu < p)$$
$$\bar{\psi}_\mu |p\rangle = 0 \quad (\mu > -p), \quad \langle p|\bar{\psi}_\mu = 0 \quad (\mu < -p)$$

where $|p\rangle$, $\langle p|$ are charged vacua defined in (1.25).

Definition (3.3). We set

$$W = W_+ \oplus W_-, \quad W_+ = \underset{\mu>0}{\oplus} \mathbb{C}\psi_\mu, \quad W_- = \underset{\mu<0}{\oplus} \mathbb{C}\psi_\mu,$$
$$\bar{W} = \bar{W}_+ \oplus \bar{W}_-, \quad \bar{W}_+ = \underset{\mu>0}{\oplus} \mathbb{C}\bar{\psi}_\mu, \quad \bar{W}_- = \underset{\mu<0}{\oplus} \mathbb{C}\bar{\psi}_\mu,$$
$$\tilde{V}_+ = \{\psi \in \tilde{V}; \; \psi|0\rangle = 0\} = W_+ \oplus \bar{W}_+,$$
$$\tilde{V}_- = \{\psi \in \tilde{V}; \; \langle 0|\psi = 0\} = W_- \oplus \bar{W}_-.$$

Elements of $\tilde{V}_+$ (resp. $\tilde{V}_-$) are called *annihilation operators* (resp. *creation operators*).

(3.4) In §1 the Fock space was defined as a semi-infinite exterior product (geometric interpretation). There is, however, another algebraic presentation of the Fock space by means of the operator algebra $\mathcal{A}$. This form is more convenient for concrete calculations.

Theorem (3.5). 1) The following homomorphism of left $\mathcal{A}$-modules

is injective and its image is dense:

$$
\begin{array}{ccc}
\mathcal{A}/\mathcal{A}\nabla_{+} & \longrightarrow & \mathcal{F} \\
\cup & & \cup \\
a & \longrightarrow & a\,|\,0\rangle
\end{array}
$$

2) The following homomrphism of right $\mathcal{A}$-modules is isomorphic:

$$
\begin{array}{ccc}
\nabla_{-}\mathcal{A}\backslash\mathcal{A} & \xrightarrow{\ \sim\ } & \mathcal{F} \\
\cup & & \cup \\
a & \longrightarrow & \langle 0\,|\,a.
\end{array}
$$

(3.6) The notion of "normal ordering",familiar to physicists

but not to mathematicians, is one of the most important techniques

in operator calculus. It can be formulated mathematically as follows:

We define a linear map (called *normal ordering*)

$$
: \quad : \quad ; \quad \wedge\,\nabla \longrightarrow \wedge(\nabla_{-}) \otimes \wedge(\nabla_{+}) \xrightarrow{\ \sim\ } \mathcal{A}
$$

such that i) it is an isomorphism as left $\wedge$ (∇_{-}), right $\wedge(\nabla_{+})$

-bimodule, and ii) $: 1 : = 1$. Such a linear map exists uniquely.

Definition (3.7). We define the following fermion field operators

$$
\psi(z) = \sum_{\mu\in\mathbb{Z}h} \psi_{\mu} z^{-\mu-1/2}
$$

$$
\overline{\psi}(z) = \sum_{\mu\in\mathbb{Z}h} \overline{\psi}_{\mu} z^{-\mu-1/2}
$$

which are operator-valued infinite formal Laurent series.

Proposition (3.8).

In the region $|z|>|w|$ the vacuum expectation value of two operators

are well defined;

$$
\overbrace{\psi(z)\overline{\psi}(w)} \equiv \langle 0\,|\,\psi(z)\overline{\psi}(w)\,|\,0\rangle = \frac{1}{z-w}
$$

$$
\overbrace{\overline{\psi}(z)\psi(w)} \equiv \langle 0\,|\,\overline{\psi}(z)\psi(w)\,|\,0\rangle = \frac{1}{z-w}
$$

$$
\overbrace{\psi(z)\psi(w)} \equiv \overbrace{\overline{\psi}(z)\overline{\psi}(w)} = 0.
$$

We extend the defining domain of $\overline{\psi}(z)\psi(w)$ from $|z| > |w|$ to

$(\mathbb{C}^{*})^{2}- \Delta = \{(z,w) \in \mathbb{C}^{*} \times \mathbb{C}^{*};\ z \neq w\}$ by analytic continuation. The

vacuum expectation value of general field operators $\langle 0 | \ldots | 0 \rangle$ can be defined in a similar way [T.K.].

Physicists know the following very convenient formula for calculations of fermion operator algebra.

Theorem (3.9) (Wick's theorem).

$$\phi(z_1) \ldots \phi(z_N) = \sum_{\text{comb.}} {\textstyle\substack{\bullet\\\bullet}}\, \phi(z_1) \ldots \overline{\phi(z_i) \ldots \phi}(z_j) \ldots \overline{\phi(z_k) \ldots \phi}(z_\ell) \ldots \phi(z_N) \,{\textstyle\substack{\bullet\\\bullet}}$$

$$= \sum_{\text{comb.}} (\pm) \prod_{\text{pairs}} \overline{\phi(z_i) \phi}(z_j) \;{\textstyle\substack{\bullet\\\bullet}} \prod_{\text{rest}} \phi(z_k) \,{\textstyle\substack{\bullet\\\bullet}}$$

where $\phi(z) = \psi(z)$ or $\bar\psi(z)$. The summation $\sum_{\text{comb.}}$ is taken over all pairs in $\{z_1, z_2, \ldots, z_N\}$ and the sign $\pm$ comes from the anticommuting relation of fermion operators (i.e. the signature of a permutation of indices).

Definition (3.10). 1) Define the filtration on W and $\overline{W}$ as

$$F^m W = \sum_{\mu > m} \mathbb{C}\psi^\mu = \sum_{\mu < -m} \mathbb{C}\psi_\mu$$

$$F^m \overline{W} = \sum_{\mu > m} \mathbb{C}\bar\psi^\mu = \sum_{\mu < -m} \mathbb{C}\bar\psi_\mu \,,$$

and introduce a topology on W by this filtration.

2) Denote by $\mathcal{W}$ (resp. $\overline{\mathcal{W}}$) the completion of W (resp. $\overline{W}$) with respect to this topology. Then we have

$$\mathcal{W} = \mathcal{W}_+ \oplus \mathcal{W}_- = \Big(\bigoplus_{\mu > 0} \mathbb{C}\psi^\mu \Big) \oplus \Big(\bigoplus_{\mu < 0} \mathbb{C}\psi^\mu \Big)$$

$$\overline{\mathcal{W}} = \overline{\mathcal{W}}_+ \oplus \mathcal{W}_- = \Big(\bigoplus_{\mu > 0} \mathbb{C}\bar\psi^\mu \Big) \oplus \Big(\bigoplus_{\mu < 0} \mathbb{C}\bar\psi^\mu \Big)$$

Proposition (3.11). 1) There are filtration-preserving linear isomorphism:

$$\overline{V} \xrightarrow{\ \sim\ } \mathcal{W} \ ; \qquad \bar{e}_\mu \longrightarrow \psi_\mu$$

$$V \xrightarrow{\ \sim\ } \overline{\mathcal{W}} \ ; \qquad e^\mu \longrightarrow \bar\psi^\mu.$$

2) The bilinear maps

$$\mathscr{W} \times \mathscr{F} \longrightarrow \mathscr{F} \ ; \ (\psi, |u\rangle) \longmapsto \psi|u\rangle$$

$$\overline{\mathscr{W}} \times \mathscr{F} \longrightarrow \mathscr{F} \ ; \ (\overline{\psi}, |u\rangle) \longmapsto \overline{\psi}|u\rangle$$

are continuous on each variable.

(3.12) For a point $U \in UGM$, we define $\overline{U} \subset \overline{V}$ to be $U^{\perp} = \{f \in Hom(V, \mathbb{C}); \ f_{|U} = 0\}$, and we use the symbol $[U]$ for the image of U in $P(\mathscr{F})$. The isomorphisms in (3.11) 1) define subspaces $\mathscr{W}_{+}(U) \subset \mathscr{W}$ and $\overline{\mathscr{W}}_{+}(U) \subset \overline{\mathscr{W}}$ corresponding to $\overline{U} \subset \overline{V}$ and $U \subset V$ respectively. The following theorem is very important. (4.16)

Theorem (3.13). We obtain the following equalities.

1) $\qquad \mathscr{W}_{+}(U) = \{\psi \in \mathscr{W}; \ \psi|u\rangle = 0, \ |u\rangle \in [U]\},$

$\qquad\qquad \overline{\mathscr{W}}_{+}(U) = \{\overline{\psi} \in \overline{\mathscr{W}}; \ \overline{\psi}|u\rangle = 0, \ |u\rangle \in [U]\}.$

2) $\qquad [U] = \{|u\rangle \in \mathscr{F}; \ \psi|u\rangle = \overline{\psi}|u\rangle = 0 \text{ for } {}^{\forall}\psi \in \mathscr{W}_{+}(U),$

$\qquad$ and ${}^{\forall}\overline{\psi} \in \overline{\mathscr{W}}_{+}(U)\}.$

We call $W_{+}(U) \oplus \overline{W}_{+}(U)$ as a space of annihilation operators associated with $U \in UGM$.

B) A Differential Operator algebra $\mathscr{D}_{z}$.

(3.14) Among endomorphisms of $K = \mathbb{C}((z^{-1}))$, we consider such differential operators which form a subalgebra $\mathscr{D}_{z}$ of $End(K)$, big enough and easy to treat. This restriction is only in order to make concrete calculation easier, and all the results in what follows can be generalized to the whole $End(K)$ in fact. The reader who knows the theory of KP eqfuations could remark that this $\mathscr{D}_{z}$ is nothing but the "Fourier transform" of the algebra of microlocal differential operators $\mathscr{E} = \mathbb{C}[[t]]((\partial^{-1}))$.

Definition (3.15). We consider the following differential operators:

$$\mathcal{D}_z = \{P(z,\tfrac{d}{dz}) = \sum_{n<<\infty} z^n a_n(\tfrac{d}{dz}) \; ; \; a_n(\tfrac{d}{dz}) \in \mathbb{C}[[\tfrac{d}{dz}]]\}.$$

We then define a valuation : $\mathrm{ord}_z : \mathcal{D}_z \to \mathbb{Z}\cup\{\infty\}$ in the following way;

$$\mathrm{ord}_z P = \begin{cases} -m & \text{if } P = \sum_{n \le m} z^n a_n(\tfrac{d}{dz}), \; a_m \neq 0 \\ \infty & \text{if } P = 0 \end{cases}$$

(i.e. $\mathrm{ord}_z z = -1$, $\mathrm{ord}_z \tfrac{d}{dz} = 0$). Define a (decreasing) filtration on $\mathcal{D}_z$ by

$$G^m \mathcal{D}_z = \{P(z,\tfrac{d}{dz}) \in \mathcal{D}_z; \; \mathrm{ord}_z P \ge m\}.$$

If we introduce a topology on $\mathcal{D}_z$ with this filtration, then $\mathcal{D}_z$ is complete Hausdorff and the product $\mathcal{D}_z \times \mathcal{D}_z \longrightarrow \mathcal{D}_z$ is continuous (i.e. $\mathcal{D}_z$ is a complete topological $\mathbb{C}$-algebra) with respect to this topology.

Proposition (3.16). 1) In case $\mathrm{ord}_z P = m$ for $P \in \mathcal{D}_z$, we have

$$P : F^{\ell}K \longrightarrow F^{\ell+m} K.$$

2) The natural map (defined from 1)) $\mathcal{D}_z \longrightarrow \mathrm{End}(K)$ is injective (and $G^m \mathcal{D}_z \subset F^m \mathrm{End}(K)$ from 1)).

(3.17) One can rewrite $P(z,\tfrac{d}{dz})$ as follows:

$$P(z,\tfrac{d}{dz}) = \sum_{n \le m} z^n a_n(\tfrac{d}{dz}) = \sum_{k \le m} z^k g_k(z\tfrac{d}{dz})$$

where g_k is a polynomial of order at most $m-k$ with respect to $z\tfrac{d}{dz}$. With this representation $\mathrm{ord}_z(P)$ is given by

$$\mathrm{ord}_z(P) = -\min\{m: \deg g_k \le m-k \text{ for } {}^{\forall}k \in \mathbb{Z}\}.$$

Now we consider another valuation defined as

$$\nu(P) = -\max\{ k \; ; \; g_k \neq 0\}$$

(i.e. $\nu(z) = -1$, $\nu(z\tfrac{d}{dz}) = 0$). Clearly we have

$$\mathrm{ord}_z(P) \le \nu(P).$$

Proposition (3.18). The filtration on $\mathcal{D}_z$ defined from ν, $F^m \mathcal{D}_z = \{P \in \mathcal{D}_z \; ; \; \nu(P) \ge m\}$, is nothing but the one induced from

the inclusion map $\mathcal{D}_z \subset \text{End}(K)$.

Proposition (3.16) implies that if a series in $\mathcal{D}_z$ converges w.r.t. ord_z, then it converges w.r.t. ν, but the converse need not hold.

C) Second quantization and anomaly

Definition (3.19). For $P \in \mathcal{D}_z$ we define the *adjoint* $P^\dagger \in \mathcal{D}_z$ as follows;

1) $\qquad z^\dagger = z$, $(\frac{d}{dz})^\dagger = -\frac{d}{dz}$,

2) $\qquad (PQ)^\dagger = Q^\dagger P^\dagger \qquad$ for $^\forall P, Q \in \mathcal{D}_z \qquad$ (anti-homomorphism).

A differential operator $P = P(z, \frac{d}{dz}) \in \mathcal{D}_z$ acts naturally on the formal parameter z of the fields operators $\psi(z), \bar{\psi}(z)$.

For $P \in \mathcal{D}_z$, P and $P^\dagger$ are related as follows

$$\int_{\mathbb{Q}} dz \; : (P\psi(z))\bar{\psi}(z) \; : \; = \int_\infty dz \; : \psi(z)(P^\dagger\bar{\psi}(z)) \; : \; ,$$

where $\int_{\mathbb{Q}}$ denotes the (clockwise) contour integral around $\mathbb{Q}$.

Definition (3.20). Given $P \in \mathcal{D}_z$, we define an operator $\Phi(P)$ acting on $\mathcal{F}$ and $\bar{\mathcal{F}}$ as follows:

$$\Phi(P) = \frac{1}{2\pi i} \int_{\mathbb{Q}} dz \; :(P\psi(z))\bar{\psi}(z): \; = \frac{1}{2\pi i} \int_{\mathbb{Q}} dz \; :\psi(z)(P^\dagger\bar{\psi}(z)):.$$

This procedure is called the *second quantization* and this $\Phi(P)$ is called the *second-quantized* operator corresponding to P. The naming is justified from the next theorem.

Theorem (3.21). 1) For $P \in \mathcal{D}_z$,

$$[\Phi(P), \psi(z)] = P\psi(z),$$

$$[\Phi(P), \bar{\psi}(z)] = -P^\dagger\bar{\psi}(z).$$

2) For $P, Q \in \Phi_z$,

$$[\Phi(P), \Phi(Q)] = \Phi([Q, P]) + c(P, Q)\text{id}$$

where

$$c(P, Q) = \left(\frac{1}{2\pi\sqrt{-1}}\right)^2 \int_{\infty} dz \int_{\infty} dw \left(P(w,\tfrac{d}{dw})\tfrac{1}{w-z}\right)\left(Q(z,\tfrac{d}{dz})\tfrac{1}{w-z}\right),$$

called the *Schwinger term*.

Proof. 1) From the convergence of the operator product we see that the contour should be chosen as follows;

$$\Phi(P)\psi(z) = \frac{1}{2\pi\sqrt{-1}} \int_{\infty\cdot z} dw \; :P\psi(w)\bar{\psi}(w): \psi(z)$$

$$\psi(z)\Phi(P) = \frac{1}{2\pi\sqrt{-1}} \int_{\infty\cdot z} dw \; \psi(z):P\psi(w)\bar{\psi}(w):$$

Then we can obtain the following relation

$$[\Phi(P), \psi(z)] = \frac{1}{2\pi\sqrt{-1}} \int_{(z)} dw \; :(P\psi(w))\bar{\psi}(w): \psi(z)$$

$$= \frac{1}{2\pi\sqrt{-1}} \int_{(z)} dw \; P\psi(w)\overline{\bar{\psi}(w)\psi(z)} = P\psi(z).$$

We can proceed in a similar way for $\bar{\psi}(z)$.

2) The commutator of $\Phi(P)$ and $\Phi(Q)$ can be expressed similarly as:

$$[\Phi(P), \Phi(Q)] = \left(\frac{1}{2\pi\sqrt{-1}}\right)^2 \int_{\infty} dz \int_{\infty} dw \; :(P\psi(w))\bar{\psi}(w): \; :(Q\psi(z))\bar{\psi}(z):$$

Using Wick's theorem for the integrand and taking the singular parts into account, we obtain

$$(\text{integrand}) = :P_w\overline{\psi(w)\bar{\psi}(w)}: \; :Q_z\psi(z)\bar{\psi}(z): +$$

$$:P_w\psi(w)\overline{\bar{\psi}(w)}::\overline{Q_z\psi}(z)\bar{\psi}(z): \; + \; :P_w\overline{\psi(w)\bar{\psi}(w)}: \; :\overline{Q_z\psi(z)\bar{\psi}}(z):.$$

Then the first two terms correspond to $\Phi([Q,P])$ and the third term provides the Schwinger term $c(P,Q)$. $\blacksquare$

Proposition (3.22) ([S.S.]).

The bilinear map $c: \mathcal{D}_z \times \mathcal{D}_z \to \mathbb{C}$ enjoys the following properties:

1) $\quad\quad c(P,Q) = -c(Q,P);$

2) $\quad\quad c(P,[Q,R]) + c(Q,[R,P]) + c(R,[P,Q]) = 0 \; ;$

3) for the basis $u_n^k \equiv z^{k+n} \frac{1}{k!} (\frac{d}{dz})^k \in \mathcal{D}_z$, $k \geq 0$, $n \in \mathbb{Z}$,

$$c(u_n{}^k, u_m{}^\ell) = (-1)^k \frac{1}{(k+\ell+1)!} \prod_{j=-\ell}^{k} (n+j)\, \delta_{m+n,0}.$$

From 1) and 2) the map c; $\mathcal{D}_z \times \mathcal{D}_z \longrightarrow \mathbb{C}$ determines a cohomology class $[c] \in H^2(\mathcal{D}_z, \mathbb{C})$, which turns out to be nonzero cohomology class.

D) Lifting of the action of $\mathcal{D}_z$

(3.23) Using $\Phi(P)$ defined in C), we can lift the infinitesimal action $\theta(P)$ on UGM to $\widetilde{\theta}(P)$ on $\widetilde{\text{UGM}}$, by defining it on $\mathcal{F}^\times$ and by checking that it preserves the image of UGM by the Plücker embedding.

Definition (3.24). For each $P \in \mathcal{D}_z$ we define a vector field; $\widetilde{\theta}(P)$ on $\mathcal{F}^\times$ in the following way;
$$\widetilde{\theta}_{|u\rangle}(P) = \Phi(P)|u\rangle \in T_{|u\rangle}\mathcal{F}^\times \overset{\sim}{\longrightarrow} \mathcal{F}$$
where $|u\rangle \in \mathcal{F}^\times$.

(3.25) Let L be a one-dimensional subspace of $\mathcal{F}$ and $[L]$ the corresponding point in $P(\mathcal{F})$. Similarly as (1.12) we have a canonical isomorphism
$$T_{[L]}P(\mathcal{F}) \overset{\sim}{\longrightarrow} \text{Hom}_{\mathbb{C}}(L, \mathcal{F}/L).$$
Then in the same way as (1.14) we define a holomorphic vector field $\theta(P) \in H^0(P(\mathcal{F}), \theta)$ for $P \in \mathcal{D}_z$ as
$$\theta_{[L]}(P): L \lhook\joinrel\longrightarrow \mathcal{F} \overset{\Phi(P)}{\longrightarrow} \mathcal{F} \longrightarrow \mathcal{F}/L.$$
For $\widetilde{\theta}$ and θ we infer readily the following propositions.

Definition (3.26). Let $F \longrightarrow B$ be a $\mathbb{C}^*$-bundle. Then the action of $\mathbb{C}^*$ defines a holomorphic vector field E on F along the fibre, called the *Euler operator*. For $\mathcal{F}^\times \longrightarrow P(\mathcal{F})$, E is nothing but the vector field corresponding to id: $\mathcal{F} \longrightarrow \mathcal{F}$ with the above description.

Proposition (3.27). Every $\widetilde{\theta}(P)$, $P \in \mathcal{D}_z$, preserves the ideal $\mathcal{I}$ of $\mathcal{O}_{\mathcal{F}}$ generated by the Plücker relation:
$$\widetilde{\theta}(P)\mathcal{I} \subset \mathcal{I}.$$

Therefore $\tilde{\theta}(P)$ (resp. $\theta(P)$) defines a holomorphic vector field on UGM (resp. UGM). This $\theta(P)$ coincides with the one defined in (1.14).

Summing up, we have obtained the following main theorem in this section.

$\ \ $Theorem (3.28). For $P \in \mathcal{D}_z$ there are holomorphic vector fields $\tilde{\theta}(P)$, $\theta(P)$ whose action is compatible with respect to the following diagram:

$$
\begin{array}{ccc}
\tilde{\theta} : & \widetilde{UGM} & \longrightarrow \mathcal{F}^{\times} \\
& \downarrow & \downarrow \\
\theta : & UGM & \longrightarrow P(\mathcal{F}),
\end{array}
$$

where θ on UGM is defined in (1.14).

Corollary (3.29). For $P, Q \in \mathcal{D}_z$ we have

$$[\tilde{\theta}(P), \tilde{\theta}(Q)] = \tilde{\theta}([Q, P]) + c(P, Q)E.$$

.E) Current and Virasoro algebra

(3.30) We observe that $\mathfrak{V} = \mathcal{D}^1(K)$,(2.7), is a Lie subalgebra of $\mathcal{D}_z$. The operators $\Phi(P)$, $P \in \mathfrak{V}$, play the most essential role in our theory.

Definition (3.31). 1) The following operators corresponding to scalar multiplications are called current operators:

$$J_n = -\Phi(z^n) = - \frac{1}{2\pi\sqrt{-1}} \oint dz \ z^n \ :\psi(z)\bar{\psi}(z): \ .$$

Then we call

$$J(z) = :\bar{\psi}(z)\psi(z): = \sum_{n \in \mathbb{Z}} J_n z^{-n-1}$$

the current operator.

2) The following operators corresponding to derivations are called Virasoro operators (for spin j):

$$L_n^{(j)} = \Phi(s_j(z^{n+1}\frac{d}{dz})) = \Phi(z^n(z\frac{d}{dz} + j(n + 1)))$$

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$$= \frac{1}{2\pi\sqrt{-1}} \int_{\oplus} dz\, z^{n+1} \; :\{(1-j)\frac{d\psi(z)}{dz}\,\overline{\psi}(z) - j\psi(z)\frac{d\overline{\psi}(z)}{dz}\}:$$

for $j \in \frac{1}{2}\mathbb{Z}$, $n \in \mathbb{Z}$ (for s_j see (2.32)). Then we call

$$T^j(z) = :(1-j)\frac{d\psi(z)}{dz}\,\overline{\psi}(z) - j\psi(z)\frac{d\overline{\psi}(z)}{dz}:$$

$$= \sum_{n\in\mathbb{Z}} L_n^{(j)} z^{-n-2}$$

the *energy momentum tensor* (for spin j-fields).

3) The Lie subalgebra generated by id and $\{J_n\}$ (resp. $L_n^{(j)}$) is

called the *current algebra* (resp. the *Virasoro algebra*). For $j = 1/2$

we omit the superindex j in the notation of L and T hereafter.

Formulae (3.32). From (3.21) we obtain the following fundamental

relations of these operators:

$$[J_n, J_m] = n\,\delta_{n+m,0} \; ,$$
$$[L_n^{(j)}, L_m^{(j)}] = (n-m)L_{n+m}^{(j)} - \frac{1}{6}(6j^2 - 6j + 1)(n^3 - n)\delta_{n+m,0} \; ,$$
$$[L_n^{(j)}, J_m] = -mJ_{n+m} - \frac{1}{2}(2j-1)(n^2+n)\delta_{n+m,0} \; ,$$
$$[J_n, \psi(z)] = -z^n\psi(z),$$
$$[J_n, \overline{\psi}(z)] = z^n\overline{\psi}(z),$$
$$[L_n^{(j)}, \psi(z)] = z^n(z\frac{d}{dz} + j(n+1))\psi(z),$$
$$[L_n^{(j)}, \overline{\psi}(z)] = z^n(z\frac{d}{dz} + (1-j)(n+1))\overline{\psi}(z).$$

Moreover between T(z) and J(z) there is an important relation.

Proposition (3.33)(Sugawara form of energy-momentum tensor).

$$T(z) = \frac{1}{2} \; {}^{\circ}_{\circ}J(z)J(z){}^{\circ}_{\circ}$$

$$= \frac{1}{2} \sum_{m,n\in\mathbb{Z}} {}^{\circ}_{\circ}J_m J_{n-m}{}^{\circ}_{\circ} \; z^{-n-2}$$

where ${}^{\circ}_{\circ} \quad {}^{\circ}_{\circ}$ is the ordering which transforms J_n $(n \geq 0)$ to the right
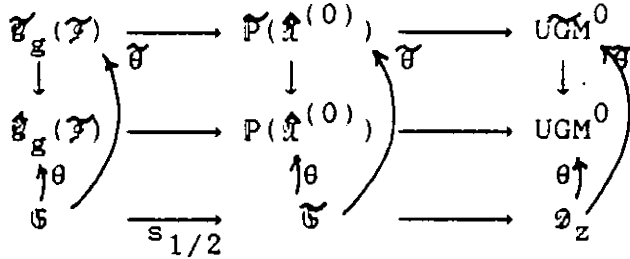
and $J_n (n < 0)$ to the left (see §4,A)).

Proposition (3.34). There exist unique maps:

$$\mathcal{T}: \mathbb{S} \longrightarrow H^0(\mathcal{X}_g(\mathcal{Y}), \theta) \text{ and } \mathcal{T}: \mathbb{T} \longrightarrow H^0(\mathbb{P}(\hat{\mathcal{X}}_g^{(0)}), \theta)$$

(holomorphic vector fields) s.t.

1) $[\bar{\vartheta}(\ell),E] = 0$, $\forall \ell \in \mathfrak{G}$ or $\bar{\mathfrak{V}}$

2) Their actions are compatible, as indicated in the following diagram:

$$
\begin{array}{ccccc}
\bar{\mathfrak{V}}_g(\mathcal{F}) & \xrightarrow{\ \bar{\vartheta}\ } & \mathbb{P}(\hat{\mathfrak{A}}^{(0)}) & \longrightarrow & \mathrm{UGM}^0 \\
\downarrow & & \downarrow & & \downarrow \\
\hat{\mathfrak{V}}_g(\mathcal{F}) & \longrightarrow & \mathbb{P}(\hat{\mathfrak{A}}^{(0)}) & \longrightarrow & \mathrm{UGM}^0 \\
\uparrow\theta & & \uparrow\theta & & \theta\uparrow \\
\mathfrak{G} & \xrightarrow{s_{1/2}} & \bar{\mathfrak{V}} & \longrightarrow & \mathfrak{D}_z
\end{array}
$$

Proposition (3.35). By the above compatibility and definition (3.24), it follows that

$$\bar{\vartheta}(\ell)\,|\vec{X}_c\rangle = \Phi(s_{1/2}\cdot\ell)\,|\vec{X}_c\rangle$$

for $\forall \vec{X}_c \in \bar{\mathfrak{V}}_g(\mathcal{F})$ and $\forall \ell \in \mathfrak{G}$.

This equation plays an essential role in §7.

§4 τ-function and Wave Function

(4.0) In this section we reformulate the Sato theory of KP equations into a convenient form for our later use. We give skechy but selfcontained proof here. More details about KP equations and the related topics, see for example [Sa.][D.J.K.M.].

A) Bosonization

(4.1) We first explain one of the most fundamental principles of two-dimensional field theory: fermion-boson correspondence. The fermion Fock space $\mathcal{F}$ is isomorphic to a corresponding boson Fock space $\mathcal{H}$ defined below and the operators acting on these spaces have also a well-defined correspondence.

The boson Fock space is defined as the following vector space;

$$\mathcal{H} = \mathbb{C}[[t_1, t_2, \ldots]] \otimes \mathbb{C}[e^{t_o}, e^{-t_o}] = \sum_{m \in \mathbb{Z}} \mathbb{C}[[t_1, t_2, \ldots]] e^{m t_o}$$

$$= \sum_{m \in \mathbb{Z}} \mathcal{H}_m$$

where t_i's are infinite number of indeterminates. An element of $\mathcal{H}$ is called a boson state vector. We define the degree and the charge as follows:

$$\deg t_i = i, \qquad \deg e^{t_o} = 0,$$

$$\text{charge } t_i = 0 \ (i \neq 0), \quad \text{charge } e^{n t_o} = n.$$

then $\mathcal{H}$ decomposes into charged sector $\mathcal{H}_n$ $(n \in \mathbb{Z})$, each of which is a complete Hausdorff vector space by the filtration: $F^d \mathcal{H}_n = \prod_{d' \geq d} \mathcal{H}_n(d')$, $\mathcal{H}_n(d) = \{f(\mathfrak{t}) \in \mathcal{H}_n; \text{ degree } f(\mathfrak{t}) = d\}$.

Definition (4.2). We define a linear map B: $\mathcal{F} \to \mathcal{H}$ as follows:

$$B|\Psi\rangle = \sum_{n \in \mathbb{Z}} e^{n t_o} \langle n | \exp\left(\sum_{m=1}^{\infty} J_m t_m\right) |\Psi\rangle$$

Theorem (4.3)(Bosonization) [D.J.K.M.].

1) The linear map $B: \mathcal{F} \rightarrow \mathcal{H}$ gives a topological isomorphism.

$$B: \quad \mathcal{F} \xrightarrow{\ \simeq\ } \mathcal{H}.$$

2) B preserves the charge and the degree

$$\mathcal{F}_n^{(d)} \xrightarrow{\ \simeq\ } \mathcal{H}_n^{(d)}.$$

(4.4) We define the following fundamental operators on $\mathcal{H}$

$$a_n = \frac{\partial}{\partial t_n} \quad (n \geq 0), \qquad a_{-n} = nt_n \quad (n > 0), \qquad q = t_0.$$

The following relations can be easily confirmed:

$$[a_m , a_n] = m\, \delta_{m+n,0}$$
$$[a_m , e^q] = e^q\, \delta_{m,0}$$

which are standard bosonic commutation relations.

The normal ordering $\stackrel{\circ}{\circ}\ \stackrel{\circ}{\circ}$ of these operators, a_m and e^{nq}, are defined as follows:

1) Inside the symbol $\stackrel{\circ}{\circ}\quad \stackrel{\circ}{\circ}$ we put polynomials of a_n ($n \in \mathbb{Z}$) and e^q, and all the operators a_n and e^q commute each other.

2) In case the operators inside the symbol $\stackrel{\circ}{\circ}\quad \stackrel{\circ}{\circ}$ are in normal order (i.e. all the creation operators (a_n ($n<0$), e^q) are located to the left of all the annihilation operators (a_n ($n\geq0$)).) the symbol can be taken away and the remaining operators are considered as those acting on $\mathcal{H}$. For example $\stackrel{\circ}{\circ} a_n a_{-m} \stackrel{\circ}{\circ} = \stackrel{\circ}{\circ} a_{-m} a_n \stackrel{\circ}{\circ} = a_{-m} a_n$, for $n,m > 0$.

Each operator $\mathcal{O}$ on the fermion Fock space has the corresponding one $\mathcal{O}_B = B\mathcal{O}B^{-1}$ on the boson Fock space.

Introducing the following field operator which is familiar in string theory;

$$Q(z) = q + a_0 \log z - \sum_{n \neq 0} \frac{a_n}{n} z^{-n}$$

and set for $k \in \mathbb{Z}$

$$V_k(z) = \; {}^{\diamond}_{\diamond}e^{kQ(z)}\; {}^{\diamond}_{\diamond}$$

$$= \exp\left(k\sum_{n=1}^{\infty} t_n z^n\right)e^{kt_0}\exp\left(k\log z\frac{\partial}{\partial t_0} - k\sum_{n=1}^{\infty}\frac{z^{-n}}{n}\frac{\partial}{\partial t_n}\right)$$

which we call a *vertex operator* of charge k.

Then we have the following:

Theorem (4.5)(Fermion-boson correspondence).

$$\psi_B(z) = V_{-1}(z),$$

$$\bar{\psi}_B(z) = V_{+1}(z),$$

$$J_B(z) = A(z) = \frac{dQ(z)}{dz} = \sum_{n\in\mathbb{Z}} a_n z^{-n-1},$$

$$T_B(z) = \frac{1}{2}\; {}^{\diamond}_{\diamond}A(z)A(z)\; {}^{\diamond}_{\diamond}$$

where $J_B(z)$ and $T_B(z)$ are the bosonized version of current operator and the energy-momentum tensor operator.

B) τ-functions and Wave Functions

(4.6) We first take an element $\widetilde{U}$ ($\in \widetilde{\mathrm{UGM}}^{\phi}$) (cf.(1.20)) of the charge zero sector of the universal Grassmann manifold and its image in the Fock space $|\widetilde{U}\rangle\in \mathcal{F}_0^{\times}$. We then define the following quantity:

Definitions (4.7).

$$\tau(t,U) = \langle 0|\; e^{H(t)}|\widetilde{U}\rangle,$$

$$\Psi(z,t,U) = \frac{\langle-1|\; e^{H(t)}\psi(z)|\widetilde{U}\rangle}{\tau(t,U)},$$

$$\bar{\Psi}(z,t,U) = \frac{\langle 1|\; e^{H(t)}\bar{\psi}(z)|\widetilde{U}\rangle}{\tau(t,U)},$$

where

$$H(t) = \sum_{n=1}^{\infty} t_n J_n.$$

We call $\tau(t,\widetilde{U})$ a τ-*function* and $\Psi(z,t,U),\bar{\Psi}(z,t,U)$ (conjugate-) *wave functions* associated with $\widetilde{U} \in \widetilde{\mathrm{UGM}}^{\phi}$. Note that $\Psi,\bar{\Psi}$ depend only on $U = \pi(\widetilde{U}) \in \mathrm{UGM}^{\phi}$

Remark. We can define $\tau(t,\widetilde{U})$ for $\widetilde{U} \in \mathrm{UGM}^{O}$ as above. Then $\tau(0,\widetilde{U})\neq 0$

if and only if $U \in UGM^{\phi}$.

We provide several important properties of the (conjugate-)wave functions.

Proposition (4.8) [D.J.K.M.].

1) The wave functions have the following expansions:

$$\Psi(z,\mathfrak{t},U) = e^{-\xi(\mathfrak{t},z)} (1 + \sum_{k=1}^{\infty} w_k(\mathfrak{t})z^{-k}),$$

$$\bar{\Psi}(z,\mathfrak{t}.U) = e^{\xi(\mathfrak{t},z)} (1 + \sum_{k=1}^{\infty} \bar{w}_k(\mathfrak{t})z^{-k})$$

where $w_k(t), \bar{w}_k(t) \in \mathbb{C}[[\mathfrak{t}]]$ and $\xi(\mathfrak{t},z) = \sum_{n=1}^{\infty} t_n z^n$.

2) If we write

$$\Psi(z,\mathfrak{t}.U) = \sum_{M} \Psi_M(z,U) \mathfrak{t}^M$$

$$(\text{resp.} \quad \bar{\Psi}(z,\mathfrak{t},U) = \sum_{M} \bar{\Psi}_M(z,U) \mathfrak{t}^M),$$

then we have $\Psi_M(z,U) \in U$ (resp. $\bar{\Psi}_M(z,U) \in \bar{U}$) where $\mathfrak{t}^M = t_1^{m_1} t_2^{m_2} t_3^{m_3}..$, $m_i (\geq 0) \in \mathbb{Z}$.

Proof. 1) The expansion 1) can be obtained easily with the bosonization rule mentioned in (4.5);

$$\Psi(z,\mathfrak{t},U) = e^{-\xi(\mathfrak{t},z)} \frac{\tau(\mathfrak{t} + [z],U)}{\tau(\mathfrak{t},U)}$$

$$\bar{\Psi}(z,\mathfrak{t},U) = e^{\xi(\mathfrak{t},z)} \frac{\tau(\mathfrak{t} - [z],U)}{\tau(\mathfrak{t},U)}$$

where $[z] = (\frac{1}{z}, \frac{1}{2z^2}, \frac{1}{3z^3}, \ldots)$.

2) Take a base $\{\xi_\mu\}_{\mu \in \mathbb{Z}}$ of $\mathcal{V}$ with

$$\xi^\mu \in U \ (\mu < 0), \quad \xi^\mu = e^\mu \ (\mu > 0),$$

and satisfying the following relation:

$$(\ldots,\xi^{\mu},\xi^{\mu+1},\ldots) = (\ldots,e^{\nu},e^{\nu+1},\ldots) \begin{vmatrix} \ddots & & 0 & & \vdots \\ & \ddots 1 & & & \vdots \\ * & 1 & & & -1/2 \\ & * & 1 & & 1/2 \\ * & * & 0 & \ddots & \vdots \\ \ldots -\frac{1}{2} & \frac{1}{2} & \cdots & & \cdot \end{vmatrix}$$

Identifying the matrix in the right hand side of the above equality with an element g of P_{-} and the corresponding operator on the Fock space (see (1.26)) $G[g]$ (i.e. $|U\rangle = G[g]|0\rangle$) we obtain the following relation;

$$G^{-1}(\sum_{\mu\in\mathbb{Z}h}\psi_{\mu}e^{\mu})G = \sum_{\mu\in\mathbb{Z}h}(G^{-1}\psi_{\mu}G)e^{\mu} = \sum_{\mu,\nu\in\mathbb{Z}h}\psi_{\nu}g^{\nu}_{\mu}e^{\mu} = \sum_{\nu\in\mathbb{Z}h}\psi_{\nu}\xi^{\nu}.$$

We then obtain

$$\psi(z)|U\rangle = (\sum_{\mu\in\mathbb{Z}h}\psi_{\mu}e^{\mu})G|0\rangle = G(\sum_{\mu\in\mathbb{Z}h}\psi_{\mu}\xi^{\mu})|0\rangle = \sum_{\mu<0}(G\psi_{\mu}|0\rangle)\xi^{\mu}.$$

Bosonizing these relations, we obtain the result:

$$\Psi_{M}(z,U) \in U.$$

The argument is the same for the conjugate sector. ∎

Since U and $\tilde{U}$ ($= U^{\perp}$) are orthogonal to each other with respect to $\mathrm{Res}_{z=\infty}$, the following relations are a natural consequence of 2):

Corollary (4.9) (Bilinear Relation).

$$\mathrm{Res}_{z=\infty} \Psi(z,t,U) \bar{\Psi}(z,t',U) = 0$$

Corollary (4.10) (Hirota's Bilinear Equation for τ-function)[H.].

$$\sum_{j=0}^{\infty} P_{j}(-2y) P_{j+1}(D_{t}) \exp(\sum_{l=1}^{\infty} y_{l}D_{t_{l}}) \tau(t)\cdot\tau(t) = 0$$

where

$$\exp(\sum_{n=1}^{\infty} t_{n}z^{n}) = \sum_{j=0}^{\infty} P_{j}(t) z^{j}, \qquad D_{t} = (D_{t_{1}}, \tfrac{1}{2}D_{t_{2}}, \tfrac{1}{3}D_{t_{3}},\ldots)$$

and

$$P(D_{t}) f(t)\cdot g(t) = [P(\partial_{y})\{f(t + y) g(t - y)\}]_{y=0}.$$

This equation is obtained as a direct consequence of the bilinear

relation. It characterizes the that τ-function is associated with an element of UGM (Theorem 4.16). (See [D.J.K.M.] for more details about (4.9) and (4.10).)

Remark (4.11). The bilinear relation (4.9) is nothing but the bosonized version of the Plücker relation, and its fermionic form is represented as

$$\text{Res}_{z=\infty} \psi(z) |U\rangle \otimes \bar{\psi}(z) |U\rangle = \sum_{\mu \in \mathbb{Z}h} \psi_\mu |U\rangle \otimes \bar{\psi}_{-\mu} |U\rangle = 0.$$

C) τ-function

(4.12) Putting aside the definitions of τ, ψ and $\bar{\psi}$ given in B) for a while, we investigate how the properties given in Proposition (4.8) of B) characterize ψ and $\bar{\psi}$.

Definition (4.13) (Generic Wave Function). We call

$$\Psi(z, \mathfrak{t}) = \sum_N \Psi_N(z) \mathfrak{t}^N \ , \quad \bar{\Psi}(z, \mathfrak{t}) = \sum_N \bar{\Psi}_N(z) \mathfrak{t}^N \in \mathbb{C}((z^{-1})) \otimes \mathbb{C}[[\mathfrak{t}]]$$

a *conjugate pair of generic wave functions* when they satisfy the following two conditions:

1) asymptotic behavior

$$\Psi(z, \mathfrak{t}) = e^{-\xi(z, \mathfrak{t})} \left(1 + \sum_{k=1}^{\infty} w_k(\mathfrak{t}) \, z^{-k}\right) , \quad w_k(\mathfrak{t}) \in \mathbb{C}[[\mathfrak{t}]],$$

$$\bar{\Psi}(z, \mathfrak{t}) = e^{\xi(z, \mathfrak{t})} \left(1 + \sum_{k=1}^{\infty} \bar{w}_k(\mathfrak{t}) \, z^{-k}\right) , \quad \bar{w}_k(\mathfrak{t}) \in \mathbb{C}[[\mathfrak{t}]];$$

2) bilinear relation

$$\text{Res}_{z=\infty} \Psi(z, \mathfrak{t}) \bar{\Psi}(z, \mathfrak{t}') = \sum_{M,N} \mathfrak{t}^M \mathfrak{t}'^N \text{Res}_{z=\infty} \Psi_M(z) \bar{\Psi}_N(z) = 0 .$$

We denote the whole set of pairs Ψ, $\bar{\Psi}$ satisfying 1) and 2) by WF^ϕ.

Definition (4.14) (Generic τ-function).

$\tau(\mathfrak{t}) \in \mathbb{C}[[\mathfrak{t}]]$ is called the *generic τ-function* when it satisfies the following conditions:

1) $\tau(\mathfrak{t})$ satisfies the Hirota's bilinear equation (Cor.(4.10)),

2) $\tau(0) \neq 0$.

We denote the whole set of generic τ-functions by TF^{ϕ}.

   Proposition (4.15).

1) For any pair $(\Psi(z,t), \bar{\Psi}(z,t)) \in WF^{\phi}$, there exists a unique element $U \in UGM^{\phi}$ such that $\Psi_N(z) \in U$, $\bar{\Psi}_N(z) \in \bar{U}$ $(= U^{\perp})$.

2) The map $WF^{\phi} \longrightarrow UGM^{\phi}$ given above is bijective.

   Proof. 1)The terms $w_n(t)$ and $\bar{w}_n(t)$ in the asymptotic expansion of $\Psi(z,t), \bar{\Psi}(z,t)$ can be expanded with respect to t_j as follows:

$$w_n(t) = c_n + \sum_{j=1}^{\infty} c_{nj} t_j + o(t^2),$$

$$\bar{w}_n(t) = \bar{c}_n + \sum_{j=1}^{\infty} \bar{c}_{nj} t_j + o(t^2).$$

Define elements in $\mathbb{C}((z^{-1}))$ as follows:

$$\varphi_0(z) \equiv \Psi_{N_0}(z) = 1 + \sum_{n=1}^{\infty} c_n z^{-n}$$

$$\varphi_j(z) \equiv \Psi_{N_j}(z) = -z^j(1 + \sum_{n=1}^{\infty} c_n z^{-n}) + \sum_{n=1}^{\infty} c_{nj} z^{-n} \qquad (j \geq 1)$$

$$\bar{\varphi}_0(z) \equiv \bar{\Psi}_{N_0}(z) = 1 + \sum_{n=1}^{\infty} \bar{c}_n z^{-n}$$

$$\bar{\varphi}_j(z) \equiv \bar{\Psi}_{N_j}(z) = z^j(1 + \sum_{n=1}^{\infty} \bar{c}_n z^{-n}) + \sum_{n=1}^{\infty} \bar{c}_{nj} z^{-n} \qquad (j \geq 1)$$

where

$$N_0 = (0,0,0,\ldots\ldots),$$

$$N_j = (0,0,\ldots,0,\overset{j}{1},0,\ldots),$$

and define the following vector subspaces:

$$U = \sum_{n=0}^{\infty} \mathbb{C}\varphi_n(z) \subsetneq K,$$

$$\bar{U} = \sum_{n=0}^{\infty} \mathbb{C}\bar{\varphi}_n(z) \subsetneq K.$$

As a direct consequence of the bilinear relation (4.10) we obtain

(1) $\bar{U} \subseteq U^\perp$, $U \subseteq \bar{U}^\perp$. The concrete expression $\varphi_n(z)$ and $\bar\varphi_n(z)$ given above naturally leads to (2) $U^\perp \cap F^0\bar{V} = 0$, $\bar{U}^\perp \cap F^0 V = 0$. Recalling $M(U) = M(\bar{U}) = \{-\frac{1}{2}, -\frac{3}{2}, \ldots\}$ (cf. (1.7),(1.11)), we can show that (3) $V = U \oplus F^0 V$, $\bar{V} = \bar{U} \oplus F^0\bar{V}$. (1), (2) and (3) imply $\bar{U} = U^\perp$, $U = \bar{U}^\perp$. Then by the bilinear relation we obtain the desired result:

$$\Psi_N(z) \in \bar{U}^\perp = U \ , \qquad \bar\Psi_N(z) \in U^\perp = \bar{U} \quad \text{for } {}^\forall N.$$

It is obvious from (2) that $U \in UGM^\phi$.

2) We now show that two pairs of wave functions $(\Psi(z,t), \bar\Psi(z,t))$, $(\Psi'(z,t), \bar\Psi'(z,t)) \in WF^\phi$ corresponding to the same $U \in UGM^\phi$ coincide with each other. For this purpose we expand the wave functions as follows:

$$\Psi(z,t) = e^{-\xi(z,t)} \sum_{n=0}^{\infty} \sum_{|N|=n} w_n(z) \, t^N$$

where $w_N(z) \in \mathbb{C}[[z^{-1}]]z^{-1} = F^0 V$ for $|N| > 0$,

i.e. $\Psi_N(z) = [e^{-\xi(z,t)} \sum_{|M| < |N|} w_N(z) t^M]_N + w_N(z)$ (*)

with $|N| = \sum_{i=1}^{\infty} n_i$. By induction with respect to $|N|$ we show $w_N(z) = w'_N(z)$. The argument works similarly for the conjugate sector.

i) For $|N| = 0$ (i.e. $N = N_0$) we obtain

$$w_{N_0}(z) = 1 + \sum_{n=1}^{\infty} c_n z^{-n} = \varphi_0(z) \in F^{-1}U$$

$$w'_{N_0}(z) = 1 + \sum_{n=1}^{\infty} c'_n z^{-n} = \varphi'_0(z) \in F^{-1}U \ .$$

Since dim $F^{-1}U = 1$, we obtain

$$w_{N_0}(z) = w'_{N_0}(z) \ .$$

ii) Assuming $w_M(z) = w'_M(z)$ for $|M| < n$, we obtain

$$\varphi \equiv \Psi_N(z) - \Psi'_N(z) = w_N(z) - w'_N(z) \in F^0 V \cap U = \{0\}.$$

for $|N| = n.$ $\lceil$ Conversely the mapping $UGM^\phi \longrightarrow WF^\phi$ is shown to be

surjective because of the correspondence of eq.'s in (4.7) and Proposition (4.8). ∎

Define the map $TF^\phi/\mathbb{C}^* \longrightarrow WF^\phi$ by

$$\Psi(\mathfrak{t},z) = e^{-\xi(z,\mathfrak{t})}\frac{\tau(\mathfrak{t} + [z])}{\tau(\mathfrak{t})}$$

$$\overline{\Psi}(\mathfrak{t},z) = e^{\xi(z,\mathfrak{t})}\frac{\tau(\mathfrak{t} - [z])}{\tau(\mathfrak{t})} . \qquad (**)$$

Theorem (4.16). The following diagram is commutative and all arrows are bijections.

Proof. Only injectivity of the mapping $TF^\phi/\mathbb{C}^* \to WF^\phi$ is nontrivial. From Proposition (4.15) $\Psi(z,\mathfrak{t})$ and $\overline{\Psi}(z,\mathfrak{t})$ are uniquely characterized by the condition $\Psi_N(z) \in U$, $\overline{\Psi}_N(z) \in \overline{U}$ for $^\forall N$. If $\Psi(z,\mathfrak{t})$ and $\overline{\Psi}(z,\mathfrak{t})$ are expressed by a τ-function $\tau(\mathfrak{t})$ as $(**)$, then the above conditions for $\Psi(z,\mathfrak{t})$ and $\overline{\Psi}(z,\mathfrak{t})$ are equivalent to the following conditions on $\tau(\mathfrak{t})$:

$$\text{Res}_{z=\infty}(\overline{\xi}(z)\psi_B(z))\tau(\mathfrak{t}) = 0 \qquad \text{for } ^\forall \overline{\xi}(z) \in \overline{U}$$

$$\text{Res}_{z=\infty}(\xi(z)\overline{\psi}_B(z))\tau(\mathfrak{t}) = 0 \qquad \text{for } ^\forall \xi(z) \in U$$

These conditions determine $\tau(\mathfrak{t})$ uniquely up to a constant by using the characterization of $|U\rangle$ by $W_+(U) \oplus \overline{W}_+(U)$ established in Theorem (3.13). Thus the mapping $TF^\phi/\mathbb{C}^* \to WF^\phi$ is injective. ∎

Remark that the τ-function $\tau(\mathfrak{t})$ corresponding to $U \in UGM^\phi$ is characterized by the following infinite order differential equations:

$$\text{Res}_{z=\infty}(\overline{\xi}(z)V_{-1}(\mathfrak{t}))\tau(\mathfrak{t}) = 0$$

$$\text{Res}_{z=\infty}(\xi(z)V_1(\mathfrak{t}))\tau(\mathfrak{t}) = 0.$$

For simplicity we have restricted the discussions in the generic case (UGM^ϕ). We can generalize the discussions to the whole UGM^0.

Theorem (4.17). The following two conditions are equivalent:

1) $f(t) \in \mathscr{K}_0^\times$ satisfies Hirota's bilinear equation (4.10)

$$\sum_{j=0}^{\infty} P_j(-2y) \, P_{j+1}(\tilde{D}_t) \, \exp(\sum_{\ell=1}^{\infty} y_\ell D_{t_\ell}) \, \tau(t) \cdot \tau(t) = 0 \; ;$$

2) there exists an element $\tilde{U} \in \widetilde{UGM}^0$ such that $f(t) = \tau(t, \tilde{U})$.

§5 The τ-Function as a Period Map

(5.0) In the preceding sections we have constructed a sequence of $\mathbb{C}^*$-bundles in the followaing (cartesian) diagram:

$$
\begin{array}{ccccccccc}
\mathcal{T}_g(\mathcal{F}) & \longrightarrow & \mathbb{P}(\mathfrak{L}_g) & \xrightarrow{\ \Gamma\ } & \widehat{\mathrm{UGM}} & \xrightarrow{\ P\ } & \mathcal{F}^\times & \xrightarrow{\ \simeq\ } & \mathcal{R}^\times \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & B & \downarrow \\
\mathcal{B}_g(\mathcal{F}) & \xrightarrow{\ \sigma\ } & \mathbb{P}(\mathfrak{L}_g) & \xrightarrow{\ \Gamma\ } & \mathrm{UGM} & \xrightarrow{\ P\ } & \mathbb{P}(\mathcal{F}) & \xrightarrow{\ B\ } & \mathbb{P}(\mathcal{R})
\end{array}
$$

$\mathcal{T}_g(\mathcal{F})$ is the pull-back bundle.

We now define a holomorphic mapping $\Delta_{1/2}\colon \mathcal{B}_g(\mathcal{F}) \longrightarrow \mathbb{P}(\mathcal{R}_0)$ by

$$
\Delta_{1/2}\colon \mathcal{B}_g(\mathcal{F}) \xrightarrow{\ \Gamma\circ\sigma\ } \mathrm{UGM}^0 \xrightarrow{\ P\ } \mathbb{P}(\mathcal{F}_0) \longrightarrow \mathbb{P}(\mathcal{R}_0).
$$

The Fundamental Problem (5.1). Construct a section of $\mathcal{T}_g(\mathcal{F})$, equivalently, a nonzero holomorphic function

$\tau\colon \mathcal{T}_g(\mathcal{F}) \longrightarrow \mathcal{R}_0^\times$ which makes the following diagram commutative

$$
\begin{array}{ccc}
 & \xrightarrow{\ \tau\ } & \mathcal{R}_0^\times \\
 & \nearrow & \downarrow \\
\mathcal{B}_g(\mathcal{F}) & \xrightarrow{\ \Delta_{1/2}\ } & \mathbb{P}(\mathcal{R}_0)
\end{array}
$$

We actually construct a lifting τ by using the theory of KP equations. Geometrically τ is a period map of the moduli space $\mathcal{B}_g(\mathcal{F})$. For a given lifting $\tau = \tau(\mathfrak{t}, X_c)$ we can construct another lifting of $\Delta_{1/2}$ by $\tau'(\mathfrak{t}, X) = A(X_c)\tau(\mathfrak{t}, X)$ for any holomorphic function $A\colon \mathcal{B}_g(\mathcal{F}) \longrightarrow \mathbb{C}^*$. In this sense the lifting τ is not unique (See §7,D)).

A) Baker-Akhiezer Function

(5.2) In this paragraph we reformulate the theory of the Baker-Akhiezer function (BA-function) of KP equations. For the original theory of the BA-function see [Kr.][D.].

Definition (5.3). For a given data $X_c = (R, (\alpha, \beta), Q, u, \mathcal{L}_c) \in \mathcal{B}_g(\mathcal{F})$,

we define $\Lambda(X_c)$, $\bar{\Lambda}(X_c) \in \mathbb{C}((z^{-1})) \hat{\otimes} \mathbb{C}[[\mathfrak{t}]]$ as follows:

$$\Lambda(X_c) \equiv \{ \; \Psi(z,\mathfrak{t}) = \sum_N \Psi_N(z)\mathfrak{t}^N \; |$$

$$\Psi_N(z) \in U(X_c) = \sqrt{du} \; H^0(R, \mathcal{L}_\Delta \otimes \mathcal{L}_c(*Q)),$$

$$\Psi(z,\mathfrak{t}) = e^{-\xi(z,\mathfrak{t})}(\sum_{k=0}^{\infty} w_k(\mathfrak{t})z^{-k}), \quad w_k(\mathfrak{t}) \in \mathbb{C}[[\mathfrak{t}]]\},$$

$$\bar{\Lambda}(X_c) \equiv \{ \; \bar{\Psi}(z,\mathfrak{t}) = \sum_N \bar{\Psi}_N(z)\mathfrak{t}^N \; |$$

$$\bar{\Psi}_N(z) \in \bar{U}(X_c) = \sqrt{du} \; H^0(R, \mathcal{L}_\Delta \otimes \mathcal{L}_c^{-1}(*Q)),$$

$$\bar{\Psi}(z,\mathfrak{t}) = e^{\xi(z,\mathfrak{t})}(\sum_{k=0}^{\infty} \bar{w}_k(\mathfrak{t})z^{-k}), \quad \bar{w}_k(\mathfrak{t}) \in \mathbb{C}[[\mathfrak{t}]]\}.$$

We call an element of $\Lambda(X_c)$ $(\bar{\Lambda}(X_c))$ as a *(conjugate) Baker-Akhiezer function* associated with the data X_c. We can then derive the following theorem. We essentially follow the proof of Krichever and Dubrovin.

Theorem (5.4) [Kr.][D.].

For a generic data $X_c = (R,(\alpha,\beta),Q,u,\mathcal{L}_c)$, i.e. $\theta(c|\Omega) \neq 0$,

1) $\Lambda(X_c)$ is a free $\mathbb{C}[[\mathfrak{t}]]$-module of rank 1 and generated by

$$\Phi(z,\mathfrak{t},X_c) = f(z) \exp(-\sum_{n=1}^{\infty} t_n \varphi^{(n)}(z)) \frac{\theta(I(\mathfrak{t})+I[z]+c|\Omega)}{\theta(I(\mathfrak{t})+c|\Omega)},$$

2) $\bar{\Lambda}(X_c)$ is a free $\mathbb{C}[[\mathfrak{t}]]$-module of rank 1 and generated by

$$\bar{\Phi}(z,\mathfrak{t},X_c) = f(z) \exp(\sum_{n=1}^{\infty} t_n \varphi^{(n)}(z)) \frac{\theta(I(\mathfrak{t})-I[z]+c|\Omega)}{\theta(I(\mathfrak{t})+c|\Omega)},$$

where $I(\mathfrak{t}) = (I^1(\mathfrak{t}),\ldots,I^g(\mathfrak{t}))$ with $I^j(\mathfrak{t}) = \sum_{n=1}^{\infty} I_n^j t_n$, and

$I[z] = (I^1[z],\ldots,I^g[z])$ with $I^j[z] = \sum_{n=1}^{\infty} I_n^j \frac{z^n}{n}$. For notations of

$f(z)$, $\varphi^{(n)}(z)$ and I_n^j see Appendix A.

For simplicity $\Phi(z,\mathfrak{t},X_c)$ will simply be written as $\Phi(z,\mathfrak{t})$ hereafter. Note that $I(\mathfrak{t})$ is formal power series of $\mathfrak{t}$, we consider $\theta(I(\mathfrak{t})|\Omega)$ as the following formal Taylor series expansion

$$\theta(I(\mathfrak{t})|\Omega) = \sum_{N=0}^{\infty} \frac{1}{N!}(I(\mathfrak{t})\frac{\partial}{\partial y})^N \theta(0|\Omega) \in \mathbb{C}[[\mathfrak{t}]]$$

where $\frac{\partial}{\partial y} = \left(\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^g} \right)$ with $\frac{\partial}{\partial y^i}$ denoting the derivation w.r.t. the i-th argument in θ.

Proof. We can show $\Phi(z,t) \in \Lambda(X_c)$ by checking the periodicity as multiplicative function associated with $\mathcal{L}_c$ and the transformation property as a half-form . Suppose there is another $\Psi(z,t)(\neq 0) \in \Lambda(X_c)$, the ratio Ψ/Φ possesses neither an essential singularity nor a pole at Q any more. Furthermore, in the ratio $\bar{\Psi}/\Phi$, the transformation factor as a form and the multiplicative factor are canceled out. Hence for the expansion of the ratio

$$\frac{\Psi(z,t)}{\Phi(z,t)} = \frac{\sum\limits_N \Psi_N(z)t^N}{\sum\limits_M \Phi_M(z)t^M} \equiv \sum\limits_N f_N(z)t^N ,$$

$f_N(z)$ is a meromorphic function on R without poles at Q. We show that $f_N(z)$ is constant by induction with respect to $|N| = 0,1,2,..$:

i) For $|N| = 0$, i.e. $N = 0 = (0,0,..)$, the pole divisor of $f_0(z) = \Psi_0(z)/\Phi_0(z)$ is at most $D = (\theta(I[z]+c|\Omega))$ the zero divisor of $\theta(I[z]+c|\Omega)$. From the assumption $\theta(c|\Omega) \neq 0$, D is nonspecial and deg $D = g$. Thus $f_0(z) \in \mathcal{L}(D) = \mathbb{C}1$.

ii) Assuming $f_M(z) = $ const. for $|M| < |N|$, we have

$$f_N(z) = t^N \text{ term of } [\Psi(z) - \Phi(z) \sum\limits_{|M|<|N|} f_M(z)t^N] / \Phi_0(z)$$

Since the pole divisor of $f_N(z)$ also divides D (non special), it follows just like i) that $f_N(z) = $ const.. We then conclude

$$^\forall \Psi(z,t) \in \Lambda(X_c) : \quad \frac{\Psi(z,t)}{\Phi(z,t)} = \sum\limits_N C_N t^N \in \mathbb{C}[[t]],$$

,namely, $\Lambda(X_c)$ is a free $\mathbb{C}[[t]]$-module of rank 1. It works similarly for the conjugate sector. ▌

Here it should be noted that $\Phi(z,t)$ and $\bar{\Phi}(z,t)$ given above are

normalized as $w_0(\mathfrak{t}) = \bar{w}_0(\text{⬛}\mathfrak{t}) = 1$. We can then identify $\Phi(z,\mathfrak{t})$ and $\bar{\Phi}(z,\mathfrak{t})$ as the wave functions associated with $U(X_c) \in UGM^\phi$.

B) τ-function for $\hat{\mathcal{E}}_g(\mathcal{F})$

Definition (5.5) (τ-function for $\hat{\mathcal{E}}_g(\mathcal{F})$)[I.M.O.][A-G.G.R.][V.]
Define $\tau: \hat{\mathcal{E}}_g(\mathcal{F}) \longrightarrow \mathcal{K}_0^\times$ as

$$\tau(\mathfrak{t},X_c) = e^{\frac{1}{2} q(\mathfrak{t})} \theta(I(\mathfrak{t})+c|\Omega)$$

where $q(\mathfrak{t}) = \sum_{n,m>0} q_{nm} t_n t_m$ (cf. Appendix A-1). We call this as τ-*function associated* with $\hat{\mathcal{E}}_g(\mathcal{F})$.

Theorem (5.6). For the τ-function (5.5) the following diagram is commutative:

Proof. By direct calculations we obtain the following:

$$\Phi(z,\mathfrak{t},X_c) = e^{-\xi(\mathfrak{t},z)} \frac{\tau(\mathfrak{t}+[z],X_c)}{\tau(\mathfrak{t},X_c)} \, ,$$

$$\bar{\Phi}(z,\mathfrak{t},X_c) = e^{\xi(\mathfrak{t},z)} \frac{\tau(\mathfrak{t}-[z],X_c)}{\tau(\mathfrak{t},X_c)} \, ,$$

which then implies that $\tau: \hat{\mathcal{E}}_g(\mathcal{F}) \longrightarrow \mathcal{K}_0^\times$ is a lifting of $\Delta_{1/2}$.

Remark (5.7). 1) In this way the fundamental problem (5.1) is solved affirmatively with the τ-function given in (5.5). Define $\lambda_{1/2}$ is the line bundle associated with $\mathbb{C}^*$-bundle $\hat{\mathcal{E}}_g(\mathcal{F}) \longrightarrow \mathcal{E}_g(\mathcal{F})$, the $\tau(\mathfrak{t},X_c)$ gives a nowhere vanishing section of $\lambda_{1/2}$.

2) The vacuum amplitude $\tau(0,X_c) = \theta(c|\Omega)$ depends only on $\mathcal{F} = \mathcal{F}_g \times \mathbb{C}^g$.

(5.8) So far we have no principle to eliminate the lifting ambiguity. From the point of view of the two-dimentional conformal field theory it is unimportant because the correlation functions

depend only on the ratios of τ-function (or its derivative). On the other hand in string theory the lifting itself is a fundamental quantity and provides the integrand of the string amplitude. It should then possess a well-behaved automorphic propeerty w.r.t. modular transformation and good boundary behavior towards the boundary of $\mathcal{B}_g$ (stable curves [F.S.][F.][N.]). We thus discuss the modular transformation property of the τ-function in the next paragraph.

C) Modular Transformation Property of the τ-Function

(5.9) A modular transformation is induced by a change of canonical homology basis (α,β), and is represented by a symplectic matrix $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2g,\mathbb{Z})$ as $\gamma\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} D & C \\ B & A \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$.

In particular $M_\Delta = \{\gamma \in M;\ \mathrm{diag}\ C^tD \equiv \mathrm{diag}\ A^tB \equiv 0\ (\mathrm{mod}\ 2)\} \subset M$ is a subgroup of M and keeps the Riemann constant invariant. Our τ-function (5.5) has the following well-behaved automorphy under the modular transformation M_Δ.

Theorem (5.10). For the τ-function (5.5) we have

$$\tau(\mathfrak{t}, \gamma(X_c))$$
$$= \varepsilon(\gamma)\ \det(C\Omega + D)^{1/2}\ \exp\left(\pi\sqrt{-1}\,{}^tc(C\Omega+D)^{-1}C(2I(\mathfrak{t})+c)\right)\tau(\mathfrak{t},X_c)$$
$$\text{for } {}^\forall\gamma \in M_\Delta.$$

where $\varepsilon(\gamma)$ is a constant with $\varepsilon(\gamma)^8 = 1$ and depending only on γ.

Proof. The modular transformation property of the following quantities on a Riemann surface are given by

$$\gamma(\Omega) = (A\Omega + B)(C\Omega + D)^{-1}$$
$$\gamma(I(\mathfrak{t})) = {}^t(C\Omega + D)^{-1}I(\mathfrak{t})$$
$$\gamma(q(\mathfrak{t})) = -2\pi\sqrt{-1}\ I(\mathfrak{t})(C\Omega + D)^{-1}C\ I(\mathfrak{t})$$
$$\gamma(c) = {}^t(C\Omega+D)^{-1}c$$

(See Appendix A for the notations). Together with the transformation

property of the θ-function [Mum.1] [A-G.M.V.] for $\gamma \in M_\Delta$:

$$\theta(^t(C\Omega+D)^{-1}z|\gamma(\Omega)) = \varepsilon(\gamma)\det(C\Omega+D)^{1/2}e^{\pi\sqrt{-1}^t z(C\Omega+D)^{-1}z}\theta(z|\Omega)$$

we obtain the desired formula.

It should be noted that the τ-function has simpler automorphic

factor than the θ-function itself (when $c = 0$) because of the

prefactor $e^{\frac{1}{2}q(\mathfrak{t})}$.

Similarly the τ-function has the following well-behaved

automorphy with respect to $c \in \mathbb{C}^g$:

Theorem (5.11).

$$\tau(\mathfrak{t},X_{c+\Omega a+b}) = \exp\left(-2\pi\sqrt{-1}\left(\tfrac{1}{2}{}^t a\Omega a + {}^t a(I(\mathfrak{t})+c)\right)\right)\tau(\mathfrak{t},X_c),$$
$$\text{for } \forall a,b \in \mathbb{Z}^g.$$

This is a direct consequence of the quasi-periodicity:

$$\theta(z+\Omega a+b|\Omega) = \exp\left(-2\pi\sqrt{-1}\left(\tfrac{1}{2}\,{}^t a\Omega a + {}^t a(z)\right)\right)\theta(z|\Omega).$$

This automorphy turns out to be very important to characterise the

τ-function, which will be discussed in §7,D).

D) Scattering operator $S(X_c)$ [I.M.O.][A-G.G.R.][V.]

(5.12) Before closing this section, we construct an operator

$S(X_c)$ which generates the state vector $|X_c\rangle$ of the Fock space

$$|X_c\rangle = S(X_c)|0\rangle\langle 0|X_c\rangle, \qquad \langle 0|S(X_c) = \langle 0|$$

where $X_c = (R,(\alpha,\beta),\mathscr{L}_c,u) \in \hat{\mathscr{E}}_g(\mathcal{F})$ and we assume $\theta(c|\Omega) \neq 0$.

This $S(X_c)$ is very convenient in practical calculations (See § 6).

For example the Wick's theorem in the calculation of various N-point

functions can be easily verified by means of of this operator.

We expand the Szegö kernel [Fay][So.] of the line bundle $\mathscr{L}_\Delta \ominus \mathscr{L}_c$

around $Q \in R$ as follows:

$$S_C(z,w) = \frac{\theta(I[z]-I[w]+c\,|\,\Omega)}{\theta(c\,|\,\Omega)\,E(z,w)} = \frac{1}{z-w} + \sum_{\mu,\nu>0} C_{\mu\nu} z^{-\mu-1/2} w^{-\nu-1/2},$$

then a basis, $\{\xi^\mu\}_{\mu\in\mathbb{Z}h}$ of V, can be given by

$$\xi^\mu = e^\mu \qquad (\mu > 0)$$

$$\xi^{-\mu} = \frac{1}{2\pi\sqrt{-1}} \int_{\infty \cdot z} dw\ w^{\mu-1/2} S_C(z,w) = e^{-\mu} + \sum_{\nu>0} C_{\nu\mu} e^\nu \qquad (\mu>0)$$

or equivalently in the matrix form

$$(\ldots,\xi^\mu,\xi^{\mu+1},\ldots) = (\ldots,e^\mu,e^{\mu+1},\ldots) \begin{pmatrix} \ddots & & & & \\ & 1 & & & \\ & & 1 & & \\ \hline & C_{\mu\nu} & & 1 & \\ & & & & 1 \\ & & & & & \ddots \end{pmatrix}$$

It is easy to check that $\xi^{-\mu}$ belongs to $U(X_C) = \sqrt{du}\ H^0(R, \mathscr{L}_\Delta \otimes \mathscr{L}_C(*Q))$
$\in \mathrm{UGM}^\phi$ for $\mu > 0$.

Define an operator $Q(X_C)$ as follows;

$$Q(X_C) = \left(\frac{1}{2\pi\sqrt{-1}}\right)^2 \int_\infty dz \cdot \int_\infty dw\ \{S_C(z,w) - \frac{1}{z-w}\}\bar{\psi}(z)\psi(w)$$

$$= \sum_{\mu\nu>0} C_{\mu\nu} \bar{\psi}_{-\mu} \psi_{-\nu}$$

Note that $Q(X_C)$ is constructed from creation operators only and

thus belongs to P_- and its exponential,

$$S(X_C) = e^{-Q(X_C)}, \qquad S(X_C)^{-1} = e^{Q(X_C)}$$

are well-defined operators on $\mathscr{F}$.

Proposition (5.13),

1) $$\psi(z,X_C) \equiv S(X_C)^{-1} \psi(z) S(X_C) = \sum_{\mu\in\mathbb{Z}h} \psi_\mu \xi^\mu$$

$$\bar{\psi}(z,X_C) \equiv S(X_C)^{-1} \bar{\psi}(z) S(X_C) = \sum_{\mu\in\mathbb{Z}h} \bar{\psi}_\mu \bar{\xi}^\mu$$

where $\bar{\xi}^\mu$'s are the dual of ξ^μ : $(\bar{\xi}^\mu|\xi^\nu) = \delta^{\mu+\nu,0}$.

2) $$|X_C\rangle = S(X_C)|0\rangle\langle 0|X_C\rangle,$$

$$\langle 0| = \langle 0|S(X_C).$$

§6 Action of Operators on the τ-function and Correlation Functions

(6.0) Using the concrete expression of the τ-function for $\mathscr{C}_g(\mathscr{Y})$ given in the previous section (5.5), we study the action of various (vertex, current, energy-momentum) operators. We can derive N-point functions of Riemann surfaces of arbitrary genus (obtained recently by several authors [I.M.O.][A-G.G.R.]). Thanks to the fermion-boson correspondence, we can derive N-point functions in two different ways to obtain nontrivial identities [E.O.2][Fay], which are closily related to the Schottky problem [Mum.3],[v.G.],[Sh.].

A) Action of Vertex Operators

(6.1) The vertex operator $V_k(z)$ $(k \in \mathbb{Z})$ has been defined as follows (4.4):

$$V_k(z) = \exp\left(k \sum_{n=1}^{\infty} t_n z^n\right) e^{k t_0} z^{k \partial_0} \exp\left(-k \sum_{n=1}^{\infty} \frac{1}{n} z^{-n} \partial_n\right).$$

Letting the vertex operators operate on the τ-function, we obtain the following result.

Theorem (6.2).

$$V_{k_1}(z_1) \cdot \ldots \cdot V_{k_N}(z_N) \tau(t, X_c)$$

$$= \prod_{i=1}^{N} f(z_i)^{k_i K} e^{t_0 K} \exp\left(\sum_{n=1}^{\infty} t_n \sum_{i=1}^{N} k_i \varphi^{(n)}(z_i)\right)$$

$$\times \prod_{1 \leq i < j \leq N} E(z_i, z_j)^{k_i k_j} e^{\frac{1}{2} q(t)} \theta\left(I(t) + \sum_{i=1}^{N} k_i I[z^i] + c \mid \Omega\right)$$

where $K = \sum_{i=1}^{N} k_i$ (total charge of operators $V_{k_1}(z_1) \ldots V_{k_N}(z_N)$).

See Appendix A for the notations. In the derivation of this formula we have used the following relations;

$$V_k(z)\, V_\ell(w) \;=\; (z-w)^{k\ell} \; \vdots \; V_k(z)\, V_\ell(w) \vdots$$

$$\left(\sum_{i=1}^{N} k_i x_i\right)^2 \;=\; -\sum_{1\le i<j\le N} k_i k_j (x_i - x_j)^2 \;+\; K\sum_{i=1}^{N} k_i x_i^2$$

In particular for $N = 1$ and $k_i = \pm 1$, we obtain the following:

Corollary (6.3).

$$\Psi(z, \mathfrak{t}.X_c) \;=\; \frac{V_{-1}(z)\tau(\mathfrak{t},X)}{\tau(\mathfrak{t},X)}\, e^{t_0}$$

$$= f(z)\exp\left(-\sum_{n=1}^{\infty} t_n \varphi^{(n)}(z)\right) \frac{\theta(I(\mathfrak{t})+I[z]+c\,|\,\Omega)}{\theta(I(\mathfrak{t})+c\,|\,\Omega)}$$

$$\bar\Psi(z, \mathfrak{t}, X_c) \;=\; \frac{V_1(z)\tau(\mathfrak{t},X)}{\tau(\mathfrak{t},X)}\, e^{-t_0}$$

$$= f(z)\exp\left(\sum_{n=1}^{\infty} t_n \varphi^{(n)}(z)\right) \frac{\theta(I(\mathfrak{t})-I[z]+c\,|\,\Omega)}{\theta(I(\mathfrak{t})+c\,|\,\Omega)} \; .$$

These are just the BA-functions $\Phi(z,\mathfrak{t})$ and $\bar\Phi(z,\mathfrak{t})$ given in Theorem (5.4).

B) Action of Current Operators

(6.4) The bosonized expression of the current operator has been given by

$$J_B(z) \;=\; \sum_{n=1}^{\infty} n t_n z^{n-1} \;+\; \sum_{n=1}^{\infty} z^{-n-1} \partial_n$$

in (4.5). When the current operator $J_B(z)$ passes through the bilinear factor $e^{\frac{1}{2}q(\mathfrak{t})}$ of the τ-function, it receives the following change;

$$J_B(z)\, e^{\frac{1}{2}q(\mathfrak{t})} \;=\; e^{\frac{1}{2}q(\mathfrak{t})}\{ \sum_{n=1}^{\infty} t_n \omega_Q^{(n)}(z) \;+\; \sum_{n=1}^{\infty} z^{-n-1}\partial_n\}$$

$$\equiv e^{\frac{1}{2}q(\mathfrak{t})}\, \vartheta_B(z)$$

Note that $\vartheta_B(z) = \vartheta_B(z,X)$ depends holomorphically on the data

$$\pi(X_c) = X = (R, (\alpha, \beta), Q, u) \in \hat{\mathscr{E}}_g.$$

Proposition (6.5)

$$J_B(z_1)\ldots J_B(z_N)\tau(\mathfrak{t}, X_c)$$

$$= \sum_{\substack{\text{comb.}}} \left(\prod_{\text{pairs}} \overbrace{\vartheta_B(z_i)\vartheta_B(z_j)} \right) \left(\prod_{\text{rests}} \overbrace{\vartheta_B(z_k)} \right) \tau(\mathfrak{t}, X)$$

where

$$\overbrace{\vartheta_B(z)\,\vartheta_B(w)} = \omega(z, w)$$

$$\overbrace{\vartheta_B(z)}\,\tau(\mathfrak{t}, X) = \{ \sum_{n=1}^{\infty} t_n\, \omega_Q^{(n)}(z) + \sum_{i=1}^{g} \omega^i(z)\, \frac{\partial}{\partial y^i} \} \, \tau(\mathfrak{t}, X_c)$$

with

$$\frac{\partial}{\partial y^i}\, \tau(\mathfrak{t}, X_c) = e^{\frac{1}{2}q(\mathfrak{t})}\, \frac{\partial}{\partial y^i}\, \theta(I(\mathfrak{t})+c\,|\,\Omega)$$

.and $\sum_{\text{comb.}}$ is a summation over all the combinations of dividing $(1, 2, ..N)$ into pairs $(i_1 i_2), (i_3 i_4), \ldots (i_{2k-1} i_{2k})$ and the rest of numbers $(i_{2k+1}, \ldots, i_N)$.

Corollary (6.6).

$$J_B(z)\,\tau(\mathfrak{t}, X) = \{ \sum_{n=1}^{\infty} t_n\, \omega_Q^{(n)}(z) + \sum_{i=1}^{g} \omega^i(z)\, \frac{\partial}{\partial c^i} \} \, \tau(\mathfrak{t}, X_c)$$

In order to see the meaning of this equality, we make use of the following definition.

Definition (6.7). For $n \in \mathbb{Z}$ with $n > 0$ we define holomorphic vector field $v^{(n)}$ on $P(\hat{\mathscr{E}}_g^{(0)})$ by

$$v_X^{(n)} = \theta_X \left(\varphi^{(n)}(z, X) \right) \in T_X P(\hat{\mathscr{E}}_g^{(0)}) \quad X \in \hat{\mathscr{E}}_g^{(0)}.$$

Note that $\varphi^{(n)}(z, X) = \int^z \omega_Q^{(n)}(X)$ depends holomorphically on the data $X = \pi(\tilde{X}) \in \hat{\mathscr{E}}_g$ (cf. Appendix A).

Theorem (6.8). Consider a diagram below

$$\begin{array}{ccc}
\mathcal{B}_g(\mathcal{Y}) & \hookrightarrow & \mathbb{P}(\mathfrak{A}_g^0) \\
\downarrow {\mathbb{C}^g} & & \downarrow {\mathcal{O}_1^*} \\
\mathcal{B}_g & \longleftarrow & \mathcal{B}_g \underset{\mathcal{J}_g}{\times} \mathcal{J} \ .
\end{array}$$

1) $v^{(n)}$ is a holomorphic vector field on $\mathbb{P}(\mathfrak{A}_g^{(0)})$ along the image of $\mathcal{B}_g(\mathcal{Y})$ and thus induces a vector field on $\mathcal{B}_g(\mathcal{Y})$.

2) The induced vector field $v^{(n)}$ on $\mathcal{B}_g(\mathcal{Y})$ is tangential to the fibre of the fibering $\pi: \mathcal{B}_g(\mathcal{Y}) \longrightarrow \mathcal{B}_g$ and the action on this direction is given by

$$v^{(n)}_{X_c} = \sum_{i=1}^{g} \frac{1}{2\pi\sqrt{-1}} \int_{\beta_i} \omega_Q^{(n)}(X) \frac{\partial}{\partial c^i} \ .$$

We can that $J_B(z)$ is an operator which generates infinitesimal deformations of line bundles.

C) Action of Energy Momentum Tensor

(6.9) Since the bosonized expression of the energy momentum tensor is given by

$$T_B(z) = \frac{1}{2} \,{}^{\circ}_{\circ} J_B(z) J_B(z) \,{}^{\circ}_{\circ} = \lim_{w \to z} \frac{1}{2} \left(J_B(z) J_B(w) - \frac{1}{(z-w)^2} \right)$$

in (4.5), we can easily obtain the action of $T_B(z)$ by using Proposition (6.5).

Theorem (6.10).

$$T_B(z)\tau(t,X_c)$$

$$= \Big[\frac{1}{2} \sum_{n,m=1}^{\infty} t_n t_m \omega_Q^{(n)}(z)\, \omega_Q^{(m)}(z) + \sum_{n=1}^{\infty}\sum_{i=1}^{g} t_n \omega_Q^{(n)}(z)\omega^i(z) \frac{\partial}{\partial y^i}$$

$$+ \sum_{i,j=1}^{g} \omega^i(z)\omega^j(z) \frac{\partial^2}{\partial y^i \partial y^j} - \frac{1}{12} S(z,X) \Big]\tau(t,X_c)$$

where $S(z,X)$ is a projective connection (Appendix A-5).

We obtain the vacuum expectation value of the energy-momentum tensor

$$\langle T(z) \rangle_{X_c} = \frac{\langle 0 | T(z) | X_c \rangle}{\langle 0 | X_c \rangle} .$$

Proposition (6.11) ([A.W.],[So.],[I]).

$$\langle T(z) \rangle_{X_c} = \frac{1}{2} \frac{1}{\theta(c|\Omega)} \sum_{i,j=1}^{g} \omega^i(z)\omega^j(z) \frac{\partial^2}{\partial y^i \partial y^j} \theta(c|\Omega) - \frac{1}{12} S(z,X).$$

Using the heat equation of the θ-function:

$$\frac{\partial^2}{\partial y^i \partial y^j} \theta(y|\Omega) = 4\pi \frac{\partial}{\partial \Omega_{ij}} \theta(y|\Omega),$$

we can recognize that the energy momentum tensor induces

infinitesimal deformations of moduli ([So.][E.O.1]).

D) N-point Functions [Ku.][E.O.2][A-G.G.R.]

In the following we fix the data set $X_c =$

$(R,Q,(\alpha,\beta),\mathcal{L}_\Delta \otimes \mathcal{L}_C,u)$ such that $\tau(0,X_c) = \theta(c|\Omega) \neq 0$ and denote $|X_c\rangle = B^{-1}\tau(t,X_c) \in \mathcal{F}$.

Definition (6.12). The *N-point function* of operators

$\Theta_1(z_1),\ldots,\Theta_N(z_N)$ on a state $|X_c\rangle \in \mathcal{F}$ is defined as follows;

$$\langle \Theta_1(z_1),\ldots,\Theta_N(z_N) \rangle_{X_c} = \frac{\langle 0 | \Theta_1(z_1),\ldots,\Theta_N(z_N) | X_c \rangle}{\langle 0 | X_c \rangle} .$$

Since we assume $\theta(c|\Omega) \neq 0$, we can represent $|X_c\rangle \in \mathcal{F}$ as

$$|X_c\rangle = S(X_c)|0\rangle \langle 0 | X_c \rangle,$$

hence the N-point function can be expressed as follows:

$$\langle \Theta_1(z_1),\ldots,\Theta_N(z_N) \rangle_{X_c} = \langle 0 | \Theta_1(z_1,X_c),\ldots,\Theta_N(z_N,X_c) | 0 \rangle,$$

where

$$\Theta(z,X_c) = S(X_c)^{-1}\Theta(z) S(X_c) ,$$

is a field operator corresponding to the Heisenberg picture.

On the other hand we can express the N-point function in the

bosonized form:

$$\langle \mathcal{O}_1(z_1), \ldots, \mathcal{O}_N(z_N) \rangle_{X_C} = \frac{\mathcal{O}_{1B}(z_1) \cdots \cdots \mathcal{O}_{NB}(z_N) \tau(\hat{t}, X)}{\tau(0, X_C)} \Bigg|_{\hat{t}=0} .$$

In the following we calculate the N-point functions of ψ, $\bar{\psi}$'s and J's in two different ways given above.

(6.13) Fermion 2N-point function is obtained by putting $k_i = \pm 1$ in the formula (6.2):

$$\langle \psi(z_1), \ldots, \psi(z_N) \bar{\psi}(w_N), \ldots, \bar{\psi}(w_1) \rangle_{X_C}$$

$$= \frac{\prod\limits_{i<j} E(z_i, z_j) \prod\limits_{i<j} E(w_j, w_i)}{\prod\limits_{i<j} E(z_i, w_j)} \frac{\theta(\sum\limits_{i=1}^{N} I[z_i] - \sum\limits_{i=1}^{N} I[w_i] + c | \Omega)}{\theta(c | \Omega)} ,$$

while in the fermionic representation

$$\langle \psi(z_1), \ldots, \psi(z_N) \bar{\psi}(w_N), \ldots, \bar{\psi}(w_1) \rangle_{X_C} = \det_{N \times N} \langle \psi(z_i) \bar{\psi}(w_j) \rangle_{X_C}$$

and

$$\langle \psi(z) \bar{\psi}(w) \rangle_{X_C} = \sum_{\mu > 0} \xi^{\mu}(z) \bar{\xi}_{\mu}(w) = S(z, w)$$

where

$$S(z, w) = \frac{\theta(I[z] - I[w] + c | \Omega)}{\theta(c | \Omega) E(z, w)}$$

is the Szegö kernel.

Identifying these two expressions, we obtain the following formula:

Proposition (6.14) (Fay's trisecant formula) [Fay][E.O.1],

$$\det_{N \times N} S(z_i, w_j)$$

$$= \frac{\prod\limits_{i<j} E(z_i, z_j) \prod\limits_{i<j} E(w_j, w_i)}{\prod\limits_{i,j} E(z_i, w_j)} \frac{\theta(\sum\limits_{i=1}^{N} I[z_i] - \sum\limits_{i=1}^{N} I[w_i] + c | \Omega)}{\theta(c | \Omega)} .$$

The bosonic representation of current N-point functions

can be obtained from (6.5):

Proposition (6.15).

$$\langle J(z_1)\ldots J(z_N)\rangle_{X_c}$$

$$= \frac{1}{\theta(c|\Omega)}\sum_{comb.}\left(\prod_{pairs}\omega(z_i,z_j)\right)\left(\prod_{rests}\left(\sum_{i=1}^{g}\omega^i(z_k)\frac{\partial}{\partial y^i}\right)\right)\theta(c|\Omega).$$

On the other hand, using the fermionic representation of $J(z,X_c)$, we obtain

$$J(z,X_c) = S(X_c)^{-1}\{\lim_{w\to z}(\bar{\psi}(z)\psi(w) - \frac{1}{z-w})\}S(X_c)$$

$$= \lim_{w\to z}\{\psi(z,X_c)\bar{\psi}(w,X_c) - \frac{1}{z-w}\}$$

and taking the vacuum expectation value $\langle 0|..|0\rangle$, we obtain the following relations.

Proposition (6.16).

$$\langle J(z_1)\ldots J(z_N)\rangle_{X_c} = \det_{N\times N}S_c^{reg}(z_i,z_j)$$

where

$$S_c^{reg\cdot}(z,z) = \lim_{w\to z}\{S_c(z,w) - \frac{1}{z-w}\} = \frac{1}{\theta(c|\Omega)}\sum_{i=1}^{g}\omega^i(z)\frac{\partial}{\partial y^i}\theta(c|\Omega)$$

$$S_c^{reg\cdot}(z_i,z_j) = S_c(z_i,z_j) \quad (i\neq j).$$

By equating these two expressions (6.15) and (6.16), we can derive various identities obtained by Fay. In order to get explicit expressions it is convenient to use the expression of the connected part of N-point functions;

$$\langle J(z_1)\ldots J(z_N)\rangle_{X_c}^{conn\cdot} \equiv J_B(z_1)\ldots J_B(z_N)\log \tau(t,X_c)|_{t=0}.$$

Proposition (6.17).

(1) Bosonic representation:

$$\langle J(z_1)...J(z_N)\rangle_{X_c}^{\text{conn.}} = \sum_{i_1,...,i_N}^{g} \frac{\partial^N \log \theta(c|\Omega)}{\partial y^{i_1}....\partial y^{i_N}} \omega^{i_1}(z_1)..\omega^{i_N}(z_N)$$

$$+ \delta_{N,2} \, \omega(z_1,z_2),$$

(2) fermionic representation:

$$\langle J(z_1).....J(z_N)\rangle_{X_c}^{\text{conn.}} = - \sum_{\Gamma} (\prod_{\iota \in \Gamma} S_\iota)$$

where Γ is a connected oriented loop which passes through the points $z_1,z_2,..,z_N$ once and for $\iota = (z_i,z_j) \in \Gamma$: $S_\iota = S_c(z_i,z_j)$.

   Proposition (6.18) (Addition Formula).

$$\sum_{i_1,...,i_N}^{g} \frac{\partial^N \log \theta(c|\Omega)}{\partial y^{i_1}....\partial y^{i_N}} \omega^{i_1}(z_1)..\omega^{i_N}(z_N) + \delta_{N,2} \, \omega(z_1,z_2)$$

$$= - \sum_{\Gamma} (\prod_{\iota \in \Gamma} S_\iota)$$

The following are special cases of this formula for $N = 2$ and $N = 4$.

   Corollary (6.19) [Fay].

1) $$\omega(z_1,z_2) + \sum_{ij=1}^{g} \omega^i(z_1)\omega^j(z_2)\frac{\partial^2}{\partial y^i \partial y^j} \log \theta(c|\Omega) = \Big(S_c(z_1,z_2)\Big)^2,$$

2) $$\sum_{ijk\ell=1}^{g} \omega^i(z_1)\omega^j(z_2)\omega^k(z_3)\omega^\ell(z_4) \frac{\partial^4}{\partial y^i \partial y^j \partial y^k \partial y^\ell} \log \theta(c|\Omega)$$

$$= - \{ S(1234) + S(1342) + S(1423) + S(1432) + S(1243) +$$

$$S(1324)\}$$

where $S(ijk\ell) = S_c(z_i,z_j)S_c(z_j,z_k)S_c(z_k,z_\ell)S_c(z_\ell,z_i)$. See Fig. 2.

§7 Fundamental Equations

(7.0) In this section we derive differential equations

for the τ-function $\tau(\mathfrak{t},X_C) = e^{\frac{1}{2}q(\mathfrak{t})}\theta(I(\mathfrak{t})+c|\Omega)$, defined on $\mathscr{E}_g(\mathscr{J})$.

Throughout this section we write $\Phi_B(s_{1/2}\cdot\mathcal{l}) = B\,\Phi(s_{1/2}\cdot \mathcal{l})\,B^{-1}$ simply

as $\Phi_B(\mathcal{l})$ for $\mathcal{l} \in \mathfrak{G}$ (see (2.32),(4.4)).

A) The Fundamental Equations

Theorem (7.1) (Fundamental equations).

The τ-function satisfies the following three equations:

1)(Equation of motion)

$$[\theta(\mathcal{l}) + a(\mathcal{l},X_C)]\tau(\mathfrak{t},X_C) = \Phi_B(\mathcal{l})\tau(\mathfrak{t},X_C) \quad \text{for } ^\forall \mathcal{l} \in \mathfrak{G}$$

where $a(\mathcal{l},X_C) = -\frac{1}{12}\,\mathrm{Res}_{z=\infty}\mathcal{l}(z)S(z,X)$;

2)(Gauge condition)

$$\sum_{i=1}^{g} \frac{1}{2\pi\sqrt{-1}} \int_{\beta_i} d\varphi\, \frac{\partial}{\partial c_i}\, \tau(\mathfrak{t},X_C) = \Phi_B(\varphi)\tau(\mathfrak{t},X_C) \quad \text{for } ^\forall\varphi(z) \in \bar{A}(X);$$

3)(Hirota's bilinear equation)

$$\sum_{j=0}^{\infty} P_j(-2y)P_{j+1}(D_\mathfrak{t})\exp(\sum_{\mathcal{l}=1}^{\infty} y_\mathcal{l} D_{t_\mathcal{l}})\tau(\mathfrak{t},X_C)\cdot\tau(\mathfrak{t},X_C) = 0.$$
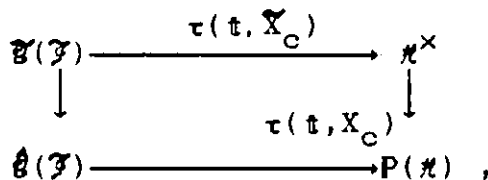
(See Appendix A) for the definitions of $A(X)$ and $\bar{A}(X)$.)

Proof. The equation 3) is already given in Theorem (4.16) and

equation 2) is a rewriting of the following.

$$v^{(n)}\tau(\mathfrak{t},X_C) = \Phi_B(\varphi^{(n)}(z,X))\tau(\mathfrak{t},X_C) \quad \text{for } ^\forall n > 0,$$

which can be derived from Theorem (6.8).

Lastly we show 1). In the following diagram,

$$- 72 -$$

$\tau(\mathfrak{t},\tilde{X}_c)$ and $\tau(\mathfrak{t},X_c)$ are related as

$$\tau(\mathfrak{t},\tilde{X}_c) = A(\tilde{X}_c)\tau(\mathfrak{t},X_c)$$

where $A:\mathcal{E}_g(\mathcal{Y})\longrightarrow \mathbb{C}^*$ is a holomorphic function satisfying $A(\lambda\tilde{X}_c) = \lambda A(\tilde{X}_c)$ ($\lambda \in \mathbb{C}^*$). Letting $\hat{\theta}(\iota),\iota \in \mathfrak{G}$, operate on $\tau(\mathfrak{t},\tilde{X}_c)$, we obtain the following relation from Proposition (3.35):

$$\hat{\theta}(\iota)\tau(\mathfrak{t},\tilde{X}_c) = \Phi_B(\iota)\tau(\mathfrak{t},\tilde{X}_c),$$

Then we obtain the desired equation. The term $a(\iota,X_c)$ is given by

$$a(\iota,X_c) = \frac{1}{A(\tilde{X}_c)}\hat{\theta}(\iota)A(\tilde{X}_c)$$

and depends only on X_c. The explicit form will be given in the next paragraph. ∎

The τ-function being considered as a period map on $\mathcal{E}_g(\mathcal{Y})$, the differential equation 1) is regarded as a defining equation of the Gauss-Manin connection, the equation of motion of the physical vacua in the interaction picture.

B) Determination of the anomaly term $a(\iota,X_c)$.

In case $\iota \in \mathfrak{G}_0$ the equation (7.1.1) can be viewed as the so-called "Ward-Takahashi identity". The term $a(\iota,X_c)$ corresponds to the anomaly term. In this view the equation in the following proposition can be identified as "Wess-Zumino consistncy condition". Those principles symbolized by these terminologies which are familiar to the physicists come very naturally into the formulation.

Proposition (7.2) (consistency condition). For ι_1, $\iota_2 \in \mathfrak{G}$ we obtain the following relations:

$$\theta(\iota_1)a(\iota_2,X_c) - \theta(\iota_2)a(\iota_1,X_c) + a([\iota_1,\iota_2],X_c) = C(\iota_1,\iota_2)$$

where

$$c(l_1, l_2) = \frac{1}{12} \text{Res}_{z=\infty} l_1''(z) l_2(z)$$

is the Schwinger term.

Proof. Consider the following equality:

$$[\Phi_B(l_1), \Phi_B(l_2)] = \Phi_B([l_2, l_1]) + c(l_1, l_2),$$

From the action of this operator on $\tau(\mathfrak{t}, X_c)$ written in two ways, we obtain the desired result. $\blacksquare$

(7.3) We now determine the form of the anomaly term $a(l, X)$ corresponding to our τ-function.

Theorem (7.4). For the τ-function the corresponding anomaly term is given by

$$a(l, X_c) = -\frac{1}{12} \text{Res}_{z=\infty} l(z) S(z, X)$$

where $S(z, X)(dz)^2$ is the projective connection (See Appendix A-5).

Proof. First we show the following:

1) $a(l, X_c) = 0$ for $l \in \mathfrak{G}_1$,

2) $a(l, X_c) = -\frac{1}{12} \text{Res}_{z=\infty} l(z) S(z, X)$ for $l \in B(X)$,

3) $a(l, X_c)$ satisfies the consistency condition.

3) is the proposition (7.2) itself, while 2) can be obtained from Theorem (7.1-1) by taking $l \in B(X)$ since $\overset{\theta}{\rule{0.5em}{0.5em}}(l) = 0$. 1) can be shown .from the following representation of τ with the bosonization operator B ((4.2)),

$$\tau(\mathfrak{t}, X_c) = \theta(c|\Omega) B(\xi^{-1/2} \wedge \xi^{-3/2} \wedge \cdots).$$

where $(\xi^{-1/2}, \xi^{-3/2}, \ldots)$ is a frame of $U(X_c)$ (cf. §5,D)).

From this expression we obtain the following for $l \in \mathfrak{G}_1$

$$\tau(\mathfrak{t}, e^{\varepsilon l} X_c) = \theta(c|\Omega) B(e^{\varepsilon l}(\xi^{-1/2}) \wedge e^{\varepsilon l}(\xi^{-3/2}) \wedge \cdots),$$

$$\theta(l)\tau(\mathfrak{t}, X_c) = \theta(c|\Omega) B\left(\sum_{\mu < 0} \xi^{-1/2} \wedge \cdots \wedge \theta(l)(\xi^{\mu}) \wedge \cdots\right)$$

$$= \theta(c|\Omega)\Phi_B(\ell)B(\xi^{-1/2} \wedge \xi^{-3/2} \wedge \cdots)$$

$$= \Phi_B(\ell)\tau(\mathfrak{t}, X_c).$$

Hence $a(\ell, X_c) = 0$ for $\ell \in \mathfrak{G}_1$.

Due to 1) $a(\ell, X_c)$ reduces to a linear function of ℓ on $\mathfrak{G}/\mathfrak{G}_1$. Fixing a sufficiently large N, we can take representatives of bases, $\hat{\ell}_n \in \mathfrak{G}$ $(n = 0,1,2,\ldots)$, of $\mathfrak{G}/\mathfrak{G}_1$ in the following:

$$\hat{\ell}_n = (z^{n+1} + \text{lower order terms})\frac{d}{dz} \in B(X), \quad n \geq N$$

$$\hat{\ell}_n = \left(\text{ad } \frac{d}{dz}\right)^{N-n} \hat{\ell}_N, \qquad \frac{d}{dz} \in \mathfrak{G}_1, \qquad 0 \leq n < N.$$

For $n \geq N$ $a(\hat{\ell}_n, X_c)$ is given in 2). For $0 \leq n < N$, one obtains the following relations by induction on n and with the consistency condition (7.2):

$$a(\hat{\ell}_{n-1}, X_c) = a\left(\left[\frac{d}{dz}, \hat{\ell}_n\right], X_c\right) = -\theta\left(\frac{d}{dz}\right)a(\hat{\ell}_n, X_c)$$

$$= \frac{1}{12} \text{Res}_{z=\infty}\hat{\ell}_n(z) \frac{d}{dz} S(z, X_c) = -\frac{1}{12}\text{Res}_{z=\infty}\hat{\ell}_{n-1}S(z, X_c).$$

In the derivation from the first line to the second we used the formula given in Appendix B). We thus obtain the desired result for all n. ∎

Corollary (7.5).

$$\theta(\ell)\tau(\mathfrak{t}, X_c) = \text{Res}_{z=\infty}\ell(z)\left[\frac{1}{2}\sum_{n,m=1}^{\infty} t_n t_m \omega_Q^{(n)}(z) \omega_Q^{(m)}(z)\right.$$

$$+ \sum_{n=1}^{\infty}\sum_{i=1}^{g} t_n\omega_Q^{(n)}(z)\omega^i(z)\frac{\partial}{\partial y^i} + \frac{1}{2}\sum_{i,j=1}^{g}\omega^i(z)\omega^j(z)\frac{\partial^2}{\partial y^i\partial y^j}\left.\right]\tau(\mathfrak{t}, X_c)$$

for $\ell \in \mathfrak{G}$.

Proof. This equation is obtained from the result of Theorem (6.10) and (7.4). ∎

Proposition (7.6). The projective connection $S(z,X)(dz)^2$ is transformed under the modular transformation $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma$ as follows

$$S(z,\gamma(X))dz^2 = S(z,X)dz^2 + 4\pi\sqrt{-1}\,{}^t\omega(z)(C\Omega+D)^{-1}C\omega(z).$$

Proof. This relation is obtained from the expression of the projective connection:

$$S(z,X)dz^2 = -6[d_z d_w \log \theta(I(z)-I(w)|\Omega)]_{w \to z}$$

and the transformation formula of θ-function.

C) Characterization of the τ-function

Main Theorem (7.7).

For a holomorphic map $f: \hat{\mathcal{B}}_g(\mathcal{Y}) \longrightarrow \mathcal{H}^\times$ the following equations determine f uniquely up to a constant: $f(\mathfrak{t},X_c) = C\tau(\mathfrak{t},X_c)$ with $C \in \mathbb{C}$.

1) $[\theta(\ell) + a(\ell,X)]f(\mathfrak{t},X_c) = \Phi_B(\ell)f(\mathfrak{t},X_c)$ for $\forall \ell \in \mathcal{G}$

where

$$a(\ell,X) = -\frac{1}{12}\operatorname{Res}_{z=\infty}\ell(z)S(z,X);$$

2) $\frac{1}{2\pi\sqrt{-1}}\sum_{i=1}^{g}\int_{\beta_i}d\varphi\,\frac{\partial}{\partial c^i}f(\mathfrak{t},X_c) = \Phi_B(\varphi)f(\mathfrak{t},X_c)$ for $\forall\varphi(z) \in \mathcal{H}(X)$;

3) $f(\mathfrak{t},X_{c+\Omega a+b}) = \exp\left(-2\pi\sqrt{-1}\left(\frac{1}{2}{}^t a\Omega a + {}^t a(I(\mathfrak{t})+c)\right)\right)f(\mathfrak{t},X_c)$

for $a,b \in \mathbb{Z}^g$.

Proof. In equation 2) the left hand side vanishes when $\varphi \in A(X)$, which specifies the form of $f(\mathfrak{t},X_c)$ as

$$f(\mathfrak{t},X_c) = e^{\frac{1}{2}q(\mathfrak{t})}f_0(I(\mathfrak{t}),X_c)$$

with $f_0: \hat{\mathcal{B}}(\mathcal{Y}) \longrightarrow \mathbb{C}[[c_1,c_2,\dots,c_g]]$.

Taking $\varphi^{(n)} \in \mathcal{H}(X)$ (See Appendix A-2)) in equation 2), we obtain equations

$$\left(\frac{\partial}{\partial t_n} - \sum_{i=1}^{g} I_n^i\frac{\partial}{\partial c^i}\right)f_0 = 0, \qquad n = 1,2,3,\dots$$

which naturally lead to the following form for $f(\mathfrak{t},X_c)$:

$$f(\mathfrak{t},X_c) = e^{\frac{1}{2}q(\mathfrak{t})}f_0(I(\mathfrak{t})+c,X).$$

From the automorphy relation 3) with respect to the Jacobian

parameter $c \in \mathbb{C}^g$, we obtain the proportionality of $f_0(I(\mathfrak{t})+c,X)$ and

$\theta(I(\mathfrak{t})+c|\Omega)$, i.e.

$$f(\mathfrak{t},X_c) = C(X)\tau(\mathfrak{t},X_c).$$

The efquations 1) shows that the factor $C(X)$ does not depend on X.   ▌

D) Principles to determine the lifting of $\Delta_{1/2}$

(7.8) We know already that our τ-function has the following

properties:

1) Automorphy under $M_\Delta \ltimes \mathbb{Z}^{2g}$

    a) $\tau(\mathfrak{t},\gamma(X_c))$

        $= \varepsilon(\gamma)\det(C\Omega+D)^{1/2}\exp\left(\pi\sqrt{-1}\,{}^t c(C\Omega+D)^{-1}C(2I(\mathfrak{t})+c)\right)\tau(\mathfrak{t},X_c)$

      · for $\gamma \in M_\Delta$;

    b) $\tau(\mathfrak{t},X_{c+\Omega a+b})$

        $= \exp\left(-2\pi\sqrt{-1}\left(\frac{1}{2}{}^t a\Omega a + {}^t a(I(\mathfrak{t}) + c)\right)\right)\tau(\mathfrak{t},X_c), \quad a,b \in \mathbb{Z}^g;$

2) $\tau(0,X_c)$ depends only on $\mathcal{T}_g \times \mathbb{C}^g$.

    Theorem (7.9).

Let f be a lifting of $\Delta_{1/2}$ satisfying 1) and 2) in (7.8):

1) Then it can be written in the form:

$$f(\mathfrak{t},X_c) = A(\pi(X_c))\tau(\mathfrak{t},X_c)$$

where A: $\mathcal{T} \longrightarrow \mathbb{C}^*$ is holomorphic and M_Δ-invariant and

$\pi: \mathcal{B}(\mathcal{T}) \longrightarrow \mathcal{T}.$

2) Furthermore $A(\pi(X_c))$ is constant if $g \geq 3$.

Proof. 1) If we write $f(\mathfrak{t}, X_c) = A(X_c) \tau(\mathfrak{t}, X_c)$, $A(X_c)$ does not depend on c from the automorphy 1)-b). From this , 1)-a) and 2) we obtain the first half of the theorem.

2) If $g \geq 3$ moreover, we know that $\mathcal{T}/M_\Delta$ has a compactification whose boundary is of codimension > 1. Hence there is no holomorphic function on it (i.e. no M_Δ-invarinat holomorphic function on $\mathcal{T}$) besides constant. ▮

Appendix

Here we list the notations and formulas on the theory of abelian

functions which we use in this article. For details see for example

[Fay][Mum.1]. We fix a Riemann surface R, a canonical bases (α,β) of

$H_1(R,\mathbb{Z})$, a point $Q \in R$ and a (formal) local coordinate z: $z(Q) = \infty$.

The whole data are denoted by $X = (R,(\alpha,\beta),Q,z)$. We set $X_c = (X,\mathscr{L}_c)$,

$c \in \mathbb{C}^g$ (cf. §2).

A) Abelian differentials and the Green function on Riemann surfaces

1) Abelian differentials

(1st kind) $\quad \omega^i = \omega^i(z)dz, \quad i = 1\ldots g,$

$$\int_{\alpha_i}\omega^j = \delta_{ij}, \quad \int_{\beta_i}\omega^j = \Omega_{ij},$$

$$\omega^i(z)dz = -d(\sum_{n>0} I_n^i \frac{z^{-n}}{n});$$

(2nd kind) $\quad \omega_Q^{(n)} = \omega_Q^{(n)}(z)dz, \quad n = 1\ 2\ldots\ ,\ Q \in R$

$$\int_{\alpha_i}\omega_Q^{(n)} = 0, \quad \int_{\beta_i}\omega_Q^{(n)} = 2\pi\sqrt{-1}\ I_n^i,$$

$$\omega_Q^{(n)}(z)dz = d(z^n - \sum_{m>0} q_{nm} \frac{z^{-m}}{m}).$$
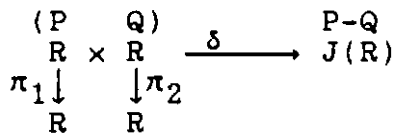
2) Green Functions

(Prime form): a holomorphic section (unique up to a constant)

of the line bundle

$\pi_1^* \mathscr{L}_\Delta^{-1} \otimes \pi_2^* \mathscr{L}_\Delta^{-1} \otimes \delta^*(\theta)$ over R × R, with a simple zero only at P = Q

where π_1, π_2, δ are defined in the following diagram:

$$
\begin{array}{ccc}
(P \quad Q) & & P-Q \\
R \times R & \xrightarrow{\delta} & J(R) \\
\pi_1\downarrow \quad \downarrow\pi_2 & & \\
R \quad R & &
\end{array}
$$

$$E(P,Q) = E(z,w)\sqrt{dz}^{-1}\sqrt{dw}^{-1}$$

$$= - E(Q,P),$$

$$\partial_z \partial_w \log \frac{E(z,w)}{z-w} = \sum_{n,m>0} q_{nm} \, z^{-n-1} w^{-m-1} \ ,$$

$$E(P+\alpha_i, Q) = E(P,Q),$$

$$E(P+\beta_i, Q) = \exp\left(2\pi\sqrt{-1}\left(\frac{\Omega_{ii}}{2} + \int_Q^P \omega^i\right)\right) E(P,Q).$$

(multiplicative meromorphic half-form with a simple pole at $Q \in R$)

$$f(z)\sqrt{dz} = \frac{\sqrt{d\zeta}}{E(\zeta,0)} \qquad (z\zeta = 1, \ \zeta(Q) = 0),$$

$$f(z+\alpha_i) = f(z)$$

$$f(z+\beta_i) = \exp\left(-2\pi\sqrt{-1}\left(\frac{\Omega_{ii}}{2} + \int_Q^z \omega^i\right)\right) f(z),$$

$$f(z)\sqrt{dz} = (1 + o(z^{-1}))\sqrt{dz} \ , \quad f(\zeta)\sqrt{d\zeta} = \left(\frac{1}{\zeta} + \text{regular}\right)\sqrt{d\zeta}.$$

(multivalued meromorphic functions with poles at $Q \in R$)

$$\varphi^{(0)}(z) = 1$$

$$\varphi^{(n)}(z) = z^n - \sum_{n=1}^{\infty} q_{nm} \frac{z^{-m}}{m} = \int^z \omega_Q^{(n)} \quad n = 1,2,3\ldots,$$

$$\varphi^{(n)}(z + \alpha_i) = \varphi^{(n)}(z),$$

$$\varphi^{(n)}(z + \beta_i) = \varphi^{(n)}(z) + 2\pi\sqrt{-1} \, I_n^i \ .$$

If we set $\bar{A}(X) = \sum_{n=0}^{\infty} \mathbb{C}\varphi^{(n)}(z) \subseteq K$ (a losed subspace),

then the following sequence is exact

$$0 \longrightarrow A(X) \longrightarrow \bar{A}(X) \longrightarrow \mathbb{C}^g \longrightarrow 0$$

$$f \longrightarrow c_i = \frac{1}{2\pi\sqrt{-1}} \int_{\beta_i} df.$$

(Fundamental normalized differential of the second kind)

$$\omega(P,Q) = \omega(z,w)dzdw = d_z d_w \log E(z,w),$$

$$\omega(z,w) = \frac{1}{(z-w)^2} + \sum_{nm>0} q_{nm} z^{-n-1} w^{-w-1}.$$

(Szego kernel)

$$S_c(P,Q) = S_c(z,w)\sqrt{dz}\sqrt{dw} ,$$

$$S_c(z,w) = \frac{\theta(I(z)-I(w)+c|\Omega)}{\theta(c|\Omega)E(z,w)} = \frac{1}{z-w} + \sum_{\mu\nu>0} C_{\mu\nu} z^{-\mu-1/2} w^{-\nu-1/2}$$

where $c \in \mathbb{C}^g$ with $\theta(c|\Omega) \neq 0$.

3) The θ-function and the Abel-Jacobi map

(θ-function)

$$\theta\begin{bmatrix}a\\b\end{bmatrix}(z|\Omega) = \sum_{n\in Z^g} \exp\{\pi i\,{}^t(n+a)\Omega(n+a)+2\pi i\,{}^t(n+a)(z+b)\} \quad z \in \mathbb{C}^g,$$

$$\theta\begin{bmatrix}a\\b\end{bmatrix}(z+e_i|\Omega) = e^{2\pi i a_i} \theta\begin{pmatrix}a\\b\end{pmatrix}(z|\Omega)$$

$$\theta\begin{bmatrix}a\\b\end{bmatrix}(z+\Omega e_i|\Omega) = \exp\{-2\pi i(\tau_{ii}/2 + z_i+b_i)\}\theta\begin{pmatrix}a\\b\end{pmatrix}(z|\Omega).$$

(Abel-Jacobi Map)

$$I : \tilde{R} \longrightarrow \mathbb{C}^g \; ; \; \tilde{R} \text{ is the universal Abelian covering of } R,$$

$$I[z] = (\int_Q^z \omega^i) = (\sum_n I_n^i z^{-n}).$$

4) Modular transformation properties

For $\gamma = \begin{pmatrix}A & B\\C & D\end{pmatrix} \in Sp(2g,Z) = M,$

$$\gamma\begin{pmatrix}\alpha\\\beta\end{pmatrix} = \begin{pmatrix}D & C\\B & A\end{pmatrix}\begin{pmatrix}\alpha\\\beta\end{pmatrix} , \qquad \gamma(\Omega) = (A\Omega+B)(C\Omega+D)^{-1},$$

$$\gamma(\omega^i) = ({}^t(C\Omega+D)^{-1})_{ij} \omega^j,$$

$$\gamma(\omega_Q^{(n)}) = \omega_Q^{(n)} - ((C\Omega+D)^{-1})_{ij} C_{jk} (\int_{\beta_i}\omega_Q^{(n)})\omega_k,$$

$$\gamma\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}D & -C\\-B & A\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} - \frac{1}{2} \text{diag}\begin{pmatrix}C\,{}^tD\\A\,{}^tB\end{pmatrix},$$

$$\theta\left[\gamma\binom{a}{b}\right]\left(^t(C\Omega+D)^{-1}z, \gamma(\Omega)\right)$$

$$= \varepsilon \det(C\Omega+D)^{1/2}\exp\{\pi iz(C\Omega+D)^{-1}Cz\} \; \theta\begin{bmatrix}a\\b\end{bmatrix}(z|\Omega)$$

where $\quad \varepsilon = \varepsilon(\gamma)$ with $\varepsilon^8 = 1$.

5) Projective connection

$$S(P) = S(z)dz^2 = -6 \lim_{w \to z} d_z d_w \; \log \frac{E(z,w)}{z-w}$$

$$\omega(z,w) = \frac{1}{(z-w)^2} + \frac{1}{6} S(z) + \text{higher order terms}$$

$$S(w)dw^2 = S(z)dz^2 + \{w,z\}dz^2$$

where

$$\{w,z\} = \frac{w'''}{w'} - \frac{3}{2}\left(\frac{w''}{w'}\right)^2 \quad \text{(Schwarzian derivetive)},$$

with $' = \frac{d}{dz}$.

B) Transformation property and $\theta(l)$-derivatives

For $l = l(z)\frac{d}{dz} \in \mathfrak{G}_0$, $\theta(l)$ is essentially a generator of coordinate transformation which fixes $z(Q) = \infty$:

$$z \longrightarrow w = e^{\varepsilon l}z = \left(1 + \varepsilon l(z)\frac{d}{dz}\right)z = z + \varepsilon l(z).$$

We can thus calculate $\theta(l)$-derivatives of geometrical objects from the data of their transformation properties.

1) Let $f(P)$ be a j-form on R and define $f(z,X): \mathcal{B}_g \longrightarrow \mathbb{C}((z^{-1}))$ by $f(P) = f(z,X)(dz)^j$ for $X = (R,Q,z)$. We have the following relation

$$f(z,e^{\varepsilon l}X)(dz)^j = f(w,X)(dw)^j.$$

On the other hand the transformation property as a j-form implies

$$f(w,X)dw^j = f(z,X)(dz)^j + \varepsilon\{\ell(z)\frac{d}{dz} + j\ell'(z)\}f(z,X)(dz)^j.$$

It then follows that

$$\theta(\ell)f(z,X) = \frac{d}{d\varepsilon}f(z,e^{\varepsilon\ell}X)\Big|_{\varepsilon=0} = \{\ell(z)\frac{d}{dz} + j\ell'(z)\}f(z,X).$$

2) Let $S(P)$ be a projective connection on R and define
$S(z,X): \hat{\mathfrak{g}}_g \longrightarrow \mathbb{C}((z^{-1}))$ by $S(P) = S(z,X)(dz)^2$, then just like 1)
we have

$$S(z,e^{\varepsilon\ell}X)(dz)^2 = S(w,X)(dw)^2,$$

$$S(w,X)(dw)^2 = S(z,X)(dz)^2 + \{w(z),z\}\, dz^2,$$

which lead to the following:

$$\theta(\ell)S(z,X) = \{\ell(z)\frac{d}{dz} + 2\ell'(z)\}S(z,X) - \ell'''(z).$$

Notes added in proof

After submition of this paper we are informed that there appered
several new papers which treat similar subjects with our paper:
L.Alvarez-Gaumé, C.Gomez and C.Reina: New methods in string theory,
CERN preprint, CERN-TH 6775/87. E.Arbarello, C.De Concini, V.Kac,
C.Procesi: Moduli space of curves and representation theory,
handwritten manuscript. E.Witten: Quntum field theory, grassmannians,
and algebraic curves, Princeton preprint, PUPT-1057.

References

[A-G.M.V] Alvarez-Gaumé L., G.Moore and C.Vafa: Theta functions, modular invariance, and strings. Commun. Math. Phys. 106,1-40 (1986)

[A-G.M.N.V.B.] Alvarez-Gaumé,L.,G.Moore,P.Nelson,C.Vafa and J.B.Bost: Bosonization in arbitrary genus. Phys.Lett.178 B, 41(1986); Alvarez-Gaumé,L.,J.-B.Bost,G.Moore,Ph.Nelson and C.Vafa: Bosonization of higher genus Riemann surfaces. CERN preprint, CERN-TH.4660/87,HUTP-86/A062

[A-G.G.R.] Alvarez-Gaumé,L.,C.Gomez and C.Reina: Loop groups, grassmanians and string theory, Phys.Lett.190B,55 (1987)

[A.W.] Alvarez,O. and P.Windey: The energy momentum tensor as geometrical datum, LBL preprint, UCB-PTH-86/36,LBL-22558

[Ar.] Arakelov,S.: An intersection theory for divisors on an arithmetic surface, Izv.Akad.Nauk.SSSR Ser.Mat.38(1974)[=Math. USSR Izv.8 1167 (1974)]

[B.K.] Belavin,A.A. and V.G.Knizhnik: Complex geometry and theory of quantum string, Sov.Phys.JETP 64 (1986) 214

[B.M.] Beilinson A.A.and Yu.I.Manin: The Mumford form and the Polyakov measure in string theory, Commun.Math.Phys. 107,359(1986)

[B.M.S.] Beilinson,A.A.,Yu.I.Manin and Y.A.Shechtman: Localization of the Virasoro and Neveu-Schwartz algebra,Moscow preprint(1986)

[B.P.Z.] Belavin,A.A.,A.M.Polyakov and A.B.Zamolodchikov: Infinite conformal symmetry in two-dimensional quantum field theory, Nucl.Phys.B241,333(1984)

[B.R.] Bernshtein,I.N. and B.I.Rozenfel'd: Homogeneous spaces of

infinite-dimensional Lie algebra and characteristic classes
of foliations, Russ.Math.Survey 107, 28(1973)

[B.Ra.] Bowick,M.J. and S.G.Rajeev: String theory as the Kähler
geometry of loop space, Phys.Rev.Lett. 58, 535 (1987)

[D.J.K.M.] Date,E.,M.Jimbo,M.Kashiwara,and T.Miwa: Transformation
groups for soliton equations, in Proceedings of RIMS Symposium
on Non-Linear Integrable Systems-Classical Theory and Quantum
Theory, Kyoto, Japan, ed. by M.Jimbo and T.Miwa, World Science
Publishing Co., Singapore, 1983

[D.M.] Deligne,P. and D.Mumford: The irreducibility of the space
of curves of given genus, Publ.Math.I.H.E.S. 36, 75 (1969)

[D.] Dubrovin,B.A.: Theta functions and non-linear equations,
Uspekhi Mat. Nauk 36:2,11 (1981)

[D.S.] Dugan,M. and H.Sonoda: Functional determinants on Riemann
surfaces, LBL preprint,LBL-22776,1986

[E.O.1] Eguchi,T. and H.Ooguri: Conformal and current algebras on a
general Riemann surface, Nucl.Phys. B282,308 (1987)

[E.O.2] Eguchi,T. and H.Ooguri: Chiral bosonization on a Riemann
surface, Phys.Lett.187B, 127 (1987)

[Fal.] Faltings,G.: Calculus on arithmetic surfaces, Ann.Math.
119,387(1984)

[F.K.] Farkas,H.M.,I.Kra: Riemann surfaces,Berlin- Heiderberg-
New York- Tokyo, Springer 1980

[Fay] Fay,J.: Theta functions on Riemann surfaces, Lecture Notes
in Math., Vol.352.Berlin- Heidelberg- New York- Tokyo, Springer
1973

[F.G.Z.] Frenkel,I.B.,H.Garland and G.J.Zuckerman: Semi-infinite

cohomology and string theory, Proc. Nat. Acad. Sci., USA, 83, 8442(1986)

[F.S.] Friedan,D. and S.Shenker: The analytic geometry of two-dimensional conformal field theory, Nucl.Phys. B281,509(1987)

[Fr.] Friedan,D.: A new formulation of string theory, Second Nobel Symposium on Elementary Particle Physics, presented at Marstrand,Sweden (1986) and Symposium on Geometry and Topology in Field Theory , Espoo,Finland (1986)

[v.G.] van der Geer: Schottky's problem. (Proc.) Arbeitstagung Bonn. 1984, Lecture notes in Math. No.1111

[H.] Hirota,R.: Direct method in soliton theory, Solitons,157 ed. by R.K.Bullough and P.J.Caudrey, Springer,1980

[I.M.O.] Ishibashi,N.,Y.Matsuo and H.Ooguri: Soliton equations and free fermions on Riemann surfaces, Mod. Phys. Lett. A2,119 (1987)

[I.] Ivanov,B.V.: On the quantum topology of strings, Phys.Lett. 189B,39 (1987)

[Kn.] Knizhnik,V.G.: Analytic fields on Riemann surfaces, Phys.Lett. 180B, 247 (1986)

[Ko.] Kodaira,K.: Complex manifolds and deformations of complex structure, Springer Verlag, 1985

[K.N.] Krichever,I.M. and S.P.Novikov: Virasoro algebra, Riemann surfaces and structure theory of soliton, Functional Analysis and its Application, 21, 46(1987)(in Russian)

[Kr.] Krichever,I.M.: Methods of algebraic geometry in the theory of non-linear equations, Russ.Math.Surveys, 32, 185 (1977)

[Mul.] Mulase,M.: Cohomological structure in soliton equations and

jacobian varieties, J.Diff.Geom. 19,403(1984)

[Mum.1] Mumford,D.: Tata lectures on theta.Vol.I,II,Boston,
Birkhauser 1983

[Mum.2] Mumford,D.: An algebro-geometric construction of commuting
operators and of solutions to the Toda lattice equation,
Korteweg-de-Vries equation and related non-linear equations,
Intern.Symp. on Algebraic Geometry, Kyoto,1977,115

[Mum.3] Mumford,D.: Curves and Jacobians, The University of Michigan
Press.1975

[N.] Namikawa,Y.: On the canonical holomorphic map from the moduli
space of stable curves to the Igusa monodial transform,
Nagoya Math.J. 52,197(1973)

[P.K.] Peterson,D.H. and V.Kac: Infinite flag varieties and
conjugacy theorems, Proc.Nat.Acad.Sci.USA,80 1778(1983)

[P.] Polyakov,A.M.: Quantum geometry of bosonic strings,
Phys.Lett.103B,207(1981)

[Sa.] Sato,M.: Soliton equations as dynamical systems on an infinite
dimensional Grassmann manifold, Res.Inst.Math.Sci.,Kyoto Univ.
-Kokyuroku 439, 30(1981)

[Sa.] Sato,M.: Soliton equations and the infinite dimensional
Grassmann manifolds, Lectures delivered at Univ. of Tokyo
,Nagoya Univ. (Notes by M.Mulase in Japanese 1981-1982) and
Kyoto Univ.(Notes by T.Umeda in Japanese 1984-1985)

[S.N.] Sato,M. and M.Noumi: Soliton equation and universal Grassmann
manifold, Sophia University Kokyuroku in Math. 18 (1984)
(in Japanese)

[S.S.] Sato,M. and Y.Sato: Solition equations as dynamical

systems on infinite dimensional Grassmann manifold,

Lecture Notes in Num.Appl.Anal.,5,259(1982), Nonlinear PDE in

Applied Science. U.S-Japan Seminar, Tokyo ,1982

[S.W.] Segal,G.B. and G.Wilson: Loop groups and equations of

KdV type, Publ.Math. I.H.E.S.,61,5(1985)

[Sh.] Shiota,T.: Characterization of jacobian varieties in terms

of soliton equations, Invent. Math.,333 (1986)

[Si.] Siegel,C.L.: Topics of complex function theory, Vol.I,II,

Wiley-Interscience, New York

[So.] Sonoda,H.: Calculation of a propagator on a Riemann surface

LBL preprint,LBL-21877, The energy-momentum tensor on a Riemann

surface, Nucl.Phys.B281 (1987)546

[T.K.1] Tsuchiya,A. and Y.Kanie: Fock space representation of the

Virasoro algebra-intertwining operators, Publ.R.I.M.S. Kyoto

Univ.259, 22(1986)

[T.K.2] Tsuchiya,A. and Y.Kanie: Vertex operators on conformal

field theory on P^1 and monodromy representations of Braid

groups, To appear in "Conformal Field Theory and Solvable

Lattice Model", Advanced Studies in Pure Mathematics,

Kinokuniya, and Lett.in Math. Phys. 13 (1987)303

[V.] Vafa,C.: Operator formulation on Riemann surfaces, Phys.Lett.

190B, 47 (1987)