HARMONIC SEQUENCES AND HARMONIC MAPS

OF SURFACES INTO COMPLEX GRASSMANN MANIFOLDS

by

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Introduction

Let $G(k,n)$ be the Grassmann manifold of all k-dimensional subspaces $\mathbb{C}^k$ in complex space $\mathbb{C}^n$ or, what is the same, all the (k-1) dimensional projective spaces $\mathbb{C}P^{k-1}$ in projective space $\mathbb{C}P^{n-1}$. $G(k,n)$ has a canonical Kähler metric. We will study the harmonic maps of a Riemann surface M into $G(k,n)$. In particular we will describe all the harmonic maps of the two-sphere S^2 into $G(k,n)$ in terms of holomorphic data and all the harmonic maps of the torus T^2 into $G(k,n)$ in terms of holomorphic data and degree zero harmonic maps. This work completes (and extends) the program for studying harmonic maps of S^2 into $G(k,n)$, first stated by the author and S.S. Chern in [4] and partially completed in [5]. The harmonic maps of $S^2 \to G(1,n) = \mathbb{C}P^{n-1}$ were first determined by Din and Zakrezwski ([6], also see [7] and [11]). The harmonic maps $S^2 \to G(2,4)$ were determined by Ramanathan [9] and the harmonic maps $S^2 \to G(2,n)$ were determined by the author and Chern [5]. Using techniques completely different from those of the papers cited above Uhlenbeck studied the harmonic maps of S^2 into the unitary group $U(n)$ [10]. In the course of the study she gave a description of the harmonic maps of S^2 into $G(k,n)$ by embedding $G(k,n)$ totally geodesically in $U(n)$. The description given in this paper is quite different from Uhlenbeck's and works intrinsically with $G(k,n)$.

The fundamental object of study in this paper is the transforms of a harmonic map of a surface M into $G(k,n)$. To define the ∂-transform (or $\bar{\partial}$-transform) consider a map

$f:M \to G(k,n)$, when M is an oriented Riemannian surface. We write the Riemannian metric of M as

$$ds_M^2 = \varphi\bar{\varphi} ,$$

where φ is a complex-valued one-form, defined up to a factor of absolute value 1. This form φ defines a complex structure on M. For $x \in M$ the space $f(x)$ has an orthogonal space $f(x)^{\perp}$ of dimension $n - k$. We denote by $[f(x)]$ and $[f(x)^{\perp}]$ their corresponding projective spaces, of dimensions $k - 1$ and $n - k - 1$, respectively. For a vector $Z(x) \in f(x)$ the orthogonal projection of ∂Z in $f(x)^{\perp}$ is multiple of φ, and hence, by cancelling out φ, defines a point of $f(x)^{\perp}$. This defines a projective collineation $\partial:[f(x)] \to [f(x)^{\perp}]$, to be called a <u>fundamental collineation.</u> The mapping defined by sending $x \in M$ to the image of $[f(x)]$ under ∂ is called the ∂-<u>transform</u>. Similarly, we define the $\bar{\partial}$-<u>transform</u>.

If the map $f:M \to G(k,n)$ is harmonic then its ∂-transform and $\bar{\partial}$-transform are also harmonic. Note that a fundamental collineation ∂ (resp. $\bar{\partial}$) may degenerate or may be zero. If it is zero than the map is antiholomorphic (resp. holomorphic). If it degenerates then the ∂-transform (resp. $\bar{\partial}$-transform) is a harmonic map $M \to G(\ell,n)$ where $\ell < k$.

By successive applications of the ∂-transform (or $\bar{\partial}$-transform) we can construct a sequence of harmonic maps

$$[f(x)] \xrightarrow{\partial} \partial[f(x)] \xrightarrow{\partial} \partial^2[f(x)] \xrightarrow{\partial} \cdots$$

called a harmonic sequence. If any of the fundamental collineations of the sequence degenerates then the sequence associates to f a harmonic map $g:M \to G(\ell,n)$, $\ell < k$. In § 4 we will show that when M has genus zero the harmonic map f can be recovered from g by iterating a construction called returning. Each returning is essentially a choice of a holomorphic subbundle of a holomorphic bundle over M. In § 5 we describe a construction different then returning, called extending, which effects the reconstruction of f from g for a surface M of any genus. Each extending, like each returning, is a choice of a holomorphic subbundle.

In § 3 we will derive an inequality relating the energy of f to the degree of f, the genus of M and the singularities of the fundamental collineations of the harmonic sequence generated by f. When the genus of M is zero or when the genus of M is one and the degree of f is nonzero this inequality implies that one of the fundamental collineations must be degenerate.

Combining the results of § 3 and § 4 and using induction we can prove.

Theorem 1 Let $f:S^2 \to G(k,n)$ be a haromic map. Then f can be constructed from holomorphic or antiholomorphic curves $S^2 \to G(\ell,n)$, where $1 \leq \ell \leq k$, using the ∂ and $\bar{\partial}$ transforms and returnings.

Combining the results of § 3 and § 5 and using induction we have

Theorem 2 Let $f:M \to G(k,n)$ be a harmonic map, where M is a surface of genus one. Then f can be constructed using the ∂ and $\bar{\partial}$ transforms and extendings from either:

(1) A holomorphic or antiholomorphic curve $T^2 \to G(\ell,n)$, $1 \le \ell \le k$.

or

(2) A degree zero harmonic map $T^2 \to G(\ell,n)$, $1 \le \ell \le k$.

In fact the statement of Theorem 2 can be made even stronger; see Theorem 5.2. Theorem 2 , with (2) deleted, holds when M is a surface of genus zero; see Theorem 5.1.

The inequality in § 3 should with more careful analysis yield much interesting information about harmonic maps and harmonic sequences in $G(k,n)$.

§ 1 and § 2 are, with some modifications, the same as § 1 and § 2 in [5]. The reader familiar with this work can probably go right to Section 3. We have included these sections to make this paper self-contained.

It is a pleasure to thank R. Bryant and D. Burghelea for some interesting conversations.

1. Geometry of $G(k,n)$

We equip $\mathbb{C}^n$ with the standard Hermitian inner product, so that, for $Z, W \in \mathbb{C}^n$,

(1.1) $$Z = (z_1,\dots,z_n) \ , \ W = (w_1,\dots,w_n) \ .$$

we have

(1.2) $$(Z,W) = \sum z_A \overline{w}_A = \sum z_A w_{\overline{A}} \ .$$

Throughout this paper we will agree on the following ranges of indices

(1.3) $$1 \leq A,B,C,\dots \leq n, \ 1 \leq \alpha,\beta,\gamma,\dots \leq k \ , \ k+1 \leq i,j,h,\dots \leq n.$$

We shall use the summation convention, and the convention

(1.4) $$\overline{z}_A = z_{\overline{A}}, \overline{t}_{\overline{A}B} = t_{A\overline{B}} \ , \text{ etc.}$$

A frame consists of an ordered set of n linearly independent vectors Z_A, so that

(1.5) $$Z_1 \wedge \dots \wedge Z_n \neq 0$$

It is called <u>unitary</u>, if

(1.6) $$(Z_A, Z_B) = \delta_{A\overline{B}} \ .$$

The space of unitary frames can be identified with the unitary group $U(n)$. Writing

(1.7) $$dZ_A = \omega_{A\overline{B}} Z_B \ ,$$

the $\omega_{A\bar{B}}$ are the Maurer-Cartan forms of $U(n)$. They are skew-Hermitian, i.e., we have

$$\omega_{A\bar{B}} + \omega_{\bar{B}A} = 0. \tag{1.8}$$

Taking the exterior derivative of (1.7), we get the Maurer-Cartan equations of $U(n)$:

$$d\omega_{A\bar{B}} = \omega_{A\bar{C}} \wedge \omega_{C\bar{B}} . \tag{1.9}$$

An element $\mathbb{C}^k$ of $G(k,n)$ can be defined by the multivector $Z_1 \wedge \ldots \wedge Z_k \neq 0$, defined up to a factor. The vectors Z_α and their orthogonal vectors Z_i are defined up to a transformation of $U(k)$ and $U(n-k)$, respectively, so that $G(k,n)$ has a G-structure, with $G = U(k) \times U(n-k)$. In particular, the form

$$ds^2 = \omega_{\alpha\bar{i}}\, \omega_{\bar{\alpha}i} \tag{1.10}$$

is a positive Hermitian form on $G(k,n)$, and defines an Hermitian metric. Its Kähler form is

$$\Omega = \frac{\sqrt{-1}}{2\pi}\, \omega_{\alpha\bar{i}} \wedge \omega_{\bar{\alpha}i} \tag{1.11}$$

By using (1.9) it can be immediately verified that Ω is closed, so that the metric ds^2 is Kählerian.

2. Harmonic maps of surfaces

Let M be an oriented surface and let $f:M \to G(k,n)$ be a non-constant harmonic map. Denote the Riemannian metric on M by $ds_M^2 = \varphi\overline{\varphi}$, where φ is a complex valued one-form: φ is defined up to a complex factor of absolute value 1. For $x \in M$ the image $f(x) \in G(k,n)$ has an orthogonal space $f(x)^{\perp} \in G(n-k,n)$. If $Z \in f(x)$, we can write

$$(2.1) \qquad dZ \equiv X \cdot \varphi + Y \cdot \overline{\varphi} \quad , \text{mod } f(x),$$

where $X, Y \in f(x)^{\perp}$. If $Z \in \mathbb{C}^n - \{0\}$, we denote by $[Z]$ the point in P_{n-1} with Z as the homogeneous coordinate vector. Then

$$(2.2) \qquad \partial:[Z] \longmapsto [X] \quad , \quad \overline{\partial}:[Z] \longmapsto [Y] \quad ,$$

if not zero, are well-defined projective collineations of the projectivized space $[f(x)]$ into $[f(x)^{\perp}]$. We shall call these the <u>fundamental collineations</u>. Dually there are adjoint fundamental collineations from $[f(x)^{\perp}]$ to $[f(x)]$. Clearly the fundamental collineation $\overline{\partial}$ (resp. ∂) is zero, if and only if f is holomorphic (resp. anti-holomorphic).

To express the situation analytically we choose, locally, a field of unitary frames Z_A, so that Z_α span $f(x)$. Then we have

$$(2.3) \qquad f^*\omega_{\alpha\bar{I}} = a_{\alpha\bar{I}}\varphi + b_{\alpha\bar{I}}\overline{\varphi} \ .$$

By (1.7) the fundamental collineations ∂ and $\overline{\partial}$ send $[Z_\alpha]$

to $[X_\alpha]$ and $[Y_\alpha]$ respectively, where

$$X_\alpha = a_{\alpha\bar{i}} Z_i \ , \ Y_\alpha = b_{\alpha\bar{i}} Z_i \ .$$

The energy of the map f is by definition

$$E(f) = \int_M \mathrm{tr}(f^* ds^2) d\,\mathrm{vol}$$

where ds^2 is the metric on $G(k,n)$ and trace is taken with respect to the metric on M. By (2.3) and (1.10) this becomes

$$(2.4) \qquad E(f) = \int_M \sum_{\alpha,i} (|a_{\alpha\bar{i}}|^2 + |b_{\alpha\bar{i}}|^2) \frac{\sqrt{-1}}{2} \varphi \wedge \bar{\varphi}$$

A map, which is a critical point of the energy functional, is called harmonic.

The pullback of the Kähler form Ω by the map f defines an integral cohomology class $[f^*\Omega] \in H^2(M,\mathbb{Z})$. Evaluating this class on the fundamental homology class of M yields an integer $[f^*\Omega]([M])$ called the <u>degree of f</u>. The degree of f can be computed from (1.11) and (2.3) as follows:

$$\deg f = \int_M f^*\Omega$$

$$(2.5) \qquad = \frac{\sqrt{-1}}{2\pi} \int_M \sum_{\alpha,i} (a_{\alpha\bar{i}}\varphi + b_{\alpha\bar{i}}\bar{\varphi}) \wedge (a_{\bar{\alpha}i}\bar{\varphi} + b_{\bar{\alpha}i}\varphi)$$

$$= \int_M \sum_{\alpha,i} (|a_{\alpha\bar{i}}|^2 - |b_{\alpha\bar{i}}|^2) \frac{\sqrt{-1}}{2\pi} \varphi \wedge \bar{\varphi}$$

The metric ds_M^2 has a connection from ρ , which is a real real one-form satisfying the equation

$$(2.6) \qquad d\varphi = -i\rho \wedge \varphi \ .$$

Its exterior derivative gives the Gaussian curvature K as follows:

$$d\rho = -i/2\ K\varphi \wedge \overline{\varphi}\ .$$

Taking the exterior derivative (2.3) and using (1.9), (2.6), we get

$$(2.7) \qquad Da_{\alpha\bar{i}} \wedge \varphi + Db_{\alpha\bar{i}} \wedge \overline{\varphi} = 0,$$

where

$$(2.8) \qquad Da_{\alpha\bar{i}} = da_{\alpha\bar{i}} - a_{\beta\bar{i}}\omega_{\alpha\bar{\beta}} + a_{\alpha\bar{j}}\omega_{j\bar{i}} - ia_{\alpha\bar{i}}\rho\ ,$$

$$Db_{\alpha\bar{i}} = db_{\alpha\bar{i}} - b_{\beta\bar{i}}\omega_{\alpha\bar{\beta}} + b_{\alpha\bar{j}}\omega_{j\bar{i}} + ib_{\alpha\bar{i}}\rho\ .$$

From (2.7) it follows that

$$(2.9) \qquad Da_{\alpha\bar{i}} = p_{\alpha\bar{i}}\varphi + q_{\alpha\bar{i}}\varphi\ ,$$

$$Db_{\alpha\bar{i}} = q_{\alpha\bar{i}}\varphi + r_{\alpha\bar{i}}\varphi\ .$$

The quadratic differential form

$$(2.10) \qquad Da_{\alpha\bar{i}}\varphi + Db_{\alpha\bar{i}}\overline{\varphi} = p_{\alpha\bar{i}}\varphi^2 + 2q_{\alpha\bar{i}}\varphi\overline{\varphi} + r_{\alpha\bar{i}}\overline{\varphi}^2$$

is the "second fundamental form" of the map f. It is well-known that the vanishing of its trace is the condition that f be harmonic, which is therefore $q_{\alpha\bar{i}} = 0$ (see [2]).

We get therefore the following criterion for the harmonicity of f, which we will apply repeatedly:

<u>Theorem</u> 2.1: <u>The property that</u> f <u>is a harmonic map is expressed by one of the following conditions, which are equivalent:</u>

(a) $Da_{\alpha\bar{i}} \equiv 0$, mod φ,

(b) $Db_{\alpha\bar{i}} \equiv 0$, mod $\bar{\varphi}$.

Theorem 2.1 allows us to study the global behavior of the maps $\partial, \bar{\partial}$ when f is harmonic.

The map $f: M \to G(k,n)$ induces over M the universal k-dimensional complex vector bundle V, with fibres $f(x), x \in M$. In terms of our frames Z_A a vector $Z \in f(x)$ can be written

$$Z = \xi^{\alpha} Z_{\alpha} , \tag{2.11}$$

and we have the natural connection defined by

$$DZ = (d\xi^{\alpha} + \xi^{\beta}\omega_{\beta\bar{\alpha}})\ Z_{\alpha} . \tag{2.12}$$

On V, which is of real dimension $2k+2$, there is an almost complex structure defined by the forms

$$\theta^{\alpha} = d\xi^{\alpha} + \xi^{\beta}\omega_{\beta\bar{\alpha}},\quad \varphi . \tag{2.13}$$

By (1.9) and (2.6) it can be immediately verified that these satisfy the Frobenius condition. Hence, by the Newlander-Nirenberg theorem there is a complex structure on V and V is a holomorphic bundle over M. Similarly, its orthogonal bundle W, with fibers $f(x)^{\perp}$, $x \in M$, is also a holomorphic bundle over M. In fact, if $Z = \eta^{i} Z_{i} \in f(x)^{\perp}$, the forms defining the complex structure on W are

$$\psi^{i} = d\eta^{i} + \eta^{j}\omega_{j\bar{i}},\quad \varphi. \tag{2.14}$$

Let $T^{(1,0)}$ be the cotangent bundle on M of type $(1,0)$, so that its sections can be written as $f\varphi$, f being a function. A section of the tensor product $W \otimes T^{(1,0)}$ can be written $\eta^i Z_j \otimes \varphi$, and its covariant differential is given by

$$D\eta^j = d\eta^j + \eta^k \omega_{k\bar{j}} - i\eta^j \rho . \tag{2.15}$$

On $W \otimes T^{(1,0)}$ there is a complex structure defined by the forms

$$\tilde{\Psi}^j = d\eta^j + \eta^k \omega_{k\bar{j}} - i\eta^j \rho,\ \varphi. \tag{2.16}$$

We define the mapping

$$\partial : V \to W \otimes T^{(1,0)} \tag{2.17}$$

by

$$\partial(\xi^\alpha Z_\alpha) = \xi^\alpha a_{\alpha\bar{I}} Z_i \otimes \varphi \tag{2.18}$$

keeping M pointwise fixed. Both sides of (2.17) being holomorphic bundles, we will prove that ∂ is a holomorphic bundle map if f is harmonic. In fact, substituting

$$\eta^j = \xi^\alpha a_{\alpha\bar{j}}$$

into $\tilde{\Psi}^j$ in (2.16), we find

$$\tilde{\Psi}^j \equiv 0 \bmod \theta^\alpha, \varphi$$

The holomorphicity of ∂ has a number of important consequences. In particular, it follows that except at isolated points the map ∂ and so the matrix $(a_{\alpha\bar{I}})$ have constant rank. The holomorphicity of $\hat{\partial}$ also implies that the image of ∂, $\partial[f(x)]$, extends continuously and smoothly over the isolated singularities of $\hat{\partial}$. Thus the image $\partial[f(x)]$ is a well-defined bundle and the fundamental collineation ∂ is a projective bundle map. Denoting $\dim \partial[f(x)] = k_1 - 1$, we define the ∂-transform of f:

(2.19) $$\partial f : M \to G(k_1, n)$$

by $(\partial f)(x) = \partial[f(x)]$, $x \in M$. Similarly $\bar{\partial}[f(x)]$ is a bundle and the fundamental collineation $\bar{\partial}$ is a projective bundle map. Also we have the $\bar{\partial}$-transform

(2.20) $$\bar{\partial} f : M \to G(k_2, n)$$

defined by $(\bar{\partial} f)(x) = \bar{\partial}[f(x)], x \in M$, where $\dim \bar{\partial}[f(x)] = k_2 - 1$.

The image of $\hat{\partial}$ is itself a holomorphic bundle which we denote $V_1 \otimes T^{(1,0)}$. Thus

(2.21) $$\hat{\partial} : V \to V_1 \otimes T^{(1,0)} .$$

Returning to (2.1) it is easy to see that

$$\hat{\partial} : Z \mapsto X \cdot \varphi$$

and so the ∂ fundamental collineation is a projective bundle map

$$\partial : [V] \longrightarrow [V_1]$$

Similarly if we define

$$\bar{\partial}: V \to W \otimes T^{(0,1)} \tag{2.22}$$

by

$$\bar{\partial}(\xi^\alpha Z_\alpha) = \xi^\alpha b_{\alpha\bar{i}} Z_i \otimes \bar{\varphi}$$

then $\bar{\partial}$ is an antiholomorphic map and

$$\bar{\partial}: Z \to Y \cdot \varphi .$$

Consider the vectors $Z \in f(x)$, such that $Y = 0$ in (2.1). They form a subspace $\ker \bar{\partial} \subset f(x)$. If f is harmonic, the above argument shows that $\ker \bar{\partial}$ is of constant dimension. We define

$$\delta_1 f: M \to G(\ell_1, n) \quad , \tag{2.23}$$

which sends $x \in M$ to the orthogonal complement of $\ker \bar{\partial}$ in $f(x)$. Similarly, we define $\delta_2 f$, using the operator ∂.

<u>Theorem 2.2.</u> <u>Let</u> $f: M \to G(k,n)$ <u>be a harmonic map. Then</u>

(a) <u>The map</u> $f^\perp: M \to G(n-k,n)$, <u>defined by</u>

$$f^\perp(x) = f(x)^\perp \ , \quad x \in M,$$

<u>is harmonic.</u>

(b) <u>The maps</u> $\partial f, \bar{\partial} f, \delta_1 f, \delta_2 f$ <u>are harmonic</u>.

(c) <u>If</u> $k_1 = k, \partial\bar{\partial} f$ <u>is</u> f <u>itself</u>.

Using the criteria in Theorem 2.1, the proof of (a) is immediate.

To prove the first statement in (b) choose frames so that Z_α span $f(x)$ and Z_u span $\partial f(x)$, when the indices have the ranges

$$k + 1 \leq u, v \leq k + k_1, k + k_1 \leq \lambda, \mu \leq n.$$

Then $a_{\alpha\bar{\lambda}} = 0$, and the matrix $(a_{\alpha\bar{u}})$ has rank k_1. Since f is harmonic, it follows from Theorem 2.1 and (2.8) that

$$(2.24) \qquad a_{\alpha\bar{u}} \omega_{u\bar{\lambda}} \equiv 0, \text{ mod } \varphi ,$$

which implies $\omega_{u\bar{\lambda}} \equiv 0$, mod φ .

We now apply to the map ∂f the criterion of harmonicity in Theorem 2.1. The space $(\partial f)(x)$ is spanned by Z_u and its orthogonal space by Z_α, Z_λ. We have

$$(2.25) \qquad \omega_{u\bar{\alpha}} = -\omega_{\bar{\alpha}u} = -b_{\bar{\alpha}u}\varphi - a_{\bar{\alpha}u}\bar{\varphi} ,$$

$$\omega_{u\bar{\lambda}} \equiv 0 , \text{ mod } \varphi.$$

By condition (b) of Theorem 2.1 we see readily that ∂f is harmonic.

In the same way we prove the other statements in (b).

The most interesting case is when $k_1 = k$. From (2.25) we see immediately that the $\bar{\partial}$-transform of $\partial f(x)$ is $f(x)$ itself. In fact (2.25) shows that the matrix of the $\bar{\partial}$ fundamental collineation of ∂f is $-a_{\bar{\alpha}u}$, minus the conjugate transpose of the ∂ fundamental collineation of f. This completes the proof of Theorem 2.2.

Repeating the constructions of Theorem 2.2 we get two sequences of harmonic maps

(2.26) $$f_0(=f) \xrightarrow{\partial} f_1 \xrightarrow{\partial} f_2 \longrightarrow \quad \dots \quad .$$

$$f_0 \xrightarrow{\bar\partial} f_{-1} \xrightarrow{\bar\partial} f_{-2} \longrightarrow \quad \dots \quad .$$

whose image spaces are connected by fundamental collineations. Such sequences will be called <u>harmonic sequences</u>.

The most intersting case is when the k_i's are equal. Then we can combine the sequences into one:

(2.27) $$\dots f_{-2} \underset{\bar\partial}{\overset{\partial}{\rightleftarrows}} f_{-1} \underset{\bar\partial}{\overset{\partial}{\rightleftarrows}} f_0 \underset{\bar\partial}{\overset{\partial}{\rightleftarrows}} f_1 \dots$$

By construction two consecutive spaces $[f_i(x)]$ and $[f_{i+1}(x)]$, $x \in M$, of a harmonic sequence are othogonal.

Example: Let $f: M \to G(1, n+1) = \mathbb{C}P^n$ be a holomorphic map. Classically there is associated to f a unitary framing $\{Z_0, \dots, Z_n\}$ of $\mathbb{C}^n$ such that $Z_0 \dots Z_k$ span the k^{th} osculating space of f. This framing is called the <u>Frenet frame</u> of the curve. Analytically each element of the Frenet frame satisfies

(2.28) $$dZ_\sigma = -\bar a_{\sigma-1}\, \bar\varphi Z_{\sigma-1} + \omega_{\sigma\bar\sigma}\, Z_\sigma + a_\sigma \varphi Z_{\sigma+1}$$

Moreover each Z_σ defines a line bundle over M, or, what is the same, a map $M \to \mathbb{C}P^n$. These line bundles (or maps) form a harmonic sequence. The ∂ fundamental collineations are given by the scalars a_σ , the $\bar\partial$ fundamental collineations by the scalars $\bar{a}_{\sigma-1}$. This sequence has length at most $n+1$ and ends in an antiholomorphic curve $M \to \mathbb{C}P^n$, the polar curve of f.

In the remainder of this paper we will abuse the notation and use ∂ and $\bar\partial$ to denote both the fundamental collineations and the maps ∂ and $\bar\partial$ of (2.21) and (2.22). This should cause no confusion. We will also adopt the convention that capital Roman letters (eg. L,V,W,etc.) we will denote rank ℓ complex subbundles of the trivial bundle $M\times\mathbb{C}^n$ and their associated maps $M \to G(\ell,n)$. We will freely identify these two corresponding objects.

§ 3 Harmonic Sequences

In this section we discuss some of the geometry of harmonic sequences over a Riemann surface and, in particular, over the two-sphere and the torus. We begin with the simplest case, the harmonic sequences of maps $M \to G(1,n) = \mathbb{C}P^{n-1}$. Let

(3.1) $$L_0 \xrightarrow{\partial_0} L_1 \xrightarrow{\partial_1} L_2 \longrightarrow \dots \xrightarrow{\partial_{s-1}} L_s \xrightarrow{\partial_s} \dots$$

be a harmonic sequence where each L_p is a map $M \to G(1,n)$ or, what is the same, a rank one vector bundle (a line bundle) over M. We have seen that the map ∂_p is a holomorphic bundle map:

(3.2) $$L_p \xrightarrow{\partial_p} L_{p+1} \otimes T^{(1,0)}$$

where $T^{(1,0)}$ is the holomorphic cotangent bundle of M. ∂_p has only isolated zeroes. The number of zeroes of ∂_p, counted according to multiplicity, is called the ramification index of ∂_p and will be denoted $r(\partial_p)$. The following formula is well-known [8]

(3.3.a) $$c_1(L_{p+1} \otimes T^{(1,0)}) = c_1(L_p) + r(\partial_p)$$

or

(3.3.b) $$c_1(L_{p+1}) = c_1(L_p) + r(\partial_p) - (2g - 2)$$

where c_1 is the Chern number of the line bundle and g is the genus of M.

On the other hand the Chern class of the line bundle L_p can be computed as follows: Choose a unitary framing

$\{Z_1,\ldots,Z_n\}$ of $\mathbb{C}$ adapted so that span $\{Z_{p-1}\} = L_{p-1}$, span $\{Z_p\} = L_p$ and span $\{Z_{p+1}\} = L_{p+1}$ (To choose such a frame requires the additional assumption that the map L_0 is conformal. However, the result to follow does not depend on this assumption. When we discuss the general case we will not make this assumption).

(3.1) and harmonicity give:

$$dZ_{p-1} = \sum_{\sigma=1}^{p-2} (\ )\varphi Z_\sigma + \omega_{p-1,\overline{p-1}} Z_{p-1} + a_{p-1}\varphi Z_p \quad + \quad (\ )\varphi Z_{p+1} + \sum_{\tau=p+2}^{n} (\ )\varphi Z_\tau$$

$$(3.4) \qquad dZ \quad = \qquad - \ \bar{a}_{p-1}\,\bar{\varphi} Z_{p-1} + \ \omega_{p\bar{p}} Z_p \quad + \quad a_p \varphi Z_{p+1}$$

$$dZ_{p+1} = \sum_{\sigma=1}^{p-2} (\ )\varphi Z_\sigma \quad + \quad (\ )\varphi Z_{p-1} \ - \ \bar{a}_p \bar{\varphi} Z_p + \omega_{p+1,\overline{p+1}}\, Z_{p+1} + \sum_{\tau=p+2}^{n} (\ )\varphi Z_\tau$$

where $\bar{a}_{p-1}$, and a_p are functions representing the $\bar{\partial}$ and ∂ fundamental collineation of L_p . $\omega_{p\bar{p}}$ is the connection 1-form of the bundle L_p. The curvature of L_p can then be computed from the Maurer-Cartan equations of $U(n)$:

$$(3.5) \qquad d\omega_{p\bar{p}} = (-\bar{a}_{p-1}\bar{\varphi}) \wedge (a_{p-1}\ ) \ + \ (a_p\varphi) \wedge (-\bar{a}_p\bar{\varphi})$$

$$= (|a_{p-1}|^2 - |a_p|^2)\varphi \wedge \bar{\varphi} \ .$$

Thus

$$(3.6) \qquad c_1(L_p) = \frac{i}{2\pi} \int_M (|a_{p-1}|^2 - |a_p|^2)\varphi \wedge \bar{\varphi}.$$

Note that from (3.4) it is immediate that the only (0,1) form among the coframing of Z_{p+1} is $\omega_{p+1,\bar{p}} = -\bar{a}_p\bar{\varphi}$. Applying the above reasoning to L_{p+1} we get

$$(3.7)\qquad c_1(L_{p+1}) = \frac{i}{2\pi}\int_M(|a_p|^2 - |a_{p+1}|^2)\varphi\wedge\bar{\varphi}\ ,$$

for some function a_{p+1} representing the ∂ fundamental collineation of L_{p+1} . It follows that

$$(3.8)\qquad \sum_{p=0}^{s} c_1(L_p) = \frac{i}{2\pi}\int_M(|a_{-1}|^2 - |a_s|^2)\varphi\wedge\bar{\varphi}$$

$$\le \frac{i}{2\pi}\int_M(|a_{-1}|^2)\varphi\wedge\bar{\varphi}$$

By (3.3)

$$(3.9)\qquad \sum_{p=0}^{s} c_1(L_p) = \sum_{p=0}^{s}\{c_1(L_0) + \sum_{q=0}^{p-1} r(\partial_q) - p(2g-2)\}$$

$$= (s+1)c_1(L_0) + \sum_{p=0}^{s}\sum_{q=0}^{p-1} r(\partial_q) - (2g-2)\,\frac{s(s+1)}{2}$$

Theorem 3.1

Let

$$(3.1)\qquad L_0 \xrightarrow{\partial_0} L_1 \xrightarrow{\partial_1} L_2 \cdots \xrightarrow{\partial_{s-1}} L_s \xrightarrow{\partial_s} \cdots$$

be a harmonic sequence for the map $L_0:M \to G(1,n)$ where M has genus g and the ramification index of ∂_p is $r(\partial_p)$. Then for any s

$$(3.10)\qquad (s+1)\ c_1(L_0) + \sum_{p=0}^{s}\sum_{q=0}^{p-1} r(\partial_q) - (2g-2)\,\frac{s(s+1)}{2} < \frac{1}{\pi}\cdot\text{energy}(L_0)$$

Proof: The energy of L_0 is $\frac{i}{2}\int_M(|a_{-1}|^2 + |a_0|^2)\varphi \wedge \bar{\varphi}$. Moreover $|a_0| = 0$ if and only if L_0 is antiholomorphic. (equivalently $a_0 = 0$ if and only if $\partial_0 = 0$).

Corollary 3.2

When $g = 0$ the harmonic sequence (3.1) must terminate.

Suppose $g = 0$ and that L_t is the last element of the harmonic sequence (3.1). Then $L_t : M \to G(1,n)$ is an antiholomorphic map. The construction of the harmonic sequence of a holomorphic or antiholomorphic curve in $\mathbb{C}P^{n-1}$ is precisely the classical construction of the curve's Frenet frame. Hence L_0 is an element of the Frenet frame of L_t and we have proved the result of Din-Zakrzewski [6] (For this version of this theorem see [11]).

Now consider the harmonic sequence

$$\text{(3.1.a)} \qquad \xleftarrow{\bar\partial} L_{-s} \xleftarrow{\bar\partial} \cdots \xleftarrow{\bar\partial} L_{-1} \xleftarrow{\bar\partial} L_0$$

The maps

$$L_{-p} \xrightarrow{\partial_{-p}} L_{-p+1} \qquad \text{and} \qquad L_{-p} \xleftarrow{\bar\partial_{-p+1}} L_{-p+1}$$

are adjoints so $r(\partial_{-p}) = r(\bar\partial_{-p+1})$. Thus (3.3) becomes

$$\text{(3.11)} \qquad c_1(L_{-p}) = c_1(L_{-p+1}) - r(\bar\partial_{-p+1}) + (2g-2)$$

So

$$\text{(3.12)} \qquad \sum_{p=0}^{s} c_1(L_{-p}) = (s+1)\, c_1(L_0) - \sum_{p=0}^{s}\sum_{q=0}^{p-1} r(\bar\partial_{-q}) + (2g-2)\frac{s(s+1)}{2}$$

Also

$$c_1(L_{-p}) = \frac{i}{2\pi}\int_M(|a_{-p-1}|^2 - |a_{-p}|^2)\ \varphi\wedge\overline{\varphi}$$

So

$$(3.13)\quad \sum_{p=0}^{s} c_1(L_{-p}) = \frac{i}{2\pi}\int_M(|a_{-s-1}|^2 - |a_0|^2)\varphi\wedge\overline{\varphi}$$

Therefore

$$(3.14)\quad -\frac{1}{\pi}\ \text{energy}\ (L_0) < -\frac{i}{2\pi}\int_M|a_0|^2\ \varphi\wedge\overline{\varphi}$$

$$\leq (s+1)c_1(L_0) - \sum_{p=0}^{s}\sum_{q=0}^{p-1} r(\overline{\partial}_{-q}) + (2g-2)\frac{s(s+1)}{2}$$

Proposition 3.3

When $g = 1$ and $\deg L_0 < 0$ then the harmonic sequence (3.1) must terminate. When $g = 1$ and $\deg L_1 > 0$ or when $g = 0$ then the harmonic sequence (3.1a) must terminate.

Proof: $\deg L_0$ is the degree of the map $L_0 : M \to \mathbb{C}P^{n-1}$. As $\deg L_0 = -c_1(L_0)$ the first statement follows from (3.10) and the second statement follows from (3.14).

Thus when $g = 1$ and $\deg L \neq 0$ there is a terminal element to the left or the right of the harmonic sequence

$$(3.15)\quad \ldots \longleftarrow L_{-1} \xleftarrow{\overline{\partial}} L_0 \xrightarrow{\partial} L_1 \xrightarrow{\partial} \ldots$$

Suppose, without loss of generality, that L_{-t}, $t > 0$, is the terminal element. Then $L_{-t} : M \to \mathbb{C}P^{n-1}$ is a holomorphic curve and the harmonic map L_0 occurs as an element of the Frenet frame of L_{-t}. This result was first proved by Eells and Wood [7].

We remark that if a harmonic sequence (3.15) terminates in one direction then it must terminate in the other direction and it contains at most n elements. This is an immediate consequence of the construction of the Frenet frame of a holomorphic or antiholomorphic curve in $\mathbb{C}P^{n-1}$.

We now turn to the general case of a harmonic sequence

$$(3.16)\quad V_0 \xrightarrow{\partial_0} V_1 \xrightarrow{\partial_1} \cdots \xrightarrow{\partial_{s-1}} V_s \xrightarrow{\partial_s} \cdots$$

$$(3.16a)\quad \cdots \xleftarrow{\bar\partial_{-s}} V_{-s} \cdots \xleftarrow{\bar\partial_{-1}} V_{-1} \xleftarrow{\bar\partial_0} V_0$$

where each V_p is a map $M \to G(k,n)$ or a rank k vector bundle over M. We would like to find conditions under which one of the ∂ or $\bar\partial$ fundamental collineations degenerates, that is, has rank less than k.

We can change the sequence (3.16) into a sequence of line bundles by taking the k^{th} exterior power of each bundle:

$$(3.17)\quad \Lambda^k V_0 \xrightarrow{\det\partial_0} \Lambda^k V_1 \xrightarrow{\det\partial_1} \Lambda^k V_2 \longrightarrow \cdots \xrightarrow{\det\partial_{s-1}} \Lambda^k V_s \longrightarrow \cdots$$

In (3.17) the map $\det \partial_p$ is a holomorphic bundle map

$$(3.18)\quad \Lambda^k V_p \xrightarrow{\det\partial_p} \Lambda^k V_{p+1} \otimes (T^{(1,0)})^k$$

Formula (3.3) can be written

$$(3.19)\qquad c_1(\Lambda^k V_{p+1}) = c_1(\Lambda^k V_p) + r(\det \partial_p) - k(2g-2)$$

We remark that (3.19) is a "Plucker formula" for harmonic maps $M \rightarrow G(k,n)$.

The Chern number $c_1(\Lambda^k V_p)$ can be computed as follows: First, it is an elementary and basic fact of k-plane bundles that if the connection form of V_p is given by $(\pi_{\alpha\beta})\, 1 \leq \alpha,\beta \leq k$, then the connection form of $\Lambda^k V_p$ is given by $\operatorname{tr}(\pi_{\alpha\beta}) = \sum_{\alpha=1}^{k} \pi_{\alpha\alpha}$. Thus

$$(3.20)\qquad c_1(V_p) = c_1(\Lambda^k V_p)$$

To compute $c_1(V_p)$ we adapt a unitary framing $\{Z_1 \dots Z_n\}$ of $\mathbb{C}^n$ to the map V_p as in § 1, that is the vectors Z_α span V_p, where $1 \leq \alpha,\beta \leq k$. Then we have

$$d\begin{pmatrix} Z_1 \\ \vdots \\ Z_k \\ \vdots \\ Z_n \end{pmatrix} = \begin{pmatrix} \pi_p & A_p\varphi + B_p\overline{\varphi} \\ \underline{t}\,\overline{A}_p\overline{\varphi} - {}^t\overline{B}_p\varphi & \cdots \end{pmatrix} \begin{pmatrix} Z_1 \\ \vdots \\ Z_k \\ \vdots \\ Z_n \end{pmatrix}$$

where π_p is a $k \times k$ skew-hermitian matrix of 1-forms and A_p and B_p are $k \times (n-k)$ matrices of functions. In fact in the notation of § 2

$$\pi_p = (\omega_{\alpha\bar{\beta}})$$

$$A_p\varphi + B_p\bar{\varphi} = (\omega_{\alpha\bar{I}})$$

$$A_p = (a_{\alpha\bar{I}}),\ B_p = (b_{\alpha\bar{I}}).$$

π_p is the connection 1-form of V_p. By the Maurer-Cartan equations, the curvature of V_p is

$$d\pi_p - \pi_p \wedge \pi_p = (-A_p{}^t\bar{A}_p + B_p^t\bar{B}_p)\varphi \wedge \bar{\varphi}\ .$$

Thus

$$\text{(3.21)} \quad \begin{aligned} c_1(V_p) &= \frac{i}{2\pi} \int \operatorname{tr}\,(d\pi_p - \pi_p \wedge \pi_p) \\ &= \frac{i}{2\pi} \int [\operatorname{tr}\,(B_p^t\bar{B}_p) - \operatorname{tr}(A_p^t\bar{A}_p)]\varphi \wedge \bar{\varphi} \end{aligned}$$

Recall that the energy of the map $M \to G(k,n)$ determined by V_p is given by

$$\text{(3.23)} \quad \begin{aligned} E(V_p) &= \frac{i}{2} \int (\sum_{\alpha,j} |a_{\alpha\bar{j}}|^2 + \sum_{\alpha,j} |b_{\alpha\bar{j}}|^2)\varphi \wedge \bar{\varphi} \\ &= \frac{i}{2} \int (\operatorname{tr}(A_p^t\bar{A}_p) + \operatorname{tr}(B_p^t\bar{B}_p))\varphi \wedge \bar{\varphi}\ . \end{aligned}$$

We define the holomorphic or ∂ energy of V_p by

$$\text{(3.24)} \quad E(\partial_p) = \frac{i}{2}\int \operatorname{tr}(A_p^t\bar{A}_p)\varphi \wedge \bar{\varphi}.$$

Similarly the antiholomorphic or $\bar{\partial}$ energy of V_p is by definition

$$\text{(3.25)} \quad E(\bar{\partial}_p) = \frac{i}{2}\int \operatorname{tr}\,(B_p^t\bar{B}_p)\varphi \wedge \bar{\varphi}\ .$$

Thus

(3.26) $E(V_p) = E(\partial_p) + E(\bar{\partial}_p)$,

and

(3.27) $c_1(V_p) = \frac{1}{\pi} E(\bar{\partial}_p) - \frac{1}{\pi} E(\partial_p)$.

Now consider the ∂-transform of V_p , namely V_{p+1}. We have, by the above argument

(3.28) $c_1(V_{p+1}) = \frac{1}{\pi} E(\bar{\partial}_{p+1}) - \frac{1}{\pi} E(\partial_{p+1})$,

where $\bar{\partial}_{p+1}$ and ∂_{p+1} are the $\bar{\partial}$ and ∂ transforms, respectively, of V_{p+1}. Recall Theorem 2.2(c). This result says that the $\bar{\partial}_{p+1}$ transform and ∂_p transform are "inverse" operations. It is an immediate consequence of the proof of this result that

(3.29) $E(\bar{\partial}_{p+1}) = E(\partial_p)$

Thus

(3.30) $c_1(V_{p+1}) = \frac{1}{\pi} E(\partial_p) - \frac{1}{\pi} E(\partial_{p+1})$

Hence we have

(3.31) $$\sum_{p=0}^{s} c_1(\Lambda^k V_p) = \frac{1}{\pi} E(\bar{\partial}_0) - \frac{1}{\pi} E(\partial_s)$$

$$< \frac{1}{\pi} E(V_0)$$

Combining (3.19) and (3.31) we have

Theorem 3.4

Let

$$(3.16)\quad V_0 \xrightarrow{\partial_0} V_1 \xrightarrow{\partial_1} V_2 \longrightarrow \cdots \xrightarrow{\partial_{s-1}} V_s \xrightarrow{\partial_s} \cdots$$

be a harmonic sequence for the map $V_0 : M \to G(k,n)$ where M has genus g and suppose that none of the fundamental collineations degenerates. Then for any s

$$(3.32)\quad (s+1)c_1(\Lambda^k V_0) + \sum_{p=0}^{s} \sum_{q=0}^{p-1} r(\det \partial_q) - k(2g-2)\frac{s(s+1)}{2}$$

$$= \frac{1}{\pi}(E(\bar{\partial}_0) - E(\partial_s))$$

$$< \frac{1}{\pi} E(V_0)$$

Remarks:

Note that by (2.5), (3.20) and (3.21) $c_1(\Lambda^k V_0) = -\deg(V_0)$, where $\deg V_0$ is the degree of the map $M \to G(k,n)$ induced by V_0. Consequently the inequality (3.32) measures the difference between the degree and energy of a harmonic map $M \to G(k,n)$.

Corollary 3.5

If $g = 0$ or if $g = 1$ and $c_1(\Lambda^k V_0) > 0$ then the harmonic sequence (3.16) must have a degenerate ∂ fundamental collineation.

Consider the harmonic sequence (3.16a). Following the above arguments it is a simple matter to prove the following result.

Theorem 3.6

Let (3.16a) be a harmonic sequence for the map $V_0 : M \to G(k,n)$ where M has genus g and suppose that none of the fundamental collineations degenerates. Then for any s

$$(3.33) \qquad -\frac{1}{\pi} E(V_0) < -\frac{1}{\pi} E(\partial_0)$$

$$\leq (s+1)c_1(\Lambda^k V_0) - \sum_{p=0}^{s} \sum_{\sigma=0}^{p-1} r(\det \bar{\partial}_{-\sigma}) + k(2g-2)\frac{s(s+1)}{2}$$

Corollary 3.7

If $g = 0$ or if $g = 1$ and $c_1(\Lambda^k V_0) < 0$ then the harmonic sequence (3.16a) must have a degenerate $\bar{\partial}$ fundamental collineation.

Since $c_1(\Lambda^k V_0) = -\deg(V_0)$, we have:

Theorem 3.8

If $g = 0$ or if $g = 1$ and $\deg V_0 \neq 0$ then the harmonic sequence generated by V_0 has a degenerate fundamental collineation.

In fact we have proved more. If M has genus 1 and V_0 is a harmonic map $M \to G(k,n)$ then the harmonic sequence generated by V_0 must have a degenerate fundamental collineation if any of the ∂ and $\bar{\partial}$ transforms of V_0 have non-zero degree. This means that the only harmonic sequences over the torus that we cannot prove have a degenerate fundamental collineation are those such that every map in the sequence has degree zero. Note that by (3.19) every fundamental collineation of such a sequence has ramification index zero. In $\mathbb{C}P^n$ every non superminimal minimal torus belongs to such a sequence. In particular, the Clifford torus in $\mathbb{C}P^2$ generates a cyclic harmonic sequence consisting of three maps all of degree zero.

Summarizing we record the following result about harmonic sequences.

<u>Theorem 3.9</u>

Let

$$V_{-t} \xleftarrow{\bar{\partial}_{-t+1}} \cdots \longleftarrow V_{-1} \xleftarrow{\bar{\partial}_0} V_0 \xrightarrow{\partial_0} V_1 \longrightarrow \cdots \xrightarrow{\partial_{s-1}} V_s$$

be a harmonic sequence and suppose that, for $-t \leq p \leq s$, V_p is a map $M \to G(k,n)$ where M is a surface of genus g. Then

(1) $\deg V_{p+1} = \deg V_p - r(\det \partial_p) + k(2g-2) \quad 0 \leq p \leq g-1$

(2) $\deg V_{-(p+1)} = \deg V_{-p} + r(\det \bar{\partial}_{-p}) - k(2g-2) \quad 0 \leq p \leq t-1$

(3) $\sum_{p=0}^{s} \deg V_p > -\frac{1}{\pi}\,\text{energy}\,(V_0)$

(4) $\sum_{p=0}^{t} \deg V_{-p} < \frac{1}{\pi}\,\text{energy}\,(V_0)$.

§ 4 Turning and harmonic maps of the two-sphere

In this section we study the degenerate harmonic maps, that is, the harmonic maps one of whose fundamental collineations is degenerate. For use later we order the Grassmann manifolds as follows. We say $G(\ell,n)$ is "smaller" than $G(k,n)$ if $\ell < k$.

Let V_0 be a harmonic map $M \to G(k,n)$ regarded as a rank k bundle. Suppose that the ∂ fundamental collineation is singular of rank ℓ where $0 < \ell < k$. Let W_0 denote the harmonic map $M \to G(\ell,n)$ determined by the image of ∂. Then have

(4.1) $$V_0 \xrightarrow{\partial} W_0$$

The vector bundle V_0 decomposes as the orthogonal direct sum of the rank $(k - \ell)$ bundle ker ∂ and the rank ℓ bundle $W_{-1} = (\ker \partial)^{\perp}$. By Theorem 2.2(b) W_{-1} describes a harmonic map $M \to G(\ell,n)$. In fact W_{-1} is the $\bar{\partial}$-transform of W_0. Let W_{-2} denote the $\bar{\partial}$-transform of W_{-1}. Define the bundle V^1 by

$$V^1 = \text{span}\,\{W_{-2}, \ker \partial\}$$

Note that in general W_{-2} and ker ∂ are not orthogonal. However we have

Lemma 4.1

V^1 is a vector bundle (i.e. V^1 has constant rank).

To prove the lemma we need the following proposition which will be used implicitly in § 5

Proposition 4.2

(1) The bundle $\ker \partial$ is a holomorphic subbundle of V_0

(2) The bundle W_{-1} is an antiholomorphic subbundle of V_0

Proof:

Because $\ker \partial \oplus W_{-1} = V_0$ the two statements in the proposition are equivalent. We will prove the first statement. Choose a unitary framing $\{Z_1, \dots, Z_n\}$ of $\mathbb{C}^n$ adapted so that Z_σ span $\ker \partial$ and Z_r span W_{-1}, where the indices have the ranges

$$1 \le \sigma,\tau \le k-\ell \ , \ k-\ell+1 \le r,s \le k \ , \ k+1 \le i,j \le n.$$

Then $a_{\sigma\bar{i}} = 0$ and the matrix $(a_{r\bar{i}})$ has rank ℓ. Since V_0 is harmonic, by Theorem 2.1 it follows that

$$\omega_{\sigma\bar{r}} \, a_{r\bar{i}} \equiv 0 \mod \varphi$$

This implies that

$$\omega_{\sigma\bar{r}} \equiv 0 \mod \varphi$$

Hence

$$dZ_\sigma \equiv 0 \mod Z_\tau, Z_i, \varphi \, .$$

Proof of the lemma:

Let $\bar{\partial}_{V_0}$ denote the $\bar{\partial}$ fundamental collineation of V_0

and $\bar{\partial}_{V_0}(W_{-1})$ denote the image of W_{-1} under $\bar{\partial}_{V_0}$. Then

$$V^1 = \bar{\partial}_{V_0}(W_{-1}) \oplus \ker \partial .$$

Since W_{-1} is an antiholomorphic subbundle, the map $\bar{\partial}_{V_0}$ restricted to W_{-1} can be regarded as an antiholomorphic map. Thus $\bar{\partial}_{V_0}(W_{-1})$ has constant rank.

Theorem 4.3.

The bundle V^1 gives a harmonic map $M \to G(k_1,n)$ where $k_1 \leq k$. If $k_1 = k$ then the ∂-transform of V^1 is W_{-1} and

$$V^1 \xrightarrow{\partial} W_{-1} \underset{\bar{\partial}}{\overset{\partial}{\rightleftarrows}} W_0 \tag{4.2}$$

is a harmonic sequence. If $k_1 < k$ then the ∂-transform of V^1 lies inside W_{-1}.

Proof: Left to the reader

The construction of (4.2) is called turning. This construction generalizes the construction of the same name described in [5].

Remarks:

(1) If $k_1 \geq \ell$ then "generically" the ∂-transform of V^1 is W_{-1} and similarly if $k_1 \leq \ell$ the $\bar{\partial}$ transform of W_{-1} is "generically" V^1 For this reason we call a turning regular if

(a) The ∂ transform of V^1 is W_{-1} when $k_1 \geq \ell$

(b) The $\bar{\partial}$ transform of W_{-1} is V when $k_1 \leq \ell$

Theorem 4.1 says that if $k_1 = k$ then the turning is regular.

(2) It is interesting (and important) to determine how to reverse the operation of turning, that is, now to recover the map V_0 from the map V^1. V^1 is a holomorphic rank k_1 bundle over M where by construction $k_1 \geq (k - \ell)$. Choose an antiholomorphic rank $(k - \ell)$ subbundle B of V^1. Then the bundle $B \oplus W_{-1}$ has rank k and its ∂ transform is W_0. For appropriate choice of B this bundle will be V_0. This operation is called <u>returning</u>. Note that when the turning is regular, the returning depends on V^1 and the choice of B, alone (because in this case W_{-1} is determined by V^1). Whereas when the turning is not regular the returning depends on V^1 , the choice of B, and W_{-1}.

It is clear that the construction of turning can be iterated to construct the sequence

$$(4.3) \qquad V^s \xrightarrow{\partial} W_{-s} \xrightarrow{\partial} W_{-s+1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} W_0.$$

Suppose that V^s is a rank k_s bundle where $k_s < k$ and that each $V^s, \sigma < s$, constructed before V^s is a rank k bundle. If the final turning is regular then V_0 can be constructed from V^s by a sequence of returnings. If the final turning is not regular then V_0 can be constructed from V^s and W_{-s} through a sequence of returnings. In both cases note that the harmonic map $V_0 : M \to G(k,n)$ can be constructed, by returnings, from harmonic maps of M into smaller Grassmann manifolds. In the non-generic (that is, the not regular) case more data (namely, W_{-s}) is required to reconstruct V_0.

Theorem 4.2

Let V_0 be a harmonic map $M \to G(k,n)$. Let W_0 denote the ∂ transform of V_0 and suppose that W_0 is a bundle of rank $\ell, \ell < k$. If M has genus zero or if M has genus one and the map W_0 has positive degree then V_0 can be constructed by returnings from maps of M into smaller Grassmann manifolds.

Proof: The hypothesis on M insure that the $\bar{\partial}$ harmonic sequence of W_0 must contain a singular $\bar{\partial}$ fundamental collineation. This in turn insures that some V^s has rank strictly less than k.

By combining Theorem 3.8 and Theorem 4.2 we have:

Theorem 4.3

If M has genus zero then any harmonic map $M \to G(k,n)$ can be constructed from either:

(1) a holomorphic or antiholomorphic curve $M \to G(k,n)$ using the ∂ or $\bar{\partial}$ transforms, or

(2) one, or possibly two harmonic maps $M \to G(k_i,n)$ $i = 1,2,$

where $k_i < k$;

using the ∂ and $\bar{\partial}$ transforms and using returnings.

Now by induction, we have

Corollary 4.4

If M has genus zero then any harmonic map $M \to G(k,n)$ can be constructed from holomorphic or antiholomorphic curves $M \to G(\ell,n)$, $1 \leq \ell \leq k$, using the ∂ and $\bar{\partial}$ transforms and returnings.

We remark that turning and returning can be formulated for the case of a harmonic map V_0 with degenerate $\bar{\partial}$ fundamental collineation. We leave this to the reader.

§ 5 Extending and harmonic maps of the two-sphere and the torus

We begin by describing another technique which, like returning, reconstructs a harmonic map from its degenerate ∂-transform (or $\bar{\partial}$-transform).

Using the same notation as in Section 4 we let V_0 denote a harmonic map $M \to G(k,n)$ with degenerate ∂ fundamental collineation and W_0 denote the ∂ transform of V_0, so that W_0 is a harmonic map $M \to G(\ell,n)$, $0 < \ell < k$. By Theorem 2.2(a) the map $W_0^{\perp}$ determined by the space orthogonal to W_0 is also harmonic. $W_0^{\perp}$ is a holomorphic vector bundle over M. Let W_{-1} denote the $\bar{\partial}$-transform of W_0. W_{-1} is a rank ℓ antiholomorphic subbundle of $W_0^{\perp}$. Now choose an antiholomorphic rank k subbundle V of $W_0^{\perp}$ satisfying the condition that W_{-1} is an antiholomorphic subbundle of V. A straightforward local computation shows that the map $M \to G(k,n)$ defined by V is harmonic. Moreover, for appropriate choice of V we have $V = V_0$. This operation is called <u>extending</u> (The bundle V "extends" the bundle W_{-1}).

Suppose V_0 has a degenerate $\bar{\partial}$ fundamental collineation and U_0 denotes its $\bar{\partial}$ transform. Let U_1 denote the ∂ transform of U_0. Then to "extend" U_1 we choose a rank k holomorphic subbundle V of $U_0^{\perp}$ satisfying the condition that U_1 is a holomorphic subbundle of V. Again V describes a harmonic map $M \to G(k,n)$ and for appropriate choice of V

we have $V = V_0$.

We have

Theorem 5.1

If M has genus zero then any harmonic map $M \to G(k,n)$ can be constructed from one holomorphic (or one antiholomorphic) curve $M \to G(\ell,n)$, $1 \leq \ell \leq k$, using the ∂ and $\bar{\partial}$ transforms and extendings.

Proof: Apply Corollary 3.7 (respectively, Corollary 3.5) repeatedly. We can also use extending to give the following description of the space of harmonic maps of the torus into $G(k,n)$.

Theorem 5.2 A harmonic map of a surface M of genus one into $G(k,n)$ can be constructed using the ∂ and $\bar{\partial}$ transforms and extendings from either

(1) a holomorphic or antiholomorphic curve $M \to G(\ell,n)$ $1 \leq \ell \leq k$

or

(2) a degree zero harmonic map $M \to G(\ell,n)$, $1 \leq \ell \leq k$.

In fact in case (2) the degree zero map can be taken to be an element of a harmonic sequence consisting only of degree zero harmonic maps.

Proof: Apply Theorem 3.8 repeatedly

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