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COMBINATORIAL RANK OF QUANTUM GROUPS OF INFINITE SERIES

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Abstract. We demonstrate that the combinatorial rank of the multiparameter version of the Lusztig "small" quantum group $u_q(\mathfrak{sp}_{2n})$, or, equivalently, of the Frobenius-Lusztig kernel of type C_n, equals $\lfloor \log_2(n-1) \rfloor + 2$ provided that q has a finite multiplicative order $t > 3$. It is known that the combinatorial rank of the multiparameter version of the Frobenius-Lusztig kernel of type A_n equals $\lfloor \log_2 n \rfloor + 1$, whereas in case B_n it is equal to $\lfloor \log_2(n-1) \rfloor + 2$, and in case D_n it is $\lfloor \log_2(n-2) \rfloor + 2$.

1. Introduction

In this article, we continue investigation of the combinatorial rank of Lusztig "small" quantum groups $u_q(\mathfrak{g})$ started in [4, 5, 6]. The combinatorial rank of a character Hopf algebra H generated by skew primitive elements $a_1, a_2 \ldots, a_n$ is the length of the sequence of bi-ideals

$$0 = J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_i \subset \ldots \subset J_\kappa = \ker \varphi,$$

where φ is the natural epimorphism $G\langle X \rangle \to H$, $x_i \to a_i$ of the free character Hopf algebra, and J_{i+1}/J_i is an ideal generated by all skew primitive elements of $\ker \varphi / J_i$.

We show that the combinatorial rank of the multiparameter version of the Lusztig "small" quantum group $u_q(\mathfrak{sp}_{2n})$, equals $\lfloor \log_2(n-1) \rfloor + 2$ provided that q has a finite multiplicative order $t > 3$. It is known from [5] that the combinatorial rank of $u_q(\mathfrak{sl}_{n+1})$ equals $\lfloor \log_2 n \rfloor + 1$, whereas in case $\mathfrak{g} = \mathfrak{so}_{2n+1}$ it has the same value $\lfloor \log_2(n-1) \rfloor + 2$ (see [6]), and in case $\mathfrak{g} = \mathfrak{so}_{2n}$ it equals $\lfloor \log_2(2n-3) \rfloor + 1 = \lfloor \log_2(n-2) \rfloor + 2$ (see [4]).

The method of calculation is based on the recently descovered explicit coproduct formula (see [7]), and it is very similar to that of previous papers [4, 5, 6]. An important difference is that in case C_n, the PBW generators $v[k, \phi(k)]$ of the explicit coproduct formula are not defined by Lyndon–Shirshov words. At the same time, the subalgebra of constants, which is very important for calculations, is generated by powers of bracketed Lyndon–Shirshov words. In fact, when t is even, the elements $v[k, \phi(k)]^{t/2}$ not always are constants. By this reason, in Section 5, we have to

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develop specific calculations for bracketed Lyndon-Shirshov words $[v_k]$ that do not appear in the coproduct formula.

In the second, third and fourth sections, we briefly recall the main concepts and basic statements which are of use later. In the fifth section we make specific calculations and demonstrate that $U_q^+(\mathfrak{sp}_{2n})$ and $u_q^+(\mathfrak{sp}_{2n})$ have the same bracketed Lyndon-Shirshov words that define the PBW bases. Then, in sixth section, we prove that every intermediate biideal is generated by powers of that bracketed words.

In the seventh section, in perfect analogy with [4, Section 8], we demonstrate that $U_q^+(\mathfrak{so}_{2n})$ is a finite extension of a skew central Hopf subalgebra GC of quantum polynomials in fixed powers of the PBW generators $y_i = [w]^{t_w}$. The elements y_i are precisely the generators of the subalgebra C of constants with respect to the non-commutative differential calculus that naturally arise on the subalgebra generated by x_i, $1 \leq i \leq n$. In the one parameter case considered by C. De Concini, V.G. Kac, and C. Procesi in [2, 3], the subalgebra C is central, and $GC = GZ_0$, where Z_0 is the smallest subalgebra invariant with respect to Lusztig braid group action containing all x_i^t, g_i^t, $1 \leq i < n$, $x_n^{t'}$, $g_n^{t'}$. Here $t' = t$ if the parameter q has odd multiplicative order, and $t' = t/2$ otherwise. We show that each Hopf ideal J_j, $j > 1$ is generated by some of the elements y_i. This allows us to find the combinatorial rank in Sections 8 using the explicit coproduct formula.

2. Preliminaries

In this section, we collect some known results on the structure of an arbitrary character Hopf algebra. Recall that a Hopf algebra H is referred to as a *character* Hopf algebra if the group G of all grouplike elements is commutative and H is generated over $\mathbf{k}[G]$ by skew primitive semi-invariants a_i, $i \in I$:

$$\Delta(a_i) = a_i \otimes 1 + g_i \otimes a_i, \quad g^{-1}a_i g = \chi^i(g)a_i, \quad g, g_i \in G, \tag{2.1}$$

where χ^i, $i \in I$ are characters of the group G.

2.1. **Skew brackets.** Let us associate a "quantum" variable x_i to a_i. For each word u in $X = \{x_i \,|\, i \in I\}$, let g_u or $\mathrm{gr}(u)$ denote an element of G that appears from u by replacing each x_i with g_i. In the same way, χ^u denotes a character that appears from u by replacing each x_i with χ^i. We define a bilinear skew commutator on homogeneous linear combinations of words in a_i or in x_i, $i \in I$ by the formula

$$[u, v] = uv - \chi^u(g_v)vu, \tag{2.2}$$

where we sometimes use the notation $\chi^u(g_v) = p_{uv} = p(u, v)$. Of course, $p(u, v)$ is a bimultiplicative map:

$$p(u, vt) = p(u, v)p(u, t), \quad p(ut, v) = p(u, v)p(t, v). \tag{2.3}$$

The brackets satisfy the following Jacobi identity:

$$[[u, v], w] = [u, [v, w]] + p_{wv}^{-1}[[u, w], v] + (p_{vw} - p_{wv}^{-1})[u, w] \cdot v. \tag{2.4}$$

The Jacobi identity (2.4) implies the following conditional identity

$$[[u, v], w] = [u, [v, w]], \text{ provided that } [u, w] = 0. \tag{2.5}$$

The brackets are related with the product by the following ad-identities

$$[u \cdot v, w] = p_{vw}[u, w] \cdot v + u \cdot [v, w], \tag{2.6}$$

$$[u, v \cdot w] = [u, v] \cdot w + p_{uv}v \cdot [u, w]. \tag{2.7}$$

2.2. **Radford biproduct and the ideal $\mathbf{\Lambda}$.** The group G acts on the free algebra $\mathbf{k}\langle X\rangle$ by $g^{-1}ug = \chi^u(g)u$, where u is an arbitrary monomial in X. The skew group algebra $G\langle X\rangle$ has the natural Hopf algebra structure

$$\Delta(x_i) = x_i \otimes 1 + g_i \otimes x_i, \quad i \in I, \quad \Delta(g) = g \otimes g, \ g \in G.$$

We fix a Hopf algebra homomorphism

$$\xi : G\langle X\rangle \to H, \quad \xi(x_i) = a_i, \quad \xi(g) = g, \quad i \in I, \quad g \in G. \tag{2.8}$$

If the kernel of ξ is contained in the ideal $G\langle X\rangle^{(2)}$ generated by x_ix_j, $i, j \in I$, then there exists a Hopf algebra projection $\pi : H \to \mathbf{k}[G]$, $a_i \to 0$, $g_i \to g_i$. Hence, by the Radford theorem [14], we have a decomposition in a biproduct, $H = A{\star}\mathbf{k}[G]$, where A is a subalgebra generated by a_i, $i \in I$ (see [1, §1.5, §1.7]).

Definition 2.1. In what follows, $\mathbf{\Lambda}$ denotes the biggest Hopf ideal in $G\langle X\rangle^{(2)}$, where, as above, $G\langle X\rangle^{(2)}$ is the ideal of $G\langle X\rangle$ generated by x_ix_j, $i, j \in I$. The ideal $\mathbf{\Lambda}$ is homogeneous in each $x_i \in X$ (see [5, Lemma 2.2]).

The algebra A has the structure of a *braided Hopf algebra*, [17], with a braiding $\tau(u\otimes v) = p(v, u)^{-1}v\otimes u$. If $\mathrm{Ker}\,\xi = \mathbf{\Lambda}$ than A is a Nichols algebra [1, §1.3, Section 2] or, equivalently, a quantum symmetric algebra.

2.3. **Differential calculi.** The free algebra $\mathbf{k}\langle X\rangle$ has two closely related differential calculi defined by the following Leibniz rules:

$$\partial_j(x_i) = \delta_i^j, \quad \partial_i(uv) = \partial_i(u) \cdot v + \chi^u(g_i)u \cdot \partial_i(v). \tag{2.9}$$

$$\partial_j^*(x_i) = \delta_i^j, \quad \partial_i^*(uv) = p(x_i, v)\partial_i^*(u) \cdot v + u \cdot \partial_i^*(v). \tag{2.10}$$

Lemma 2.2. ([12, Lemma 2.10]). *Let $u \in \mathbf{k}\langle X\rangle$ be an element homogeneous in each x_i, $1 \le i \le n$. If p_{uu} is a t^{th} primitive root of 1, then*

$$\partial_i(u^t) = p(u, x_i)^{t-1}\,\underbrace{[u, [u, \ldots [u}_{t-1}, \partial_i(u)] \ldots]]. \tag{2.11}$$

The partial derivatives and coproduct are related by

$$\Delta(\partial_i(u)) = \sum_{(u)} g_i^{-1}u^{(1)} \otimes \partial_i(u^{(2)}), \tag{2.12}$$

where we use the Sweedler notations $\Delta(u) = \sum_{(u)} u^{(1)}\otimes u^{(2)}$. Recall that the *braided antipode* $\sigma^b : \mathbf{k}\langle X\rangle \to \mathbf{k}\langle X\rangle$ acts on a word u as follows

$$\sigma^b(u) = \mathrm{gr}(u)\sigma(u),$$

where σ is the antipode of $G\langle X\rangle$. The braided antipode satisfies

$$\sigma^b([u, v]) \sim [\sigma^b(v), \sigma^b(u)] \tag{2.13}$$

and

$$\sigma^b(\partial_i^*(u)) \sim \partial_i(\sigma^b(u)), \tag{2.14}$$

where $\sim$ is the projective equality: $a \sim b \iff a = \alpha b, \ 0 \neq \alpha \in \mathbf{k}$.

Lemma 2.3. (Milinski—Schneider criterion). *Suppose that $Ker\,\xi = \mathbf{\Lambda}$. If a polynomial $f \in \mathbf{k}\langle X\rangle$ is a constant in A (that is, $\partial_i(f) \in \mathbf{\Lambda}$, $i \in I$), then there exists $\alpha \in \mathbf{k}$ such that $f - \alpha = 0$ in A.*

See details in [4, Section 2].

3. Combinatorial representation

Recall that the quantum groups $U_q(\mathfrak{sp}_{2n})$, $u_q(\mathfrak{sp}_{2n}$, are generated as $\mathbf{k}$-algebras by the grouplike elements

$$g_1, g_2, \dots, g_n, f_1, f_2, \dots, f_n, \quad \Delta(g_i) = g_i \otimes g_i, \quad \Delta(f_i) = f_i \otimes f_i,$$

by their inverses, and by the skew primitive elements $x_1, x_2, \dots, x_n, x_1^-, x_2^-, \dots, x_n^-$,

$$\Delta(x_i) = x_i \otimes 1 + g_i \otimes x_i; \ \Delta(x_i^-) = x_i^- \otimes 1 + f_i \otimes x_i^-.$$

Whereas all the grouplike elements commute each with other, the skew primitive generators commute with the grouplikes via

$$x_i g_j = p_{ij} g_j x_i, \ x_i^- g_j = p_{ij}^{-1} g_j x_i^-, \ x_i f_j = p_{ji} f_j x_i, \ x_i^- f_j = p_{ji}^{-1} f_j x_i^-,$$

where p_{ij} are arbitrary parameters satisfying the following relations

$$p_{ii} = q, \ 1 \leq i < n; \ \ p_{i\,i-1} p_{i-1\,i} = q^{-1}, \ 1 < i < n; \ \ p_{ij} p_{ji} = 1, \ j > i+1;$$
$$p_{nn} = q^2 \quad \ p_{n-1\,n}\, p_{n\,n-1} = q^{-2}.$$

The group G of the grouplike elements may satisfy arbitrary additional relations that are compatible with the above commutation rules (for example, if $p_{ij} = p_{ji}$ for all i, j then one may additionally suppose that $f_i = g_i$).

Let $G\langle X \cup X^- \rangle$ be the Hopf algebra defined as above with free skew primitives x_i, x_i^-. Then we have Hopf algebra morphisms

$$\varphi : G\langle X \cup X^- \rangle \to U_q(\mathfrak{sp}_{2n}), \quad \varphi_1 : G\langle X \cup X^- \rangle \to u_q(\mathfrak{sp}_{2n}).$$

By its definition, the ideal $\ker \varphi$ is generated by the elements

$$\mu_{ij} \stackrel{df}{=} x_i x_j^- - p_{ji} x_j^- x_i - \delta_i^j (1 - g_i f_i)$$

and the quantum Serre polynomials

$$S_{ij}(x_i, x_j), \quad S_{ij}(x_i^-, x_j^-), \quad 1 \leq i, j \leq n.$$

It is important here that all of these elements are skew primitive in $G\langle X \cup X^- \rangle$; see [11, Theorem 6.1]. In the explicit form the polynomials $S_{ij}(x_i, x_j)$ are:

$$[x_i, [x_i, x_{i+1}]] = [[x_i, x_{i+1}], x_{i+1}] = 0, \ 1 \leq i < n-1; \ \ [x_i, x_j] = 0, \ \ j > i+1; \tag{3.1}$$

$$[[x_{n-1}, x_n], x_n] = [x_{n-1}, [x_{n-1}, [x_{n-1}, x_n]]] = 0. \tag{3.2}$$

The ideal $\ker \varphi_1$ is generated by μ_{ij} and the two sets $\mathbf{\Lambda}$, $\mathbf{\Lambda}^-$, see Subsection 2.2.

4. Hard super-letters and PBW basis

We shall concentrate on the positive quantum Borel subalgebra, the subalgebra generated over G by by the x_i's.

On the set of all words in the x_i's we fix the lexicographical order with the priority from the left to the right considering $x_1 > x_2 > \ldots > x_n$, where a proper beginning of a word is considered to be greater than the word itself.

A non-empty word u is called a *standard* word (Lyndon word, Lyndon-Shirshov word) if $vw > wv$ for each decomposition $u = vw$ with non-empty v, w. A *nonassociative* word is a word where brackets $[,]$ somehow arranged to show how multiplication applies. If $[u]$ denotes a nonassociative word, then u denotes an associative word obtained from $[u]$ by removing the brackets. The set of *standard nonassociative* words is the biggest set SL that contains all variables x_i and satisfies the following properties.

1) If $[u]=[[v][w]]\in SL$, then $[v],[w]\in SL$, and $v>w$ are standard.
2) If $[u]=[\,[[v_1][v_2]]\,[w]\,]\in SL$, then $v_2\leq w$.

Every standard word has only one arrangement of brackets such that the appeared nonassociative word is standard (Shirshov theorem [15]). In order to find this arrangement one may use the following inductive procedure:

Algorithm. The factors v,w of the nonassociative decomposition $[u]=[[v][w]]$ are the standard words such that $u=vw$ and v has the minimal length ([16], see also [13]).

Definition 4.1. A *super-letter* is a polynomial that equals a nonassociative standard word where the brackets are defined as follows $[u,v]=uv-p(u,v)vu$, while $p(u,v)$ is a bimultiplicative map, see (2.3), defined on words so that $p(x_i,x_j)=p_{ij}$. A *super-word* is a word in super-letters.

By Shirshov's theorem every standard word u defines only one super-letter, in what follows we shall denote it by $[u]$. The order on the super-letters is defined in the natural way: $[u]>[v]\iff u>v$.

Definition 4.2. In what follows we fix a homogeneous in each x_i bi-ideal J of $G\langle X\rangle$ such that $\ker\varphi\cap G\langle X\rangle\subseteq J\subseteq \mathbf{\Lambda}$.

By [5, Lemma 2.2] the bi-ideal $\mathbf{\Lambda}$ itself is homogeneous.

Definition 4.3. A super-letter $[u]$ is called *hard in $G\langle X\rangle/J$* provided that its value in $G\langle X\rangle/J$ is not a linear combination of values of super-words in smaller than $[u]$ super-letters.

Definition 4.4. We say that a *height* of a hard super-letter $[u]$ in $G\langle X\rangle/J$ equals $h=h([u])$ if h is the smallest number such that: first, $p(u,u)$ is a primitive s-th root of 1 and either $h=s$ or $h=sl^r$, where $l=$char($\mathbf{k}$); and next the value of $[u]^h$ in $G\langle X\rangle/J$ is a linear combination of super-words in less than $[u]$ super-letters. If there exists no such a number, the height is infinite.

Theorem 4.5. ([10, Theorem 2]). *The values of all hard in $G\langle X\rangle/J$ super-letters with the above defined height function form a set of PBW-generators for $G\langle X\rangle/J$ over* $\mathbf{k}[G]$; *that is, the set of all products*

$$g[u_1]^{n_1}[u_2]^{n_2}\cdots[u_k]^{n_k},\quad [u_1]<[u_2]<\ldots<[u_k],\ n_i<h([u_i]),g\in G$$

form a basis of $G\langle X\rangle/J$.

The hard super-letters for $U_q^+(\mathfrak{sp}_{2n})$ are described in [9, Theorem C_n].

Definition 4.6. In what follows, x_i, $n<i<2n$ denotes the generator x_{2n-i}. Respectively, $v(k,m)$, $1\leq k\leq m<2n$ is the word $x_kx_{k+1}\cdots x_{m-1}x_m$. If $1\leq i<2n$, then $\phi(i)$ denotes the number $2n-i$, so that $x_i=x_{\phi(i)}$.

The word $v(k,m)$ is standard if and only if $k\leq m<\phi(k)$ or $k=m=n$. The standard arrangement of brackets, $[v(k,m)]$, is described in [9, Lemma 7.18]. In [7, Proposition 4.1], it is shown that the value of $[v(k,m)]$ in $U_q^+(\mathfrak{sp}_{2n})$ coincides with the value of the long skew commutator

$$v[k,m]=[[\ldots[[x_k,x_{k+1}],x_{k+2}]\ldots x_{m-1}],x_m],\quad k\leq m<\phi(k).$$

Following [7, p. 21] we define the bracketing of $v(k,m)$, $k \le m < 2n$ as follows:

$$v[k,m] = \begin{cases} [[[\ldots[x_k, x_{k+1}], \ldots], x_{m-1}], x_m], & \text{if } m < \phi(k); \\ [x_k, [x_{k+1}, [\ldots, [x_{m-1}, x_m]\ldots]]], & \text{if } m > \phi(k); \\ [\![v[k,m-1], x_m]\!], & \text{if } m = \phi(k), \end{cases} \tag{4.1}$$

where in the latter term, $[\![u,v]\!] \stackrel{df}{=} uv - q^{-1}p(u,v)vu$.

By [7, Theorem 5.1] the coproduct on the elements $v[k,m]$, $k \le m < 2n$ has the following explicit form:

$$\Delta(v[k,m]) = v[k,m] \otimes 1 + g_{km} \otimes v[k,m] \tag{4.2}$$

$$+ \sum_{i=k}^{m-1} \tau_i (1-q^{-1}) g_{ki}\, v[i+1,m] \otimes v[k,i],$$

where $\tau_i = 1$ with two exceptions, being $\tau_{n-1} = 1 + q^{-1}$ if $m = n$, and $\tau_n = 1 + q^{-1}$ if $k = n$. Here $g_{ki} = \mathrm{gr}(v(k,i)) = g_k g_{k+1} \cdots g_i$.

Conditional identity (2.5) implies that the value of $v[k,m]$ in $U_q^+(\mathfrak{sp}_{2n})$ is independent of the precise arrangement of brackets, provided that $m \le n$ or $k \ge n$. In general, the value of bracketed word $v[k,m]$ is almost independent of the precise arrangement of brackets.

Lemma 4.7. ([7, Lemma 3.6]). *If $k \le n \le m < \phi(k)$, then the value in $U_q^+(\mathfrak{sp}_{2n})$ of the bracketed word $[y_k x_n x_{n+1} \cdots x_m]$, where $y_k = v[k, n-1]$, is independent of the precise arrangement of brackets.*

Lemma 4.8. ([7, Lemma 3.7]). *If $k \le n$, $\phi(k) < m$, then the value in $U_q^+(\mathfrak{sp}_{2n})$ of the bracketed word $[x_k x_{k+1} \cdots x_n y_m]$, where $y_m = v[n+1, m]$, is independent of the precise arrangement of brackets.*

Definition 4.9. We define the words v_k, $1 \le k \le n$ as follows:

$$v_k = v(k, n-1)v(k,n) = x_k x_{k+1} \ldots x_{n-1} x_k x_{k+1} \ldots x_{n-1} x_n, \quad v_n = x_n.$$

Certainly, these are standard words and the standard arrangement of brackets is $[v_k] = [[v(k,n-1)][v(k,n)]]$.

Proposition 4.10. ([9, Theorem C_n]). *If $q^3 \ne 1$, $q \ne -1$, then the set*

$$\mathfrak{C} = \{[v(k,m)],\ [v_s] \mid k \le m < \phi(k),\ 1 \le s \le n\} \tag{4.3}$$

is the set of all hard super-letters for $U_q^+(\mathfrak{sp}_{2n})$.

5. PBW-generators for $u_q^+(\mathfrak{sp}_{2n})$

Now we are going to demonstrate that the set $\mathfrak{C}$ are hard super-letters for $u_q^+(\mathfrak{sp}_{2n})$ as well.

Recall that $\deg_i(w)$, $1 \le i \le n$ denotes a degree in x_i of the word w, the number of occurrences of x_i in w. Respectively, $D(w) = (\deg_1(w), \deg_2(w), \ldots, \deg_n(w))$ is a *multidegree* of w.

Lemma 5.1. *If $[w] \in \mathfrak{C}$, then $sD(w)$ is not a sum of multidegrees $D(u)$ of lesser than $[w]$ elements $[u] \in \mathfrak{C}$. Here s is an arbitrary natural number.*

Proof. Let $w = v(k,m)$, $k \leq m < \phi(k)$. If $[u] \in \mathfrak{C}$ and $u < v(k,m)$, then either $\deg_k(u) \leq 1$ or u contains a letter that does not occur in w. Certainly, in the latter case $D(u)$ may not appear in the decomposition of $sD(w)$ in the sum.

Hence, at least s super-letters $[u]$ with $\deg_k(u) = 1$ appear in the decomposition. At the same time, all such super-letters are in the list

$$[v(k,m+1)],\ [v(k,m+2)], \dots,\ [v(k,\phi(k)-1)]. \tag{5.1}$$

If $m < n$, then $[v(k,m)]$ belongs to the Hopf subalgebra generated by the elements $x_1, x_2, \dots, x_{n-1}$. In this case, the above list of hard super-letters reduces to

$$[v(k,m+1)],\ [v(k,m+2)], \dots,\ [v(k,n-1)].$$

All elements in this list, if any, depend on x_{m+1}, whereas $[v(k,m)]$ is independent of it.

If $m \geq n$, then all super-letters from the list (5.1) are of degree 2 in x_{m+1}. Therefore, the $(m+1)$th component of the sum of s multidegrees of that type is $2s$. However, the $(m+1)$th component of $sD(v(k,m))$ equals s. Thus, the decomposition is still impossible.

Assume $w = v_k$, $1 \leq k < n$. In this case the list (5.1) take the form

$$[v(k,n)],\ [v(k,n+1)], \dots,\ [v(k,\phi(k)-1)].$$

All words in this list are linear in x_k and in x_n. Hence the kth component of $\sum D(u)$ is less than or equal to the nth component of it. At the same time, the kth component of $sD(v_k)$ equals $2s$, and the nth component is s. □

Lemma 5.2. *If a standard word w is independent of x_n, then either $w = v(k,m)$, $k \leq m < n$ or $[w] = 0$ in $U_q^+(\mathfrak{sp}_{2n})$.*

Proof. Subalgebra generated by $x_1, x_2, \dots x_{n-1}$ is the quantization of type A_{n-1}. Hence [9, Theorem A_n case 3] applies. □

Lemma 5.3. *If $k+1 < i \leq n$, then $[v[k,\phi(i)-1], x_{i-1}] = 0$.*

Proof. If $i = n$, then

$$[v[k,\phi(i)-1], x_{i-1}] = [[v[k,n-2], x_{n-1}], x_{n-1}] = 0$$

due to the relations $[[x_{n-2}, x_{n-1}], x_{n-1}] = 0$ and $[x_j, x_{n-1}] = 0$, $j < n-2$.

Assume $i < n$. By Lemma 4.8, we have $v[k,\phi(i)-1] = [v[k,n-1], y]$, where $y = v[n+1,\phi(i)-1]$. In this case, $[y, x_{i-1}] = 0$ due to the defining relations $[x_j, x_{i-1}] = 0$, $i < j \leq n$, whereas

$$[v[k,n-1], x_{i-1}] = [v[k,i-3], [v[i-2,n-1], x_{i-1}]] = 0$$

because the standard word $v(i-2,n-1)x_{i-1}$ is independent of x_n and the standard bracketing is precisely $[v[i-2,n-1], x_{i-1}]$. □

Lemma 5.4. *If $i \neq k$, then $\partial_i(v[k,m]) = 0$, $k \leq m < \phi(k)$.*

Proof. If an element u is independent of x_i then, of course, $\partial_i(u) = 0$. The Leibniz formula (2.9) implies

$$\partial_i([u,x_i]) = \partial_i(ux_i) - p(u,x_i)\partial_i(x_i u) = p(u,x_i)u \cdot \partial_i(x_i) - p(u,x_i)\partial_i(x_i) \cdot u = 0,$$

provided that $\partial_i(u) = 0$. It remains to apply definition (4.1) of the bracketed word $v[k,m]$. □

Lemma 5.5. *We have $\partial_k(v[k,m]) \sim v[k+1,m]$ provided that $k < m < \phi(k)$.*

Proof. If $m=k+1$, then

$$\partial_k([x_k,x_{k+1}])=\partial_k(x_kx_{k+1})-p(x_k,x_{k+1})\partial_k(x_{k+1}x_k)$$
$$=x_{k+1}-p(x_k,x_{k+1})p(x_{k+1},x_k)x_{k+1}=(1-q^{-\varepsilon})x_{k+1},$$

where $\varepsilon=1$ or $\varepsilon=2$. In the general case, we may perform the evident induction:

$$\partial_k(v[k,m])=\partial_k([v[k,m-1],x_m])$$
$$=\partial_k(v[k,m-1]x_m)-p(v(k,m-1),x_m)\partial_k(x_mv[k,m-1])$$
$$=\alpha v[k+1,m-1]x_m-p(v(k,m-1),x_m)p(x_m,x_k)x_m(\alpha v[k+1,m-1]).$$

If $k+1<m<\phi(k)-1$, then $p(x_k,x_m)p(x_m,x_k)=1$ and the above linear combination equals $\alpha[v[k+1,m-1],x_m]=\alpha v[k+1,m]$, which is required. If $m=\phi(k)-1$, then $p(x_k,x_m)p(x_m,x_k)=q^{-1}$ and we arrive to $\alpha v[k+1,\phi(k+1)]$ due to (4.1). □

Lemma 5.6. *In the algebra $U_q^+(\mathfrak{sp}_{2n})$, we have*

$$[\,v[k,m],[v[k,m],\partial_k(v[k,m])]\,]=0,\quad k\leq m\leq\phi(k)-2. \tag{5.2}$$

Proof. Consider the word $w=v(k,m)v(k,m)v(k+1,m)$. This is a standard word, and the standard arrangement of brackets given by the Algorithm p. 5 is precisely

$$[\,[v(k,m)]\,[\,[v(k,m)]\,[v(k+1,m)]\,]\,]\,.$$

By Proposition 4.10, all hard super-letters in $U_q^+(\mathfrak{sp}_{2n})$ are

$$[v(s,r)],\ 1\leq s\leq r<\phi(s);\quad [v_s],\ 1\leq s\leq n. \tag{5.3}$$

In particular, $[w]$ is not hard. If $m<n$, then Lemma 5.2 applies. Assume $m\geq n$. The multiple use of Definition 4.3 demonstrates that the value of $[w]$ is a linear combination of super-words in smaller than $[w]$ hard super-letters. Because $U_q^+(\mathfrak{sp}_{2n})$ is homogeneous, the hard super-letters that may appear in the linear combination are

$$[v(k,r)],\ m<r<\phi(k);\ [v(s,r)],\ k<s\leq r<\phi(s),\quad [v_r],\ k<r\leq n.$$

In the above list, all super-letters that depend on x_k have degree 2 in x_{m+1} and degree 1 in x_k. At the same time w has degree 2 in x_k and degree 3 in x_{m+1}. Therefore the linear combination is empty, $[w]=0$. □

Lemma 5.7. *In the algebra $U_q^+(\mathfrak{sp}_{2n})$, we have*

$$[v[k,\phi(k)-1],\partial_k(v[k,\phi(k)-1])]=0,\quad 1\leq k<n. \tag{5.4}$$

Proof. If $k=n-1$, then the required relation reduces to the defining relation $[[[x_{n-1}]x_n],x_n]=0$.

Assume $k<n-1$. By Lemma 5.5 the element $\partial_k([v(k,\phi(k)-1)])$ is proportional to

$$v[k+1,\phi(k+1)]=[\![v[k+1,\phi(k)-2],x_{k+1}]\!].$$

Hence, it suffices to prove two equalities:

$$[v[k,\phi(k)-1],x_{k+1}]=0,\quad [v[k,\phi(k)-1],v[k+1,\phi(k)-2]]=0.$$

Consider the word $w=v(k,\phi(k)-1)x_{k+1}$. This is a standard word, and it does not appear in the list (5.3). Therefore the value of $[w]$ in $U_q^+(\mathfrak{sp}_{2n})$ is a linear combination of super-words in smaller than w hard super-letters. Since w is linear in x_k, it follows that each term of the linear combination has a hard super-letter linear in x_k which is less than $[w]$. But in the list (5.3) there does not exist such a super-letter. Thus $[w] = 0$ in $U_q^+(\mathfrak{sp}_{2n})$. According to the Algorithm p. 5, the standard arrangement of brackets in w is precisely $[[v(k, \phi(k)-1)]x_{k+1}]$.

Consider the word $w_1 = v(k, \phi(k)-1)v(k+1, \phi(k)-2)$. This is also a standard word. The super-letter $[w_1]$ is not in the list (5.3). Therefore, the value of $[w_1]$ in $U_q^+(\mathfrak{sp}_{2n})$ is a linear combination of super-words in smaller than w_1 hard super-letters. Since w is linear in x_k, it follows that each term of the linear combination has a hard super-letters linear in x_k which is less than $[w_1]$. In the list (5.3), there does not exist such a super-letter, hence $[w_1] = 0$ in $U_q^+(\mathfrak{sp}_{2n})$.

According to the Algorithm, the standard arrangement of brackets in w is

$$[v(k, \phi(k)-2)][x_{k+1}[v(k+1, \phi(k)-2)]]. \tag{5.5}$$

To correct the arrangement of brackets, we may use the conditional identity (2.5). To this end, consider the word

$$w_2 = v(k, \phi(k)-2)v(k+1, \phi(k)-2).$$

The super-letter $[w_2]$ is not in the list (5.3). Therefore the value of $[w_2]$ in $U_q^+(\mathfrak{sp}_{2n})$ is a linear combination of super-words in smaller than w_2 hard super-letters.

Since in the list (5.3) there does not exist a linear in x_k super-letter which is less than $[w_2]$, it follows that $[w_2] = 0$ in $U_q^+(\mathfrak{sp}_{2n})$. It remains to note that according to the Algorithm, the standard arrangement of brackets in w_2 has the required form $[[v(k, \phi(k)-2)][v(k+1, \phi(k)-2)]]$. □

Lemma 5.8. *In the algebra $U_q^+(\mathfrak{sp}_{2n})$, we have*

$$[[v_k], \partial_k([v_k])] = 0, \quad 1 \le k < n. \tag{5.6}$$

Proof. Denote for short $u = [v(k, n-1)]$, $v = [v(k, n)]$, $u' = \partial_k(u)$, $v' = \partial_k(v)$. In this case by definition $[v_k] = [u, v]$. By means of Leibniz formula (2.9), we have

$$\partial_k([v_k]) = \partial_k(uv) - p(u, v)\partial_k(vu)$$

$$= u'v + p(u, x_k)uv' - p(u, v)v'u - p(u, v)p(v, x_k)vu'. \tag{5.7}$$

First, we note that $[v_k]$ skew commutes with the first an the last terms of the above linear combination. To this end, it suffices to check the equalities

$$[[v_k], v] = [[v_k], u'] = 0. \tag{5.8}$$

Consider the word $v_k v = v(k, n-1)v(k, n)v(k, n)$. This is a standard word, and the standard arrangement of brackets (see Algorithm p. 5) is precisely

$$[[v(k, n-1)v(k, n)]v(k, n)].$$

This word does not belong to the list (5.3), hence $[[v_k]v]$ is not hard. The multiple use of Definition 4.3 demonstrates that the value of $[v_k v]$ is a linear combination of super-words in smaller than $[v_k v]$ hard super-letters. Because $U_q^+(\mathfrak{sp}_{2n})$ is homogeneous, the hard super-letters that may appear in the linear combination are

$$[v(k, m)],\ n \le m < \phi(k); \quad [v(s, m)],\ k < s \le m < \phi(s), \quad [v_r],\ k < r \le n. \tag{5.9}$$

Each of that super-letter is either independent of x_k or linear both in x_k and x_n. Since $v_k v$ is of degree 3 in x_k and of degree 2 in x_n, it follows that the linear combination is empty; that is, $[[v_k]v] = 0$ in $U_q^+(\mathfrak{sp}_{2n})$.

Consider the word $w_1 = v_k v(k+1, n-1)$. This is also a standard non hard word, therefore its value is a linear combination of super-words in smaller than w_1 hard super-letters. The super-letters that may appear in the linear combination are the same (5.9). As we have noted before, each of that super-letter is either independent of x_k or linear both in x_k and x_n. Since $v_k v(k+1, n-1)$ is of degree 2 in x_k and of degree 1 in x_n, it follows that the linear combination is empty; that is, $[v_k v(k+1, n-1)] = 0$ in $U_q^+(\mathfrak{sp}_{2n})$.

In this case, the standard arrangement of brackets due to the Algorithm p. 5 is $[v(k, n-1)[v(k, n)v(k+1, n-1)]]$ but not the required

$$[[v(k, n-1)v(k, n)]\,[v(k+1, n-1)]].$$

Nevertheless, we may apply the conditional identity (2.5) because

$$[[v(k, n-1)], [v(k+1, n-1)]] = 0$$

due to Lemma 5.2. Since $\partial_k(u) \sim [v(k+1, n-1)]$, it follows that $[[v_k], u'] = 0$.

Let us turn to the second and the third terms of (5.7). We have

$$p(u, x_k)uv' - p(u, v)v'u = p(u, x_k)[u, v'].$$

Therefore, it remains to prove the equality $[[v_k], [u, v']] = 0$.

Consider the word $w_2 = v(k, n-1)v(k, n)v(k, n-1)v(k+1, n)$. This is a standard word. Therefore the value of $[w_2]$ in $U_q^+(\mathfrak{sp}_{2n})$ is a linear combination of super-words in smaller than w_2 hard super-letters. The super-letters that may appear in the linear combination are precisely (5.9), that are either independent of x_k or linear both in x_k and x_n. Since w_2 is of degree 3 in x_k and of degree 2 in x_n, it follows that the linear combination is empty; that is, $[w_2] = 0$ in $U_q^+(\mathfrak{sp}_{2n})$.

The standard arrangement of brackets $[w_2]$ is

$$[[v(k, n-1)v(k, n)][x_k[v(k+1, n-1)v(k+1, n)]]]$$

that differs from the required

$$[[v(k, n-1)v(k, n)][[x_k v(k+1, n-1)][v(k+1, n)]]].$$

Using the quantum Jacobi identity (2.4) with

$$u \leftarrow x_k, \quad v \leftarrow [v(k+1, n-1)], \quad w \leftarrow [v(k+1, n)],$$

we see that

$$[[x_k, [v(k+1, n-1)]], [v(k+1, n)]]$$

is a linear combination of the following three terms

$$[x_k[v(k+1, n-1)v(k+1, n)]],\ [v(k, n)] \cdot [v(k+1, n-1)],\ [v(k+1, n-1)] \cdot [v(k, n)].$$

Equality $[w_2] = 0$ implies that the element $[v_k]$ skew commutes with the first term. Relations (5.8) demonstrate that $[v_k]$ skew commutes with $[v(k, n)]$ and also with $[v(k+1, n-1)]$. □

Lemma 5.9. *If* $1 \leq i \leq n$, $k < m < \phi(k)$, *then we have*

$$\partial_i^*(v[k, m]) \sim \begin{cases} v[k, m-1], & \text{if } m = i \text{ or } m = \phi(i); \\ 0, & \text{otherwise.} \end{cases}$$

Proof. If an element u is independent of x_i, then, of course, $\partial_i^*(u)=0$. If $\partial_i^*(u)=0$, then the Leibniz formula (2.10) implies

$$\partial_i^*([u,x_i])=\partial_i^*(ux_i)-p(u,x_i)\partial_i^*(x_iu)=(1-p(u,x_i)p(x_i,u))u. \tag{5.10}$$

In particular, taking $u=v[k,i-1]$, we obtain the required formula for $m\leq i$.

If $x_j\neq x_i$, then again by the Leibniz formula (2.10), we have

$$\partial_i^*([u,x_j])=\partial_i^*(ux_j)-p(u,x_j)\partial_i^*(x_ju)=p(x_i,x_j)[\partial_i^*(u),x_j]. \tag{5.11}$$

In particular, taking $u=v[k,i]$, $j=i+1$ and using the evident relation

$$[v[k,i-1],x_{i+1}]=0,\ i\leq n,$$

we obtain the required $\partial_i^*(v[k,i+1])=0$, and then step-by-step the relations $\partial_i^*(v[k,m])=0$, $i<m<\phi(i)$.

Further, the substitution $u\leftarrow v[k,\phi(k)-1]$ in (5.10) implies the required formula with $m=\phi(i)$. Then, substitution $u\leftarrow v[k,\phi(i)]$, $j=i-1$ in (5.11) demonstrate that $\partial_i^*(v[k,\phi(i)+1])$ is proportional to $[v[k,\phi(i)-1],x_{i-1}]$, which is equal to zero due to Lemma 5.3 because condition $\phi(i)+1<\phi(k)$ implies $k+1<i$. Thus the required formula is valid for $m=\phi(i)+1<\phi(k)$. Now, step-by-step, using (5.11), we have $\partial_i^*(v[k,m])=0$, $\phi(i)<m<\phi(k)$. □

Lemma 5.10. *If* $1\leq k<n$, *then*

$$\partial_i^*([v_k])\sim\begin{cases}v[k,n-1]^2, & \text{if } i=n;\\ 0, & \text{otherwise.}\end{cases}$$

Proof. Denote $u=v[k,n-1]$, and $v=v[k,n]$. In this case, $[v_k]=[u,v]$. By Lemma 5.9 and the Leibniz formula (2.10), we have $\partial_i^*([v_k])=0$ if $i<n-1$.

If $i=n-1$, then $\partial_i^*(u)\sim v[k,n-2]$ and $\partial_i^*(v)=0$. The Leibniz formula (2.10) yields

$$\partial_i^*([v_k])\sim p(x_{n-1},v)v[k,n-2]\cdot v-p(u,v)v\cdot v[k,n-2]=p(x_n,v)[v[k,n-2],v].$$

In this formula, if $k=n-1$, one has to replace $v[k,n-2]$ with 1, and the resulting expression becames 0. In general case, still

$$[v[k,n-2],v]=[v[k,n-2],[v[k,n-1],x_n]]=[[v[k,n-2],v[k,n-1]],x_n]=0$$

because, first, $[[v[k,n-2],x_n]=0$, and, next, $[v[k,n-2],v[k,n-1]]=0$ as the word $v(k,n-2)v(k,n-1)$ is standard and independent of x_n (see, Lemma 5.2).

Assume $i=n$. In this case, $\partial_i^*(u)=0$ and $\partial_i^*(v)\sim u$. Using the Leibniz formula (2.10), we obtain the required relation $\partial_i^*([v_k])\sim u^2$. □

Lemma 5.11. *If* $1\leq i\leq n$, $k<m<\phi(k)$, $m\neq n$, *then*

$$[\partial_i^*(v[k,m]),v[k,m]]=0.$$

Proof. By Lemma 5.9, we have to demonstrate that $[v[k,m-1],v[k,m]]=0$. The word $w=v(k,m-1)v(k,m)$ is standard with standard arrangement of brackets $[w]=[[v(k,m-1)][v(k,m)]]$. If $m<n$, then w is independent of x_n and Lemma 5.2 implies $[w]=0$.

If $m > n$, then $[w]$ still is not a hard super-letter. Therefore its value is a linear combination of super-words in smaller than $[w]$ hard super-letters. The smaller hard super-letters are:

$$[v(k,r)],\ m \leq r < \phi(k); \quad [v(s,r)],\ k < s < r < \phi(s); \quad [v_r], k < r < n.$$

Each of them is either independent of x_k or linear in x_k and of degree 2 in x_m. At the same time w has degree 2 in x_k and degree 3 in x_m. hence the linear combination is empty, and $[w] = 0$. □

Lemma 5.12. *If* $1 \leq i \leq n$, *then*

$$[[\partial_i^*(v[k,n]), v[k,n]], v[k,n]] = 0.$$

Proof. By Lemma 5.9, we have to demonstrate that $[[v[k,n-1], v[k,n]], v[k,n]] = 0$. This equality is already proven, see (5.8). □

Lemma 5.13. *If* $1 \leq i \leq n$, *then*

$$[\partial_i^*([v_k]), [v_k]] = 0.$$

Proof. By Lemma 5.10, we have to consider just one case $i = n$. In this case, $\partial_i^*([v_k]) \sim v[k,n-1]^2$. Therefore it suffices to demonstrate that $[v[k,n-1], [v_k]] = 0$.

Consider the word $w = v(k,n-1)v(k,n-1)v(k,n)$. This is a standard word with the following standard arrangement of brackets:

$$[w] = [v(k,n-1)[v(k,n-1)v(k,n)]].$$

Since $[w]$ is not a hard super-letter, it follows that its value is a linear combination of super-words in smaller than $[w]$ hard super-letters:

$$[v(k,r)],\ n \leq r < \phi(k); \quad [v(s,r)],\ k < s < r < \phi(s); \quad [v_r],\ k \leq r < n.$$

The word w is linear in x_n and of degree 3 in x_k, whereas all words in the above list which depend on x_k have degree 1 or 2 in x_2 and degree 1 in x_n. Thus, the linear combination is empty, and $[w] = 0$. □

Theorem 5.14. *If the multiplicative order* t *of* q *is finite,* $t > 3$, *then the values of*

$$v[k,m],\ k \leq m < \phi(k), \quad [v_s],\ 1 \leq s \leq n$$

form a set of PBW-generators for $u_q^+(\mathfrak{sp}_{2n})$ *over* $\mathbf{k}[G]$. *The height of* $v[k,m]$, *equals* t. *The height* h *of* $[v_s]$, $1 \leq s \leq n$ *equals* t *if* t *is odd, otherwise* $h = t/2$. *In all cases* $v[k,m]^t = 0$, $[v_s]^h = 0$ *in* $u_q^+(\mathfrak{sp}_{2n})$.

Proof. First, we note that Definition 4.3 implies that a non-hard super-letter in $U_q^+(\mathfrak{sp}_{2n})$ remains non-hard in $u_q^+(\mathfrak{sp}_{2n})$. Hence, all hard super-letters for $u_q^+(\mathfrak{sp}_{2n})$ are in the list (4.3).

Next, if $[v(k,m)]$, $k \leq m < \phi(k)$ is not hard in $u_q^+(\mathfrak{sp}_{2n})$, then by the multiple use of Definition 4.3, the value of $[v(k,m)]$ is a linear combination of super-words in hard super-letters smaller than given $v(k,m)$. Because $u_q^+(\mathfrak{sp}_{2n})$ is homogeneous, each of the super-words M in that decomposition has a hard super-letter smaller than $v(k,m)$ and of degree 1 in x_k. At the same time, all such super-letters are in the list

$$[v(k,m+1)],\ [v(k,m+2)], \ldots,\ [v(k,\phi(k)-1)].$$

Each of them has degree 2 in x_{m+1} if $m \geq n$, and at least 1 if $m < n$. Hence the super-word M has degree of at least 2 if $m \geq n$, and at least 1 if $m < n$. However $u(k,m)$ is of degree 1 in x_{m+1} if $m \geq n$, and is independent of x_{m+1} if $m < n$. Therefore the decomposition is empty, and $[v(k,m)] = 0$ in $u_q^+(\mathfrak{sp}_{2n})$. Nevertheless, all elements $v[k,m]$ are nonzero in $u_q^+(\mathfrak{sp}_{2n})$ because by Lemma 5.9 we have $\partial_m^*(v[k,m]) \sim v[k,m-1]$, and evident induction applies.

Similarly, if $[v_k]$ is not hard in $u_q^+(\mathfrak{sp}_{2n})$, then its value is a linear combination of super-words in hard super-letters smaller than v_k. Each of the super-words M in that decomposition has a hard super-letter of degree 2 in x_k or at least two hard super-letters of degree 1 in x_k because the degree of v_k in x_k equals 2 (unless $k = n$). All possible super-letters of M are in the list

$$[v(k,n)],\ [v(k,n+1)], \ldots,\ [v(k,\phi(k)-1)].$$

No one of them has degree 2 in x_k, and each of them has degree 1 in x_n. Hence M is of degree at least 2 in x_n. However $[v_k]$ is of degree 1 in x_n. Therefore the decomposition is empty, and $[v_k] = 0$ in $u_q^+(\mathfrak{sp}_{2n})$.

By Lemma 5.10 we obtain $v[k,n-1]^2 \sim \partial_n^*([v_k]) = 0$. Using Leibniz formula (2.10), we have $0 = (\partial_{n-1}^*)^2(v[k,n-1]^2) \sim v[k,n-2]^2$ unless $q = -1$. Applying operator $(\partial_{n-2}^*)^2$ we find $v[k,n-3]^2 = 0$, and so on. Finally, $x_k^2 \neq 0$ in $u_q^+(\mathfrak{sp}_{2n})$ because $\partial_k(x_k^2) \sim x_k \neq 0$.

Let us find the heights of hard super-letters in $u_q^+(\mathfrak{sp}_{2n})$.

For short we put $v = v[k,m]$, $u = [v_k]$. Applying [7, Lemma 3.5], we have $p_{vv} = q$ and $p_{uu} = q^2$. By Definition 4.4 the minimal possible value for the height is precisely the h given in the proposition. It remains to show that $v^t = 0$, $u^h = 0$ in $u_q^+(\mathfrak{sp}_{2n})$.

By Lemma 2.3 it suffices to prove that all partial derivatives of the related elements are zero. Lemma 2.2 yields

$$\partial_i(v^t) = p(v,x_i)^{t-1}\,\underbrace{[v,[v,\ldots[v}_{t-1},\partial_i(v)]\ldots]].$$

$$\partial_i(u^h) = p(u,x_i)^{h-1}\,\underbrace{[u,[u,\ldots[u}_{h-1},\partial_i(u)]\ldots]].$$

It remains to apply Lemma 5.4, Lemma 5.6, Lemma 5.7 and Lemma 5.8. □

6. Defining relations for $G\langle X\rangle/J$

We are reminded that an ideal J is given in Definition 4.2 as an arbitrary homogeneous intermediate ideal, whereas the set $\mathfrak{C}$ of hard in $U_q^+(\mathfrak{sp}_{2n})$ and $u_q^+(\mathfrak{sp}_{2n})$ super-letters is defined in (4.3).

Proposition 6.1. *The values in $G\langle X\rangle/J$ of the set $\mathfrak{C}$ form a set of PBW-generators for $G\langle X\rangle/J$ over $\mathbf{k}[G]$. The ideal J is uniquely defined by the heights of $\mathfrak{C}$. More precisely, J is generated by relations of $U_q^+(\mathfrak{sp}_{2n})$ and $[w]^h$, where $[w] \in \mathfrak{C}$ and h is the height of $[w]$ in $G\langle X\rangle/J$.*

Proof. The definition of the hard super-letter implies that a hard in $G\langle X\rangle/\mathbf{\Lambda} = u_q^+(\mathfrak{sp}_{2n})$ super-letter is hard in $G\langle X\rangle/J$, whereas a hard in $G\langle X\rangle/J$ one is hard in $U_q^+(\mathfrak{sp}_{2n})$. Hence, $\mathfrak{C}$ is the set of all hard in $G\langle X\rangle/J$ super-letters.

By Theorem 4.5, the values of $\mathfrak{C}$ form a set of PBW-generators (with some height function) for $G\langle X\rangle/J$ over $\mathbf{k}[G]$; that is, the set of all products

$$g[w_1]^{n_1}[w_2]^{n_2}\cdots[w_k]^{n_k},\quad [w_1] < [w_2] < \ldots < [w_k],\ n_i < h([w_i]),\ g \in G$$

form a basis of $G\langle X\rangle/J$.

If $h = h([w])$ is the height of $[w]$ in $G\langle X\rangle/J$, then by definition, the value of $[w]^h$ in $G\langle X\rangle/J$ is a linear combination of super-words in hard super-letters smaller than $[w]$. Because J is homogeneous, the multidegree of each super-word M in that decomposition equals the multidegree of w^h. Lemma 5.1 states that this is impossible. Therefore the decomposition is empty, and $[w]^h = 0$ in $G\langle X\rangle/J$; that is, $[w]^h \in J$.

Finally, let J' be the ideal generated by relations of $U_q^+(\mathfrak{so}_{2n})$ and $[w]^h$, where $[w] \in \mathfrak{C}$, while h is the height of $[w]$ in $G\langle X\rangle/J$. By Theorem 4.5 applied to $U_q^+(\mathfrak{sp}_{2n})$, the set of all products

$$g[w_1]^{n_1}[w_2]^{n_2}\cdots[w_k]^{n_k}, \quad [w_1] < [w_2] < \ldots < [w_k], \ [w_i] \in \mathfrak{C}, \quad g \in G$$

form a basis of $U_q^+(\mathfrak{sp}_{2n})$. Hence values of that elements with additional restriction, $n_i < h([w_i])$, span $G\langle X\rangle/J'$. Since the products with that restriction are linearly independent in $G\langle X\rangle/J$, and $J' \subseteq J$, it follows that they are still linearly independent in $G\langle X\rangle/J'$. In other words, $G\langle X\rangle/J'$ and $G\langle X\rangle/J$ have the same basis, therefore $J' = J$. □

Corollary 6.2. *Each homogeneous skew-primitive in $G\langle X\rangle/J$ element u of total degree > 1 has the form $u = \alpha[w]^h$, $\alpha \in \mathbf{k}$, where $[w] \in \mathfrak{C}$. If t is odd or $w = v(k,m)$, then $h = t$ or $h = tl^s$ in the case of characteristic $l > 0$. If t is even and $w = v_k$, then $h = t/2$ or $h = (t/2)l^s$ in the case of characteristic $l > 0$.*

Proof. By [9, Lemma 4.9] (essentially proven in [10, Lemmas 12, 13]), the decomposition of u in the PBW-basis has the form $u = \alpha[w]^h + \sum_i \alpha_i W_i$, where W_i are basis super-words in less than $[w]$ hard super-letters. In Proposition 5.1, it is proven that the multidegree of $[w]^h$ is not a sum of multidegrees of lesser than $[w]$ super-letters from $\mathfrak{C}$. Therefore, $u = \alpha[w]^h$.

If t is odd or $w = v(k,m)$, then $p(w,w) = q$ (see [7, Lemma 3.5]) and due to sited above [9, Lemma 4.9] the exponent h is t, or 1, or tl^s in the case of characteristic $l > 0$. If t is even and $w = v_k$, then $p(w,w) = q^2$ which implies that the exponent h may have only values $t/2$, $(t/2)l^s$, or 1. No one of $[w]$, $w \neq x_i$ is skew-primitive in $G\langle X\rangle/\mathbf{\Lambda}$ because each of them has nonzero partial derivatives; that is, $h \neq 1$. □

7. Constants of differential calculi

Let $\mathfrak{A}$ be a subalgebra of $U_q^+(\mathfrak{sp}_{2n})$ generated by the x_i's, and let

$$C = \{u \in \mathfrak{A} \mid \partial_i(u) = 0, 1 \leq i \leq n\}$$

be the subalgebra of constants for calculus (2.9), whereas

$$C^* = \{u \in \mathfrak{A} \mid \partial_i^*(u) = 0, 1 \leq i \leq n\}$$

be the subalgebra of constants for calculus (2.10). Because the operators ∂_i ∂_i^* diminish degree in x_i of every monomial by one and do not change the degree in other variables, both subalgebras are homogeneous in each variable. By means of the substitution $u \leftarrow C^*$ in (2.14) we have $\sigma^b(C^*) \subseteq C$. Similarly the substitution $u \leftarrow (\sigma^b)^{-1}(C)$ implies $(\sigma^b)^{-1}(C) \subseteq C^*$; that is, $C = \sigma^b(C^*)$.

Theorem 7.1. *The following statements are valid.*

1. *The algebra C is generated by the elements $v[k,m]^t$, $k \leq m < \phi(k)$, $[v_k]^h$, $1 \leq k < n$, where $h = t$ if t is odd and $h = t/2$ if t is even.*

2. *We have* $[f, v[k,m]^t] = [v[k,m]^t, f] = 0$ *and* $[f, [v_k]^h] = [[v_k]^h, f] = 0$ *for all homogeneous* $f \in \mathfrak{A}$.

3. $C = C^*$.

4. *A subalgebra* GC *of* $U_q^+(\mathfrak{sp}_{2n})$ *generated by* G *and* C *is a Hopf subalgebra.*

5. $U_q^+(\mathfrak{sp}_{2n})$ *is a free finitely generated module over* GC *of rank* t^{n^2} *if* t *is odd and of rank* $t^{n^2-n}(t/2)^n$ *if* t *is even.*

Proof. The proof is very similar to that of [4, Theorem 2]. Let us note, first, that $v[k,m]^t, [v_k]^h \in C \cap C^*$. By Lemma 2.2 we have

$$\partial_i([w]^{t'}) \sim \underbrace{[[w],[[w],\dots[[w]}_{t'-1},\partial_i([w])]\dots]],$$

where $t' = t$ if $w = v(k,m)$ or t is odd, and $t' = t/2$ if $w = v_k$ and t is even. Applying Lemma 5.4, Lemma 5.6, and Lemma 5.7, we obtain $\partial_i(v[k,m]^t) = 0$, whereas Lemma 5.8 implies $\partial_i([v_k]^h) = 0$.

Similarly, Lemma 2.2 and (2.13), (2.14) imply

$$\partial_i^*([w]^{t'}) \sim [\dots[[\partial_i^*([w]),\underbrace{[w]],[w]],\dots[w]]}_{t'-1}.$$

Applying Lemma 5.9, Lemma 5.11, and Lemma 5.12, we have $\partial_i^*(v[k,m]^t) = 0$. Lemma 5.13 implies $\partial_i^*([v_k]^h) = 0$. Thus $[w]^{t'} \in C \cap C^*$ for all $[w] \in \mathfrak{C}$.

Using (2.12) let us note that C is a left coideal: $\Delta(C) \subseteq G\mathfrak{A} \otimes C$. If $c \in C$, then

$$0 = \Delta(\partial_i(c)) = \sum_{(c)} g_i^{-1} c^{(1)} \otimes \partial_i(c^{(2)}).$$

Because $g_i^{-1}c^{(1)}$ are linearly independent, we have $\partial_i(c^{(2)}) = 0$, and $c^{(2)} \in C$.

This implies that $GC^* = G(\sigma^b)^{-1}(C) = G\sigma^{-1}(C)$ is a right coideal subalgebra which contains the coradical $\mathbf{k}[G]$. By [8, Theorem 4.1], the subalgebra GC^* has a PBW basis that can be extended up to a PBW basis of $U_q^+(\mathfrak{sp}_{2n})$.

The PBW generators may be chosen in the following way. For every $[w] \in \mathfrak{C}$, we choose an arbitrary element, if any, with the minimal possible a of the form

$$c_w = [w]^a + \sum_i \alpha_i W_i R_i \in GC^*, \quad \alpha_i \in \mathbf{k}, \tag{7.1}$$

where the W_i's are nonempty basis super-words in less than $[w]$ super-letters, whereas the R_i's are basis super-words in greater than or equal to $[w]$ super-letters. According to [8, Lemma 4.3], the number a either equals 1, or $p(w,w)$ is a primitive rth root of 1 and $a = r$ or (in the case of positive characteristic) $a = r(\mathrm{char}\mathbf{k})^s$.

In our case, $p(w,w) = q$ is a primitive tth root of 1 if t is odd or $w = v(k,m)$, and $p(w,w) = q^2$ is a primitive $(t/2)$th root of 1 if t is even and $w = v_k$ (see [7, Lemma 3.5]). In both cases, $a \neq 1$ because otherwise due to Milinski–Schneider criterion (Lemma 2.3), we have $c_w - \alpha \in \mathbf{\Lambda}$, $\alpha \in \mathbf{k}$. However, this is impossible as (7.1) with $m = 1$ is a linear combination of elements from PBW basis of $u_q^+(\mathfrak{sp}_{2n})$ with the leading term $[w]$.

Thus, we can choose $c_w = [w]^h$ because we already know that $[w]^h \in C^*$. In particular, $C^* = GC^* \cap \mathfrak{A}$ as an algebra is generated by the elements $c_w = [w]^h$ with $[w] \in \mathfrak{C}$. Since all of that elements belong to C, it follows that $C^* \subseteq C$.

The map σ^2 is an automorphism such that $\sigma^2(x_i) = p_{ii}x_i$, $\sigma^2(g_i) = g_i$. This implies $\sigma^2(f) \sim f$ for every homogeneous polynomial. Moreover,

$$(\sigma^b)^2(f) = \mathrm{gr}(f)\sigma(\mathrm{gr}(f)\sigma(f)) = \mathrm{gr}(f)f\mathrm{gr}(f)^{-1} \sim f$$

as well. Applying this proportion to $f = [w]^h$, we have $(\sigma^b)^2(C^*) = C^*$. Let us apply σ^b to the the already proven relation $C^* \subseteq C = \sigma^b(C^*)$. We obtain

$$C = \sigma^b(C^*) \subseteq \sigma^b(C) = (\sigma^b)^2(C^*) = C^*.$$

This completes the proof of 3 and 1.

Let $a \in C$. Using Leibniz rule (2.9), we have $\partial_i(x_i a) = a$, whereas $\partial_i(ax_i) = p(a, x_i)a$. Hence, $\partial_i([a, x_i]) = p(a, x_i)a - p(a, x_i)a = 0$. Because certainly $\partial_k([a, x_i]) = 0$, $x_k \neq x_i$, we get $[a, x_i] \in C$ and also $[x_i, a] = \sigma^b([a, x_i]) \in C$. At the same time, the degree in x_i of each generator $[w]^h$ is a multiple of h (which is t or $t/2$). Therefore the degree in x_i of each homogeneous constant is a multiple of t or $t/2$ also. However, $\deg_i([a, x_i]) = \deg_i([x_i, a]) \equiv 1 (\mathrm{mod}\ t/2)$. This is possible only if $[a, x_i] = [x_i, a] = 0$. Hence, all constants, particularly $[w]^h$, $[w] \in \mathfrak{C}$, are skew central. This proves 2.

We have seen above that C is a left coideal, and GC^* is a right coideal. Since $C^* = C = \sigma^b(C)$, it follows that the subalgebra GC is both a left and a right coideal and it is invariant with respect to the antipode; that is, GC is a Hopf subalgebra, which completes 4.

Finally, each element $[w]^r$ has a decomposition $[w]^r = [w]^{r_0} \cdot ([w]^{h_w})^g$, $0 \leq r_0 < h_w$. Hence, the products $\prod_{[w]\in\mathfrak{C}} [w]^{k_w}$, $0 \leq k_w < h_w$, form a basis of $U_q^+(\mathfrak{sp}_{2n})$ over GC. The total number of such products is t^{n^2} if t is odd and it is $t^{n^2-n}(t/2)^n$ if t is even. □

Corollary 7.2. *If $[w] \in \mathfrak{C}$, then the ideal J_w of $U_q^+(\mathfrak{sp}_{2n})$ generated by all elements $[u]^{h_u}$ with $[u] \in \mathfrak{C}$, $u \neq w$ contains no one of the elements $[w]^b$, $b \geq 1$.*

The proof literally coincides with the proof of [4, Corolary 2].

8. Combinatorial rank

Consider the chain that defines the combinatorial rank

$$J_0^+ = G\langle X\rangle \cap \ker\varphi \subset J_1^+ \subset J_2^+ \subset \ldots \subset J_\kappa^+ = \mathbf{\Lambda}.$$

Proposition 8.1. *All J_i^+ are homogeneous Hopf ideals. Let $[w] \in \mathfrak{C}$.*
If $w = v(k, m)$, then $[w]^h \in J_i^+$ if and only if $h \geq t$ and $m - k < 2^i - 1$.
If $w = v_k$, and t is odd, then $n - k < 2^{i-1}$ implies $[w]^t \in J_i^+$.
If $w = v_k$, and t is even, then $n - k < 2^i - 1$ implies $[w]^{t/2} \in J_i^+$.

Proof. We perform induction on i. Let $t' = t/2$ if t is even, and $t' = t$ otherwise.

Theorem C_n from [9] describes all skew primitive elements of $U_q^+(\mathfrak{sp}_{2n})$. They are x_i, x_i^t, $x_i^{t_i l^r}$, $1 \leq i < n$, x_n $x_n^{t'}$, $x_n^{t' l^r}$, $1 - g$, $g \in G$ and possibly some linear combinations of these elements. Because all that elements are skew-primitive in $G\langle X\rangle$, the ideal J_1^+ is generated by $x_n^{t'}$, x_i^t, $1 \leq i < n$ and quantum Serre relations $S_{ij}(x_i, x_j)$ given in (3.1) and (3.2). As a result, Corollary 7.2 implies that $[v(k, m)]^h \in J_1^+$ if and only if $k = m < n$, $h \geq t$, whereas $[v_k]^h \in J_1^+$ if and only if $k = n$, $h \geq t'$. At the same time, $m - k < 2^1 - 1$ & $m < \phi(k)$ is equivalent to $k = m < n$; and $n - k < 2^{1-1}$ means $k = n$. If t is even and $w = v_k$, then $n - k < 2^{1-2}$ implies $k = n$, and $[v_n]^{t/2} = x_n^{t/2} \in J_1$.

Suppose that the statement is valid for J_{i-1}^+. Corollary 6.2 implies that every homogeneous skew primitive element of $G\langle X\rangle/J_{i-1}^+$ is proportional to $[w]^h$ with $[w]\in\mathfrak{C}$, $h=t$ or $h=t/2$. Because J_{i-1}^+ is a homogeneous Hopf ideal, each homogeneous component of a skew primitive element of $G\langle X\rangle/J_{i-1}^+$ is again skew primitive. Thus, J_i^+ is generated by both J_{i-1}^+ and all elements $[w]^h$ that are skew primitive in $G\langle X\rangle/J_{i-1}^+$. In particular, J_i^+ is a homogeneous Hopf ideal.

Moreover, Corollary 7.2 implies that $[w]^h\in J_i^+$ if and only if $[w]^{h'}$, $h'\leq h$ is in the list of the skew primitives of $G\langle X\rangle/J_{i-1}^+$.

Let us demonstrate, first, that if $m-k<2^i-1$ then $[w]^t$ with $w=v(k,m)$ is skew primitive in $G\langle X\rangle/J_{i-1}^+$. By Theorem 7.1 the subalgebra GC generated over G by the elements $T_u=[u]^{t_u}$, $[u]\in\mathfrak{C}$ is a Hopf subalgebra (here $t_u=t$ if $u=v(k,m)$ or t is odd, and $t_u=t/2$ othrwise). Therefore there exists a decomposition

$$\Delta([w]^t)=\sum_{(c)}\mathrm{gr}(c^{(2)})c^{(1)}\otimes c^{(2)}, \tag{8.1}$$

such that $c^{(1)},c^{(2)}$ are words (products) in T_u.

Assume that $[w]^t$ is not skew primitive in $G\langle X\rangle/J_{i-1}^+$. Let us fix a tensor $c^{(1)}\otimes c^{(2)}$ with nonempty $c^{(1)}$ and $c^{(2)}$ which is not zero in $G\langle X\rangle/J_{i-1}^+\otimes G\langle X\rangle/J_{i-1}^+$. Certainly, no one of the factors in $c^{(1)}\sim\prod_{\mu\in M_1}[u_\mu]^{t_\mu}$ and $c^{(2)}\sim\prod_{\mu\in M_2}[u_\mu]^{t_\mu}$ is zero in $G\langle X\rangle/J_{i-1}^+$. Hence, by the inductive supposition we have $m_\mu-k_\mu\geq 2^{i-1}-1$, if $u_\mu=v(k_\mu,m_\mu)$, and $n-k_\mu\geq 2^{i-2}$ if $u_\mu=v_{k_\mu}$ and t is odd, whereas $n-k_\mu\geq 2^{i-1}-1$ if $u_\mu=v_{k_\mu}$ and t is even.

The total degree of the tensor equals the total degree of $[w]^t$. At the same time, the total degree of $[u_\mu]^{t_\mu}$ equals $(m_\mu-k_\mu+1)t$ if $u_\mu=v(k_\mu,m_\mu)$, and it is

$$(2n-2k_\mu+1)t'=(m_\mu-k_\mu+1)t',$$

where by definition we put $m_\mu=\phi(k_\mu)$ provided that $u_\mu=v_{k_\mu}$. Hence, we have

$$(m-k+1)t=\sum_{\mu\in M_1\cup M_2}(m_\mu-k_\mu+1)t_\mu. \tag{8.2}$$

If t is odd, then $t_\mu=t$, and the above equality with conditions on $m-k$ and $m_\mu-k_\mu$ imply

$$2^i>m-k+1\geq |M_1\cup M_2|\cdot 2^{i-1}. \tag{8.3}$$

This is a contradiction because no one of the sets M_1, M_2 is empty.

If t is even and no one of u_μ has the form v_{k_μ}, then we arrive to the same contradiction (8.3). If $u_\mu=v_\mu$, then the degree in x_n of $[u_\mu]^{t/2}$ equals $t/2$. Since degree in x_n of $[w]^t$ either is zero or equals t, it follows that there exists a unique $\nu\in M_1\cup M_2$, $\nu\neq\mu$, such that $u_\nu=v_{k_\nu}$. In this case, the equality (8.2) implies

$$2^it>(m-k+1)t\geq(2n-2k_\mu+1)\frac{t}{2}+(2n-2k_\nu+1)\frac{t}{2}\geq(2^i-1)t, \tag{8.4}$$

and hence $2^i>m-k+1\geq 2^i-1$. This inequality means $m-k+1=2^i-1$, whereas (8.4) becomes the equality

$$(2n-2k_\mu+1)\frac{1}{2}+(2n-2k_\nu+1)\frac{1}{2}=2^i-1,$$

which is possible only if $n-k_\mu$ and $n-k_\nu$ take minimal possible value (that is, $k_\mu=k_\nu=n-2^{i-1}+1$) and $M_1\cup M_2=\{\mu,\nu\}$. Since $w=v(k,m)$ depends on

x_k, and is independent of x_i, $i<k$, it follows that $k_\mu=k_\nu=k$. But this is still impossible because $[v(k,m)]^t$ is of degree t in x_k, whereas $[v_k]^{t/2}\otimes[v_k]^{t/2}$ is of degree $2t$ in x_k.

Let $n-k<2^{i-1}$ and t is odd. We have to demonstrate that $[v_k]^t$ is skew primitive in $G\langle X\rangle/J^+_{i-1}$. The decomposition (8.1) with $w=v_k$ implies

$$(2n-2k+1)t=\sum_{\mu\in M_1\cup M_2}(m_\mu-k_\mu+1)t\geq|M_1\cup M_2|2^{i-1}t\geq 2^it. \tag{8.5}$$

Inequality $2^i>2n-2k$ implies $2^i>2n-2k+1$ because 2^i is even, and we obtain a contradiction $2^i>2^i$.

Let $n-k<2^i-1$ and t is even. We shall prove that $[v_k]^{t/2}$ is skew primitive in $G\langle X\rangle/J^+_{i-1}$. Consider the decomposition (8.1) with $w=v_k$. The degree of $[v_k]^{t/2}$ in x_n equals $t/2$, therefore there exists one and only one μ, such that $u_\mu=v_{k_\mu}$. Since no one of sets M_1, M_2 is empty, it follows that there exists at least one ν, such that $u_\nu=v(k_\nu,m_\nu)$. In this case the equality (8.2) takes the form

$$(2n-2k+1)\frac{t}{2}=(2n-2k_\mu+1)\frac{t}{2}+(m_\nu-k_\nu+1)t+\cdots .$$

By the induction hypothesis, we have $n-k_\mu\geq 2^{i-1}-1$ and $m_\nu-k_\nu\geq 2^{i-1}-1$. This implies a contradiction:

$$\frac{2^{i+1}-1}{2}>\frac{2n-2k+1}{2}\geq\frac{2^i-1}{2}+2^{i-1}.$$

Next, we show that if $m-k\geq 2^i-1$, $k\leq m<\phi(k)$, then $[w]^t$ with $w=v(k,m)$ is not skew primitive in $G\langle X\rangle/J^+_{i-1}$. Let s be an arbitrary number less than n. We shall analyze all tensors of the decomposition

$$\Delta([w]^t)=(\Delta([w]))^t=\sum_{(c)}c^{(1)}\otimes c^{(2)}$$

such that $\deg_s(c^{(2)})=t$, $\deg_{s+1}(c^{(2)})=0$. By the coproduct formula (4.2) each tensor of that decomposition has the form

$$\alpha g a_1a_2\cdots a_t\otimes b_1b_2\cdots b_t,$$

where $a_\lambda=v[1+i_\lambda,m]$, $b_\lambda=v[k,i_\lambda]$. Because $\deg_{s+1}(b_\lambda)=0$, we have $i_\lambda\leq s$. Therefore the inequality $s<n$ implies $\deg_s(b_\lambda)\leq 1$. At the same time,

$$\sum_{\lambda=1}^t\deg_s(b_\lambda)=\deg_s(c^{(2)})=t.$$

Hence, $\deg_s(b_\lambda)=1$, all λ. In particular $i_\lambda\geq s$. Thus, $i_\lambda=s$ for all λ, and there is only one tensor of the required degrees in the decomposition:

$$\alpha g_k^tg_{k+1}^t\cdots g_s^t\,v[s+1,m]^t\otimes v[k,s]^t,\ \ \alpha\neq 0. \tag{8.6}$$

By the inductive supposition $v[k,s]^h\notin J^+_{i-1}$ if $s-k\geq 2^{i-1}-1$. At the same time, either $v[s+1,m]$ or $\sigma^b(v[s+1,m])=v[\phi(m),\phi(s+1)]$ belongs to $\mathfrak{C}$, unless $m=\phi(s{+}1)$. Hence, again by the inductive supposition, $v[s{+}1,m]^t\notin J^+_{i-1}$ provided that $m-s-1\geq 2^{i-1}-1$, $m\neq\phi(s{+}1)$. Denote $s_{\min}=2^{i-1}-1+k$, $s_{\max}=m-2^{i-1}$ for short.

To show that $[w]^t$ is not skew primitive in $G\langle X\rangle/J^+_{i-1}$, it suffices to find at least one point s satisfying $s<n$, $m\neq\phi(s+1)$ in the interval $[s_{\min},s_{\max}]$.

This interval is not empty: $s_{\max}-s_{\min} = m-k-2^i+1 \geq 0$. We have $s_{\min}+s_{\max} = k+m-1 \leq 2n-2$ because $m < \phi(k)$.

If the interval contains at least two points, $s_{\min} \leq s_{\max} - 1$, then $2s_{\min} \leq s_{\min}+s_{\max}-1 \leq 2n-3$. Hence, $s_{\min} \leq n-2$; that is, the interval contains at least two points satisfying $s < n$. One of them satisfies $m \neq \phi(s+1)$.

If the interval contains just one point, $s = s_{\min} = s_{\max}$, then $m+s = m+s_{\max} = 2m-2^{i-1}$ is an even number (here of course $i > 1$). At the same time, $m = \phi(s+1)$ is equivalent to $m+s+1 = 2n$. Hence, $m \neq \phi(s+1)$. □

Theorem 8.2. *The combinatorial rank of $u_q^+(\mathfrak{sp}_{2n})$ equals $\lfloor \log_2(n-1) \rfloor + 2$.*

Proof. First, we note that J_κ^+ with $\kappa = \lfloor \log_2(n-1) \rfloor + 2$ contains all elements $[v(k,m)]^t$, and $[v_k]^{t'}$.

Using the evident inequality $a < 1 + \lfloor a \rfloor$, we have

$$m-k \leq (\phi(1)-1)-1 = 2n-3 = 2^{1+\log_2(n-1)} - 1 < 2^{2+\lfloor \log_2(n-1) \rfloor} - 1,$$

and Proposition 8.1 implies that $[v(k,m)]^t \in J_\kappa^+$.

Similarly,

$$n-k \leq n-1 = 2^{\log_2(n-1)} < 2^{1+\lfloor \log_2(n-1) \rfloor} = 2^{\kappa-1} \leq 2^\kappa - 1.$$

Hence, Proposition 8.1 implies $[v_k]^t \in J_\kappa^+$ if t is odd, and $[v_k]^{t/2} \in J_\kappa^+$ if t is even.

Next, we note that $[v(1,2n-2)]^t \notin J_{\kappa-1}^+$. Using inequality $a \geq \lfloor a \rfloor$, we have

$$(2n-2)-1 = 2^{1+\log_2(n-1)} - 1 \geq 2^{\kappa-1} - 1,$$

and Proposition 8.1 applies. □

Theorem 8.3. *The combinatorial rank of $u_q(\mathfrak{sp}_{2n})$ is $\lfloor \log_2(n-1) \rfloor + 2$.*

The proof almost literally coincides with the proof of [6, Theorem 8.1] or [4, Theorem 4].

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References

[1] N. Andruskiewitsch, H.-J. Schneider, Pointed Hopf algebras, in: S. Montgomery, H.-J. Schneider (Eds.) New Directions in Hopf Algebras, MSRI Publications, 43(2002), 1–68.

[2] C. De Concini and V. G. Kac, *Representations of quantum groups at roots of* 1, Colloque Dixmier 1989, pp. 471-506; Progress in Math., vol. 92, Birkhauser, Boston, MA, 1990.

[3] C. De Concini, V.G. Kac, C. Procesi, *Quantum coadjoint action*, Journal of AMS, 5(1992), 151–189.

[4] M.L. Díaz Sosa, V. Kharchenko, Combinatorial rank of $u_q(\mathfrak{so}_{2n})$, Journal of Algebra, 448(2016), 48–73.

[5] V.K. Kharchenko, A. Andrade Álvarez, *On the combinatorial rank of Hopf algebras*, Contemporary Mathematics, v. 376(2005), 299–308.

[6] V.K. Kharchenko, M. L. Diaz Sosa, *Computing of the combinatorial rank of $u_q(\mathfrak{so}_{2n+1})$*, Communications in Algebra, v. 39(2011) 4705–4718.

[7] V.K. Kharchenko, *Explicit coproduct formula for quantum groups of infinite series*, Israel Journal of Mathematics, 208(2015), 13-43.

[8] V.K. Kharchenko, *PBW-basis of coideal subalgebras and a freeness theorem*, TAMS, v. 300, N10(2008), 5121–5143.

[9] V.K. Kharchenko, *A combinatorial approach to the quantification of Lie algebras*, Pacific Journal of Mathematics 203 N1(2002) 191–233.

[10] V.K. Kharchenko, *A quantum analogue of the Poincaré-Birkhoff-Witt theorem*, Algebra i Logika 38 N4(1999) 476–507; English translation: Algebra and Logic, 38, N4(1999), 259-276; arXiv: math. QA/0005101.

[11] V.K. Kharchenko, *An algebra of skew primitive elements*, Algebra i Logica, 37, N2(1998),181–224,English translation: Algebra and Logic, 37, N2(1998),101–127, arXiv: math. QA/0006077.

[12] V.K. Kharchenko, *Right coideal subalgebras of* $U_q^+(\mathfrak{so}_{2n+1})$, Journal of the European Mathematical Society, 13(2011), 1677–1735.

[13] M. Lothaire, Algebraic Combinatorics on Words, Cambridge Univ. Press, 2002.

[14] D.E. Radford, *The structure of Hopf algebras with projection*, Journal of Algebra, 92(1985), 322–347.

[15] A.I. Shirshov, *On free Lie rings*, Matem. Sbornic 45, 87(2)(1958), 113–122.

[16] A.I. Shirshov, *Some algorithmic problems for Lie algebras*, Sibirskii Math. J., 3(2)(1962), 292–296.

[17] M. Takeuchi, *Survey of braided Hopf algebras,* in: New Trends in Hopf Algebra Theory, Contemp. Math., vol. 267, AMS, Providence RI, 2000, pp. 301-324.

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