ON THE IMAGE OF GALOIS L-ADIC REPRESENTATIONS FOR ABELIAN VARIETIES OF TYPE III

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Abstract. In this paper we investigate the image of the l-adic representation attached to the Tate module of an abelian variety defined over a number field. We consider simple abelian varieties of type III in the Albert classification cf.[**M**, Th.2, p. 201]. We compute the image of the l-adic and mod l Galois representations and we verify the Mumford-Tate conjecture for a wide class of simple abelian varieties of type III.

1. Introduction. In this paper we compute the image of l-adic and mod l Galois representations attached to certain abelian varieties of type III according to Albert classification list. We also prove the Mumford-Tate conjecture for these varieties. To be more precise, the main results of this paper concern the following class of abelian varieties:

Definition of class $\mathcal{B}$.
Abelian variety A/F, defined over a number field F is of class $\mathcal{B}$, if the following conditions hold:

(i) *A is a simple, abelian variety of dimension g*
(ii) *$\mathcal{R} = End_{\bar{F}}(A) = End_F(A)$ and the endomorphism algebra $D = \mathcal{R} \otimes_{\mathbb{Z}} \mathbb{Q}$, is of type III in the Albert list of division algebras with involution (cf. [M], Th.2, p. 201).*
(iii) *the field F is such that for every l the Zariski closure G_l^{alg} of $\rho_l(G_F)$ in $GL_{V_l(A)}/\mathbb{Q}_l$ is a connected algebraic group*
(iv) *$g = hed$, where h is an odd integer, $e = [E : \mathbb{Q}]$ is the degree of the center E of D and $d^2 = [D : E]$.*

In sections 2 and 3 we give an explicit description of the endomorphism algebra and its involution for an abelian variety of type III as well as the relation to various bilinear forms coming from Weil pairing. This detailed treatment of endomorphism algebras and bilinear forms differs significantly from the approach of [C2] and [BGK2]. Due to our approach the proof of Theorem 3.29, in section 3, is achieved in a very simple and explicit way. Theorem 3.29 is an important tool which shows how to extract a symmetric form out of the Weil pairing, which is a symplectic form. In section 4 we compute Lie algebras that lead to the proof of Theorem 4.19. In section 5 we apply Theorem 4.19 in the proof of the Mumford-Tate conjecture for the abelian varieties of class $\mathcal{B}$.

Theorem 5.11. *If A is an abelian variety of class $\mathcal{B}$, then*

$$G_l^{\mathrm{alg}} = MT(A) \otimes \mathbb{Q}_l,$$

for every prime number l, where $MT(A)$ denotes the Mumford-Tate group of A, i.e., the Mumford Tate conjecture holds true for A.

This generalizes the result of Tankeev [Ta] who proved the Mumford-Tate conjecture for abelian varieties of type III, with similar dimension restrictions, such that $End(A) \otimes \mathbb{Q}$ has center equal to $\mathbb{Q}$.

On the way of the proof of Mumford-Tate conjecture we also compute explicitly the Hodge group and prove that it is equal to the Lefschetz group. However it is not enough to get directly the Hodge conjecture for abelian varieties of type III of class $\mathcal{B}$ cf. [Mu]. The proof of Mumford-Tate conjecture and equality of Hodge and Lefschetz groups for abelian varieties of type I and II of class $\mathcal{A}$ in [BGK2] gave us the Hodge and Tate conjectures for these abelian varieties. In section 6 (Theorem 6.22, Corollary 6.26) we compute the images of l-adic and mod l representations for abelian varieties of class $\mathcal{B}$. Finally in section 7 (Theorem 7.2) we prove an analogue of the open image Theorem of Serre cf. [Se1-Se4], [Se6] for abelian varieties of class $\mathcal{B}$.

Theorem 7.2. *If A is an abelian variety of class $\mathcal{B}$, then for every prime number l, $\rho_l(G_F)$ is open in the group $C_{\mathcal{R}}(GSp_{(\Lambda,\,\kappa)})(\mathbb{Z}_l)$. In addition, for $l \gg 0$ the subgroup $\rho_l(\overline{G_F'})$ of $C_{\mathcal{R}}(Sp_{(\Lambda,\,\kappa)})(\mathbb{Z}_l)$ is of index dividing $2^{r(l)}$, where $r(l)$ is the number of primes over l in $\mathcal{O}_E$. More precisely (cf. 6.23)*

$$\prod_{\lambda | l} \pi_\lambda(Spin_{(T_\lambda,\,\psi_\lambda)}(\mathcal{O}_\lambda)) \quad \subset \quad \rho_l(\overline{G_F'}) \quad \subset \quad \prod_{\lambda | l} SO_{(T_\lambda,\,\psi_\lambda)}(\mathcal{O}_\lambda),$$

2. Abelian varieties of type III and their endomorphisms algebras.

Let A/F be a simple abelian variety of dimension g such that $D = End_{\bar{F}}(A) \otimes_{\mathbb{Z}} \mathbb{Q} = End_F(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ and the polarization of A is defined over F. We assume that A/F is an abelian variety over F of type III according to the Albert's classification list (see [M], p. 201). Hence D is an indefinite quaternion algebra over E with center E, a totally real extension of $\mathbb{Q}$ of degree e such that for every imbedding $E \subset \mathbb{R}$

$$D \otimes_E \mathbb{R} = \mathbb{H}.$$

Observe that in this case $[D : E] = 4$ so $g = 2eh$ where $e = [E : \mathbb{Q}]$ and h is an integer. We take $l \gg 0$ such that A has good reduction at all primes over l (cf. [ST]) and the algebra D splits over all primes over l and l does not divide the degree of the polarization. Let $\mathcal{R}_D$ be a maximal order in D. Since $\mathcal{R} = End_{\bar{F}}(A)$ is an order in D, we observe that $\mathcal{R} \otimes_{\mathbb{Z}} \mathbb{Z}_l = \mathcal{R}_D \otimes_{\mathbb{Z}} \mathbb{Z}_l$ for l that does not divide the index $[\mathcal{R}_D : \mathcal{R}]$. Since $\mathcal{R}$ is finitely generated, free $\mathbb{Z}$ module we check that $\mathcal{R} \cap E = \mathcal{O}_E^0$ is an order in $\mathcal{O}_E$.

To get explicit information about the algebra D we start with a more general framework. Let D be a division algebra with two involutions $*_1$ and $*_2$ and center E. For each $x \in D$ we will denote x^{*_i} to be the image of the involution $*_i$ acting on x. By Skolem-Noether Theorem [R] p. 103 there is an element $a \in D$ such that for each $x \in D$ we have:

$$x^{*_2} = a\, x^{*_1}\, a^{-1}. \tag{2.1}$$

Because $*_i \circ *_i = id_D$, applying $*_2$ to (2.1) we get

$$a^{*_1} = \epsilon a \tag{2.2}$$

for $\epsilon \in E$ and applying $*_1$ we check that $\epsilon^2 = 1$ hence $\epsilon = 1$ or $\epsilon = -1$ c.f. [M] p. 195. Let $E_0 = \{c \in E;\ c^* = c\}$. Then E/E_0 is an extension of degree at most 2.

For a simple abelian variety of type III, $E = E_0$ and E is totally real cf. [M] p. 194. Also in this case $\epsilon = 1$ in (2.2) (cf. [M] pp. 193-196). Hence $a \in E$ and $*_2 = *_1$. Therefore the division algebra D coming from a simple abelian variety of type III has a unique positive involution $*$, the Rosati involution. Moreover the map $D \to D$ given by $\alpha \to \alpha^*$ is an isomorphism of E algebras so by [R] Cor. 7.14 p. 96 the algebra D gives an element of order 1 or 2 in $Br(E)$. Since D is a noncommutative division algebra, it gives an element of order 2 in $Br(E)$.
By theorem of Suslin and Merkurjev [SM] for any field L such that $\mu_m \subset L$ and $(char\, L,\ m) = 1$ there is a natural isomorphism

$$K_2(L)/m\, K_2(L) \xrightarrow{\cong} Br(L)[m] \otimes \mu_m \tag{2.3}$$

If L is a number field, then by a result of Lenstra [Le] every element of $K_2(L)$ is a single Steinberg symbol $\{c,\ d\}$ for some $c, d \in L$.

Therefore for $m = 2$ and $L = E$ the isomorphism (2.3) shows that D is isomorphic as an E-algebra to a division algebra

$$D(c,d) := \{a_0 + a_1\alpha + a_2\beta + a_3\alpha\beta;\ \ \alpha^2 = c,\ \beta^2 = d,\ \alpha\beta = -\beta\alpha\}$$

This isomorphism induces the unique positive involution on $D(c,d)$ which will also be denoted by $*$. Therefore $*$ must be the following natural positive involution

$$(a_0 + a_1\alpha + a_2\beta + a_3\alpha\beta)^* = a_0 - a_1\alpha - a_2\beta - a_3\alpha\beta$$

on $D(c,d)$. From now on we identify D with $D(c,d)$. Put $L = E(\alpha)$. Let $\eta = a_0 + a_1\alpha$ and $\gamma = a_2 + a_3\alpha$. Hence

$$\eta + \gamma\beta = a_0 + a_1\alpha + a_2\beta + a_3\alpha\beta.$$

For an element $\delta = e + f\alpha \in L$, with $e, f \in E$, put $\bar{\delta} = e - f\alpha$. The field L splits the algebra $D(c,d)$. Namely we have an isomorphism of L algebras:

$$D(c,d) \otimes_E L \to M_{2,2}(L) \tag{2.4}$$

$$(\eta + \gamma\beta) \otimes 1 \ \to \begin{bmatrix} \eta & \gamma \\ d\bar{\gamma} & \bar{\eta} \end{bmatrix}$$

from this isomorphism it is clear that

$$(\eta+\gamma\beta)^* = Tr^0(\eta+\gamma\beta) - (\eta+\gamma\beta)$$

because by definition

$$Tr^0(\eta+\gamma\beta) = Tr \begin{bmatrix} \eta & \gamma \\ d\bar{\gamma} & \bar{\eta} \end{bmatrix} = 2a_0,$$

where Tr^0 denotes the reduced trace (see [R] pp.112-116) from $D(c,d)$ to E. The involution on $M_{2,2}(L)$ induced by $*$ is of the following form:

$$B^* = J\,B^t\,J^{-1} \tag{2.5}$$

where $B \in M_{2,2}(L)$ and

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Remark 2.6. *It is clear that if we take in the above computations instead of $L = E(\alpha)$ the field $E(\beta)$ or the field $E(\alpha\beta)$ then they also split the algebra D by a formula similar to (2.4) and the involution $*$ will induce on $M_{2,2}(E(\beta))$ and $M_{2,2}(E(\alpha\beta))$ the involution given by formula (2.5).*

Note that any maximal commutative subfield of $D(c,d)$ has form $E(a_1\alpha + a_2\beta + a_3\alpha\beta)$ for some $a_1, a_2, a_3 \in E$ not all equal to zero. If $Nr^0 : D(c,d) \rightarrow E$ denotes the reduced norm, then for every $\eta+\gamma\beta \in D(c,d)$:

$$Nr^0(\eta+\gamma\beta) = det \begin{bmatrix} \eta & \gamma \\ d\bar{\gamma} & \bar{\eta} \end{bmatrix} = (\eta+\gamma\beta)^*(\eta+\gamma\beta) =$$

$$= a_0^2 - a_1^2 c - a_2^2 d + a_3^2 cd = a_0^2 - (a_1\alpha + a_2\beta + a_3\alpha\beta)^2. \tag{2.7}$$

For some $a_1, a_2, a_3 \in E$ not all equal to zero put $\alpha' := a_1\alpha + a_2\beta + a_3\alpha\beta$. If $\beta' := b_1\alpha + b_2\beta + b_3\alpha\beta$, is an element of $D(c,d)$, put $c_1 := a_3b_2 - a_2b_3,\ \ c_2 := a_1b_3 - a_3b_1$, $c_3 := a_1b_2 - a_2b_1$. Then

$$\alpha'\beta' = a_1b_1c + a_2b_2d - a_3b_3cd + c_1d\alpha + c_2c\beta + c_3\alpha\beta \tag{2.8}$$

and

$$det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ dc_1 & cc_2 & c_3 \end{bmatrix} = -d\,c_1^2 - c\,c_2^2 + c_3^2 \geq 0 \tag{2.9}$$

Since $c < 0$ and $d < 0$, the determinat in (2.9) is zero if and only if elements α' and β' are linearly dependent over E. Hence it is possible to find β' in such a way that $a_1b_1c + a_2b_2d - a_3b_3cd = 0$ and the determinant in (2.9) is nonzero. With this choice of β' we see that $c' := {\alpha'}^2 < 0$, $d' := {\beta'}^2 < 0$ and $\alpha'\beta' = -\beta'\alpha'$. We observe that for any $a_0', a_1', a_2', a_3' \in E$

$$(a_0' + a_1'\alpha' + a_2'\beta' + a_3'\alpha'\beta')^* = a_0' - a_1'\alpha' - a_2'\beta' - a_3'\alpha'\beta'. \tag{2.10}$$

Hence $D(c,d) = D(c',d')$ and we can use the field $L = E(\alpha')$ and the isomorphism (2.4) for this field to split our algebra $D(c',d')$.

For a prime number l throughout the paper λ will denote an ideal in $\mathcal{O}_E$ such that $\lambda | l$ and w will denote an ideal of $\mathcal{O}_L$ such that $w | \lambda$.

3. Bilinear forms associated with abelian varieties of type III.
Put $\mathcal{R}_l = \mathcal{R} \otimes \mathbb{Z}_l$ and $D_l = D \otimes \mathbb{Q}_l$. The polarization of A gives a $\mathbb{Z}$-bilinear, non-degenerate, alternating pairing

$$\kappa \,:\, \Lambda \times \Lambda \to \mathbb{Z}. \tag{3.1}$$

which upon tensoring with $\mathbb{Z}_l$ ([Mi], diagram on page 133) becomes $\mathbb{Z}_l$-bilinear, non-degenerate, alternating pairing

$$\kappa_l \,:\, T_l(A) \times T_l(A) \to \mathbb{Z}_l, \tag{3.2}$$

easily derived from the Weil pairing. By assumption on l, for any $\alpha \in \mathcal{R}_l$ we get $\alpha^* \in \mathcal{R}_l$, (see [Mi] chapter 13 and 17) where α^* is the image of α via the Rosati involution. Hence for any $v, w \in T_l(A)$ we have $\kappa_l(\alpha v, w) = \kappa_l(v, \alpha^* w)$ (see loc. cit.). In case of an abelian variety which is not principally polarized, we take l that does not divide the degree of the polarisation of A to get $\alpha^* \otimes 1 \in \mathcal{R} \otimes \mathbb{Z}_l$ for $\alpha \in \mathcal{R}$. Let $V_l(A) = T_l(A) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$, and let $\kappa_l^0 \,:\, V_l(A) \times V_l(A) \to \mathbb{Q}_l$ be the bilinear form $\kappa_l \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$. By [BGK2] lemma 3.1 there is a unique $\mathcal{O}_{E_l}$-bilinear form

$$\phi_l \,:\, T_l(A) \times T_l(A) \to \mathcal{O}_{E_l} \tag{3.3}$$

such that $Tr_{E_l/\mathbb{Q}_l}(\phi_l(v_1, v_2)) = \kappa_l(v_1, v_2)$ for all $v_1, v_2 \in T_l(A)$. Put $\phi_l^0 = \phi_l \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$,

$$\phi_l^0 \,:\, V_l(A) \times V_l(A) \to E_l. \tag{3.4}$$

By uniqueness of the form ϕ_l for each $\alpha \in \mathcal{R}_l$ and for all $v_1, v_2 \in T_l(A)$

$$\phi_l(\alpha v_1,\, v_2) = \phi_l(v_1,\, \alpha^* v_2) \tag{3.5}$$

hence $\phi_l^0(\alpha v_1,\, v_2) = \phi_l^0(v_1,\, \alpha^* v_2)$ for each $\alpha \in D_l$ and for all $v_1, v_2 \in V_l(A)$. Define $\tilde{T}_l(A) = T_l(A) \otimes_{\mathcal{O}_E^0} \mathcal{O}_L$, $\tilde{V}_l(A) = V_l(A) \otimes_E L$ and $\tilde{\phi}_l = \phi_l \otimes_{\mathcal{O}_E^0} \mathcal{O}_L$

$$\tilde{\phi}_l \,:\, \tilde{T}_l(A) \times \tilde{T}_l(A) \,\to\, \mathcal{O}_{L_l}. \tag{3.6}$$

Hence $\tilde{\phi}_l^0 = \tilde{\phi}_l \otimes_{\mathcal{O}_L} L$ is the L_l bilinear form:

$$\tilde{\phi}_l^0 \,:\, \tilde{V}_l(A) \times \tilde{V}_l(A) \,\to\, L_l. \tag{3.7}$$

The form $\tilde{\phi}_l$ is non-degenerate iff ϕ_l is non-degenerate.

By (2.4) we get the following isomorphism

$$D \otimes_E L_l \cong M_{2,2}(L_l). \tag{3.8}$$

We treat $\mathcal{R} \otimes_{\mathcal{O}_E^0} \mathcal{O}_{L_l}$ as the subring of $M_{2,2}(L_l)$ via isomorphism (3.8). Since $\mathcal{R}$ is a finitely generated $\mathcal{O}_E^0$ module and $\mathcal{O}_w$ is a PID we note that for all $l \gg 0$ and all $w|l$ the isomorphism (3.8) gives an imbedding $\mathcal{R} \otimes_{\mathcal{O}_E^0} \mathcal{O}_w \subset M_{2,2}(\mathcal{O}_w)$ of free $\mathcal{O}_w$ modules of rank 4. Since $\mathcal{R}$ is an order of D the matrices $e_{ij} \in M_{2,2}(L_w)$ are actually in $\mathcal{R} \otimes_{\mathcal{O}_E^0} \mathcal{O}_w$ for $l \gg 0$ by (2.4), where e_{ij} has ij entry equal to 1 and all other entries equal to 0. So (3.8) induces an isomorphism $\mathcal{R} \otimes_{\mathcal{O}_E^0} \mathcal{O}_w \cong M_{2,2}(\mathcal{O}_w)$ hence also an isomorphism

$$\mathcal{R} \otimes_{\mathcal{O}_E^0} \mathcal{O}_{L_l} \cong M_{2,2}(\mathcal{O}_{L_l}) \tag{3.9}$$

From now on $l \gg 0$ be such that (3.9) holds. In particular for such l the matrix $J \in M_{2,2}(L_l)$ is an element of $\mathcal{R} \otimes_{\mathcal{O}_E^0} \mathcal{O}_{L_l}$.

Remark 3.10. *We should note that an isomorphism between both sides of (3.9) can be obtained by Corollary 11.6 p. 134 and Theorem 17.3 p. 171 of [R] for all* $l \gg 0$. *However these results give an isomorphism which comes from a conjugation by an element of* $D \otimes_E L_l \cong M_{2,2}(L_l)$ *(see [R] loc. cit.). To keep track of the action of the involution* $*$ *we prefer to use the isomorphism (3.9) induced by (3.8).*

Hence by (2.5) for each $B \in \mathcal{R} \otimes_{\mathcal{O}_E^0} \mathcal{O}_{L_l}$ and for all $v_1, v_2 \in \tilde{T}_l(A)$

$$\tilde{\phi}_l(Bv_1,\, v_2) = \tilde{\phi}_l(v_1, B^* v_2) = \tilde{\phi}_l(v_1,\, J\, B^t\, J^{-1} v_2).$$

Therefore for each $B \in M_{2,2}(L_l)$ and for all $v_1, v_2 \in \tilde{V}_l(A)$

$$\tilde{\phi}_l^0(Bv_1,\, v_2) = \tilde{\phi}_l^0(v_1, B^* v_2) = \tilde{\phi}_l^0(v_1,\, J\, B^t\, J^{-1} v_2).$$

Definition 3.11. *Define a new bilinear form* $\tilde{\psi}_l$ *as follows.*

$$\tilde{\psi}_l \,:\, \tilde{T}_l(A) \times \tilde{T}_l(A) \,\rightarrow\, \mathcal{O}_{L_l}$$
$$\tilde{\psi}_l(v_1, v_2) = \tilde{\phi}_l(Jv_1, v_2).$$

Let $\tilde{\psi}_l^0 = \tilde{\psi}_l \otimes_{\mathcal{O}_L} L$.

Proposition 3.12. *The form* $\tilde{\psi}_l$ *(*$\tilde{\psi}_l^0$ *resp.) is non-degenerate iff* $\tilde{\phi}_l$ *(*$\tilde{\phi}_l^0$ *resp.) is non-degenerate. For each* $v, w \in \tilde{T}_l(A)$ *and each* $B \in \mathcal{R} \otimes_{\mathcal{O}_E^0} \mathcal{O}_{L_l}$

$$\tilde{\psi}_l(Bv_1,\, v_2) = \tilde{\psi}_l(v_1,\, B^t v_2). \tag{3.13}$$

Similarly, for each $v_1, v_2 \in \tilde{V}_l$ *and each* $B \in M_{2,2}(L_l)$

$$\tilde{\psi}_l^0(Bv_1,\, v_2) = \tilde{\psi}_l^0(v_1,\, B^t v_2). \tag{3.14}$$

Moreover $\tilde{\psi}_l$ *(*$\tilde{\psi}_l^0$ *resp.) is symmetric (resp. antisymmetric) if and only if* $\tilde{\phi}_l$ *(*$\tilde{\phi}_l^0$ *resp.) is antisymmetric (resp. symmetric).*

Proof.

$$\tilde{\psi}_l(Bv_1, v_2) = \tilde{\phi}_l(JBv_1, v_2) = \tilde{\phi}_l(v_1,\, JB^t J^t J^{-1} v_2) = \tilde{\phi}_l(v_1,\, -JB^t v_2) =$$
$$= \tilde{\phi}_l(v_1,\, JJ^t J^{-1} B^t v_2) = \tilde{\phi}_l(Jv_1,\, B^t v_2) = \tilde{\psi}_l(v_1,\, B^t v_2).$$

The remaining claim follows by Definition 3.11 and by the observation that $J^t = J^{-1} = -J$ and $JJ^tJ^{-1} = -J$. □

Lemma 3.15. *For* $l \gg 0$ *the ideal* $\lambda \,|\, l$ *splits completely in one of the maximal commutative subfields of* $D = D(c, d)$.

Proof. Inverting a finite set S of primes of $\mathbb{Z}$ if necessary we can assume that $\mathcal{R}$ is a maximal order of D with $\mathcal{R} \cap E = \mathcal{O}_{E,S}$ and $\mathcal{R} = \mathcal{O}_{E,S} + \mathcal{O}_{E,S}\alpha + \mathcal{O}_{E,S}\beta + \mathcal{O}_{E,S}\alpha\beta$. We assume that $2 \in S$. Take $l \gg 0$ such that $l \notin S$ and such that all primes of $\mathcal{O}_{E,S}$ are unramified primes for the algebra D cf. [R], Th. 32.1. Let $t > 0$ be an element of $\lambda - \lambda^2$. By [R], Th. 22.4, Th. 22.15 and Th. 24.13 there is a maximal ideal $M \subset \mathcal{R}$ such that $Nr^0(M) = \lambda$. Let $m = a_0 + a_1\alpha + a_2\beta + a_3\alpha\beta \in M$ be such that $Nr^0(m) = t$. By formula (2.7) $t = a_0^2 - (a_1\alpha + a_2\beta + a_3\alpha\beta)^2$. Put $\alpha' := (a_1\alpha + a_2\beta + a_3\alpha\beta)$. Let $L := E(\alpha')$. The ring of S-integers of L is $\mathcal{O}_{L,S} = \mathcal{O}_{E,S}[\alpha']$. It follows by [R], Th. 32.1 that $a_0 \notin \lambda$. Indeed, if $a_0 \in \lambda$, then $(\alpha')^2 \in \lambda - \lambda^2$. Hence λ ramifies in $\mathcal{O}_{L,S}$. But this contradicts [R], Th. 32.1 (ii). Hence $(\alpha')^2$ is a square in $\mathcal{O}_\lambda^\times$. Hence λ splits in L. □

Remark 3.16. *We need Lemma 3.15 to have very explicit formula for the splitting isomorphism for our algebra D (cf. 2.4) and for the extension of the involution $*$ to this split algebra cf. (2.5).*

Proposition 3.17. *Let w be a prime of L over a prime λ which splits in L. Then the involution $*$ induced on $\mathcal{R} \otimes_{\mathcal{O}_E^0} \mathcal{O}_\lambda$ (on $M_{2,2}(E_\lambda)$ resp.) from D has the form $B^* = JB^tJ^{-1}$ for any $B \in \mathcal{R} \otimes_{\mathcal{O}_E^0} \mathcal{O}_\lambda$ (for any $B \in M_{2,2}(E_\lambda)$ resp.)*

Proof. By (2.4) and (2.5) for any $B \in M_{2,2}(L)$ we get $B^* = JB^tJ^{-1}$. But also by (2.4) and (3.8) we get:

$$D \otimes_E L_w \cong M_{2,2}(L) \otimes_L L_w \cong M_{2,2}(L_w), \tag{3.18}$$

hence for any $B \in M_{2,2}(L_w)$ we get $B^* = JB^tJ^{-1}$. Since λ splits completely in L the isomorphism (3.18) is just the isomorphism $D \otimes_E E_\lambda \cong M_{2,2}(E_\lambda)$ induced by (2.4). Hence for any $B \in M_{2,2}(E_\lambda)$ we get $B^* = JB^tJ^{-1}$ so identifying $\mathcal{R} \otimes_{\mathcal{O}_E^0} \mathcal{O}_\lambda$ with a subring of $M_{2,2}(E_\lambda)$ we get $B^* = JB^tJ^{-1}$ for any $B \in \mathcal{R} \otimes_{\mathcal{O}_E^0} \mathcal{O}_\lambda$. $\square$

Let e_λ be the idempotent corresponding to the decomposition $\mathcal{O}_{E_l} \cong \prod_{\lambda | l} \mathcal{O}_\lambda$. Put $T_\lambda(A) = e_\lambda T_l(A) \cong T_l(A) \otimes_{\mathcal{O}_{E_l}} \mathcal{O}_\lambda$, $V_\lambda(A) = T_\lambda(A) \otimes_{\mathcal{O}_\lambda} E_\lambda$. Define $\mathcal{O}_\lambda$- bilinear form $\phi_\lambda = \phi_l \otimes_{\mathcal{O}_E^0} \mathcal{O}_\lambda$. Put $\tilde{\phi}_w := \tilde{\phi}_l \otimes_{\mathcal{O}_{L_l}} \mathcal{O}_w$ and $\tilde{\psi}_w := \tilde{\psi}_l \otimes_{\mathcal{O}_{L_l}} \mathcal{O}_w$. For $l \gg 0$ such that λ splits in L, we have $\mathcal{O}_\lambda = \mathcal{O}_w$. Hence for such an l we get $\phi_\lambda = \tilde{\phi}_w$. This allows us to define the $\mathcal{O}_\lambda$- bilinear form

$$\begin{aligned} \psi_\lambda \,&:\, T_\lambda(A) \times T_\lambda(A) \to \mathcal{O}_\lambda, \\ \psi_\lambda(v_1, v_2) \,&=\, \phi_\lambda(Jv_1, v_2) \end{aligned} \tag{3.19}$$

for all $v_1, v_2 \in T_\lambda(A)$. Observe that $\psi_\lambda = \tilde{\psi}_w$. This gives us corresponding k_λ-bilinear form $\overline{\psi}_\lambda = \psi_\lambda \otimes_{\mathcal{O}_\lambda} k_\lambda$ and E_λ- bilinear form $\psi_\lambda^0 = \psi_\lambda \otimes_{\mathcal{O}_\lambda} E_\lambda$ respectively:

$$\overline{\psi}_\lambda \,:\, A[\lambda] \times A[\lambda] \to k_\lambda, \tag{3.20}$$

$$\psi_\lambda^0 \,:\, V_\lambda(A) \times V_\lambda(A) \to E_\lambda. \tag{3.21}$$

Corollary 3.22. *Let λ be a prime of E such that λ splits in L. Then for any $v_1, v_2 \in T_\lambda(A)$ and any $B \in \mathcal{R} \otimes_{\mathcal{O}_E^0} \mathcal{O}_\lambda$*

$$\psi_\lambda(Bv_1, v_2) \;=\; \psi_\lambda(v_1, B^t v_2)$$

Hence for any $v_1, v_2 \in A[\lambda]$ and any $B \in \mathcal{R} \otimes_{\mathcal{O}_E^0} k_\lambda \cong M_{2,2}(k_\lambda)$ (resp. for any $v_1, v_2 \in V_\lambda(A)$ and any $B \in M_{2,2}(E_\lambda)$)

$$\overline{\psi}_\lambda(Bv_1, v_2) \;=\; \psi_\lambda(v_1, B^t v_2)$$

$$(\textit{ resp. } \psi_\lambda^0(Bv_1, v_2) \;=\; \psi_\lambda(v_1, B^t v_2)\ )$$

Proof. The Corollary follows from Propositions 3.12 and 3.17. $\square$

Remark 3.23. *All bilinear forms $\psi_\lambda, \overline{\psi}_\lambda$ and ψ_λ^0 are symmetric and non-degenerate. This follows by results of this section, Lemmas 3.1 and 3.2 of [BGK2] and by the non-degeneracy of the independent of l pairing (3.1).*

We proceed to investigate some natural Galois actions. From now on, we assume that $\mathcal{R} = End_{\bar{F}}(A) = End_F(A)$. Consider the representations:

$$\rho_l \,:\, G_F \to GL(T_l(A))$$

$$\rho_l^0 \,:\, G_F \to GL(V_l(A))$$

$$\overline{\rho}_l \,:\, G_F \to GL(A[l])$$

Let $\mathcal{G}_l^{alg}$ be the Zariski closure of $\rho_l(G_F)$ in $GL_{T_l(A)}$ and let G_l^{alg} be the Zariski closure of $\rho_l^0(G_F)$ in $GL_{V_l(A)}$. Let $G(l)^{alg}$ be the special fiber of $\mathcal{G}_l^{alg}/\mathbb{Z}_l$. Note that G_l^{alg} is the general fiber of $\mathcal{G}_l^{alg}/\mathbb{Z}_l$. This gives natural representations:

$$\rho_\lambda \,:\, G_F \to GL(T_\lambda(A))$$

$$\rho_\lambda^0 \,:\, G_F \to GL(V_\lambda(A))$$

$$\overline{\rho}_\lambda \,:\, G_F \to GL(A[\lambda])$$

We can also define $\mathcal{G}_\lambda^{alg}$ to be the Zariski closure of $\rho_\lambda(G_F)$ in $GL_{T_\lambda(A)}$ and let G_λ^{alg} be the Zariski closure of $\rho_\lambda^0(G_F)$ in $GL_{V_\lambda(A)}$. Let $G(\lambda)^{alg}$ be the special fiber of $\mathcal{G}_\lambda^{alg}/\mathcal{O}_\lambda$. Then, G_λ^{alg} is the general fiber of $\mathcal{G}_\lambda^{alg}/\mathcal{O}_\lambda$.

Lemma 3.24. *Let $\chi_\lambda \,:\, G_F \to \mathbb{Z}_l \subset \mathcal{O}_\lambda$ be the composition of the cyclotomic character with the natural imbedding $\mathbb{Z}_l \subset \mathcal{O}_\lambda$. Let $l \gg 0$ be such that $\lambda | l$ be a prime of E which splits in L.*

(i) *For any $\sigma \in G_F$ and all $v_1, v_2 \in T_\lambda(A)$*

$$\psi_\lambda(\sigma v_1, \sigma v_2) = \chi_\lambda(\sigma)\psi_\lambda(v_1, v_2).$$

(ii) *For any $B \in \mathcal{R} \otimes_{\mathcal{O}_E^0} \mathcal{O}_\lambda$ and all $v_1, v_2 \in T_\lambda(A)$*

$$\psi_\lambda(Bv_1, v_2) = \psi_\lambda(v_1, B^t v_2).$$

Proof. The part (i) follows by [C2 Lemma 2.3] or [BGK2, Lemma 4.7] and also by (3.19) and by definition 3.11 because the G_F action commutes with the action of $\mathcal{R}$ on $T_l(A)$. Indeed [C2 Lemma 2.3] and [BGK2, Lemma 4.7] concern the pairing ϕ_λ but $\psi_\lambda(v, w) = \phi_\lambda(Jv, w)$ and J commutes with the G_F-action by assumption so we get immediately statement (i) for ψ_λ. The part (ii) of the Lemma follows by Corollary 3.22. □

Let $D_\lambda := D \otimes_E E_\lambda$ $\mathcal{R}_\lambda := \mathcal{R} \otimes_{\mathcal{O}_E^0} \mathcal{O}_\lambda$. By [Fa], Th. 3 and [BGK2] lemma 4.17 G_F acts on both $V_\lambda(A)$ and $A[\lambda]$ semi-simply and G_λ^{alg} and $G(\lambda)^{alg}$ are reductive algebraic groups. Hence $\mathcal{G}_\lambda^{alg}$ is a reductive group scheme over $\mathcal{O}_\lambda$ for $l \gg 0$ by [LP] Prop. 1.3 cf. [Wi] Th. 1.

Take $\lambda | l$ a prime of E which splits in L. Then by (2.7) $D_\lambda \cong M_{2,2}(E_\lambda)$. By (2.8) $R_\lambda \cong M_{2,2}(\mathcal{O}_\lambda)$, so $R_\lambda \otimes_{\mathcal{O}_\lambda} k_\lambda \cong M_{2,2}(k_\lambda)$. Let

$$t = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let $f = \frac{1}{2}(1+u)$, $\mathcal{X} = f\,T_\lambda(A)$, $\mathcal{Y} = (1-f)\,T_\lambda(A)$. Put $X = \mathcal{X} \otimes_{\mathcal{O}_\lambda} E_\lambda$, $Y = \mathcal{Y} \otimes_{\mathcal{O}_\lambda} E_\lambda$, $\overline{\mathcal{X}} = \mathcal{X} \otimes_{\mathcal{O}_\lambda} k_\lambda$, $\overline{\mathcal{Y}} = \mathcal{Y} \otimes_{\mathcal{O}_\lambda} k_\lambda$. Because $tft = 1-f$, the matrix t gives an $\mathcal{O}_\lambda[G_F]$-isomorphism between $\mathcal{X}$ and $\mathcal{Y}$, hence it gives an $E_\lambda[G_F]$-isomorphism between X and Y and a $k_\lambda[G_F]$-isomorphism between $\overline{\mathcal{X}}$ and $\overline{\mathcal{Y}}$. Using the computations of endomorphism algebras by [Fa], Satz 4 and [Za], Cor. 5.4.5 we get:

$$End_{\mathcal{O}_\lambda[G_F]}(\mathcal{X}) = \mathcal{O}_\lambda \tag{3.25}$$

$$End_{E_\lambda[G_F]}(X) = E_\lambda \tag{3.26}$$

$$End_{k_\lambda[G_F]}(\overline{\mathcal{X}}) = k_\lambda. \tag{3.27}$$

So the representations of G_F on the spaces X and Y (resp. $\overline{\mathcal{X}}$ and $\overline{\mathcal{Y}}$) are absolutely irreducible over E_λ (resp. over k_λ). Hence, the bilinear form ψ_λ^0 (resp. $\overline{\psi}_\lambda$) when restricted to any of the spaces X, Y (resp. spaces $\overline{\mathcal{X}}$ and $\overline{\mathcal{Y}}$) is either non-degenerate or isotropic.

Lemma 3.28. *The modules $\mathcal{X}$ and $\mathcal{Y}$ are orthogonal with respect to ψ_λ. Consequently the modules X and Y (resp. $\overline{\mathcal{X}}$ and $\overline{\mathcal{Y}}$) are orthogonal with respect to ψ_λ^0 (resp. $\overline{\psi}_\lambda$).*

Proof. Note that $uf = f$ and $u(1-f) = -(1-f)$. Hence for every $v \in \mathcal{X}$ and for every $w \in \mathcal{Y}$ we get $uv = v$ and $uw = -w$. Hence

$$\psi_\lambda(v,w) = \psi_\lambda(uv,w) = \psi_\lambda(v,u^t w) = \psi_\lambda(v,uw) = \psi_\lambda(v,-w) = -\psi_\lambda(v,w).$$

Hence $\psi_\lambda(v,w) = 0$ for every $v \in \mathcal{X}$ and for every $w \in \mathcal{Y}$. $\square$

Theorem 3.29. *Let A be of type III and $l \gg 0$. Then there is a free $\mathcal{O}_\lambda$-module $\mathcal{W}_\lambda(A)$ of rank $2h$ such that*

(i) *$T_\lambda(A) \cong \mathcal{W}_\lambda(A) \oplus \mathcal{W}_\lambda(A)$ as $\mathcal{O}_\lambda[G_F]$- modules*
(ii) *there exists a symmetric, non-degenerate pairing* $\psi_\lambda : \mathcal{W}_\lambda(A) \times \mathcal{W}_\lambda(A) \to \mathcal{O}_\lambda$
(ii') *For $W_\lambda(A) = \mathcal{W}_\lambda(A) \otimes_{\mathcal{O}_\lambda} E_\lambda$ the induced symmetric pairing $\psi_\lambda^0 : W_\lambda(A) \times W_\lambda(A) \to E_\lambda$ is non-degenerate. The G_F module $W_\lambda(A)$ is absolutely irreducible.*
(ii") *For $\overline{\mathcal{W}}_\lambda(A) = \mathcal{W}_\lambda(A) \otimes_{\mathcal{O}_\lambda} k_\lambda$ the induced symmetric pairing $\overline{\psi}_\lambda : \overline{\mathcal{W}}_\lambda(A) \times \overline{\mathcal{W}}_\lambda(A) \to k_\lambda$ is non-degenerate. The G_F module $\overline{\mathcal{W}}_\lambda(A)$ is absolutely irreducible.*

Pairings (ii), (ii') and (ii") are compatible with the G_F-action in the same way as the pairing in Lemma 3.24 (i).

Proof. Part (i) follows by taking $\mathcal{W}_\lambda(A) = \mathcal{X}$. We get part (ii) restricting ψ_λ to $\mathcal{X}$. To finish the proof observe that the form (3.2) is non-degenerate so $\overline{\psi}_l = \psi_l \otimes \mathbb{F}_l$ is non-degenerate for any abelian variety with polarization degree prime to l. By Lemma 3.2 [BGK2] the form $\overline{\psi}_\lambda$ is non-degenerate for all λ hence the forms ψ_λ^0 and $\overline{\psi}_\lambda$ are simultaneously non-degenerate. Hence (ii') and (ii") follow by and (3.26) and (3.27) and also by Remark 3.23, Lemma 3.28. $\square$

4. Representations associated with Abelian varieties of type III.

Let A/F be an abelian variety of type III. The field of definition F is such that G_l^{alg} is a connected algebraic group. Let us put $T_\lambda = \mathcal{W}_\lambda(A)$, $V_\lambda = T_\lambda \otimes_{\mathcal{O}_\lambda} E_\lambda$ and $A_\lambda = V_\lambda / T_\lambda$. With this notation by Theorem 3.29 we have $V_l(A) = \bigoplus_{\lambda|l} \big(V_\lambda \oplus V_\lambda\big)$. We put

$$V_l \;=\; \bigoplus_{\lambda|l} V_\lambda \tag{4.1}$$

Let V_{Φ_λ} be the space V_λ considered over $\mathbb{Q}_l$. Then there is the following equality of $\mathbb{Q}_l$-vector spaces:

$$V_l \;=\; \bigoplus_{\lambda|l} V_{\Phi_\lambda} \tag{4.2}$$

The l-adic representation

$$\rho_l^0 \,:\, G_F \longrightarrow GL(V_l(A)) \tag{4.3}$$

induces the following representations (note that we use the notation ρ_l^0 for both representations (4.4) and (4.5) cf. Remark 5.13 [BGK2]):

$$\rho_l^0 : G_F \longrightarrow GL(V_l) \tag{4.4}$$

$$\rho_\lambda^0 : G_F \longrightarrow GL(V_\lambda) \tag{4.5}$$

The representation ρ_{Φ_λ} was defined in [BGK2]:

$$\rho_{\Phi_\lambda} : G_F \longrightarrow GL(V_{\Phi_\lambda}). \tag{4.6}$$

By Theorem 3.29 (cf. [BGK2], Remark 5.13) the group scheme G_l^{alg} (resp. G_λ^{alg}) is naturally isomorphic to the Zariski closure in GL_{V_l} (resp. GL_{V_λ}) of the image of the representation ρ_l of (4.4) (resp. ρ_λ of (4.5)). Let $G_{\Phi_\lambda}^{alg}$ denote the Zariski closure in $GL_{V_{\Phi_\lambda}}$ of the image of the representation ρ_{Φ_λ} of (4.6). Let $\mathfrak{g}_l = Lie(G_l^{alg})$, $\mathfrak{g}_\lambda = Lie(G_\lambda^{alg})$ and let $\mathfrak{g}_{\Phi_\lambda} = Lie(G_{\Phi_\lambda}^{alg})$. By definition $G_l^{alg} \subset \prod_{\lambda|l} G_{\Phi_\lambda}^{alg}$ so $\mathfrak{g}_l \subset \bigoplus_{\lambda|l} \mathfrak{g}_{\Phi_\lambda}$. This implies:

$$(G_l^{alg})' \,\subset\, \prod\nolimits_{\lambda|l} (G_{\Phi_\lambda}^{alg})' \tag{4.7}$$

$$\mathfrak{g}_l^{ss} \,\subset\, \bigoplus\nolimits_{\lambda|l} \mathfrak{g}_{\Phi_\lambda}^{ss}. \tag{4.8}$$

Similarly to previos section we work with abelian varieties of type III. In this section we compute the Lie algebras corresponding to representations we consider. Some results that we proved in [BGK2] for abelian varieties of type I and II work as well for abelian varieties of type III. Since the detailed proofs of these results were given in [BGK2] we will merely reformulate them for abelian varieties of type III.

Lemma 4.9. *The following natural map of Lie algebras:*

$$\mathfrak{g}_l^{ss} \to \mathfrak{g}_{\Phi_\lambda}^{ss}$$

is surjective.

Proof. The proof is the same as the proof of Lemma 5.20 of [BGK2]. □

Lemma 4.10. *Let A/F be an abelian variety over F of type III such that $End_F\,(A) = End_{\overline{F}}\,(A)$. Then*

$$End_{\mathfrak{g}_\lambda}\,(V_\lambda) \,\cong\, End_{E_\lambda[G_F]}\,(V_\lambda) \,\cong\, E_\lambda$$

$$End_{\mathfrak{g}_{\Phi_\lambda}}\,(V_{\Phi_\lambda}) \,\cong\, End_{\mathbb{Q}_l[G_F]}\,(V_{\Phi_\lambda}) \,\cong\, E_\lambda.$$

Proof. The same proof as the proof of Lemma 5.22 of [BGK2]. □

Define the following subgroups of $GL(V_\lambda)$:

$$GO_{(V_\lambda,\psi_\lambda)} = \{A \in GL_{V_\lambda} : \psi_\lambda(Av_1, Av_2) = c_\lambda(A)\psi_\lambda(v_1, v_2) \text{ for all } v_1, v_2 \in V_\lambda\}$$

$$O_{(V_\lambda,\psi_\lambda)} = \{A \in GL_{V_\lambda} : \psi_\lambda(Av_1, Av_2) = \psi_\lambda(v_1, v_2) \text{ for all } v_1, v_2 \in V_\lambda\}$$

Denote by $SO_{(V_\lambda,\psi_\lambda)}$ the connected component of identity in $O_{(V_\lambda,\psi_\lambda)}$ By Lemma 3.24 we see that $\rho_\lambda(G_F) \subset GO_{(V_\lambda,\psi_\lambda)}$ and therefore $G_\lambda^{alg} \subset GO_{(V_\lambda,\psi_\lambda)}$. This of course implies that $(G_\lambda^{alg})' \subset O_{(V_\lambda,\psi_\lambda)}$. Extending the base field F if necessary one can assume that G_λ^{alg} and hence $(G_\lambda^{alg})'$ are connected (cf. [C1] Prop.(3.6)). This gives the inclusions

$$(G_\lambda^{alg})' \subset SO_{(V_\lambda,\psi_\lambda)} \tag{4.11}$$

and

$$\mathfrak{g}_\lambda^{ss} \subset so_{(V_\lambda,\psi_\lambda)}. \tag{4.12}$$

Lemma 4.13. $\mathfrak{g}_\lambda^{ss} = so_{(V_\lambda,\psi_\lambda)}$.

Proof. The proof is similar to the proof of Lemma 3.2 [BGK1] and Lemma 5.33 [BGK2]. Since type III is more exotic then type I and II, we will give here a complete proof. Observe that the minuscule conjecture for the λ-adic representations $\rho_F : G_F \to GL(V_\lambda)$ holds. Namely by [P], Corollary 5.11, we know that $\mathfrak{g}_l^{ss} \otimes \bar{\mathbb{Q}}_l$ may only have simple factors of types A, B, C or D with minuscule weights. By Lemma 4.9 the natural map of Lie algebras

$$\mathfrak{g}_l^{ss} \to \mathfrak{g}_{\Phi_\lambda}^{ss} \tag{4.14}$$

is surjective. Hence by the semisimplicity of $\mathfrak{g}_l^{ss}$ the simple factors of $\mathfrak{g}_{\Phi_\lambda}^{ss} \otimes \bar{\mathbb{Q}}_l$ are are also of types A, B, C or D with minuscule weights. By Proposition 2.12 of [BGK2]and Lemmas 2.21, 2.22, 2.23 of [BGK2] there is an isomorphism of $\mathbb{Q}_l$ Lie algebras:

$$\mathfrak{g}_{\Phi_\lambda}^{ss} \cong R_{E_\lambda/\mathbb{Q}_l}\mathfrak{g}_\lambda^{ss}. \tag{4.15}$$

The isomorphisms $\mathfrak{g}_{\Phi_\lambda}^{ss} \otimes_{\mathbb{Q}_l} \overline{\mathbb{Q}}_l \cong \mathfrak{g}_\lambda^{ss} \otimes_{E_\lambda} E_\lambda \otimes_{\mathbb{Q}_l} \overline{\mathbb{Q}}_l \cong \bigoplus_{E_\lambda \hookrightarrow \overline{\mathbb{Q}}_l} \mathfrak{g}_\lambda^{ss} \otimes_{E_\lambda} \overline{\mathbb{Q}}_l$ imply that simple factors of $\mathfrak{g}_\lambda^{ss} \otimes_{E_\lambda} \overline{\mathbb{Q}}_l$ are of types A, B, C or D with minuscule weights. Put $\bar{V}_\lambda = V_\lambda \otimes \overline{\mathbb{Q}}_l$. We have the decomposition:

$$\bar{V}_\lambda = E(\omega_1) \otimes_{\overline{\mathbb{Q}}_l} \cdots \otimes_{\overline{\mathbb{Q}}_l} E(\omega_r),$$

where $E(\omega_i)$, for all $1 \leq i \leq r$, are the irreducible Lie algebra modules of the highest weight ω_i. The modules $E(\omega_i)$ correspond to simple Lie algebras $\mathfrak{g}_i$ which are summands of the image

$$Im\left(\mathfrak{g}_\lambda^{ss} \otimes \overline{\mathbb{Q}}_l \rightarrow so_{2h}(\overline{V}_\lambda) \right) = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r.$$

By [B] Chap.VIII Proposition 12 $E(\omega_i)$ are symplectic or orthogonal. By Corollary 5.11 [P] all simple factors of $\mathfrak{g}_\lambda^{ss} \otimes \overline{\mathbb{Q}}_l$ are of classical type A, B, C and D and the weights $\omega_1, \ldots, \omega_r$ are minuscule. All minuscule weights and dimensions of representations are listed in [B], Chap. VIII, Tables 1 and 2 and in [H2], p. 72. Since h is odd, the investigation of the tables of minuscule weights and the dimensions of associated representations shows that the tensor product can contain only one factor which is orthogonal and is either of type D_n, weight w_1 and dimension $2n$ or of type A_{4k+3}, weight w_{2k+2} and dimension $\binom{4k+4}{2k+2}$. Hence V_λ is an irreducible g_λ^{ss}-module and we get

$$g_\lambda^{ss} = so_{(V_\lambda, \psi_\lambda)}.$$

□

Lemma 4.16. *There are natural isomorphisms of $\mathbb{Q}_l$-algebras.*

$$End_{\mathfrak{g}_{\Phi_\lambda}^{ss}}(V_{\Phi_\lambda}) \cong End_{\mathfrak{g}_\lambda^{ss}}(V_\lambda) \cong E_\lambda$$

Proof. The same proof as the proof of Lemma 5.35 of [BGK2]. □

Proposition 4.17. *There is an equality of Lie algebras:*

$$\mathfrak{g}_l^{ss} = \bigoplus_{\lambda | l} \mathfrak{g}_{\Phi_\lambda}^{ss} \tag{4.18}$$

Proof. By use of (4.8) and Lemma 4.16 the proof is the same as the proof of Proposition 5.39 of [BGK2]. □

Theorem 4.19. *There is an equality of group schemes over $\mathbb{Q}_l$:*

$$(G_l^{alg})' = \prod_{\lambda | l} R_{E_\lambda/\mathbb{Q}_l} SO_{(V_\lambda, \psi_\lambda)}. \tag{4.20}$$

Proof. By [BGK2], Prop. 2.12 we get:

$$G_{\Phi_\lambda}^{alg} \cong R_{E_\lambda/\mathbb{Q}_l} G_\lambda^{alg} \subset R_{E_\lambda/\mathbb{Q}_l} GO_{(V_\lambda, \psi_\lambda)}. \tag{4.21}$$

Hence it follows from [BGK2], Lemma 2.23 that

$$(G_{\Phi_\lambda}^{alg})' \subset R_{E_\lambda/\mathbb{Q}_l} SO_{(V_\lambda, \psi_\lambda)}. \tag{4.22}$$

By (4.7) and (4.22) we have an imbedding of two connected group schemes over $\mathbb{Q}_l$:

$$(G_l^{alg})' \subset \prod_{\lambda | l} R_{E_\lambda/\mathbb{Q}_l} SO_{(V_\lambda, \psi_\lambda)}.$$

But this imbedding induces the Lie algebra isomorphism of Proposition 4.17 hence the theorem follows by Prop. 4.17 and [H1], Theorem on page 87 and [H1], Prop. on page 110. □

5. Mumford-Tate conjecture for abelian varieties of type III.

Choose an imbedding of F into the field of complex numbers $\mathbb{C}$. Let $V = H^1(A(\mathbb{C}), \mathbb{Q})$ be the singular cohomology group with rational coefficients. Consider the Hodge decomposition

$$V \otimes_{\mathbb{Q}} \mathbb{C} = H^{1,0} \oplus H^{0,1},$$

where $H^{p,q} = H^p(A; \Omega^q_{A/\mathbb{C}})$ and $\overline{H^{p,q}} = H^{q,p}$. Observe that $H^{p,q}$ are invariant subspaces with respect to $D = End_{\overline{F}}(A) \otimes \mathbb{Q}$ action on $V \otimes_{\mathbb{Q}} \mathbb{C}$. Hence, in particular $H^{p,q}$ are E-vector spaces. Let

$$\kappa : V \times V \to \mathbb{Q}$$

be the $\mathbb{Q}$-bilinear, nondegenerate, alternating form coming from the Riemann form of A. Define the cocharacter

$$\mu_\infty : \mathbb{G}_m(\mathbb{C}) \to GL(V \otimes_{\mathbb{Q}} \mathbb{C}) = GL_{2g}(\mathbb{C})$$

such that, for any $z \in \mathbb{C}^\times$, the automorphism $\mu_\infty(z)$ is the multiplication by z on $H^{1,0}$ and the identity on $H^{0,1}$.

Definition 5.1. *The Mumford-Tate group of the abelian variety A/F is the smallest algebraic subgroup $MT(A) \subset GL_{2g}$, defined over $\mathbb{Q}$, such that $MT(A)(\mathbb{C})$ contains the image of μ_∞. The Hodge group $H(A)$ is by definition the connected component of the identity in $MT(A) \cap SL_V$.*

$MT(A)$ is a reductive group (see [D], [G]). Since, by definition

$$\mu_\infty(\mathbb{C}^\times) \subset GSp_{(V,\kappa)}(\mathbb{C}),$$

it follows that the group $MT(A)$ is a subgroup of the group of symplectic similitudes $GSp_{(V,\kappa)}$ and that

$$H(A) \subset Sp_{(V,\kappa)}. \tag{5.2}$$

Definition 5.3. *The algebraic group $L(A) = C_D^\circ(Sp_{(V,\,\kappa)})$ is called the Lefschetz group of an abelian variety A. Note that the group $L(A)$ does not depend on the form κ cf. [R2].*

Before investigating further Mumford-Tate group let us make two general remarks concerning centralizers of group schemes which we will often use.

Remark 5.4. Let $B_1 \subset B_2$ be two commutative rings with identity. Let Λ be a free, finitely generated B_1-module such that it is also an R-module for a B_1-algebra R. Let G be a B_1-group subscheme of GL_Λ. Then $C_R(G)$ will denote the centralizer of R in G. The symbol $C_R^\circ(G)$ will denote the connected component of identity in $C_R(G)$. Let $\beta \,:\, \Lambda \times \Lambda \to B_1$ be a bilinear form and let $G_{(\Lambda,\beta)} \subset GL_\Lambda$ be the subscheme of GL_Λ of the isometries with respect to the form β. Then we check that $C_R(G_{(\Lambda,\beta)}) \otimes_{B_1} B_2 \cong C_{R\otimes_{B_1} B_2}(G_{(\Lambda\otimes_{B_1} B_2,\, \beta\otimes_{B_1} B_2)})$.

Remark 5.5. Let L/K be a finite separable field extension. Let V be a finite dimensional vector space and let $\phi \,:\, V \times V \to L$ be a nondegenerate bilinear form. Assume that $G_{(V,\phi)}$ is a connected algebraic group. Then there is natural isomorphism $R_{L/K} G_{(V,\phi)} \cong C_L^\circ(G_{(V,\, Tr_{L/K}\phi)})$.

By [D], Sublemma 4.7, there is a unique E-bilinear, nondegenerate, alternating pairing

$$\phi \,:\, V \times V \to E$$

such that $Tr_{E/\mathbb{Q}}(\phi) = \kappa$. Since the actions of $H(A)$ and $L(A)$ on V commute with the D-structure, and since $R_{E/\mathbb{Q}}(Sp_{(V,\,\phi)}) = C_E(Sp_{(V,\,\kappa)})$ by Remark 5.5 we get

$$H(A) \subset L(A) = C_D^\circ(R_{E/\mathbb{Q}}(Sp_{(V,\,\phi)})) \subset C_D(R_{E/\mathbb{Q}}(Sp_{(V,\,\phi)})). \tag{5.6}$$

If $L/\mathbb{Q}$ is a field extension of $\mathbb{Q}$ we put

$$MT(A)_L := MT(A) \otimes_{\mathbb{Q}} L, \quad H(A)_L := H(A) \otimes_{\mathbb{Q}} L, \quad L(A)_L := L(A) \otimes_{\mathbb{Q}} L.$$

Conjecture 5.7 (Mumford-Tate cf. [Se5], C.3.1). *If A/F is an abelian variety over a number field F, then for any prime number l*

$$(G_l^{alg})^\circ = MT(A)_{\mathbb{Q}_l}, \tag{5.8}$$

where $(G_l^{alg})^\circ$ denotes the connected component of the identity.

Theorem 5.9 (Deligne [D], I, Prop. 6.2). *If A/F is an abelian variety over a number field F and l is a prime number, then*

$$(G_l^{alg})^\circ \subset MT(A)_{\mathbb{Q}_l}. \tag{5.10}$$

Theorem 5.11. *The Mumford-Tate conjecture holds true for abelian varieties of class $\mathcal{B}$.*

Proof. It is enough to verify (5.8) for a single prime l only due to [LP], Theorem 4.3. Hence we can use the equality (4.20) for a big enough prime l. The proof goes similarly to the proof of Theorem 7.12 in [BGK2], although we need to make careful transition, at certain point, from symplectic form to symmetric form to be able to apply the results of previous sections of this paper. It is known that $H(A)$ is semisimple (cf. [G], Prop. B.63) and the center of $MT(A)$ is $\mathbb{G}_m$ (cf. [G], Cor. B.59). In addition $MT(A) = \mathbb{G}_m H(A)$, hence

$$(MT(A)_{\mathbb{Q}_l})' = (H(A)_{\mathbb{Q}_l})' = H(A)_{\mathbb{Q}_l}. \tag{5.12}$$

By (4.20), (5.9) and (5.12)

$$\prod_{\lambda | l} R_{E_\lambda/\mathbb{Q}_l}(SO_{(V_\lambda, \psi_\lambda^0)}) \subset H(A)_{\mathbb{Q}_l}. \tag{5.13}$$

By (5.6) and Remark 5.5 we have:

$$H(A)_{\mathbb{Q}_l} \subset L(A)_{\mathbb{Q}_l} \subset$$

$$\subset C_D(R_{E/\mathbb{Q}}(Sp_{(V,\,\phi)})) \otimes_{\mathbb{Q}} \mathbb{Q}_l \cong \prod_{\lambda | l} C_{D_\lambda}(R_{E_\lambda/\mathbb{Q}_l}(Sp_{(V_\lambda(A),\, \phi_\lambda^0)})). \tag{5.14}$$

where $\kappa_l = \kappa \otimes_{\mathbb{Q}} \mathbb{Q}_l$ and κ_l is essentially the Weil pairing cf. [Mi], diagram on p. 133. By definition of the forms ϕ_λ and ψ_λ we have:

$$C_{D_\lambda}(Sp_{(V_\lambda(A),\, \phi_\lambda^0)}) \cong C_{D_\lambda}(SO_{(V_\lambda(A),\, \psi_\lambda^0)}). \tag{5.15}$$

So by (5.13), (5.14) and (5.15) we have

$$\prod_{\lambda | l} R_{E_\lambda/\mathbb{Q}_l}(SO_{(V_\lambda, \psi_\lambda^0)}) \subset H(A)_{\mathbb{Q}_l} \subset$$

$$\subset L(A)_{\mathbb{Q}_l} \subset \prod_{\lambda | l} C_{D_\lambda}(R_{E_\lambda/\mathbb{Q}_l}(SO_{(V_\lambda(A),\, \psi_\lambda^0)})). \tag{5.16}$$

Observe that $V_\lambda(A) \cong V_\lambda \oplus V_\lambda$ by Theorem 3.29. Moreover $D_\lambda = M_{2,2}(E_\lambda)$ by assumption on λ. Hence evaluating left and right sides of the inclusion (5.16) on the $\overline{\mathbb{Q}}_l$-points, we get equality with both sides equal to

$$\prod_{\lambda | l} \prod_{E_\lambda \hookrightarrow \overline{\mathbb{Q}}_l} (SO_{(V_\lambda, \psi_\lambda)})(\overline{\mathbb{Q}}_l)$$

which is an irreducible algebraic variety over $\overline{\mathbb{Q}}_l$. Then we use Prop. II, 2.6 and Prop. II, 4.10 of [H] in order to conclude that the groups $H(A)_{\overline{\mathbb{Q}}_l}$, $L(A)_{\overline{\mathbb{Q}}_l}$ and $\prod_{\lambda|l} C_{D_\lambda}(R_{E_\lambda/\mathbb{Q}_l}(SO_{(V_\lambda(A),\, \psi_\lambda^0)})) = \prod_{\lambda|l} C_{D_\lambda}(SO_{(V_\lambda(A),\, \psi_\lambda^0)}))$ are connected and (5.16) becomes the following equality by use of Remark 5.5:

$$\prod_{\lambda | l} R_{E_\lambda/\mathbb{Q}_l}(SO_{(V_\lambda, \psi_\lambda^0)}) = H(A)_{\mathbb{Q}_l} = L(A)_{\mathbb{Q}_l} = \prod_{\lambda | l} C_{D_\lambda}(SO_{(V_\lambda(A),\, \psi_\lambda^0)})). \tag{5.17}$$

The equalities (4.20), (5.17) and [Bo], Corollary 1. p. 702 give

$$MT(A)_{\mathbb{Q}_l} = \mathbb{G}_m H(A)_{\mathbb{Q}_l} = \mathbb{G}_m (G_l^{alg})' \subset G_l^{alg}. \tag{5.18}$$

The Theorem follows by (5.10) and (5.18). $\square$

Corollary 5.19. *If A is an abelian variety of class $\mathcal{B}$ then*

$$H(A) = L(A) = C_D^{\circ}(R_{E/\mathbb{Q}}(Sp_{(V,\,\phi)})) = C_D(R_{E/\mathbb{Q}}(Sp_{(V,\,\phi)})) \tag{5.20}$$

Proof. By (5.6) and (5.17) we get equality of Lie algebras

$$\mathcal{L}ie\, H(A) = \mathcal{L}ie\, L(A) = \mathcal{L}ie\, C_D^{\circ}(R_{E/\mathbb{Q}}(Sp_{(V,\,\phi)})) = \mathcal{L}ie\, C_D(R_{E/\mathbb{Q}}(Sp_{(V,\,\phi)}))$$

of connected group schemes. Hence (5.20) follows by (5.6) and Theorem p. 87 [H1].

Conjecture 5.21 (Lang). *Let A be an abelian variety over a number field F. Then for $l \gg 0$ the group $\rho_l(G_F)$ contains the group of all homotheties in $GL_{T_l(A)}(\mathbb{Z}_l)$.*

Theorem 5.22 (Wintenberger [Wi], Cor. 1, p.5). *Let A be an abelian variety over a number field F. The Lang conjecture holds for such abelian varieties A for which the Mumford-Tate conjecture holds or if $\dim A < 5$.*

Theorem 5.23. *The Lang's conjecture holds true for abelian varieties of class $\mathcal{B}$.*

Proof. It follows by Theorem 5.11 and Theorem 5.22. $\square$

Consider the bilinear form (3.1). Abusing notation sligthly, we will denote also by κ the Riemann form $\kappa \otimes_{\mathbb{Z}} \mathbb{Q}$, i.e.:

$$\kappa \ :\ V \times V \ \to\ \mathbb{Q}. \tag{5.24}$$

We have:

$$H^1(A(\mathbb{C});\, \mathbb{C}) \,\cong\, V \otimes_{\mathbb{Q}} \mathbb{C} \,\cong\, \bigoplus_{\sigma : E \hookrightarrow \mathbb{C}} V \otimes_{E,\sigma} \mathbb{C}.$$

Put $V_\sigma(A) = V \otimes_{E,\sigma} \mathbb{C}$ and let κ_σ be the form

$$\kappa \otimes_{E,\sigma} \mathbb{C} \,:\, V_\sigma(A) \otimes_{\mathbb{C}} V_\sigma(A) \,\to\, \mathbb{C}.$$

Since A is of type III there is an isomorphism $D \otimes_E \mathbb{C} \cong D \otimes_E \mathbb{R} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong M_{2,2}(\mathbb{C})$. Define the bilinear form:

$$\psi_\sigma \,:\, V_\sigma(A) \times V_\sigma(A) \to \mathbb{C}$$

by the formula

$$\psi_\sigma(v_1, v_2) := \kappa_\sigma(Jv_1, v_2).$$

Lemma 5.25. *If A is simple abelian variety of type III, then for each $\sigma : E \hookrightarrow \mathbb{C}$ there is an $\mathbb{C}$-vector space $W_\sigma(A)$ of dimension $\frac{g}{e} = \frac{4\,dim A}{[D\colon \mathbb{Q}]}$ such that:*

(i) $V_\sigma(A) \cong W_\sigma(A) \oplus W_\sigma(A)$,
(ii) *the restriction of ψ_σ to $W_\sigma(A)$ gives a nondegenerate, alternating pairing*

$$\psi_\sigma \,:\, W_\sigma(A) \times W_\sigma(A) \to \mathbb{C}.$$

Proof. The idea of the proof is the same as the proof of Theorem 3.29. Namely, using some arguments that gave the proof of Corollary 3.22 we show that: $\psi_\sigma(Bv_1, v_2) = \psi_\sigma(v_1, B^t v_2)$ for every $B \in M_{2,2}(\mathbb{C})$. Let $t, u, f, e \in M_{2,2}(\mathbb{C})$ be the matrices defined in section 3. Define $W_\sigma(A) := fV_\sigma(A)$. Repeating the argument of Lemma 3.28 finishes the proof. $\square$

Corollary 5.26. *If A is an abelian variety of class $\mathcal{B}$, then*

$$H(A)_{\mathbb{C}} = L(A)_{\mathbb{C}} = \prod_{\sigma E \hookrightarrow \mathbb{C}} SO_{(W_\sigma(A),\, \psi_\sigma)}. \tag{5.27}$$

Proof. With use of Lemma 5.25 and the argument similar to the proof of formula (5.17) we obtain

$$C_D^{\circ}(R_{E/\mathbb{Q}}(Sp_{(V,\,\phi)})) \otimes_{\mathbb{Q}} \mathbb{C} \;\cong\; \prod_{\sigma E \hookrightarrow \mathbb{C}} SO_{(W_\sigma(A),\, \psi_\sigma)}.$$

Hence (5.27) follows by (5.20). □

6. Images of the Galois representations ρ_l and $\overline{\rho}_l$.

In this section we explicitly compute the images of the l-adic representations induced by the action of the absolute Galois group on the Tate module of abelian varieties of types III .

By Theorem 3.29 (i) the representation ρ_λ induces naturally the representation (denoted in the same way)

$$\rho_\lambda \,:\, G_F \to GL(T_\lambda).$$

Moreover by Theorem 3.29 (ii)

$$\rho_l(G_F) \subset \prod_{\lambda | l} GO_{(T_\lambda, \psi_\lambda)}(\mathcal{O}_\lambda) = \prod_{\lambda | l} R_{\mathcal{O}_\lambda/\mathbb{Z}_l}(GO_{(T_\lambda, \psi_\lambda)})(\mathbb{Z}_l). \tag{6.1}$$

By (6.1) there is a closed immersion

$$\mathcal{G}_l^{alg} \subset \prod_{\lambda | l} R_{\mathcal{O}_\lambda/\mathbb{Z}_l}(GO_{(T_\lambda, \psi_\lambda)}) \tag{6.2}$$

which implies:

$$\rho_l(G_F) \subset \mathcal{G}_l^{alg}(\mathbb{Z}_l) \subset \prod_{\lambda | l} R_{\mathcal{O}_\lambda/\mathbb{Z}_l}(GO_{(T_\lambda, \psi_\lambda)})(\mathbb{Z}_l) \tag{6.3}$$

Since l is unramified in E, there is natural isomorphism $R_{\mathcal{O}_\lambda/\mathbb{Z}_l}(.) \otimes_{\mathbb{Z}_l} \mathbb{F}_l \cong R_{k_\lambda/\mathbb{F}_l}(.)$ To see this isomorphism elementary use [BGK2] Remark 2.8 and modification of Lemma 2.1 of [BGK2] to the case of $R_{\mathcal{O}_\lambda/\mathbb{Z}_l}$. Changing base in (6.2) we get a natural closed immersion of group schemes:

$$G(l)^{alg} \subset \prod_{\lambda | l} R_{k_\lambda/\mathbb{F}_l}(GO_{(A_\lambda[\lambda],\, \overline{\psi}_\lambda)}). \tag{6.4}$$

Where $A_\lambda[\lambda] = \overline{\mathcal{W}}_\lambda(A)$ and $A[\lambda] \cong A_\lambda[\lambda] \oplus A_\lambda[\lambda]$ cf. Theorem 3.13 (i), (ii"). Hence going mod l in (6.3) brings us to:

$$\overline{\rho}_l(G_F) \subset G(l)^{alg}(\mathbb{F}_l) \subset \prod_{\lambda | l} R_{k_\lambda/\mathbb{F}_l}(GO_{(A_\lambda[\lambda],\, \overline{\psi}_\lambda)})(\mathbb{F}_l). \tag{6.5}$$

Because extracting derived subgroup commutes with base change (see [BGK2], Remark 6.8), and because $(\mathcal{G}_l^{alg})'$ (resp. $(G(l)^{alg})'$ are connected, by (6.2) (resp. by (6.4)) we get

$$(\mathcal{G}_l^{alg})' \subset \prod_{\lambda | l} R_{\mathcal{O}_\lambda / \mathbb{Z}_l}(SO_{(T_\lambda, \psi_\lambda)}) \tag{6.6}$$

$$(G(l)^{alg})' \subset \prod_{\lambda | l} R_{k_\lambda / \mathbb{F}_l}(SO_{(A_\lambda[\lambda], \overline{\psi}_\lambda)}). \tag{6.7}$$

Proposition 6.8. *Let A/F be an abelian variety of class $\mathcal{B}$. Then for all $l \gg 0$, we have equalitiy of ranks of group schemes over $\mathbb{F}_l$:*

$$rank\ (G(l)^{alg})' = rank\ \prod_{\lambda | l} R_{k_\lambda / \mathbb{F}_l}(SO_{(A_\lambda[\lambda], \overline{\psi}_\lambda)}) \tag{6.9}$$

Proof. Taking into account Lemma 6.1 of [BGK2] and Th. 4.19, the proof is the same as the proof of Theorem 6.6 of [BGK2] by use of (6.7). □

Theorem 6.10. *Let A/F be an abelian variety of class $\mathcal{B}$. Then for all $l \gg 0$, we have the equality of group schemes:*

$$(G(l)^{alg})' = \prod_{\lambda | l} R_{k_\lambda / \mathbb{F}_l}(SO_{(A_\lambda[\lambda], \overline{\psi}_\lambda)}) \tag{6.11}$$

Proof. Projecting onto the λ component in (6.7) we obtain the representation

$$\underline{\rho}_{\Phi_\lambda} : (G(l)^{alg})' \to R_{k_\lambda / \mathbb{F}_l}(SO_{(A_\lambda[\lambda], \overline{\psi}_\lambda)}), \tag{6.12}$$

This gives the representation:

$$(G(l)^{alg})' \otimes_{\mathbb{F}_l} k_\lambda \to SO_{(A_\lambda[\lambda], \overline{\psi}_\lambda)}. \tag{6.13}$$

By (3.27) we have the natural isomorphism:

$$(End_{k_\lambda[G_F]}\ A_\lambda[\lambda]) \otimes_{k_\lambda} L \cong L \tag{6.14}$$

for any field extension L/k_λ. Hence by (6.13), (6.14) and the Schur's Lemma it follows that:

$$\underline{\rho}_{\Phi_\lambda}(Z((G(l)^{alg})' \otimes_{\mathbb{F}_l} k_\lambda)) \subset k_\lambda^\times Id_{A_\lambda[\lambda]}.$$

Hence by (6.13)

$$\underline{\rho}_{\Phi_\lambda}(Z((G(l)^{alg})' \otimes_{\mathbb{F}_l} k_\lambda)) \subset \mu_2$$

which implies that

$$\underline{\rho}_{\Phi_\lambda}(Z((G(l)^{alg})')) \subset R_{k_\lambda / \mathbb{F}_l}(\mu_2).$$

Hence

$$Z((G(l)^{alg})') \subset \prod_{\lambda|l} R_{k_\lambda/\mathbb{F}_l}(\mu_2) \subset Z(\prod_{\lambda|l} R_{k_\lambda/\mathbb{F}_l}(SO_{(A_\lambda[\lambda],\,\overline{\psi}_\lambda)})).$$

Since both groups $(G(l)^{alg})'$ and $\prod_{\lambda|l} R_{k_\lambda/\mathbb{F}_l}(SO_{(A_\lambda[\lambda],\,\overline{\psi}_\lambda)})$ are reductive, the proof is finished in the same way as the proof of Lemma 3.4 in [BGK1]. $\square$

Let $\widetilde{G}$ denote the universal cover for a group scheme G see [Ch], [St], [SGA3].

Remark 6.15. Let G_1 and G_2 be group schemes over a field L. Then $\widetilde{G_1 \times_L G_2} \cong \widetilde{G_1} \times_L \widetilde{G_2}$. In addition for a finite separable extension L/K there is a natural isomorphism $\widetilde{R_{L/K}(G)} \cong R_{L/K}(\widetilde{G})$. We can see these isomorphisms by comparing root lattices.

Let $Spin_{(T_\lambda,\,\psi_\lambda)}$, $(Spin_{(V_\lambda,\,\psi_\lambda)}$ resp.), $(Spin_{(A_\lambda[\lambda],\,\overline{\psi}_\lambda)}$ resp.) denote the universal cover of the group scheme $SO_{(T_\lambda,\,\psi_\lambda)}$, $(SO_{(V_\lambda,\,\psi_\lambda)}$ resp.), $(SO_{(A_\lambda[\lambda],\,\overline{\psi}_\lambda)}$ resp.). Consider the following, short exact sequences of group schemes:

$$1 \longrightarrow \mu_2 \longrightarrow Spin_{(T_\lambda,\,\psi_\lambda)} \xrightarrow{\pi_\lambda} SO_{(T_\lambda,\,\psi_\lambda)} \longrightarrow 1 \tag{6.16}$$

$$1 \longrightarrow \mu_2 \longrightarrow Spin_{(V_\lambda,\,\psi_\lambda)} \xrightarrow{\pi_\lambda} SO_{(V_\lambda,\,\psi_\lambda)} \longrightarrow 1 \tag{6.17}$$

$$1 \longrightarrow \mu_2 \longrightarrow Spin_{(A_\lambda[\lambda],\,\overline{\psi}_\lambda)} \xrightarrow{\overline{\pi}_\lambda} SO_{(A_\lambda[\lambda],\,\overline{\psi}_\lambda)} \longrightarrow 1 \tag{6.18}$$

The sequences (6.17) and (6.18) are obtained by base change from the sequence (6.16). Evaluating the exact sequence (6.16), (resp. (6.18)) on $\mathcal{O}_\lambda$, (resp. on k_λ) points we get

$$SO_{(T_\lambda,\,\psi_\lambda)}(\mathcal{O}_\lambda)\,/\,\pi_\lambda(Spin_{(T_\lambda,\,\psi_\lambda)}(\mathcal{O}_\lambda)) \cong \mathbb{Z}/2 \tag{6.19}$$

$$\text{resp. } (\; SO_{(A_\lambda[\lambda],\,\overline{\psi}_\lambda)}(k_\lambda)\,/\,\overline{\pi}_\lambda(Spin_{(A_\lambda[\lambda],\,\overline{\psi}_\lambda)}(k_\lambda)) \cong \mathbb{Z}/2 \;) \tag{6.20}$$

Evaluating the exact sequence (6.17) on E_λ points we get

$$SO_{(V_\lambda,\,\psi_\lambda)}(E_\lambda)\,/\,\pi_\lambda(Spin_{(V_\lambda,\,\psi_\lambda)}(E_\lambda)) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2. \tag{6.21}$$

Indeed it follows by a theorem of Steinberg (cf [Kn], Th. 2.1) that $H^1(k_\lambda,\, Spin_{(A_\lambda[\lambda],\,\overline{\psi}_\lambda)}) = 0$ and by a theorem of Kneser (cf. [Kn], Th. 22) that $H^1(E_\lambda,\, Spin_{(V_\lambda,\,\psi_\lambda)}) = 0$. In addition by a theorem of Tits (cf. Th. 4.1 [N]) the natural map

$$H^1_{et}(\mathcal{O}_\lambda,\, Spin_{(T_\lambda,\,\psi_\lambda)}) \to H^1(E_\lambda,\, Spin_{(V_\lambda,\,\psi_\lambda)})$$

is an imbedding. Hence $H^1_{et}(\mathcal{O}_\lambda,\, Spin_{(T_\lambda,\,\psi_\lambda)}) = 0$.

Theorem 6.22. *Let A/F be an abelian variety of class $\mathcal{B}$. Then for $l \gg 0$:*

$$\prod_{\lambda | l} \pi_\lambda(Spin_{(T_\lambda,\,\psi_\lambda)}(\mathcal{O}_\lambda)) \quad \subset \quad \rho_l(\overline{G_F'}) \quad \subset \quad \prod_{\lambda | l} SO_{(T_\lambda,\,\psi_\lambda)}(\mathcal{O}_\lambda), \tag{6.23}$$

$$\prod_{\lambda | l} \overline{\pi}_\lambda(Spin_{(A_\lambda[\lambda],\,\overline{\psi}_\lambda)}(k_\lambda)) \quad \subset \quad \overline{\rho_l}(G_F') \quad \subset \quad \prod_{\lambda | l} SO_{(A_\lambda[\lambda],\,\overline{\psi}_\lambda)}(k_\lambda) \tag{6.24}$$

where $\overline{\rho_l}$ is the representation ρ_l mod l and $\overline{G_F'}$ is the closure of the commutator subgroup $G_F' \subset G_F$ computed with respect to the natural profinite topology of G_F.

Proof. By (6.5) and (6.11)

$$\overline{\rho_l}(G_F') \;=\; (\overline{\rho_l}(G_F))' \quad \subset \quad \prod_{\lambda | l} SO_{(A_\lambda[\lambda],\,\overline{\psi}_\lambda)}(k_\lambda).$$

By a theorem of Serre (cf. [Wi], Th.4), [Wi], Lemma 5 and Remark 6.15 we get

$$\prod_{\lambda | l} \pi_\lambda(Spin_{(A_\lambda[\lambda],\,\overline{\psi}_\lambda)}(k_\lambda)) \;\subset\; \overline{\rho_l}(G_F).$$

Since $Spin_{(A_\lambda[\lambda],\,\overline{\psi}_\lambda)}(k_\lambda)$ is a perfect group [St], chap. 7, Corollary 2 (b), then

$$\prod_{\lambda | l} \pi_\lambda(Spin_{(A_\lambda[\lambda],\,\overline{\psi}_\lambda)}(k_\lambda)) \;\subset\; \overline{\rho_l}(G_F').$$

This proves (6.24). By (6.20) and (6.24) we note that $\overline{\rho_l}(G_F') = \prod_{\lambda | l} H_\lambda$ where H_λ is either $SO_{(A_\lambda[\lambda],\,\overline{\psi}_\lambda)}(k_\lambda)$ or $\overline{\pi}_\lambda(Spin_{(A_\lambda[\lambda],\,\overline{\psi}_\lambda)}(k_\lambda))$. Let $\mathcal{H}_\lambda := SO_{(T_\lambda,\psi_\lambda)}(\mathcal{O}_\lambda)$ if $H_\lambda = SO_{(A_\lambda[\lambda],\,\overline{\psi}_\lambda)}(k_\lambda)$ and $\mathcal{H}_\lambda := \pi_\lambda(Spin_{(T_\lambda,\psi_\lambda)}(\mathcal{O}_\lambda))$ if $H_\lambda = \overline{\pi}_\lambda(Spin_{(A_\lambda[\lambda],\,\overline{\psi}_\lambda)}(k_\lambda))$. Consider the following commutative diagram.

$$\begin{array}{ccccc}
\prod_{\lambda|l} Spin_{(T_\lambda,\psi_\lambda)}(\mathcal{O}_\lambda) & \longrightarrow & \prod_{\lambda|l} \mathcal{H}_\lambda & \longrightarrow & \prod_{\lambda|l} SO_{(T_\lambda,\psi_\lambda)}(\mathcal{O}_\lambda) \\
{\scriptstyle r_\lambda}\downarrow & & \downarrow & & \downarrow \\
\prod_{\lambda|l} Spin_{(A_\lambda[\lambda],\,\overline{\psi}_\lambda)}(k_\lambda) & \longrightarrow & \prod_{\lambda|l} H_\lambda & \longrightarrow & \prod_{\lambda|l} SO_{(A_\lambda[\lambda],\,\overline{\psi}_\lambda)}(k_\lambda).
\end{array} \tag{6.25}$$

From (6.6) the group $\rho_l(\overline{G_F'}) = \overline{(\rho_l(G_F))'}$ is a closed subgroup of $\prod_{\lambda|l} SO_{(T_\lambda,\psi_\lambda)}(\mathcal{O}_\lambda)$. Taking the proimage of $\rho_l(\overline{G_F'})$ to the left top corner of the diagram (6.25) we obtain a closed subgroup $\mathcal{K}$. By (6.23) $\mathcal{K}$ surjects onto the group $\prod_{\lambda|l} H_\lambda$ via the left square of the diagram (6.25). Because the group in the lower, left corner of the diagram (6.25) is a perfect group [St], chap. 7, Corollary 2 (b), we observe by (6.18) that $r_\lambda(\mathcal{K}) = \prod_{\lambda|l} Spin_{(A_\lambda[\lambda],\,\overline{\psi}_\lambda)}(k_\lambda)$. Hence by [Lar], Prop. 2.6 we get $\mathcal{K} = \prod_{\lambda|l} Spin_{(T_\lambda,\,\psi_\lambda)}(\mathcal{O}_\lambda)$. This shows that

$$\prod_{\lambda | l} \pi_\lambda(Spin_{(T_\lambda,\,\psi_\lambda)}(\mathcal{O}_\lambda)) \quad \subset \quad \rho_l(\overline{G_F'}). \quad \square$$

Corollary 6.26. *Let A/F be an abelian variety of class $\mathcal{B}$. Let $l \gg 0$. Then:*

$$\overline{\rho_l}(G'_F) = \prod_{\lambda | l} H_\lambda \tag{6.27}$$

$$\rho_l(\overline{G'_F}) = \prod_{\lambda | l} \mathcal{H}_\lambda \tag{6.28}$$

Proof. The isomorphism (6.27) is established in the proof of Theorem 6.22. Observe that for each $\lambda | l$ there is a natural isomorphism (cf. (6.19) and (6.20)) $\mathcal{H}_\lambda / \, \pi_\lambda(Spin_{(T_\lambda, \psi_\lambda)}(\mathcal{O}_\lambda)) \;\cong\; H_\lambda / \, \overline{\pi}_\lambda(Spin_{(A_\lambda[\lambda], \overline{\psi}_\lambda)}(k_\lambda))$. Hence $\rho_l(\overline{G'_F}) = \prod_{\lambda | l} \mathcal{H}_\lambda$. □

7. Open image property of ρ_l.

Consider the group scheme $C_{\mathcal{R}}(Sp_{(\Lambda, \kappa)})$ over $Spec\, \mathbb{Z}$. Since $C_{\mathcal{R}}(Sp_{(\Lambda, \kappa)}) \otimes_{\mathbb{Z}} \mathbb{Q} = C_D(Sp_{(V, \kappa^0)})$ (see Remark 5.4), there is an open imbedding in the l-adic topology:

$$C_{\mathcal{R}}(Sp_{(\Lambda, \kappa)})(\mathbb{Z}_l) \subset C_D(Sp_{(V, \kappa^0)})(\mathbb{Q}_l). \tag{7.1}$$

Theorem 7.2. *If A is an abelian variety of class $\mathcal{B}$, then for every prime number l, $\rho_l(G_F)$ is open in the group $C_{\mathcal{R}}(GSp_{(\Lambda, \kappa)})(\mathbb{Z}_l)$. In addition, for $l \gg 0$ the subgroup $\rho_l(\overline{G'_F})$ of $C_{\mathcal{R}}(Sp_{(\Lambda, \kappa)})(\mathbb{Z}_l)$ is of index dividing $2^{r(l)}$, where $r(l)$ is the number of primes over l in $\mathcal{O}_E$.*

Proof. The group $GSp_{(\Lambda, \kappa)}(\mathbb{Z}_l)$ is generated by $Sp_{(\Lambda, \kappa)}(\mathbb{Z}_l)$ and subgroup which in the Frobenius bases of Λ has the following form:

$$\{ \begin{pmatrix} aI_g & 0 \\ 0 & I_g \end{pmatrix} ; \, a \in {\mathbb{Z}_l}^{\times} \}.$$

The group $\mathbb{Z}_l^{\times} Sp_{(\Lambda, \kappa)}(\mathbb{Z}_l)$ has index 2 (index 4 resp.) in $GSp_{(\Lambda, \kappa)}(\mathbb{Z}_l)$, for $l > 2$ (for $l = 2$ resp.). By [Bo], Cor. 1. on p. 702, there is an open subgroup $U \subset \mathbb{Z}_l^{\times}$ such that $U \subset \rho_l(G_F)$. Hence $U \, C_{\mathcal{R}}(Sp_{(\Lambda, \kappa)})(\mathbb{Z}_l) = C_{\mathcal{R}}(U \, Sp_{(\Lambda, \kappa)}(\mathbb{Z}_l))$ is an open subgroup of $C_{\mathcal{R}}(GSp_{(\Lambda, \kappa)})(\mathbb{Z}_l) = C_{\mathcal{R}}(GSp_{(\Lambda, \kappa)}(\mathbb{Z}_l))$. By [Bo], Th. 1, p. 701, the group $\rho_l(G_F)$ is open in $G_l^{alg}(\mathbb{Q}_l)$. By Theorem 5.11, and Corollary 5.19

$$U \, C_{\mathcal{R}}(Sp_{(\Lambda, \kappa)})(\mathbb{Z}_l) \;\subset\; \mathbb{Q}_l^{\times} \, C_D(Sp_{(V, \kappa)})(\mathbb{Q}_l) \;=$$

$$= \;\mathbb{G}_m(\mathbb{Q}_l) H(A)(\mathbb{Q}_l) \subset MT(A)(\mathbb{Q}_l) = G_l^{alg}(\mathbb{Q}_l). \tag{7.3}$$

Hence, $U \, C_{\mathcal{R}}(Sp_{(\Lambda, \kappa)})(\mathbb{Z}_l) \cap \rho_l(G_F)$ is open in $U \, C_{\mathcal{R}}(Sp_{(\Lambda, \kappa)})(\mathbb{Z}_l)$ and we get that $\rho_l(G_F)$ is open in $C_{\mathcal{R}}(GSp_{(\Lambda, \kappa)})(\mathbb{Z}_l)$. By Remark 5.4 and the universality of the fiber product:

$$C_{\mathcal{R}}(Sp_{(\Lambda, \kappa)})(\mathbb{Z}_l) = C_{\mathcal{R} \otimes_{\mathbb{Z}} \mathbb{Z}_l}(Sp_{(T_l(A), \kappa_l)})(\mathbb{Z}_l). \tag{7.4}$$

For $l \gg 0$ we get

$$C_{\mathcal{R}\otimes_{\mathbb{Z}}\mathbb{Z}_l}(Sp_{(T_l(A),\,\kappa_l)}) \;\cong\; C_{\mathcal{R}\otimes_{\mathbb{Z}}\mathbb{Z}_l}(C_{\mathcal{O}_E\otimes_{\mathbb{Z}}\mathbb{Z}_l}(Sp_{(T_l(A),\,\kappa_l)})) \;\cong$$

$$\cong\; C_{\mathcal{R}\otimes_{\mathbb{Z}}\mathbb{Z}_l}(\prod_{\lambda | l} R_{\mathcal{O}_\lambda/\mathbb{Z}_l}(Sp_{(T_\lambda(A),\,\psi_\lambda)})). \tag{7.5}$$

By definition of the forms ψ_λ, κ_λ we have:

$$C_{\mathcal{R}_\lambda}(Sp_{(T_\lambda(A),\,\kappa_\lambda)}) \cong C_{\mathcal{R}_\lambda}(SO_{(T_\lambda(A),\,\psi_\lambda)}). \tag{7.6}$$

Evaluating the group schemes in (7.5) on $Spec\,\mathbb{Z}_l$ we get

$$C_{\mathcal{R}\otimes_{\mathbb{Z}}\mathbb{Z}_l}(Sp_{(T_l(A),\,\psi_l)})(\mathbb{Z}_l) \;\cong\; C_{\mathcal{R}\otimes_{\mathbb{Z}}\mathbb{Z}_l}(\prod_{\lambda | l} R_{\mathcal{O}_\lambda/\mathbb{Z}_l}(Sp_{(T_\lambda(A),\,\psi_\lambda)}))(\mathbb{Z}_l) \;\cong$$

$$\cong\; \prod_{\lambda | l} C_{\mathcal{R}_\lambda} Sp_{(T_\lambda(A),\,\psi_\lambda)}(\mathcal{O}_\lambda) \;\cong\; \prod_{\lambda | l} C_{\mathcal{R}_\lambda} SO_{(T_\lambda(A),\,\psi_\lambda)}(\mathcal{O}_\lambda) \;\cong$$

$$\cong\; \prod_{\lambda | l} SO_{(T_\lambda,\,\psi_\lambda)}(\mathcal{O}_\lambda). \tag{7.7}$$

The Theorem for $l \gg 0$ follows by (6.23), (7.4), (7.7) $\square$

Theorem 7.8. *If A is an abelian variety of class $\mathcal{B}$, then for every prime number l, the group $\rho_l(G_F)$ is open in the group $\mathcal{G}_l^{alg}(\mathbb{Z}_l)$ in the l-adic topology.*

Proof. By Theorem 7.2 the group $\rho_l(G_F)$ is open in $C_{\mathcal{R}\otimes_{\mathbb{Z}}\mathbb{Z}_l}(GSp_{(T_l(A),\,\kappa_l)})(\mathbb{Z}_l)$ in the l-adic topology, so $\rho_l(G_F)$ has a finite index in the group $C_{\mathcal{R}\otimes_{\mathbb{Z}}\mathbb{Z}_l}(GSp_{(T_l(A),\,\kappa_l)})(\mathbb{Z}_l)$. By the definition of $\mathcal{G}_l^{alg}$, we have:

$$\rho_l(G_F) \subset \mathcal{G}_l^{alg}(\mathbb{Z}_l) \subset C_{\mathcal{R}\otimes_{\mathbb{Z}}\mathbb{Z}_l}(GSp_{(T_l(A),\,\kappa_l)})(\mathbb{Z}_l).$$

Hence, $\rho_l(G_F)$ has a finite index in $\mathcal{G}_l^{alg}(\mathbb{Z}_l)$, and the claim follows since $C_{\mathcal{R}\otimes_{\mathbb{Z}}\mathbb{Z}_l}(GSp_{(T_l(A),\,\kappa_l)})(\mathbb{Z}_l)$ is a profinite group. $\square$

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