ON THE ISOTROPY OF LOW DIMENSIONAL FORMS OVER THE FUNCTION FIELD OF A QUADRIC

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Abstract. Let ϕ be a field of characteristic different from 2 and let ϕ and ψ be two anisotropic quadratic form over F. An important problem in the algebraic theory of quadratic forms is to find conditions on ϕ and ψ so that $\phi_{F(\psi)}$ is isotropic. We prove, that under certain conditions on the quadratic form ϕ ($\dim\phi \leqslant 8$, and ϕ contains no a quadratic form with trivial determinant) the isotropy of $\phi_{F(\psi)}$ implies that there exists a homomorphism of even parts of theirs Clifford algebras, $C_0(\psi) \to C_0(\phi)$.

Let F be a field of characteristic different from 2 and let ϕ be an anisotropic quadratic form over F. For any field extension L/F, we can consider the quadratic form $\phi_L = \phi \otimes_F L$ over L. Of special interest here is the case $L = F(\psi)$, where $F(\psi)$ is the field of rational functions on the quadric hypersurface defined by the equation $\psi = 0$, ψ being an anisotropic quadratic form over F.

An important problem in the algebraic theory of quadratic forms is to find conditions on ϕ and ψ so that $\phi_{F(\psi)}$ is isotropic. The fundamental Cassels–Pfister subform theorem states that if ϕ is a Pfister neighbour of an anisotropic Pfister form π, then $\phi_{F(\psi)}$ is isotropic if and only if ψ is similar to a subform in π (see [Lam], [Sch]). In particular, this completely solves the problem of isotropy of $\phi_{F(\psi)}$ in the cases where $\dim(\phi) \leqslant 3$, or $\dim(\phi) = 4$ and $\det(\phi) = 1$.

In the case $\dim(\phi) = 4$, $\det(\phi) \neq 1$ the problem was solved by Shapiro [Sh], and in the case $\dim(\phi) = 5$ by Hoffmann [H1]. The case $\dim(\phi) = 6$, $\dim(\psi) \neq 4$ was completely investigated by Hoffmann, Leep, Merkurev and Laghribi (see [H2, Le, Me, La1]), and in the case $\dim(\phi) = 8$, $\det(\phi) = 1$, a partial answer has recently been obtained by Laghribi [La2, La3].

The proofs of the mentioned results are based on a combination on the two following methods:

(A) Construct a field extension L/F such that ϕ_L is an anisotropic Pfister neighbour of a certain Pfister form π over L. Then the isotropy of $\phi_{F(\psi)}$ implies that ψ_L is similar to a subform of π (by the Cassels–Pfister subform theorem cited above). After that a descent to the ground field is conducted in this or that way.

(B) Use the following corollary of the index reduction formula (Tignole): if $C_0(\phi)$ is a division algebra or has the form $D \times D$, where D is a division algebra, then the

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isotropy of $\phi_{F(\psi)}$ implies that there is a homomorphism of algebras, $C_0(\psi) \to C_0(\phi)$ (see [Ti]).

Already in dimension seven there are forms to which neither of these methods can be applied. Indeed, let $\phi = q \perp \langle a \rangle$, where q is an Albert form. Then there exists no field extension L such that ϕ_L is an anisotropic Pfister neighbour. Moreover, $C_0(\phi) \cong M_2(C_0(q_0))$, where q_0 is a 5-dimensional subform of q, and hence we cannot use the above-mentioned corollary of the index reduction formula.

It seems a natural idea to try to generalize the second method (B) by eliminating the restriction on the algebra $C_0(\phi)$, namely, to try to prove the following conjecture: if ϕ and ψ are two anisotropic forms such that $\phi_{F(\psi)}$ is isotropic, then there is a homomorphism of algebras, $C_0(\psi) \to C_0(\phi)$. This conjecture is true if $\dim \phi \leqslant 4$. However, it is not true in the full generality. The simplest counterexample can be given by the 3-fold Pfister form $\psi = \langle\langle a, b, c \rangle\rangle$ and 5-dimensional form $\phi = \langle\langle a, b \rangle\rangle \perp \langle -c \rangle$ (or more generally $(n+5)$-dimensional form $\psi = \langle\langle a, b \rangle\rangle \perp \langle -c, t_1, \dots, t_n \rangle$ over the field $F(t_1, \dots, t_n)$). Note that in this counterexample the quadratic form ϕ contains a 4-dimensional subform with trivial determinant (namely $\langle\langle a, b \rangle\rangle$).

This makes natural to modify our conjecture, by means of introducing certain restrictions on ϕ. Counterexample considered above shows that one of the conditions on ϕ must be following: quadratic form ϕ contains no quadratic forms with trivial determinant. The main result of the present paper (Theorem 6.3) states that if $\dim(\phi) \leqslant 8$ then this restriction on the form ϕ is sufficient for our conjecture:

Theorem. *Let ϕ and ψ be anisotropic quadratic forms over F. Suppose that ϕ contains no 4-dimensional subforms with trivial determinant and $\dim \phi \leqslant 8$. Then the isotropy of $\phi_{F(\psi)}$ implies that there is a homomorphism of algebras, $C_0(\psi) \to C_0(\phi)$.*

Among the corollaries of this theorem we point out the following (see Corollary 6.4): *Let ϕ be a quadratic form over F with $\dim \phi \leqslant 8$. Then the following conditions are equivalent*:

1) *There exists a 3-fold Pfister form ψ such that $\phi_{F(\psi)}$ is isotropic,*
2) *ϕ contains a 5-dimensional Pfister neighbour (i.e., $\dim \phi > 4$ and there is a 4-dimensional subform in ϕ with trivial determinant).*

In section §§5–6 we apply our main theorem to study of 7- and 8-dimensional forms ϕ over $F((t))$ and $F(t)$ such that $\phi = q \perp t\mu$, where q is an anisotropic Albert form over F and μ is an anisotropic form of dimension one or two (obviuosly that for these forms neither of the methods (A) and (B) can be applied). In particular, we prove the following assertion (Proposition 5.3, Corollary 5.5): *Let q be an anisotropic Albert form over F. Consider the quadratic form $\phi = q \perp \langle t \rangle$ over $E = F((t))$. If ψ is a quadratic form over E such that $\phi_{E(\psi)}$ is isotropic, then $\dim(\psi) \leqslant 7$. If, in addition, $\dim(\psi) = 7$, then ψ is similar to ϕ.*

The paper is organized as follows. In §1 the notation is introduced. In §2 we study the problem of existence of a homomorphism from one semisimple algebra to another one. In §3 we introduce a certain class of forms ($\mathcal{G}$-forms) and is §4 prove our main technical result (Theorem 4.5). Applications to 7- and 8-dimensional forms are considered in §§5 and 6, respectively.

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§1. Some basic facts and notation

1.1. Quadratic forms. We will use the following notation throughout the paper: by $\phi \perp \psi$, $\phi \cong \psi$, and $[\phi]$ we denote respectively orthogonal sum of forms, isometry of forms, and the class of ϕ in the Witt ring $W(F)$ of the field F. The maximal ideal of $W(F)$ generated by the classes of even dimensional forms is denoted by $I(F)$. We write $\phi \subset \psi$ if ϕ is isometric to a subform in ψ. We say that ψ contains ϕ if $\phi \subset \psi$. With $\det\phi, \det_\pm \phi \in F^*/F^{*2}$ we denote the determinant and signed determinant of ϕ respectively. We write $\phi \sim \psi$ if ϕ is similar to ψ, i.e., $k\phi = \psi$ for some $k \in F^*$. The anisotropic part of ϕ is denoted by ϕ_{an} and $i_W(\phi)$ denotes the Witt index of ϕ. We denote by $\langle\langle a_1, \dots, a_n\rangle\rangle$ the n-fold Pfister form

$$\langle 1, -a_1\rangle \otimes \cdots \otimes \langle 1, -a_n\rangle .$$

Note that this definition is different from the one given in the Lam's book [Lam]. The set of all n-fold Pfister forms (corr. all forms similar to n-fold Pfister forms) we denote by $P_n(F)$ (corr. $GP_n(F)$). For any field extension L/F we put $\phi_L = \phi \otimes L$, and $D_L(\phi) = \{x \in L^* \mid \langle x\rangle \subset \phi_L\}$.

Let ϕ be a quadratic form such that $\dim\phi \geqslant 2$, $\phi \ncong \mathbb{H}$. The function field $F(\phi)$ of a form ϕ over F is the function field of the projective variety X_ϕ given by equation $\phi = 0$. In the case where $\dim\phi \leqslant 1$ or $\phi \cong \mathbb{H}$, we set $F(\phi) \overset{\mathrm{def}}{=} F$.

Let ϕ be a form over F. In [Kn1], Knebusch have defined inductively fields $F_0, F_1, \dots$, and forms ϕ_i over F_i as follows. Let $\phi_0 = \phi_{\mathrm{an}}$ and $F_0 = F$. For $i \geqslant 1$ let $F_i = F_{i-1}(\phi_{i-1})$ be the function field of F_{i-1} and $\phi_i = (\phi_{F_i})_{\mathrm{an}}$. The smallest h for which $\dim\phi_h \leqslant 1$ is called the height of ϕ. The sequence of the fields $F_0, F_1, \dots, F_h$ is called a generic splitting tower of ϕ. We need the following property of the field F_i (see [Kn1]): Let K/F be a field extension such that $\dim(\phi_K)_{\mathrm{an}} \leqslant \dim(\phi_{F_i})_{\mathrm{an}}$. Then the free composite KF_i of K and F_i over F is purely transcendental over K.

1.2. Algebras. Let F be a field of characteristic different from 2. In our paper an F-algebra will always mean a finite-dimensional F-algebra. Following [Lam] we say that A is a SA (corr. CSA) over F if A is a simple algebra (corr. central simple algebra) over F. If A is an SA over F, then its center $E = Z(A)$ is a subfield of A, and it is clear that A is a CSA over $Z(A)$. The number $\sqrt{\dim_{Z(A)} A}$ is always an integer and is denoted by $\deg(A)$.

By the Wedderburn Main Theorem, every simple algebra over F has the form $M_m(D)$, where D is a division SA. Moreover the algebra A determines integer m and algebra D uniquely (up to isomorphism). The integer m we denote by $\operatorname{Coind} A$. The *Schur index* of A is $\operatorname{ind}(A) = \deg(D)$ (or more detail $\operatorname{ind}_E A$, where $E = Z(A)$). Clearly, $\deg(A) = \operatorname{Coind}(A)\operatorname{ind}(A)$.

In this paper we consider mostly the following three types of F-algebras:

Type 1: Central simple algebras of exponent 2 over F,

Type 2: Central simple algebra of exponent 2 over L, where L/F is a quadratic field extension.

Type 3: Algebras of the form $A \times A$, where A is a central simple algebra of exponent 2 over F.

If A is algebra of one of the above types, we write $A \in \mathrm{Alg}(F)$. For algebra of "third type" we set $\mathrm{ind}(A \times A) \stackrel{\mathrm{def}}{=} \mathrm{ind}\, A$, $\mathrm{Coind}(A \times A) \stackrel{\mathrm{def}}{=} \mathrm{Coind}\, A$. If A belongs to $\mathrm{Alg}(F)$ and E/F is a field extension, then $A_E = A \otimes_F E \in \mathrm{Alg}(E)$ and $\mathrm{ind}\, A_E \leqslant \mathrm{ind}\, A$, $\mathrm{Coind}\, A_E \geqslant \mathrm{Coind}\, A$. Moreover, if E/F either unirational, or has the form $E = F(\phi)$, where ϕ is a quadratic form with $\dim \phi \geqslant 3$, then the "type" of A_E is the same as that of A. If E/F is an unirational field extension, then $\mathrm{ind}\, A_E = \mathrm{ind}\, A$ and $\mathrm{Coind}\, A_E = \mathrm{Coind}\, A$.

If A is one of the three types, and B is a CSA over F, then the tensor product $A \otimes_F B$ is an algebra of the same type as that of A.

We shall say that A is similar to A', if there is n, n' such that $M_n(A) \cong M_{n'}(A')$. The similarity is an equivalence relation, and the equivalence class of A will be denoted by $[A]$. Obviously that the condition $[A] = [A']$ implies that Algebra A' has the same type as that of A, and $\mathrm{ind}\, A = \mathrm{ind}\, A'$. If $[A'] = [A]$ and $k = \deg A'/\deg A$ is an integer, then $A' \cong M_k(A)$.

If $A \in \mathrm{Alg}(F)$ and B is a CSA, then we denote by $[A] + [B]$ the class of tensor product $[A \otimes_F B]$. Obviously this operation is associative. The classes of equivalence of CSAs over F and the binary operation $[A] + [B]$ constitute the Brauer group $\mathrm{Br}(F)$ of F.

1.3. Clifford algebras. Let ϕ be a quadratic form. We denote by $C(\phi)$ (corr. $C_0(\phi)$) the Clifford algebra (corr. the even part of the Clifford algebra) of the quadratic form ϕ. We need the following properties of Clifford algebras (see [Lam, Ch. 5,Th. 2.5]).

1) If $\dim \phi$ is odd, then $C_0(\phi)$ is a CSA over F.
2) If $\dim \phi$ is even and $\det_\pm \phi \neq 1$, then $C_0(\phi)$ is a CSA over the field $L = F(\sqrt{\det_\pm \phi})$.
3) If $\dim \phi$ is even and $\det_\pm \phi = 1$, then $C_0(\phi)$ has the form $A \times A$, where A is a CSA over F. Moreover, in this case we have $C(\phi) \cong M_2(A)$.

It follows from these properties that $C_0(\phi) \in \mathrm{Alg}(F)$. For any field extension E/F, we have $C_0(\phi_E) = C_0(\phi)_E$. If ϕ and ψ are quadratic forms over F such that $\phi \equiv \psi \pmod{I^3(F)}$, then $[C_0(\phi)] = [C_0(\psi)]$. If $\gamma = \phi \perp k\langle\langle a, b\rangle\rangle$, then $[C_0(\gamma)] = [C_0(\phi)] + [(a, b)]$, where (a, b) denotes the biquaternion algebra $\left(\frac{a,b}{F}\right)$ (see [Lam, Ch.5, Corollary 2.9 and Corollary 3.3]).

1.4. Index reduction formula. Let A be a CSA over F and let E/F be a field extension. The quotient $\frac{\mathrm{ind}(A)}{\mathrm{ind}(A_E)}$ is called the *index reduction factor* of A relative to E/F (see [Dr]). The index reduction factor is always an integer. If the field extension E/F is unirational, then the index reduction factor is equal to one. If E/F is a finite extension, then the index reduction factor divides $[E : F]$.

Let ϕ be a quadratic form over F and A be a CSA over F. By definition of $F(\phi)$, there is subfield $E \subset F(\phi)$ such that E/F is purely transcendental and $[F(\phi) : E] = 2$. Hence the index reduction factor of A relative to $F(\phi)/F$ divides 2. Therefore either $\mathrm{ind}(A_{F(\phi)}) = \mathrm{ind}(A)$ or $\mathrm{ind}(A_{F(\phi)}) = \frac{1}{2}\,\mathrm{ind}(A)$.

Theorem 1.5 [Me]. *Let A be a CSA over F, and ϕ a quadratic form over F*

of dimension d. Then $\operatorname{ind}(A_{F(\phi)}) = \frac{1}{2}\operatorname{ind}(A)$ *if and only if one of the following conditions is fulfilled:*

1) $\dim\phi$ *is odd and* $\operatorname{ind}(A \otimes_F C_0(\phi)) = \frac{\operatorname{ind} A}{2^{(d-1)/2}}$;
2) $\dim\phi$ *is even,* $\phi \in I^2(F)$*, and* $\operatorname{ind}(A \otimes_F C(\phi)) = \frac{\operatorname{ind} A}{2^{(d-2)/2}}$;
3) $\dim\phi$ *is even,* $\phi \notin I^2(F)$*, and* $\operatorname{ind}(A \otimes_F C_0(\phi)) = \frac{\operatorname{ind} A}{2^{d/2}}$. $\square$

In the case where the central simple algebra is a division algebra, we have the following

Theorem 1.6(see [Ti]). *Let D be a division CSA over F, and* ϕ *a quadratic form over F. Then the following conditions are equivalent:*

1) $D_{F(\phi)}$ *is not a division algebra.*
2) $\operatorname{ind} D_{F(\phi)} = \frac{1}{2}\operatorname{ind} D$.
3) *There is a homomorphism of F-algebras,* $C_0(\phi) \to D$.

Corollary 1.7. *Let* ϕ *be an anisotropic quadratic form over F. Suppose that* $C_0(\phi)$ *is either a division algebra or has the form* $D \times D$ *where D is a division algebra. Let* ψ *be a quadratic form such that* $\phi_{F(\psi)}$ *is isotropic. Then there is a homomorphism of F-algebras,* $C_0(\psi) \to C_0(\phi)$. $\square$

If ϕ is a quadratic form of dimension less or equal to 6, then the condition that $C_0(\phi)$ is either a division algebra or has the form $D \times D$, where D is a division algebra can be rewriting as follows: ϕ contains no 5-dimensional Pfister neighbours.

Thus we have the following

Corollary 1.8. *Let* ϕ *and* ψ *be anisotropic quadratic forms over F. Suppose that* ϕ *contains no 5-dimensional Pfister neighbours and* $\dim\phi \leqslant 6$*. Then the isotropy of* $\phi_{F(\psi)}$ *implies that there is a homomorphism of F-algebras,* $C_0(\psi) \to C_0(\phi)$ $\square$

§2. Homomorphisms of algebras

Let A and B be two semisimple algebras over F. Here we consider the question of whether there is a homomorphism of algebras, $B \to A$.

Theorem 2.1. *Let L/F be a field extension, A be a CSA over F, and B be a CSA over L. Then the following conditions are equivalent:*

1) *There is a homomorphism of F-algebras,* $B \to A$.
2) *The number*

$$k(A,B) \overset{\text{def}}{=} \frac{\deg(A)}{\deg(B)\operatorname{ind}(B^{\mathrm{op}} \otimes_F A)\cdot[L:F]}$$

is an integer.

3) *There is a homomorphism of L-algebras* $M_n(B) \to A_L$*, where* $n = [L:F]$.

Proof. 1) $\Rightarrow$ 2). Since B is a simple algebra, the homomorphism $B \to A$ is injective. So we can suppose that $B \subset A$. By the "Double Centralizer theorem", we have $\dim_F C_A(B) \cdot \dim_F B = \dim_F A = (\deg A)^2$ (see [Pi, 12.7]). Since B and $C_A(B)$ both are CSAs over L, we have $\dim_F(C_A(B)) = [L:F](\deg C_A(B))^2$, and

$\dim_F(B) = [L:F](\deg B)^2$. Hence $[L:F](\deg C_A(B))^2[L:F](\deg B)^2 = (\deg A)^2$, i.e.,

$$\deg C_A(B) \deg B \cdot [L:F] = \deg A.$$

Using again the Centralizer theorem, we get $[C_A(B)] = [B^{\mathrm{op}} \otimes_F A] \in \mathrm{Br}(L)$ (see [Dr, p. 42]). Hence

$$\begin{aligned} k(A,B) =& \frac{\deg(A)}{\deg(B)\,\mathrm{ind}(C_A(B)) \cdot [L:F]} = \\ =& \mathrm{Coind}(C_A(B)) \cdot \frac{\deg(A)}{\deg(B)\deg(C_A(B)) \cdot [L:F]} = \mathrm{Coind}(C_A(B)). \end{aligned}$$

Hence $k(A,B)$ is an integer.

2) $\Rightarrow$ 3). Let D be a division algebra over L such that $[D] = [B^{\mathrm{op}} \otimes_F A] \in \mathrm{Br}(L)$. Let $k = k(A,B)$. We have

$$\begin{aligned} \deg(A_L) = \deg(A) &= k \deg(B)\,\mathrm{ind}(B^{\mathrm{op}} \otimes_F A) \cdot [L:F] \\ &= kn \deg(B)\deg(D) = \deg(M_n(B) \otimes_L M_k(D)). \end{aligned}$$

Let $A' = M_n(B) \otimes_L M_k(D)$. We have

$$[A'] = [B] + [D] = [B] + [B^{\mathrm{op}} \otimes_F A] = [B] + [B^{\mathrm{op}} \otimes_L A_L] = [A_L] \in \mathrm{Br}(L).$$

Since $\deg(A_L) = \deg(A')$, and $[A_L] = [A']$, we conclude that there is an isomorphism $A' \xrightarrow{\sim} A_L$. Hence one get a homomorphism $M_n(B) \to A_L$ as the composition $M_n(B) \hookrightarrow M_n(B) \otimes_L M_k(D) = A' \xrightarrow{\sim} A_L$.

3) $\Rightarrow$ 1). Since $M_n(B)$ is a simple algebra, we can suppose that $M_n(B) \subset A_L$. Let $D = C_{A_L}(M_n(B)) \otimes_L B$. By the Centralizer theorem, we have $M_n(D) \cong A_L$. Let E be an arbitrary subfield of D such that $[D_E] = 0$ (one can take E as a maximal subfield of D). Then $[E:L]$ divides $\deg(D)$. We have $[A_E] = [M_n(D_E)] = 0 \in \mathrm{Br}(E)$. Hence there exists a central simple algebra A' over F such that $[A] = [A']$ and E is a strictly maximal subfield of A' (see [Pi, 13.3]). Since

$$s = \frac{\deg(A)}{\deg(A')} = \frac{\deg(A_L)}{[E:F]} = \frac{\deg(M_n(D))}{[E:L]\cdot[L:F]} = \frac{n\deg(D)}{n[E:L]} = \frac{\deg(D)}{[E:L]}$$

is an integer, we have $A \cong M_s(A')$. Hence there is a homomorphism $L \to A$, defined as the composition $L \hookrightarrow E \hookrightarrow A' \hookrightarrow M_s(A') \xrightarrow{\sim} A$.

Thus we can suppose that L is a subfield of A. We have $M_n(C_A(L)) \cong A_L$ (see [Dr, p. 42]). Since $M_n(D) \cong A_L \cong M_n(C_A(L))$, we have $D \cong C_A(L)$. Therefore we get a homomorphism $B \to A$ as the composition $B \hookrightarrow D \xrightarrow{\sim} C_A(L) \hookrightarrow A$. $\square$

Lemma 2.2. *Let A and B be semisimple algebras over F:*

$$A = A_1 \times \cdots \times A_s, \qquad B = B_1 \times \cdots \times B_r,$$

where $A_1, \ldots, A_s, B_1, \ldots, B_r$ are simple algebras over F. Then the following conditions are equivalent:

1) *There is a homomorphism $B \to A$ of F-algebras.*
2) *For any j $(1 \leqslant j \leqslant s)$ there is a homomorphism $B \to A_j$ of F-algebras.*
3) *For any j $(1 \leqslant j \leqslant s)$ there is i $(1 \leqslant i \leqslant r)$ and a homomorphism $B_i \to A_j$ of F-algebras.*

Proof. Obvious. $\square$

Corollary 2.3. *Let A be a CSA over F, and $B = B_1 \times \cdots \times B_r$ be a semisimple algebra over F, where $B_1, \dots, B_r$ are simple algebras over F. Then the following conditions are equivalent:*

1) *There is a homomorphism of F-algebras, $B \to A$.*
2) *There is i $(1 \leqslant i \leqslant r)$ such that $k(A, B_i)$ is an integer.*

Proof. Obvious. □

Lemma 2.4. *Let A be a simple algebra and let $L = Z(A)$. Let B be a semisimple algebra over F. Let $B_L = B \otimes_F L$. Then the following conditions are equivalent:*

1) *There is a homomorphism $B \to A$ of F-algebras,*
2) *There is a homomorphism $B_L \to A$ of L-algebras.*
3) *There is a homomorphism $B_L / \operatorname{Rad}(B_L) \to A$ of semisimple L-algebras.*

Proof. 1) $\Rightarrow$ 2). Let $f\colon B \to A$ be a homomorphism of F-algebras. We define $\widetilde{f}\colon B_L = B \otimes_F L \to A$ by the rule $\widetilde{f}(b \otimes l) = f(b)l$.

2) $\Rightarrow$ 3). Since A is a simple algebra, the homomorphism $B_L \to A$ factors through the algebra $B_L / \operatorname{Rad}(B_L)$.

3) $\Rightarrow$ 1). If $B_L / \operatorname{Rad}(B_L) \to A$ is a homomorphism of L-algebras, one can define $B \to A$ as the composition $B \to B_L / \operatorname{Rad}(B_L) \to A$. □

Lemma 2.5. *Let A a central simple algebra and B be a simple algebra over F. Then:*

1) *For any positive integer n we have $k(M_n(A), M_n(B)) = k(A, B)$;*
2) *If Q is a CSA over F, then $k(A \otimes_F Q, B \otimes_F Q) = k(A, B)$;*
3) *If E/F is an unirational field extension, then $k(A \otimes_F E, B \otimes_F E) = k(A, B)$;*

Proof. Obvious. □

Proposition 2.6. *Let A and B be semisimple algebras over F, and Q be a CSA over F.*

Then the following conditions are equivalent:

1) *There is a homomorphism of F-algebras $B \otimes_F Q \to A \otimes_F Q$,*
2) *There is a homomorphism of F-algebras $B \to A$.*

Proof. 1) $\Rightarrow$ 2). Let $f\colon B \otimes_F Q \to A \otimes_F Q$ be a homomorphism of F-algebras. Then there is a homomorphism

$$B \otimes_F Q \otimes_F Q^{\mathrm{op}} \to A \otimes_F Q \otimes_F Q^{\mathrm{op}}.$$

Since Q is a CSA over F, we conclude that $Q \otimes_F Q^{\mathrm{op}} \cong M_n(F)$, with $n = \dim_F Q$. Hence we get a homomorphism $M_n(B) \to M_n(A)$. Using Lemma 2.2 one can reduce our problem to the case where A is a simple algebra. After this using Lemma 2.4 we reduce the problem to the case where A is a CSA. Now the required result follows immediately from Lemma 2.5 and Corollary 2.3.

2) $\Rightarrow$ 1). Obvious. □

Proposition 2.7. *Let A and B be semisimple algebras over F. Let E/F be a unirational field extension. Then the following conditions are equivalent:*

1) *There is a homomorphism of E-algebras, $B \otimes_F E \to A \otimes_F E$;*
2) *There is a homomorphism of F-algebras, $B \to A$.*

Proof. 1) $\Rightarrow$ 2). First of all, it is sufficient to consider only the case where $E = F(t)$. As in a Proposition 2.6, we can reduce the proof to the case where A is a CSA over F, and the required result follows from Lemma 2.5 and Corollary 2.3.

2) $\Rightarrow$ 1) Obvious. $\square$

§3. $\mathcal{G}$-FORMS

I this section we introduce a certain classes of qudararic form (see Definitions 3.1 and 3.3 bellow) and study properties of these classes.

Definition 3.1. *Let F be a field, and n be a positive integer. Let $\Gamma_n(F)$ be the set of all $(2n+1)$-dimensional forms ϕ satisfying the following condition:*

There are $a, b \in F^*$ and an n-dimensional quadratic form γ such that $\langle\langle a, b\rangle\rangle\lambda$ is anisotropic and $\phi \subset \langle\langle a, b\rangle\rangle\lambda$.

Example 3.2. *Let F be a field and ϕ be an anisotropic quadratic form over F. Then:*

1) *$\phi \in \Gamma_1(F)$ if and only if $\dim\phi = 3$;*
2) *$\phi \in \Gamma_2(F)$ if and only if ϕ is a 5-dimensional Pfister neighbour.*

Definition 3.3. *Let ϕ be an anisotropic quadratic form over F. We say that ϕ is a $\mathcal{G}$-form if it contains a subform belonging to one of the sets $\Gamma_2(F)$, $\Gamma_4(F)$, ..., $\Gamma_{2i}(F)$, ...*

We denote by $\mathcal{G}(F)$ the set of all $\mathcal{G}$-forms over F:

$$\mathcal{G}(F) = \{\phi \text{ is anisotropic over } F \mid \exists\psi \subset \phi, \psi \in \cup_{i=1}^{\infty}\Gamma_{2i}(F)\}.$$

Example 3.4. *Let F be a field and ϕ be an anisotropic quadratic form over F. Then:*

1) *If $\dim\phi \leqslant 4$, then $\phi \notin \mathcal{G}(F)$.*
2) *Let $5 \leqslant \dim\phi \leqslant 8$. Then $\phi \in \mathcal{G}(F)$ if and only if there is 4-dimensional subform $\tau \subset \phi$ such that $\tau \in GP_2(F)$.*

In other words, for any form ϕ with $\dim\phi \leqslant 8$ the condition $\phi \in \mathcal{G}(F)$ holds if and only if ϕ contains a 5-dimensional Pfister neighbour.

Example 3.5. *Let q be an anisotropic Albert form over F. Then:*

1) *q is not a $\mathcal{G}$-form,*
2) *The quadratic form $q \perp \langle t\rangle$ over $F((t))$ is not a $\mathcal{G}$-form,*
3) *If $a \in F^*$ is such that $q_{F(\sqrt{a})}$ is anisotropic, then the quadratic form $q \perp t\langle\langle a\rangle\rangle$ over $F((t))$ is not a $\mathcal{G}$-form.*

Proof. Let $\mu = 0$ in the first case, $\mu = \langle 1\rangle$ in the second case, and $\mu = \langle\langle a\rangle\rangle$ in the third case. Suppose that $\phi = q \perp t\mu$ is a $\mathcal{G}$-form. Then there is $\tau \subset \phi$

such that $\tau \in GP_2(\widehat{E})$ (see Example 3.4). Since $\widehat{E} = F((t))$, we can write τ in the form $\tau = \tau_0 \perp t\tau_1$, where the quadratic forms τ_0, τ_1 are defined over F. Since $\max(\dim\tau_0, \dim\tau_1) \geqslant 2$, there is $b \in F^*$ such that $\tau_{\widehat{E}(\sqrt{b})}$ is isotropic. Since $\tau \in GP_2(\widehat{E})$, we see that $\tau_{\widehat{E}(\sqrt{b})}$ is hyperbolic. Hence $i_W(\phi_{\widehat{E}(\sqrt{b})}) \geqslant i_W(\tau_{\widehat{E}(\sqrt{b})}) = 2$. Obviously, $i_W(\phi_{\widehat{E}(\sqrt{b})}) = i_W(q_{F(\sqrt{b})}) + i_W(\mu_{F(\sqrt{b})})$. Hence $i_W(q_{F(\sqrt{b})}) + i_W(\mu_{F(\sqrt{b})}) \geqslant 2$. Since q is an anisotropic Albert form, we have $i_W(q_{F(\sqrt{b})}) \leqslant 1$. Since $\dim\mu \leqslant 2$, we have $i_W(\mu_{F(\sqrt{b})}) \leqslant 1$. Therefore $i_W(q_{F(\sqrt{b})}) = i_W(\mu_{F(\sqrt{b})}) = 1$ and moreover $\dim\mu = 2$. Hence $\mu = \langle\langle a\rangle\rangle$ and $a \equiv b \in F^*/F^{*2}$. Therefore $i_W(q_{F(\sqrt{a})}) = i_W(q_{F(\sqrt{b})}) = 1$, i.e., the quadratic form $q_{F(\sqrt{a})}$ is isotropic, a contradiction. □

Lemma 3.6. *Let ϕ be an anisotropic quadratic form over field F. The following conditions are equivalent:*

1) *ϕ is a $\mathcal{G}$-form.*
2) *There exist a quadratic form τ and $a, b \in F^*$ such that*
   (a) *$\dim\tau < \dim\phi$,*
   (b) *$\phi \equiv \tau \pmod{\langle\langle a, b\rangle\rangle I(F)}$.*

Proof. 1) ⇒ 2). Let $\phi \in \mathcal{G}(F)$. By the definition of $\mathcal{G}(F)$, we can write ϕ in the form $\phi = \psi \perp \xi$ where $\psi \in \Gamma_{2k}(F)$.

It follows from $\psi \in \Gamma_{2k}(F)$ that $\dim\psi = 4k+1$ and there are $a, b \in F^*$ and a $2k$-dimensional quadratic form λ over F such that $\psi \subset \langle\langle a, b\rangle\rangle\lambda$. Let ψ' be such that $\psi \perp -\psi' = \langle\langle a, b\rangle\rangle\lambda$. Obviously $\dim\psi' = 4k-1$ and $\psi' \equiv \psi \pmod{\langle\langle a, b\rangle\rangle I(F)}$. Now it is sufficient to set $\tau = \psi' \perp \xi$.

2) ⇒ 1). We claim that $[\langle\langle a, b\rangle\rangle] \neq 0$. Indeed, otherwise $[\phi] = [\tau]$, and hence $\dim\phi = \dim\phi_{\text{an}} = \dim\tau_{\text{an}} \leqslant \dim\tau$, a contradiction.

Since $\phi \perp -\tau \in \langle\langle a, b\rangle\rangle I(F)$ it follows that $(\phi \perp -\tau)_{F(\langle\langle a,b\rangle\rangle)}$ is hyperbolic. Hence there is a quadratic form λ such that $(\phi \perp -\tau)_{\text{an}} = \langle\langle a, b\rangle\rangle\lambda$ (see [Sch, Ch.4, Theorem 5.4(iv)]).

Suppose that $\lambda \notin I(F)$. Then $[\phi] - [\tau] = [\langle\langle a, b\rangle\rangle\lambda] \equiv [\langle\langle a, b\rangle\rangle] \not\equiv 0 \pmod{I^3(F)}$. On the other hand, $\phi \equiv \tau \pmod{\langle\langle a, b\rangle\rangle I(F)}$ implies that $[\phi] - [\tau] \in \langle\langle a, b\rangle\rangle I(F) \subset I^3(F)$, a contradiction.

Therefore, $\lambda \in I(F)$, i.e., $\dim\lambda$ is even. Let $\dim\lambda = 2k$. It is clear, that there are quadratic forms ϕ_0, ρ_0, and ξ such that

$$\phi = \phi_0 \perp \xi, \quad \tau = \tau_0 \perp \xi, \quad (\phi \perp -\tau)_{\text{an}} = \phi_0 \perp -\tau_0.$$

Then $\phi_0 \perp -\tau_0 = \langle\langle a, b\rangle\rangle\lambda$. Hence $\dim\phi_0 + \dim\tau_0 = 8k$. Since $\dim\phi > \dim\tau$, we conclude that $\dim\phi_0 > \dim\tau_0$. Hence $\dim\phi_0 \geqslant 4k+1$. Let ψ be any $(4k+1)$-dimensional subform of ϕ_0. Since $\psi \subset \phi_0 \subset \langle\langle a, b\rangle\rangle\lambda$ and $\dim\psi = 4k+1$, we see that $\psi \in \Gamma_{2k}$. Finally since $\psi \subset \phi_0 \subset \phi$, it follows that $\phi \in \mathcal{G}(F)$. □

Lemma 3.7. *Let ϕ be an anisotropic form over F. Then the following conditions are equivalent:*

1) *$\phi \in \mathcal{G}(F)$,*
2) *$\phi_{F(t)} \in \mathcal{G}(F(t))$,*
3) *$\phi_{F((t))} \in \mathcal{G}(F((t)))$.*

Proof. 1)⇒2)⇒3). Obvious.

3)⇒ 1). It follows from Lemma 3.6, that there are quadratic forms $\widetilde{\tau}$ and $\widetilde{\lambda}$ over $F((t))$ and $\widetilde{a}, \widetilde{b} \in F((t))^*$ such that $\dim \widetilde{\tau} < \dim \phi$, $[\widetilde{\phi} \perp -\widetilde{\tau}] = [\langle\langle \widetilde{a}, \widetilde{b} \rangle\rangle \widetilde{\lambda}]$, and $\widetilde{\lambda} \in I(F((t)))$. Since $F((t))^*/F((t))^{*2} = F^*/F^{*2} \times \{1, t\}$, we can assume that

$$\widetilde{a} = at^p, \quad \widetilde{b} = bt^q, \quad \widetilde{\tau} = \left\langle k_1 t^{i_1}, k_2 t^{i_2}, \ldots \right\rangle, \quad \widetilde{\lambda} = \left\langle l_1 t^{j_1}, l_2 t^{j_2}, \ldots \right\rangle,$$

where $\quad k_1, k_2, \ldots, l_1, l_2, \ldots, a, b \in F^*, \quad i_1, i_2, \ldots, j_1, j_2, \ldots, p, q \in \{0, 1\}$.

Setting $\tau = \langle k_1, k_2, \ldots \rangle$, $\lambda = \langle l_1, l_2, \ldots \rangle$, one can see that $\dim \tau < \dim \phi$, $[\phi \perp -\tau] = [\langle\langle a, b \rangle\rangle \lambda]$, and $\lambda \in I(F)$. By Lemma 3.6, we have $\phi \in \mathcal{G}(F)$. □

Corollary 3.8. *Let ϕ be an anisotropic form over F and E/F be an unirational field extension. Suppose that $\phi_E \in \mathcal{G}(E)$. Then $\phi \in \mathcal{G}(F)$.* □

§4. Generalization of Tignol's theorem

The purpose of this section is to generalizate a Crollory 1.7.

Proposition 4.1. *Let ϕ be an anisotropic quadratic form over F and $\pi = \langle\langle a, b \rangle\rangle$. Suppose that $\phi_{F(\pi)}$ is isotropic. Then at least one of the following conditions holds:*

1) *ϕ is a $\mathcal{G}$-form.*
2) $\operatorname{Coind}(C_0(\phi) \otimes_F (a, b)) \geqslant 4$.

Proof. Since the extension $F(\pi)/F$ is excellent (see [ELW, Appendix II], or [Ro], [LWG]) there is a quadratic form τ over F such that $(\phi_{F(\pi)})_{\mathrm{an}} = \tau_{F(\pi)}$. Hence $(\phi \perp -\tau)_{\mathrm{an}} = \langle\langle a, b \rangle\rangle \lambda$.

If $\lambda \in I(F)$, then condition 2) in Lemma 3.6 holds. Then $\phi \in \mathcal{G}(F)$.

Now let $\lambda \notin I(F)$. Then

$$[\phi] - [\tau] = [\langle\langle a, b \rangle\rangle] \cdot [\lambda] \equiv [\langle\langle a, b \rangle\rangle] \pmod{I^3(F)}.$$

Therefore, $[C_0(\phi) \otimes_F (a, b)] = [C_0(\tau)]$.

Let $\dim \phi = n$. Then $\dim C_0(\phi) = 2^{n-1}$, and hence $\dim(C_0(\phi) \otimes (a, b)) = 2^{n+1}$. Since $\dim \tau \leqslant \dim \phi - 2 = n - 2$, we have $\dim(C_0(\tau)) \leqslant 2^{n-3}$. Hence $\dim(C_0(\phi) \otimes_F (a, b)) / \dim C_0(\tau) \geqslant 16$. Since $[C_0(\phi) \otimes_F (a, b)] = [C_0(\tau)]$, we have $C_0(\phi) \otimes (a, b) = M_k(C_0(\tau))$, where $k \geqslant \sqrt{16} = 4$. Hence $\operatorname{Coind}(C_0(\phi) \otimes_F (a, b)) \geqslant 4$. □

Corollary 4.2. *Let ϕ be an anisotropic quadratic form such that either $C_0(\phi) = M_2(A)$, or $C_0(\phi) = M_2(A) \times M_2(A)$ where A is a simple algebra. Let $\pi = \langle\langle a, b \rangle\rangle$. Suppose that $\phi_{F(\pi)}$ is isotropic. Then at least one of the following conditions holds:*

1) *ϕ is a $\mathcal{G}$-form.*
2) *$A \otimes_F (a, b)$ is not a division algebra.*

Proof. We can suppose that $\phi \notin \mathcal{G}(F)$. By Proposition 4.1, we have $\operatorname{Coind}(C_0(\phi) \otimes_F (a, b)) \geqslant 4$. Since $\operatorname{Coind}(C_0(\phi) \otimes_F (a, b)) = 2 \operatorname{Coind}(A \otimes (a, b))$, we have $\operatorname{Coind}(A \otimes_F (a, b)) \geqslant 2$, i.e., $A \otimes_F (a, b)$ is not a division algebra. □

Proposition 4.3. *Let ϕ be an anisotropic quadratic form such that:*

1) $\phi \notin \mathcal{G}(F)$,
2) $\mathrm{Coind}(C_0(\phi)) = 2$, *i.e., either* $C_0(\phi) = M_2(A)$ *or* $C_0(\phi) = M_2(A) \times M_2(A)$, *where A is a division simple algebra.*

Let ψ be a quadratic form such that $\phi_{F(\psi)}$ is isotropic and $\dim \psi \geqslant 3$.

Then there exist an extension E of the field F, a quadratic form γ over E and a quaternion algebra Q over E such that:

1) *$E(\gamma)/F$ is an unirational field extension,*
2) *$A \otimes_F Q$ is a division E-algebra,*
3) *$(A \otimes_F Q)_{E(\gamma)}$ is not a division $E(\gamma)$-algebra,*
4) $\dim \gamma \geqslant \dim \psi$, $\gamma \equiv \psi_E \pmod{I^2(E)}$, *and* $[C_0(\gamma)] = [C_0(\psi) \otimes_F Q]$.

Proof. Let $E_0 = F(X, Y, Z)$ be the field of rational function in three variables, and $\gamma_0 = \psi \perp -Z\langle\langle X, Y\rangle\rangle$ be a quadratic form over E_0. Let $E_0, E_1, \dots, E_h$ be the generic splitting tower of γ_0 and let $\gamma_i = ((\gamma_0)_{L_i})_{\mathrm{an}}$.

Let $n = \dim \psi$. Since $\dim \gamma_0 = n + 4 > n - 2 \geqslant 1 \geqslant \dim \gamma_h$, there exists s ($0 < s \leqslant r$) such that $\dim \gamma_s \leqslant n - 2 < \dim \gamma_{s-1}$.

Lemma 4.4.

1) $\dim \gamma_s \leqslant n - 2$ *and* $\dim \gamma_0, \dots, \dim \gamma_{s-1} \geqslant n$.
2) *For any field extension K/E_0 such that* $\dim((\gamma_0)_K)_{\mathrm{an}} \leqslant n - 2$, *the field extension $(KE_s)/K$ is purely transcendental.*
3) *The field extension E_s/F is unirational.*
4) *The quadratic form $\psi_{E_s(\langle\langle X,Y\rangle\rangle)}$ is isotropic.*

Proof. 1) By the definition of $\gamma_0, \dots, \gamma_s$, we see that $\dim \gamma_0 > \dots > \dim \gamma_{s-1} > n-2 \geqslant \dim \gamma_s$. Since $\dim \gamma_{s-1} \equiv \dim \gamma_0 = n+4 \equiv n \pmod 2$ and $\dim \gamma_{s-1} > n-2$, we conclude that $\dim \gamma_{s-1} \geqslant n$.

2) Since $\dim \gamma_{s-1} \geqslant n \geqslant 3$, the field extension $E_s = E_{s-1}(\gamma_{s-1})$ contains no nontrivial algebraic extensions of the field F. Therefore there exists a free composite KE_s of the fields K and E_s over F. Now it follows from properties of generic splitting tower that $(KF_s)/K$ is purely transcendental.

3) Since $\dim \psi \geqslant 3$, we can write ψ in the form $\psi = \langle a_1, a_2, a_3\rangle \perp \rho$. Let $K = F(X, Y, Z)(\sqrt{\frac{Z}{a_1}}, \sqrt{\frac{-ZX}{a_2}}, \sqrt{\frac{-ZY}{a_3}})$. It is clear that K/F is purely transcendental. We have $\langle a_1, a_2, a_3\rangle_K = \langle Z, -ZX, -ZY\rangle_K$. Hence

$$[\gamma_K] = [\langle a_1, a_2, a_3\rangle_K] + [\rho_K] - [Z\langle\langle X, Y\rangle\rangle_K] = [(\rho \perp \langle -ZXY\rangle)_K].$$

Therefore $\dim(\gamma_K)_{\mathrm{an}} \leqslant \dim(\rho \perp \langle -ZXY\rangle) = n - 3 + 1 = n - 2$. It follows from assertion 2), that $(KE_s)/K$ is purely transcendental. Since K/F is purely transcendental, we conclude that $(KE_s)/F$ is purely transcendental. Hence E_s/F is unirational.

4) Since $[\gamma_0] = [\psi_{E_0}] - [Z\langle\langle X, Y\rangle\rangle]$, we have $[\gamma_s] = [\psi_{E_s}] - [Z\langle\langle X, Y\rangle\rangle_{E_s}]$. Hence $[(\gamma_s)_{E_s(\langle\langle X,Y\rangle\rangle)}] = [\psi_{E_s(\langle\langle X,Y\rangle\rangle)}]$. Therefore

$$\dim(\psi_{E_s(\langle\langle X,Y\rangle\rangle)})_{\mathrm{an}} = \dim(\gamma_{E_s(\langle\langle X,Y\rangle\rangle)})_{\mathrm{an}} \leqslant \dim(\gamma_s)_{\mathrm{an}} \leqslant n - 2 < \dim \psi.$$

Hence $\psi_{E_s(\langle\langle X,Y\rangle\rangle)}$ is isotropic. The lemma is proved. $\square$

Now we proceed to the proof of Proposition 4.3. Since $\phi_{F(\psi)}$ and $\psi_{E_s(\langle\langle X,Y\rangle\rangle)}$ are isotropic, $\phi_{E_s(\langle\langle X,Y\rangle\rangle)}$ is isotropic too. Since E_s/F is unirational, we conclude that ϕ_{E_s} is anisotropic, and A_{E_s} is a division algebra. Thus anisotropic quadratic form ϕ_{E_s} becomes isotropic over the function field of quadratic form $\langle\langle X,Y\rangle\rangle_{E_s}$. Therefore all the conditions in Corollary 4.2 hold for the field E_s, quadratic forms ϕ_{E_s} and $\pi = \langle\langle X,Y\rangle\rangle_{E_s}$, and algebra A_{E_s}. Therefore at least one of the following conditions holds:

1) $\phi_{E_s} \in \mathcal{G}(E_s)$,
2) $A \otimes_F Q_s$ is not a division E_s-algebra, where Q_s denotes the quaternion algebra (X,Y) over the field E_s.

Consider first the case $\phi_{E_s} \in \mathcal{G}(E_s)$. Since E_s/F is unirational, it follows from Corollary 3.8 that $\phi \in \mathcal{G}(F)$, i.e., in this case Proposition 4.3 is proved.

Suppose now that $\phi_{E_s} \notin \mathcal{G}(E_s)$. Then $A \otimes_F Q_s$ is not a division algebra. Since $A \otimes_F Q_0$ is a division algebra over $E_0 = F(X,Y,Z)$, we conclude that there is k $(0 \leqslant k \leqslant s-1)$ such that $A \otimes_F Q_k$ is a division algebra, and $A \otimes_F Q_{k+1}$ is not a division algebra.

Let $E = E_k$, $\gamma = \gamma_k$, and $Q = Q_k$. We claim that the condition 1)–4) in Proposition 4.3 hold for E, γ and Q. Indeed:

1) $E(\gamma)/F$ is unirational because $E(\gamma) = E_{k+1} \subset E_s$, and E_s/F is unirational.

2),3) $A \otimes_F Q$ is a division algebra and $(A \otimes_F Q)_{E(\gamma)}$ is not a division algebra by definition of $E = E_k$.

4) $\dim \gamma = \dim \gamma_k \geqslant \dim \gamma_{s-1} \geqslant n = \dim \psi$. Since $[\gamma] = [\psi_E] - [Z\langle\langle X,Y\rangle\rangle_E]$, we conclude that $\gamma \equiv \psi_E \pmod{I^2(E)}$ and $[C_0(\gamma)] = [C_0(\psi) \otimes_F Q]$.

Thus Proposition 4.3 is proved. $\square$

Theorem 4.5. *Let ϕ and ψ be quadratic forms over F. Suppose that $\phi \notin \mathcal{G}(F)$ and* $\operatorname{Coind}(C_0(\phi)) \leqslant 2$. *Then the isotropy of $\phi_{F(\psi)}$ implies that there is a homomorphism of algebras, $C_0(\psi) \to C_0(\phi)$.*

Proof. If $\operatorname{Coind}(C_0(\phi)) = 1$, then $C_0(\phi)$ is either a division algebra or has the form $A \times A$, where A a division algebra. In this case theorem is always proved (see Corollary 1.7).

Thus we can suppose that $\operatorname{Coind}(C_0(\phi)) = 2$.

Consider the case $\dim \psi = 2$. Since $\phi_{F(\psi)}$ is isotropic, we conclude that there is $k \in F^*$ such that $k\psi \subset \phi$. Hence $C_0(\psi) \cong C_0(k\psi) \subset C_0(\psi)$.

Thus we can suppose that $\dim \psi \geqslant 3$. Then the conditions of Proposition 4.3 hold. Let A, E, γ, and Q be such as in Proposition 4.3. Let $D = A \otimes_F Q$. It follows from Proposition 4.3 that D is a division CSA over E. However $D_{E(\gamma)}$ is not a division algebra.

By Theorem 1.6, there is a homomorphism of E-algebras, $C_0(\gamma) \to D$. Since

$$[C_0(\psi_E) \otimes_E Q] = [C_0(\gamma)] = [C_0(\gamma) \otimes_E M_2(E)],$$

and

$$\dim(C_0(\psi_E) \otimes_E Q) = 2^{\dim \psi - 1} 4 \leqslant 2^{\dim \gamma - 1} 4 = \dim(C_0(\gamma) \otimes_E M_2(E)),$$

we conclude that there is a homomorphism of E algebras:

$$C_0(\psi_E)\otimes_E Q \to C_0(\gamma)\otimes_E M_2(E)$$

Hence there is a homomorphism $C_0(\psi_E)\otimes_E Q \to C_0(\phi_E)\otimes_E Q$ as the composition

$$C_0(\psi_E)\otimes_E Q \to C_0(\gamma)\otimes_E M_2(E) \to \\ \to D\otimes_E M_2(E) \cong M_2(A_E)\otimes_E Q \to C_0(\phi_E)\otimes_E Q.$$

It follows from Corollary 2.6 that there is a homomorphism $C_0(\psi_E)\to C_0(\phi_E)$ of E-algebras.

Since E/F is unirational, it follows from Corollary 2.7, that there is a homomorphism $C_0(\psi)\to C_0(\phi)$ of F-algebras. $\square$

§5. 7-dimensional forms

Theorem 5.1. *Let ϕ and ψ be quadratic forms over F, and $\dim\phi = 7$. Suppose that the quadratic form $\phi_{F(\psi)}$ is isotropic. Then at least one of the following conditions holds.*

1) *There is a 4-dimensional subform in ϕ with trivial determinant.*
2) *There is a homomorphism of F-algebras, $C_0(\psi)\to C_0(\phi)$.*

Proof. Let $\widetilde{\phi} = \phi \perp \langle\det\phi\rangle$. If $\operatorname{ind}(C_0(\phi)) = \operatorname{ind}(C(\widetilde{\phi})) \leqslant 2$, then there is $a\in F^*$ such that $\widetilde{\phi}_{F(\sqrt{a})}$ is hyperbolic (see [Kn2, Example 9.12]). Hence $[\phi_{F(\sqrt{a})}] = [\langle -\det\phi\rangle]$. Therefore $i_W(\phi_{F(\sqrt{a})}) = 3 > 2$. Hence there is a 2-dimensional quadratic form μ such that $\langle\langle a\rangle\rangle\mu\subset\phi$. Thus condition 1) holds.

Let now $\operatorname{ind}(C_0(\phi)) > 2$. Then $\operatorname{Coind}(C_0(\phi))\leqslant 8/4 = 2$. If $\phi\in\mathcal{G}(F)$, then condition 1) holds (see Example 3.4). If $\phi\notin\mathcal{G}(F)$, then the conditions of Theorem 4.5 hold. Hence there is a homomorphism, $C_0(\psi)\to C_0(\phi)$. $\square$

Remark. In the case where $\widetilde{\phi} = \phi\perp\langle\det\phi\rangle$ is anisotropic, $\dim\psi\geqslant 5$ and ψ is not a Pfister neighbour of a 3-fold Pfister form, Theorem 5.1 can be easy proved by using results of Laghribi [La2,La3].

Corollary 5.2. *Let ϕ be a quadratic form over F with $\dim\phi\leqslant 7$. Then the following conditions are equivalent:*

1) *There is a 3-fold Pfister form ψ such that $\phi_{F(\psi)}$ is isotropic.*
2) *ϕ contains a 5-dimensional Pfister neighbour (i.e., $\dim\phi > 4$ and there is a 4-dimensional subform in ϕ with trivial determinant).*

Proof. In the cases where $\dim\phi\leqslant 6$ the required result follows immediately from results of Hoffmann [H1, H2]. Suppose now that $\dim\phi = 7$.

(1) $\Rightarrow$ (2). Let ψ be a 3-fold Pfister form such that $\phi_{F(\psi)}$ is isotropic.

Suppose first that there is a homomorphism, $C_0(\psi)\to C_0(\phi)$. Since $C_0(\psi) = M_8(F)\times M_8(F)$ and $\deg C_0(\phi) = 8$, we conclude that $M_8(F)\cong C_0(\phi)$. Hence ϕ is a Pfister neighbor. Then ϕ contains a 5-dimensional Pfister neighbour.

If there is no a homomorphism from $C_0(\psi)$ to $C_0(\phi)$, then by Theorem 5.1, we conclude that there is a 4-dimensional subform in ϕ with trivial determinant.

(2) $\Rightarrow$ (1). Obvious. $\square$

Proposition 5.3. *Let q be an anisotropic Albert form over F and $\phi = q \perp \langle t \rangle$ be a quadratic form over $E = F((t))$. Let ψ be a 7-dimensional quadratic form over E such that $\phi_{E(\psi)}$ is isotropic. Then ψ is similar to ϕ.*

To proof this proposition we need the following lemma.

Lemma 5.4. *Let ϕ_0, ϕ_1, ψ_0, and ψ_1 be anisotropic quadratic forms over F. Let $E = F(t)$ and $\widehat{E} = F((t))$. Let $\phi = \phi_0 \perp t\phi_1$, $\psi = \psi_0 \perp t\psi_1$ be quadratic forms over $E = F(t)$.*

1) *The following conditions are equivalent:*
   (a) *$\phi_{E(\psi)}$ is isotropic,*
   (b) *$\phi_{\widehat{E}(\psi)}$ is isotropic.*
2) *If the both forms ψ_0, ψ_1 are nonzero, then the following conditions are equivalent:*
   (a) *$\phi_{E(\psi)}$ is isotropic,*
   (b) *$\phi_{\widehat{E}(\psi)}$ is isotropic,*
   (c) *For any field extension K/F, we have*

$$D_K(\psi_0)D_K(\psi_1) \subset D_K(\phi_0)D_K(\phi_1).$$

3) *If $\phi_{E(\psi)}$ (or $\phi_{\widehat{E}(\psi)}$) is isotropic, then:*
   (a) *if $\dim \psi_0 \geqslant 2$, then at least one of the forms ϕ_0, ϕ_1 is isotropic over the field $F(\psi_0)$,*
   (b) *if $\dim \psi_1 \geqslant 2$, then at least one of the forms ϕ_0, ϕ_1 is isotropic over the field $F(\psi_1)$,*
4) *If $\phi_{E(\psi)}$ (or $\phi_{\widehat{E}(\psi)}$) is isotropic, and $\dim \phi_1 = \dim \psi_1 = 1$, then ψ is similar to a subform in ϕ.*

Proof. We start the proof of the lemma from assertions 2), and 3).

2), $(a) \Rightarrow (b)$. Obvious.

2), $(b) \Rightarrow (c)$. Obviously we can suppose that $K = F$. Let $a \in D_F(\psi_0)D_F(\psi_1)$. Then $\psi_0 \perp -a\psi_1$ is isotropic. Obviously $\psi_{\widehat{E}(\sqrt{-at})} = (\psi_0 \perp -a\psi_1)_{\widehat{E}(\sqrt{-at})}$. Hence $\psi_{\widehat{E}(\sqrt{-at})}$ is isotropic. By the assumption, $\phi_{\widehat{E}(\psi)}$ is isotropic. Therefore $\phi_{\widehat{E}(\sqrt{-at})}$ is isotropic too. Hence $(\phi_0 \perp -a\phi_1)_{\widehat{E}(\sqrt{-at})} = \phi_{\widehat{E}(\sqrt{-at})}$ is isotropic. Since the quadratic form $\phi_0 \perp -a\phi_1$ is defined over F, and $\widehat{E}(\sqrt{-at})$ is a complete discrete valuation field with the residue field F, we conclude that $\phi_0 \perp -a\phi_1$ is isotropic. Hence $a \in D_F(\phi_0)D_F(\phi_1)$.

2), $(c) \Rightarrow (a)$. Set $K = E(\psi)$. Since $(\psi_0 \perp t\psi_1)_K$ is isotropic, we obviously have $-t \in D_K(\psi_0)D_K(\psi_0)$. Hence $-t \in D_K(\phi_0)D_K(\phi_0)$. Therefore $(\phi_0 \perp t\phi_1)_K$ is isotropic, i.e., $\phi_{E(\psi)}$ is isotropic.

3(a). By condition, $\phi_{E(\psi)}$ (or $\phi_{\widehat{E}(\psi)}$) is isotropic. In any case $\phi_{\widehat{E}(\psi)}$ is isotropic. Therefore $\phi_{\widehat{E}(\psi_0)}$ is isotropic. Since $\widehat{E}(\psi_0) \subset F(\psi_0)((t))$, we see that $(\phi_0)_{F(\psi_0)} \perp t(\phi_1)_{F(\psi_0)}$ is isotropic over $F(\psi_0)((t))$. Using Springer theorem, one can see that at least one of the forms $(\phi_0)_{F(\psi_0)}$ or $(\phi_1)_{F(\psi_0)}$ is isotropic.

3(b). The proof is the same as that of 3(a).

Assertion 1) of the lemma follows immediately from 2) and 3).

4). By our assumption and in view of assertion 1) of the lemma, we can suppose that $\phi_{E(\psi)}$ is isotropic. After scalaring, we can suppose that $\phi_1 = \psi_1 = \langle 1\rangle$. For any field extension K/F we have

$$D_K(\psi_0) = D_K(\psi_0)D_K(\psi_1) \subset D_K(\phi_0)D_K(\phi_1) = D_K(\phi_0).$$

It follows from *Third Representation Theorem* [Lam, Th. 2.8], that $\psi_0 \subset \phi_0$. Therefore $\psi = \psi_0 \perp \langle t\rangle \subset \phi_0 \perp \langle t\rangle = \phi$. $\square$

Proof of Proposition 5.3. It follows from Theorem 5.1 and Example 3.5 that there is a homomorphism $C_0(\psi) \to C_0(\phi)$. Since $\dim(C_0(\psi)) = \dim(C_0(\phi))$ and $C_0(\psi)$ is a SA, this homomorphism is isomorphism. Therefore $[C_0(\psi)] = [C_0(\phi)]$.

Since $E = F((t))$, we can write ψ in the form $\rho \perp t\tau$, where ρ and τ are quadratic forms over F. We can suppose that $\dim\tau$ is odd and $\dim\rho$ is even (otherwise we change ψ by $t\psi$).

Let $\widetilde{\tau} = \tau \perp \langle\det(\tau\perp\rho)\rangle$, $\widetilde{\psi} = \rho \perp t\widetilde{\tau}$. By the definition we have $\widetilde{\psi} \in I^2(E)$, $[C(\widetilde{\psi})] = [C_0(\psi)]$. Hence $[C(\widetilde{\psi})] = [C_0(\phi)] = [C(q)]$. Therefore $\widetilde{\psi} \equiv q \pmod{I^3(E)}$. Hence $[\rho] = \partial_t^0(\psi) \in I^2(F)$, and $[\widetilde{\tau}] = \partial_t^1(\psi) \in I^2(F)$.

By the assumption, ρ is an even-dimensional anisotropic quadratic form with $\dim\rho \leqslant \dim\phi = 7$. Therefore $\dim\rho$ is equal to one of the numbers 0, 2, 4, or 6.

Suppose first that $\dim\rho = 0$. Then $\psi = t\tau$ and $E(\psi) = F(\tau)(t)$. Since $\dim\psi = 7$, the quadratic form $q_{F(\psi)}$ is anisotropic (see [Le, Me]). Hence by Lemma 5.4, we conclude that $\phi_{E(\psi)}$ is anisotropic, a contradiction.

The case $\dim\rho = 2$ is impossible, because of $\rho \in I^2(F)$.

Suppose now that $\dim\rho = 4$. It follows from $\rho \in I^2(F)$, that $\rho \in GP_2(F)$. Therefore $\psi = \rho \perp t\tau$ contains a 5-dimensional Pfister neighbour. Hence there is $\pi \in P_3(E)$ such that $\psi_{E(\pi)}$ is isotropic. Since $\phi_{E(\psi)}$ and $\psi_{E(\pi)}$ are both isotropic, we conclude that $\phi_{E(\pi)}$ is isotropic. But this is impossible in view of Corollary 5.2 and Example 3.5.

Thus we see that $\dim(\rho) = 6$. Therefore $\dim\tau = 1$. It follows from the assertion 4) of Lemma 5.4 that $\phi \sim \psi$. $\square$

Corollary 5.5. *Let q be an anisotropic Albert form over F and $\phi = q \perp \langle t\rangle$ be a quadratic form over $\widehat{E} = F((t))$. Let ψ be a 8-dimensional quadratic form over $\widehat{E}$. Then $\phi_{\widehat{E}(\psi)}$ is anisotropic.*

Proof. Suppose at the moment that $\phi_{\widehat{E}(\psi)}$ is isotropic. Write ψ in the form $\psi = \rho \perp t\tau$, where ρ and τ are defined over F. We can suppose that that $\dim\rho \geqslant 4 \geqslant \dim\tau$ (otherwise we change ψ by $t\psi$). It follows from Lemma 5.4 that $q_{F(\rho)}$ is isotropic. Hence $\dim\rho \leqslant 6$ (see [Me, Le]).

Let ρ_0 subform of ρ of dimension $\dim\rho_0 = \dim\rho - 1$. Let $\psi_0 = \rho_0 \perp t\tau$. Since $\psi_0 \subset \psi$, we see that $\phi_{E(\psi_0)}$ is isotropic. Hence by Proposition 5.3 we have $\phi \sim \psi_0$. Hence either $\dim\rho_0 = 6$ or $\dim\rho_0 = 1$. On the other hand, it follows from $4 \leqslant \dim\rho \leqslant 6$ that $3 \leqslant \dim\rho_0 \leqslant 5$. This contradiction completes the proof. $\square$

Corollary 5.6. *Let q be an anisotropic Albert form over F and $\phi = q \perp \langle t\rangle$ be a quadratic form over $E = F(t)$. Let ψ be a 8-dimensional quadratic form over E. Then $\phi_{E(\psi)}$ is anisotropic.* $\square$

§6. 8-dimensional forms

Theorem 6.1. *Let ϕ and ψ be quadratic forms over F, and $\dim\phi = 8$. Suppose that the quadratic form $\phi_{F(\psi)}$ is isotropic. Then at least one of the following conditions holds.*

1) *There is a 4-dimensional subform in ϕ with trivial determinant.*
2) *There is a homomorphism of F-algebras, $C_0(\psi) \to C_0(\phi)$.*

Proof. If $\operatorname{Coind} C_0(\phi) \leqslant 2$, the required result follows from Theorem 4.5 and Example 3.4. Thus we can suppose that $\operatorname{Coind} C_0(\phi) \geqslant 4$, i.e., $\operatorname{ind} C_0(\phi) \leqslant 2$. Now the required result follows immediately from the following lemma.

Lemma 6.2. *Let ϕ be a 8-dimensional quadratic form such that* $\operatorname{ind}(C_0(\phi)) \leqslant 2$. *Then:*

1) *There is $a \in F^*$ such that $i_W(\phi_{F(\sqrt{a})}) \geqslant 2$.*
2) *There is a 4-dimensional subform in ϕ with trivial determinant.*

Proof. 1) If $\phi \in I^2(F)$, then the condition $\operatorname{ind} C_0(\phi) \leqslant 2$ implies that ϕ has the form $\langle\langle a\rangle\rangle q$, with q a 4-dimensional quadratic form [Kn2, Example 9.12]. Hence $i_W(\phi_{F(\sqrt{a})}) = 4 \geqslant 2$.

Suppose now that $\phi \notin I^2(F)$. Let $d = \det_\pm \phi$ and $L = F(\sqrt{d})$. Let $\gamma = \phi \perp \langle\langle d\rangle\rangle$. We have $\gamma \in I^2(F)$. The condition $\operatorname{ind}(C(\gamma)_L) = \operatorname{ind}(C_0(\phi)) \leqslant 2$ implies that $\operatorname{ind}(C(\gamma)) \leqslant 4$. Hence there is a biquaternion algebra A such that $[C(\gamma)] = [A]$. Since $\operatorname{ind}(A_L) = \operatorname{ind}(C(\gamma)_L) \leqslant 2$, we can write A in the form $A = (u, v) \otimes (d, s)$. Let $q = s\,\langle -u, -v, uv, d, s, -sd\rangle$. We have $[C(\gamma)] = A = [C(q)]$. Hence $\gamma - q \in I^3(F)$.

Let $\rho = \phi \perp -s\,\langle -u, -v, uv, d\rangle$. We have

$$\rho = \phi - s\,\langle -u, -v, uv, d\rangle = \phi + \langle\langle d\rangle\rangle - s\,\langle -u, -v, uv, d, s, -sd\rangle = \gamma - q \in I^3(F)$$

Since $\dim(\rho) = 4 + 8 = 12$ and $\rho \in I^3(F)$, we can write ρ in the form $\rho = \langle\langle a\rangle\rangle\tau$, where τ a 6-dimension Albert form (see [Pf]). Then $\rho_{F(\sqrt{a})}$ is hyperbolic and hence $[\phi_{F(\sqrt{a})}] = [s\,\langle -u, -v, uv, d\rangle_{F(\sqrt{a})}]$. Therefore $i_W(\phi_{F(\sqrt{a})}) \geqslant 2$.

2) Since $i_W(\phi_F(\sqrt{a})) \geqslant 2$, there is 2-dimensional quadratic form λ such that $\langle\langle a\rangle\rangle\lambda \subset \phi$. Since $\dim(\langle\langle a\rangle\rangle\lambda) = 4$ and $\det(\langle\langle a\rangle\rangle\lambda) = 1$, we see that ϕ contains a 4-dimensional subform with trivial determinant. $\square$

Remark. In the case where $\phi \in I^2(F)$, $\dim\psi \geqslant 5$, Theorem 6.1 follows imediately from results of Laghribi [La2], [La3].

Now collected together the results of Corollary 1.8, Theorem 5.1 and Theorem 6.1 we get the following

Theorem 6.3. *Let ϕ and ψ be anisotropic quadratic forms over F. Suppose that ϕ contains no 4-dimensional subforms with trivial determinant and $\dim\phi \leqslant 8$. Then the isotropy of $\phi_{F(\psi)}$ implies that there is a homomorphism of algebras, $C_0(\psi) \to C_0(\phi)$.* $\square$

Corollary 6.4. *Let ϕ be a quadratic form over F with $\dim\phi \leqslant 8$. Then the following conditions are equivalent:*

1) *There is a 3-fold Pfister form ψ such that $\phi_{F(\psi)}$ is isotropic.*
2) *ϕ contains a 5-dimensional Pfister neighbour (i.e., $\dim\phi > 4$ and there is a 4-dimensional subform in ϕ with trivial determinant).*

Proof. 1)⇒2). In view of Corollary 5.2 it is sufficient to consider only the case $\dim\phi = 8$. By Theorem 6.3 we can suppose that there is a homomorphism, $C_0(\psi) \to C_0(\phi)$. Since ψ is a 3-fold Pfister form, we have $C_0(\psi) \cong M_8(F) \times M_8(F)$. Thus we get a homomorphism $M_8(F) \to C_0(\phi)$. Therefore $\mathrm{Coind}(C_0(\phi)) \geqslant 3$ and hence $\mathrm{ind}(C_0(\phi)) = 1$. It follows from Lemma 6.2 that ϕ contains a 5-dimensional Pfister neighbour.

2)⇒1). Obvious. □

Remark. Corollary 6.4 gives rise to the following conjecture: if a quadratic form ϕ and a 3-fold Pfister form ψ are such that $\phi_{F(\psi)}$ is isotropic then ϕ contains a 5-dimensional Pfister neighbour of ψ. In [H2] D. Hoffmann proved that this conjecture is true if $\dim\phi \leqslant 6$. However the conjecture is not true in the case where $\dim\phi = 8$ (see [Izh]). Therefore for any $n \geqslant 8$ there is a counterexample with $\dim\phi = n$. Thus the only unsolved problem is: if the conjecture true in the case $\dim\psi = 7$?

Proof of the following result is similar to that of Proposition 5.3 and we omit it.

Example 6.5. *Let q be anisotropic Albert form over F and $a \in F^*$ be such that $q_{F(\sqrt{a})}$ is anisotropic. Let $\phi = q \perp t\langle\langle a\rangle\rangle$ be a quadratic form over $E = F((t))$. Let ψ be a 8-dimensional quadratic form over E such that $\phi_{E(\psi)}$ is isotropic. Then there is $k \in F^*$ such that $\psi \sim q \perp tk\langle\langle a\rangle\rangle$.* □

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