On linear forms with coefficients in $N\zeta(1+N)$

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Linear forms with coefficients in $\mathbb{N}\zeta(1+\mathbb{N})$

Symphonic drama in memory of professor Alexander Buchstab.

This is a brief exposition of my work "The lower estimate for some linear forms with coefficients prpotional to the values of $\zeta(s)$ for integer s".

VINITI,1997,No 3072-B97,pp. 1-77.

It is proved

Theorem. *If* $\phi_i(x_1,x_2)=\sum_{k=1}^{2}(2ik-i-k+2)(\zeta(i+k+1))x_k$ $(i=1,2)$, $||d||$ $(d\in\mathbb{R})$ *is*

the distance between d *and* $\mathbb{Z}$, $\gamma=43,464412$, *then there exist such* $c>0$, *that*

$||\phi_1(x_1,x_2)||+||\phi_2(x_1,x_2)||\geq c(|x_1|+|x_2|)^{-\gamma}$ $(|x_1|\in\mathbb{Z},x_2\in\mathbb{Z},|x_1|+|x_2|>0))$.

Corollary. *If* $p\in\mathbb{Q},q\in\mathbb{Q},p^2+q^2>0$, *then* $\{2p\zeta(3)+3q\zeta(4),3p\zeta(4)+6q\zeta(5)\}\not\subseteq\mathbb{Q}$

and therefore $\{\zeta(3+2k)/\zeta(4),(12\zeta(3)\zeta(5)-9(\zeta(4))^2)/\zeta(4)\}\not\subseteq\mathbb{Q}$ $(k=0,1)$.

In this paper are used the G - functions of C.S.Meijer.

(Invitation)Let $|z|\geq 1,-3\pi/2<arg(z)\leq\pi/2,log(z)=ln(|z|)+iarg(z)$. We take further z in the domain $Re(z)>0$;then $ln(-z)=ln(z)-i\pi$.

Let $\Delta\in\mathbb{N},1<\Delta,d_l=\Delta+(-1)^l$ $(l=1,2),\nu\in\mathbb{N}$,

(1) $f_1(z,\nu)=G_{6,6}^{(1,3)}\left(z\left|\begin{matrix}-\nu d_1, & -\nu d_1, & -\nu d_1, & 1+\nu d_2, & 1+\nu d_2, & 1+\nu d_2\\ 0, & 0, & 0, & \nu, & \nu, & \nu\end{matrix}\right.\right)\times$

$(-(-1)^{\nu(\Delta+1)})=-(-1)^{\nu(\Delta+1)}(2\pi i)^{-1}\int_{L_1}g_{6,6}^{(1,3)}(s)ds,$

where $g_{6,6}^{(1,3)}(s)=(z)^s\Gamma(-s)(\Gamma(1+s))^2(\Gamma(1+\nu d_1+s))/(\Gamma(1-\nu+s)\Gamma(1+\nu d_2-s)))^3$,

curve L_1 goes from $+\infty$ to $+\infty$, passing around the set $\mathbb{N}-1$ in negative direction but not including any point of the set $-\mathbb{N}$,

$f_2(z,\nu)=-(-1)^{\nu\Delta}(2\pi i)^{-1}\int_{L_2}g_{6,6}^{(4,3)}(s)ds=-(-1)^{\nu\Delta}\times$

$G_{6,6}^{(4,3)}\left(-z\left|\begin{matrix}-\nu d_1, & -\nu d_1, & -\nu d_1, & 1+\nu d_2, & 1+\nu d_2, & 1+\nu d_2\\ 0, & 0, & 0, & \nu, & \nu, & \nu\end{matrix}\right.\right),$

where $Re(z)>0,|z|>1,\nu\in\mathbb{N},g_{6,6}^{(4,3)}=g_{6,6}^{(4,3)}(s)=$

$((-z)^s(\Gamma(-s))^3\Gamma(-s-\nu))(\Gamma(1-\nu+s))^{-2}(\Gamma(1+\nu d_1+s))^3(\Gamma(1+\nu d_2-s))^{-3}$, curve L_2 goes from $-\infty$ to $-\infty$,passing around the set $-\mathbb{N}$ in the positive direction,

but not including any point of the set $\mathbb{N}-1$,

$f_3 = (-1)^{\nu(\Delta+1)}(2\pi i)^{-1} \int\limits_{L_2} g_{6,6}^{(5,3)}(s)ds = (-1)^{\nu(\Delta+1)}\times$

$$G_{6,6}^{(5,3)}\left(-z\Big|\begin{matrix}-\nu d_1, & -\nu d_1, & -\nu d_1, & 1+\nu d_2, & 1+\nu d_2, & 1+\nu d_2\\ 0, & 0, & 0, & \nu, & \nu, & \nu\end{matrix}\right),$$

where $g_{6,6}^{(5,3)} = g_{6,6}^{(5,3)}(s) = ((z)^s\times$

$(\Gamma(-s))^3\Gamma(-s-\nu))^2(\Gamma(1-\nu+s))^{-1}(\Gamma(1+\nu d_1+s))^3(\Gamma(1+\nu d_2-s))^{-3}$,

$f_5^{\vee} = -(-1)^{\nu\Delta}(2\pi i)^{-1}\int\limits_{L_2} g_{6,6}^{(6,3)}(s)ds = -(-1)^{\nu\Delta}\times$

$$G_{6,6}^{(6,3)}\left(-z\Big|\begin{matrix}-\nu d_1, & -\nu d_1, & -\nu d_1, & 1+\nu d_2, & 1+\nu d_2, & 1+\nu d_2\\ 0, & 0, & 0, & \nu, & \nu, & \nu\end{matrix}\right),$$

where $g_{6,6}^{(6,3)} = g_{6,6}^{(6,3)}(s) = ((-z)^s\times$

$(\Gamma(-s))^3\Gamma(-s-\nu))^3(\Gamma(1+\nu d_1+s))^3(\Gamma(1+\nu d_2-s))^{-3}$,

$f_4(z,\nu) = f_3(z,\nu) - (log(z))f_2(z,\nu), f_5(z,\nu) =$

$f_5^{\vee}(z,\nu) + i\pi f_3(z,\nu), f_6(z,\nu) = f_5(z,\nu) - f_2(z,\nu)(log(z))^2/2 - f_4(z,\nu)log(z)$,

$f_k^*(z,\nu) = ((\nu\Delta)!)/((\nu d_1)!))^3 f_k(z,\nu)\ (k=1,...,6)$,

$R(a;b;t) = \frac{b!}{(b-a)!}\ (\prod\limits_{\kappa=a+1}^{b}(t-\kappa))(\prod\limits_{\kappa=0}^{b}\frac{1}{t+\kappa}), R_0(t;\nu) = R(\nu;\nu\Delta;t)$

Therefore

$$R_0(t;\nu)^3 = \sum_{i=0}^{2}\sum_{k=0}^{\nu\Delta}\alpha_{\nu,k,i}^*(t+k)^{i-3}$$

where

$\alpha_{\nu,k,0}^* = \alpha_{\nu,k}^* = (-1)^{\nu\Delta+\nu+k}\binom{\nu\Delta}{k}^3\binom{\nu\Delta+k}{\nu d_1}^3 (k=0,...,\nu\Delta)$,

$\alpha_{\nu,k,1}^* = \beta_{\nu,k}^* = 3\alpha_{\nu,k}^*(\sum\limits_{\kappa=\nu+k+1}^{\nu\Delta+k}\kappa^{-1} - \sum\limits_{\kappa=1}^{\nu\Delta-k}\kappa^{-1} + \sum\limits_{\kappa=1}^{k}\kappa^{-1})(k=0,...,\nu\Delta)$,

$\alpha_{\nu,k,2}^* = \gamma_{\nu,k}^* = 2^{-1}9\alpha_{\nu,k}^*\ \times$

$$(\sum_{\kappa=\nu+k+1}^{\nu\Delta+k}\kappa^{-1} - \sum_{\kappa=1}^{\nu\Delta-k}\kappa^{-1} + \sum_{\kappa=1}^{k}\kappa^{-1})^2 - 2^{-1}3\alpha_{\nu,k}^*(\sum_{\kappa=\nu+k+1}^{\nu\Delta+k}\kappa^{-2} - \sum_{\kappa=1}^{\nu\Delta-k}\kappa^{-2} - \sum_{\kappa=1}^{k}\kappa^{-2})$$

$$(k=0,...,\nu\Delta),$$

$f_{2+2k}^*(z,\nu) = (-1)^{\nu}(k!)^{-1}\sum\limits_{t=\nu\Delta+1}^{\infty}(((d/dt)^k((R_0)^3))(t,\nu))z^{-t+\nu}\ (k=0,1,2)$,

$f_1^*(z,\nu) = z^\nu(-1)^{\nu\Delta}\sum_{k=0}^{\nu\Delta}(-z)^k\binom{\nu\Delta}{k}^3\binom{\nu\Delta+k}{\nu d_1}^3.$

Since $R_0(t;\nu) = 0$ $(t = \nu+1, ..., \nu\Delta)$,then

$f_{2+2k}^*(z,\nu) = (-1)^\nu(k!)^{-1}\sum_{t=\nu+1}^{\infty}(((d/dt)^k((R_0)^3))(t;\nu))z^{-t+\nu}$ $(k = 0,1,2)$.

Let $\alpha_i^*(z;\nu) = (-z)^\nu\sum_{k=0}^{\nu\Delta}\alpha_{\nu,k,i}^* z^k \in \mathbb{Q}[z]$ $(i = 0,1,2; \nu \in \mathbb{N})$, $L_i(z) = \sum_{t=1}^{+\infty} z^t t^{-i}$ $(i \in \mathbb{Z})$,

$\alpha_{3+i}^*(z;\nu) = (-1)^\nu\sum_{k=0}^{\nu\Delta}\sum_{t=1}^{k+\nu} z^{k+\nu-t}\sum_{s=0}^{2}\binom{2+i-s}{i}\alpha_{\nu,k,i}^* t^{-3-i+s}$ $(i = 0,1,2, \nu \in \mathbb{N})$.

Then $f_1^*(z,\nu) = \alpha_0^*(z;\nu)$,

$f_{2+2i}^*(z,\nu) = ((2+i)(1+i)\alpha_0^*(z;\nu)L_{3+i}(1/z))/2 +$

$(1+i)\alpha_1^*(z;\nu)L_{2+i}(1/z) + \alpha_2^*(z;\nu)L_{1+i}(1/z) - \alpha_{3+i}^*(z;\nu)$ $(i = 0,1,2; \nu \in \mathbb{N})$.

Let $D^\wedge(z,\nu,w) = z(w+1+\nu d_1)^3(w-\nu d_2)^3 - w^3(w-3\nu)^3, \delta = zd/dz$.

According to general properties of functions of C.S.Meijer ,

if $k = 1,2,3,5, \nu \in \mathbb{N}-1, Re(z) > 0, |z| > 1$, then $D^\wedge(z,\nu,\delta)f_k^*(z,\nu) = 0$,

$((\delta-\nu)\prod_{l=1}^{d_1}(\nu d_1+l)^{-1}(\delta+\nu d_1+l))^3 f_k^*(z,\nu) = (\prod_{l=1}^{d_2}(\delta-1+(\nu+1)d_2+l)^3)f_k^*(z,\nu+1)$.

Let (first bars of main theme),ν^{-1} is variable,taking values in $\mathbb{C}$ including $0, i = 0,1$,

$D_i^*(z,\nu^{-1},w) =$

$\nu^{-6}D^\wedge(z,\nu+i,\nu w) = b_{i,0}^\vee(z,\nu^{-1})w^0 + ... + b_{i,5}^\vee(z,\nu^{-1})w^5 + (z-1)w^6$,

$P_i^* = P_i^*(\nu^{-1},w) = (w-1)^{3-3i}(\prod_{k=1}^{\Delta-i}(\Delta-i+k\nu^{-1})^3)\prod_{k=1}^{\Delta-1+2i}(w+(-1)^i(d_{1+i}+k\nu^{-1})^3 =$

$p_{i,0}(\nu^{-1})w^0 + ... + p_{i,3\Delta+3i}(\nu^{-1})w^{3\Delta+3i} \in \mathbb{Z}[(\nu^{-1},w], P_i^\sim(w) = P_i^*(0,w)$.

Then $D_i^*(z,\nu^{-1},w) \in Q[z,\nu^{-1},w], D_0^*(z,0,w) = D_1^*(z,0,w)$. If $\nu \in \mathbb{N}, Im(z) > 0$,

$|z| > 1$, then $D_i^*(z,\nu^{-1},\nu^{-1}\delta)f_k^*(z,\nu+i) = 0, (P_1^*(\nu^{-1},\nu^{-1}\delta)f_k^*)(z,\nu+1) =$

$(P_0^*(\nu^{-1},\nu^{-1}\delta)f_k^*)(z,\nu)$ $(k = 1,2,3,5)$. Let $X_k(z,\nu)$ $(k = 1,2,3,5)$ is the column

with 6 elements $((\nu^{-1}\delta)^{j-1}f_k^*)(z,\nu)$ $(j = 1,...,6), i^2 = i, B_i(z,\nu^{-1})$ is the 6×6-matrix

with the unit matrix on its first 5 rows and last 5 columns,$b_{i,j-1}^\vee(z,\nu^{-1})/(1-z)$ in its

j - th column of its last row and 0 in its first 5 rows and first column,$D^\sim(z,w) =$

$D_0^*(z,0,w) = D_1^*(z,0,w), B^\sim(z) = B_0(z,0) = B_1(z,0)$. If $i^2 = i, k = 1,2,3,5$,

$\nu \in \mathbb{N}, Im(z) > 0, |z| > 1$,then $(\nu^{-1}\delta)X_k(z,\nu+i) = B_i(z,\nu^{-1})X_k(z,\nu+i)$,

(2) $Q^*(\nu^{-1}, \nu^{-1}\delta)X_k(z, \nu+1) = P^*(\nu^{-1}, \nu^{-1}\delta)X_k(z, \nu)$.

(Development of main theme)**Lemma 1.** *Let* $m \in \mathbb{N}, K = \mathbb{Q}[z](z-1)^{-m}] \cap \mathbb{Q}[(z-1)^{-1}]$,

$H^{\sim} \in Mat_6(K), H^* \in Mat_6(K, \nu^{-1}]), a_0 \in \mathbb{Q}, a_1 \in \mathbb{Q}, H = H^{\sim} + \nu^{-1}H^*$. *Then there is*

such $R_i(z, \nu) \in \nu^{-1}Mat_6(K, \nu^{-1}])$ $(i = 0, 1)$,*that*$(\kappa^{-1}\delta + a_0 + a_1\kappa^{-1})HX_k(z, \kappa + i) =$

$(H^{\sim}(B^{\sim}(z) + a_0E) + R_i(z, \nu)X_k(z, \kappa+i)$ $(k = 1, 2, 3, 5, \kappa \in \mathbb{N}, Im(z) > 0, |z| > 1)$.

It is proved in [5],p. 12. In [5],p.13 may be found the proof of the following

Lemma 2. *Let* $\mathfrak{W}_i = Mat_6((\mathbb{Q}[z, \nu^{-1}](z-1)^{-3\Delta-3i}] \cap \mathbb{Q}[(z-1)^{-1}, \nu^{-1}]))$ $(i = 0, 1)$.

Then $\mathfrak{W}_i$ $(i = 0, 1)$ *contains such* $U_i(z, \nu^{-1}))$,*that* $U_0(z, \nu^{-1}) - P^{\sim}(B^{\sim}(z)) \in \nu^{-1}\mathfrak{W}_0$,

$U_1(z, \nu^{-1}) - Q^{\sim}(B^{\sim}(z)) \in \nu^{-1}\mathfrak{W}_1, P^*(\kappa^{-1}, \kappa^{-1}\delta)X_k(z, \kappa) = U_0(z, \kappa^{-1})X_k(z, \kappa)$

$Q^*(\kappa^{-1}, \kappa^{-1}\delta)X_k(z, \kappa+1) = U_1(z, \kappa^{-1})X_k(z, \kappa+1)$,*if* $k = 1, 2, 3, 5; \kappa \in \mathbb{N}$.

Let $\varepsilon \in]0, 1[, \mathfrak{N}(\epsilon) = \{\kappa^{-1} \in \mathbb{C} \mid |\kappa^{-1}| < \varepsilon\}, F$ is a compact in $\mathbb{C}, \mathfrak{H}(F, \varepsilon)$ is

the ring of all defined on $F \times \mathfrak{N}(\varepsilon)$ elements from $\mathbb{Q}(z, \nu^{-1}), \mathfrak{H}_0(F)$ is the ring

of all defined on F elements from $\mathbb{Q}(z)$. In [5],p. 14, is proved the following

Lemma 3. *For each compact* $F \in \mathbb{C}\backslash\{1\}$ *there is such* $\varepsilon_0 = \varepsilon_0(F) \in]0, 1[\}$ *,that*

(3) $A(z, \nu^{-1}) = (U_1(z, \nu^{-1}))^{-1}U_0(z, \nu^{-1}) \in Mat_6(\mathfrak{H}(F, \varepsilon_0)$,

and $A(z, 0) = A^{\sim}(z) = (Q^{\sim}(B^{\sim}(z)))^{-1}P^{\sim}(B^{\sim}(z))$ $(z \in F)$.

According to (2),lemmas 2 and 3,if F is a compact in $\{z \in \mathbb{C} | 0 < Im(z), |z| > 1\}$,then

(4) $X_k(z; \nu+1) = A(z, \nu^{-1})X_k(z; \nu)$ $(k = 1, 2, 3, 5; z \in F, \nu \in \mathbb{N} + [1/\varepsilon_0(F)])$.

Lemma 4. (It is proved in [5],pp. 15 - 19.) *Let* F *is a compact in* $\mathbb{C}, \nu \in \mathbb{N} + [1/\varepsilon_0]$,

$n \in \mathbb{N}, X_\nu(z) \in (\mathbb{C}^F)^n$ $A(z, \nu^{-1}) \in Mat_n(\mathfrak{H}(F, \varepsilon_0)), X_{\nu+1}(z) = A(z, \nu^{-1})X_\nu(z)$ $(z \in F)$,

the multiplicity of any root of $T_0(z, \lambda) = det(\lambda E - A(z, 0)) \in \mathbb{C}[\lambda]$ $(z \in F)$ *is*

equal to 1,$C(z) \in Mat_n(\mathbb{C}^F), det(C(z)) \neq 0$ $(z \in F), (C(z))^{-1}A(z, 0)C(z)$ $(z \in F)$ *is*

a diagonal matrix and all elements of the first row in $C(z)$ *aren't equal to 0. Then for*

some $\varepsilon_2 \in]0, \varepsilon_0[, q_j^*(z, \nu^{-1}) \in \mathfrak{H}(F, \varepsilon_2)$ $(j = 0, ..., n-1)$, *if* $z \in F, \nu \in \mathbb{N} + [1/\varepsilon_2]$,

the equation $-x_{\nu+n}(z) = q_0^*(z, \nu^{-1})x_{\nu+0}(z) + ... + q_{n-1}^*(z, \nu^{-1})x_{\nu+n-1}(z)$ *is satisfied* *by the first element in* $X_\nu(z)$;*besides* $\lambda^n + q_0^*(z,0)\lambda^0 + ... + q_{n-1}^*(z,0)\lambda^{n-1} = T_0(z,\lambda)$.

Lemma 5. *Let* $A(z,\nu^{-1}) \in Mat_n\mathfrak{H}(F,\varepsilon_0), b_j^{\check{}}(z) \in \mathfrak{H}_0\ (j = 0,...,n), b_0^{\check{}}(z) = 1$,*for*

each $z \in F$ *the multiplicity of any root of* $D^{\check{}}(z,\lambda) = b_0^{\check{}}(z)\lambda^0 + ... + b_n^{\check{}}(z)\lambda^n$ *is equal to 1,*

$B^{\check{}}(z)$ *is a* $n \times n$ *- matrix,having* $-b_{j-1}^{\check{}}(z)\ (j = 1,...,n)$ *as the* j*-th element of its*

last row,(if $n > 1$*) the unit matrix on its first* $n-1$ *rows and last* $n-1$ *columns and 0*

in first $n-1$ *rows of its first column,*$P^{\check{}}(z,\lambda) \in \mathfrak{H}_0(F)[\lambda], Q^{\check{}}(z,\lambda) \in \mathfrak{H}_0(F)[\lambda]$),

if $z \in F, \lambda \in \mathbb{C}$ *and* $D^{\check{}}(z,\lambda) = 0$,*then* $Q^{\check{}}(z,\lambda) \neq 0$,*on* $\{\lambda \in \mathbb{C} | D^{\check{}}(z,\lambda) = 0\}\ (z \in F)$

the map $\lambda \to P^{\check{}}(z,\lambda)/Q^{\check{}}(z,\lambda)$ *is injective and* $Q^{\check{}}(z,B^{\check{}}(z))A(z,0) = P^{\check{}}(z,B^{\check{}}(z))$.

Then the multiplicity of any root of the polynomial $T_0(z,\lambda)\ (z \in F)$ *is equal to 1,there is*

such $C(z) \in Mat_n(\mathbb{C}^F)$,*that for each* $z \in F$ *all elements of its first row aren't equal to 0,*

$det(C(z)) \neq 0$ *and* $(C(z))^{-1}A(z,0)C(z)$*is a diagonal matrix.* It is proved in [31],pp. 20 - 21.

Let $\delta_0 = 1/\Delta, \gamma_1 = (1-\delta_0)/(1+\delta_0)$. The substitution $w = (\eta d_2 + d_1)/(\eta - 1)$ turns

the polynomials $D^{\check{}}(z,w), P_i^{\check{}}(w)\ (i = 0,1)$ and rational function $P_0^{\check{}}(w)/P_1^{\check{}}(w)$into

respectively $D^{\check{}}(z,w) = -(\eta-1)^{-6}\Delta^3(\Delta+1)^3((\eta+1)^3(\eta+\gamma_1)^3 - 2^3(1+\gamma_1)^3 z\eta^3), P_i^{\check{}}(w) =$

$P_{i,1}^{\check{}}(\eta) = \Delta^{3\Delta}(\eta-1)^{-3\Delta-3i}(d_1)^{-3d_1}(\eta+1)^{3-3i}2^{3d_1+i}\eta^{(3d_1)(1-i)}$ and $P_0^{\check{}}(w)/P_1^{\check{}}(w) =$

$h^{\check{}}(\eta) = P_{0,1}^{\check{}}(\eta)/P_{1,1}^{\check{}}(\eta) = (\eta^2-1)^3(1-\delta_0)^{-3d_1}2^{-6}\eta^{3d_1}$;this substitution and inverse to it

connects the roots of $D^{\check{}}(z,w)$ and $D^{\wedge}(z,\eta) = (\eta+1)^3(\eta+\gamma_1)^3 - 2^3(1+\gamma_1)^3 z\eta^3$.

Let $z = -r^3e^{3i\varphi}\ (1 \le r, -\pi/3 < \varphi \le \pi/3), \psi = \varphi + 2k\pi/3\ (k^3 = k)$. Then $D^{\wedge}(z,\eta) =$

$(\eta+\gamma_1)(\eta+1) + 2re^{i\psi}(1+\gamma_1)\eta\ (1 \le r, -\pi < \psi \le \pi)$. Let $\eta = \theta(1+\gamma_1), \theta^{\vee} = -e^{-i\psi}\theta$,

$D_1^{\vee}(r,\psi,\theta) = \theta^2 + 2(\frac{1}{2} + re^{i\psi})\theta + \frac{\gamma_1}{(1+\gamma_1)^2} = ((\theta^{\vee})^2 - 2(e^{-i\psi}/2 + r)\theta^{\vee} + e^{-2i\psi}\frac{\gamma_1}{(1+\gamma_1)^2})e^{2i\psi}$.

Then $D^{\vee}(r,\psi,\eta) = (1+\gamma_1)^2 D_1^{\vee}(r,\psi,\theta)$.Let $1 \le r, |\psi| \le \pi, 0 \le \delta_0 \le 1, D_0(r,\psi,\delta_0) =$

$r^2 + re^{-i\psi} + (\delta_0/2)^2 e^{-2i\psi}, R_0(r,\psi,\delta_0) = (|D_0(r,\psi,\delta_0)| + Re(D_0(r,\psi,\delta_0))^{1/2} -$

$i(sign(\psi))(|D_0(r,\psi,\gamma)| - Re(D_0(r,\psi,\delta_0))^{1/2}$. Then $R_0(r,\psi,1) = e^{-i\psi}/2 + r$, $(R_0(r,\psi,\delta_0)^2 = D_0(r,\psi,\delta_0)$ and $-e^{i\psi}(R_0(r,\psi,1) + (-1)^k R_0(r,\psi,\delta_0)) = \theta_k^{\wedge}(r,\psi,\delta_0)$ $(k = 0,1), \eta_k^{\wedge}(r,\psi,\delta_0) = \theta_k^{\wedge}(r,\psi,\delta_0)(1+\gamma_1)$ $(k = 0,1)$ forms the set of

all roots of respectively $D_1^{\vee}(r,\psi,\theta), D^{\vee}(r,\psi,\eta)$. Let $1 \le r, -\pi/3 < \varphi \le \pi/3, z = -r^3 e^{3i\varphi}, \mathbb{Q}[z,\eta] \ni F(r,\phi,\eta) = \prod_{l=-1}^{1} \prod_{k=0}^{1} (\eta - h^{\sim}(\eta_k^{\wedge}(r,(\varphi+2\pi l)/3))$ (we use the theorem

about symmetric polynomials) and $\mathbb{Q}[z]$ includes its discriminant $\mathfrak{D}(z)$ respectively to η.

(Exercise) **Lemma 6.** *If* $0 < \delta_0 < 1$,*then* $\mathfrak{D}(1) \ne 0$. It is proved in [4],pp. 25 - 30.

According to the lemma 6,there is such $\varepsilon_1 \in]0,1[$,that $\mathfrak{D}(z) \ne 0$ $(|z-1| \le \varepsilon_1)$.

If $\varepsilon \in]0,\varepsilon_1[$,then $\Omega(\varepsilon) = \{z \in \mathbb{C} \mid \varepsilon + 1 < |z|, |z-1| < \varepsilon_1, 2^{-1}arctg(\frac{\varepsilon}{2}) < arg(z)\} \ne \emptyset$.

The closure $F_1(\varepsilon)$of the domain $\Omega(\varepsilon)$ is a compact in $\mathbb{C}\backslash\{1\}$. It follows from (3),(4),

the inequality $\mathfrak{D}(z) \ne 0$ $(|z-1| \le \varepsilon_1)$,lemmas 3 and 5,that for $n = 6$,any $\varepsilon \in]0,\varepsilon_1[$,

$A(z,\nu^{-1})$ from (30),with $F_1(\varepsilon), \varepsilon_0(F_1(\varepsilon))$ and $X_k(z,\nu)$ $(k = 1,2,3,5)$ respectively in

the role of F, ε_0 and $X_\nu(z)$ are fullfilled all conditions of the lemma 4;then there are

such $\varepsilon_2 \in]0,\varepsilon_0[$, $q_j^{\vee}(z,\nu-1) \in \mathfrak{H}(F_1(\varepsilon),\varepsilon_2)$ $(z \in F_1(\varepsilon), j = 0,...,5)$,that $T_0(z,\lambda) = q_0^{\vee}(z,0)\lambda^0 + ... + q_5^{\vee}(z,0)\lambda^5 + \lambda^6 = F(r,\phi,\lambda)$ and the first element $y(\nu) = f_k^*(z;\nu)$ in $X_k(z,\nu)$ $(k = 1,2,3,5)$ for $z \in F_1(\varepsilon), \nu \in \mathbb{N} + [1/\varepsilon_2]$ satisfies to the equation

(5) $q_0(z,\nu^{-1})y(\nu) + q_1(z,\nu^{-1})y(\nu+1) + ... + q_6(z,\nu^{-1})y(\nu+6) = 0$,

where $q_j(z,\nu^{-1}) \in \mathbb{Q}[z,\nu^{-1}]$ $(j = 0,...,6)$,they haven't common divisors,exepting inequal to 0 constants,$q_j(z,\nu^{-1})/q_6(z,\nu^{-1}) = q_j^{\vee}(z,\nu^{-1})$ $(j = 0,...,6)$.

Lemma 7 ([3],pp.36 - 37). The rank over $\mathbb{C}(z)$of $\{L_k(z) | k = 1,...,n\}$ $(n \in \mathbb{N})$ equals n.

Since $f_k^*(z,\nu)(k = 4,6)$ also are solutions of the equation (5) in $\Omega(\varepsilon)$ for $\nu \in [1/\varepsilon_2] + \mathbb{N}$,

according to the lemma 7,$\alpha_i^*(z,\nu)$ $(i = 0,...,5)$ are its solutions for $\nu \in [1/\varepsilon_2] + \mathbb{N}, z$

in $\Omega(\varepsilon)$ and then for all $z \in \mathbb{C}$. But $\mathfrak{N} = \{j \in \{1,2,3,4,5,6\} | q_j(1;\nu) \ne 0_{\mathbb{Q}[\nu^{-1}]}\} \ne \emptyset$.

Let $n = sup(\mathfrak{N})$. There is such $\varepsilon_3 \in]0,\varepsilon_2[$,that $q_n(1,\nu^{-1}) \ne 0$ $(\nu \in [1/\varepsilon_3] + \mathbb{N})$.

Let $q_j^*(\nu^{-1}) = q_j(1,\nu^{-1})/q_n(1,\nu^{-1})$ $(\nu \in [1/\varepsilon_3] + \mathbb{N}; j = 0, ..., n)$.

Then $y(\nu) = \alpha_i^*(1,\nu)$ $(i = 0, ..., 5)$,if $\nu \in [1/\varepsilon_3] + \mathbb{N}$, are solutions of the equation

(6) $q_0^*(\nu^{-1})(\nu+0) + ... + q_n^*(\nu^{-1})(\nu+n) = 0$.

(For the left hand) **Lemma 8.** *For each $m \in \mathbb{N}$ five sequences $\alpha_k^*(1,\nu)$ $(\nu \in m+\mathbb{N})$ with k = 0,1,3,4,5 form a linear independent system over $\mathbb{C}$.* It is proved in [4],p. 30 - 38.

The lemma 8 implies for n in (6) the ineqality $n > 4$.

(Ballad) **Lemma 9** **([5],pp. 27 - 35).** *The equality $q_0^*(1,\nu^{-1})\} = 0_{\mathbb{Q}(\nu^{-1})}$ is true.*

It follows now from lemma 8,that $n = 6$ and $q_j^{\vee}(1,\nu^{-1}) = q_j^*(1,\nu^{-1})$ $(j = 1, ..., 5)$.

If $q_j^{\wedge *}(\nu^{-1}) = q_{j+1}^*((\nu-1)^{-1})(j = 0, ..., 4)$,then $q_0^{\wedge *}(1,\nu^{-1}) \neq 0$ $(\nu \in [1/\varepsilon_3] + \mathbb{N}+1)$

and $\alpha_j^*(1;\nu)$ $(j = 0,1,3,4,5)$ forms a basis of space of solutions of the equation

(7) $q_0^{\wedge *}(\nu^{-1})(\nu+0) + ... + q_4^{\wedge *}(\nu^{-1})(\nu+4) + y(\nu+5) = 0$ $(\nu \in [1/\varepsilon_3] + \mathbb{N}+1)$.

(Pizzicato) **Lemma 10.** *Let $\varepsilon_3 > 0, \beta_j(\nu)$ $(j = 1, ..., n)$, is a basis of space of solutions of*

the equation $a_0(\nu^{-1})(\nu+0)+...+a_n(\nu^{-1})(\nu+n) = 0$ $(\nu \in [1/\varepsilon_3]+\mathbb{N}), a_n(\nu^{-1}) = 1$,

$a_0(\nu^{-1}) \neq 0$,if $\nu \in [1/\varepsilon_3]+\mathbb{N}$, and there are such $C > 0$, > 0,that $|\beta_j(\nu)| \leq CM^{\nu}$,

when $j = 1, ..., n, \nu \in [1/\varepsilon_3] + \mathbb{N}$. If,besides, $a_j(\nu^{-1}) \in \mathbb{Q}(\nu^{-1})$ $(j = 0, ..., n-1)$,

then they are defined at the point $\nu^{-1} = 0$. It is proved in [5],p. 35 - 37.

Since ([5],p. 35 - 37) for the equation (7) are fullfiled all conditions of the lemma 10,

it is the equation of Poincaré type;we use the inclusion $f_1^*(z,\nu) \in \mathbb{Z}[z]$ $(\nu \in \mathbb{N})$to study

its characteristic polinomial $F_0(\eta) = q_0^{\wedge *}(1,0)\eta^0 + ... + q_4^{\wedge *}(1,0)\eta^4 + \eta^5$. Let $\mathfrak{D}$

is the ring of all rational functions in $\mathbb{Q}(z;\nu^{-1})$,defined at the point $(1,0)$, $\mathfrak{D} < v >=$

$\bigcup_{n=0}^{\infty} \mathfrak{D}_n < v >$,where $\mathfrak{D}_n < v >$ is the set of all differential polinomials $D = \sum_{k=0}^{\infty} \mathfrak{c}_k(D)v^k$,

with $v = \nu^{-1}zd/dz, \mathfrak{c}_k(D) \in \mathfrak{D}$ $(k \in \mathbb{N}-1), \mathfrak{c}_k(D) = 0$ $(k \in \mathbb{N}+n)$. Since $vf + f \circ v =$

$v \circ f \in \mathfrak{D} < v >$ $(f \in \mathfrak{D})$,the set $\mathfrak{D} < v >$ is a $\mathbb{Q}$ - subalgebra of a $\mathbb{Q}$ - algebra $\mathcal{M}_{\mathbb{Q}}(\mathfrak{D})$

of all $\mathbb{Q}$ - linear operators acting on $\mathfrak{D}$. For each $n \in \mathbb{N}-1$ let $\mathfrak{D}_n^* < v >$ is the set of all $D \in \mathfrak{D}_n < v >$ with $(\mathfrak{c}_0(D))(1,0) \neq 0\}, \mathfrak{D}^* < v >$ is the join of all $\mathfrak{D}_n^* < v >$.

(Metamorphosis) For all $k \in \mathbb{N}-1, D \in \mathfrak{D}_n^* < v >$ let $\mathfrak{b}_k(D) = \mathfrak{c}_k(D) + v\mathfrak{c}_{k+1}(D) - (v\mathfrak{c}_0(D))\mathfrak{c}_{k+1}(D)/\mathfrak{c}_0(D)$; then $\mathfrak{b}_k(D) \in \mathfrak{D}$ $(k \in \mathbb{N}-1)$,

(8) $\mathfrak{b}_k(D) = \mathfrak{c}_k(D)$ $(k \in n + \mathbb{N} - 1), \mathfrak{b}_k(D)(z,0) = \mathfrak{c}_k(D)(z,0)$ $(k \in \mathbb{N}-1)$.

Let $\mathfrak{d}$ denote the map of the set $\mathfrak{D}^* < v >$ in $\mathfrak{D}^* < v >$,for which $\mathfrak{c}_k(\mathfrak{d}D) = \mathfrak{b}_k(D)$, if $D \in \mathfrak{D}^* < v >, k \in \mathbb{N}-1$. Then $\mathfrak{d}(\mathfrak{D}_n^* < v >) \in \mathfrak{D}_n^* < v >$ $(n \in \mathbb{N}-1)$.

Lemma 11 (proved in [5],p. 41). *If* $D \in \mathfrak{D}^* < v >$,*then* $(v - \frac{v\mathfrak{c}_0(D)}{\mathfrak{c}_0(D)}) \circ D = (\mathfrak{d}D) \circ v$.

Because $v\mathfrak{D} \in \nu^{-1}\mathfrak{D}$, then $D \circ v^n - v^n \circ D \in \nu^{-1}\mathfrak{D} < v > (D \in \mathfrak{D} < v >, n \in \mathbb{N}-1)$,

$D_1 \circ D_2 - D_2 \circ D_1 \in \nu^{-1}\mathfrak{D} < v > (D_i \in \mathfrak{D} < v >, i = 1,2)$ and for each $D \in \mathfrak{D} < v >$

the right ideal $D \circ \mathfrak{D} < v > + \nu^{-1}\mathfrak{D} < v >$ is two-sided. In particular,two-sided is the right

ideal $(z-1)\mathfrak{D} < v > + \nu^{-1}\mathfrak{D} < v >$. If $k = 1$,then,according to lemma 11 and (15),

(9) $v^k Ker(D) \subset Ker(\mathfrak{d}^k D)(D \in \mathfrak{D} < v >), \mathfrak{d}^k D - D \subset \nu^{-1}\mathfrak{D} < v > (D \subset \mathfrak{D}^* < v >)$;

the induction shows,that (9) are true for all $k \in \mathbb{N}$. Let $\mathfrak{r}_n$ $(n \in \mathbb{N})$ is the projection,for

which $\mathfrak{r}_n(D) = D - \mathfrak{c}_n(D)v^n$ $(D \in \mathfrak{D} < v >)$. Then $\mathfrak{r}_n\mathfrak{D}^* < v > \subset \mathfrak{D}^* < v >$ $(n \in \mathbb{N})$.

Let $D \in \mathfrak{D}_n^* < v >$ and $\mathfrak{c}_n(D) \in (z-1)\mathfrak{D}$. According to (8),$\mathfrak{d}^k D - \mathfrak{r}_n(\mathfrak{d}^k D) = \mathfrak{c}_n(\mathfrak{d}^k D) =$

$\mathfrak{c}_n(D) \in (z-1)\mathfrak{D} < v > (k \in \mathbb{N}-1), \mathfrak{r}_n(\mathfrak{d}^k D)Ker(\mathfrak{d}^k D) \subset (z-1)\mathfrak{D}$ $(k \in \mathbb{N}-1)$ and,

according to (9),$(\mathfrak{r}_n(\mathfrak{d}^k D) \circ v^k)Ker(D) \subset \mathfrak{r}_n((\mathfrak{d}^k D))Ker(\mathfrak{d}^k D) \subset (z-1)\mathfrak{D}$,if $k \in \mathbb{N}-1$.

If,besides, $(\mathfrak{c}_{n-1}(D))(1;0) \neq 0$,then, according to (8),$\mathfrak{c}_{n-1}(\mathfrak{d}^k D)(1;0) \neq 0$ $(k \in \mathbb{N}-1)$.

Moreover,for each $s \in \mathbb{N}-1$ there is such $\varepsilon(D,s) > 0$,that all $\mathfrak{c}_i((\mathfrak{d}^k D))(z;\nu^{-1})$

with $i = 0,....,n, k = 0,...,s$ and $(\mathfrak{c}_{n-1}(\mathfrak{d}^k D)(z;\nu^{-1}))^{-1}$ $(k = 0,...,s)$ are defined

in the set $|z-1| < \varepsilon(D,s), |\nu^{-1}| < \varepsilon(D,s)$. Let $\mu \in \mathbb{Q}, \mathfrak{D}^{(\mu)}$ is the ring of all elements

in $\mathfrak{D}$,defined at the point $(1;\mu)$.Thus,$\mathfrak{D}^{(0)} = \mathfrak{D}$ and $\mathfrak{c}_i(\mathfrak{d}^k D) \in \mathfrak{D}^{(\mu)}, (\mathfrak{c}_{n-1}(\mathfrak{d}^k D))^{-1}$

is in $\mathfrak{D}^{(\mu)}$, if $i = 0,....,n, k = 0,...,s, \mu \in [0, \varepsilon(D,s)[$.We denote by $\theta_\mu$the natural

extensions of the homomorphism of $\mathfrak{D}^{(\mu)}$ in the ring $\mathfrak{B}$ of all elements in $\mathbb{Q}(z)$,defined at the point $z = 1$,for which $(\theta_\mu f)(z) = f(z;\mu)$ to homomorphism of $Mat_n(\mathfrak{D}^{(\mu)})$ in $Mat_n(\mathfrak{B})$ $(n \in \mathbb{N})$,to homomorphism of $Mat_m(\mathfrak{D}^{(\mu)})$ - module $A_{m,n}(\mathfrak{D}^{(\mu)})$ of all $m\times n$ – matrices with all elements in $\mathfrak{D}^{(\mu)}$ in $Mat_m(\mathfrak{B})$ - module $A_{m,n}(\mathfrak{B})$,where m and n are in $\mathbb{N}$,to homomorphism of $\mathfrak{D}^{(\mu)} < v >$ in $\mathfrak{B} < \mu\delta >$,for which $\theta_\mu(D) = \sum_{i=0}^{\infty}(\theta_\mu(\mathfrak{c}_i(D))(\mu\delta)^i \in \mathfrak{B} < \mu\delta >$ $(D = \sum_{i=0}^{\infty} \mathfrak{c}_i(D)v^i \in \mathfrak{D}^{(\mu)} < v >))$. Then θ_μ turns $\mathfrak{B}$ into $\mathfrak{D}^{(\mu)} < v >$ - module,$\mathfrak{r}_n(\mathfrak{B} < \theta_\mu(v) >) \subset \mathfrak{B} < \theta_\mu(v) >$ $(n \in \mathbb{N})$and $\mathfrak{r}_n\theta_\mu = \theta_\mu\mathfrak{r}_n$.

Let $D \in \mathfrak{D}_n^* < v >, (\mathfrak{c}_{n-1}(D))(1;0) \neq 0, \mathfrak{c}_n(D) \in (z-1)\mathfrak{D}, s \in \mathbb{N}-1, k = 0,...,s,$ $\mu \in [0,\varepsilon(D,s)[$. If $f(z) \in \mathbb{Q}[z] \cap Ker(\theta_\mu(D))$,then $(\theta_\mu(\mathfrak{r}_n(\partial D)))((\theta_\mu(v^k))f) \in (z-1)\mathfrak{B}$,

$0 = (\theta_\mu((\mathfrak{c}_0(\partial^k D))(1;\mu))(\theta_\mu(v^{k+0})f)(1) +...+(\theta_\mu((\mathfrak{c}_{n-1}(\partial^k D))(1;\mu))(\theta_\mu(v^{k+n-1})f)(1).$

Let $n > 1, B_k^\vee(D;z;\nu^{-1})$ is the $(n-1) \times (n-1)$- matrix,having,(if $n > 2$) the unit matrix on its first $n-2$ rows and last $n-2$ columns,0 in first $n-2$ rows of its first column and $-(\mathfrak{c}_{j-1}(\partial^k D))(z;\nu^{-1})/(\mathfrak{c}_{n-1}(\partial^k D))(z;\nu^{-1})$ $(j = 1,...,n-1)$ in its last row and the j-th column ,$C_k^\vee(D;z;\nu^{-1}) = \prod_{i=1}^{k} B_{k-i}^\vee(D;z;\nu^{-1})$ $(k \in \mathbb{N}-1)$. According to (8),$B_k^\vee(D;z;0) = B_0^\vee(D;z;0), (B_0^\vee(D;z;0))^k = C_k^\vee(D;z;0)$ $(k \in \mathbb{N}-1)$;if $s \in \mathbb{N}$, $\mu \in [0,\varepsilon(D,s)[, k = 0,...,s$,then $\{B_k^\vee(D;z;\nu^{-1}), C_k^\vee(D;z;\nu^{-1})\} \subset Mat_{n-1}(\mathfrak{D}^{(\mu)})$.

Let $X_{m,T} = X_{m,T}(z;\nu^{-1})$ $(m \in \mathbb{N}, T = T(z) \in \mathbb{C}[z])$ is the column in $(\mathbb{C}[z;\nu^{-1}])^m$, having $v^{i-1}T$ as its i -th $(i = 1,...,m)$ element. If $H = \mathfrak{c}_0(H)v^0 + ... + \mathfrak{c}_s(H)v^s$ is in $\mathfrak{D} < v >$,let $F(H;D;z;\nu^{-1}) = \mathfrak{c}_0(H)_0^\vee(D;z;\nu^{-1}) + ... + \mathfrak{c}_s(H)_s^\vee(D;z;\nu^{-1})$. Then $F(H;D;1;0) = \mathfrak{c}_0(H)(1;0)(B_0^\vee(D;1;0))^0 +...+\mathfrak{c}_s(H)(1;0)(B_0^\vee(D;1;0))^s$,

$\theta_\mu(B_k^\vee(D;z;\nu^{-1})v^k X_{n-1,f})(1) = \theta_\mu(v^{k+1}X_{n-1,f})(1), \theta_\mu(C_k^\vee(D;z;\nu^{-1})X_{n-1,f})(1) = \theta_\mu(v^k X_{n-1,f})(1), (\theta_\mu(H)X_{n-1,f})(1) = (F(H;D;1;\mu)X_{n-1,f})(1)$ $(H \in \mathfrak{D}^{(\mu)} < v >)$,

if $f(z) \in \mathbb{Q}[z] \cap Ker(\theta_\mu(D)), k = 0,...,s$. We connect with $D_i^*(z,\nu^{-1},w), P_i^*(\nu^{-1},w)$ the $< v >$ - polinomials $D_i = b_{i,0}(z,\nu^{-1})v^0 + ... + b_{i,5}(z,\nu^{-1})v^5 + (z-1)v^6$, $P_i = p_{i,0}(\nu^{-1})v^0 + ... + p_{i,3\Delta+3i}(\nu^{-1})v^{3\Delta+3i}$,where $i^2 = i$ Then $\mathfrak{c}_6(D_i) \in (z-1)\mathfrak{D}, 0 \neq$

$(c_5(D_i))(1;0), D_i \in \mathfrak{D}_6^* < v >, P_i \in \mathfrak{D}_{3\Delta+3} < v > \cap \mathfrak{D}^{(\mu)} < v >$ $(\mu \in \mathbb{Q})$,if $i^2 = i$. We

take $6, 3d_2$,for $i = 0, 1$ $f_1^*(z; \nu + i), D_i, P_i$ respectively in the role of n, s, f, D, H.

Then $(\theta_{1/\nu}(P_i)X_5, f_1^*(z; \nu + i))(1) = (F(P_i; D_i; 1; 1/\nu)X_{5, f_1^*(z;\nu+i)})(1)$,if $i^2 = i, \nu \in \mathbb{N}$,

$1/\nu < min\{\varepsilon(D_i, 3d_2)|i = 0, 1\}$. But $(Q[\nu^{-1}]) < v > \ni P_i$ $(i = 0, 1)$ is commutative;

then $P_i X_{5, f_1^*(z;\nu+i)} = X_{5, P_i f_1^*(z;\nu+i)}$ $(i = 0, 1), (F(P_0; D_0; 1; 1/\nu)X_{5, f_1^*(z;\nu)})(1) =$

$(F(P_1; D_1; 1; 1/\nu)X_{5, f_1^*(z;\nu+1)})(1)$,if $\nu \in \mathbb{N}, 1/\nu < \varepsilon(D_i, s)$ $(i^2 = i)$. This admits

(repeating of the main theme) to apply the lemmas 5 and 4,since $w = (\eta d_2 + d_1)/(\eta - 1)$ runs

over all roots of $D^{\check{}}(1, w)$,if η runs over all inequal to 1 roots of $D^{\wedge}(1, \eta)$;then $y(\nu) =$

$f_1^*(1; \nu)$ for some $\varepsilon > 0, q_j^{\wedge}(1, \nu^{-1}) \in \mathfrak{H}(\{1\}, \varepsilon)$ $(j = 0, ..., 4)$ satisfies to the equation

(10) $y(\nu + 5) + q_4^{\wedge}(1, \nu^{-1})y(\nu + 4) + ... + q_0^{\wedge}(1, \nu^{-1})y(\nu + 0) = 0$ $(\nu \in \mathbb{N} + [1/\varepsilon])$

of Poincaré type,having $F_1(\eta) = (\eta - h(\gamma_1)) \prod_{k=0}^{1} \prod_{l=0}^{1} (\eta - h(\eta_k^{\wedge}(1, (2l-1)\pi/3, \delta_0)))$ as its

characteristic polynomial. According to Perron's theorem and lemma 8,since $f_1^*(1; \nu) \in \mathbb{Z}$,

(11) $|h(\eta_0^{\wedge}(1, \pi/3), \delta_0))| = |h(\eta_0^{\wedge}(1, -\pi/3, \delta_0))| > 1 > |h(\eta_1^{\wedge}(1, \pi, \delta_0))| =$

$$|h(\gamma_1)| > |h(\eta_1^{\wedge}(1, \pi/3, \delta_0))| = |h(\eta_1^{\wedge}(1, -\pi/3, \delta_0))| > 0,$$

the solution $y(\nu) = f_1^*(1; \nu)$ corresponds to the root $h(\eta_0^{\wedge}(1, \pi/3, \delta_0))$.

Lemma 12 ([5],pp 51 - 58). *The polynomial* $F_1^{\wedge}(\eta) = \frac{F_1(\eta)}{(\eta - h(\gamma_1))}$ *is irreducible over* $\mathbb{Q}$.

Let $(\mathbb{Q}(\nu))_n < \nabla >$ $(n \in \mathbb{N} - 1)$ is the set of all difference polinomials $D = \sum_{k=0}^{\infty} c_k(D)\nabla^k$,

with $c_k(D) \in \mathbb{Q}(\nu)(k \in \mathbb{N} - 1), c_k(D) = 0$ $(k \in \mathbb{N} + n), \nabla(z) = z$ $(z \in \mathbb{Q}), \nabla(\nu) = \nu + 1$,

$(\mathbb{Q}(\nu)) < \nabla >$ is the join of all $(\mathbb{Q}(\nu))_n < \nabla >$;since $\nabla \circ f = (\nabla f) \circ \nabla \in (\mathbb{Q}(\nu)) < \nabla >$,

where $f \in \mathbb{Q}(\nu)$,the set $(\mathbb{Q}(\nu)) < \nabla >$ is a $\mathbb{Q}$ - subalgebra of the $\mathbb{Q}$ - algebra $\mathcal{M}_{\mathbb{Q}}(\mathbb{Q}(\nu))$

of all $\mathbb{Q}$ - linear operators acting on $\mathfrak{X} = \mathbb{Q}(\nu)$.In [5],pp 60 - 62 is proved the following

Lemma 13. *If* $B \in \mathfrak{X}_m < \nabla >, Q \in \mathfrak{X}_s < \nabla >, \mathfrak{c}_m(B) = \mathfrak{c}_s(Q) = 1, Q \circ B \in \mathfrak{D} < \nabla >$,

then $B \in \mathfrak{D} < \nabla >$. *The ring* $\mathfrak{X} < \nabla >$ *is the ring of main left (respectively right) ideals.*

Let $i^2 = i, D_i \in \mathfrak{X}_5 < \nabla >$with $\mathfrak{c}_5(D_i) = 1, \mathfrak{c}_j(D_i) = q_{i,j}(1, \nu^{-1})$ $(j = 0, ..., 4)$ conform

to (7) and (10) respectively. This equations for sufficient small $\varepsilon > 0$ and $\nu \in [1/\varepsilon] + \mathbb{N}$

have common solution $f_1^*(1; \nu)$. According to lemma 13, D_0 and $D_1$for some $n \in \mathbb{N}$

have common right divisor $D_2 \in \mathfrak{D}_n < \nabla >$with $\mathfrak{c}_n(D_2) = 1$.According to lemma 12

and (11),$\theta_0(\mathfrak{c}_0(D_2))\eta^0 + ... + \theta_n(\mathfrak{c}_j(D_2))\eta^n \neq \eta - h(\gamma_1)$,(first culmination)$F_0(\eta) \in F_1^{\wedge}(\eta)\mathbb{Q}[\eta]$.

(On one string) **Lemma 14.** *If* $z \geq 1$ *then* $\lim_{\nu \to \infty} (f_2^*(z, \nu))^{1/\nu} = h(\eta_1^{\wedge}(z, \pi, \delta_0))$,

$\lim_{\nu \to \infty} f_{2+2k}^*(z, \nu)/f_2^*(z, \nu) = 2((logz)/2)^k (k = 1, 2)$. It is proved in [4],pp,48 - 62.

(Variations on a theme by Poincaré - Perron) **Lemma 15.** *Let* $\{s - 1, n\} \subset \mathbb{N}, i = 0, ..., n, a_i^{\sim} \in \mathbb{C}$,

$\nu \in \mathbb{N}, a_i(\nu) \in \mathbb{C}, a_n(\nu) = 1, a_i(\nu) - a_i^{\sim} = O(1/\nu), V_m$ $(m \in \mathbb{N})$ *is the set of all solutions of*

the equation $a_0(\nu)y(\nu+0)+...+a_n(\nu)y(\nu+n) = 0$ $(\nu \in m+\mathbb{N}), \rho_i$ $(i \in \mathbb{N}, i \leq 1+s)$ *are*

the modules of all roots of the polinomial $a_0^{\sim}z^0 + ... + a_n^{\sim}z^n, \rho_{s+1} = 0 < \rho_j < \rho_i$ $(i < j \leq s)$,

e_i, k_i *are respectively the sum and the maximum of the multiplicities of this roots with*

module ρ_i $(1 \leq i \leq s + 1)$. *Then* V_m *for some* $m \in \mathbb{N}$ *is such direct sum of* $V_{m,i}^{\vee} \subset V_m$,

where i = 1,...,s + 1,that,if $V_{m,j}^{\wedge}$ $(j = 1, ..., s+1)$*is the sum of* $V_{m,i}^{\vee}$ $(i = j, ..., s+1)$

and $V_{m,\theta}^{\wedge} \backslash V_{m,\theta+1}^{\wedge}$ *for some* $\theta \in \{1, ..., s\}$ *includes a solution* $y = y(\nu), \nu \in m+\mathbb{N}$,

$\omega_{n,y}(\nu) = max(|y(\nu)|, ..., |y(\nu+n-1)|)$, *then* $e^{-O(1)(ln(\nu)+\nu^{1-1/k_\theta})}\beta \leq \omega_{n,y}(\nu)(\rho_\theta)^{-\nu} \leq$

$e^{O(1)(ln(\nu)+\nu^{1-1/k_\theta})}\omega_{n,y}(m)$,*where* $\beta = \beta(y) > 0$ *dusn't depend on* $\nu, O(1)$ *dusn't depend*

on ν *and* y;*if* $y \in V_{m,s+1}, \mathfrak{k} = k_{s+1} > 0$,*then* $y(\nu) = (O(1)/\nu)^{\nu/\mathfrak{k}}\omega_{n,y}(m)$,*where* $O(1)$

dusn't depend on ν *and* y;*besides* $dim_{\mathbb{C}}(V_{m,i}) = e_i$ $(i = 1, ..., s + 1)$.It is proved in [6].

Lemma 16 ([2],theorem 3). *Let $m \in \mathbb{N}, y_j(\nu)$ $(j = 1, ..., r; \nu \in \mathbb{N} + m)$ is a basis*

of such subspace V of V_m from lemma 15,that $V \cap V_{m,s+1} = \{0\}$,

$k_3(V) = max(\{k \in [1, s] \cap \mathbb{Z} | V \subset V_{m,k}), k_4(V) = min(\{k \in [1, s] \cap \mathbb{Z} | V \cap V_{m,k+1} = \{0\})$.

Then for each $\varepsilon \in]0, 1[$ there are such $C_3(\varepsilon) > 0, C_4(\varepsilon) > 0$,that, if for all $\nu \in \mathbb{N} + m$

and $X \in \mathbb{C}^r$,having x_i $(i = 1, ..., r)$ as its $i-th$ coordinate,we let

(12) $h(X) = max(|x_1|, ..., |x_r|), y = y^{\vee}(X, \nu) = x_1 y^{\vee}_1(\nu) + ... + x_r y^{\vee}_r(\nu)$,

then $C_3(\rho_{k_4}(1-\varepsilon))^{\nu} h(X) \leq \omega_{n,y}(\nu) \leq C_4(\rho_{k_2}+\varepsilon)^{\nu} h(X)$.

(Second culmination). According to (11),lemma 15 with $s = 3, n = 5$ and lemma 14,$F_0(\eta) =$

$F_1^{\wedge}(\eta)(\eta - \lambda)$ with $|\lambda| = |h(\gamma_1)|$ and $V_{m,3}$ for some $m \in \mathbb{N}$ contains the solutions $y(\nu) =$

$f^*_{2k}(z, \nu)$ $(k = 2, 3)$ of the equation (7). According to [3] and [1],$\mathbb{N}$ contains (love to two

oranges) such $D^{**}(\nu), d^*_1(\nu)$,that $\alpha^*_i(z, \nu)(D^{**}(d_2\nu, 2\Delta\nu))^5(d^*_1(\nu))^{-3} \in \mathbb{Z}[z]$,where $\nu \in \mathbb{N}$,

$ln(D^{**}(\nu)) = (\Delta+1)(1+o(1))\nu$ $(\nu \to +\infty), ln(d^*_1(\nu)) = (1+o(1))\nu\xi$ $(\nu \to +\infty)$,if $\xi =$

$$\sum_{i=0}^{1}(-((\Delta+(-1)^i)/2)log(1+(-1)^i/\Delta)+(-1)^i\pi \sum_{\kappa=1}^{[(\Delta+(-1)^i)/2]}(ctg(\pi\kappa/(\Delta+(-1)^i)))/2).$$

(Fortissimo). **Lemma 17 ([5],pp.64 - 67).** *Let $a_{j,k} \in \mathbb{R}$ $(j = 1, ..., q, k = 1, ..., r), l \in \mathbb{N}$,*

where $\{q, r\} \subset \mathbb{N}, m \in \mathbb{N}, \alpha_j(\nu)$ $(\nu \in \mathbb{N} - 1, j = 1, ..., r + q)$ are maps of $m + \mathbb{N}$ in $\mathbb{Z}$,

there are such $\gamma_0, r_1^{\wedge} \geq 1, ..., r_q^{\wedge} \geq 1$,that $|\alpha_j(\nu)| < \gamma_0(r_j^{\wedge})^{\nu}$ $(\nu \in \mathbb{N} + m, j = 1, ..., q)$,

$y_k^{\wedge}(\nu) = \alpha_1(\nu)a_{1,k} + ... + \alpha_q(\nu)a_{q,k} - \alpha_{q+k}(\nu)$ $(\nu \in \mathbb{N}+m, k = 1, ..., r)$, $||x||$ *denotes*

the distance from $x \in \mathbb{R}$ to $\mathbb{Z}$, x_i $(i = 1, ..., r)$ is $i-th$ coordinate of $X \in \mathbb{C}^r$,

$R_1 \geq R_2 > 1, \gamma_p > 0, (\gamma_p(R_p)^{-\nu}h(X) - |y^{\wedge}(X; \nu+0)| - ... - |y^{\wedge}(X; \nu+l)|)(-1)^p \geq 0$,

where $p = 1, 2$. Let $\alpha_j = (log(r_j R_1/R_2))/log(R_2)$ $(j = 1, ..., q)$. Then there is

such $\gamma_4 > 0$,that $||\varphi_1(X)||(h(X))^{\alpha_1} + ... + ||\varphi_q(X)||(h(X))^{\alpha_q} \geq \gamma_4$ $(X \in \mathbb{Z}^r \backslash \{0\})$. Since

the residue in $t = \infty$ of $(R_0(t, \nu))^3$ equals $0, \alpha^*_3(z, \nu) \in (z-1)\mathbb{Q}[z]$;therefore $0 =$

$\alpha^*_2(z, \nu)L_k(z)|_{z=1}$ $(k \in \mathbb{N})$;*then* (third culmination)$f^*_{2+2k}(1, \nu) = (1+k)\alpha^*_1(1; \nu)\zeta(2+k)+$

$3k\alpha_0^*\ (1;\nu)\zeta(3+k) - \alpha_{3+k}^*(1;\nu)\ (k=1,2,\nu \in \mathbb{N})$. Taking $2, 2, (2ik-i-k+2)\times$

$\zeta(i+k+1)\ (\{i,k\} \subset \{1,2\}), (D^{**}(\nu))^5(d_1^*(\nu))^{-3}(\alpha_j^*(z;\nu)\ (\nu \in \mathbb{N}+m, j = 0,1,4,5)$,

$\phi_i(x_1,x_2) = (i+1)\zeta(i+2)x_1 + 3i\zeta(i+3)x_2\ (i=1,2), ln(|h(\eta_0^\wedge(1,\pi/3,\delta_0))|) + 5\Delta + 5-$

$3\xi+\varepsilon, 3\xi - ln(|h(\eta_1^\wedge(1,\pi/3,\delta_0))|) - 5\Delta - 5 - \varepsilon$ with sufficiently small $\varepsilon > 0$ and $\Delta = 13$

respectively in the role of $q, r, a_{j,k}\ (j=1,...,q, k=1,...,r), \alpha_j(\nu)\ (j=1,...,r+q)$,

$y_k^\wedge(\nu)\ (\nu)\ (\nu \in \mathbb{N}+m, k=1,...,r), ln(r_j)\ (j=1,...,q), ln(R_j)\ (j=1,2)$ of

the lemma 17 and applying the lemmas 8,16,17,after some computations we verify,

that assertion of our theorem is true. The theorem is proved.

Aknowledgenents.

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