A note on geodesics on ellipsoid

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Abstract

The aim of this note is to show that the Jacobi problem of geodesics on ellipsoid [Ja 1839a], [Ja 1839b], [Ja 1866] may be reduced to a more simple case, namely, to the Clebsch problem [Cl 1871]. The last one is the problem with quadratic nonlinearity and may be solved directly by using of Weber's approach [We 1878] in terms of multi-dimensional theta functions.

1. The problem of geodesics on ellipsoid is a classical one. For two-dimensional ellipsoid, its solution was announced by Jacobi at 28 December, 1838 [Ja 1839a] (see Appendix). Using the remarkable substitution, he reduced this problem to quadratures. These results were published in the paper [Ja 1839b] and then considered in details by using of elliptic coordinates in his lectures at Königsberg University in 1841/42. They can be found in the book [Ja 1866].

Later Weierstrass described another approach and succeed to integrate the equations for geodesics explicitely in terms of two-dimensional theta functions [We 1861].

Another solution of this problem was given by Moser ([Mo 1980a], [Mo 1980b]) who included it to the modern scheme of isospectral deformations and described the relation of this problem to the geometry of quadrics and algebraic geometry of spectral curve. For n-dimensional case, the algebraic-geometrical approach was given by Knörrer [Kn 1980].

Here we show that this problem may be considered without using of elliptic coordinates and the algebraic geometry as a projection of a more simple system (for a general description of this method, see [OP 1981], [Pe 1990]). Namely, it may be considerd as the projection of the system with a larger number of degrees of freedom and with the simplest (quadratic) nonlinearity, the so-called Clebsch system [Cl 1871]. The equations of motion for this system were integrated in terms of multi-dimensional theta functions by Weber [We 1878] (for modern exposition, see [Du 1981]).

2. The $(n-1)$-dimensional ellipsoid in n-dimensional Euclidean space $\mathbb{R}^n$ is defined by the equation

$$Q_0(x) = \sum_{j=1}^{n} \frac{x_j^2}{a_j} = 1. \tag{1}$$

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The standard metric in Euclidean space induces the metric on ellipsoid. The free motion of a point particle of unit mass on such an ellipsoid is described by Hamiltonian

$$H = \frac{1}{2} \sum_{j=1}^{n} y_j^2 \,. \tag{2}$$

The equations of motion have the form

$$\dot{x}_j = y_j, \qquad \dot{y}_j = \ddot{x}_j = -\,\nu(x,y)\,\frac{x_j}{a_j}\,, \qquad j = 1, \ldots, n. \tag{3}$$

To obtain $\nu(x,y)$, we shall differentiate the equation (1) twice with respect to time. As a result, we get

$$\sum_{j=1}^{n} \frac{x_j y_j}{a_j} = 0 \tag{4}$$

and

$$\nu(x,y) = \frac{B(y)}{A(x)}, \tag{5}$$

where

$$B(y) = \sum_{j=1}^{n} \frac{y_j^2}{a_j}\,, \qquad A(x) = \sum_{j=1}^{n} \frac{x_j^2}{a_j^2}\,. \tag{6}$$

Theorem. *Equations* (3), (5), (6) *for geodesics on ellipsoid are equivalent to the equations of motion for the Clebsch system,*

$$\begin{aligned} \frac{d}{d\tau}\, y_j &= -\sum_k \omega_{jk}\, y_k, \\ \frac{d}{d\tau}\, l_{jk} &= \sum_m (l_{jm}\,\omega_{mk} - \omega_{jm}\, l_{mk}) - \left(a_j^{-1} - a_k^{-1}\right) y_j\, y_k. \end{aligned} \tag{7}$$

Here we introduced additional variables related to *angular momentum tensor*

$$l_{jk} = x_j y_k - x_k y_j \quad \text{and} \quad \omega_{jk} = \frac{1}{a_j a_k}\, l_{jk} \tag{8}$$

and τ is a *local time* defined by the formula

$$d\tau = A^{-1}\, dt, \qquad \frac{d}{d\tau} = A\,\frac{d}{dt}\,. \tag{9}$$

Proof. After the change of a standard time t by a local time τ, the right-hand side of equations (3) take a polynomial form

$$\frac{d}{d\tau}\,x_j = A(x)\,y_j, \qquad \frac{d}{d\tau}\,y_j = -\,B(y)\,\frac{x_j}{a_j}\,. \tag{10}$$

One can see that there are following identities:

$$\begin{aligned} B(y)\,\frac{x_j}{a_j} &= \sum_k \omega_{jk}\,y_k, \qquad B(y) = \sum_{j<k} \frac{1}{a_j a_k}\,l_{jk}^2, \\ \left(a_j^{-1} - a_k^{-1}\right) B(y)\,x_j x_k &= \sum_n (l_{jm}\omega_{mk} - \omega_{jm}\,l_{mk}) - \left(a_j^{-1} - a_k^{-1}\right) y_j y_k. \end{aligned} \tag{11}$$

From here, it follows equations (7). They are the equations with the simplest, namely, quadratic nonlinearity and they are exactly equations of motion for the Clebsch system [Cl 1871]. Content-Length: 5392 X-Lines:

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3. It is easy to check that equations (7) have n independent integrals of motion,

$$F_j = y_j^2 + \frac{1}{a_j - a_k}\,l_{jk}^2, \qquad j,k = 1,\ldots,n, \tag{12}$$

in involution, $\{F_j, F_k\} = 0$. So, the system under consideration is completely integrable.

The generating function of integrals of motion has the form

$$G_\lambda(y,l) = \sum_{j=1}^n (a_j - \lambda)^{-1} F_j = \sum_{j=1}^n \frac{y_j^2}{a_j - \lambda} - \sum_{j=1}^n \frac{l_{jk}^2}{(a_j - \lambda)(a_k - \lambda)},$$

and we have

$$\{G_\lambda, G_\mu\} = 0.$$

Let us rewrite these equations in the Hamiltonian form. For this, use the Poisson structure related to the Lie algebra $e(n)$ of motion of Euclidean space $\mathbb{R}^n$,

$$\begin{aligned} \{l_{ij}, l_{km}\} &= \delta_{ik}\,l_{jm} - \delta_{im}\,l_{jk} - \delta_{jk} l_{im} + \delta_{jm} l_{ik}, \\ \{l_{ij}, y_k\} &= \delta_{jk}\,y_i - \delta_{ik}\,y_j, \qquad \{y_j, y_k\} = 0. \end{aligned} \tag{13}$$

So, the quantities y_j, l_{km} generate the Lie algebra $e(n)$ of motion in n-dimensional Euclidean space.

Hamiltonian H should be the function of F_j and in fact, due to the quadratic form of equations of motion, it should be a linear combination of F_j,

$$H = \frac{1}{2}\, b_j F_j. \tag{14}$$

The simple calculations show that $b_j = a_j^{-1}$. Hence

$$\begin{aligned} \dot{y}_j &= \{H, y_j\}, \qquad \dot{l}_{jk} = \{H, l_{jk}\}, \\ H &= \frac{1}{2}\left(\sum_j \frac{y_j^2}{a_j} - \sum_{i,j} \frac{l_{ij}^2}{a_i a_j}\right) = G_0(y,l). \end{aligned} \tag{15}$$

Eqs. (15) are the Hamiltonian equations related to the Poisson structure for the Lie algebra $e(n)$.

For two-dimensional case, these equations were integrated explicitly by Weber [We 1878] in terms of theta functions of two variables. For multi-dimensional case, see [Kn 1980] and [Du 1981].

Note that as it was shown by Joachimsthal [Jo 1843], the quantity

$$I(x,y) = A(x)\, B(y)$$

is an integral of motion, $I(x,y) = c^2$.

Now we may obtain the expressions for the quantities x_j and $A(x)$:

$$x_j(\tau) = -\, B^{-1}(y)\, a_j \,\frac{d}{d\tau}\, y_j, \qquad A(x) = \sum_j \frac{x_j^2}{a_j^2} = c^2\, B^{-1}(y). \tag{16}$$

Finally, the expression for t as the function of τ is given by a quadrature

$$t(\tau) = \int_0^\tau A(\tau)\, d\tau. \tag{17}$$

So, we get the formulae for all dynamical variables.

Appendix

Lettre de M. Jacobi à M. Arago, concernant les lignes géodésiques tracées sur un ellipsoide à trois axes [Ja 1839a].

Königsberg, le 28 décembre 1838

Monsieur,
Je suis parvenu à ramener aux quadratures la ligne géodésique sur un ellipsoide à trois axes inégaux. Ce sont des formules extrémement simples, des intégrales abéliennes qui se changent dans les intégrales elliptiques connues, en égalant entre eux deux de ces trois axes. Ce probleme m'ayant paru long-temps très difficile, je crois que sa solution pourra intéresser peut-ètre quelques-uns des illustres membres de l'Académie des Sciences.

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References

[Cl 1871] Clebsch A., *Über die Bewegung eines Körpers in einer Flüssigkeit*, Math. Ann. **3**, 238–262

[Du 1981] Dubrovin B.A., *Theta functions and nonlinear equations*, Russ. Math. Surv. **36**, 11–92

[Ja 1839a] Jacobi C., *Lettre de M. Jacobi à M. Arago concernant les lignes géodésiques tracées sur un ellipsoide à trois axes*, Compt. Rend. **8**, 284

[Ja 1839b] Jacobi C., *Note von der geodätischen Linie auf einem Ellipsoid und den verschidenen Anwendungen*, JRAM **19**, 309–313

[Ja 1866] Jacobi C., in: *Vorlesungen über Dynamik* which Jacobi gave at 1842/43 at Königsberg University, Berlin: Reimer

[Jo 1843] Joachimsthal F., *Observationes de lineis brevissimis et curvis curvaturae in superficiebus secundi gradus*, JRAM **26**, 155–171

[Kn 1980] Knörrer H., *Geodesics on the ellipsoid*, Invent. Math. **59**, 119–143

[Mo 1980a] Moser Ju., *Various aspects of integrable Hamiltonian systems*, in: *Progress in Mathematics* **8**, *Dynamical Systems*, Birkhäuser, 233–287

[Mo 1980b] Moser Ju., *Geometry of quadrics and spectral theory* in: *The Chern Symposium* 1979, Springer–Verlag, 147–188

[OP 1981] Olshanetsky M.A. and Perelomov A.M., *Classical integrable finite-dimensional systems related to Lie algebras*, Phys. Reps. **71**, No.5, 313–400

[Pe 1990] Perelomov A.M., *Integrable Sytems of Classical Mechanics and Lie Algebras.* **I**, Basel: Birkhäuser

[We 1861] Weierstrass K., *Über die geodätischen Linien auf dem dreiachsigen Ellipsoid*, Monatsber. Königl. Akad. Wiss., 257–273

[We 1878] Weber H., *Anwendung der Thetafunctionen zweier Veründlicher und die Theorie der Bewegung eines festen Körpers in einer Flussigkeit*, Math. Ann. **14**, 173–206