Smoothness in Relative Geometry

Florian Marty
fmarty@math.ups-tlse.fr

Université Toulouse III - Laboratoire Emile Picard

Abstract

In [TVa], Bertrand Toën and Michel Vaquié defined a scheme theory for a closed monoidal category $(\mathcal{C}, \otimes, 1)$. In this article, we define a notion of smoothness in this relative (and not necessarily additive) context which generalizes the notion of smoothness in the category of rings. This generalisation consists practically in changing homological finiteness conditions into homotopical ones using Dold-Kan correspondence. To do this, we provide the category $s\mathcal{C}$ of simplicial objects in a monoidal category and all the categories $sA - mod$, $sA - alg$ ($A \in sComm(\mathcal{C})$) with compatible models structures using the work of Rezk in [R]. We give then a general notions of smoothness in $sComm(\mathcal{C})$. We prove that this notion is a generalisation of the notion of smooth morphism in the category of rings, is stable under compositions and homotopic pushouts and we provide some examples of smooth morphisms, in particular in $\mathbb{N} - alg$ and $Comm(Set)$.

Contents

Introduction

In [TVa], Bertrand Toën and Michel Vaquié defined a scheme theory for a closed monoidal category $(\mathcal{C}, \otimes, 1)$. In this article, we define a notion of smoothness in this relative context which generalizes the notion of smoothness in the category of rings. The motivations for this work are that interesting objects in the non additive contexts $\mathcal{C} = Ens$ or $\mathbb{N} - mod$ are expected not to be schemes but Stacks. The first step is to get a definition for smooth morphism.
The following theorem gives the good definition of smoothness that can be generalised to a relative context:

THEOREM 1. Assume $\mathcal{C} = \mathbb{Z} - mod$. A morphism of rings $A \to B$ is smooth if and only if

i. The ring B is finitely presented in $A - alg$.

ii. The morphism $A \to B$ is flat.

iii. The ring B is a perfect complex of $B \otimes_A B$-modules.

The flatness of $A \to B$ is in fact equivalent to $Tordim_A B = 0$ hence the two last conditions are homological finiteness conditions. By the correspondence of Dold-Kan, the second condition can then be translated in an homotopical condition. Finally, a result from [TV] asserts that B is a perfect complex in $ch(B \otimes_A B)$ if and only if it is homotopically finitely presented in $sB \otimes_A B - mod$.
We provide then the category $s\mathcal{C}$ of simplicial objects in a monoidal category and all the categories $sA - mod$, $sA - alg$ with models structures using the work of Rezk in [R]. The classical functors between the categories $A - mod, A - alg,\ A \in sComm(\mathcal{C})$ induce Quillen functors between the corresponding simplicial categories. We give the following general definition for smoothness

Definition 2. A morphism $A \to B$ is smooth if and only if

i. The simplical algebra B is homotopically finitely presented in $sA - alg$.

ii. The simplicial algebra B has Tor dimension 0 on A.

iii. The morphism $B \otimes^h_A B \to B$ is homotopically finitely presented in $sB \otimes^h_A B - mod$.

The first condition implies the first condition of 1 ([TV], 2.2.2.4) and there is equivalence for smooth morphisms of rings. When A, B are rings, the Tor dimension 0 imply that the derived tensor product is weakly equivalent to the tensor product. The equivalence with previous theorem for rings is then clear.

We prove that relative smooth morphisms are stable under composition and pushouts of Algebras. We finally provide examples of smooth morphism in relative non-additive contexts , for $\mathcal{C} = \mathbb{N} - mod$ or $\mathcal{C} = Set$. In particular the affine line $\mathbb{F}_1 \to \mathbb{N}$ and the scheme $G_{m,F_1} \simeq Spec(\mathbb{Z})$ are smooth in $sComm(Set)$. Similarly, the affine line $\mathbb{N} \to \mathbb{N}[X]$ and the scheme $G_{m,\mathbb{N}}$ are smooth in $s\mathbb{N} - mod$. We conclude by proving that a cofibration with cofibrant source and goal which is a Zariski open immersion is smooth.

Preliminaries

Let $(\mathcal{C}, \otimes, 1)$ be a complete and cocomplete closed symmetric monoidal category. In the category $\mathcal{C}$, there exists a notion of commutative monoid and for a given commutative monoid A, of A-module. Let $Comm(\mathcal{C})$ denotes the category of commutative monoids (with unity) in $\mathcal{C}$. For $A \in Comm(\mathcal{C})$, $A - mod$ denotes the category of A-modules. It is well known that the category $A - mod$ is a closed monoidal tensored and cotensored category, complete and cocomplete. The category $Comm(A - mod)$ will be denoted by $A - alg$ and is described by the equivalence $A/Comm(\mathcal{C}) \simeq A - alg$. A pushout in $A - alg$ is a tensor product in the sense that for commutative monoids $B, C \in A - alg$ $B \otimes_A C \simeq B \coprod_A C$.

All along this work, $(\mathcal{C}, \otimes, 1)$ is a locally finitely presentable monoidal category i.e. verifies that the full subcategory of finitely presented objects, denoted $\mathcal{C}_0$, is essentially small, that the (Yoneda) functor $i : \mathcal{C} \to Pr(\mathcal{C}_0)$, is fully faithful, that $\mathcal{C}_0$ is stable under tensor product and contains the unity 1. The functor $Hom_{\mathcal{C}}(1, -)$, denoted $(-)_0 : X \to X_0$ is called "underlying set functor". For $k \in \mathcal{C}_0$, the functors $Hom_{\mathcal{C}}(k, -)$, denoted $(-)_k : X \to X_k$ are called "weak underlying set functor". It is known that if $\mathcal{C}$ is a locally finitely presentable monoidal category, so are its categories of modules $A - mod$, $A \in sComm(\mathcal{C})$

There are two fundamental adjunctions:

$$\mathcal{C} \underset{i}{\overset{(-\otimes A)}{\rightleftarrows}} A - mod \qquad\qquad \mathcal{C} \underset{i}{\overset{L}{\rightleftarrows}} Comm(\mathcal{C})$$

where the forgetful functor i is a right adjoint and the functor "free associated monoid" L is defined by $L(X) := \coprod_{n \in \mathbb{N}} X^{\otimes n}/S_n$. In these adjunctions, $\mathcal{C}$ can be replaced by $B - mod$ for $B \in Comm(\mathcal{C})$, and L by L_B defined by $L_B(M) := \coprod_{n \in \mathbb{N}} M^{\otimes_B n}/S_n$. Let φ (resp φ_B) and ψ (resp ψ_B) denote these adjunctions for the category $\mathcal{C}$ (resp $B - mod$). For $X \in \mathcal{C}$ and $M \in A - mod$, $\varphi : Hom_{\mathcal{C}}(X, M) \to Hom_{A-mod}(X \otimes A, M)$ is easy to describe :

$$\varphi : f \to \mu_M \circ Id_A \otimes f$$
$$\varphi^{-1} : g \to g \circ (Id_X \otimes i_A) \circ r_X^{-1}$$

Let $s\mathcal{C}$ denotes the category of simplicial objects in $\mathcal{C}$. There is a functor "constant simplicial object" denoted k from $\mathcal{C}$ to $s\mathcal{C}$ which is right adjoint to the functor π_0 from $s\mathcal{C}$ to $\mathcal{C}$ defined by $\pi_0(X) := Colim(\ X[1] \rightrightarrows X[0]\ )$. The tensor product of $\mathcal{C}$ induces a tensor product on $s\mathcal{C}$, its unity element is $k(1)$. For A in $Comm(\mathcal{C})$, $sA-mod$ and $sA-alg$ will denote respectively the simplicial categories $sk(A)-mod$ and $sk(A)-alg$. As $sComm(\mathcal{C}) \simeq Comm(s\mathcal{C})$, we will always refer to simplicial category of commutative monoids in $s\mathcal{C}$ as $sComm(\mathcal{C})$. The functor induced by L on simplical categories will be denoted sL. The functor $i : \mathcal{C} \to Pr(\mathcal{C}_0)$ induces a functor $si : s\mathcal{C} \to sPr(s\mathcal{C}_0)$.

We need finally hypotheses to endow $s\mathcal{C}$, $sComm(\mathcal{C})$, and for $A \in sComm(\mathcal{C})$, $sA-mod$ and $sA-alg$ with compatible model structures. Let J denotes the family of isomorphism classes of the objects of $\mathcal{C}_0$. As $\mathcal{C}_0$ is essentially small, it is a set. One solution of this question is to assume that the natural functors from $s\mathcal{C}$ and $sComm(\mathcal{C})$ to $sSet^J$ are monadic. The characterisation of monadic functors of [Bc] implies that for any commutative simplicial monoid A, the induced functors from $sA-mod$ to $sSet^J$ is also monadic.

1 General Theory

1.1 Simplicial Categories and Simplicial Theories

Definition 1.1. A simplicial theory is a monad (on $sSet^J$) commuting with filtered colimits.

THEOREM 1.2. *(Rezk)*
Let T be a simplicial theory in $sSet^J$, then $T-alg$ admits a simplicial model structure. f is a Weak equivalence or a fibration in $T-alg$ if and only if so is its image in $sSet^J$ (for the projective model structure). Moreover, this Model category is right proper.

Proposition 1.3. *Model structures on the simplicial categories.*

i. Let $A = (A_p)$ be a commutative monoid in $s\mathcal{C}$. The monadic adjunctions $A_p - mod \rightleftarrows Set^J$ induce a monadic adjunction $sA - mod \rightleftarrows sSet^J$ i.e. there is an equivalence $sA - mod \simeq T_A - alg =$ where T_A is the monad induced by adjunction. In particular $s\mathcal{C} \simeq T_1 - alg$.

ii. Let $A = (A_p)$ be a commutative monoid in $s\mathcal{C}$. The monadic adjunctions $A_p - alg \rightleftarrows Ens^J$ induce a monadic adjunction $sA - alg \rightleftarrows sSet^J$ i.e. there is an equivalence $sA - alg \simeq T^c_A - alg$ where T^c_A is the monad induced by adjunction. In particular $sComm(\mathcal{C}) \simeq T^c_1 - alg$.

Proof
As explained in the preliminaries, this is due to the characterisation of monadic functors ([Bc]).

♦

Remark 1.4. The right adjoints functors all commute with filtered colimits. So do the monads which are then simplicial theories on $sSet^J$.

Corollary 1.5. *Let A be a commutative monoid in $s\mathcal{C}$. The categories $s\mathcal{C}$ and $sA - mod$ and $sComm(\mathcal{C})$ are models categories. Moreover, the functors $(A \otimes -)$ and sL are left Quillen and their adjoints preserve by construction weak equivalences and fibrations.*

THEOREM 1.6. *The Category $s\mathcal{C}$ (resp $sA - mod$) is a monoidal model category*

Proof:
The proof for $s\mathcal{C}$ and $sA - mod$ are similar, so let us prove it for $s\mathcal{C}$. Let I, I' be respectively the sets of generating cofibration and generating trivial cofibration. I and I' are the image by the left adjoint functor respectively of generating cofibration and generating trivial cofibration in $sSet^J$. We juste have to prove (cf [H] chap IV) that $I\square I$ is a set of cofibrations and $I\square I'$ and $I'\square I$ are sets of trivial cofibrations. It is true for generating cofibration and generating trivial cofibration in $sSet^J$, which are all morphisms concentrated in one level. Moreover, it is easy to verify that the functor sK commutes with $\square$ of morphisms concentrated in one level. So it is true in $s\mathcal{C}$. The second axiom is clearly verified, as 1 is cofibrant.

♦

1.2 Compactly Generated Model categories

Definition 1.7. Let $\mathcal{M}$ be a cofibrantly generated simplicial model category and I be the set of generating cofibrations.

i. An object $X \in I - cell$ is strictly finite if and only if there exists a finite sequence

$$\emptyset = X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_n = X$$

and $\forall\ i$ a pushout diagram:

$$\begin{array}{ccc} X_i & \longrightarrow & X_{i+1} \\ \big\downarrow & & \big\downarrow \\ A & \xrightarrow[u_i]{} & B \end{array}$$

with $u_i \in I$.

ii. An object $X \in I - cell$ is finite if and only if it is weakly equivalent to a strictly finite object.

iii. An object X is homotopically finitely presented if and only if for any filtered diagram Y_i, the morphism :

$$Hocolim_i Map(X, Y_i) \to Map(X, Hocolim_i Y_i)$$

is an isomorphism in $Ho(sSet)$

iv. A model category $\mathcal{M}$ is compactly generated if it is cellular, cofibrantly generated and if the domains and codomains of generating cofibration and generating trivial cofibration are cofibrant, ω-compact and ω-small (Relative to $\mathcal{M}$).

Proposition 1.8. *([TV])*
Let $\mathcal{M}$ be a compactly generated model category.

i. For any filtered diagram X_i, the natural morphism $Hocolim_i X_i \to Colim_i X_i$ is an isomorphism in $Ho(\mathcal{M})$.

ii. Assume that filtered colimits are exact in $\mathcal{M}$. Then homotopically finitely presented objects in $\mathcal{M}$ are exactly objects equivalent to weak retracts of strictly finite $I - cell$ objects.

Proposition 1.9. *i. The simplicial model category $sSet^J$ is compactly generated.*

ii. The categories of simplicial algebras over a simplicial theory are compactly generated.

Lemma 1.10. *Let A be in $sComm(\mathcal{C})$. Let u^j be the family of images by the left adjoint functor in $sA - mod$ (resp $sA - alg$) of elements $*_j$ of $sSet^J$ defined by $*$ on level j and $\emptyset$ on other levels. Any codomains of a generating cofibrations of $sA - mod$ (resp $sA - alg$) is weakly equivalent to an object u^j. Any domain of a generating cofibration is weakly equivalent, for a given element j in J, to an object obtained from the initial object (denoted $\emptyset$) and u^j in a finite number of homotopic pushouts.*

Proof:
Generating cofibrations of $sA - mod$ are images of generating cofibrations of $sSet^J$ by the left adjoint functor. Generating cofibrations of $sSet$ are morphisms $\delta\Delta^p \to \Delta^p$. Their codomain is contractible, thus so are the codomains of generating cofibrations of $sSet^J$ for the projective model structure, and their image by the left adjoint is weakly equivalent to the unity 1. For the domains, consider the relation $\delta\Delta^{p+1} \backsimeq \Delta^{p+1} \coprod^h_{\delta\Delta^p} \Delta^{p+1} \backsimeq * \coprod^h_{\delta\Delta^p} *$ and $\delta\Delta^0 = \emptyset$. Domains of generating cofibration in $sSet^J$ for the projective model structure are objects $(\delta\Delta^{p,j})_{p\in N,\ j\in V}$ defined in level $i \neq j$ by $\emptyset$ and in level j by $\delta\Delta^p$ and verifying the relation

$$(\delta\Delta^{p,j}) \backsimeq *_j \coprod^h_{(\delta\Delta^{p-1,j})} *_j$$

Clearly $\delta\Delta^{0,j} = \emptyset$ and $\delta\Delta^{1,j} = *_j$. Let $u^{p,j}$ denote the image of $\delta\Delta^{p,j}$. For all j, $u^{p,j}$ is obtained in a finite number of pushouts from $\emptyset$ and u^j.

♦

Corollary 1.11. *of proposition 1.9 and lemma 1.10.*

i. The Simplicial Model categories $s\mathcal{C}$, $sA-mod$ ($A \in sComm(\mathcal{C})$), $sComm(\mathcal{C})$ and $sA-alg$ ($a \in sComm(\mathcal{C})$) are compactly generated.

ii. Homotopically finitely presented objects of $sA-mod$ ($respsA-alg$) are exactly objects weakly equivalent to weak retracts of strictly finite $I-Cell$ objects.

iii. The sub-category of $Ho(sA-mod)$ (resp $Ho(sA-alg)$) $Ho(sA-mod)_c$ (resp $Ho(sA-alg)_c$) Consisting of homotopically finitely presented objects is the smallest full sub-category of $Ho(sA-mod)$ (resp $Ho(sA-alg)$) containing the family $(u^j)_{j\in J}$ (resp $(u^j_A := L_A(u^j))_{j\in J}$), and stable under retracts and homotopic pushouts.

Proof of *iii*:

Let $\mathcal{D}$ be the smallest full sub-category of $Ho(s\mathcal{C})$ containing $(u^j)_{j\in J}$ ($resp$ $(u^j_A)_{j\in J}$), the initial object $\emptyset$ and stable under retracts and homotopic pushouts. Clearly, by *ii*, as $(\emptyset \to u^j)_{j\in J}$ are generating cofibrations of $s\mathcal{C}$, $Ho(s\mathcal{C})_c$ contains the family $(u^j)_{j\in J}$, and is stable under retracts and homotopic pushouts. Thus $\mathcal{D} \subset Ho(s\mathcal{C})_c$. Reciprocally, let X be an object of $Ho(s\mathcal{C})_c$, by *ii*, X is isomorphic to a weak retract of a strictly finite $I-cell$ object. Therefore, there exists n and $X_0 ... X_n$ such that:

$$\emptyset = X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_n = X$$

and $\forall k \in \{0, .., n-1\}$, $\exists K \to L$, a generating cofibration such that:

$$\begin{array}{ccc} X_k & \longrightarrow & X_{k+1} \\ \uparrow & & \uparrow \\ K & \longrightarrow & L \end{array}$$

is a pushout diagram. Now, as domains and codomains of generating cofibrations are in $\mathcal{D}$, X is in $\mathcal{D}$.

♦

1.3 Categories of Modules and Algebras

Proposition 1.12. *Let A be in $sComm(\mathcal{C})$ and B be a simplicial monoid in $sA-alg$, cofibrant in $sA-mod$.*

i. The forgetful functor from $sB-mod$ to $sA-mod$ preserves cofibrations

ii. The forgetful functor from $sA-alg$ to $sA-mod$ preserves cofibrations whose domain is cofibrant in $sA-mod$. In particular, it preserves cofibrant objects.

Proof

In each case, we just have to prove it for generating cofibrations and then generalize it to any cofibration by the small object argument.

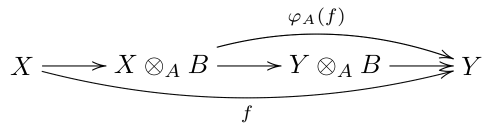
Proof of *i*: First, we choose a generating cofibration in $sB-mod$. As generating cofibration of $sB-mod$ are images of generating cofibrations of $sSet^J$, we juste have to set $L \to M$, a generating cofibration in $sSet^J$. Let K_A, K_B denotes respectively the left adjoint functors (from $sSet^J$) for $sA-mod$ and $sB-mod$. The axiom of stability under $\square$ implies that the morphism $(\emptyset \to B)\square(K_A(L) \to K_A(M))$ is a cofibration in $sA-mod$. This morphism is in fact $K_B(L) = B \otimes_A K_A(L) \to B \otimes_A K_A(M) = K_B(M)$, hence generating cofibrations of $sB-mod$ are cofibrations in $sA-mod$.

Proof of *ii*: As for *i*, let $N \to M$ be a generating cofibration in $sSet^J$. Let L^s denotes the functor " free associated commutative monoid" of $sSet^J$. The functors L (resp sL_A in $sA-mod$) and K_A are defined by colimits and so commute up to isomorphisms. That means that $K_A \circ L^s \backsimeq sL_A \circ K_A$. So the generating cofibration of $sA-alg$ corresponding to $N \to M$ is isomorphic to $K_A(L(N)) \to K_A(L(M))$. To prove that it is a cofibration in $sA-mod$, we have then to prove that the morphism $L(N) \to L(M)$ is injective levelwise and this is clear as any morphism $N^{\otimes n}/S_n \to M^{\otimes n}/S_n$ is injective. Thus any generating cofibration of $sA-alg$ is a cofibration in $sA-mod$. In fact it is a generating cofibration of $sA-mod$. To use the small object argument (of $sA-alg$), we need to verify that it preserves cofibrations in $sA-mod$. In fact, we need to check that an homotopic pushout in $sA-alg$ of a cofibration in $sA-mod$ is still a cofibration in $sA-mod$. We let the reader verify that it is a consequence of the axiom of stability by $\square$. Finally, the forgetful functor preserves cofibrations and, as A is cofibrant in $sA-mod$, any cofibrant object of $sA-alg$ is also cofibrant in $sA-mod$.

♦

Lemma 1.13. *Let $A \to B \in sComm(\mathcal{C})$ be a trivial cofibration between cofibrant objects. The categories of modules are equivalent i.e. $Ho(sA-mod) \backsimeq Ho(sB-mod)$.*

Proof:
We must prove that for X cofibrant in $sA-mod$ and Y fibrant in $sB-mod$, $\varphi_A(f) : X \otimes_A B \to Y$ is a weak equivalence in $sB-mod$ if and only if so is $f : X \to Y$ in $sA-mod$. By previous lemma, $A \to B$ is a trivial cofibration in $sA-mod$. Thus as X is cofibrant, using the axiom of stability under $\square$, $g : X \to B \otimes_A X$ is a weak equivalence in $sA-mod$. By construction of the adjunction φ_A, the following diagram is commutative :

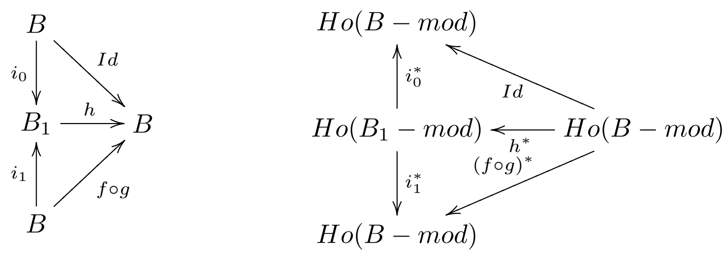


Thus $f = g \circ \varphi_A(f)$. Finally, $\varphi_A(f)$ is a weak equivalence in $sA-mod$ if and only if so is it in $sB-mod$ and the *two out of three* axiom ends the proof.

♦

Proposition 1.14. *Let $f : A \to B \in sComm(\mathcal{C})$ be a weak equivalence between cofibrant objects. The categories of modules are equivalent i.e. $Ho(sA-mod) \backsimeq Ho(sB-mod)$.*

Let r_c be the fibrant replacement of $sComm(\mathcal{C})$, then by previous lemma, the homotopical categories of modules over A and r_cA (resp B and r_cB) are equivalent. Thus A and B can be taken fibrant and f is an homotopy equivalence i.e. $\exists$ g such that $f \circ g$ and $g \circ f$ are homotopic to identity. The following diagrams are commutative:



where i_0 and i_1 are cofibrations and have the same right inverse p i.e. such that $p \circ i_1 = p \circ i_0 = Id_B$. the morphism h is a trivial fibration thus i_0 is a weak equivalence. By previous lemma, i_0^* is an equivalence of categories. Thus so is p^*. As i_1^* and i_0^* are both inverses of p^*, they are isomorphic and i_1^* is also an equivalence. Finally, h^* is an equivalence and so is $(f \circ g)^*$. The same method prove that $(g \circ f)^*$ is an equivalence.

♦

1.4 Finiteness Conditions

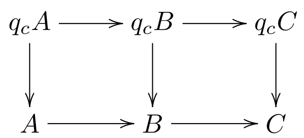
Definition 1.15. Let q_c be a cofibrant replacement in $sComm(\mathcal{C})$ and $f : A \to B$ be a morphism in $sComm(\mathcal{C})$.

- ▷ The morphism f is homotopically finite (denoted hf) if B is homotopically finitely presented in sq_cA-mod.
- ▷ The morphism f is homotopically finitely presented (denoted hfp) if B is homotopically finitely presented in sq_cA-alg.

Remark 1.16. The morphism $A \to B$ is hf (resp hfp) if and only if the morphism $q_cA \to q_cB$ is hf (resp hfp). The morphism $q_cB \to B$ is always hf.

Lemma 1.17. *The hf (resp hfp) morphisms are stable under composition.*

Proof
The proofs for hf morphisms and hfp morphisms are analogous so let us prove it for hf morphisms. Let $A \to B \to C$ be the composition of two hf morphisms. There is a diagram



and forgetful functors $F_1 : sq_cC - mod \to sq_cB - mod$ and $F_2 : sq_cB - mod \to sq_cA - mod$. The image $F_1(q_cC)$ of q_cC is homotopically finitely presented in $sq_cB - mod$ hence weakly equivalent to a retract of a finite homotopical colimit of q_cB in $Ho(sq_cB - mod)$. The forgetful functor F_2 preserves retracts, equivalences, finite colimits, cofibrant objects and cofibrations whose domain is cofibrant. Thus it also preserves finite homotopical colimit and sends q_cC to a retract of a finite homotopical colimit of q_cB in $Ho(sq_cA - mod)$. As q_cB is homotopically finitely presented in $sq_cA - mod$, and as homotopically finitely presented objects are stable under retracts, equivalences and finite homotopical colimit, C is sent by $F_2 \circ F_1$ in $sq_cA - mod_c$. Hence $A \to C$ is finite.

♦

Lemma 1.18. *The hf (resp hfp) morphims are stable under homotopic pushout of simplicial monoids.*

Proof:
The proofs for hf morphisms and hfp morphisms are analogous so let us prove it for hf morphisms. Let $A \to B$ and $A \to C$ be in $sComm(\mathcal{C})$ such that the first is finite. Let q_{cA} be the cofibrant replacement of $q_cA - alg$, it is weakly equivalent to q_c and the object $q_{c_A}B$ is homotopically finitely presented in $sq_cA - mod$. Let us prove that $B \otimes^h_A C \simeq q_{cA}B \otimes_{q_cA} q_cC$ (in $Ho(q_cA - mod)$, Reedy lemma) is homotopically finitely presented in $q_cC - mod$. The forgetful functor $sq_cC - mod \to sq_cA - mod$ preserves filtered colimits and weak equivalences hence it preserves homotopical filtered colimits. Thus the derived functor $- \otimes_{q_cA} q_cC$ preserves homotopically finitely presented objects. So $B \otimes^h_A C$ is homotopically finitely presented in $Ho(q_cC)$.

♦

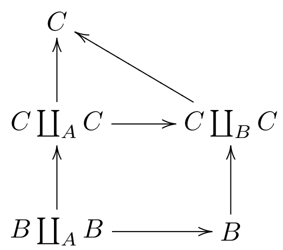
1.5 A Definition for Smoothness

Definition 1.19. A morphism $A \to B$ in $sComm(\mathcal{C})$ is formally smooth if the morphism $B \otimes^h_A B \to B$ is hf.

Remark 1.20. This definition does not generalise the definition of formal smoothness in the sense of rings. However, the corresponding notion of smoothness is a generalisation of the classical notion of smoothness, as it will be proved in this article.

Proposition 1.21. *Formally smooth morphisms are stable under composition.*

Proof:
By previous remarks, it can be assumed that A is cofibrant in $sComm(\mathcal{C})$, B is cofibrant in $sA - alg$ and C is cofibrant in $sB - alg$. Let $A \to B \to C$ be the composition of two formally smooth morphisms . The morphisms $B \coprod_A B \to B$ and $C \coprod_B C \to C$ are hf. The following diagram commutes and is clearly cocartesian :



Thus, if it is cofibrant for the Reedy stucture, it will be homotopically cocartesian. The morphisms $B \simeq B \coprod_A A \to B \coprod_A B$ and $B \coprod_A B \to C \coprod_A C$ are images by the left Quillen functor *colim*, of clear Reedy cofibrations (see [A] for a descriptions of these cofibrations), thus are cofibrations. In particular $B \coprod_A B$ and $C \coprod_A C$ are cofibrant and the diagram considered is Reedy cofibrant. Finally, the morphism $C \coprod_A C \to C \coprod_B C$ is hf as a pushout of hf morphisms and $C \coprod_A C \to C$ is hf as a composition of hf morphisms.

♦

Proposition 1.22. *Formally smooth morphism are stable under homotopic pushout.*

Proof:
Let $u : A \to B$ be a formally smooth morphism and C be a commutative A-algebra. By previous remarks it can be assumed that A and c are cofibrants in $sComm(\mathcal{C})$ and that B is cofibrant in $sA - alg$. Let D denote the homotopic pushout of $B \otimes_A C$ and u' denote the morphism from B to D. Clearly:

$$D \otimes_C D \simeq B \otimes_A C \otimes_C B \otimes_A C \simeq B \otimes_A D$$

Thus the following diagram commutes :

$$\begin{array}{ccccc} & & B \otimes_A B & \xrightarrow{m_B} & B \\ & \swarrow & \downarrow{\scriptstyle Id \otimes_A f} & & \downarrow{\scriptstyle f} \\ D \otimes_C D & \longrightarrow & B \otimes_A D & \xrightarrow[m_D]{} & D \end{array}$$

And is cocartesian :

$$B \otimes_{B \otimes_A B} B \otimes_A B \otimes_A C \simeq B \otimes_A C \simeq D$$

Moreover it is clearly cofibrant as $B \otimes_A -$ preserve cofibrations. Finally by stability of hf morphism under homotopic pushout, the morphism $C \to D$ is formally smooth.

♦

Definition 1.23. Let A be in $sComm(\mathcal{C})$ and M be in $sA - mod$.

i. The object M is n-truncated if $Map_{s\mathcal{C}}(X, M)$ is n-truncated in $sSet$, $\forall\ X \in s\mathcal{C}$.

ii. The Tor-Dimension of M in $sA - mod$ is defined by

$$Tordim_A(M) = inf\{n\ st\ M \otimes^h_A X\ is\ n + p - truncated\ \forall\ X \in\ sA - mod\ p - truncated\}$$

iii. A morphism of monoids $A \to B$ has *Tor dimension* n if $Tordim_A(B) = n$.

Lemma 1.24. *Tor dimension zero morphisms are stable under composition and homotopic pushout.*

Proof:
Let $A \to B \to C$ be the composition of two *Tor dimension zero* morphisms. Let M be a p truncated A-module,

$$M \otimes_A C \simeq M \otimes_A B \otimes_B C.$$

As $Tordim_A(B) = 0$, $M \otimes_A B$ is a p truncated B-module. As $Tordim_B(C) = 0$, $(M \otimes_A B) \otimes_B C$ is a p truncated C-module. Thus $Tordim_A(C) = 0$.

Let $A \to B$ be a *Tor dimension zero* morphism and $A \to C$ be a morphism in $Comm(\mathcal{C})$. Let M be in C-mod and let D denote the pushout $B \otimes_A C$. We have

$$M \otimes_C D \simeq M \otimes_A B.$$

Thus, $Tordim_A(B) = 0$ implies $Tordim_C(D) = 0$.

♦

Definition 1.25. A morphism $A \to B$ in $Comm(\mathcal{C})$ is smooth if it is formally smooth, hfp and has Tor-Dimension zero. A morphism of affine scheme is smooth if the corresponding morphism of monoids is smooth. We say that an affine scheme X is smooth if the morphism $X \to Spec(1)$ is smooth.

Theorem 1.26. *Smooth morphisms are stable under composition and homotopic pushout.*

Proof:
This a a corollary of 1.24, 1.21, 1.22, 1.17 and 1.18.

♦

2 Simplicial Presheaves Cohomology

In the article [T1], B. Toën defines a cohomology for a connected and pointed simplicial presheaf. We will define here a cohomology for a general simplicial presheaf. This theory will be used to find examples of smooth morphisms of commutative monoids (in sets). The references cited in this section are [T1], [GJ] and [J].

2.1 Definitions

In this section, $\mathcal{D}$ is a category and $sPr(\mathcal{D})$ is the category of simplicial presheaves over $\mathcal{D}$.

Definition 2.1. ([GJ]VI.3)
Soit $F \in sPr(\mathcal{D})$. The tower of n-truncations of F is a Postnikov tower:

$$\cdots \longrightarrow \tau_{\leq n}F \longrightarrow \tau_{\leq n-1}F \longrightarrow \cdots \longrightarrow \tau_{\leq 1}F \longrightarrow \tau_{\leq 0}F \ .$$

Definition 2.2. Let F be a simplicial presheaf.

- ▷ The functor $\pi_0(F) : \mathcal{D} \to Ens$ is defined by $\pi_0(F) := X \to \pi_0(F(X))$.
- ▷ The category $(\mathcal{D}/F)_0$ is the full subcategory of $sPr(\mathcal{D})/F$ whose objects are in $\mathcal{D}$.
- ▷ The functor $\pi_n(F) : (\mathcal{D}/F)_0 \to Ens$ is defined by $\pi_n(F)(X,u) := \pi_n(F(X),u)$.

Definition 2.3. Let G be a simplicial group.

- ▷ The bisimplicial set $E(G,1)$ is defined by $E(G,1)_{p,q} = G_p^q$.
- ▷ The classifying space of G, denoted $K(G,1)$, is given by the diagonal of the bisimplicial set $E(G,1)/G$. More precisely $K(G,1)_n = G_n^n/G_n$. It is abelian if G is abelian.
- ▷ The endofunctor of abelien groups $K(G,1)^{\circ n}$ is denoted $K(G,n)$.

Remarks 2.4. As the diagonal of $E(G,1)$ is pointed (by identity), the simplicial set $K(G,1)$ is also pointed. In particular, $\pi_n(K(G,1),*) \simeq \pi_{n-1}(G,e_g)$. This construction is functorial (in G) and then extends to presheaves of simplicial groups.

2.2 Simplicial presheaves Cohomology

It is necessary to work in the proper category to construct a cohomology for a simplicial presheaf F which is not connected or pointed. In fact the 1-truncation of F is the nerve NG of a groupoid G and in the category $sPr(\mathcal{D})/NG$, F becomes connected and pointed. But in this category, there is no clear construction for classifying spaces. The solution of this problem is given by a Quillen equivalence with the category $sPr(\mathcal{D}/G)$, for a well chosen category $\mathcal{D}/G$. We choose now a simplicial presheaf F.

The Category of Presheaves

The left adjoint functor $\widetilde{(-)}$

Definition 2.5. The category $\mathcal{D}/G$ is the Grothendieck construction associated to G, i.e. is the category whose objects are couples (X,x), $x : X \to NG$, and whose morphisms from (X,x) to (Y,y) are couples (f,u) where $f : X \to Y$ and $u : y \circ f \simeq x$ in $G(X) \simeq \pi_1 F(X)$.

Next step is to construct a functor $\widetilde{(-)} : \mathcal{D}/G \to sPr(\mathcal{D})/NG$.

Definition 2.6. Let $(X,x,)$ be in $\mathcal{D}/G$. Define a presheaf of groupoids $G_{X,x}$ on $\mathcal{D}$. The image of $S \in \mathcal{D}$ is the groupoid described as follow

- The objects are triples (u,y,h), $u : S \to X$, $y \in G(S)$, and $h : x \circ u \simeq y \in G(S)$.
- A morphism from (u,y,h) to (u',y',h') is an endomorphism k of S such that $k^*(h : x \circ u \to y) = h' : x' \circ u' \to y'$.

Let $\check{X}$ denote the nerve of this groupoid.

Remark 2.7. There is a commutative diagram of presheaves of groupoids

$$\begin{array}{ccc} X & \xrightarrow{x} & G \\ & {\scriptstyle j}\searrow \quad \nearrow{\scriptstyle l} & \\ & G_{X,x} & \end{array}$$

where l is the projection on G and j is given for $S \in \mathcal{D}$ by $j(S) : u \in Hom_{\mathcal{D}}(S, X) \to (u, x \circ u, Id) \in G_{X,x}$. Applying the functor nerve, one get a morphism $\breve{x} := Nl : \breve{X} \to G$. It defines a functor

$$\begin{gathered} \breve{(-)} : \mathcal{D}/G \to sPr(\mathcal{D})/NG \\ (X, x) \to (\breve{X}, \breve{x}) \end{gathered}$$

Definition 2.8. The functor $\widetilde{(-)} : \mathcal{D}/G \to sPr(\mathcal{D})/NG$ is defined by

$$\widetilde{(-)} : (X, x) \to (\widetilde{X}, \widetilde{x}) := Q(\breve{X}, \breve{x})$$

where Q is a cofibrant replacement in $sPr(\mathcal{D})/NG$.

Remarks 2.9. This functor has a kan extension to $sPr(\mathcal{D}/G)$, still denoted

$$\widetilde{(-)} : sPr(\mathcal{D}/G) \to sPr(\mathcal{D})/NG.$$

In facts, the category $sPr(\mathcal{D}/G)$ is equivalent to the category $sPr(\mathcal{D})^{NG}$ defined in [J] and the equivalence of category we are constructing is constructed in a different way and a more general situation in [J].

The right adjoint functor $(-)_1$ We construct now the (right) adjoint of $\widetilde{(-)}$, denoted $(-)_1$.

Definition 2.10. The functor $(-)_1 : sPr(\mathcal{D})/NG \to sPr(\mathcal{D}/G)$ is defined by

$$(-)_1 : (H, u) \to H_1 := (X, x) \to \underline{Hom}^{\Delta}_{sPr(\mathcal{D})/NG}((\widetilde{X}, \widetilde{x}), (H, u))$$

where $\underline{Hom}^{\Delta}$ is the simplicial Hom. As $(\widetilde{X}, \widetilde{x})$ is constructed cofibrant, the functor $(-)_1$ is right Quillen and its adjoint is then left Quillen. We prove now that $R(-)_1$ commutes with homotopy colimits. We need to recall first some properties.

Definition 2.11. Let (H, h) be in $sPr(\mathcal{D})/NG$ and (X, x) be in $\mathcal{D}/G$. Define an object (H_X, h_x) by the homotopy pullback diagram

$$\begin{array}{ccc} H_X & \longrightarrow & H \\ \downarrow & & \downarrow{\scriptstyle h} \\ X & \xrightarrow[x]{} & NG \end{array}$$

Lemma 2.12. *Let (H, f) be an homotopy colimit, $H \simeq Hocolim(H_i)$, in $sPr(\mathcal{D})/NG$ and let (X, x) be in $\mathcal{D}/G$.*

- ⋄ *There is an isomorphism $H_X \simeq Hocolim(H_i)_X$ in $Ho(sPr(\mathcal{D})/NG)$.*
- ⋄ *There is an isomorphism $RH_1(X) \simeq Map_{sPr(\mathcal{D})/X}((X, Id), (H_X, h_x))$.*

Corollary 2.13. *The functor $R(-)_1$ commutes with homotopy colimits.*

Proof:
Let H be isomorphic to $Hocolim(H_i)$ and (X, x) be in $\mathcal{D}/G$.

$$\begin{gathered} RH_1(X) \simeq Map_{sPr(\mathcal{D})/X}((X, Id), ([Hocolim(H_i)]_X, [Hocolim(h_i)]_x)) \\ \simeq Map_{sPr(\mathcal{D})/X}((X, Id), (Hocolim[(H_i)_X], Hocolim[(h_i)_x])) \simeq Hocolim(R(H_i)_1(X)). \end{gathered}$$

♦

The Equivalence

Proposition 2.14. *The Quillen functors* $\widetilde{(-)}$ *and* $(-)_1$ *define a Quillen equivalence.*

Proof:
The functor $\widetilde{(-)}$ commutes with homotopy colimits and as any object in $sPr(\mathcal{D}/G)$ is an homotopy colimit of representable objects $H \simeq Hocolim(X_i)$, its image can be computed in terms of representable objects, i.e. $\widetilde{H} \simeq Hocolim(\widetilde{X_i})$. The short exact sequence $H_1 \to H \to \tau_{\leq 1}H$ proves that $(-)_1$ preserves weak equivalences. Then

$$(\widetilde{H})_1 \simeq (Hocolim(\widetilde{X_i}))_1 \simeq Hocolim(X_i) \simeq H.$$

If H is cofibrant in $sPr(\mathcal{D}/G)$ and H' is fibrant in $sPr(\mathcal{D})/NG$, we consider a morphism between short exact sequences

$$\begin{array}{ccccc} H & \longrightarrow & \widetilde{H} & \longrightarrow & NG \\ \downarrow & & \downarrow & & \downarrow {\scriptstyle Id} \\ H'_1 & \longrightarrow & H' & \longrightarrow & NG \end{array}$$

Applying the functors π_i, it is clear that if $\widetilde{H} \to H'$ is an equivalence, so is $H \to H'_1$ and reciprocally, if $H \to H'_1$ is an equivalence, the homotopic fibers of $\widetilde{H} \to H'$ upon NG are equivalences thus so is $\widetilde{H} \to H'$.

♦

The Cohomology

Definition 2.15. Let F be in $sPr(\mathcal{D})$, a local system on F is a presheaf of abelian groups on $\mathcal{D}/G$, where G verifies $NG \simeq \tau_{\leq 1}F$. A Morphism of local systems is a morphism of presheaves of abelian groups. The category of local systems on F will be denoted $sysloc(F)$. The n-th classifying space of M is denoted $K(M,n)$ and its image by $L\widetilde{(-)}$ is denoted $L\tilde{K}(M,n)$.

Remark 2.16. The object $L\tilde{K}(M,n)$ is characterised up to equivalence by the isomorphisms

$$\pi_n(L\tilde{K}(M,n)) \simeq M,\ \pi_1(L\tilde{K}(M,n)) \simeq \pi_1(F)$$
$$\pi_0(L\tilde{K}(M,n)) \simeq \pi_0(F),\ \pi_k(L\tilde{K}(M,n)) \simeq *, k \neq 0,1,n$$

Definition 2.17. Let F be in $sPr(\mathcal{D})$ and M be a local system on F. The n-th cohomology group of F with coefficient in M is

$$H^n(F,M) := \pi_0 Map_{sPr(\mathcal{D})/NG}(F, L\tilde{K}(M,n))$$

The standard example of local system is π_n. Indeed, it has been defined on $(\mathcal{D}/F)_0$ but it clearly lifts to $\mathcal{D}/G$.

The important theorem is here.

THEOREM 2.18. *Let G be a groupoid. For all m, the functor*

$$H^m(NG,-) : Sysloc(NG) \to Ab$$
$$M \to H^m(NG,M)$$

is isomorphic to the n-th derived functor of the functor $H^0(NG,-)$.

Proof:
There is an equivalence between the category of simplicial abelian group presheaves, denoted $sAb(\mathcal{D}/G)$, on $\mathcal{D}/G$ and the category of complexes of abelian group presheaves with negative or zero degree, denoted $C^-(\mathcal{D}/G, Ab)$. This is a generalisation of Dold-Kan correspondence. There is a correspondence between quasi-isomorphisms of complexes and weak equivalences of simplicial presheaves, and then an induced equivalence between the homotopical categories :

$$\Gamma : D^-(\mathcal{D}/G, Ab) \simeq Ho(sAb(\mathcal{D}/G))$$

The derived functors of H^0 are then given by

$$H^m_{der}(\mathcal{D}/G, M) \simeq Hom_{D^-(\mathcal{D}/G,Ab)}(\mathbb{Z}, M[m])$$

Where $\mathbb{Z}$ is regarded as a complex concentrated in degree zero and $M[m]$ is concentrated in degree $-m$, with value M. As $\Gamma(\mathbb{Z})$ is the constant presheaf with fiber $\mathbb{Z}$, still denoted $\mathbb{Z}$, and as $\Gamma(M[m])$ is equivalent to $K(M,m)$, Γ induces an isomorphism :

$$Hom_{D^-(\mathcal{D}/G,Ab)}(\mathbb{Z}, M[m]) \simeq Hom_{Ho(sAb(\mathcal{D}/G))}(\mathbb{Z}, K(M,m))$$

Finally, the adjunction between the abelianisation functor,denoted $\mathbb{Z}(-)$ from $sPr(\mathcal{D}/G)$ to $sAb(\mathcal{D}/G)$ and the forgetful functor gives

$$Hom_{D^-(\mathcal{D}/G,Ab)}(\mathbb{Z}, M[m]) \simeq Hom_{Ho(sPr(\mathcal{D}/G))}(*, K(M,m)) \simeq H^m(NG, M).$$

♦

Obstruction Theory

There is an homotopic pullback diagram in $sPr(\mathcal{D}/G)$:

$$\begin{array}{ccc} \tau_{\leq n}F_1 & \longrightarrow & * \\ \downarrow & & \downarrow \\ \tau_{\leq n-1}F_1 & \longrightarrow & K(\pi_n(F), n+1) \end{array}$$

As F_1 is 1-connex, this pullback diagram is a (functorial) generalisation to presheaves of the diagram given by the proposition 5.1 of [GJ]. By the Quillen equivalence $((-)_1, \widetilde{(-)})$, there is an homotopic pullback diagram:

$$\begin{array}{ccc} \tau_{\leq n}F & \longrightarrow & NG \\ \downarrow & & \downarrow \\ \tau_{\leq n-1}F & \longrightarrow & L\widetilde{K}(\pi_n(F), n+1) \end{array}$$

If $H \to \tau_{\leq n-1}F$ is a morphism in $Ho(sPr(\mathcal{D})/NG)$, it has a lift to $\tau_{\leq n}F$ if and only if it is sent to a zero element in the group

$$\pi_0 Map_{sPr(\mathcal{D})/NG}(H, L\widetilde{K}(\pi_n(F), n+1))$$

This group can be described in terms of cohomology. Indeed, if G' is a groupoid such that $NG' \simeq \tau_{\leq 1}H$. Let u denote the morphism $u : NG' \to NG$. To simplify the notations, we still write H for what we should call u^*H. There is a Quillen adjunction:

$$sPr(\mathcal{D})/NG' \underset{-\times_{NG}NG'}{\overset{u^*}{\rightleftarrows}} \mathcal{D}/NG$$

which induces an isomorphism

$$Map_{sPr(\mathcal{D})/NG}(H, L\widetilde{K}(\pi_n(F), n+1)) \simeq Map_{sPr(\mathcal{D})/NG'}(H, L\widetilde{K}(\pi_n(F), n+1) \times_{NG} NG').$$

There is a clear weak equivalence $L\widetilde{K}(\pi_n(F), n+1) \times_{NG} NG' \simeq L\widetilde{K}(\pi_n(F) \circ u^*, n+1)$, thus :

$$\pi_0 Map_{sPr(\mathcal{D})/NG}(H, L\widetilde{K}(\pi_n(F), n+1)) \simeq H^{n+1}(H, \pi_n(F) \circ u^*)$$

2.3 Simplicial Modules Cohomology

It is well known that for a commutative monoid B in $(Set, \times, \mathbb{F}_1)$, there is an equivalence

$$sPr(\mathcal{B}B) \simeq sB-mod$$

where $\mathcal{B}B$ is the category with one object with a set of endomorphisms isomorphic to B. We will identify these two categories in this part. Let now A be a commutative monoid in sets and $B \to A$ be a morphism of commutative monoids. We are in a particular case of previous section, the category $\mathcal{D}$ is $\mathcal{B}B$ and the presheaf of groupoids G is just A. Let M be a local system on $\mathcal{B}B$, there is an isomorphism

$$H^n(A, M) \simeq \pi_0 Map_{sB-mod/A}(A, L\widetilde{K}(M, n+1)).$$

Let Z denote the abelianization functor from $B-mod/A$ to the category of abelian group objects in $B-mod/A$, denoted $Ab(B-mod/A)$. There is an equivalence between $Ab(B-mod/A)$ and the category of A graduated $Z(B)$-modules, denoted $Z(B)-mod^{A-grad}$. The following functor realizes this equivalence, its inverse is the forgetful functor.

$$\Theta : (\ M \xrightarrow{f} A\ ) \in Ab(sB-mod/A) \to \oplus_{m\in A} f^{-1}(m) \in Z(B)-mod^{A-grad}$$

This equivalence lifts to simplicial categories and it is easy to see that

$$H^{n+1}(A,M) \backsimeq \pi_0 Map_{Z(B)-mod^{A-grad}}(Z(A), L\widetilde{K}(M,n+1)) \quad (1)$$

Here is the proposition that interrests us.

Proposition 2.19. *Let $B \to A$ be a morphism of commutative monoids in sets. The morphism $B \to A$ is hf if and only if*

- *$Z(A)$ is homotopically finitely presented in $Z(B)-mod^{A-grad}$.*
- *A is homotopically finitely presented for the 1-truncated model structure i.e. in the category $B-Gpd$.*

Proof

Let us prove first the easiest part. Let A be an homotopically finitely presented object in $sB-mod$. Let $sB-mod^{\leq 1}$ denotes the category $sB-mod$ endowed with its 1-truncated model structure. In the adjunctions

$$sB-mod \underset{Id}{\overset{Id}{\rightleftarrows}} sB-mod^{\leq 1}$$

$$sB-modZ \underset{i}{\rightleftarrows} sZ(B)-mod/A$$

$$sB-mod/AZ \underset{i}{\overset{Z}{\rightleftarrows}} sZ(B)-mod^{A-grad}$$

the left adjoint functors preserve weak equivalences and cofibrations thus the right adjoints preserve homotopically finitely presentable objects.

Let us now prove the hardest part. We start with this lemma:

Lemma 2.20. *There exists $m_0 \in \mathbb{N}$ such that for any local system M and all $n \geq m_0$*

$$H^n(A,M) \backsimeq *.$$

Proof

The isomorphism 1 proves that the cohomology of A is isomorphic to the Ext functors of $Z(A)$ in $sZ(B)-mod^{A-grad}$. Moreover, there is an equivalence of abelian categories

$$sZ(B)-mod^{A-grad} \backsimeq C^-(\mathcal{B}B/A, Ab)$$

which induces by 2.18 an equivalence with the derived functors of H^0. In particular as $Z(A)$ is homotopically finitely presented, the derived functors of H^0 vanished after a set rank denoted m_0.

♦

Remark 2.21. Two corollaries comes now. They are a consequence of this lemma and the following short exact sequence, $C \in A/sB-mod$

$$\begin{array}{ccc} Map_{sB-mod/\tau_{\leq n-1}C}(A,\tau_{\leq n}C) & \longrightarrow & Map_{sB-mod}(A,\tau_{\leq n}C) \\ & & \downarrow \\ Map_{sB-mod/NG}(A, L\widetilde{K}(\pi_n(C),n+1)) & \longleftarrow & Map_{sB-mod}(A,\tau_{\leq n-1}C) \end{array}$$

Corollary 2.22. *Let $A \xrightarrow{v} C$ be in $A/sB-mod$. For all $i \geq 1$, for all $n \geq n_i = n_0 + i + 1$*

$$\pi_0 Map_{sB-mod}(A,\tau_{\leq n-1}C) \backsimeq \pi_0 Map_{sB-mod}(A,\tau_{\leq n}C)$$
$$\pi_i(Map_{sB-mod}(A,\tau_{\leq n-1}C),v) \backsimeq \pi_i(Map_{sB-mod}(A,\tau_{\leq n}C),v)$$

Proof
We first prove that the simplicial set $Map_{sB-mod/\tau_{\leq n-1}C}(A, \tau_{\leq n}C)$ is not empty. There are pushout squares :

$$\begin{array}{ccccc} A \times^h_{L\widetilde{K}(\pi_n(C),n+1)} NG & \longrightarrow & \tau_{\leq n}C & \longrightarrow & NG \\ \downarrow & & \downarrow & & s \downarrow \curvearrowright p \\ A & \longrightarrow & \tau_{\tau_{\leq n-1}C} & \longrightarrow & L\widetilde{K}(\pi_n(C), n+1) \end{array}$$

where $p \circ s = Id$. There are then equivalences

$$Map_{sB-mod/\tau_{\leq n-1}F}(A, \tau_{\leq n}C) \backsimeq Map_{sB-mod/L\widetilde{K}(\pi_n(C),n+1)}(A, NG) \backsimeq Map_{sB-mod/A}(A, A \times^h_{L\widetilde{K}(\pi_n(C),n+1)} NG).$$

Let f be the morphism from A to $L\widetilde{K}(\pi_n(C), n+1)$. There is a morphism $p \circ f : A \to NG$. As the cohomology of A vanished for $n \geq n_0$, the elements $s \circ p \circ f$ and f of the cohomology group are equals and thus

$$p \circ f \in \pi_0 Map_{sB-mod/L\widetilde{K}(\pi_n(C),n+1)}(A, NG).$$

Then, for $i = 0$, the corollary is a clear consequence of lemma 2.20 and the short exact sequence of remark 2.21. Now, Let us study the case $i > 0$. As $NG \times^h_{L\widetilde{K}(\pi_n(C),n+1)} NG \backsimeq L\widetilde{K}(\pi_n(C), n+1)$, we obtain

$$A \times^h_{L\widetilde{K}(\pi_n(C),n+1)} NG \backsimeq L\widetilde{K}(\pi_n(C) \circ v^*, n)$$

Thus

$$\pi_i(Map_{sB-mod/\tau_{\leq n-1}C}(A, \tau_{\leq n}C), v) \backsimeq \pi_i((Map_{sB-mod/A}(A, L\widetilde{K}(\pi_n(C) \circ v^*, n)), q) \backsimeq H^{n-i}(A, \pi_n(C))$$

where q is the natural morphism from A to $L\widetilde{K}(\pi_n(C) \circ v^*, n)$. We deduce then the result from lemma 2.20 and the short exact sequence of remark 2.21.

♦

Corollary 2.23. *Let* $A \xrightarrow{v} C$ *be in* $A/sB-mod$. *The pointed tower of fibrations*

$$(Map_{sB-mod}(A, \tau_{\leq n}C), v)$$

converges completly in the sense of [GJ].

Proof
It can be checked with the corollary 2.21 of the complete convergence lemma of [GJ].

♦

Corollary 2.24. *For all* $i \geq 0$, *all* $n \geq n_i$ *and all* $A \xrightarrow{v} C$ *in* $A/sB-mod$, *there are isomorphisms*

$$\pi_i(Map_{sB-mod}(A, C), v) \backsimeq lim_{n\in\mathbb{N}}\pi_i(Map_{sB-mod}(A, \tau_{\leq n}C), v) \backsimeq \pi_i(Map_{sB-mod}(A, \tau_{\leq n_i}C), v)$$

Proof
The first isomorphism is a consequence of Milnor exact sequence ([GJ], 2.15) and the vanishing of the lim^1 induced by the complete convergence. The second isomorphism is a consequence of corollary 2.22

♦

Let us now recall a well known lemma with which we will prove the last technical lemma necessary for the proof of 2.19.

Lemma 2.25. *Let*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ Z & \xrightarrow[g]{} & T \end{array}$$

be a commutative square in sSet where g *is a weak equivalence. The morphism* f *is a weak equivalence if and only if for all* $z \in Z$, *the homotopic fibers* X_z *and* $Y_{g(z)}$ *are simultaneously empty and equivalent when not empty.*

Here is the last technical lemma:

Lemma 2.26. *Let $C \simeq Hocolim_{\alpha\in\Theta}(C_\alpha)$ be an homotopical filtered colimit. There is a weak equivalence in $sSet$*

$$Map_{sB-mod}(A,C) \simeq HocolimMap_{sB-mod}(A,C_\alpha).$$

Proof
By induction on the truncation level n of C. This is an hypothesis of 2.19 for $n=1$. Let us assume that is is true for $n-1$. Let C be an n-truncated object in $sB-mod$ and $\bar{u}$ be in $HocolimMap_{sB-mod}(A,\tau_{n-1}C_\alpha)$, represented by $u \in Map_{sB-mod}(A,\tau_{n-1}C_{\alpha_0})$. Let $\tilde{u}$ denote its image in $Map_{sB-mod}(A,C)$. The filtered hocolimit along Θ is weak equivalent to the hocolimit along α_0/θ. We will use previous lemma, computing the fibers along u as in the following diagram:

$$\begin{array}{ccc}
Hocolim_{\alpha_0/\Theta}Map_{sB-mod/\tau_{n-1}C_\alpha}((A,u_\alpha),C_\alpha) & \longrightarrow & Map_{sB-mod/\tau_{n-1}C}((A,\tilde{u}_{n-1}),C) \\
\downarrow & & \downarrow \\
Hocolim_{\alpha_0/\Theta}Map_{sB-mod}(A,C_\alpha) & \longrightarrow & Map_{sB-mod}(A,C) \\
\downarrow & & \downarrow \\
Hocolim_{\alpha_0/\Theta}Map_{sB-mod}(A,\tau_{n-1}C_\alpha) & \longrightarrow & Map_{sB-mod}(A,\tau_{n-1}C)
\end{array}$$

Where $u_\alpha : \ A \xrightarrow{u} C_{\alpha_0} \longrightarrow C_\alpha$ and $\tilde{u}_{n-1} : \ A \xrightarrow{\tilde{u}} C \longrightarrow \tau_{n-1}C$.

Let us show first that the fibers are simultaneously empty. The naturel morphism

$$Hocolim_{\alpha_0/\Theta}Map_{sB-mod}(A,\tau_{n-1}C_\alpha) \to Map_{sB-mod}(A,\tau_{n-1}C)$$
$$\bar{u} \to \tilde{u}$$

induces the naturel morphism on cohomology groups

$$Hocolim_{\alpha_0/\Theta}H^{n+1}(A,\pi_nC_\alpha) \to H^{n+1}(A,\pi_nC)$$

which is a weak equivalence. Indeed, the H^n are isomorphic to Ext functors in $sZ(B)-mod^{A-grad}$ which commute with filtered hocolimits by the first hypothesis of 2.19. The images of $\bar{u}$ and $\tilde{u}$ in the cohomology groups vanish then simultaneously, and the fibers are simultaneously empty.

Let us assume now that the fibers are unempty and prove that they are equivalent. The functors π_i commute with homotopical filtered colimits, applying them on the fibers, we get the following natural morphism

$$colim_{\alpha_0/\Theta}\pi_iMap_{sB-mod/\tau_{n-1}C_\alpha}((A,u_\alpha),C_\alpha) \to \pi_iMap_{sB-mod/\tau_{n-1}C}((A,\tilde{u}_{n-1}),C)$$

As these π_i are in fact isomorphic to H^{n-1}, these morphisms are isomorphisms. By 2.25, this ends the proof of the lemma.

♦

Let us now end the proof of 2.19.
Let $v : A \to C$ be in $A/sB-mod$ such that $C \simeq Hoclolim(C_\alpha)$. Let us prove that the morphism

$$Hocolim(Map_{sB-mod}(A,C_\alpha)) \to Map_{sB-mod}(A,C)$$

is a weak equivalence. Let i be a positive integer, to check if the image of this morphism by π_i is an isomorphism, we can just consider the case C n-truncated by 2.24. As the truncation commuta with homotopical filtered colimits, this is a consequence of 2.26. This ends the proof of 2.19.

♦

3 Examples

3.1 The Category $(\mathbb{Z}-mod, \otimes_{\mathbb{Z}}, \mathbb{Z})$

In classical algebraic geometry, the notion of (projective) resolution is obtained using chain complex of modules or rings. In facts, considering the correspondence of Dold-Kan this method is equivalent to taking cofibrant resolution in the simplical category (cf [Q]).

Theorem 3.1. *(Dold-Kahn correspondance)*
Let A be a ring. There is an equivalence of categories:

$$sA-mod \simeq Ch(A-mod)^{\geq 0} \text{ and } \forall i\ \pi_i(Map(\mathbb{Z}, X)) \simeq H_i(X).$$

In particular, it induces a correspondence between weak equivalences and quasi-isomorphisms.

Remark 3.2. Let A be a ring. Generating cofibrations of $Ch(A-mod)^{\geq 0}$ are levelwise equal to $\{0\} \to A$ or Id_A.

Definition 3.3. Let A be a rings, M, N be two A-modules.

i. Define $Tor_*^A(M, N) := H_*(M \otimes_A^L N)$.

ii. Define $Ext_A^*(M, N) := H^*(R\underline{Hom}_{A-mod}(M, N))$.

iii. Define the projective dimension of M by:

$$ProjDim_A(M) := inf\{n\ st\ Ext_A^{n+1}(M, -) = \{0\}\}.$$

iv. Define the Tor-dimension of M by:

$$TorDim_A(M) := inf\{n\ st\ \forall\ X\ p-truncated\ Tor_i^A(M, X) = \{0\}\ \forall i > n+p\}$$

Remark 3.4. The functor of Dold-Kan correspondence is a strong monoidal functor, as a consequence the Tor dimension can be computed with π_i instead of H_i.

Lemma 3.5. *Let X be in $Ho(sSet)$ and M be in $s\mathbb{Z}-mod$ (resp $sA-mod$, for A a ring)*

- ⋄ *The object X is n-truncated if and only if $Map(*, X) \simeq Map(S^i, X)\ \forall\ i > n$ in $Ho(sSet)$.*
- ⋄ *The object M is n-truncated if and only if $Map_{s\mathbb{Z}-mod}(\mathbb{Z}, M)$ (resp $Map_{sA-mod}(A, Z)$) is n-truncated in $Ho(sSet)$.*

Proof
For the first statement, by2.25, we can consider equivalently the homotopic fibers of this morphism upon $Map(*, X)$. The fiber of $Map(*, X)$ is a point and the fiber of $Map(S^i, X)$ is $Map_{sSet/*}(S^i, X)$. As $\pi_j Map_{sSet/*}(S^i, X) \simeq \pi_{i+j}(X)$, the equivalence is clear.

For the second statement, any object in $s\mathbb{Z}-mod$ is an homotopical colimit of free objects, i.e. $\forall\ N \in s\mathbb{Z}-mod$ there exists a family of sets $(\lambda_i)_{i\in I}$ such that $qN \simeq hocolim_I \coprod_{\lambda_i} \mathbb{Z}$ in $Ho(s\mathbb{Z}-mod)$. Assume that $Map_{s\mathbb{Z}-mod}(\mathbb{Z}, M)$ is n-truncated. $Map_{s\mathbb{Z}-mod}(N, M) \simeq holim_I \prod_{\lambda_i}(Map_{s\mathbb{Z}-mod}(\mathbb{Z}, M))$, hence is an homotopical limit of n-truncated objects. by i, n-truncated objects in $sSet$ are clearly stable under homotopical limits.

♦

Lemma 3.6. *(cf [TV]) Let $u : A \to B$ be in $s\mathbb{Z}-mod$. The morphism u is flat if and only if*

i. The natural morphism $\pi_*(A) \otimes_{\pi_0(A)} \pi_0(B) \to \pi_0(B)$ is an isomorphism.

ii. The morphism $\pi_0(u)$ is flat.

In particular, if A is cofibrant and n-truncated, u flat implies B n-truncated.

Remark 3.7. [TV] Let $A \to B$ be in $\mathbb{Z}-alg$. The morphism $A \to B$ is flat if and only if $TorDim_A(B) = 0$.

We give now the lemmas necessary to the theorem of comparison of the notions of smoothness in rings and relative smoothness.

Lemma 3.8. *Let $A \to B$ be a smooth morphism of rings. There exists a pushout square*

$$\begin{array}{ccc} A' & \longrightarrow & B' \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array}$$

such that $A' \to B'$ is a smooth morphism of noetherian rings.

Proof:
This is the affine case in the corollary 17.7.9(b) of [EGAIV].

♦

Lemma 3.9. *Let $A \to B$ and $A \to C$ be two morphisms in $\mathbb{Z} - alg$. If B is a perfect complex of $B \otimes_A B$ modules then $D := B \otimes_A C$ is a perfect complex of $D \otimes_C D$ modules.*

Proof:
Perfect complexes are clearly stable under base change. As $D \otimes_C D \backsimeq B \otimes_A D$, the natural morphism $D \otimes_C D \to D$ is a pushout of $B \otimes_A B \to B$ hence D is a perfect complex.

♦

Lemma 3.10. *Let A be a noetherian ring. Every flat A-module of finite type is projective.*

Lemma 3.11. *Assume that A is a noetherian ring and consider $A \to B \in \mathbb{Z} - alg$, B of finite type. There is an equivalence between*

i. The ring B is of finite Tor-dimension on A.

ii. The ring B is of finite projective dimension on A.

The part $ii \Rightarrow i$ is clear, if B has a finite projective resolution $0 \to P_n \to ... \to B$, then for $i \geq n$, $Tor^{i+1}(M, -) \backsimeq Tor^{i-n}(P_{n+1}, -)$ and $P_{n+1} = 0$.

Reciprocally, if $TorDim_A b < +\infty$, let $... \to P_n \to ... \to B$ be a free resolution of B. The module $P_n/im(P_{n+1})$ has Tor dimension 0 by previous formula hence is flat by 3.7. As A is noetherian and B is of finite type, it is projective and we have a clear finite projective resolution.

♦

Lemma 3.12. *Let $u : A \to B$ be in rings. Assume that A is an algebraically closed field, then there is an equivalence*

- ◇ *The morphism u is formally smooth in the sense of rings.*
- ◇ *Any morphism $x : B \to A$ in rings provides A with a structure of B-module of finite projective dimension over B.*

Lemma 3.13. *Let $u : A \to B$ be a finitely presented flat morphism in rings. The morphism u is smooth if and only if for all algebraically closed field K under A, $K \to K \otimes_A B$ is smooth.*

Theorem 3.14. *A morphism $A \to B$ in $\mathbb{Z} - alg$ is smooth in the sense of rings if and only if*

i. The ring B is finitely presented in $A - alg$.

ii. The morphism $A \to B$ is flat.

iii. The ring B is a perfect complex of $B \otimes_A B$-modules.

Proof:
Let us now prove the first part of the theorem. Assume that $A \to B$ is smooth. i and ii are clear.

Let us prove iii. By 3.8, as iii is stable under pushout, we just have to prove it for A and B noetherian. Let us prove first that $B \otimes_A B \to B$ is of finite Tor dimension (hence of finite projective dimension by 3.11).
Let L be an algebraically closed field in $A - alg$. Set $B_L := B \otimes_A L$. Clearly

$$B \otimes_{B \otimes_A B} L \backsimeq B_L \otimes_{B_L \otimes_L B_L} L$$

hence computing the Tor dimension of B over $B \otimes_A B$ is equivalent to compute the Tor dimension of B_L over $B_L \otimes_L B_L$. The morphism $L \to B_L \to B_L \otimes_L B_L$ is smooth, by composition of smooth morphisms, over an algebraically closed field. The ring $B_L \otimes_L B_L$ is then smooth on afield, hence regular. Now, B_L is a module of finite type on this regular ring thus it is a perfect complex on it. In particular, it is of finite projective dimension hence of finite Tor dimension. Finally, B is of finite Tor dimension hence of finite projective dimension over $B \otimes_A B$. As previously, B of finite type over $B \otimes_A B$. As these rings are noetherian, B is a perfect complex. Indeed, B has a finite projective resolution by (P_i). Each P_i is of finite Tor dimension hence of finite projective dimension.

Let us prove the second part of the theorem. Let $A \to B$ be a morphism of rings verifying i, ii and iii. Let K be an algebraically closed field under A. We will use 3.13 and 3.12.
Let $x : B \to K$ be in $\mathbb{Z} - alg$. The following commutative diagram is an homotopic pushout:

$$
\begin{array}{ccc}
B \otimes_K B & \xrightarrow{Id \otimes_K x} & B \otimes_K K \simeq B \\
\downarrow & & \downarrow x \\
B & \xrightarrow[x]{} & K
\end{array}
$$

Thus K has finite projective dimension in $B-mod$. Finally, by 3.13, $K \to B$ is smooth in the sense of rings. As it is true for any K, by 3.12, $A \to B$ is smooth in the sense of rings.

♦

Here is now the comparison theorem.

THEOREM 3.15. *Let $A \to B$ be a morphism of rings. It is smooth if and only if it is smooth in the sense of rings.*

Proof
The two following lemmas, and remark 3.7 prove the theorem.

Lemma 3.16. *[TV] Let $A \to B$ be a morphism in $\mathbb{Z}-alg$.*

i. if $A \to B$ is hfp, then it is finitely presented in $\mathbb{Z}-alg$.

ii. if $A \to B$ is smooth and finitely presented, then it is hfp.

Lemma 3.17. *[TV] Let $A \to B$ be a morphism of rings. The ring B is a perfect complex of B-modules if and only if $A \to B$ is hf.*

♦

3.2 The Category *Set*

The most difficult problem consists in finding examples of formally smooth morphisms. The Lemma 2.19 gives us a characterisation of these morphisms in the relative context $\mathcal{C} = Set$.
The functor nerve and the functor "fundamental groupoid" define a Quillen equivalence between the category $sB-mod$ endowed with its 1-truncated model structure and the category $B-Gpd$. Moreover, this last category is compactly generated and thus its filtered $Hocolim$ can be computer as filtered colimits. Here is the formula to do this

Lemma 3.18. *Let $\mathcal{I}$ be a filtered diagram and $F : \mathcal{I} \to Gpd$. The colimit of F consists of*

- *On objects*

$$(ColimF)_0 := Colim(u \circ F)$$

where u is the forgetful functor from Gpd to Set.

- *On morphisms, for $\bar{x}, \bar{y} \in Colim(u \circ F)$ represented by $x \in F(i)$ and $y \in F(i')$. There exists k under i and i' such that*

$$Hom_{Hocolim(F)}(\bar{x}, \bar{y}) := Colim_{k/\mathcal{I}}(Hom_{F(j)}((l_{i,j})_*)(x), (l_{i',j})_*)(y))$$

where $l_{i,j} : i \to j$ and $l_{i',j} : i' \to j$.

We also need to describe the derived enriched Homs.

Lemma 3.19. *Let B be a monoid in Set. There is an equivalence of categories between $Ho(B-Gpd)$ and the category $[B-Gpd]$ whose objects are B-groupoids and morphisms are isomorphism classes of functors. In particular, for two B-groupoids G and G', $R\underline{Hom}^{\Delta \leq 1}_{B-gpd}(G, G') \simeq \underline{Hom}^{\Delta \leq 1}_{[B-gpd]}(G, G')$ in $Ho(Gpd)$, where the exponent $\Delta \leq 1$ means that the Homs are enriched on groupoids.*

Lemma 3.20. *The commutative monoid $\mathbb{N}$ is homotopically finitely presented for the 1-truncated model structure i.e. in the category $(\mathbb{N} \times \mathbb{N}) - Gpd$.*

Let $\mathbb{N}^2$ denotes $\mathbb{N} \times \mathbb{N}$. Let $F : \mathcal{J} \to Gpd$ be a funnctor from a filtered diagram $\mathcal{I}$ to Gpd. We have to prove

$$Hocolim(\underline{Hom}^{\Delta \leq 1}_{[\mathbb{N}^2-Gpd]}(\mathbb{N}, F(-))) \simeq \underline{Hom}^{\Delta \leq 1}_{[\mathbb{N}-Gpd]}(\mathbb{N}, Hocolim(F))$$

We let the reader verify that the following functor denoted φ define an equivalence of groupoids.

Let $\bar{H}$ be in $Hocolim(\underline{Hom}^{\Delta \leq 1}_{[\mathbb{N}^2-Gpd]}(\mathbb{N}, F(-)))$ represented by $H \in \underline{Hom}_{[\mathbb{N}^2-gpd]}(\mathbb{N}, F(j))$. We define φ on objects by

$$\varphi : \bar{H} \to \hat{H} := n \to \bar{H(n)}$$

Now, by construction, any morphism $\bar{\eta}$ in $Hocolim(\underline{Hom}^{\Delta\leq 1}_{[\mathbb{N}^2-Gpd]}(\mathbb{N}, F(-)))$ has a representant $\eta : G \to G' \in Hom_{\underline{Hom}_{[\mathbb{N}^2-gpd]}(\mathbb{N},F(j))}(G, G')$. We define φ on morphisms by

$$\varphi : \bar{\eta} \to \hat{\eta} := n \to \bar{\eta_n}$$

♦

Lemma 3.21. *The commutative group $\mathbb{Z}$ is homotopically finitely presented for the 1-truncated model structure i.e. in the category $(\mathbb{Z} \times \mathbb{Z}) - Gpd$.*

Proof
This is the same proof as previous lemma, replacing $\mathbb{N}$ by $\mathbb{Z}$.

♦

Corollary 3.22. *The morphisms $\mathbb{F}_1 \to \mathbb{N}$ and $\mathbb{F}_1 \to \mathbb{Z}$ are smooth. In particular, the affine scheme $Gl_{1,\mathbb{F}_1} \simeq Spec(\mathbb{Z})$, also denoted $\mathbb{G}_{m,\mathbb{F}_1}$ in [TVa], is smooth.*

Proof
They are clearly hfp and of Tor dimension zero. Their diagonal is hf for the 1-truncated model structure, thus, we just have to check that the diagonal of their abelianisation is hf in the simplicial graduated category given in 2.19. The abelianisation of $\mathbb{N}$ is $\mathbb{Z}[X]$ and the abelianisation of $\mathbb{Z}$ is $\mathbb{Z}(X)$, and the morphisms $\mathbb{Z}[X] \otimes_{\mathbb{Z}} \mathbb{Z}[X] \to \mathbb{Z}[X]$ and $\mathbb{Z}(X) \otimes_{\mathbb{Z}} \mathbb{Z}(X) \to \mathbb{Z}(X)$ are hf respectively in $s(\mathbb{Z}[X] \otimes_{\mathbb{Z}} \mathbb{Z}[X]) - Mod^{\mathbb{N}-grad}$ and $s(\mathbb{Z}(X) \otimes_{\mathbb{Z}} \mathbb{Z}(X)) - Mod^{\mathbb{Z}-grad}$.

♦

Remark 3.23. The scheme $Gl_{n,\mathbb{F}_1}$ is not affine but, according to its description in [TVa] and last corollary, it will be possible to call it smooth as soon as a proper definition of smooth morphisms for relative schemes (or schemes over F_1) is written.

3.3 Some Other examples

If $(\mathcal{C}, \otimes, 1)$ is a symmetric monoidal category as described in the preliminaries, its associated category of simplcial objects has simplicial Homs,denoted $\underline{Hom}^{\Delta}$, and there is an adjunction

$$s\mathcal{C} \underset{sK_0}{\overset{\underline{Hom}^{\Delta}(1,-)}{\rightleftarrows}} sSet$$

where $sK_0((X_n)_{n\in\mathbb{N}}) = (\coprod_{X_n} 1)_{n\in\mathbb{N}}$. One verifies easily that as 1 is cofibrant, finitely presentable, and as $\underline{Hom}^{\Delta}(1,-)$ preserves weak equivalences (by construction of the model structure on $\mathcal{C}$), the functor sK_0 preserve homotopically finitely presentable objects. In particular, sK_0 preserves hf morphisms and formally smooth morphisms. Restricting the adjunction to the categories of algebra, where weak equivalences and homotopical filtered colimits are obtained with the forgetful functor, it is also clear that $sK_0(u)$ preserves hfp morphisms. We write then the following proposition.

Proposition 3.24. *Let $u : A \to B$ be a smooth morphism in $Comm(Set)$, then $sK_0(u)$ is smooth if and only if $sK_0(B)$ is of finite Tor dimension over $sK_0(A)$.*

This gives particular examples. Indeed, in every context the affine line correspond to the morphism $1 \to 1[X] := \coprod_{\mathbb{N}} 1$ and the scheme $\mathbb{G}_m$ to the morphism $1 \to 1(X) := \coprod_{\mathbb{Z}} 1$. We write then the following theorem.

THEOREM 3.25. *The affine line and the scheme $\mathbb{G}_m$ are smooth in any context where, respectively, $1[X]$ and $1(X)$ are of finite Tor dimension over 1.*

This theorem can be applied in particular to the context $\mathbb{N} - mod$. The following lemma provides us, in this context, examples of morphisms of Tor-dimension 0.

Lemma 3.26. *Let $A \to B$ be in $Comm(\mathbb{N} - mod)$ such that B is free over A. The monoid B has Tor-dimension 0 over A.*

Proof :
Let $M \in A - mod$ be a n-truncated module. There exists a set λ such that $B \simeq \coprod_{\lambda} A$. Thus $B \otimes^L_A M' \simeq Coprod_{\lambda} QM$ in $Q_cA - mod$ where Q, Q_c are cofibrant replacement respectively in $Q_cA - mod$ and $Comm(\mathbb{N} - mod)$. Thus as this coproduct is a product in set, we get

$$B \otimes_A^L M' \simeq Colim_{\lambda' fini \subset \lambda} \prod_{\lambda'} QM$$

As functors π_i commute with products in sets and filtered colimits, the Tor dimenson of B over A is zero.

♦

Theorem 3.27. *Examples in* $\mathbb{N} - mod$.

- ◇ *The affine line in* $\mathbb{N} - mod$, $\mathbb{A}^1_{\mathbb{N}}$, *is smooth.*
- ◇ *The scheme* $\mathbb{G}_{m,\mathbb{N}}$ *relative to* $\mathbb{N} - mod$ *is smooth.*

We conclude with a last theorem

Theorem 3.28. *Let* $\mathcal{C}$ *be a relative context in the sense of [M] and* $A \to B$ *be a Zariski open immersion in* $Comm(\mathcal{C})$, *with* A *cofibrant in* $Comm(\mathcal{C})$ *and* B *cofibrant in* $A - alg$. *The morphism* $A \to B$ *is smooth.*

Proof
A Zariski open immersion is always formally smooth, its diagonal is even an isomorphism. Thus we will need to prove that it is hfp and of Tor dimension zero. First, if there exists $f \in A_0$,an object of the underlying set of A, such that $B \simeq A_f$, the result is clear. Indeed, A_f is given by a filtered colimit of A thus is of Tor dimension zero. Let us prove that it is hfp. It is clear that $A \to A[X]$ is homotopically finitely presented, then as everything is cofibrant, we can write A_f as a finite colimit of $A[X]$ ([M]) which is in facts a finite homotopical colimit and thus finally $A \to A_f$ is hfp.

Now if B define a Zariski open object of A, we can write B it as a cokernel of products of A_f. As functors π_i commute with products, the products preserve weak equivalences and it is then clear that $A \to B$ is hfp. For the Tor dimension, recall that there is a finite family of functor reflecting isomorphisms $B - mod \to A_f - mod$. Let M be a p-truncated A-module. This family sends $M \otimes_A^L B$ and its n-truncations,$n > p$ to the same module QM_f (Q is the cofibrant replacement of $A - mod$) thus clearly $M \otimes_A^L B$ is p truncated and $TorDim_A(B) = 0$.

♦

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