On l-adic iterated integrals, II
Functional equations and l-adic polylogarithms.

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Abstract. We continue to study l-adic iterated integrals introduced in the first part. We shall show that the l-adic iterated integrals satisfy essentially the same functional equations as the classical complex iterated integrals. Next we are studying l-adic analogs of classical polylogarithms.

The plan of the paper.

Part II

Part III

Mathematics Subject Classification 11 Number theory

§9. Introduction to Part II.

9.1. The classical complex iterated integrals satisfy functional equations (see [W1]). We shall show that l-adic iterated integrals satisfy the same functional equations as the classical complex iterated integrals.

First we introduce the following notation which we shall use in this paper. Let π (resp. L) be a group (resp. a Lie algebra). We denote by $\{\Gamma^k\pi\}_{k\geq 1}$ (resp. $\{\Gamma^k\mathrm{L}\}_{k\geq 1}$) the lower central series of the group π (resp. the Lie algebra L).

We set

$$gr^k_\Gamma\pi := \Gamma^k\pi/\Gamma^{k+1}\pi \quad (\text{resp. } gr^k_\Gamma\mathrm{L} := \Gamma^k\mathrm{L}/\Gamma^{k+1}\mathrm{L}).$$

Before we formulate our main result we make a following remark. Let $Y = \mathbf{P}^1_K \setminus \{b_1, \dots, b_{m+1}\}$. Then

$$\bigoplus_{k=1}^{\infty} gr^k_\Gamma\pi_1(Y(\mathbf{C}); x) \otimes \mathbf{Q}$$

is canonically isomorphic to a free Lie algebra over $\mathbf{Q}$ on m generators $Y_1, \dots, Y_m$, which we denote by $\mathrm{Lie}(Y_1, \dots, Y_m)$. Hence any linear form φ on $gr^k_\Gamma\pi_1(Y(\mathbf{C}); x) \otimes \mathbf{Q}$ corresponds to a linear form φ on $\mathrm{Lie}(Y_1, \dots, Y_m)$. Now we formulate our main result concerning functional equations.

Theorem D. *Let $X = \mathbf{P}^1_K \setminus \{a_1, \dots, a_{n+1}\}$ and let $Y = \mathbf{P}^1_K \setminus \{b_1, \dots, b_{m+1}\}$. Let $f_i : X \to Y$ be a smooth morphism and let $\varphi_i \in \mathrm{Lie}(Y_1, \dots, Y_m)^\diamond$ be a linear form of degree q defined over $\mathbf{Q}$ for $i = 1, \dots, N$. Let $n_1, \dots, n_N$ be rational numbers. If*

$$\sum_{i=1}^{N} n_i \varphi_i \circ (f_i)_* = 0$$

in $\mathrm{Hom}\big(gr^q_\Gamma\pi_1(X(\mathbf{C}); x); \mathbf{Q}\big)$, where $(f_i)_* : gr^q_\Gamma\pi_1(X(\mathbf{C}); x) \to gr^q_\Gamma\pi_1(Y(\mathbf{C}); f_i(x))$ is the map induced by f_i on fundamental groups for $i = 1, \dots, N$, then we have a functional equation

$$\sum_{i=1}^{N} n_i \mathcal{L}^{\varphi_i}(f_i(z), f_i(v)) = 0.$$

Next we generalize well known formulas

$$\int_a^b \omega + \int_b^a \omega = 0 \quad \text{and} \quad \int_a^c \omega = \int_a^b \omega + \int_b^c \omega$$

from the elementary calculus (ω is a one-form). We show the following result.

Theorem E. *Let $z, y, v \in \hat{X}(K)$ and let $\varphi \in \mathrm{Lie}(X_1, \ldots, X_n)^\diamond$. Then we have*

$$\mathcal{L}^\varphi(z, v) + \mathcal{L}^\varphi(v, z) = 0$$

and

$$\mathcal{L}^\varphi(z, v) = \mathcal{L}^\varphi(z, y) + \mathcal{L}^\varphi(y, v).$$

Let ω_1, ω_2 be one-forms. The classical complex iterated integrals satisfy the following relations written here for two one-forms (see [Ch]).

i) $\int_\gamma \omega_1, \omega_2 + \int_\gamma \omega_2, \omega_1 = \int_\gamma \omega_1 \cdot \int_\gamma \omega_2$,

ii) $\int_{\alpha\beta} \omega_1, \omega_2 = \int_\alpha \omega_1, \omega_2 + \int_\alpha \omega_1 \cdot \int_\beta \omega_2 + \int_\beta \omega_1, \omega_2$,

iii) $\int_\gamma \omega_1, \omega_2 = (-1)^2 \int_{\gamma^{-1}} \omega_2, \omega_1$.

The anolog of the formula i) is satisfied by "l-adic iterated integrals" (coefficients of the power series $\Lambda_p(\sigma)$) by the very definition because the image of the inclusion map of the fundamental group into the algebra of formal non-commutative power series is of the form $\exp L(\mathbf{X})$, where $L(\mathbf{X})$ is the set of Lie elements in the algebra of formal non-commutative power series.

The formula

$$\mathfrak{f}_{pq}(\sigma) = q^{-1} \cdot \mathfrak{f}_p(\sigma) \cdot q \cdot \mathfrak{f}_q(\sigma)$$

(see Part I Lemma 1.0.6), which after using suitable embeddings implies

$$\Lambda_{pq}(\sigma) = \Lambda_p(\sigma) \cdot \Lambda_q(\sigma)$$

is the analog of the formula ii).

We do not know how to show an analog of the formula iii) for "l-adic iterated integrals" (coefficients of the power series $\Lambda_p(\sigma)$). To complete the picture we are still missing several l-adic analogs in the following table.

classical iterated integrals	**l-adic iterated integrals**
values of Riemann zeta function	Soulé classes for $\mathbf{Q}$ at positive integers
multivalue zeta numbers	values of l-adic iterated integrals at 1 and at roots of 1
multivalue zeta functions	?

shuffle relations for multivalue zeta ?

numbers and multivalue zeta functions ?

The classical polylogarithms are the most important examples of iterated integrals. In section 11 we introduce l-adic polylogarithms and we study their properties. We prove a theorem saying when a linear combination of l-adic polylogarithms is a cocycle. The reader can compare our result with Proposition in section 4.6 of [BD]. In section 11 we study functional equations of l-adic polylogarithms. We show that the l-adic dilogarithm satisfies the distribution relation

$$m(\sum_{i=0}^{m-1} l_2(\xi_m^i z)) = l_2(z^m)$$

on the Galois group $G_{\mathbf{Q}(\mu_m)}$ and the Abel five term functional equation on $G_{\mathbf{Q}(\mu_{l^\infty})}$.

These results are stronger than those in Theorem D in the sense that we get functional equations on the Galois groups $G_{\mathbf{Q}(\mu_n)}$ and $G_{\mathbf{Q}(\mu_{l^\infty})}$, while the functional equations from Theorem D hold on the subgroup $\bigcap_{i=1}^N H_q(Y_i; f_i(z), f_i(x))$ of G_K.

In the preprint version there is no picture. Pictures are not necessary to read and to understand this paper though they can help.

§10. Functional equations.

10.0. Let $X = \mathbf{P}^1_K \setminus \{a_1, \dots, a_{n+1}\}$ and let $Y = \mathbf{P}^1_K \setminus \{b_1, \dots, b_{m+1}\}$. Let $f : X \to Y$ be a smooth morphism. Let $z, v \in \hat{X}(K)$. The morphism f induces

$$f_* : \pi_1(X_{\bar{K}}; v) \to \pi_1(Y_{\bar{K}}; f(v))$$

and

$$f_* : \pi(X_{\bar{K}}; z, v) \to \pi(Y_{\bar{K}}; f(z), f(v)).$$

Let us fix a a path p from v to z. We recall that for $\sigma \in G_K$ we have defined

$$\mathfrak{f}_p(\sigma) := p^{-1} \cdot \sigma(p).$$

Then we have

10.0.1. $$f_*(\mathfrak{f}_p(\sigma)) = \mathfrak{f}_{f(p)}(\sigma).$$

Let $x = (x_1, \dots, x_{n+1})$ (resp. $y = (y_1, \dots y_{m+1})$) be a sequence of geometric generators of $\pi_1(X(\mathbf{C}); v)$ (resp. $\pi_1(Y(\mathbf{C}); f(v))$. We set $\mathbf{X} := \{X_1, \dots, X_n\}$ and $\mathbf{Y} := \{Y_1, \dots, Y_n\}$. We recall that we have embeddings $k_x : \pi_1(X(\mathbf{C}); v) \to \mathbf{Q}_l\{\{\mathbf{X}\}\}$ and $k_y : \pi_1(Y(\mathbf{C}); f(v)) \to \mathbf{Q}_l\{\{\mathbf{Y}\}\}$ associated with a choice of sequences of geometric generators x of $\pi_1(X(\mathbf{C}); v)$ and y of $\pi_1(Y(\mathbf{C}); f(v))$. There is a homomorphism of $\mathbf{Q}_l$-algebras

$$f_\diamond : \mathbf{Q}_l\{\{\mathbf{X}\}\} \to \mathbf{Q}_l\{\{\mathbf{Y}\}\}$$

such that

10.0.2. $$f_\diamond \circ k_x = k_y \circ f_* \text{ and } f_\diamond \circ k_{x,p} = k_{y,f(p)} \circ f_*.$$

Let $\sigma \in G_{K(\mu_{l^\infty})}$. The equations 10.0.1 and 10.0.2 imply that

$$f_\diamond \circ \sigma_{x,p} = \sigma_{y,f(p)} \circ f_\diamond.$$

Hence we have

$$f_\diamond \circ \log\sigma_{x,p} = \log\sigma_{y,f(p)} \circ f_\diamond$$

and

10.0.3. $$f_\diamond((\log\sigma_{x,p})(1)) = (\log\sigma_{y,f(p)})(1).$$

The map $f_\diamond$ induces a homomorphism of Lie algebras

$$f_\diamond : L(\mathbf{X}) \to L(\mathbf{Y}).$$

Let

$$f_\bullet : \oplus_{i=1}^{\infty} gr^i_\Gamma L(\mathbf{X}) \to \oplus_{i=1}^{\infty} gr^i_\Gamma L(\mathbf{Y})$$

be the map induced by $f_\diamond$ on associated graded Lie algebras. The associated graded Lie algebras are canonically isomorphic to $\mathrm{Lie}(\mathbf{X})$ and $\mathrm{Lie}(\mathbf{Y})$. Hence the map $f_\diamond$ induces

$$f_\bullet : \mathrm{Lie}(\mathbf{X}) \to \mathrm{Lie}(\mathbf{Y}).$$

Let $\varphi \in \mathrm{Lie}(\mathbf{Y})^\diamond$ be a linear form of degree q. Let us set $a^{\varphi\circ f_\diamond}_{x,p} := \varphi(f_\diamond((\log\sigma_{x,p})(1)))$. (In Part I we defined coefficients $a^\varphi_{x,p}$ only for homogenous forms, hence we introduce this new definition.) It follows from 10.0.3 that

10.0.4. $$a^{\varphi\circ f_\diamond}_{x,p} = a^\varphi_{y,f(p)}.$$

The map $f_\diamond$ is not homogenous. Therefore we have

10.0.5. $$a^{\varphi\circ f_\diamond}_{x,p} = a^{\varphi\circ f_\bullet}_{x,p} + \sum_{deg\chi<q} a^\chi_{x,p}.$$

It follows from 10.0.4 and 10.0.5 that

10.0.6. $$\mathcal{L}^{\varphi\circ f_\bullet}(z,v) = \mathcal{L}^\varphi(f(z), f(v))$$

on the subgroup $H_q(X; z, v)$ of G_K.

Below we shall use fundamental groups of X or Y with various base points. Sequences of geometric generators and embeddings into algebras of non-commutative formal power series will be chosen as above.

Let v and v' belong to $\hat{X}(K)$. If $x = (x_1, \ldots, x_{n+1})$ is a sequence of geometric generators of $\pi_1(X(\mathbf{C}); v)$ and q is a path from v' to v then $q^{-1} \cdot x \cdot q := (q^{-1} \cdot x_1 \cdot q, \ldots, q^{-1} \cdot x_{n+1} \cdot q)$ is a sequence of geometric generators of $\pi_1(X(\mathbf{C}); v')$. Then we have embeddings

$$k_x : \pi_1(X(\mathbf{C}); v) \to \mathbf{Q}_l\{\{\mathbf{X}\}\}$$

given by $k_x(x_i) = e^{X_i}$ for $i = 1, \ldots, n$ and

$$k_{q^{-1} \cdot x \cdot q} : \pi_1(X(\mathbf{C}); v') \to \mathbf{Q}_l\{\{\mathbf{X}\}\}$$

given by $k_{q^{-1} \cdot x \cdot q}(q^{-1} \cdot x_i \cdot q) = e^{X_i}$ for $i = 1, \ldots, n$.

Theorem 10.0.7. *Let $f_i : X \to Y$ be a smooth morphism and let $\varphi_i \in L(Y_1, \ldots, Y_m)^\diamond$ be a linear form of degree q defined over $\mathbf{Q}$ for $i = 1, \ldots, N$. Let $n_1, \ldots, n_N$ be rational numbers. If*

$$\sum_{i=1}^{N} n_i \varphi_i \circ (f_i)_* = 0$$

in $\mathrm{Hom}\big(gr^q_\Gamma \pi_1(X(\mathbf{C}); x); \mathbf{Q}\big)$, where $(f_i)_* : gr^q_\Gamma \pi_1(X(\mathbf{C}); x) \to gr^q_\Gamma \pi_1(Y(\mathbf{C}), f_i(x))$ is the map induced by f_i for $i = 1, \ldots, N$, then we have functional equations

$$\sum_{i=1}^{N} n_i \mathcal{L}^{\varphi_i}(f_i(z); f_i(v)) = 0$$

on the subgroup $H_q(X; z, v)$ of G_K and

$$\sum_{i=1}^{N} n_i a^{\varphi_i}_{y, f_i(p)} = \text{lower degree terms}$$

on G_K, where "lower degree terms" means a linear combination of $a^\chi_{x,p}$ with degree of χ strictly smaller than q.

Proof. It follows from 10.0.6 that $\sum_{i=1}^N n_i \mathcal{L}^{\varphi_i}(f_i(z); f_i(v)) = \sum_{i=1}^N n_i \mathcal{L}^{\varphi_i \circ (f_i)_\bullet}(z, v) = \mathcal{L}^{\sum_{i=1}^N n_i \varphi_i \circ (f_i)_\bullet}(z, v) = 0$. It follows from 10.0.4 and 10.0.5 that $\sum_{i=1}^N n_i a^{\varphi_i}_{y, f_i(p)} = \sum_{i=1}^N n_i a^{\varphi_i \circ (f_i)_\diamond}_{x,p} = \sum_{i=1}^N n_i a^{\varphi_i \circ (f_i)_\bullet}_{x,p} + \text{lower degree terms} = a^{\sum_{i=1}^N n_i \varphi_i \circ (f_i)_\bullet}_{x,p} + \text{lower degree terms} = \text{lower degree terms}$.

10.1. Let p be a path from v to z. Let $x = (x_1, \ldots, x_{n+1})$ be a sequence of geometric generators of $\pi_1(X(\mathbf{C}); v)$. Then $x' := (p \cdot x_1 \cdot p^{-1}, \ldots, p \cdot x_{n+1} \cdot p^{-1})$ is a sequence of geometric generators of $\pi_1(X(\mathbf{C}); z)$. The action of $\sigma_{p^{-1}}$ on $\pi_1(X_{\bar{K}}; z)$ can be expressed in the following way by the action of σ_p on $\pi_1(X_{\bar{K}}; v)$. Let $\omega \in \pi_1(X_{\bar{K}}; z)$. Then $\sigma_{p^{-1}}(\omega) = p \cdot \sigma(p^{-1} \cdot \omega \cdot p) \cdot \mathfrak{f}_p(\sigma)^{-1} \cdot p^{-1}$. This implies that on $\mathbf{Q}_l\{\{\mathbf{X}\}\}$ we have

10.1.0. $$\sigma_{x,p} = L_{\Lambda_p(\sigma)} \circ \sigma_x \quad \text{and} \quad \sigma_{x',p^{-1}} = R_{\Lambda_p(\sigma)^{-1}} \circ \sigma_x.$$

Lemma 10.1.1. *Let D be a derivation of the algebra $\mathbf{Q}_l\{\{\mathbf{X}\}\}$ and let $\omega \in L(\mathbf{X})$. Then*

$$L_\omega \bigcirc D = L_\zeta + D \;\text{ and }\; R_{-\omega} \bigcirc D = R_{-\zeta} + D$$

for some $\zeta \in L(\mathbf{X})$.

Proof. The lemma follows from the identities

$$[L_\omega, D] = L_{-D(\omega)}, \qquad [R_{-\omega}, D] = R_{D(\omega)}$$

and

$$[L_\omega, L_{-D(\omega)}] = L_{-[\omega, D(\omega)]}, \qquad [R_{-\omega}, R_{D(\omega)}] = R_{[\omega, D(\omega)]}.$$

Theorem 10.1.2. *Let $z, v \in \hat{X}(K)$ and let p be a path from v to z. Then we have*

$$i) \;\; \mathcal{L}^e(z, v) + \mathcal{L}^e(v, z) = 0,$$

$$ii) \;\; a^e_{x,p} + a^e_{pxp^{-1},p^{-1}} = 0.$$

Proof. It follows from 10.1.0 that

$$(\log\sigma_{x',p^{-1}})(1) = (R_{-\log\Lambda_p(\sigma)} \bigcirc \log\sigma_x)(1).$$

It follows from Lemma 10.1.1 that

$$(R_{-\log\Lambda_p(\sigma)} \bigcirc \log\sigma_x)(1) = -(L_{\log\Lambda_p(\sigma)} \bigcirc \log\sigma_x)(1).$$

Hence we get that

$$(\log\sigma_{x',p^{-1}})(1) = -(\log\sigma_{x,p})(1).$$

Evaluating a linear form on both sides of the equation we get the theorem.

Theorem 10.1.3. *Let $z, y, v \in \hat{X}(K)$. Then we have*

$$\mathcal{L}^e(z,v) = \mathcal{L}^e(z,y) + \mathcal{L}^e(y,v).$$

Proof. Let p be a path from v to y, let r be a path from y to z and let $q = r \cdot p$. We have $\sigma_p = L_{\mathfrak{f}_p(\sigma)} \circ \sigma$ and $\sigma_q = L_{\mathfrak{f}_q(\sigma)} \circ \sigma$ on $\pi_1(X_{\bar{K}}; v)$ and $\sigma_r = L_{\mathfrak{f}_r(\sigma)} \circ \sigma$ on $\pi_1(X_{\bar{K}}, y)$. It follows from Lemma 1.0.6 that $\sigma_q = L_{p^{-1}\mathfrak{f}_r(\sigma)p} \circ \sigma_p$. Let us choose a sequence x of geometric generators of $\pi_1(X_{\bar{K}}, y)$. Then $x' = p^{-1} \cdot x \cdot p$ is a sequence of geometric generators of $\pi_1(X_{\bar{K}}; v)$. Observe that

$$\sigma_{x',q} = \sigma_{x,r} \circ \sigma_x^{-1} \circ \sigma_{x',p}.$$

Hence we get

$$\log\sigma_{x',q} = \log\sigma_{x,r} \bigcirc \log\sigma_x^{-1} \bigcirc \log\sigma_{x',p}.$$

Let σ belongs to the degree m step of the filtration defined in section 3, i.e., $\sigma \in \mathcal{K}_m^T(X)$ for some finite subset $T \subset \hat{X}(K)^2$. Then

$$(\log\sigma_{x',q})(1) \equiv (\log\sigma_{x,r})(1) + (\log\sigma_{x',p})(1) \mod \Gamma^{m+1}L(\mathbf{X}).$$

Evaluating a linear form of degree m on both sides of the congruence we get the theorem.

10.2. It follows from Proposition 7.1.10 that relations between functions $\mathcal{L}^e(z,v)$ imply relations between symbols $\{z,v\}_e$. Hence we get the following result

Corollary 10.2.1. *Assume that Conjectures D_n are true for all n. Assume that for all n the maps realization* : $\mathrm{Ext}^1_{\mathcal{MM}_K}(\mathbf{Q}(0), \mathbf{Q}(n)) \otimes \mathbf{Q} \to H^1(G_K, \mathbf{Q}_l(n))$ *are injective. Then we have*

$$\{z,v\}_e + \{v,z\}_e = 0$$

and

$$\{z,v\}_e = \{z,y\}_e + \{y,v\}_e$$

in $\mathcal{L}^K(X)$.

Proof. The corollary follows from Theorems 10.1.2 and 10.1.3 and Proposition 7.1.10.

10.3. Let π be a group. If π is nilpotent then we denote by $\pi \otimes \mathbf{Q}$ its rationalization. For an arbitrary group π, $\pi \otimes \mathbf{Q} := \varprojlim_n ((\pi/\Gamma^n\pi) \otimes \mathbf{Q})$ is a rational completion of π. The group $\pi_1(X_{\bar{K}}; v)$ is equipped with a pro-finite topology. Hence every quotient $\pi_1(X_{\bar{K}}; v)/\Gamma^n\pi_1(X_{\bar{K}}; v)$ is equipped with a pro-finite topology. Therefore rationalization $(\pi_1(X_{\bar{K}}; v)/\Gamma^n\pi_1(X_{\bar{K}}; v)) \otimes \mathbf{Q}$ is a $\mathbf{Q}_l$-Lie group. Hence $\pi_1(X_{\bar{K}}; v) \otimes \mathbf{Q} = \varprojlim_n ((\pi_1(X_{\bar{K}}; v)/\Gamma^n\pi_1(X_{\bar{K}}; v)) \otimes \mathbf{Q})$ is equipped with a topology of the inverse limit of $\mathbf{Q}_l$-Lie groups. The action of G_K on $\pi_1(X_{\bar{K}}; v)$ extends uniquely to a continous action of G_K on $\pi_1(X_{\bar{K}}; v) \otimes \mathbf{Q}$.

Now we shall define a rational completion of $\pi_1(X_{\bar{K}}; v)$-torsor $\pi(X_{\bar{K}}; z, v)$. We introduce an equivalence relation on the product $\pi(X_{\bar{K}}; z, v) \times \pi_1(X_{\bar{K}}; v) \otimes \mathbf{Q}$. We say that a pair (p, S) is equivalent to a pair (q, T) and we write $(p, S) \sim (q, T)$ if $S = (p^{-1} \cdot q) \cdot T$ in $\pi_1(X_{\bar{K}}; v) \otimes \mathbf{Q}$.

We set

$$\pi(X_{\bar{K}}; z, v) \otimes \mathbf{Q} := \big(\pi(X_{\bar{K}}; z, v) \times \pi_1(X_{\bar{K}}; v) \otimes \mathbf{Q}\big)/\sim .$$

The Galois group G_K acts on the product $\pi(X_{\bar{K}}; z, v) \times \pi_1(X_{\bar{K}}; v) \otimes \mathbf{Q}$ component wise. The group $\pi_1(X_{\bar{K}}; v) \otimes \mathbf{Q}$ acts on the product $\pi(X_{\bar{K}}; z, v) \times \pi_1(X_{\bar{K}}; v) \otimes \mathbf{Q}$ by the right multiplication on the second factor. Both actions are compatible with the equivalence relation $\sim$ and continous. Hence G_K acts on the set of equivalence classes $\pi(X_{\bar{K}}; z, v) \otimes \mathbf{Q}$. The action of $\pi_1(X_{\bar{K}}; v) \otimes \mathbf{Q}$ on the product $\pi(X_{\bar{K}}; z, v) \times \pi_1(X_{\bar{K}}; v) \otimes \mathbf{Q}$ induces a structure of $\pi_1(X_{\bar{K}}; v) \otimes \mathbf{Q}$-torsor on the set of equivalence classes $\pi(X_{\bar{K}}; z, v) \otimes \mathbf{Q}$. Elements of $\pi(X_{\bar{K}}; z, v) \otimes \mathbf{Q}$ have the form $p \cdot S$, where p is in $\pi(X_{\bar{K}}; z, v)$ and $S \in \pi_1(X_{\bar{K}}; v) \otimes \mathbf{Q}$. We shall call them $\mathbf{Q}_l$-paths.

Lemma 10.3.1. *The embedding $k_x : \pi_1(X_{\bar{K}}; v) \to \mathbf{Q}_l\{\{\mathbf{X}\}\}$ extends uniquely to a continous multiplicative embedding $\bar{k}_x : \pi_1(X_{\bar{K}}; v) \otimes \mathbf{Q} \to \mathbf{Q}_l\{\{\mathbf{X}\}\}$.*

Proof. The image of k_x is contained in $\mathbf{Q}_l\{\{\mathbf{X}\}\}^*$. The group $\mathbf{Q}_l\{\{\mathbf{X}\}\}^*$ is a pro-unipotent group with exponents in $\mathbf{Q}_l$. Hence k_x extends to $\bar{k}_x : \pi_1(X_{\bar{K}}; v) \otimes \mathbf{Q} \to \mathbf{Q}_l\{\{\mathbf{X}\}\}$.

Further we shall denote the embedding $\bar{k}_x$ by k_x. One shows that the formulas

$$\mathfrak{f}_{p\cdot q}(\sigma) = q^{-1} \cdot \mathfrak{f}_p(\sigma) \cdot q \cdot \mathfrak{f}_q(\sigma), \quad \Lambda_{p\cdot q}(\sigma) = \Lambda_p(\sigma) \cdot \Lambda_q(\sigma), \quad \Lambda_p(\tau \cdot \sigma) = \Lambda_p(\tau) \cdot \tau(\Lambda_p(\sigma))$$

and $g_*(\mathfrak{f}_p) = \mathfrak{f}_{g(p)}$, where $g : X_K \to X_K$ is a regular map, are valid also for $\mathbf{Q}_l$-paths p and q.

§11. l-adic polylogarithms.

11.0. In this subsection we introduce l-adic polylogarithms. We give sufficient conditions when a linear combination of l-adic polylogarithms is a cocycle. Next we are studying a relative version of l-adic polylogarithms. We also show that l-adic polylogarithms are special case of l-adic iterated integrals introduced in section 5.

Let K be a number field . Let $V = \mathbf{P}^1_K \setminus \{0, 1, \infty\}$. Let x and y be standard generators of $\pi_1(V_{\bar{K}}; \overrightarrow{01})$ - loops around 0 and 1 respectively (see the Picture 1).

Picture 1

Let $k : \pi_1(V_{\bar{K}}; \overrightarrow{01}) \otimes \mathbf{Q} \to \mathbf{Q}_l\{\{X, Y\}\}$ be a multiplicative continous embedding given by $k(x) = e^X$ and $k(y) = e^Y$. We denote by $\mathrm{Lie}(X, Y)$ a free Lie algebra over $\mathbf{Q}_l$ on X and Y and by $L(X, Y)$ a completion of $\mathrm{Lie}(X, Y)$ with respect to the lower central series. We identify $L(X, Y)$ with the Lie algebra of Lie elements in $\mathbf{Q}_l\{\{X, Y\}\}$.

Let us set $E_1 := Y,\ E_{k+1} := [E_k, X]$. Let $\mathcal{B}$ be a base of $\mathrm{Lie}(X, Y)$ given by basic Lie elements. We assume that $E_k \in \mathcal{B}$ for $k = 1, 2, \dots$.

Let $z \in \hat{V}(K)$ and let p be a $\mathbf{Q}_l$-path from $\overrightarrow{01}$ to z. We recall that $\mathfrak{f}_p(\sigma) = p^{-1} \cdot \sigma(p) \in \pi_1(V_{\bar{K}}; \overrightarrow{01}) \otimes \mathbf{Q}$ and $\Lambda_p(\sigma) := k(\mathfrak{f}_p(\sigma)) \in \mathbf{Q}_l\{\{X, Y\}\}$ for any $\sigma \in G_K$.

If $e \in \mathcal{B}$ we denote by e^* the dual linear form to e with respect to $\mathcal{B}$.

Definition 11.0.1. Let $\sigma \in G_K$. We set

$$l_n(z)(\sigma) := E_n^*(\log\Lambda_p(\sigma)) \text{ and } l(z)(\sigma) := X^*(\log\Lambda_p(\sigma)).$$

The coefficient $l_n(z)$ is an l-adic polylogarithm (n-th order l-adic polylogarithm) evaluated at z. It is a function from G_K to $\mathbf{Q}_l(n)$. It depends on a choice of p in $\pi(V_{\bar{K}}; z, \overrightarrow{01}) \otimes \mathbf{Q}$. The coefficient $l(z)$ is an l-adic logarithm evaluated at z. If we are using various paths and it is important to indicate the dependence of $l_n(z)$ (resp. $l(z)$) on a path p we shall write $l_n(z)_p$ (resp. $l(z)_p$).

Definition 11.0.2. We set

$$\mathcal{L}_n(z) := l_n(z)_{|H_n(V;z,\overrightarrow{01})}.$$

Observe that $\mathcal{L}_n(z)$ depends only on z.

Let us set $e_1 := y$ and $e_{k+1} := (e_k, x)$. Observe that any element $g \in \pi_1(V_{\bar{K}}; \overrightarrow{01}) \otimes \mathbf{Q}$ can be written in the following form

$$g = x^{\alpha^0(g)} \cdot y^{\alpha^1(g)} \cdot e_2^{\alpha^2(g)} \cdot e_3^{\alpha^3(g)} \cdot ((y,x)y)^{\beta(g)} \cdot e_4^{\alpha^4(g)} \cdot f_4 \cdot \ldots \cdot e_n^{\alpha^n(g)} \cdot f_n \cdot \ldots,$$

where the exponents are in $\mathbf{Q}_l$ and each f_n is a product of powers of commutators of length n, which contain y at least twice.

Definition 11.0.3. Let $\sigma \in G_K$. We define functions $\kappa_z^n : G_K \to \mathbf{Q}_l$ by the identity

$$\mathfrak{f}_p(\sigma) = x^{\kappa_z^0(\sigma)} \cdot y^{\kappa_z^1(\sigma)} \cdot e_2^{\kappa_z^2(\sigma)} \cdot e_3^{\kappa_z^3(\sigma)} \cdot f_3 \cdot e_4^{\kappa_z^4(\sigma)} \cdot f_4 \cdot \ldots \cdot e_n^{\kappa_z^n(\sigma)} \cdot f_n \cdot \ldots.$$

Let $n \geq 1$. Then κ_z^n we view as a function from G_K to $\mathbf{Q}_l(n)$. κ_z^0 we view as a function from G_K to $\mathbf{Q}_l(1)$. We shall also use the notation $\kappa_0(z) := \kappa_z^0$ and $\kappa_1(z) := \kappa_z^1$. If we are using various paths and it is important to indicate the dependence of $\kappa_z^n(\sigma)$ on a path p we shall write $\kappa_z^n(\sigma)_p$.

We shall express l-adic polylogarithms in terms of functions κ_z^n.

Let $f \in L(X,Y)$. We define a derivation $ad\, f$ of $L(X,Y)$ setting $(ad\, f)(g) = [f,g]$ for any $g \in L(X,Y)$.

Let I_k be a Lie ideal of $L(X,Y)$ generated topologically by Lie brackets which contain Y at least k-times.

Lemma 11.0.4. *We have*

$$\log(k(e_{n+1})) = (-1)^n \sum_{k_1,\ldots,k_n=1}^{\infty} \frac{1}{k_1! \cdot \ldots \cdot k_n!} (ad\, X)^{k_1+\ldots+k_n}(Y) \mod I_2.$$

Lemma 11.0.5. (see [B] chapitre II). *We have*

$$\log(e^X \cdot e^Y) = X + Y + \frac{1}{2}[X,Y] + \sum_{n=1}^{\infty} \frac{1}{(2n)!} B_{2n} (ad\, X)^{2n}(Y) \mod I_2.$$

Proposition 11.0.6. *Let $\sigma \in G_K$. We have*

$$\log\Lambda_p(\sigma) = \kappa_z^0(\sigma)X + \sum_{i=1}^{\infty}(-1)^{i-1}\kappa_z^i(\sigma)\big(\sum_{k_1,\dots,k_{i-1}=1}^{\infty}\frac{1}{k_1!\cdot\ldots\cdot k_{i-1}!}(ad\,X)^{k_1+\ldots+k_{i-1}}(Y)\big)$$

$$+\frac{1}{2}[\kappa_z^0(\sigma)X, \sum_{i=1}^{\infty}(-1)^{i-1}\kappa_z^i(\sigma)\big(\sum_{k_1,\dots,k_{i-1}=1}^{\infty}\frac{1}{k_1!\cdot\ldots\cdot k_{i-1}!}(ad\,X)^{k_1+\ldots+k_{i-1}}(Y))]+$$

$$\sum_{n=1}^{\infty}\frac{(\kappa_z^0(\sigma))^{2n}}{(2n)!}B_{2n}(ad\,X)^{2n}\big(\sum_{i=1}^{\infty}(-1)^{i-1}\kappa_z^i(\sigma)\big(\sum_{k_1,\dots,k_{i-1}=1}^{\infty}\frac{1}{k_1!\cdot\ldots\cdot k_{i-1}!}(ad\,X)^{k_1+\ldots+k_{i-1}}(Y)\big)\big)$$

mod I_2.

Proof. The proposition follows from Lemmas 11.0.4 and 11.0.5.

Using Proposition 11.0.6 we can easily calculate l-adic polylogarithms in terms of functions κ_z^n. For example in small degrees we get the following result.

Corollary 11.0.7. *We have*

$$l(z) = \kappa_z^0, \quad l_1(z) = \kappa_z^1, \quad l_2(z) = \kappa_z^2 - \frac{1}{2}\kappa_z^0\cdot\kappa_z^1$$

and

$$l_3(z) = \kappa_z^3 - \frac{1}{2}\kappa_z^0\cdot\kappa_z^2 + \frac{1}{12}(\kappa_z^0)^2\cdot\kappa_z^1 - \frac{1}{2}\kappa_z^2.$$

Proposition 11.0.8. *Let $\zeta \in \hat{V}(K)$ and let p be a $\mathbf{Q}_l$-path from $\overrightarrow{01}$ to ζ. Let q be the standard path from $\overrightarrow{01}$ to $\overrightarrow{10}$ (an interval $[0,1]$). Let $g : V_K \to V_K$ be given by $g(z) = 1-z$. Then we have*

$$l_1(\zeta)_p = l(1-\zeta)_{g(p)\cdot q}.$$

Proof. It follows from Corollary 11.0.7 that

$$\mathfrak{f}_p \equiv x^{l(\zeta)_p}\cdot y^{l_1(\zeta)_p} \mod \Gamma^2(\pi_1(V_{\bar{K}};\overrightarrow{01})\otimes\mathbf{Q}).$$

Observe that $g(p)\cdot q$ is a $\mathbf{Q}_l$-path from $\overrightarrow{01}$ to $1-\zeta$. Hence we have

$$\mathfrak{f}_{g(p)\cdot q} \equiv x^{l(1-\zeta)_{g(p)\cdot q}}\cdot y^{l_1(1-\zeta)_{g(p)\cdot q}} \mod \Gamma^2(\pi_1(V_{\bar{K}};\overrightarrow{01})\otimes\mathbf{Q}).$$

On the other side

$$\mathfrak{f}_{g(p)\cdot q} = q^{-1}\cdot\mathfrak{f}_{g(p)}\cdot q\cdot\mathfrak{f}_q = q^{-1}\cdot g_*(\mathfrak{f}_p)\cdot q\cdot\mathfrak{f}_q \equiv x^{l_1(\zeta)_p}\cdot y^{l(\zeta)_p} \mod \Gamma^2(\pi_1(V_{\bar{K}};\overrightarrow{01})\otimes\mathbf{Q})$$

because $q^{-1}\cdot g_*(x)\cdot q = y$, $q^{-1}\cdot g_*(y)\cdot q = x$ and $\mathfrak{f}_q \equiv 1 \mod \Gamma^2(\pi_1(V_{\bar{K}};\overrightarrow{01})\otimes\mathbf{Q})$. The proposition follows from the last two congruences.

Theorem 11.0.9. *Let $z_i \in \hat{V}(K)$, let $p_i \in \pi(V_{\bar{\mathbf{Q}}}; z_i, \overrightarrow{01}) \otimes \mathbf{Q}$ and let $n_i \in \mathbf{Q}_l$ for $i = 1, \dots, N$. Let us assume that l-adic polylogarithms $l_k(z_i)$ calculated along the $\mathbf{Q}_l$-paths p_i for $i = 1, \dots, N$ satisfy the following conditions*

i) $\sum_{i=1}^N n_i (l(z_i)(\tau))^\alpha \cdot (l(z_i)(\sigma))^\beta \cdot \big(l(z_i)(\tau) \cdot l_1(z_i)(\sigma) - l(z_i)(\sigma) \cdot l_1(z_i)(\tau)\big) = 0$ *for any* $\tau, \sigma \in G_K$ *and for any* α *and* β *such that* $\alpha + \beta = n - 2$,

ii) $\sum_{i=1}^N n_i (l(z_i)(\tau))^\alpha \cdot (l(z_i)(\sigma))^\beta \cdot l_k(z_i)(\sigma) = 0$ *for any* $\tau, \sigma \in G_K$, *for* $k = 2, \dots, n-1$ *and for any* α *and* β *such that* $\alpha + \beta = n - k$.

Then $\sum_{i=1}^N n_i l_n(z_i)$ *is a cocycle on* G_K *with values in* $\mathbf{Q}_l(n)$.

Proof. The equality $\Lambda_p(\tau\sigma) = \Lambda_p(\tau) \cdot \tau(\Lambda_p(\sigma))$ implies

$$\log\Lambda_p(\tau\sigma) = \log\Lambda_p(\tau) + \log\tau(\Lambda_p(\sigma)) + \frac{1}{2}[\log\Lambda_p(\tau), \log(\tau(\Lambda_p(\sigma)))] -$$

$$\frac{1}{12}[[\log\Lambda_p(\tau), \log(\tau(\Lambda_p(\sigma)))], \log\Lambda_p(\tau)] + \frac{1}{12}[[\log\Lambda_p(\tau), \log(\tau(\Lambda_p(\sigma)))], \log(\tau(\Lambda_p(\sigma)))]$$

$$-\frac{1}{24}[[[\log\Lambda_p(\tau), \log(\tau(\Lambda_p(\sigma)))], \log(\tau(\Lambda_p(\sigma)))]\log\Lambda_p(\tau)] + \dots.$$

Comparing coefficients at E_n we get

$$l_n(z)(\tau\sigma) = l_n(z)(\tau) + \chi(\tau)^n l_n(z)(\sigma) + \frac{1}{2}\big(l_{n-1}(z)(\tau)\chi(\tau)l(z)(\sigma) - \chi(\tau)^{n-1}l_{n-1}(z)(\sigma)l(z)(\tau)\big) -$$

$$\tfrac{1}{12}\big(l_{n-2}(z)(\tau)\chi(\tau)l(z)(\tau)l(z)(\sigma) - \chi(\tau)^{n-2}l_{n-2}(z)(\sigma)(l(z)(\tau))^2\big) +$$

$$\frac{1}{12}\big(l_{n-2}(z)(\tau)\chi(\tau)^2(l(z)(\sigma))^2 - \chi(\tau)^{n-1}l_{n-2}(z)(\sigma)l(z)(\tau)l(z)(\sigma)\big) -$$

$$\frac{1}{24}\big(l_{n-3}(z)(\tau)\chi(\tau)^2 l(z)(\tau)(l(z)(\sigma))^2 - \chi(\tau)^{n-2}l_{n-3}(z)(\sigma)(l(z)(\tau))^2 l(z)(\sigma)\big) + \dots.$$

The assumptions of the theorem imply that

$$\sum_{i=1}^N n_i l_n(z_i)(\tau\sigma) = \sum_{i=1}^N n_i l_n(z_i)(\tau) + \chi(\tau)^n \sum_{i=1}^N n_i l_n(z_i)(\sigma).$$

The l-adic polylogarithm $l_n(z)_p$ depends on a choice of a $\mathbf{Q}_l$-path from $\overrightarrow{01}$ to z. We have the following elementary result.

Lemma 11.0.10. *Let p be a $\mathbf{Q}_l$-path from $\overrightarrow{01}$ to z and let $S \in \pi_1(V_{\bar{K}}, \overrightarrow{01}) \otimes \mathbf{Q}$. If $S = x^\alpha \cdot y^\beta$ mod $\Gamma^2(\pi_1(V_{\bar{K}}; \overrightarrow{01}) \otimes \mathbf{Q})$ then $l(z)_{pS} = l(z)_p + \alpha(\chi - 1)$ and $l_1(z)_{pS} = l_1(z)_p + \beta(\chi - 1)$.*

Proof. We have $\mathfrak{f}_{pS}(\sigma) = S^{-1} \cdot \mathfrak{f}_p(\sigma) \cdot \sigma(S)$. Hence $\Lambda_{pS}(\sigma) = k(S)^{-1} \cdot \Lambda_p(\sigma) \cdot k(\sigma(S))$. Let $S = x^\alpha \cdot y^\beta \cdot e_2^{\beta_2} \cdot e_3^{\beta_3} \cdot f_3 \cdot e_4^{\beta_4} \cdot \ldots$. Therefore $\log\Lambda_{pS}(\sigma) = \big(-\log(e^{\alpha X} \cdot e^{\beta Y} \cdot (e^X \cdot e^Y \cdot e^{-X} \cdot e^{-Y})^{\beta_2} \cdot \ldots)\big) \bigcirc \log\Lambda_p(\sigma) \bigcirc \log(e^{\alpha\chi(\sigma)X} \cdot e^{\beta\chi(\sigma)Y} \cdot (e^{\chi(\sigma)X} \cdot e^{\chi(\sigma)Y} \cdot e^{-\chi(\sigma)X} \cdot e^{-\chi(\sigma)Y})^{\beta_2} \cdot \ldots) \equiv -\alpha X - \beta Y + l(z)_p X + l_1(z)_p Y + \alpha\chi(\sigma)X + \beta\chi(\sigma)Y \bmod \Gamma^2 L(X,Y)$. The lemma follows from the congruence.

Theorem 11.0.11. *Let $z_i \in V(K)$, let $p_i \in \pi(V_{\bar{K}}; z_i, \overrightarrow{01}) \otimes \mathbf{Q}$ and let $n_i \in \mathbf{Q}$ for $i = 1, \ldots, N$. Let $\mathcal{S}$ be a subgroup of $K^* \otimes \mathbf{Q}$ generated by z_i and $1 - z_i$ for $i = 1, \ldots, N$. Assume that*

- *i) the map $\varphi : \mathcal{S} \to Z^1(G_K; \mathbf{Q}_l(1))$ given by $\varphi(z_i) = l(z_i)_{p_i}$ and $\varphi(1 - z_i) = l_1(z_i)_{p_i}$ is well defined and it is a homomorphism;*
- *ii) $\sum_{i=1}^N n_i \nu_1(z_i) \cdot \ldots \cdot \nu_{n-2}(z_i)(z_i) \wedge (1 - z_i) = 0$ in $(\mathcal{S} \wedge \mathcal{S}) \otimes \mathbf{Q}_l$ for any homomorphisms $\nu_1, \ldots, \nu_{n-2}$ from $\mathcal{S}$ to $\mathbf{Q}_l$;*
- *iii) $\sum_{i=1}^N n_i \cdot \nu_1(z_i)^\alpha \cdot \nu_2(z_i)^\beta \cdot l_k(z_i)(\sigma) = 0$ for any homomorphisms ν_1 and ν_2 from $\mathcal{S}$ to $\mathbf{Q}_l$, for any $\sigma \in G_K$, for $k = 2, \ldots, n-1$ and for any α and β such that $\alpha + \beta = n - k$.*

Then $\sum_{i=1}^N n_i l_n(z_i)_{p_i}$ is a cocycle on G_K with values in $\mathbf{Q}_l(n)$.

Proof. Let us fix $\tau \in G_K$. The map $\mathcal{S} \to \mathbf{Q}_l(1)$ given by $s \to \varphi(s)(\tau)$ $(z_i \to l(z_i)(\tau))$ is a homomorphism. Let us fix $\tau, \sigma \in G_K$. The map $\mathcal{S} \otimes \mathcal{S} \to \mathbf{Q}_l(2)$, $x \otimes y \to \varphi(x)(\tau) \cdot \varphi(y)(\sigma) - \varphi(x)(\sigma) \cdot \varphi(y)(\tau)$ $\big(z_i \otimes (1 - z_i) \to l(z_i)(\tau) \cdot l_1(z_i)(\sigma) - l(z_i)(\sigma) \cdot l_1(z_i)(\tau)\big)$ factors through $\mathcal{S} \wedge \mathcal{S}$. Hence the theorem follows from Theorem 11.0.9.

Corollary 11.0.12. *Let ξ_m be a m-th root of 1 different from 1. There is a $\mathbf{Q}_l$-path p from $\overrightarrow{01}$ to ξ_m such that $l_n(\xi_m)_p$ is a cocycle on $G_{\mathbf{Q}(\mu_m)}$. If l does not divide m then one can choose the path p in $\pi(V_{\bar{\mathbf{Q}}}; \xi_m, \overrightarrow{01})$.*

Proof. Let $m = l^{k_0} \cdot r$, where l does not divide r. Let q be a path from $\overrightarrow{01}$ to ξ_m. There are α, β and γ in $\mathbf{Z}_l$ such that $(\xi^\alpha_{l^{k_0+n}} \cdot \xi_r^{\frac{\beta}{l^n}} \cdot \xi^\gamma_{l^n})_{n \in \mathbf{N}}$ is a compatible family of l^n-th roots of ξ_m determined by the path q. Hence $l(\xi_m)_q = (\frac{\alpha}{k_0} + \gamma)(\chi - 1)$. Lemma 11.0.10 implies that there is a $\mathbf{Q}_l$-path p from $\overrightarrow{01}$ to ξ_m such that $l(\xi_m)_p = 0$. Theorem 11.0.9 implies that $l_n(\xi_m)_p$ is a cocycle. Observe that if $k_0 = 0$ then one can choose p in $\pi(V_{\bar{\mathbf{Q}}}; \xi_m, \overrightarrow{01})$.

The classical polylogarithms are iterated integrals defined by $\int_0^z \frac{dz}{1-z}, \frac{dz}{z}, \ldots, \frac{dz}{z}$. The iterated integral $\int_a^b \frac{dz}{1-z}, \frac{dz}{z}, \ldots, \frac{dz}{z}$ can be express by classical polylogarithms. Now we shall define a normalized analog of the iterated integral $\int_a^b \frac{dz}{1-z}, \frac{dz}{z}, \ldots, \frac{dz}{z}$.

Let $z, v \in \hat{V}(K)$. Let q be a path from $\overrightarrow{01}$ to v and let p be a path from v to z. We shall define relative polylogarithms $l_n(z, v)$. Let us set $x_1 := q \cdot x \cdot q^{-1},\ y_1 := q \cdot y \cdot q^{-1}$. Observe that x_1, y_1 are generators of $\pi_1(V_{\bar{K}}; v)$. Let $G_{n+1} \subset \pi_1(V_{\bar{K}}; \overrightarrow{01})$ (resp. $G'_{n+1} \subset \pi_1(V_{\bar{K}}; v)$) be a closed normal subgroup generated by $\Gamma^{n+1}\pi_1(V_{\bar{K}}; \overrightarrow{01})$ (resp. $\Gamma^{n+1}\pi_1(V_{\bar{K}}; v)$) and all commutators which contain y (resp. y_1) at least twice. Let $\pi := \pi_1(V_{\bar{K}}; \overrightarrow{01})/G_{n+1}$ and $\pi' := \pi_1(V_{\bar{K}}; v)/G'_{n+1}$.

It follows from Proposition 2.2.1 that the action of G_K on π' is given by

$$\sigma(x_1) = (q \cdot \mathfrak{f}_q(\sigma) \cdot q^{-1}) \cdot x_1^{\chi(\sigma)} \cdot (q \cdot (\mathfrak{f}_q(\sigma))^{-1} \cdot q^{-1}) \mod G'_{m+1}$$

and

$$\sigma(y_1) = (q \cdot \mathfrak{f}_q(\sigma) \cdot q^{-1}) \cdot y_1^{\chi(\sigma)} \cdot (q \cdot (\mathfrak{f}_q(\sigma))^{-1} \cdot q^{-1}) \mod G'_{m+1}.$$

Lemma 11.0.13. *The action of G_K on $\pi_1(V_{\bar{K}}; v)$ induced from the action on the torsor $\pi(V_{\bar{Q}}; z, v)$ by the isomorphism t_p* (see Part I Section 1) *is given by*

$$\sigma_p(w) = (q \cdot \mathfrak{f}_{pq}(\sigma) \cdot q^{-1}) \cdot \bar{\sigma}(w) \cdot (q \cdot (\mathfrak{f}_q(\sigma))^{-1} \cdot q^{-1}) \mod G'_{n+1},$$

where $\bar{\sigma}(x_1) = x_1^{\chi(\sigma)}$, $\bar{\sigma}(y_1) = y_1^{\chi(\sigma)}$ and $\bar{\sigma}$ is continous and multiplicative.

Proof. The formula for $\sigma_p(w)$ follows from Lemma 1.0.2 and Lemma 1.0.6.

Let I be the augmentation ideal of $\mathbf{Q}_l\{\{X, Y\}\}$ and let J_{n+1} be a closed ideal of $\mathbf{Q}_l\{\{X, Y\}\}$ generated by I^{n+1} and all monomials which contain Y at least twice. We define two maps

$$k : \pi_1(V_{\bar{K}}; \overrightarrow{01}) \to \mathbf{Q}_l\{\{X, Y\}\}/J_{n+1} \text{ and } k' : \pi_1(V_{\bar{K}}; v) \to \mathbf{Q}_l\{\{X, Y\}\}/J_{n+1}$$

by $k(x) = e^X,\ k(y) = e^Y$ and $k'(x_1) = e^X,\ k'(y_1) = e^Y$.

Let $(\ )_p : G_K \to GL\left(\mathbf{Q}_l\{\{X,Y\}\}/J_{n+1}\right)$ be the action of G_K induced from the action of G_K on the torsor $\pi(V_{\bar{\mathbf{Q}}}; z, v)$ by the isomorphism t_p and the embedding k'.

Let us set

$$\psi_p(\sigma) := \sigma_p \circ \rho(\chi(\sigma)^{-1}).$$

We recall that $E_1 := Y$ and $E_{k+1} := [E_k, X]$ for $k = 1, \dots, n-1$. Then any Lie element of $\mathbf{Q}_l\{\{X,Y\}\}/J_{n+1}$ is a linear combination with $\mathbf{Q}_l$ coefficients of $X, E_1, \dots, E_n$. If $g \in \pi'$ then $\log k'(g)$ is a Lie element of $\mathbf{Q}_l\{\{X,Y\}\}/J_{n+1}$.

Definition 11.0.14. Let $\sigma \in G_K$. We set

$$(\log\psi_p(\sigma))(1) = l(z,v)_p(\sigma)X + \sum_{k=1}^{n} l_k(z,v)_p(\sigma)E_k.$$

Proposition 11.0.15. *We have*

$$l_n(z,v)_p = l_n(z)_{pq} - l_n(v)_q.$$

Proof. Observe that $k'(q \cdot \mathfrak{f}_{pq}(\sigma) \cdot q^{-1}) = k(\mathfrak{f}_{pq}(\sigma)) = \Lambda_{pq}(\sigma)$ and $k'(q \cdot (\mathfrak{f}_q(\sigma))^{-1} \cdot q^{-1}) = k((\mathfrak{f}_q(\sigma))^{-1}) = (\Lambda_q(\sigma))^{-1}$. Let $\sigma \in G_K$. It follows from Lemma 11.0.13 that

$$\psi_p(\sigma) = L_{\Lambda_{pq}(\sigma)} \circ R_{(\Lambda_q(\sigma))^{-1}}.$$

This implies that

$$\log\psi_p(\sigma) = L_{\log\Lambda_{pq}(\sigma)} \bigcirc R_{-\log\Lambda_q(\sigma)}.$$

The operators $L_{\log\Lambda_{pq}(\sigma)}$ and $R_{-\log\Lambda_q(\sigma)}$ commute. Hence $\log\psi_p(\sigma) = L_{\log\Lambda_{pq}(\sigma)} + R_{-\log\Lambda_q(\sigma)}$. This implies the proposition.

Corollary 11.0.16. *We have*

$$l_n(z, \overrightarrow{01})_p = l_n(z)_p.$$

Proof. It follows from Proposition 11.0.15 that $l_n(z,\overrightarrow{01})_p = l_n(z)_p - l_n(\overrightarrow{01})_c$, where c is a constant path. For such a path $l_n(\overrightarrow{01})_c = 0$.

Remark. The relative polylogarithm $l_n(z,v)$ is the function $a_p^{E_n}$ from section 5. Hence the l-adic polylogarithm $l_n(z)_p$ is also a special case of l-adic iterated integrals defined in section 5.

We finish this subsection with a result expressing coefficiets of $\mathfrak{f}_p$ in degree one for an arbitrary X by l-adic logarithms.

Proposition 11.0.17. *Let* $X = \mathbf{P}^1_K \setminus \{a_1, \dots, a_n, \infty\}$*, let* $z, v \in \hat{X}(K)$ *and let* p *be a path from* v *to* z*. Let* $g_i : X \to \mathbf{P}^1_K \setminus \{0, \infty\}$ *be given by* $g_i(z) = z - a_i$ *for* $i = 1, \dots, n$*. Then*

$$\mathfrak{f}_p \equiv x_1^{l(z-a_1)_{g_1(p)\cdot q_1} - l(v-a_1)_{q_1}} \cdot \ldots \cdot x_n^{l(z-a_n)_{g_n(p)\cdot q_n} - l(v-a_n)_{q_n}} \mod \Gamma^2\pi_1(X_{\bar{K}}; v),$$

where q_i *is any path from* $\overrightarrow{01}$ *to* $v - a_i$ *on* $\mathbf{P}^1_{\bar{K}} \setminus \{0, \infty\}$ *for* $i = 1, \dots, n$*.*

Proof. Without loss of generality we can suppose that $X = \mathbf{P}^1_K \setminus \{a, \infty\}$ and $g : X \to \mathbf{P}^1_K \setminus \{0, \infty\}$ is given by $g(z) = z - a$. Let p be a path from v to z on $X_{\bar{K}}$. Then $g(p)$ is a path from $v - a$ to $z - a$ on $\mathbf{P}^1_K \setminus \{0, \infty\}$. Let q be any path from $\overrightarrow{01}$ to $v - a$ on $\mathbf{P}^1_{\bar{K}} \setminus \{0, \infty\}$. We have

$$\mathfrak{f}_{g(p)\cdot q} = q^{-1} \cdot \mathfrak{f}_{g(p)} \cdot q \cdot \mathfrak{f}_q = q^{-1} \cdot g_*(\mathfrak{f}_p) \cdot q \cdot \mathfrak{f}_q.$$

It follows from Corollary 11.0.7 that

$$x^{l(z-a)_{g(p)\cdot q}} = q^{-1} \cdot g_*(\mathfrak{f}_p) \cdot q \cdot x^{l(v-a)_q},$$

where x is a loop around 0. Hence we get that $\mathfrak{f}_p = x_a^{l(z-a)_{g(p)\cdot q} - l(v-a)_q}$, where $g_*(x_a) = q \cdot x \cdot q^{-1}$.

11.1. In this subsection we shall study functional equations of l-adic polylogarithms. We shall prove the distribution relation and the Abel five term equation for l-adic dilogarithms. We shall show that l-adic dilogarithms satisfy these functional equations without lower degree terms.

We start with the discussion of the l-adic analog of the functional equation

$$\log(x \cdot y) = \log x + \log y$$

of the classical logarithm.

Proposition 11.1.0. *Let $\zeta, y \in \mathbf{P}^1(K) \setminus \{0,\infty\}$. Then there exist paths γ from $\overrightarrow{01}$ to ζ, δ from $\overrightarrow{01}$ to y and φ from $\overrightarrow{01}$ to $y \cdot \zeta$ such that*

$$l(y\cdot\zeta)_\varphi = l(y)_\delta + l(\zeta)_\gamma$$

on G_K.

Proof. Let $g : \mathbf{P}^1_K \setminus \{0,\infty\} \to \mathbf{P}^1_K \setminus \{0,\infty\}$ be given by $g(z) = y \cdot z$. Let p be a path from $\overrightarrow{01}$ to ζ. Then $g(p)$ is a path from $\overrightarrow{0y}$ to $y\cdot\zeta$. We recall that x is a standard generator of $\pi_1(\mathbf{P}^1_{\bar K} \setminus \{0,\infty\}, \overrightarrow{01})$. Let us fix a path q from $\overrightarrow{01}$ to $\overrightarrow{0y}$. Let us set $x' = q\cdot x\cdot q^{-1}$. Then x' is a generator of $\pi_1(\mathbf{P}^1_{\bar K} \setminus \{0,\infty\}, \overrightarrow{0y})$. Observe that $g_*(x) = x'$. It follows from Corollary 11.0.7 that

$$\mathfrak{f}_p(\sigma) = x^{l(\zeta)_p(\sigma)} \quad \text{and} \quad \mathfrak{f}_{g(p)\cdot q}(\sigma) = x^{l(y\cdot\zeta)_{g(p)\cdot q}(\sigma)}.$$

On the other side we have

$$\mathfrak{f}_{g(p)\cdot q}(\sigma) = q^{-1}\cdot\mathfrak{f}_{g(p)}(\sigma)\cdot q\cdot\mathfrak{f}_q(\sigma) = q^{-1}\cdot g_*(\mathfrak{f}_p(\sigma))\cdot q\cdot\mathfrak{f}_q(\sigma) = x^{l(\zeta)_p(\sigma)}\cdot x^{l(\overrightarrow{0y})_q(\sigma)} = x^{l(\zeta)_p(\sigma)+l(\overrightarrow{0y})_q(\sigma)}.$$

Comparing exponents we get

$$l(y\cdot\zeta)_{g(p)\cdot q} = l(\zeta)_p + l(\overrightarrow{0y})_q.$$

Let t be the canonical path from $\overrightarrow{0y}$ to y. Then $t\cdot q$ is a path from $\overrightarrow{01}$ to y (see Picture 2).

Picture 2

We have $x^{l(y)_{t\cdot q}(\sigma)} = \mathfrak{f}_{t\cdot q}(\sigma) = q^{-1}\cdot\mathfrak{f}_t(\sigma)\cdot q\cdot\mathfrak{f}_q(\sigma) = q^{-1}\cdot\mathfrak{f}_t(\sigma)\cdot q\cdot x^{l(\overrightarrow{0y})_q(\sigma)}$. Without loss of generality we can suppose that $y = 1$ and t is the canonical path from $\overrightarrow{01}$ to 1. Then it is clear that $\mathfrak{f}_t(\sigma) = 1$. Hence $l(\overrightarrow{0y})_q = l(y)_{t\cdot q}$. This finishes the proof of the proposition.

Now we shall discuss the l-adic analog of the functional equation

$$Li_2(z^m) = m\left(\sum_{i=0}^{m} Li_2(\xi_m^i z)\right)$$

of the classical dilogarithm. Let $Y = \mathbf{P}^1_{\mathbf{Q}(\mu_m)} \setminus \{0,\mu_m,\infty\}$ and $V = \mathbf{P}^1_{\mathbf{Q}(\mu_m)} \setminus \{0,1,\infty\}$. We choose generators $x', y_0, \ldots, y_{m-1}$ of $\pi_1(Y_{\bar{\mathbf{Q}}}, \overrightarrow{01})$ as on the picture.

Picture 3

Let $f : Y \to V$ be given by $f(z) = z^m$. We have $f_*(x') = x^m$, $f_*(y_0) = y$ and $f_*(y_i) = x^{-i} \cdot y \cdot x^i$. Let $z \in \hat{Y}(\mathbf{Q}(\mu_m))$ and let p be a path from $\overrightarrow{01}$ to z. We define functions $\lambda(z)$, $\mu_0(z), \ldots, \mu_{m-1}(z)$, $\nu_0(z), \ldots, \nu_{m-1}(z)$ from $G_{\mathbf{Q}(\mu_m)}$ to $\mathbf{Z}_l$ by the following congruence

11.1.1. $$\mathfrak{f}_p \equiv x'^{\lambda(z)} \cdot y_0^{\mu_0(z)} \cdot y_1^{\mu_1(z)} \cdot \ldots \cdot y_{m-1}^{\mu_{m-1}(z)} \cdot (y_0, x')^{\nu_0(z)} \cdot \ldots \cdot (y_{m-1}, x')^{\nu_{m-1}(z)} \cdot$$

$\prod_{i<j} (y_i, y_j)^{\alpha_{ij}(z)} \mod \Gamma^3 \pi_1(Y_{\bar{\mathbf{Q}}}; \overrightarrow{01})$.

Observe that $f(p)$ is a path from $\overrightarrow{01}$ to z^m. Hence we have

$$\mathfrak{f}_{f(p)} \equiv x^{\kappa^0_{z^m}} \cdot y^{\kappa^1_{z^m}} \cdot (y, x)^{\kappa^2_{z^m}} \mod \Gamma^3 \pi_1(V_{\bar{\mathbf{Q}}}; \overrightarrow{01}),$$

(see Definition 11.0.3.)

Lemma 11.1.2. *We have* $\kappa^0_{z^m} = m\lambda(z)$, $\kappa^1_{z^m} = \mu_0(z) + \mu_1(z) + \ldots + \mu_{m-1}(z)$ *and* $\kappa^2_{z^m} = m(\nu_0(z) + \ldots + \nu_{m-1}(z)) + \mu_1(z) + \ldots + i\mu_i(z) + \ldots + (m-1)\mu_{m-1}(z)$.

Proof. We have

$$f_*\mathfrak{f}_p \equiv x^{m\lambda(z)} \cdot y^{\mu_0(z)} \cdot x^{-1} \cdot y^{\mu_1(z)} \cdot x \cdot \ldots \cdot x^{-(m-1)} \cdot y^{\mu_{m-1}(z)} \cdot x^{m-1} \cdot (y, x)^{m(\nu_0(z) + \ldots + \nu_{m-1}(z))} \equiv$$

$$x^{m\lambda(z)} \cdot y^{\mu_0(z) + \ldots + \mu_{m-1}(z)} \cdot (y, x)^{m(\nu_0(z) + \ldots + \nu_{m-1}(z)) + \sum_{i=0}^{m-1} i\mu_i(z)} \mod \Gamma^3 \pi_1(V_{\bar{\mathbf{Q}}}; \overrightarrow{01}).$$

Observe that $f_*\mathfrak{f}_p = \mathfrak{f}_{f(p)}$. Comparing exponents of $f_*\mathfrak{f}_p$ and $\mathfrak{f}_{f(p)}$ we get the equalities of the lemma.

Let q_i be a path from $\overrightarrow{0\xi_m^i}$ to $\overrightarrow{01}$ as on Picture 4.

Picture 4

Let us set $x_i := q_i^{-1} \cdot x' \cdot q_i$ and $y_k^{(i)} := q_i^{-1} \cdot y_k \cdot q_i$. Let $f_i : Y \to V$ be given by $f_i(z) = \xi_m^{-i} \cdot z$. Observe that $(f_i)_* \overrightarrow{0\xi_m^i} = \overrightarrow{01}$, $(f_i)_*(x_i) = x$, $(f_i)_*(y_i^{(i)}) = y$ and $(f_i)_*(y_k^{(i)}) = 1$ for $k \neq i$.

Lemma 11.1.3. *We have*

$$\kappa^0_{\xi_m^{-i}z} = \lambda(z) + \frac{i}{m}(1-\chi), \;\; \kappa^1_{\xi_m^{-i}z} = \mu_i(z) \;\text{ and }\; \kappa^2_{\xi_m^{-i}z} = \nu_i(z) + \frac{i}{m}(1-\chi)\mu_i(z).$$

Proof. $f_i(pq_i)$ is a path from $\overrightarrow{01}$ to $\xi_m^{-i}z$. Hence we have

$$\mathfrak{f}_{f_i(pq_i)} \equiv x^{\kappa^0_{\xi_m^{-i}z}} \cdot y^{\kappa^1_{\xi_m^{-i}z}} \cdot (y,x)^{\kappa^2_{\xi_m^{-i}z}} \mod \Gamma^3\pi_1(V_{\bar{\mathbf{Q}}}; \overrightarrow{01}),$$

by the Definition 11.0.3. On the other side

$$(f_i)_*\mathfrak{f}_{pq_i} = (f_i)_*(q_i^{-1} \cdot \mathfrak{f}_p \cdot q_i) \cdot (f_i)_*(\mathfrak{f}_{q_i}).$$

Hence it follows from 11.1.1 that

$$(f_i)_*\mathfrak{f}_{pq_i} \equiv x^{\lambda(z)} \cdot y^{\mu_i(z)} \cdot (y,x)^{\nu_i(z)} \cdot x^{\frac{i}{m}(1-\chi)} \equiv x^{\lambda(z)+\frac{i}{m}(1-\chi)} \cdot y^{\mu_i(z)} \cdot (y,x)^{\nu_i(z)+\frac{i}{m}(1-\chi)\mu_i(z)}$$

mod $\Gamma^3\pi_1(V_{\bar{\mathbf{Q}}}; \overrightarrow{01})$.
We have the identity

$$(f_i)_*\mathfrak{f}_{pq_i} = \mathfrak{f}_{f_i(pq_i)}.$$

Hence comparing exponents of $(f_i)_*\mathfrak{f}_{pq_i}$ and $\mathfrak{f}_{f_i(pq_i)}$ we get the equalities of the lemma.

Proposition 11.1.4. *Let $l_2(z^m)$ be calculated along the path $f(p)$ and let $l_2(\xi_m^{-i}z)$ be calculated along the $\mathbf{Q}_l$-path $f_i(pq_i) \cdot x^{\frac{i}{m}}$ for $i = 0, 1, \ldots, m-1$. Then we have*

$$l_2(z^m) = m(\sum_{i=0}^{m-1} l_2(\xi_m^{-i}z)).$$

Proof. It follows from Corollary 11.0.7 that

$$l_2(z)_p(\sigma) = \kappa_z^2(\sigma)_p - \frac{1}{2}\kappa_z^0(\sigma)_p \cdot \kappa_z^1(\sigma)_p.$$

Hence it follows from Lemma 11.1.2 that

$$l_2(z^m)_{f(p)} = m(\sum_{i=0}^{m-1} \nu_i(z)) + \sum_{i=0}^{m-1} i\mu_i(z) - \frac{1}{2}m\lambda(z)(\sum_{i=0}^{m-1} \mu_i(z)).$$

Let us calculate $l_2(\xi_m^{-i}z)_{f_i(pq_i)\cdot x^{\frac{i}{m}}}$. We have

$$\mathfrak{f}_{f_i(pq_i)\cdot x^{\frac{i}{m}}}(\sigma) = x^{-\frac{i}{m}} \cdot \mathfrak{f}_{f_i(pq_i)}(\sigma) \cdot x^{\frac{i}{m}\chi(\sigma)}.$$

Hence it follows from Lemma 11.1.3 that

$$l_2(\xi_m^{-i}z)_{f_i(pq_i)x^{\frac{i}{m}}} = \nu_i(z) + \frac{i}{m}\mu_i(z) - \frac{1}{2}\lambda(z)\mu_i(z).$$

Comparing formulas for $l_2(z^m)_{f(p)}$ and $l_2(\xi_m^{-i}z)_{f_i(pq_i)\cdot x^{\frac{i}{m}}}$ we get

$$l_2(z^m)_{f(p)} = m(\sum_{i=0}^{m-1} l_2(\xi_m^{-i}z)_{f_i(pq_i)\cdot x^{\frac{i}{m}}}).$$

The classical dilogarithm satisfy the functional equation

$$Li_2(\frac{(1-y)z}{z-1}) - Li_2(yz) + Li_2(\frac{(z-1)y}{1-y}) - Li_2(\frac{y}{y-1}) + Li_2(z) = \textit{lower degree terms.}$$

We shall prove its l-adic analog.

Let $V = \mathbf{P}^1_K \setminus \{0,1,\infty\}$ and let $Y = \mathbf{P}^1_K \setminus \{0,1,\frac{1}{y},\infty\}$, where $y \in K \setminus \{0,1\}$. Let $\xi, \zeta \in \hat{V}(K)$ and let π be a path from ξ to ζ and let ρ be a path from $\overrightarrow{01}$ to ξ (see Picture 5).

Picture 5

Let us set

$$x' = \rho \cdot x \cdot \rho^{-1} \text{ and } y' = \rho \cdot y \cdot \rho^{-1}$$

where x, y are generators of $\pi_1(V_{\bar{K}}; \overrightarrow{01})$ as in 11.0. We define functions $\mathfrak{k}(\pi)$, $\mathfrak{k}_1(\pi)$ and $\mathfrak{k}_2(\pi)$ from G_K to $\mathbf{Z}_l$ by the following congruence

$$\mathfrak{f}_\pi \equiv x'^{\mathfrak{k}(\pi)} \cdot y'^{\mathfrak{k}_1(\pi)} \cdot (y', x')^{\mathfrak{k}_2(\pi)} \mod \Gamma^3 \pi_1(V_{\bar{K}}; \xi).$$

Lemma 11.1.5.

i). We have

$$l_2(\zeta)_{\pi\rho} - l_2(\xi)_\rho = \mathfrak{k}_2(\pi) - \frac{1}{2}\mathfrak{k}(\pi)\mathfrak{k}_1(\pi) - \frac{1}{2}\kappa^0_\zeta \kappa^1_\xi + \frac{1}{2}\kappa^0_\xi \kappa^1_\zeta.$$

ii). If we replace ρ by $\rho_1 = \rho \cdot x^a$ then in terms of new generators $x'' = \rho_1 \cdot x \cdot \rho_1^{-1}$, $y'' = \rho_1 \cdot y \cdot \rho_1^{-1}$ the triple $\mathfrak{k}(\pi)$, $\mathfrak{k}_1(\pi)$, $\mathfrak{k}_2(\pi)$ is replaced by the triple $\mathfrak{k}(\pi)$, $\mathfrak{k}_1(\pi)$, $\mathfrak{k}_2(\pi) + a\mathfrak{k}_1(\pi)$.

Proof. It follows from the formula $\mathfrak{f}_{\pi\rho} = \rho^{-1}\mathfrak{f}_\pi \rho \cdot \mathfrak{f}_\rho$ (see Lemma 1.0.6) that $\Lambda_{\pi\rho}(\sigma) = \Lambda_\pi(\sigma) \cdot \Lambda_\rho(\sigma)$, where $\Lambda_\pi(\sigma)$ is the image of $\mathfrak{f}_\pi$ by the embedding of $\pi_1(V_{\bar{K}}; \xi)$ into $\mathbf{Q}_l\{\{X, Y\}\}$ sending x' to e^X and y' to e^Y. Applying logarithm we get

$$\log\Lambda_{\pi\rho}(\sigma) \bigcirc (-\log\Lambda_\rho(\sigma)) = \log\Lambda_\pi(\sigma).$$

Comparing coefficient at $[Y, X]$ we get

$$l_2(\zeta)_{\pi\rho} - l_2(\xi)_\rho = \mathfrak{k}_2(\pi) - \frac{1}{2}\mathfrak{k}(\pi)\mathfrak{k}_1(\pi) - \frac{1}{2}\kappa^0_\zeta \kappa^1_\xi + \frac{1}{2}\kappa^0_\xi \kappa^1_\zeta.$$

The second part of the lemma follows from the congruence $y'' = x'^a \cdot y' \cdot x'^{-a} \equiv y' \cdot (y', x')^{-a}$ mod $\Gamma^3 \pi_1(V_{\bar{K}}; \xi)$.

Definition 11.1.6. Let us set

$$K(\zeta, \xi) := -\kappa^0_\zeta \kappa^1_\xi + \kappa^0_\xi \kappa^1_\zeta.$$

Observe that $K(\zeta, \xi)$ is a function from G_K to $\mathbf{Q}_l$. After the restriction to $G_{K(\mu_{l^\infty})}$ the function $K(\zeta, \xi)$ does not depend on a choice of paths from $\overrightarrow{01}$ to ξ and ζ.

Now we start to look for l-adic analog of the 5-term functional equation of the classical dilogarithm. Let $f(z) = \frac{(1-y)z}{z-1}$, $g(z) = yz$, $h(z) = \frac{(z-1)y}{1-y}$ and $k(z) = z$. Observe that f, g, h and k define regular maps from Y to V. Let v be a tangential base point at 0 corresponding to the local parameter yz at 0. Let x_0, x_1, x_y, x_∞ be geometric generators of $\pi_1(Y_{\bar{\mathbf{Q}}}; v)$ - loops around $0, 1, \frac{1}{y}$ and ∞ respectively (see Picture 6).

Picture 6

We assume that

$$x_\infty \cdot x_y \cdot x_1 \cdot x_0 = 1.$$

Let $z \in \hat{Y}(K)$ and let $p \in \pi(Y_{\bar{K}}; z, v)$. We introduce functions $\lambda(z)$, $\mu(z)$, $\nu(z)$, $\alpha(z)$, $\beta(z)$ and $\gamma(z)$ from G_K to $\mathbf{Z}_l$ by the following congruence

11.1.7. $\mathfrak{f}_p \equiv x_0^{\lambda(z)} \cdot x_1^{\mu(z)} \cdot x_y^{\nu(z)} \cdot (x_1, x_0)^{\alpha(z)} \cdot (x_y, x_0)^{\beta(z)} \cdot (x_y, x_1)^{\gamma(z)} \mod \Gamma^3\pi_1(Y_{\bar{K}}; v).$

We recall that $f : Y \to V$ is given by $f(z) = \frac{(1-y)z}{z-1}$. Observe that $f_*(v) = w$, where w is a tangential base point at 0 corresponding to the local parameter $\frac{y}{y-1} \cdot z$ at 0. Let us set $x' := f_*(x_0)$ and $y' := f_*(x_y)$. Observe that $f(\infty) = 1 - y$. This implies that $f_*(x_\infty) = 1$. Therefore $f_*(x_1) = y'^{-1} \cdot x'^{-1}$. Let q be a path from $\overrightarrow{01}$ to $f_*(v)$ such that $q \cdot x \cdot q^{-1} = x'$ and $q \cdot y \cdot q^{-1} = y'$ (see Picture 7).

Picture 7

By the definition of functions $\mathfrak{k}$ and $\mathfrak{k}_i$ we have

$$\mathfrak{f}_{f(p)} \equiv x'^{\mathfrak{k}(f(p))} \cdot y'^{\mathfrak{k}_1(f(p))} \cdot (y', x')^{\mathfrak{k}_2(f(p))} \mod \Gamma^3\pi_1(V_{\bar{K}}, f_*(v)).$$

Applying f_* to 11.1.7 we get

$$f_*\mathfrak{f}_p \equiv x'^{\lambda(z)-\mu(z)} \cdot y'^{\nu(z)-\mu(z)} \cdot (y', x')^{-\alpha(z)+\beta(z)-\gamma(z)+\frac{1}{2}\mu(z)^2+\frac{1}{2}\mu(z)} \mod \Gamma^3\pi_1(V_{\bar{K}}, f_*(v)).$$

The equality $f_*\mathfrak{f}_p = \mathfrak{f}_{f(p)}$ implies

11.1.8. $$\mathfrak{k}(f(p)) = \lambda(z) - \mu(z),\ \mathfrak{k}_1(f(p)) = \nu(z) - \mu(z)$$

and

11.1.8. $$\mathfrak{k}_2(f(p)) = -\alpha(z) + \beta(z) - \gamma(z) + \frac{1}{2}\mu(z)^2 + \frac{1}{2}\mu(z).$$

We recall that $g : Y \to V$ is given by $g(z) = yz$. Observe that $g_*(v) = \overrightarrow{01},\ g_*(x_0) = x,\ g_*(x_1) = 1$ and $g_*(x_y) = y$. Comparing coefficients of $\mathfrak{f}_{g(p)}$ and $g_*\mathfrak{f}_p$ we get

11.1.9. $$\mathfrak{k}(g(p)) = \lambda(z),\ \ \mathfrak{k}_1(g(p)) = \nu(z),\ \ \mathfrak{k}_2(g(p)) = \beta(z).$$

We recall that $h : Y \to V$ is given by $h(z) = \frac{(z-1)y}{1-y}$. Observe that $h_*(v) = \frac{y}{y-1}$. Let us set $x_h := h_*(x_1)$ and $y_h := h_*(x_y)$. Notice that $h_*(x_0) = 1$. Let r be a path from $\overrightarrow{01}$ to $\frac{y}{y-1}$ such that $r \cdot x \cdot r^{-1} = x_h$ and $r \cdot y \cdot r^{-1} = y_h$ (see Picture 8).

Picture 8

Comparing coefficients of $\mathfrak{f}_{h(p)}$ and $h_*\mathfrak{f}_p$ we get

11.1.10. $$\mathfrak{k}(h(p)) = \mu(z),\ \ \mathfrak{k}_1(h(p)) = \nu(z),\ \ \mathfrak{k}_2(h(p)) = \gamma(z).$$

We recall that $k : Y \to V$ is given by $k(z) = z$. Observe that $k_*(v) = v$. Let us set $x_k := k_*(x_0)$ and $y_k := k_*(x_1)$. We have $k_*(x_y) = 1$. Let s be a path from $\overrightarrow{01}$ to v such that

$$s \cdot x \cdot s^{-1} = x_k \ \text{ and } \ s \cdot y \cdot s^{-1} = y_k$$

(see Picture 9).

Picture 9

Comparing coefficients of $\mathfrak{f}_{k(p)}$ and $k_*\mathfrak{f}_p$ we get

11.1.11. $$\mathfrak{k}(k(p)) = \lambda(z),\ \ \mathfrak{k}_1(k(p)) = \mu(z),\ \ \mathfrak{k}_2(k(p)) = \alpha(z).$$

It follows from the equalities 11.1.8 - 11.1.11 that

11.1.12. $$\mathfrak{k}_2(f(p)) - \frac{1}{2}\mathfrak{k}(f(p))\mathfrak{k}_1(f(p)) - \mathfrak{k}_2(g(p)) + \frac{1}{2}\mathfrak{k}(g(p))\mathfrak{k}_1(g(p)) +$$
$$\mathfrak{k}_2(h(p)) - \frac{1}{2}\mathfrak{k}(h(p))\mathfrak{k}_1(h(p)) + \mathfrak{k}_2(k(p)) - \frac{1}{2}\mathfrak{k}(k(p))\mathfrak{k}_1(k(p)) = \frac{1}{2}\mu(z).$$

Lemma 11.1.13. *On $G_{K(\mu_{l^\infty})}$ we have the following equality*

$$K(f(z),f_*(v))-K(g(z),g_*(v))+K(h(z),h_*(v))+K(k(z),k_*(v))=0.$$

Proof. We recall that $\kappa_0(z)=\kappa_z^0$ and $\kappa_1(z)=\kappa_z^1$. Hence we have

$$K(f(z),f_*(v))-K(g(z),g_*(v))+K(h(z),h_*(v))+K(k(z),k_*(v))=K(f(z),w)-K(g(z),\overrightarrow{01})+$$

$$K(h(z),\frac{y}{y-1})+K(k(z),v)=-\kappa_0(\frac{(1-y)z}{z-1})\kappa_1(w)+\kappa_0(w)\kappa_1(\frac{(1-y)z}{z-1})+\kappa_0(yz)\kappa_1(\overrightarrow{01})$$

$$-\kappa_0(\overrightarrow{01})\kappa_1(yz)-\kappa_0(\frac{(z-1)y}{1-y})\kappa_1(\frac{y}{y-1})+\kappa_0(\frac{y}{y-1})\kappa_1(\frac{(z-1)y}{1-y})-\kappa_0(z)\kappa_1(v)+\kappa_0(v)\kappa_1(z).$$

One checks that $\kappa_0(\overrightarrow{0a})=\kappa_0(a)$ and $\kappa_1(\overrightarrow{0a})=0$. The lemma follows from the fact that $\kappa_0(x\cdot y)=\kappa_0(x)+\kappa_0(y)$ and $\kappa_1(z)=\kappa_0(1-z)$ on $G_{K(\mu_{l^\infty})}$.

Theorem 11.1.14. *There are paths ($\mathbf{Q}_2$-paths if $l=2$) from $\overrightarrow{01}$ to points $\frac{(1-y)z}{z-1}$, yz, $\frac{(z-1)y}{1-y}$, $\frac{y}{y-1}$ and z such that on $G_{K(\mu_{l^\infty})}$ for l-adic dilogarithms calculated along these paths we have*

$$l_2(\frac{(1-y)z}{z-1})-l_2(yz)+l_2(\frac{(z-1)y}{1-y})-l_2(\frac{y}{y-1})+l_2(z)=0.$$

Proof. It follows from Lemma 11.1.5, the equality 11.1.12 and Lemma 11.1.13 that

$$l_2(f(z))-l_2(f_*(v))-l_2(g(z))+l_2(g_*(v))+l_2(h(z))-l_2(h_*(v))+l_2(k(z))-l_2(k_*(v))=\frac{1}{2}\mu(z).$$

To eliminate $\frac{1}{2}\mu(z)$ we replace the path s by $s'=s\cdot x^{-\frac{1}{2}}$. Then $x'_k=s'\cdot x\cdot s'^{-1}=x_k$ and $y'_k=(s\cdot x^{-\frac{1}{2}})\cdot y\cdot(s\cdot x^{-\frac{1}{2}})^{-1}=s\cdot y\cdot(y,x)^{\frac{1}{2}}\cdot s^{-1}=y_k\cdot(y_k,x_k)^{\frac{1}{2}}$. In terms of generators x'_k and y'_k of $\pi_1(V_{\bar{K}};v)$ we have

$$\mathfrak{l}_2(k(p))=\alpha(z)-\frac{1}{2}\mu(z).$$

Observe that $l_2(\overrightarrow{0a})=0$. Hence we get

$$l_2(\frac{(1-y)z}{z-1})-l_2(yz)+l_2(\frac{(z-1)y}{1-y})-l_2(\frac{y}{y-1})+l_2(z)=0$$

for l-adic dilogarithms calculated along the paths $f(p)\cdot q$, $g(p)$, $h(r)\cdot r$, r and $k(p)\cdot s\cdot x^{-\frac{1}{2}}$ respectively.

It would be interesting to choose paths in such a way that we get the Abel equation on G_K without lower degree terms.

11.2. Now we shall discuss functional equations of arbitrary l-adic polylogarithms. The next result is a corollary of Theorem 10.0.7. We recall that a subgroup G_{n+1} of $\pi_1(V_{\bar{\mathbf{Q}}};\overrightarrow{01})$ was defined at the end of subsection 11.0.

We are not able to show that after a suitable choice of paths l-adic polylogarithms satisfy functional equations without lower degree terms. We have only the following result.

Theorem 11.2.1. *Let K be a number field and let $V = \mathbf{P}^1_K \setminus \{0, 1, \infty\}$. Let $a_1, \ldots, a_{m+1}$ be K-points of $\mathbf{P}^1_K$ and let $Y = \mathbf{P}^1_K \setminus \{a_1, \ldots, a_{m+1}\}$. Let $n_i \in \mathbf{Z}$ for $i = 1 \ldots, N$ and let $f_i : Y \to V$ be regular maps defined over K for $i = 1, \ldots, N$. Let $z, v \in \hat{Y}(K)$. Let us assume that $\sum_{i=1}^N n_i(f_i)_* = 0$ in*

$$\mathrm{Hom}(\Gamma^n\pi_1(Y_{\bar{K}};*)/\Gamma^{n+1}\pi_1(Y_{\bar{K}};*);\Gamma^n\pi_1(V_{\bar{\mathbf{Q}}};\overrightarrow{01})/G_{n+1}).$$

Then we have a functional equation

$$\sum_{i=1}^{N} n_i(\mathcal{L}_n(f_i(z)) - \mathcal{L}_n(f_i(v)) = 0$$

on the subgroup $H_n(Y; z, v)$ of G_K.

Proof. The theorem follows from Theorem 10.0.7 and Proposition 11.0.15.

Corollary 11.2.2. *Let ξ_m be a primitive m-th root of 1. Then we have*

$$m^{n-1}(\sum_{k=0}^{m-1}\mathcal{L}_n(\xi_m^k z)) = \mathcal{L}_n(z^m)$$

on the subgroup $H_n(\mathbf{P}^1_{\mathbf{Q}(\mu_m)} \setminus \{0, \mu_m, \infty\}; z, \overrightarrow{01})$ of $G_{\mathbf{Q}(\mu_m)}$.

In Part III we shall need a special case of the equality from Corollary 11.2.2.

Corollary 11.2.3. *Let ξ_m be a primitive m-th root of* 1*. Then we have*

$$m^{n-1}(\sum_{k=0}^{m-1}\mathcal{L}_n(\xi_m^k)) = \mathcal{L}_n(1)$$

on the subgroup $H_n(\mathbf{P}_{\mathbf{Q}(\mu_m)}\setminus\{0,\mu_m,\infty\};\overrightarrow{10},\overrightarrow{01})$ of $G_{\mathbf{Q}(\mu_m)}$, where $\mathcal{L}_n(1) := \mathcal{L}_n(\overrightarrow{10})$.

Both corollaries follow immediately from Theorem 11.2.1. We give however a detailed proof of Corollary 11.2.3 because of it importance in Part III.

Proof of Corollary 11.2.3.

We shall use the notation of section 11.1, where we discussed the l-adic analog of the functional equation $Li_2(z^m) = m(\sum_{i=0}^m Li_2(\xi_m^i z))$. We shall use also the following notation. If a and b are elements of a group then $(a,b^1) := (a,b) = a\cdot b\cdot a^{-1}\cdot b^{-1}$ and $(a,b^n) := ((a,b^{n-1}),b)$ for $n>1$.

We recall that $Y = \mathbf{P}^1_{\mathbf{Q}(\mu_m)}\setminus\{0,\mu_m,\infty\}$ and $f: Y\to V$ is given by $f(z) = z^m$. Let p be a path from $\overrightarrow{01}$ to $\overrightarrow{10}$, the interval $[0,1]$. Let $\sigma\in H_n(Y;\overrightarrow{10},\overrightarrow{01})$. Then we have

$$\mathfrak{f}_p(\sigma) \equiv (y_0, x'^{n-1})^{\nu_0^n(\overrightarrow{10})(\sigma)}\cdot\ldots\cdot(y_{m-1},x'^{n-1})^{\nu_{m-1}^n(\overrightarrow{10})(\sigma)}$$

modulo a subgroup generated by $\Gamma^{n+1}\pi_1(Y_{\bar{K}};\overrightarrow{01})$ and commutators which contain at least two y's. Observe that $f(p)$ is a path from $\overrightarrow{01}$ to $m\cdot\overrightarrow{10}$. Then for any $\sigma\in H_n(V;\overrightarrow{10},\overrightarrow{01})$, and therefore also for any $\sigma\in H_n(Y;\overrightarrow{10},\overrightarrow{01})$ we have

$$\mathfrak{f}_{f(p)}(\sigma)\equiv (y,x^{n-1})^{\kappa^n_{\overrightarrow{10}}(\sigma)} \mod G_{n+1}.$$

It follows from the equality $f_*\mathfrak{f}_p = \mathfrak{f}_{f(p)}$ that

11.2.4. $$m^{n-1}(\nu_0^n(\overrightarrow{10})+\ldots+\nu_{m-1}^n(\overrightarrow{10})) = \kappa^n_{\overrightarrow{10}}$$

on $H_n(Y;\overrightarrow{10},\overrightarrow{01})$. We recall that $f_i: Y\to V$ is given by $f_i(z) = \xi_m^{-i}\cdot z$. Observe that $(f_i)_*\mathfrak{f}_{pq_i}(\sigma)\equiv(y,x^{n-1})^{\nu_i(\overrightarrow{10})(\sigma)} \mod G_{n+1}$ for $\sigma\in H_n(Y;\overrightarrow{10},\overrightarrow{0\xi_m^i}) = H_n(Y;\overrightarrow{10},\overrightarrow{01})$ and $\mathfrak{f}_{f_i(pq_i)}(\sigma)\equiv(y,x^{n-1})^{\kappa^n_{\xi_m^{-i}}(\sigma)} \mod G_{n+1}$ for $\sigma\in H_n(Y;\overrightarrow{\xi_n^{-i}0},\overrightarrow{01}) = H_n(Y;\overrightarrow{10},\overrightarrow{01})$, where q_i is a path from $\overrightarrow{0\xi_m^i}$ to $\overrightarrow{01}$ as on Picture 4. Hence we get

11.2.5. $$\nu_i^n(\overrightarrow{10}) = \kappa^n_{\xi_m^{-i}}$$

on $H_n(Y;\overrightarrow{\xi_n^{-i}0},\overrightarrow{01}) = H_n(Y;\overrightarrow{10},\overrightarrow{01})$. It follows from 11.2.4 and 11.2.5 that

$$m^{n-1}(\sum_{i=0}^{m-1}\kappa^n_{\xi_m^{-i}}) = \kappa^n_{\overrightarrow{10}}$$

on $H_n(Y;\overrightarrow{\xi_n^{-i}0},\overrightarrow{01}) = H_n(Y;\overrightarrow{10},\overrightarrow{01})$. For $\sigma \in H_n(Y;\overrightarrow{10},\overrightarrow{01})$ we have $\kappa^n_{\xi_m^{-i}}(\sigma) = \mathcal{L}_n(\xi_m^{-i})(\sigma)$ and $\kappa^n_{\overrightarrow{10}}(\sigma) = \mathcal{L}_n(\overrightarrow{10})(\sigma)$. This finishes the proof of Corollary 11.2.3.

One of the most useful functional equations of classical polylogarithms is the relation between $Li_n(z)$ and $Li_n(\frac{1}{z})$. For l-adic polylogarithms we have the following result.

Corollary 11.2.6. *For any $z \in V(K)$, we have*

$$\mathcal{L}_n(z) + (-1)^n\mathcal{L}_n(\frac{1}{z}) = 0$$

on the subgroup $H_n(V_{\bar{\mathbf{Q}}}; z, \overrightarrow{01})$.

Proof. It follows from Theorem 11.2.1 that

$$\mathcal{L}_n(z) - \mathcal{L}_n(\overrightarrow{01}) + (-1)^n(\mathcal{L}_n(\frac{1}{z}) - \mathcal{L}_n(\overrightarrow{\infty 1})) = 0.$$

$\mathcal{L}_n(\overrightarrow{01})$ vanishes. Hence we have to calculate $\mathcal{L}_n(\overrightarrow{\infty 1})$. Let p a path from $\overrightarrow{01}$ to $\overrightarrow{10}$ and let s a path from $\overrightarrow{10}$ to $\overrightarrow{1\infty}$ as on the picture.

Picture 10

Let $g : V \to V$ be given by $g(z) = \frac{1}{z}$. Let us set $q := g(p)^{-1} \cdot s \cdot p$. We denote by π'' the subgroup $[\Gamma^2\pi_1(V_{\bar{K}};\overrightarrow{01}), \Gamma^2\pi_1(V_{\bar{K}};\overrightarrow{01})]$ of $\pi_1(V_{\bar{K}};\overrightarrow{01})$. Let $(\Gamma^{n+1}\pi_1(V_{\bar{K}};\overrightarrow{01}), \pi'')$ be a normal subgroup of $\pi_1(V_{\bar{K}};\overrightarrow{01})$ generated by $\Gamma^{n+1}\pi_1(V_{\bar{K}};\overrightarrow{01})$ and π''.

Let $\sigma \in H_n(V_{\bar{\mathbf{Q}}}; z, \overrightarrow{01})$. Then we have

11.2.7. $$\mathfrak{f}_q(\sigma) = \prod_{i+j=n, i\geq 1, j\geq 1}(((y,x)x^{i-1})y^{j-1})^{\kappa_{i,j}(\overrightarrow{\infty 1})(\sigma)} \bmod \left(\Gamma^{n+1}\pi_1(V_{\bar{K}}, \overrightarrow{01}), \pi''\right)$$

for some $\kappa_{i,j}(\overrightarrow{\infty 1})(\sigma) \in \mathbf{Z}_l$. It follows from Lemma 1.0.6 and from equality 10.0.1 that

11.2.8. $$\mathfrak{f}_q = q^{-1} \cdot g_*(\mathfrak{f}_p)^{-1} \cdot q \cdot p^{-1} \cdot \mathfrak{f}_s \cdot p \cdot \mathfrak{f}_p.$$

Observe that

11.2.9. $$q^{-1} \cdot g_*(y) \cdot q = y \ \text{ and } \ q^{-1} \cdot g_*(x) \cdot q = x^{-1} \cdot y^{-1}.$$

Let $\sigma \in H_n(V_{\bar{\mathbf{Q}}}; z, \overrightarrow{01})$. Then we have

11.2.10. $$\mathfrak{f}_p(\sigma) = \prod_{i+j=n, i\geq 1, j\geq 1} (((y,x)x^{i-1})y^{j-1})^{\kappa_{i,j}(\overrightarrow{10})(\sigma)} \bmod (\Gamma^{n+1}\pi_1(V_{\bar{K}}, \overrightarrow{01}), \pi'')$$

for some $\kappa_{i,j}(\overrightarrow{10})(\sigma) \in \mathbf{Z}_l$. It follows from 11.2.7 - 11.2.10 that

$$\kappa_{n-1,1}(\overrightarrow{\infty 1}) = (-1)^n \kappa_{n-1,1}(\overrightarrow{10})(\sigma) + \kappa_{n-1,1}(\overrightarrow{10})(\sigma).$$

Hence $\kappa_{n-1,1}(\overrightarrow{\infty 1}) = 0$ if n is odd.

We shall show that $\kappa_{n-1,1}(\overrightarrow{10})$ vanishes for n even. Let x, y and z be generators of $\pi_1(V_{\bar{K}}; \overrightarrow{01})$ as on the picture.

Picture 11

Then we have $z \cdot y \cdot x = 1$. It follows from Proposition 2.2.1 that

$$(\mathfrak{f}_q(\sigma)(x,y))^{-1} \cdot z^{\chi(\sigma)} \cdot (\mathfrak{f}_q(\sigma)(x,y)) \cdot (\mathfrak{f}_p(\sigma)(x,y))^{-1} \cdot y^{\chi(\sigma)} \cdot (\mathfrak{f}_p(\sigma)(x,y)) \cdot x^{\chi(\sigma)} = 1.$$

Let $\sigma \in H_n(V_{\bar{K}}; z, \overrightarrow{01})$. It follows from 11.2.8 and 11.2.9 that

$$\mathfrak{f}_q(\sigma)(x,y) = (\mathfrak{f}_p(\sigma)(x^{-1}y^{-1}, y))^{-1} \cdot (\mathfrak{f}_p(\sigma)(x,y)).$$

Hence we get

$$(\mathfrak{f}_p(\sigma)(x,y))^{-1} \cdot (\mathfrak{f}_p(\sigma)(x^{-1}y^{-1}, y)) \cdot x^{-1} \cdot y^{-1} \cdot (\mathfrak{f}_p(\sigma)(x^{-1}y^{-1}, y))^{-1} \cdot (\mathfrak{f}_p(\sigma)(x,y)) \cdot$$

$$(\mathfrak{f}_p(\sigma)(x,y))^{-1} \cdot y \cdot (\mathfrak{f}_p(\sigma)(x,y)) \cdot x = 1.$$

Comparing exponents at (y, x^n) we get $(1 + (-1)^n)\kappa_{n-1,1}(\overrightarrow{10}) = 0$. Hence $\kappa_{n-1,1}(\overrightarrow{10}) = 0$ for n even (see also [I1], [I2] and [D], where the element $\mathfrak{f}_p(\sigma)$ is studied). Therefore $\kappa_{n-1,1}(\overrightarrow{\infty 1}) = 0$ for any n. The equality $\kappa_{n-1,1}(\overrightarrow{\infty 1}) = \mathcal{L}_n(\overrightarrow{\infty 1})$ implies the corollary.

The fact that $\kappa_{n-1,1}(\overrightarrow{10})$ vanishes for n even implies the following well known result.

Corollary 11.2.11.

$$\mathcal{L}_{2n}(\overrightarrow{10}) = 0.$$

§12. Monodromy of l-adic iterated integrals and l-adic polylogarithms.

12.0. We shall show here that suitable defined l-adic polylogarithms form a local system with the similar shape of the monodromy representation as the local system of classical polylogarithms given in [BD]. We start with the discussion of the monodromy of arbitrary l-adic iterated integrals. The notation is the same as in section 10.

Let p be a path from v to z on $X_{\bar{K}}$ and let $S \in \pi_1(X_{\bar{K}}; v)$. Then we have

12.0.0. $$\mathfrak{f}_{pS}(\sigma) = S^{-1} \cdot \mathfrak{f}_p(\sigma) \cdot S \cdot \mathfrak{f}_S(\sigma).$$

Let $\mathrm{Map}(G_K; \pi_1(X_{\bar{K}}; v))$ be the set of all maps from G_K to $\pi_1(X_{\bar{K}}; v)$. We define a map

$$r_{z,v;p} : \pi_1(X_{\bar{K}}; v) \to \mathrm{Aut}_{set}(\mathrm{Map}\,(G_K; \pi_1(X_{\bar{K}}; v)))$$

setting

$$r_{z,v;p}(S)(w)(\sigma) := S^{-1} \cdot w(\sigma) \cdot S \cdot \mathfrak{f}_S(\sigma),$$

for $S \in \pi_1(X_{\bar{K}}; v)$, $w \in \mathrm{Map}\,(G_K; \pi_1(X_{\bar{K}}; v))$ and $\sigma \in G_K$.

Further we drop the indices ${}_{z,v;p}$ to simplify the notation.

Lemma 12.0.1. *The map $r_{z,v;p}$ is a representation of $\pi_1(X_{\bar{K}}; v)$.*

Proof. Let $S, T \in \pi_1(X_{\bar{K}}; v)$. We have $r(T)(r(S)w)(\sigma) = T^{-1}(S^{-1} \cdot w(\sigma) \cdot S \cdot \mathfrak{f}_S(\sigma)) \cdot T \cdot \mathfrak{f}_T(\sigma) = (S \cdot T)^{-1} \cdot w(\sigma) \cdot (S \cdot T) \cdot (T^{-1} \cdot \mathfrak{f}_S(\sigma) \cdot T \cdot \mathfrak{f}_T(\sigma)) = (S \cdot T)^{-1} \cdot w(\sigma) \cdot (S \cdot T) \cdot \mathfrak{f}_{ST}(\sigma) = r(S \cdot T)(w)(\sigma)$. We recall that in our notation $S \cdot T$ means that first we go along T and then along S. Therefore r is a representation of $\pi_1(X_{\bar{K}}; v)$.

We recall that $k_x : \pi_1(X_{\bar{K}}; v) \to \mathbf{Q}_l\{\{\mathbf{X}\}\}$ is a continous multiplicative embedding given by $k_x(x_i) = e^{X_i}$ for $i = 1, \ldots, n$ and that for a path p from v to z we set $\Lambda_p(\sigma) := k_x(\mathfrak{f}_p(\sigma))$.

Let $\mathrm{Map}(G_K; \mathbf{Q}_l\{\{\mathbf{X}\}\})$ be the set of all maps from G_K to $\mathbf{Q}_l\{\{\mathbf{X}\}\}$. Observe that $\mathrm{Map}(G_K; \mathbf{Q}_l\{\{\mathbf{X}\}\})$ is a vector space over $\mathbf{Q}_l$. We denote by $GL(\mathrm{Map}\,(G_K; \mathbf{Q}_l\{\{\mathbf{X}\}\}))$ the group of linear automorphisms of the vector space $\mathrm{Map}(G_K; \mathbf{Q}_l\{\{\mathbf{X}\}\})$.

Let us define a map

$$R_{z,v;p} : \pi_1(X_{\bar{K}}; v) \to GL(\mathrm{Map}\,(G_K; \mathbf{Q}_l\{\{\mathbf{X}\}\}))$$

setting

$$R_{z,v;p}(S)(W)(\sigma) := k_x(S)^{-1} \cdot W(\sigma) \cdot k_x(S) \cdot \Lambda_S(\sigma).$$

Proposition 12.0.2. *The map $R_{z,v;p}$ is a representation of $\pi_1(X_{\bar{K}}; v)$.*

Proof. To simplify the notation let us set $R = R_{z,v;p}$. Let $S, T \in \pi_1(X_{\bar{K}}; v)$. We have $R(T)(R(S)(W))(\sigma) = k_x(T)^{-1} \cdot (R(S)(W)(\sigma)) \cdot k_x(T) \cdot \Lambda_T(\sigma) = k_x(T)^{-1} \cdot (k_x(S)^{-1} \cdot W(\sigma) \cdot k_x(S) \cdot \Lambda_S(\sigma)) \cdot k_x(T) \cdot \Lambda_T(\sigma) = k_x(S \cdot T)^{-1} \cdot W(\sigma) \cdot k_x(S \cdot T) \cdot k_x(T)^{-1} \cdot \Lambda_S(\sigma) \cdot k_x(T) \cdot \Lambda_T(\sigma) = R(S \cdot T)(W)(\sigma)$.

It follows from Lemma 10.3.1 that the embedding $k_x : \pi_1(X_{\bar{K}}; v) \to \mathbf{Q}_l\{\{\mathbf{X}\}\}$ extends uniquely to a continous multiplicative embedding $\bar{k}_x : \pi_1(X_{\bar{K}}; v) \otimes \mathbf{Q} \to \mathbf{Q}_l\{\{\mathbf{X}\}\}$.

Proposition 12.0.3. *The representation $R_{z,v;p}$ extends to the representation*

$$\bar{R}_{z,v;p} : \pi_1(X_{\bar{K}}; v) \otimes \mathbf{Q} \to GL(\mathrm{Map}\,(G_K; \mathbf{Q}_l\{\{\mathbf{X}\}\})).$$

Let $S \in \pi_1(X_{\bar{K}}; v) \otimes \mathbf{Q}$. Then we have

$$\bar{R}_{z,v;p}(S)(W)(\sigma) = \bar{k}_x(S)^{-1} \cdot W(\sigma) \cdot \bar{k}_x(S) \cdot \bar{R}_{z,v;p}(S)(1)(\sigma).$$

Proof. We define an increasing filtration $\{\mathcal{W}_{-i}\}_{i \in \mathbf{N}}$ of the $\mathbf{Q}_l$-vector space $\mathrm{Map}\,(G_K; \mathbf{Q}_l\{\{\mathbf{X}\}\})$ setting

$$\mathcal{W}_{-2k} = \mathcal{W}_{-2k-1} \text{ to be a set of all maps from } G_K \text{ to } I^k,$$

where I^k is a k-th power of the augmentation ideal of $\mathbf{Q}_l\{\{\mathbf{X}\}\}$. Let $S \in \pi_1(X_{\bar{K}}; v)$ and let $W \in \mathcal{W}_{-2k}$. Then we have

$$R_{z,v;p}(S)(W) \equiv W \mod \mathcal{W}_{-2(k+1)}.$$

Hence the image of $R_{z,v;p}$ is in the subgroup of pro-unipotent automorphisms of the vector space $\mathrm{Map}\,(G_K; \mathbf{Q}_l\{\{\mathbf{X}\}\})$. This implies that the representation $R_{z,v;p}$ extends to the representation

$$\bar{R}_{z,v;p} : \pi_1(X_{\bar{K}}; v) \otimes \mathbf{Q} \to GL(\mathrm{Map}\,(G_K; \mathbf{Q}_l\{\{\mathbf{X}\}\})).$$

Let $S \in \pi_1(X_{\bar{K}}; v) \otimes \mathbf{Q}$ be such that $S^{l^m} \in \pi_1(X_{\bar{K}}; v)$. Then we have

$$R_{z,v;p}(S^{l^m})(W)(\sigma) = k_x(S^{l^m})^{-1} \cdot W(\sigma) \cdot k_x(S^{l^m}) \cdot \Lambda_{S^{l^m}}(\sigma),$$

where $\Lambda_{S^{l^m}}(\sigma) = R_{z,v;p}(S^{l^m})(1)(\sigma)$. This implies that

$$\bar{R}_{z,v;p}(S)(W)(\sigma) = \bar{k}_x(S)^{-1} \cdot W(\sigma) \cdot \bar{k}_x(S) \cdot \bar{R}_{z,v;p}(S)(1)(\sigma).$$

The elements $S \in \pi_1(X_{\bar{K}}; v) \otimes \mathbf{Q}$ such that $S^{l^m} \in \pi_1(X_{\bar{K}}; v)$ for some m are dense in $\pi_1(X_{\bar{K}}; v) \otimes \mathbf{Q}$ hence the last formula holds for any $S \in \pi_1(X_{\bar{K}}; v) \otimes \mathbf{Q}$. This finishes the proof of the proposition.

12.1. Now we shall study monodromy of l-adic polylogarithms, more exactly, we shall study monodromy of coefficients at $X^{n-1}Y$ of the power series $\Lambda_p(\sigma)$. Let $V = \mathbf{P}^1_K \setminus \{0, 1, \infty\}$ and let p be a path from $\overrightarrow{01}$ to z. From now on the notation is the same as in section 11.0.

We define functions $\lambda_i(z)_p$, $\mu_j(z)_p$ and $\nu_{i,j}(z)_p$ from G_K to $\mathbf{Q}_l$ by the congruence

$$\Lambda_p(\sigma) \equiv 1 + \sum_{k=1}^{\infty} \frac{(l(z)_p(\sigma))^k}{k!} X^k + \sum_{i=1}^{\infty} \lambda_i(z)_p(\sigma) X^{i-1} Y + \sum_{j=2}^{\infty} \mu_j(z)_p(\sigma) Y X^{j-1} + \sum_{i,j=1}^{\infty} \nu_{i,j}(z)_p(\sigma) X^i Y X^j$$

modulo the ideal generated by monomials with at least two Y's.

The function $\lambda_1(z)_p = l_1(z)_p$ and the l-adic polylogarithms $l_k(z)_p$ can be expressed by the function $\lambda_k(z)_p$ and the functions $l(z)_p$ and $\lambda_i(z)_p$ with $i < k$.

Proposition 12.1.1. *The monodromy transformation of functions $l(z)_p$ and $\lambda_n(z)_p$ is as follows:*

$$x : l(z)_p \to l(z)_p + (\chi - 1), \qquad \lambda_n(z)_p \to \lambda_n(z)_p + \sum_{i=1}^{n-1} \frac{(-1)^{n-i}}{(n-i)!} \lambda_i(z)_p,$$

$$\mu_n(z)_p \to \mu_n(z)_p + \sum_{i=2}^{n-1} \frac{\chi^{n-i}}{(n-i)!} \mu_i(z)_p + \frac{\chi^{n-1}}{(n-1)!} \lambda_1(z)_p$$

and

$$y : l(z)_p \to l(z)_p, \quad \lambda_1(z)_p \to \lambda_1(z)_p + (\chi - 1), \quad \lambda_n(z)_p \to \lambda_n(z)_p + \chi \frac{(l(z)_p)^{n-1}}{(n-1)!}$$

for $n > 1$ *and* $\mu_n(z)_p \to \mu_n(z)_p - \frac{(l(z)_p)^{n-1}}{(n-1)!}$.

Proof. The proposition follows from the formula

$$\Lambda_{p\cdot S}(\sigma) = k(S)^{-1} \cdot \Lambda_p(\sigma) \cdot k(S) \cdot \Lambda_S(\sigma),$$

which for $S = x$ gives

$$\Lambda_{p\cdot x}(\sigma) = e^{-X} \cdot \Lambda_p(\sigma) \cdot e^{\chi(\sigma)X}.$$

For $S = y$ the formula is more complicated, however when we restrict our attention to coefficients with only one Y then the formula have the same simple form.

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