Multiresolution wavelet analysis of integer scale Bessel functions

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Abstract

We identify multiresolution subspaces giving rise via Hankel transforms to Bessel functions. They emerge as orthogonal systems derived from geometric Hilbert-space considerations, the same way the wavelet functions from a multiresolution scaling wavelet construction arise from a scale of Hilbert spaces. We study the theory of representations of the C^*-algebra $O_{\nu+1}$ arising from this multiresolution analysis. A connection with Markov chains and representations of $O_{\nu+1}$ is found. Projection valued measures arising from the multiresolution analysis give rise to a Markov trace for quantum groups SO_q.

1 Introduction

The starting point for the multiresolution analysis from wavelet theory is a system U, $\{T_j\}_{j\in\mathbf{Z}}$, of unitary operators with the property that the underlying Hilbert space $\mathcal{H}$ with norm $\|.\|$, contains a vector $\varphi \in \mathcal{H}$, $\|\varphi\| = 1$, satisfying

$$U\varphi = \sum_j a_j T_j \varphi \tag{1}$$

for some sequence $\{a_j\}$ of complex scalars, such that, in particular (1) converges in $\mathcal{H}$. In addition, the operator system $\{U, T_j\}$ must satisfy a non-trivial commutation relation. In the case of wavelets, it is

$$UT_jU^{-1} = T_N, \qquad j \in \mathbf{Z}, \tag{2}$$

where N is the *scaling number*, or equivalently the number of *subbands* in the corresponding multiresolution. When this structure is present, there is a way to recover the spectral theory of the problem at hand from representations of an associated C^*-algebra. In the case of orthogonal wavelets, we may take this C^*-algebra to be the Cuntz algebra. In that case, the operators T_j may be represented on $L^2(\mathbf{R})$ as translations,

$$(T_j\xi)(x) = \xi(x-j), \qquad \xi \in L^2(\mathbf{R}),$$

and U may be taken as the scaling $(U\xi)(x) = N^{-1/2}\xi(x/N)$, $N \in \mathbf{N}$. This system clearly satisfies (2). (For a variety of other examples of these relations, the reader is referred to Ref. [33]. The setup there applies to dynamical systems of N-to-1 Borel measurable self-maps: for example, those of complex dynamics and Julia sets.) In the wavelet case, a multiresolution is built from a solution $\varphi \in L^2(\mathbf{R})$ to the scaling identity (1). The numbers $\{a_j\}_{j\in\mathbf{Z}}$ from (1) must then satisfy the "orthogonality relations"'

$$\sum_{k\in\mathbf{Z}} a_k = 1, \qquad \sum_{k\in\mathbf{Z}} \bar{a}_k a_{k+2m} = \delta_{0,m}, m \in \mathbf{Z} \tag{3}$$

In this case, the analysis is based on the Fourier transform: define m_0 as a map from S^1 to $\mathbf{C}$ by

$$m_0\left(e^{it}\right) = \sum_k a_k e^{ikt}, t \in \mathbf{R} \tag{4}$$

(of course we assume here and below convergence of the series and products involved). Then (in the wavelet case, following Ref. [1]) a solution to (1) will have the product form

$$\hat{\varphi}(t) = \prod_{j=1}^{\infty} m_0\left(t/N^j\right), \tag{5}$$

up to a constant multiple. The Cuntz algebra O_N enters the picture as follows: Formula (5) is not practical for computations, and the analysis of orthogonality relations is done better by reference to the Cuntz relations, see (11)–(12) below. Setting, for $\xi \in \mathbf{C}$, $j \in \mathbf{Z}$

$$W\left(\{\xi_j\}\right) := \sum_{j\in\mathbf{Z}} \xi_j \varphi\left(x-j\right), \tag{6}$$

and using (3), we get an isometry W of ℓ^2 into a subspace of $L^2\left(\mathbf{R}\right)$, the resolution subspace. Setting

$$\left(S_0 f\right)(z) := \sqrt{N} m_0\left(z\right) f\left(z^N\right), \qquad f \in L^2\left(\mathbf{T}\right), \text{Borel measurable} \tag{7}$$

and using $L^2\left(\mathbf{T}\right) \cong \ell^2$ by the Fourier series, we establish the following crucial intertwining identity:

$$WS_0 = UW, \tag{8}$$

so that U is a unitary extension of the isometry S_0. We showed in Refs. [2] and [3] that functions $m_1, \ldots, m_{N-1} \in L^\infty\left(\mathbf{T}\right)$ may then be chosen such that the corresponding matrix

$$\left(m_j\left(e^{i(t+k2\pi/N)}\right)\right)_{j,k=0}^{N-1} \tag{9}$$

is in $\mathrm{U}_N\left(\mathbf{C}\right)$ for Lebesgue a.a. t. Then it follows that the operators

$$S_j f\left(z\right) := \sqrt{N} m_j\left(z\right) f\left(z^N\right), \qquad f \in L^2\left(\mathbf{T}\right), \tag{10}$$

will yield a representation of the Cuntz relations; see (11)–(12) below. Conversely, if (10) is given to satisfy the Cuntz relations, then the matrix in (9) takes values in $\mathrm{U}_N\left(\mathbf{C}\right)$.

The present paper aims at an analogous construction, but based instead on the Bessel functions, i.e., we use the Bessel functions in (4) in place of the usual Fourier basis $\left\{e^{ikt}\right\}_{k\in\mathbf{Z}}$; see (20) below. If $\nu \in \mathbf{N}$ is the parameter of the Bessel function J_ν, then we show that $N = \nu + 1$ is an admissible scaling for a multiresolution construction.

The motivation for doing a multiresolution construction based on a wider variety of special functions, other than the Fourier basis, derives in part from the rather restrictive axiom system dictated by the traditional setting [31, 32, 33, 35, 34, 36]. It is namely known[1] that many applications require a more general mathematical setup. Moreover, our present approach also throws some new light on special-function theory, and may be of independent interest for that reason.

We will apply multiresolutions to the Hankel transform and the Bessel functions of integer parameter ν. Our analysis is especially well suited for the introduction of a quantum variable q,($0 < q < 1$), in such a way that variations in q lead to a better understanding of an associated family of deformations. Our use of the Cuntz algebra is motivated by Refs. [2] and [4]. The Cuntz algebras [5] have been used independently in operator algebra theory and in the study of multiresolution wavelets, and our present paper aims to both make this connection explicit, and as well make use of it in the analysis of special functions. The q-deformations of the special functions[6, 7, 8, 9, 10, 11] may be of independent interest. This deformation is related to, but different from, those which have appeared in Refs. [11, 12, 13, 14, 15]. In the last sections of the paper we construct a Markov chain which turns out to be related to the representations of $O_{\nu+1}$ discussed in the previous sections via projection valued measures. Random walks on quantum group $SO_q(N)$ are then constructed via representations of the braid groups.

2 The Cuntz algebra and iterated function systems

We shall consider representations π of the Cuntz algebra $O_{\nu+1}$ coming from multiresolution analysis based on Hankel transforms. In Section 3 we give some preliminaries on Hankel transforms on $L^2(\mathbf{R})$. We then construct wavelets arising from multiresolutions with scaling $\nu+1$ using Hankel transforms on $L^2(\mathbf{C})$, relative to an appropriate measure on the field of complex numbers $\mathbf{C}$. The map from wavelets into representations is described. We establish connections between certain representations of $O_{\nu+1}$ and Hankel wavelets arising from that multiresolution analysis.

Recall that $O_{\nu+1}$ is the C^*-algebra generated by $\nu+1$, $\nu \in \mathbf{N}$, isometries $S_0, \ldots, S_\nu$ satisfying

$$S_i^* S_j = \delta_{ij} \mathbf{1} \tag{11}$$

and

$$\sum_{i=0}^{\nu} S_i S_i^* = \mathbf{1}. \tag{12}$$

The representations we will consider are realized on the Hilbert spaces $H = L^2(\Omega, d\mu)$ where Ω is a measure space (to be specified below) and μ is a probability measure on Ω.

We define the representations in terms of certain maps

$$\sigma_i \colon \Omega \longrightarrow \Omega \quad \text{such that} \quad \Omega = \bigcup_{i=0}^{\nu} \sigma_i(\Omega) \quad \text{and} \quad \mu(\sigma_i(\Omega) \cap \sigma_j(\Omega)) = 0 \tag{13}$$

for all $i \neq j$. We will apply this in Section 5 to the Riemann surface of $\sqrt[N]{z}$.

In Section 4 we develop a q-parametric multiresolution wavelet analysis in $L^2(\mathbf{C}, \mu_q)$ where μ_q is a q-measure, as in Refs. [15, 21] by using q-Hankel transforms.

A class of q-parametric representations of the C^*-algebra $O_{\nu+1}$ is found. We further identify a class of representations of the Cuntz algebra which has the structure of compact quantum groups of type B.[29]

3 Hankel transforms and a multiresolution analysis

In this section we construct a multiresolution using Hankel transforms. We start by giving some basic definitions on Hankel tranforms.

Let us recall that the Hankel transform of order $\alpha \in \mathbf{R}$ of a function f, denoted by $\tilde{f}$, is defined, for $t \in (0,\infty)$ and $x \in (0,\infty)$, by

$$\tilde{f}(t) = \int_0^\infty J_\alpha(xt) f(x)\, x\, dx, \tag{14}$$

where

$$J_\alpha(x) = \left(\frac{x}{2}\right)^\alpha \sum_{k=0}^\infty \frac{(-1)^k}{k!\Gamma(\alpha+k+1)} \left(\frac{x}{2}\right)^{2k}$$

is the Bessel function of order α, $\alpha \in \mathbf{R}$ and

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt, \qquad \mathrm{Re}(z) > 0,$$

is the classical gamma function. If we multiply both sides of (14) by $J_\alpha(yt)\, t$ and integrate from $t = 0$ to $+\infty$ we obtain

$$\int_0^\infty J_\alpha(yt)\, \tilde{f}(t)\, t\, dt = f(y) = \int_0^\infty J_\alpha(yt)\, t \int_0^\infty J_\alpha(xt) f(x)\, x\, dx\, dt, y \in (0,\infty) \tag{15}$$

The integral transform on the left-hand side of (15) is equal to $f(y)$ for suitable functions f, by the Hankel inversion theorem.[17] The resulting double integral is called the Hankel Fourier-Bessel integral

$$f(y) = \int_0^\infty J_\alpha(yt) \left(\int_0^\infty J_\alpha(xt) f(x)\, x\, dx \right) t\, dt. \tag{16}$$

It can be written as the following transform pair

$$\begin{aligned} g(t) &= \int_0^\infty J_\alpha(yt) f(y)\, y\, dy, \qquad (17) \\ f(y) &= \int_0^\infty J_\alpha(yt)\, g(t)\, t\, dt. \end{aligned}$$

A Plancherel type result can be easily derived for this transform: if $F(\rho)$ and $G(\rho)$, $\rho \in (0,\infty)$, are Hankel transforms of $f(x)$ and $g(x)$, $x \in (0,\infty)$, respectively, then we have

$$\begin{aligned} \int_0^\infty \rho F(\rho) G(\rho)\, d\rho &= \int_0^\infty \rho F(\rho) \int_0^\infty x g(x) J_\nu(\rho x)\, dx\, d\rho \\ &= \int_0^\infty x g(x) \left(\int_0^\infty \rho F(\rho) J_\nu(\rho x)\, d\rho \right) dx \\ &= \int_0^\infty x f(x) g(x)\, dx. \end{aligned}$$

Let us give some preliminaries on the standard multiresolution wavelet analysis of scale ν, $\nu \in \mathbf{N}$. Following Refs. [1, 18] we define scaling by ν on $L^2(\mathbf{R})$ by

$$(U\xi)(x) = (\nu+1)^{-\frac{1}{2}}\,\xi\left(\frac{x}{\nu+1}\right)$$

and translation by 1 on $L^2(\mathbf{R})$ by

$$(T\xi)(x) = \xi(x-1)\,, x \in \mathbf{R}.$$

As mentioned in the Introduction, it is our aim here to adapt the theory of multiresolutions from wavelet theory[1, 19] to the analysis of the Bessel functions via the Hankel transform. The classical theory[20] is based on recurrence algorithms which we show adapt very naturally to the multiresolutions. But our analysis will still be based on the "classical" identities for the special functions (see, e.g., Refs. [21, 22, 23, 24, 25, 26, 37]).

A scaling function is a Borel measurable function $\varphi \in L^2(\mathbf{R})$ such that if V_0 is the closed linear span of all translates $T^k\varphi$, $k \in \mathbf{Z}$, then φ has the following four properties

i) $\left\{T^k\varphi : k \in \mathbf{Z}\right\}$ is an orthonormal set in $L^2(\mathbf{R})$;

ii) $U\varphi \in V_0$;

iii) $\bigwedge_{n\in\mathbf{Z}} U^n V_0 = \{0\}$;

iv) $\bigvee_{n\in\mathbf{Z}} U^n V_0 = L^2(\mathbf{R})$.

The simplest example of a scaling function is the characteristic function of the interval $[0,1]$, i.e., the zeroth Haar function. By i) we may define an isometry

$$F_\varphi\colon V_0 \longrightarrow L^2(\mathbf{R})\,, \qquad \xi \longmapsto m,$$

as follows. The scaling by ν on $L^2(\mathbf{R})$ is defined by the unitary operator U given by $(U\xi)(x) = (\nu+1)^{-\frac{1}{2}}\,\xi\left((\nu+1)^{-1}x\right)$ for $\xi \in L^2(\mathbf{R})$, $x \in \mathbf{R}$, and the translation as the following operator $(T\xi)(x) = \xi(x-1)$.

We consider the scaling Haar function φ given as the sum $\varphi(x) = h(1-x) - h(-x)$ of Heaviside functions h, $h(x) = 1$ for $x \geq 0$ and $h(x) = 0$ for $x < 0$

Let V_0 be the linear span of $\left\{\varphi_\nu^{(k)}(x) \equiv x^\nu \varphi(x-k)\right\}_{k\in\mathbf{Z}}$. Then V_0 is a closed subspace of $L^2(\mathbf{R})$ with respect to the following scalar product: $\langle f \mid g\rangle = \int \overline{f(x)}g(x)\,x\,dx$. We have $\bigcap_{n\in\mathbf{Z}} U^n V_0 = \{0\}$ and $\bigvee U^n V_0 = L^2(\mathbf{R})$.

Let $\xi \in L^2(\mathbf{R})$, and assume that $\xi(x) = \sum_k b_k \left\{\varphi_\nu^{(k)}(x)\right\}$, $b_k \in \mathbf{C}$, $x \in \mathbf{R}$

By applying the Hankel transform $H_\nu(\,\cdot\,,t)$ to both sides of the above equality and using the definition of φ we get for $t \geq 0$, $x \in \mathbf{R}$:

$$H_\nu(\xi(x),t) = \sum_k b_k H_\nu\left(\varphi(x-k)\,x^\nu, t\right) \tag{18}$$

$$
\begin{aligned}
&= \sum_k b_k H_\nu \left(h\left(k+1-x\right)x^\nu, t\right) - \sum_k b_k H_\nu \left(h\left(k-x\right)x^\nu, t\right) \\
&= \left[\sum_k b_k \left(k+1\right)^{\nu+1} J_{\nu+1}\left(t\left(k+1\right)\right) - \sum_k b_k k^{\nu+1} J_{\nu+1}\left(tk\right)\right] \\
&\times \ H_0\left(\frac{1}{x}, t\right).
\end{aligned}
$$

All series converge in $L^2\left(\mathbf{R}\right)$

To write the above expression in a more compact form we use the addition formula for Bessel functions

$$
J_n\left(x+y\right) = \sum_{k=-\infty}^{\infty} J_k\left(x\right) J_{n-k}\left(y\right).
$$

Then we get

$$
\begin{aligned}
H_\nu\left(\xi\left(x\right), t\right) &= \sum_k b_k \left[\left(k+1\right)^{\nu+1} J_{\nu+1}\left(kt+t\right) - k^\nu J_{\nu+1}\left(kt\right)\right] H_0\left(\frac{1}{x}, t\right) \\
&= \sum_k b_k \left[\left(k+1\right)^{\nu+1} \sum_h J_h\left(tk\right) J_{\nu+1-h}\left(t\right) - k^\nu J_{\nu+1}\left(kt\right)\right] \\
&\times \ H_0\left(\frac{1}{x}, t\right). \qquad (19)
\end{aligned}
$$

Define

$$
m_0\left(t\right) = \sum_k b_k \left[\left(k+1\right)^{\nu+1} \sum_h J_h\left(tk\right) J_{\nu+1-h}\left(t\right) - k^\nu J_{\nu+1}\left(kt\right)\right]. \qquad (20)
$$

Here we consider $L^2\left(\mathbf{R}, \mu\right)$ with $d\mu(x) = x\,dx$. By using the Plancherel Theorem, and the orthogonality of the Haar functions, we get

$$
\begin{aligned}
\frac{\delta_{k,0}}{2\left(\nu+1\right)} &= \int_0^\infty \varphi_\nu^{(k)}\left(x\right) \varphi_\nu^{(0)}\left(x\right) x\,dx \qquad (21) \\
&= \int_0^\infty \left[H_\nu\left(x^\nu\left[h\left(k+1-x\right) - h\left(k-x\right)\right], t\right)\right. \qquad (22) \\
&\times \ \left. H_\nu\left(x^\nu\left[h\left(1-x\right) - h\left(-x\right)\right], t\right)\right] t\,dt \\
&= \sum_{j\in\mathbf{Z}} \int_j^{j+1} [H_\nu(x^\nu[h(k+1-x) - h(k-x)], t) \\
&\times \ H_\nu(x^\nu[h(1-x) - h(-x)], t)] t\,dt.
\end{aligned}
$$

Thus the latter, upon a change of variables, can be rewritten as

$$
\int_0^1 \sum_{j\in\mathbf{Z}} [H_\nu(x^\nu[h(k+1-x)-h(k-x)], t+j) H_\nu(x^\nu[h(1-x)-h(-x)], t+j)](t+j)\,dt. \qquad (23)
$$

We used the following obvious fact:

$$\frac{1}{2(\nu+1)} = \int_0^1 \frac{dt}{2(\nu+1)}. \tag{24}$$

On comparing (24) and (23) for $k = 0$, we get

$$\sum_{j\in\mathbf{Z}} H_\nu^2 \left(x^\nu \left[h(1-x) - h(-x)\right], t+j\right) - \frac{1}{2(\nu+1)} = 0, \qquad \text{Lebesgue a.e.}$$

On the other hand, in view of (21), (20), (19) and (18) the left-hand side of this equality can be rewritten in terms of m_0 as follows

$$\sum_{j\in\mathbf{Z}} \left|m_0(t+j)\right|^2 \left|H_0(1/z, t+j)\right|^2 = \frac{1}{2(\nu+1)}, \qquad \text{Lebesgue a.e.}$$

To get a direct connection with representations of $O_{\nu+1}$, we need to consider our new multiresolutions on the complex plane $\mathbf{C}$. Assume φ to be a step function on $\mathbf{C}$, defined for $|z| \le 1$ by

$$\varphi\left(|z|\, e^{i\,\mathrm{Arg}(z)}\right) = \begin{cases} 1 & \text{if } 0 \le \mathrm{Arg}(z) \le \alpha, \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha = \frac{2\pi}{m}$, for a fixed $m \in \mathbf{N}$. With $k, m \in \mathbf{N}$, $1 \le N \le m$ take then V_0 to be the span of $\{\varphi\left[(|z|+k)\exp\left(i\left(\mathrm{Arg}(z) + N\alpha\right)\right)\right]\}$. Let

$$U\xi(z) = (\nu+1)^{-1/2}\, \xi\left(\frac{z}{\nu+1}\right) \tag{25}$$

be the scaling operator. For $j \in \mathbf{Z}$ let V_j be the closed span in $L^2(\mathbf{C}, \nu)$ of

$$\left\{\varphi\left[\left(\frac{|z|}{(\nu+1)^j} + k\right) e^{i(\mathrm{Arg}(z)+N\alpha)}\right]\right\}_{k\in\mathbf{Z},\ 1\le N\le m},$$

Consider $L^2(\mathbf{C}, \nu)$ where the measure $\nu(z) = z^\nu\, dz$, and dz denotes the planar measure on $\mathbf{C}$. Assume $U\varphi \in V_0$, i.e.,

$$(U\varphi)(z) = \sum_k a_k \varphi\left[(|z|+k)\exp\left(i\left(\mathrm{Arg}(z) + N\alpha\right)\right)\right], k \in \mathbf{Z}$$

Proposition 1 *With the assumptions above, the properties* i)–iv) *of a multiresolution are satisfied.*

Proof. i) follows from the fact that the φ's have disjoint support on $L^2(\mathbf{C}, \nu)$. ii) holds for Haar functions and iii) follows from i). By the density of step functions in $L^2(\mathbf{C}, \nu)$ also iv) follows.

If $\xi \in V_{-j}$ and $\nu \ne -1$, then $U^j\xi \in V_0$. Since

$$\varphi \in V_0 \subset V_{-1} \quad \text{and} \quad \left\{\varphi\left[\left(\frac{|z|}{\nu+1} + k\right)\exp\left(i\left(\mathrm{Arg}(z) + N\alpha\right)\right)\right]\right\}$$

are orthonormal in V_{-1}, we have

$$\varphi(z)=\sum_k a_k\varphi\left[\left(\frac{|z|}{\nu+1}+k\right)\exp\left(i\left(\operatorname{Arg}(z)+N\alpha\right)\right)\right], z\in\mathbf{C},$$

so by applying the Hankel transform of order ν , we get

$$H_\nu\left(\varphi\left[\left(\frac{|z|}{\nu+1}+k\right)\exp\left(i\left(\operatorname{Arg}(z)+N\alpha\right)\right)\right];t\right)=m_0\left(t\right)H_0\left(\frac{1}{z};t\right).$$

Using the orthogonality of $\varphi\left[\left(|z|+k\right)\exp\left(i\left(\operatorname{Arg}(z)+N\alpha\right)\right)\right]_{k\in\mathbf{Z}}$ in $L^2\left(\mathbf{C},\nu\right)$we have

$$\begin{aligned}
\langle\varphi^{(k,N)}\mid\varphi^{(0,0)}\rangle &\equiv \iint_{\mathbf{C}}\varphi^{(k,N)}\left(z\right)\overline{\varphi^{(0,0)}\left(z\right)}z\,d\mu\left(z\right) \qquad (26)\\
&= \int_0^\infty\int_0^{2\pi}\varphi\left[\left(|z|+k\right)\exp\left(i\left(\operatorname{Arg}\left(z\right)+N\alpha\right)\right)\right]\\
&\times \overline{\varphi\left[|z|\exp\left(i\operatorname{Arg}\left(z\right)\right)\right]}\\
&\times |z|^{\nu+1}\exp\left(i\operatorname{Arg}\left(z\right)\left(\nu+1\right)\right)\,d\left|z\right|\,d\operatorname{Arg}\left(z\right)\\
&= \int_0^1|z|^{\nu+1}\,\delta_{k,0}\,d\left|z\right|\int_0^\alpha\exp\left(i\operatorname{Arg}\left(z\right)\left(\nu+1\right)\right)\\
&\times \delta_{N,0}\,d\operatorname{Arg}\left(z\right)\\
&= \frac{1}{\nu+2}\delta_{k,0}\frac{e^{i\alpha(\nu+1)}-1}{i\left(\nu+1\right)}\delta_{N,0}.
\end{aligned}$$

By the Plancherel theorem, we then have

$$\frac{1}{\nu+2}\delta_{k,0}\frac{e^{i\alpha(\nu+1)}-1}{i\left(\nu+1\right)}\delta_{N,0}=\iint_{\mathbf{C}}H_\nu\left(\varphi^{(k,N)}\left(z\right);t\right)\overline{H_\nu\left(\varphi^{(0,0)}\left(z\right);t\right)}t\,d\nu\left(t\right).$$

The left-hand side can then be rewritten as

$$\int_0^1\int_0^\alpha H_\nu\left(\varphi^{(k,N)}\left(z\right);t\right)\overline{H_\nu\left(\varphi^{(0,0)}\left(z\right);t\right)}t^{\nu+1}\,dt\,d\operatorname{Arg}\left(t\right).$$

Upon a change of variable letting $\theta=\operatorname{Arg}\left(t+2\pi j\right)$, the latter equals

$$\int_0^1|t|^{\nu+1}\,d\left|t\right|\int_0^\alpha e^{i2\pi(\nu+1)}\sum_j H_\nu\left(\varphi^{(k,N)}\left(z\right);|t|\,e^{i\theta}\right)\overline{H_\nu\left(\varphi^{(0,0)}\left(z\right);|t|\,e^{i\theta}\right)}\,d\theta.$$

Comparing the previous two formulae for $k=N=0$ we get

$$\sum_j\left|H_\nu\left(\varphi^{(0,0)};|t|\,e^{i\theta}\right)\right|^2-\frac{1}{\nu+2}\frac{e^{i\alpha(\nu+1)}-1}{i\left(\nu+1\right)}=0\qquad\text{for Lebesgue a.e.}$$

Rewriting the above in terms of m_0 we have

$$\sum_j\left|m_0\left(te^{2\pi ij}\right)\right|^2\left|H_0\left(\frac{1}{z};|t|\,e^{i\theta}\right)\right|^2=\frac{1}{\nu+2}\frac{e^{i\alpha(\nu+1)}-1}{i\left(\nu+1\right)},$$

since

$$\int_0^1 |t|^{\nu+1}\, d|t| \int_0^\alpha e^{i2\pi(\nu+1)} \sum_j \left|m_0\left(te^{2\pi ij}\right)\right|^2 \left|H_0\left(\frac{1}{z};|t|\,e^{i\theta}\right)\right|^2 d\theta$$
$$= \int_0^1 |t|^{\nu+1}\, d|t| \int_0^\alpha e^{i2\pi(\nu+1)} \sum_j \left|m_0\left(te^{2\pi ij}\right)\right|^2 \frac{1}{|t|^2}\, d\theta$$
$$= \int_0^1 |t|^{\nu-1}\, d|t| \int_0^\alpha e^{i2\pi(\nu+1)} \sum_j \left|m_0\left(te^{2\pi ij}\right)\right|^2 d\theta.$$

From (26) we get

$$\begin{aligned}\int_0^1 |t|^{\nu-1}\, d|t| \int_0^\alpha e^{i2\pi(\nu+1)} \sum_j \left|m_0\left(te^{2\pi ij}\right)\right|^2 d\theta &= \frac{\delta_{k,0}}{\nu+2}\frac{e^{i\alpha(\nu+1)}-1}{i(\nu+1)}\delta_{N,0} \\ &= \frac{\nu}{\nu+2}\int_0^1 |t|^{\nu-1}\, d|t| \\ &\times \int_0^\alpha e^{i\theta(\nu+1)}\, d\theta.\end{aligned}$$

Thus

$$\left(\frac{1}{\nu+1}\right)\sum_j \left|m_0\left(te^{2\pi ij/(\nu+1)}\right)\right|^2 = \frac{\nu}{\nu+2}. \tag{27}$$

Set $c=\frac{\nu}{\nu+2}$; then $\left(\frac{1}{c(\nu+1)}\right)\sum_j \left|m_0\left(te^{2\pi ij/(\nu+1)}\right)\right|^2 = 1.$ □

In fact, as in Ref. [1], Thm. 5.1.1, we have proved a part of the following result.

Theorem 2 *If the ladder of the closed subspaces $\{V_j\}_{j\in\mathbf{Z}}$ in $L^2(\mathbf{C},\nu)$ satisfies properties* i)–iv), *then there exists an associated orthonormal wavelet basis $\{\psi_{jk} : j,k\in\mathbf{Z}\}$ for $L^2(\mathbf{C},\nu)$ such that*

$$(U\varphi)(z) = \sum_k a_k \varphi\left[(|z|+k)\exp\left(i\left(\operatorname{Arg}(z)+N\alpha\right)\right)\right]$$

holds. One possibility for construction of the wavelet corresponding to φ is that

$$H_\nu\left(\varphi\left[(|z|+k)\exp\left(i\left(\operatorname{Arg}(z)+N\alpha\right)\right)\right];(\nu+1)\,t\right) = m_0(t)\,H_0\left(\frac{1}{z};t\right)$$

be satisfied.

Completion of proof. . We observe that the Bessel functions have a "multiplicative periodicity" on the unit circle in the following sense:

$$J_\nu\left(ze^{\pi ik}\right) = e^{\pi ik\nu} J_\nu(z)$$

From the above (27), this implies that

$$c^{-1}\sum_{j=0}^{\nu}\left|m_0\left(ze^{2\pi ij/(\nu+1)}\right)\right|^2=(\nu+1)\,.$$

Given m_0 satisfying (27) there exists $\{m_i,\ i=1,\ldots,\nu\}$ from Corollary 4.2 of Ref. [3] such that

$$\sum_{j=0}^{\nu}c^{-1}\overline{m_k\left(z\exp\left(2\pi ij/\left(\nu+1\right)\right)\right)}m_{k'}\left(z\exp\left(2\pi ij/\left(\nu+1\right)\right)\right)=\delta_{kk'}\left(\nu+1\right).$$

Thus, reformulating the orthogonality conditions in $L^2(\mathbf{C},\nu)$, we get that the following matrix,

$$M(z)=\frac{1}{\sqrt{c\,(\nu+1)}}\begin{pmatrix} m_0(\sigma_0(z)) & m_0(\sigma_1(z)) & \ldots & m_0(\sigma_\nu(z)) \\ m_1(\sigma_0(z)) & m_1(\sigma_1(z)) & \ldots & m_1(\sigma_\nu(z)) \\ \vdots & \vdots & \ddots & \vdots \\ m_\nu(\sigma_0(z)) & m_\nu(\sigma_1(z)) & \ldots & m_\nu(\sigma_\nu(z)) \end{pmatrix},$$

is unitary for Lebesgue almost all $z\in\mathbf{C}$.

Let $O_{\nu+1}$ be the C^*-algebra generated by $\nu+1$ isometries $S_0,S_1,\ldots,S_\nu$, $\nu\in\mathbf{N}$ satisfying:

$$S_i^*S_j=\delta_{i,j}1,\qquad \sum_{i=0}^{\nu}S_iS_i^*=1.$$

The representations we consider are now realized on the Hilbert space $H=L^2(\mathbf{C},\nu)$ where the measure ν is given by $d\nu(z)=z^\nu\,dz$.

As in Ref. [18] the representation of the Cuntz algebra is defined in terms of certain maps

$$\sigma_i\colon\Omega\longrightarrow\Omega,$$

such that $\mu(\sigma_i(\Omega)\cap\sigma_j(\Omega))=0$ for $i\neq j$, as in (13), and of measurable functions $m_0,\ldots,m_\nu\colon\mathbf{C}\longrightarrow\mathbf{C}$. Also we have, for $L^2(\mathbf{C},\nu)$:

$$\int_{\mathbf{C}}f(z)\;d\nu(z)=\sum_{r\in\mathbf{Z}_{\nu+1}}\rho_r\int_{\mathbf{C}}f(\sigma_r(z))\;d\nu(z)\,,\tag{28}$$

where $\{\rho_r\}$ is a (finite) probability distribution on the cyclic group $\mathbf{Z}_{\nu+1}$.

The representations take the following form on $L^2(\mathbf{C},\nu)$

$$(S_k\xi)(z)=m_k(z)\,\xi\left(z^{\nu+1}\right),\xi\in L^2(\mathbf{C},\nu):$$

where the functions m_k are obtained from the above multiresolution construction. It is easy to verify that S_k is a representation of $O_{\nu+1}$, $z\in\mathbf{C}$ and that

$$(S_k^*\xi)(z)=\sum_{r\in\mathbf{Z}_{\nu+1}}c^{-1}\rho_r\overline{m_k(\sigma_r(z))}\xi(\sigma_r(z))\,.$$

In fact we have

$$\begin{aligned}\left(S_k^* S_{k'}\xi\right)(z) &= \sum_{r\in\mathbf{Z}_{\nu+1}} c^{-1}\rho_r\overline{m_k\left(\sigma_r\left(z\right)\right)}m_{k'}\left(\sigma_r\left(z\right)\right)\xi\left(\sigma\sigma_r\left(z\right)\right)\\ &= \delta_{k,k'}\xi\left(z\right),\end{aligned}$$

by the unitarity of the matrix $M(z)$. Similarly we may verify that

$$\sum_{k\in\mathbf{Z}_{\nu+1}}\left(S_k S_k^*\xi\right)(z) = \xi\left(z\right), \xi\in L^2\left(\mathbf{C},\nu\right)$$

As a result, we then have indeed a representation of $O_{\nu+1}$. □

4 A q-parametric construction of m_0

Let us now turn to a q-parametric construction of m_0. We start by giving a q-extension of the Hankel Fourier-Bessel integral. We use the orthogonality relations from the following result (Theorem 3.10, p. 35 of Ref. [15]) and [21].

Theorem 3 *For $x\in\boldsymbol{C}$ and $|x|<q^{-\frac{1}{2}}$, $n,m\in\mathbf{Z}$, $0<q<1$, we have*

$$\begin{aligned}\delta_{m,n} &= \sum_{k=-\infty}^{\infty} x^{k+n}q^{\frac{1}{2}(k+n)}\frac{\left(x^2q;q\right)_\infty}{(q;q)_\infty}\Phi_{1,1}\left(\begin{array}{c}0\\ x^2q\end{array}\middle| q, q^{n+k+1}\right)\\ &\quad\times x^{k+m}q^{\frac{1}{2}(k+m)}\frac{\left(x^2q;q\right)_\infty}{(q;q)_\infty}\Phi_{1,1}\left(\begin{array}{c}0\\ x^2q\end{array}\middle| q, q^{m+k+1}\right)\end{aligned}$$

where the sum is absolutely convergent, uniformly on compact subsets of the open disk $|x|<q^{-1/2}$.

We prove that the orthogonality relation of the above theorem is a q-analogue of the Hankel Fourier-Bessel integral (16). To simplify notations, we replace q by q^2 and x by q^α. For $\mathrm{Re}(\alpha)>-1$ this gives

$$\begin{aligned}\delta_{m,n} &= \sum_{k=-\infty}^{\infty} q^{(\alpha+1)(k+n)}\frac{\left(q^{2\alpha+2};q^2\right)_\infty}{(q^2;q^2)_\infty}\Phi_{1,1}\left(\begin{array}{c}0\\ q^{2\alpha+2}\end{array}\middle| q^2, q^{2n+2k+2}\right) \qquad (29)\\ &\quad\times q^{(\alpha+1)(k+n)}\frac{\left(q^{2\alpha+2};q^2\right)_\infty}{(q^2;q^2)_\infty}\Phi_{1,1}\left(\begin{array}{c}0\\ q^{2\alpha+2}\end{array}\middle| q^2, q^{2n+2k+2}\right).\end{aligned}$$

Now rewrite (29) as the transform pair

$$\begin{aligned}g\left(q^n\right) &= \sum_{k=-\infty}^{\infty} q^{(\alpha+1)(k+n)}\frac{\left(q^{2\alpha+2};q^2\right)_\infty}{(q^2;q^2)_\infty}\\ &\times\ \Phi_{1,1}\left(\begin{array}{c}0\\ q^{2\alpha+2}\end{array}\middle| q^2, q^{2n+2k+2}\right)f\left(q^k\right),\end{aligned}$$

$$
\begin{aligned}
f\left(q^{k}\right) &= \sum_{k=-\infty}^{\infty} q^{(\alpha+1)(k+n)} \frac{\left(q^{2\alpha+2};q^{2}\right)_{\infty}}{\left(q^{2};q^{2}\right)_{\infty}} \\
&\times \quad \Phi_{1,1}\left(\begin{array}{c} 0 \\ q^{2\alpha+2} \end{array} \middle| q^{2}, q^{2n+2k+2}\right) g\left(q^{n}\right),
\end{aligned}
$$

where f, g are L^2-functions on the set $\left\{q^k : k \in \mathbf{Z}\right\}$ with respect to the counting measure. Insert in the above formulae $J_\alpha(x;q)$, i.e., the q-Bessel function given by

$$
\begin{aligned}
J_{\alpha}(x;q) &= \frac{\left(q^{\alpha+1};q\right)_{\infty}}{(q;q)_{\infty}} x^{\alpha} \sum_{k=0}^{\infty} \frac{(-1)^{k} q^{\binom{k+1}{2}} x^{2k}}{\left(q^{\alpha+1};q\right)_{k}(q;q)_{k}} \\
&= \frac{\left(q^{\alpha+1};q\right)_{\infty}}{(q;q)_{\infty}} x^{\alpha} \Phi_{1,1}\left(\begin{array}{c} 0 \\ q^{\alpha+1} \end{array} \middle| q, x^{2} q\right), x \in \mathbf{R}
\end{aligned}
$$

instead of $\frac{\left(q^{\alpha+1};q\right)_{\infty}}{(q;q)_{\infty}} x^{\alpha} \Phi_{1,1}$ and replace $f\left(q^k\right)$ and $g\left(q^n\right)$ respectively by $q^k f\left(q^k\right)$ and $q^n g\left(q^n\right)$. This implies that $xf(x)$ and $xg(x)$ have to be L^2-functions respect to the d_q measure on the set $\left\{q^k : k \in \mathbf{Z}\right\}$, see [21]. Hence we have

$$
\begin{aligned}
g\left(q^{n}\right) &= \sum_{k=-\infty}^{\infty} q^{2k} J_{\alpha}\left(q^{k+n};q^{2}\right) f\left(q^{k}\right), \\
f\left(q^{k}\right) &= \sum_{n=-\infty}^{\infty} q^{2n} J_{\alpha}\left(q^{k+n};q^{2}\right) g\left(q^{n}\right),
\end{aligned} \tag{30}
$$

and the result follows.

Remark 4 *When $q \longrightarrow 1$ with the condition*

$$
\frac{\log(1-q)}{\log q} \in 2\mathbf{Z},
$$

we can replace q^k and q^n in (30) *by $(1-q)^{\frac{1}{2}} q^k$ and $(1-q)^{\frac{1}{2}} q^n$ respectively. By using the following q-integral notation,* [21], [15]

$$
\int_{0}^{\infty} f(t)\, d_{q}t = (1-q) \sum_{k=-\infty}^{\infty} f\left(q^{k}\right) q^{k}, \tag{31}
$$

then (30) *takes the form*

$$
\begin{aligned}
g(\lambda) &= \int_{0}^{\infty} f(x)\, J_{\alpha}\left((1-q)\lambda x; q^{2}\right) x\, d_{q}(x), \\
f(x) &= \int_{0}^{\infty} g(\lambda)\, J_{\alpha}\left((1-q)\lambda x; q^{2}\right) \lambda\, d_{q}(\lambda),
\end{aligned}
$$

where λ in the first identity, and x in the second identity, take the values q^n, $n \in \mathbf{Z}$. For $q \longrightarrow 1$ we therefore obtain, at least formally, the Hankel transform pair

$$\begin{aligned} g(\lambda) &= \int_0^\infty f(x) J_\alpha(\lambda x) x\, dx, \\ f(x) &= \int_0^\infty g(\lambda) J_\alpha(\lambda x) \lambda\, d\lambda. \end{aligned}$$

We construct a q-analogue of a multiresolution via q-Hankel transforms. To achieve that, let us proceed as we did in the previous section; but now we replace the Hankel transform by the deformed one using a q-measure. Let us consider as before the space $L^2(\mathbf{C})$, but with the measure $d\nu(z)$ replaced by the q-measure $d\nu_q(z)$, i.e., $d\mu_q(z) = z^\nu\, d_q(z)$, see [15],[21]. Assume φ to be the function on $\mathbf{C}$ defined for $|z| \leq 1$ by

$$\varphi\left(|z|\, e^{i\,\mathrm{Arg}(z)}\right) = \begin{cases} 1 & \text{if } 0 \leq \mathrm{Arg}(z) \leq \alpha, \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha = \frac{2\pi}{m}$, $m \in \mathbf{N}$. Take then V_0 to be the closed span in $L^2(\mathbf{C}, \nu_q(z))$ of $\{\varphi[(|z|+k)\exp(i(\mathrm{Arg}(z)+N\alpha))]\}$, with $k, m \in \mathbf{Z}$, $1 \leq N \leq m$. Let U be the scaling operator (25). Let

$$V_j = \mathrm{span}\left\{\varphi\left[\left(\frac{|z|}{(\nu+1)^j} + k\right) e^{i(\mathrm{Arg}(z)+N\alpha)}\right]\right\}_{j\in\mathbf{Z},\ 1\leq N\leq m}.$$

Let ξ be a function on $L^2(\mathbf{C}, \nu_q(z))$ given by

$$\xi(z) = \sum_k a_k \{\varphi[(|z|+k)\exp(i(\mathrm{Arg}(z)+N\alpha))]\}.$$

Assume $U\varphi \in V_0$, i.e.,

$$(U\varphi)(z) = \sum_k a_k \varphi[(|z|+k)\exp(i(\mathrm{Arg}(z)+N\alpha))].$$

Proposition 5 *With the assumptions above, we conclude that the properties* i)–iv) *of a multiresolution are satisfied.*

Proof. i) follows from the fact that the φ's have disjoint support on $L^2(\mathbf{C}, \nu_q)$. ii) holds as before and iii) follows from i). By the density of step functions on $L^2(\mathbf{C}, d\nu_q)$ also iv) follows.

If $\xi \in V_{-j}$ then $U^j \xi \in V_0$. By applying the q-Hankel transform of order ν we get

$$H_\nu^q(\varphi[(|z|+k)\exp(i(\mathrm{Arg}(z)+N\alpha))]; t) = m_0(t)\, H_0^q\left(\frac{1}{z}; t\right),$$

where we denote by $H_\nu^q(z;t)$ the q-Hankel transform to avoid confusion wth the usual non deformed transform. The Plancherel Theorem for Hankel transforms extends in a natural way to the case of q-Hankel transforms where it takes the following form:

$$\iint_{\mathbf{C}} tF(t)\,G(t)\;d\nu_q(t) = \iint_{\mathbf{C}} zf(z)\,g(z)\;d\nu_q(z),$$

where $d\nu_q(z)$ is the q-measure (31).

Then by using the orthogonality of $\varphi\left[(|z|+k)\exp\left(i\left(\operatorname{Arg}(z)+N\alpha\right)\right)\right]_{k\in\mathbf{Z}}$ and the following fact:

$$\int_0^1 |z|^{\nu+1}\;d_q|z| = \frac{1-q}{1-q^{\nu+2}},\, 0<q<1$$

we have

$$\begin{aligned}
\langle\varphi^{(k,N)}\mid\varphi^{(0,0)}\rangle &\equiv \iint_{\mathbf{C}}\varphi^{(k,N)}(z)\,\overline{\varphi^{(0,0)}(z)}z\,d\mu_q(z) \qquad (32)\\
&= \int_0^\infty\int_0^{2\pi}\varphi\left[(|z|+k)\exp\left(i\left(\operatorname{Arg}(z)+N\alpha\right)\right)\right]\\
&\times\ \overline{\varphi\left[|z|\exp\left(i\operatorname{Arg}(z)\right)\right]}\\
&\times\ |z|^{\nu+1}\exp\left(i\operatorname{Arg}(z)(\nu+1)\right)\;d|z|\;d\operatorname{Arg}(z)\\
&= \int_0^1|z|^{\nu+1}\,\delta_{k,0}\,d|z|\\
&\times\ \int_0^\alpha\exp\left(i\operatorname{Arg}(z)(\nu+1)\right)\delta_{N,0}\,d\operatorname{Arg}(z)\\
&= \frac{1-q}{1-q^{\nu+2}}\delta_{k,0}\frac{e^{i\alpha(\nu+1)}-1}{i(\nu+1)}\delta_{N,0}.
\end{aligned}$$

By the Plancherel theorem, we then have

$$\begin{aligned}
&\frac{1-q}{1-q^{\nu+2}}\delta_{k,0}\frac{e^{i\alpha(\nu+1)}-1}{i(\nu+1)}\delta_{N,0}\\
&= \iint_{\mathbf{C}} H_\nu^q\left(\varphi^{(k,N)}(z);t\right)\overline{H_\nu^q\left(\varphi^{(0,0)}(z);t\right)}t\,d\nu_q(t).
\end{aligned}$$

The left-hand side can then be rewritten as

$$\int_0^1\int_0^\alpha H_\nu^q\left(\varphi^{(k,N)}(z);t\right)\overline{H_\nu^q\left(\varphi^{(0,0)}(z);t\right)}t^{\nu+1}\,d_qt\,d\operatorname{Arg}(t).$$

Upon a change of variable setting $\theta=\operatorname{Arg}(t+2\pi j)$,$j\in\mathbf{Z}$ the latter expression is equal

$$\int_0^1|t|^{\nu+1}\;d_q|t|\int_0^\alpha e^{i\theta(\nu+1)}\sum_j H_\nu^q\left(\varphi^{(k,N)}(z);|t|\,e^{i\theta}\right)\overline{H_\nu^q\left(\varphi^{(0,0)}(z);|t|\,e^{i\theta}\right)}\,d\theta.$$

Comparing the previous two formulae for $k = N = 0$ we get

$$\sum_j \left| H_\nu^q \left(\varphi^{(0,0)}; |t| \, e^{i\theta} \right) \right|^2 - \frac{1-q}{1-q^{\nu+2}} \frac{e^{i\alpha(\nu+1)} - 1}{i(\nu+1)} = 0 \qquad \nu_q - \text{ a.e.}$$

Rewriting the above in terms of m_0 we have

$$\sum_j \left| m_0 \left(t e^{2\pi i j} \right) \right|^2 \left| H_0^q \left(\frac{1}{z}; |t| \, e^{i\theta} \right) \right|^2 = \frac{1-q}{1-q^{\nu+2}} \frac{e^{i\alpha(\nu+1)} - 1}{i(\nu+1)}.$$

From (32) we have

$$\begin{aligned} \int_0^1 |t|^{\nu-1} \; d_q |t| \int_0^\alpha e^{i\theta(\nu+1)} \sum_j \left| m_0 \left(t e^{2\pi i j} \right) \right|^2 \, d\theta & \\ = \frac{1-q}{1-q^{\nu+2}} \delta_{k,0} \frac{e^{i\alpha(\nu+1)} - 1}{i(\nu+1)} \delta_{N,0} & \\ = \frac{1-q^\nu}{1-q^{\nu+2}} \int_0^1 |t|^{\nu-1} \; d_q |t| \int_0^\alpha e^{i\theta(\nu+1)} \, d\theta, & \end{aligned}$$

thus

$$\left(\frac{1}{\nu+1} \right) \sum_j \left| m_0 \left(t e^{2\pi i j/(\nu+1)} \right) \right|^2 = \frac{1-q^\nu}{1-q^{\nu+2}}.$$

Set $c_q = \frac{1-q^\nu}{1-q^{\nu+2}}$; thus we get $\frac{1}{c_q(\nu+1)} \sum_j \left| m_0 \left(t e^{2\pi i j/(\nu+1)} \right) \right|^2 = 1$.

We notice that the Bessel functions have a "multiplicative periodicity" on the unit circle in the following sense:

$$J_\nu \left(z e^{\pi i k} \right) = e^{\pi i k \nu} J_\nu (z) .$$

This implies that

$$c_q^{-1} \sum_{j=0}^{\nu} \left| m_0 \left(t e^{2\pi i j/(\nu+1)} \right) \right|^2 = (\nu+1) .$$

Thus a q-analogue of Theorem 2 holds. As in the previous section we construct representations of the Cuntz algebra in terms of the functions m_i whose existence is guaranteed from Corollary 4.2 of Ref. [3].

As before we construct representations of the algebra $O_{\nu+1}$ associated to the above multiresolution for the q-deformed case. The representations are realized on a Hilbert space $H = L^2 (\mathbf{C}, d\nu_q)$ where the measure is given by $d\nu_q (z) = z^\nu \, d_q z$. □

We now turn to the representation of the Cuntz algebra $O_{\nu+1}$. It is given in terms of certain maps

$$\sigma_k \colon \Omega \longrightarrow \Omega, \qquad \sigma_k (z) = \sigma_0 (z) \, e^{ik2\pi/(\nu+1)}, \nu \in \mathbf{N} \tag{33}$$

where

$$\sigma_0(z)^{\nu+1} = z, z \in \mathbf{C},$$

such that $\mu_q(\sigma_i(\Omega) \cap \sigma_j(\Omega)) = 0$ for $i \neq j$. Also, for $f \in L^2(\mathbf{C}, d\nu_q)$

$$\int_{\mathbf{C}} f(z)\, d\nu_q(z) = \sum_{r \in \mathbf{Z}_{\nu+1}} \rho_r \int_{\mathbf{C}} f(\sigma_r(z))\, d\nu_q(z). \tag{34}$$

In fact, we have $\mu_q(\sigma_r(E)) = \rho_r \mu_q(E)$, for Borel subsets $E \subset \mathbf{C}$.

The representation takes the following form on $L^2(\mathbf{C}, d\nu_q)$:

$$(S_k \xi)(z) = m_k(z)\, \xi\left(z^{\nu+1}\right),$$

where the functions m_k are obtained from the above multiresolution construction. Then we have

$$(S_k^* \xi)(z) = \sum_{r \in \mathbf{Z}_{\nu+1}} c_q^{-1} \rho_r \overline{m_k(\sigma_r(z))} \xi(\sigma_r(z)).$$

Thus:

$$\begin{aligned}
(S_k^* S_{k'} \xi)(z) &= \sum_{r \in \mathbf{Z}_{\nu+1}} c_q^{-1} \rho_r \overline{m_k(\sigma_r(z))} m_{k'}(\sigma_r(z))\, \xi(\sigma \sigma_r(z)) \\
&= \delta_{k,k'} \xi(z),
\end{aligned}$$

by the unitarity of the matrix $M(z)$. We have used the convention $\sigma(z) = z^{\nu+1}$ and the fact that $\sigma \circ \sigma_r = \mathrm{id}$ for all r. It is easy similarly to verify that

$$\sum_{k \in \mathbf{Z}_{\nu+1}} (S_k S_k^* \xi)(z) = \xi(z).$$

As a result, we then have a representation of $O_{\nu+1}$.

5 Multiresolution analysis

We study now a particular case of a construction of a multiresolution. We then see how to construct a representation of the Cuntz algebra. It is interesting to see that for the corresponding representation so constructed we get a q-number related to the modulus of a Markov trace [27] for compact quantum groups of type B.[29]

Let us consider

$$V_0 = \text{closed span} \left\{ \left[h\left(q^k - z\right) - h\left(q^{k+1} - z\right) \right] \right\}_{k \in \mathbf{Z}}, 0 < q < 1$$

Consider the step function given by

$$\varphi_\nu^{(k)}(z, q) = \left[h\left(q^k - z\right) - h\left(q^{k+1} - z\right) \right]. \tag{35}$$

Set

$$[\nu+1]_{q^2}=\frac{1-q^{2(\nu+1)}}{1-q^2}$$

and define the scaling

$$Uf(z)=(\nu+1)^{-\frac{1}{2}}f\left((\nu+1)^{-1}z\right),f\in L^2(\mathbf{C},\nu_q)$$

Assume $U\varphi_\nu^{(k)}\in V_0$, then

$$U\varphi_\nu^{(k)}(z,q)=\sum_k a_k\left[h\left(q^k-z\right)-h\left(q^{k+1}-z\right)\right].$$

It follows

$$U^j\varphi_\nu^{(k)}(z,q)=\sum_k a_k\left[h\left(q^k-\frac{z}{(\nu+1)^j}\right)-h\left(q^{k+1}-\frac{z}{(\nu+1)^j}\right)\right].$$

Let $V_j=U^jV_0$, so that if $f\in V_j$, $U^{-j}f\in V_0$. The set $\left\{\varphi_\nu^{(k)}\right\}_{k\in Z}$ is an orthonormal set in $L^2(\mathbf{C})$. In fact

$$h\left(q^k-z\right)-h\left(q^{k+1}-z\right)=\begin{cases}1 & \text{if } q^{k+1}<|z|<q^k,\\ 0 & \text{otherwise,}\end{cases}$$

are defined for $q^{k+1}<|z|<q^k$ in the annulus of $r=q^{k+1}$, $R=q^k$. It follows that the set (35) $\left\{\varphi_\nu^{(k)}(z,q)\right\}_{k\in\mathbf{Z}}$ is orthogonal in $L^2(\mathbf{C})$ since the functions $\varphi_\nu^{(k)}$ have disjoint support. Actually the set is orthogonal in $L^2(\mathbf{T})$ since for $k\longrightarrow\infty$, we have $q^k\longrightarrow 0$, and for $k\longrightarrow 0$, we have $q^k\longrightarrow 1$. Let

$$\xi(z)=\sum_k a_k\left[h\left(q^k-z\right)-h\left(q^{k+1}-z\right)\right]. \tag{36}$$

By applying the q-Hankel transform $\xi\to\hat{\xi}$ to both sides of (36) we get then

$$\hat{\xi}(t)=\sum_k a_k H_\nu^q\left[\left(h\left(q^k-z\right)-h\left(q^{k+1}-z\right)\right);t\right],$$

which implies

$$\begin{aligned}\hat{\xi}(t)=H_0^q\left(\frac{1}{z};t\right)\times&\\ \left[\sum_k a_k\left(q^{k(\nu+1)}J_{\nu+1}\left((1-q)tq^k;q\right)-q^{(k+1)(\nu+1)}J_{\nu+1}\left((1-q)tq^{k+1};q\right)\right)\right]&\\ =\left[\sum_k a_k q^{k(\nu+1)}\right]\left[J_{\nu+1}\left((1-q)tq^k;q\right)-q^{\nu+1}J_{\nu+1}\left((1-q)tq^{k+1};q\right)\right]&\\ \times H_0^q\left(\frac{1}{z};t\right).&\end{aligned}$$

Using the Plancherel theorem for q-Hankel transforms and orthogonality of $\left\{\varphi_\nu^{(k)}\right\}_{k\in\mathbf{Z}}$ as before, since we have

$$\delta_{k,0} = 1 = \frac{1}{1-q}\int_q^1 t\, d_q t,$$

the left-hand side becomes then

$$\begin{aligned} 0 &= \int_q^1 \left[\sum_{j\in\mathbf{Z}} q^{2j} H_\nu^q \left(\left(h\left(q^k - z\right) - h\left(q^{k+1} - z\right); q^j s \right) \right) \right. \\ &\times \left. \overline{H_\nu^q \left(\left(h\left(1 - z\right) - h\left(q^1 - z\right); q^j s \right) \right)} - \frac{1}{1-q^{2(\nu+1)}} \right] s\, d_q s, \end{aligned}$$

so that almost everywhere with respect to d_q,

$$\begin{aligned} &\sum_{j\in\mathbf{Z}} q^{2j} H_\nu^q \left(\left(h\left(q^k - z\right) - h\left(q^{k+1} - z\right); q^j s \right) \right) \\ \times\ &\overline{H_\nu^q \left(\left(h\left(1 - z\right) - h\left(q^1 - z\right); q^j s \right) \right)} = \frac{1}{1-q^{2(\nu+1)}}. \end{aligned}$$

Now we have by using the above

$$\begin{aligned} &\sum_{j\in\mathbf{Z}} q^{2j} H_\nu^q \left(\left(h\left(q^k - z\right) - h\left(q^{k+1} - z\right); q^j t \right) \right) \\ \times\ &\overline{H_\nu^q \left(\left(h\left(1 - z\right) - h\left(q^1 - z\right); q^j t \right) \right)} = \sum_{j\in\mathbf{Z}} \left| m_0\left(tq^j\right) \right|^2 \left| H_0^q\left(\frac{1}{z}; tq^j\right) \right|^2. \end{aligned}$$

Hence we have:

$$\sum_{j\in\mathbf{Z}} \left| m_0\left(tq^j\right) \right|^2 \left| H_0^q\left(\frac{1}{z}; tq^j\right) \right|^2 = \frac{1}{1-q^{2(\nu+1)}}.$$

By a similar argument as above we get the special property for the function m_0:

$$\sum_{j\in\mathbf{Z}} \left| m_0\left(tq^j\right) \right|^2 \left| H_0^q\left(\frac{1}{z}; tq^j\right) \right|^2 = \frac{1}{1-q^{2(\nu+1)}}. \tag{37}$$

Observe that in this case since $q \le |t| \le 1$ and then from $q \le q^{1-j} \le |t|\, q^{-j} \le q^{-j} \le 1$ we have $|t| \le q^j \le 1$ and then

$$j \le \frac{\log|t|}{\log q}.$$

For $|t| = q$, $j = 1$ and for $|t| = 1$, $j = 0$. Hence the sum in (37) reduces to a finite sum, by using a similar argument as for the Haar wavelet multiresolution.[1] For a scale $\nu + 1$, $\nu \in \mathbf{N}$ we thus have

$$\sum_{j=0}^{\nu} \left| m_0 \left(tq^j \right) \right|^2 \left| H_0^q \left(\frac{1}{z}; tq^j \right) \right|^2 = \frac{1}{1 - q^{2(\nu+1)}}.$$

In this case we should note that $\left| H_0 \left(\frac{1}{z}; tq^j \right) \right|^2 = q^{-2j}$. Thus it follows:

$$\sum_{j=0}^{\nu} q^{-2j} \left| m_0 \left(tq^j \right) \right|^2 = \frac{1}{1 - q^{2(\nu+1)}}.$$

Set $d_q = \frac{1}{1-q^{2(\nu+1)}}$; then $d_q^{-1} \sum_{j=0}^{\nu} q^{-2j} \left| m_0 \left(tq^j \right) \right|^2 = 1$.

With the function m_0 given choose $m_1, \ldots, m_\nu$ in $L^2 \left(\mathbf{T},, \nu_q \right)$ such that

$$\sum_{j=0}^{\nu} q^{-2j} m_r \left(tq^j \right) \overline{m_{r'} \left(tq^j \right)} = \delta_{r,r'} \frac{1}{1 - q^{2(\nu+1)}}. \tag{38}$$

Define the functions $\psi_1, \psi_2, \ldots, \psi_\nu$ by the formula:

$$H_\nu^q \left(\psi_r^{(j,m)} (z) ; t (\nu + 1) \right) = m_r (t) \, H_0^q \left(\frac{1}{z}; t \right). \tag{39}$$

Concretely the functions in (39) are $\psi_r^{(j,m)} (z) = \psi_r \left((\nu + 1)^{-m} z - q^j \right)$. Then using (38) and (39) it follows that

$$\left\{ (\nu + 1)^{\frac{-m}{2}} \psi_r^{(j,m)} (z) \right\}_{j,m}$$

is an orthogonal basis for the space $V_{-1} \cap V_0^\perp$ and then by iii) and iv) they form an orthogonal basis for $L^2 \left(\mathbf{C}, \nu_q \right)$.

Now reformulating (38), the orthonormality of $\left\{ (\nu + 1)^{\frac{-m}{2}} \psi_r^{(j,m)} (z) \right\}_{j,m}$ is equivalent to the following matrix $M (t)$ with entries,

$$\frac{1}{\sqrt{d_q (\nu + 1)}} \begin{pmatrix} \sqrt{\rho_0} m_0 (\sigma_0 (t)) & \sqrt{\rho_1} m_0 (\sigma_1 (t)) & \ldots & \sqrt{\rho_\nu} m_0 (\sigma_\nu (t)) \\ \sqrt{\rho_0} m_1 (\sigma_0 (t)) & \sqrt{\rho_1} m_1 (\sigma_1 (t)) & \ldots & \sqrt{\rho_\nu} m_1 (\sigma_\nu (t)) \\ \vdots & \vdots & \ddots & \vdots \\ \sqrt{\rho_0} m_\nu (\sigma_0 (t)) & \sqrt{\rho_1} m_\nu (\sigma_1 (t)) & \ldots & \sqrt{\rho_\nu} m_\nu (\sigma_\nu (t)) \end{pmatrix},$$

being unitary, where $\rho_j = q^{-2j}$.

The class of representations of the algebra $O_{\nu+1}$ associated to the above multiresolution construction is given as in the previous cases in terms of the functions m_i and of the maps σ_i. The representations are realized on the Hilbert space $H = L^2 \left(\mathbf{C}, \nu_q \right)$, where as before $d\nu_q (z) = z^\nu \, d_q z$. A similar construction works for the case $q = 1$ where we use classical Bessel functions and the usual Hankel transform.

Define the representation of the Cuntz algebra in terms of certain maps (analogous to (33)):

$$\sigma_i \colon \Omega \longrightarrow \Omega, \qquad \sigma_i(z) = \sigma_0(z)\, q^i,$$

where

$$\sigma_0(z)^{\nu+1} = z,$$

such that $\mu_q(\sigma_i(\Omega) \cap \sigma_j(\Omega)) = 0$ for $i \neq j$. Hence, the system (13) here will be the N-sheeted Riemann surface of $\sqrt[N]{z}$. Also for $L^2(\mathbf{C}, d\nu_q)$

$$\int_{\mathbf{C}} f(z)\, d\nu_q(z) = \sum_{r \in \mathbf{Z}_{\nu+1}} \rho_r \int_{\mathbf{C}} f(\sigma_r(z))\, d\nu_q(z)\,, \tag{40}$$

which is the analogue of (28). In fact, $\mu_q(\sigma_r(E)) = \rho_r \mu_q(E)$ with $\rho_i = q^{-2i}$, for Borel sets $E \subset \mathbf{C}$.

The representation takes the following form on $L^2(\mathbf{C}, \nu_q)$:

$$(S_k \xi)(z) = m_k(z)\, \xi\left(z^{\nu+1}\right),$$

where the functions m_k are obtained from the above multiresolution construction. By using (40) we have

$$(S_k^* \xi)(z) = \sum_{r \in \mathbf{Z}_{\nu+1}} d_q^{-1} \rho_r \overline{m_k(\sigma_r(z))} \xi(\sigma_r(z))\,.$$

Thus:

$$\begin{aligned} (S_k^* S_{k'} \xi)(z) &= \sum_{r \in \mathbf{Z}_{\nu+1}} d_q^{-1} \rho_r \overline{m_k(\sigma_r(z))} m_{k'}(\sigma_r(z))\, \xi(\sigma \sigma_r(z)) \\ &= \delta_{k,k'} \xi(z)\,, \end{aligned}$$

by the unitarity of the matrix $M(z)$. It is easy to verify that

$$\sum_{k \in \mathbf{Z}_{\nu+1}} (S_k S_k^* \xi)(z) = \xi(z)\,.$$

We then have a representation of $O_{\nu+1}$. The interesting feature in this case is the fact that the q-number

$$\frac{1}{1 - q^{2\nu+2}} = \frac{1}{1 - q^2} [\nu + 1]_{q^2}^{-1}$$

appearing in the orthogonality relations is exactly a multiple of the modulus of the Markov trace[27] associated to the compact quantum group of type B.

Then we can perform a Fourier-type analysis over the cyclic group $Z_{\nu+1}$ introducing

$$A_{i,j}(z) = \left(\frac{1}{1-q^{2(\nu+1)}}\right)^{-1} \sum_{\omega:\omega^{\nu+1}=z} \omega^{-j} m_i(\omega)$$

and the inverse transform

$$m_i(z) = \sum_{j=0}^{\nu} z^j A_{i,j}\left(z^{\nu+1}\right).$$

6 Tight Frames, deformed Tight Frames and representations of $O_{\nu+1}$

In this section we construct tight frames giving rise to certain representations of the Cuntz algebra.

The representations we will consider are realized on a Hilbert space $H = L^2(\Omega, \mu)$ where Ω is a measure space and μ is a probability measure on Ω.

A frame is a set of non-independent vectors which can be used to construct an explicit and complete expansion for every vector in the space. Thus we have the following definition:

Definition 6 *A family of functions* $\{\varphi_j\}_{j\in J}$ *in a Hilbert space* H *is called a frame if there exist* $0 < A < \infty$, $0 < B < \infty$ *so that for all* f *in* H *we have:*

$$A\|f\|^2 \leq \sum_{j\in J} |\langle f \mid \varphi_j\rangle|^2 \leq B\|f\|^2.$$

We call A and B the frame bounds. If the two frame bounds are equal then as in Ref. [1] the frame will be called a tight frame. Thus in a tight frame we have, for all $f \in H$,

$$\sum_{j\in J} |\langle f \mid \varphi_j\rangle|^2 = A\|f\|^2,$$

where $\langle f \mid \varphi_j\rangle$ are the Fourier coefficients.

We construct tight frames but instead of a Fourier transform we use the Hankel transform (defined in the previous sections). We will see that the construction will then extend to a q-deformed tight frame.

Let us start with functions $m_0, m_1, \ldots, m_\nu \colon \mathbf{T} \to \mathbf{C}$ such that the following $\nu+1 \times \nu+1$ matrix

$$M(t) = \frac{1}{\sqrt{\nu+1}} \begin{pmatrix} m_0(\sigma_0(t)) & m_0(\sigma_1(t)) & \ldots & m_0(\sigma_\nu(t)) \\ m_1(\sigma_0(t)) & m_1(\sigma_1(t)) & \ldots & m_1(\sigma_\nu(t)) \\ \vdots & \vdots & \ddots & \vdots \\ m_\nu(\sigma_0(t)) & m_\nu(\sigma_1(t)) & \ldots & m_\nu(\sigma_\nu(t)) \end{pmatrix}$$

is unitary for almost all $z \in \mathbf{T}$. Assume that $m_0(0) = 1$ and that the following infinite product:

$$H_k(\varphi(z); t) = \prod_{l=1}^{\infty} m_0\left((\nu+1)^{-l} t\right)$$

converges pointwise almost everywhere. By Ref. [1] it follows from the condition

$$\sum_{j=0}^{\nu} \left| m_0\left(te^{\pi ij}\right) \right|^2 = 1$$

that $H_k(\varphi(z); t) \in L^2(\mathbf{T})$, and that $\|\varphi\|_2 \leq 1$. Let us now define $\psi_1, \psi_2, \ldots, \psi_\nu$ by the formula:

$$H_{k+j}\left(\psi_r^{(j,m)}(z); t(\nu+1)\right) = m_r(t) H_0\left(\frac{1}{z}; t\right).$$

Then we have that the system

$$\left\{\psi_r^{(j,m)}(z)\right\}_{j,m}$$

is not an orthogonal set with respect to Lebesgue measure on $\mathbf{R}$, so

$$\left\{\psi_r^{(j,m)}(z)\right\}_{j,m}$$

is not an orthogonal basis for $L^2(\mathbf{R})$, but only a tight frame in the sense that

$$\sum_{m,j,r} \left| \left\langle f \;\middle|\; \psi_r^{(j,m)}(z) \right\rangle \right| = \|f\|^2$$

for all $f \in L^2(\mathbf{R})$.

Let us specialize to the following case on the space $L^2(\mathbf{T},\mu)$ with $d\mu(z) = z^{-1}\,dz$:

$$m_0(z) = \sum_{k \in \mathbf{Z}} b_k J_k(z) \quad \text{and} \quad m_r(\sigma_j(z)) = \sum_{k \in \mathbf{Z}} b_k J_{k+r}\left(ze^{\pi ij}\right)$$
$$\text{where } \sigma_j(z) = \sigma_0(z) e^{\pi ij}.$$

The unitarity of the matrix $M(z)$ implies the following conditions:

1. For the diagonal entries we have:

$$\sum_{r=0}^{\nu} \left| m_r\left(ze^{\pi ir}\right) \right|^2 = \sum_{r=0}^{\nu} \sum_{k,l} b_k \overline{b_l} J_{k+r}(z) \overline{J_{l+r}(z)} e^{\pi ir(k-l)}.$$

Since we have the following:

$$\int_{|z|=1} \sum_{r=0}^{\nu} |m_0(\sigma_r(z))|^2 \, d\mu(z) = 1 = \frac{1}{2\pi i} \int_{|z|=1} d\mu(z),$$

for $k = l$ the Residue Theorem gives the following:

$$1 = (\nu + 1) \sum_k |b_k|^2 \frac{1}{(k!2^k)^2}.$$

2. For the off-diagonal entries, i.e., for $k' \neq l'$, we have:

$$\sum_{r=0}^{\nu} m_{k'} \left(te^{\pi ir}\right) \overline{m_{l'} \left(te^{\pi ir}\right)} = 0\,;$$

then we get $\sum_{k',l'} b_{k'} \overline{c_{l'}} = 0$ and then $\sum_{k'} b_{k'} \overline{c_{n+k'}} = 0$, $l' = n+k'$, by using the "multiplicative periodicity" of the Bessel functions with respect to the argument. Define now $\psi_1, \psi_2, \ldots, \psi_\nu$ such that

$$(\nu + 1)^{\frac{1}{2}} H_{k+j} \left(\psi_r \left((z)\,; (\nu + 1)\, t\right)\right) = m_j\,(t)\, H_0 \left(\frac{1}{z}; t\right). \tag{41}$$

(such ψ exist by the above wavelet construction). Then the $\left\{\psi_r^{(j,m)}\,(z)\right\}_{j,m}$ are not orthogonal in $L^2\,(\mathbf{R})$ but they satisfy:

$$\sum_{m,j,r} \left| \left\langle f \,\middle|\, \psi_r^{(j,m)}\,(z) \right\rangle \right| = \|f\|^2$$

for all $f \in L^2\,(\mathbf{R})$. This follows as in Ref. [1], Prop. 6.2.3, from the unitarity of the matrix of the $(m_{i,j})_{i,j} = (m_i\,(\sigma_j\,(z)))_{i,j}$ and from the formula (41). It then follows that the set $\left\{\psi_r^{(j,m)}\,(z)\right\}_{j,m}$ is a tight frame.

Let us look at the case of the deformed representations of the algebra $O_{\nu+1}$. See Ref. [30] for a class of deformed representations of the Cuntz algebra related to the Jackson q-Bessel functions. We consider the space $L^2\,(\mathbf{T},\nu)$ as before, where we take the measure given by

$$d\nu\,(z) = z^{-1}\, dz,$$

and using the q-Bessel functions previously defined instead of classical Bessel functions.

Define the operators S_k on $L^2\,(\mathbf{T}, d\mu)$ by:

$$(S_k \xi)\,(z) = m_k\,(z)\, \xi \left(z^{\nu+1}\right),$$

where

$$m_0(z) = \sum_{k \in \mathbf{Z}} b_k J_k(z; q)$$

and

$$m_r\,(\sigma_j\,(z)) = \sum_{k \in \mathbf{Z}} b_k J_{k+r} \left(zq^j; q\right).$$

Hence we have:

$$\left(S_k^*\xi\right)(z) = \sum_{r\in\mathbf{Z}_{\nu+1}} \rho_r \overline{m_k\left(\sigma_r\left(z\right)\right)}\xi\left(\sigma_r\left(z\right)\right).$$

Thus:

$$\begin{aligned}\left(S_k^*S_{k'}\xi\right)(z) &= \sum_{r\in\mathbf{Z}_{\nu+1}} \rho_r \overline{m_k\left(\sigma_r\left(z\right)\right)} m_{k'}\left(\sigma_r\left(z\right)\right)\xi\left(\sigma\sigma_r\left(z\right)\right) \\ &= \delta_{k,k'}\xi\left(z\right)\end{aligned}$$

by using the unitarity of the matrix $M\left(z\right)$.

7 Markov chains and representations of O_N and $SO_q(N)$

Let (Ω, F, P) be a given probability space and let $S = \mathbf{Z}_N$ be the finite set $0, 1, ...N$. An S-valued sequence of random variables ξ_n, $n \in \mathbf{N}$ is called an S-valued Markov chain if for every $n \in \mathbf{N}$ and all $s \in S$ we have:

$$P(\xi_{n+1} = s \mid \xi_0, \dots \xi_n) = P(\xi_{n+1} = s \mid \xi_n). \tag{42}$$

where $P(\xi_{n+1} = s \mid \xi_0, \dots \xi_n)$ denotes the conditional probability of the event $(\xi_n = s)$ with respect to the random variable ξ_n and respectively to the field generated by the ξ_n which we denote by $\sigma(\xi_n)$. Similarly, $P(\xi_{n+1} = s \mid \xi_0, \dots \xi_n)$ is the conditional probability of $\xi_{n+1} = s$ with respect to $\sigma(\xi_0, \dots \xi_n)$, the σ-field generated by $\xi_0, \dots \xi_n$ Formula (42)is the Markov property of the chain ξ_n, $n \in \mathbf{N}$. The set S is called the state space and the elements of S are called the states. We construct a model associated to representations of the Cuntz algebra O_N which is a Markov chain. The transition probabilities depend on a parameter $0 < q < 1$. The Markov chain $\mathbf{P}$ gives rise to a random walk on the quantum group $SO_q(N)$. Let us start by constructing the Markov chain we are interested in. Denote by $M := \{\xi_n\}$ the following process where the ξ_n are random variables with state space $S = \mathbf{Z}_N = 0, \dots, N$. We define the following transition probabilities:

$$p(r \mid s) = P(\xi_1 = r \mid \xi_0 = s). \tag{43}$$

as in the following transition matrix:

$$\mathbf{P} = \left([N]_q\right)^{-1} \begin{pmatrix} q & q^N & q^{N-1} & \dots & q^2 \\ q^2 & q & q^N & \dots & q^3 \\ \vdots & \vdots & \vdots & \vdots & \\ q^N & q^{N-1} & q^{N-2} & \dots & q \end{pmatrix}$$

The matrix P is doubly stochastic since

$$\sum_{s\in S} p(r \mid s) = 1$$

and

$$\sum_{r\in S} p(r \mid s) = 1$$

The Markov property is clearly satisfied by construction. The transition probabilities can be written as

$$p(r \mid s) = q^{\sigma_s(r)}$$

where $\sigma_s(r) = N + r - s + 1$ mod N

8 Iterated subdivisions and projection valued measures

Let us consider the family of representations of the Cuntz algebra O_N where $N = \nu + 1$ previously constructed. A given representation of O_N restricted to its canonical maximal abelian subalgebra $C(X)$ for X a Gelfand space induces naturally a projection-valued measure on X. The isometries generating O_N provide subdivisions of the Hilbert space H in view of

$$S_i^* S_j = \delta_{ij} \mathbf{1}$$

and

$$\sum_{i=0}^{N} S_i S_i^* = \mathbf{1}.$$

In particular for every $k \in \mathbf{N}$ the subspaces :

$$H(a_1, a_2, \ldots, a_k) := S_{a_1} S_{a_2} \ldots S_{a_k} H \tag{44}$$

are mutually orthogonal and

$$\sum_{i_1, i_2, \ldots, i_k} H(a_1, a_2, \ldots, a_k) := H \tag{45}$$

If $f \in H$ and $\|f\| = 1$ then

$$\mu_f(.) :=< f, E(.)f >= \|E(.)f\| \tag{46}$$

is a probability measure on the unit interval $[0, 1]$. We want to specialize $E(.)$ to our case and compute this measure which turns out to be related to the Markov chain constructed before. Let us observe that the index labels $(a_1, a_2, \ldots, a_k)$ are used to assign N-adic partitions (e.g. the intervals $\left[\frac{a_1}{N} + \ldots + \frac{a_k}{N^k}, \frac{a_1}{N} + + \ldots + \frac{a_k}{N^k} + \frac{1}{N^k}\right]$), then we have the mapping

$$(a_1, \ldots, a_k) \to H(a_1, a_2, \ldots, a_k) \tag{47}$$

where the $(a_1, a_2, \ldots, a_k) \in \{(0, 1, \ldots, N)\}$ and the length of the interval is $\frac{1}{N^k}$. These partitions are a special case of endomorphisms

$$\sigma : X \to X \tag{48}$$

where X is a compact Hausdorff space and σ is continuous and onto. Then for every $x \in X$ we have that $card(\sigma^{-1}(x)) = \{x \in X/\sigma(y) = x\} = N$. There exists branches of the inverse, i.e. maps

$$\sigma_0, \ldots, \sigma_{N-1} : X \to X \tag{49}$$

such that

$$\sigma \circ \sigma_i = 1_X \tag{50}$$

for each $0 \leq i < N$ the above intervals written in terms of the maps are:

$$\begin{aligned} I_k(a) = \left[\frac{a_1}{N} + \frac{a_2}{N^2} + \ldots + \frac{a_k}{N^k}, \frac{a_1}{N} + \frac{a_2}{N^2} + \ldots + \frac{a_k}{N^k} + \frac{1}{N^k}\right] \\ = \sigma_{a_1} \circ \sigma_{a_2} \ldots \circ \sigma_{a_k}(X) \end{aligned}$$

The system $\sigma_a = \sigma_{a_1} \circ \sigma_{a_2} \ldots \circ \sigma_{a_k}$ forms a set of branches for $\sigma^k = \sigma \circ \sigma \ldots \circ \sigma$. and is called an N- adic systems of partitions of X. Thus for every $k \in \mathbf{Z}_+$ $\{J_k(a)\}$ is a partition indexed by $a \in \Gamma_N^k : \Gamma_N \times \Gamma_N \times \ldots \times \Gamma_N$. On the other hand, given an Hilbert space H, a partition of projections in H is a system $P(i)_{i \in I}$ of projections, i.e. $P(i) = P(i)^* = P(i)^2$ such that

$$P(i)P(j) = 0$$

if $i \neq j$ and

$$\sum_{i \in \mathbf{I}} P(i) = 1_H$$

Let $N \in \mathbf{N}$, $N \geq 2$. Suppose that for every $k \in \mathbf{N}$, there is a partition of projections $P_k(a)_{a \in \Gamma_N^k}$ such that every $P_{k+1}(a)$ is contained in some $P_{k+1}(b)$ i.e. $P_k(b)P_{k+1}(a) = P_{k+1}(a)$ then $P_k(a)_{a \in \Gamma_N^k}$ is a system of partitions of 1_H. By Lemma 3.5 [28] given an N-adic system of projections of X and $P_k(a)_{k \in \mathbf{Z}_+, a \in \Gamma_N^k}$ an N adic system of projections there is a unique normalized orthogonal projection-valued measure $E(.)$ defined on the Borel subsets of X with values in the orthogonal projections of H such that $E(J_k a) = P_k(a)$ for every $k \in \mathbf{Z}_+$, $a \in \Gamma_N^k$. Let S_i be a representation of O_N on H and let $a = (a_1, a_2, \ldots, a_k) \in \Gamma_N^k$ and $S_a := S_{(a_1)} \ldots S_{(a_k)}$ then $P_k(a) = S_{(a)} S_{(a)}^*$. Assuming then unitarity condition on the filters m_j we get

$$\begin{aligned} \mu_f(I_k(a)) = |E(I_k(a))|^2 = \left|S_{(a)} S_{(a)^*} f\right|^2 \\ = < f, S_{(a)} S_{(a)}^* f > = \left\|S_{(a)}^* f\right\|^2 \\ = \sum \left|< e_n, S_{(a)}^* f >\right|^2 = \sum \left|< S_{(a)} e_n, f >\right|^2 \end{aligned}$$

Using Plancherel theorem for Hankel transforms for $a = a_1$ we get that

$$\mu_f\left(I_1(a)\right) = \sum_{j\in \mathbf{Z}_N} \left|< H^q\psi_a(z), q^j tN, f >\right|^2 = \left([N]_q^2\right)^{-1} \tag{51}$$

Choosing $2j = N - r - s + 1 \ mod\ N$ we get that $< S_{a_{(j)}} e_n, f >$ gives transition probabilities of the Markov chain constructed in the previous section.

9 Markov trace and representations of the braid group B_∞

Let $\mathcal{F}$ be a category whose objects we denote by ρ, σ, τ, ... The set of arrows between a pair ρ, σ of objects will be denoted by (ρ, σ) and the identity of ρ by 1_ρ. A BWM symmetry is a linear operator G on $\rho\otimes\rho$ satisfying the Yang–Baxter equation

$$G_1G_2G_1 = G_2G_1G_2,$$

and the following BWM condition: let $E = 1 - (q - q^{-1})^{-1}(G - G^{-1})$. Then

$$EG = p^{-1}E, \qquad EGE = pE, \qquad EG^{-1}E = p^{-1}E,$$

where $p, q \in \mathbf{C}-\{0\}$ (to be specified later) and G satisfy the cubic equation

$$(G - q)(G + q^{-1})(G - p^{-1}) = 0.$$

Then E is a complex multiple of a projection:

$$E^2 = (1 + (p - p^{-1})(q - q^{-1})^{-1})E.$$

In particular for our purpose let us consider $(\mathcal{F}, G)$ a braided tensor C*-category associate to the quantum group $SO(N)$ [29]. Let $g \in B_\infty$ be an element of the infinite braid group and let $p = p(g)$ be its associated permutation written as a product of disjoint cycles of length $k_1, \ldots, k_m$ with $k_1 + k_2 + \ldots + k_m = n$. Denote by θ the braiding in the category. Then

$$\omega^{(n)}\left(\theta^{(n)}(g)\right) = \varphi^{(n)}\theta^{(n)}(g) = d_q(\rho)^{(m)}\left(\varphi\left(\theta\left(\rho,\rho\right)\right)\otimes 1_{\rho^{n-1}}\right)^n \tag{52}$$

where $\varphi(T) = C^*\otimes 1_{\rho^{n-1}} \circ 1_{\bar\rho}\otimes T\circ C\otimes 1_{\rho^{n-1}}$, $(d_q(\rho) = C^*\circ C$, $C \in (i, \rho\overline{\rho})$ and $\overline{C} \in (i, \overline{\rho}\rho)$ are intertwiners. For our purpose we let $(\mathcal{F}, G)$ be a braided tensor C*-category associate to the quantum group $SO(N)$ [29] generated by a single object $\rho = H$ and having conjugate $\bar\rho$. By $\omega^{(n)}$ we denote the Markov trace for the BWM symmetries which has modulus $q^{(2m)}(d_q(\rho)^{(-1)}$ where $(d_q(\rho)$ is the quantum dimension.

For the quantum SO(N) (see [29]), $N = 2m + 1$, the operator G has the form

$$\begin{aligned} G &= \sum_{i\neq 0}(qe_{i,i}\otimes e_{i,i} + q^{-1}e_{i,-i}\otimes e_{-i,i}) + e_{0,0}\otimes e_{0,0} + \sum_{i\neq j} e_{i,j}\otimes e_{j,i} \\ &\quad +(q - q^{-1})(\sum_{i<j} e_{i,i}\otimes e_{j,j} - \sum_{j<i} q^{\frac{i+j}{2}} e_{i,j}\otimes e_{-i,-j}). \end{aligned} \tag{53}$$

Here $\{e_{i,j}\}$ is the $N \times N$ matrix with 1 in the (i,j) position and 0 elsewhere; G acts on a finite-dimensional Hilbert space H with basis indexed by $I = \{-2m+1, -2m+3, ... -3, -1, 0, 1, 3, ..., 2m-1\}$. The element $E = 1 - (q - q^{-1})^{-1}(G - G^{-1})$ has the form

$$E = \sum_{i,j} q^{\frac{i+j}{2}} e_{i,j} \otimes e_{-i,-j}.$$

Then it is easy to see that $E^2 = xE$, $x = \sum_i q^i$.

By [29]

(i) *There exists a faithful Markov trace w given by a left inverse via a conjugate $C \in (i, \bar{\rho}\rho)$ such that $w(G) = \frac{q^{2m}}{d_q(\rho)}$ and $E = C \circ C^*$ such that $E = (q - q^{-1})^{-1}(G - G^{-1})$.*

(ii) *There exists $\tau_q \in (\mathbf{C}, \rho^2)$, a group-like element, and non-degenerate mapping given by $\tau_q \lambda = \lambda \sum_i e_i \otimes J^{-1} e_i$, where $J = (q^{j/2} \delta_{i,\bar{j}})$, $\bar{j} = N + 1 - j$. Furthermore there exists an antisymmetric tensor $\epsilon_{i_1 \ldots i_N} : \mathbf{C} \longrightarrow H^N$ which gives a non-degenerate form.*

Thus we construct a random walk on $SO_q(N)$ induced from the Markov chain as follows: choose $2m = N + j - i + 1 \ mod \ N$ and $d_q(\rho) = [N]_{(q^2)}$ presented in section 7. Thus

$$w(G) = \frac{q^{2m}}{d_q(\rho)} = p(j,i) \tag{54}$$

Thus the transition probabilities $p(j,i)$ of the Markov chain give rise to a Markov trace on $SO_q(N)$ with $N = 2m + 1$

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