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# SPECTRAL HIRZEBRUCH-MILNOR CLASSES OF SINGULAR HYPERSURFACES

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Abstract. We prove formulas for the localized Hirzebruch-Milnor class of a projective hypersurface in the case where the multiplicity of a generic hyperplane section is not 1. These formulas are necessary for the calculation of the localized Hirzebruch-Milnor class in the hyperplane arrangement case. To formulate them, we introduce the spectral Hirzebruch class transformation which may be viewed as a homology class version of (the dual of) Steenbrink spectrum for a mixed Hodge structure with a finite order automorphism. Here we need the Thom-Sebastiani theorem for the underlying filtered $D$-modules of vanishing cycles, and its rather simple proof is explained. From this we can deduce the Thom-Sebastiani theorem for the localized spectral Hirzebruch-Milnor classes in the case of hypersurfaces defined by global functions on smooth varieties. We also explain some applications to multiplier ideals, log canonical thresholds, and Du Bois singularities, etc.

## Introduction

Let $Y$ be a smooth complex projective variety with $L$ a very ample line bundle on $Y$. Let $X$ be a hypersurface section of $Y$ defined by $s \in \Gamma(Y, L^{\otimes m}) \setminus \{0\}$ for some $m \in \mathbf{Z}_{>0}$. When $m = 1$, a formula for the Hirzebruch-Milnor class (which expresses the difference between the Hirzebruch class and the virtual one) was given in [MaSaSc1] by using a sufficiently general section of $L$. By specializing to $y = -1$ and using [Sch2, Proposition 5.21], this implies a formula for the Chern-Milnor class conjectured by S. Yokura [Yo2], and proved by A. Parusiński and P. Pragacz [PaPr] (where $m = 1$). In the case of hyperplane arrangements as in [MaSaSc2], however, it is desirable to generalize this to the case $m > 1$ (since a *hypersurface* section of a hyperplane arrangement is not a hyperplane arrangement). In order to realize this, we need an inductive argument as follows.

Let $s'_1, \ldots, s'_{n+1}$ be sufficiently general sections of $L$ with $n := \dim X$. Let $a_1, \ldots, a_{n+1}$ be sufficiently general non-zero complex numbers with $|a_j|$ sufficiently small. For $j \in [1, n+1]$, set

$$\begin{aligned} &s_{a,j} := s - a_1 s_1'^m - \cdots - a_{j-1} s_{j-1}'^m, \quad X_{a,j} := s_{a,j}^{-1}(0), \quad X'_j := s_j'^{-1}(0),\\ &f_{a,j} := \big(s_{a,j}/s_j'^m\big)|_{U_j}, \quad U_j := Y \setminus X'_j, \quad \Sigma_j := \operatorname{Sing} X_{a,j} \ (= \textstyle\bigcap_{k<j} X'_k \cap \Sigma_1). \end{aligned}$$

Put $\Sigma := \operatorname{Sing} X = \Sigma_1$, and $r := \max\big\{j \mid \Sigma_j \neq \emptyset\big\} \leqslant n+1$. We denote the Hirzebruch class and the virtual one by $T_{y*}(X)$, $T_{y*}^{\mathrm{vir}}(X) \in \mathbf{H}_\bullet(X)[y]$ as in [MaSaSc1], where $\mathbf{H}_k(X) = H_{2k}^{\mathrm{BM}}(X, \mathbf{Q})$ or $\mathrm{CH}_k(X)_{\mathbf{Q}}$, see (1.1) below. In this paper, we denote by $\varphi_{f_{a,j}} \mathbf{Q}_{h,U_j}$ the mixed Hodge module on $\Sigma_j \setminus X'_j$ up to a shift of complex such that its underlying $\mathbf{Q}$-complex is the vanishing cycle complex $\varphi_{f_{a,j}} \mathbf{Q}_{U_j}$ in the sense of [De2], see [Sa3], [Sa5]. For the definition of $T_{y*}(\mathcal{M}^\bullet)$ with $\mathcal{M}^\bullet$ a bounded complex of mixed Hodge modules, see (1.1) below. In this paper we show the following.

**Theorem 1.** *We have the localized Hirzebruch-Milnor class $M_y(X) \in \mathbf{H}_\bullet(\Sigma)[y]$, satisfying*

$$T_{y*}^{\mathrm{vir}}(X) - T_{y*}(X) = (i_{\Sigma,X})_* M_y(X) \qquad \text{with}$$

$$M_y(X) = \textstyle\sum_{j=1}^{r} T_{y*}\big((i_{\Sigma_j \setminus X'_j, \Sigma})_! \, \varphi_{f_{a,j}} \mathbf{Q}_{h,U_j}\big), \tag{0.1}$$

*where $i_{A,B} : A \hookrightarrow B$ denotes the inclusion for $A \subset B$ in general.*

The assertion (0.1) can be viewed as an inductive formula, since we have the following.

**Proposition 1.** *For $j \in [1,r]$, there are equalities in $\mathbf{H}_\bullet(X_{a,j})[y]$ :*

$$\lim_{a_j\to 0} T_{y*}(X_{a,j+1}) - T_{y*}(X_{a,j}) = T_{y*}\big((i_{\Sigma_j\setminus X'_j, X_{a,j}})_!\,\varphi_{f_{a,j}}\mathbf{Q}_{h,U_j}\big). \tag{0.2}$$

The limit in (0.2) is defined by using the nearby cycle functor $\psi$ for mixed Hodge modules, see (2.3.1) below. Note that $X_{a,1} = X$, and $X_{a,r+1}$ is smooth. (If $r = n+1$, then $s_{a,n+2}$ and $X_{a,n+2}$ can be defined in the same way as above, and is smooth.) We assumed $m = 1$ in [MaSaSc1], where the formula was rather simple in the hypersurface case (since $X_{a,2}$ is smooth and $r = 1$).

In order to express more explicitly the right-hand side of (0.1–2) for the summand with $j \in [2,r]$, we use the variable $\widetilde{y} = -y$ together with the *spectral Hirzebruch class*

$$T^{\rm sp}_{\widetilde{y}*}(\mathcal{M}, T_s) \in \mathbf{H}_\bullet(X)\big[\widetilde{y}^{1/e}, \widetilde{y}^{-1/e}\big],$$

for mixed Hodge modules $\mathcal{M}$ on $X$ endowed with an action of $T_s$ of finite order $e > 0$, see also [CaMaScSh, Remark 1.3(4)]. This is defined by extending the definition of the "dual" $\mathrm{Sp}'(f,x)$ of the Steenbrink spectrum $\mathrm{Sp}(f,x)$ in [Sa8, Section 2.1]. Note that the former is called the Hodge spectrum in the definition before [DeLo, Corollary 6.24], see also [GeLoMe, Section 6.1]. (In the case $X$ is a point, the spectral Hirzebruch class is identified with the Hodge spectrum as is explained in [CaMaScSh, Remark 3.7].)

We need this refinement of Hirzebruch classes, since there is a shift of the Hodge filtration $F$ in the Thom-Sebastiani theorem for filtered $\mathcal{D}$-modules depending on the eigenvalues of the semisimple part of the monodromy $T_s$ (see Theorem (3.2) below). We can prove the following.

**Proposition 2.** *For $j \in [1,r]$, we have*

$$T^{\rm sp}_{\widetilde{y}*}\big((i_{\Sigma_j\setminus X'_j,\Sigma_j})_!\,\varphi_{f_{a,j}}\mathbf{Q}_{h,U_j}, T_s\big) \in \mathbf{H}_\bullet(\Sigma_j)\big[\widetilde{y}^{1/e}\big], \tag{0.3}$$

*with $T_s$ the semisimple part of the monodromy $T$, and moreover*

$$T^{\rm sp}_{\widetilde{y}*}\big((i_{\Sigma_j\setminus X'_j,\Sigma_j})_!\,\varphi_{f_{a,j}}\mathbf{Q}_{h,U_j}, T_s\big)^{\rm int} = T_{y*}\big((i_{\Sigma_j\setminus X'_j,\Sigma_j})_!\,\varphi_{f_{a,j}}\mathbf{Q}_{h,U_j}\big) \quad \text{in } \mathbf{H}_\bullet(\Sigma_j)[y]. \tag{0.4}$$

Here $(*)^{\rm int}$ is defined by the tensor product of the $\mathbf{Q}$-linear morphism

$$\mathbf{Q}\big[\widetilde{y}^{1/e}\big] \ni \textstyle\sum_{i\in\mathbf{N}} a_i\,\widetilde{y}^{i/e} \mapsto \sum_{i\in\mathbf{N}} a_i(-y)^{[i/e]} \in \mathbf{Q}[y],$$

with $a_i \in \mathbf{Q}$ $(i \in \mathbf{N})$ and $[i/e]$ the integer part of $i/e$. This corresponds to forgetting the action of $T_s$. For $j \in [2,r]$, set

$$Z_j := \textstyle\bigcap_{k<j} X'_k \cap U_j, \quad f'_j := (s/s'^{\,m}_j)|_{Z_j}.$$

Note that $Z_j \cap \Sigma = \Sigma_j \setminus X'_j$. We have the following.

**Theorem 2.** *For $j \in [2,r]$, there are equalities in $\mathbf{H}_\bullet(\Sigma_j)\big[\widetilde{y}^{1/e}\big]$:*

$$T^{\rm sp}_{\widetilde{y}*}\big((i_{\Sigma_j\setminus X'_j,\Sigma_j})_!\,\varphi_{f_{a,j}}\mathbf{Q}_{h,U_j}, T_s\big) = T^{\rm sp}_{\widetilde{y}*}\big((i_{\Sigma_j\setminus X'_j,\Sigma_j})_!\,\varphi_{f'_j}\mathbf{Q}_{h,Z_j}, T_s\big)\cdot\big(-\textstyle\sum_{i=1}^{m-1}\widetilde{y}^{i/m}\big)^{j-1}, \tag{0.5}$$

*where $e$ is replaced by an appropriate multiple of it if necessary.*

This follows from Thom-Sebastiani theorem for filtered $\mathcal{D}$-modules (see Theorem (3.2) below), to which we give a rather simple proof using the algebraic microlocalization as is mentioned in [Sa7, Remark 4.5]. Note that, in the proof of Theorem 2, we can apply the Thom-Sebastiani theorem only by restricting to $U_j = Y \setminus X'_j$ and moreover only after passing to the normal bundle of $Z_j \subset U_j$ by using the deformation to the normal bundle, see Section 3 below.

The above Thom-Sebastiani theorem also implies the following Thom-Sebastiani theorem for the *localized spectral Hirzebruch-Milnor classes*

$$M_{\widetilde{y}}^{\rm sp}(X) := T_{\widetilde{y}}^{\rm sp}(\varphi_f\mathbf{Q}_{h,Y}, T_s) \in \mathbf{H}_{\bullet}(\Sigma)\big[\widetilde{y}^{1/e}\big],$$

in the case where $X = f^{-1}(0)$ with $f$ a non-constant function on a smooth complex variety or a connected complex manifold $Y$ (that is, $f \in \Gamma(Y, \mathcal{O}_Y) \setminus \mathbf{C}$), and $\Sigma := \operatorname{Sing} X$.

**Theorem 3.** *Let $X_a := f_a^{-1}(0)$ with $f_a$ a non-constant function on a smooth complex variety or a connected complex manifold $Y_a$ $(a = 1, 2)$. Set $X := f^{-1}(0) \subset Y := Y_1 \times Y_2$ with $f := f_1 + f_2$. Put $\Sigma_a := \operatorname{Sing} X_a$ $(a = 1, 2)$. Then we have the equality*

$$M_{\widetilde{y}}^{\rm sp}(X) = -M_{\widetilde{y}}^{\rm sp}(X_1) \boxtimes M_{\widetilde{y}}^{\rm sp}(X_2) \quad in \;\; \mathbf{H}_{\bullet}(\Sigma)\big[\widetilde{y}^{1/e}\big], \tag{0.6}$$

*by replacing $Y_a$ with an open neighborhood of $X_a$ $(a = 1, 2)$ so that $\Sigma = \Sigma_1 \times \Sigma_2$ if necessary, where $\boxtimes$ is defined by using cross products or Künneth maps.*

This assertion holds at the level of Grothendieck groups (more precisely, in $K_0(\Sigma)\big[\widetilde{y}^{1/e}\big]$, see (3.5.1) below). Here $\Sigma \neq \Sigma_1 \times \Sigma_2$ if and only if there are non-zero critical values $c_a$ of $f_a$ $(a = 1, 2)$ with $c_1 + c_2 = 0$. Note that $X$ is always non-compact even if $X_1, X_2$ are compact, see Remarks (3.5)(iii) below. In the case of isolated hypersurface singularities, Theorem 3 is equivalent to the Thom-Sebastiani theorem for the spectrum as in [ScSt], [Va]. We can calculate the localized spectral Hirzebruch-Milnor class $M_{\widetilde{y}}^{\rm sp}(X)$ by improving the arguments in [MaSaSc1, Section 5] (keeping track of the action of $T_s$), see also [CaMaScSh, Remark 1.3]

It is known (see [CaMaScSh, Theorem 3.2], [Sch4, Corollary 3.12]) that we have

$$(i_{\Sigma,X})_*\big(M_{\widetilde{y}}^{\rm sp}(X)\big)^{\rm int} = (i_{\Sigma,X})_* M_y(X) = T_{y*}^{\rm vir}(X) - T_{y*}(X). \tag{0.7}$$

Here the first equality follows from the definition, and the last one from the short exact sequence associated with the nearby and vanishing cycles together with [Sch3] (or [MaSaSc1, Proposition 3.3]) and [Ve, Theorem 7.1]. (We may also need [MaSaSc1, Proposition 1.3.1] to show some compatibility of definitions, see the remark about $T_{y*}^{\rm vir}(X)$ after (1.1.9) below.)

Specializing to $y = -1$ (that is, $\widetilde{y} = 1$), Theorems 1 and 2 imply the corresponding assertions for the Chern classes. In fact, $T_{y*}(X)$ and $T_{y*}^{\rm vir}(X)$ respectively specialize at $y = -1$ to the MacPherson-Chern class $c(X)$ (see [Mac]) and the virtual Chern class $c^{\rm vir}(X)$ (called the Fulton or Fulton-Johnson class, see [Fu], [FJ]) with rational coefficients, see [Sch2, Proposition 5.21]. The specialization of Theorem 3 to $\widetilde{y} = 1$ is already known, see [OhYo, Section 4] (where a different sign convention is used).

We have an application of the Thom-Sebastiani type theorem to multiplier ideals $\mathcal{J}(\alpha X)$ and their graded quotients $\mathcal{G}(\alpha X)$ (see (5.2) below) as follows:

**Theorem 4.** *With the notation and the assumption of Theorem 3, we have the equalities for $\alpha \in (0, 1)$ :*

$$\mathcal{J}(\alpha X) = \textstyle\sum_{\alpha_1+\alpha_2=\alpha} \mathcal{J}(\alpha_1 X_1) \boxtimes \mathcal{J}(\alpha_2 X_2) \quad in \;\; \mathcal{O}_Y = \mathcal{O}_{Y_1} \boxtimes \mathcal{O}_{Y_2}, \tag{0.8}$$

*together with the canonical isomorphisms of $\mathcal{O}_X$-modules for $\alpha \in (0, 1)$ :*

$$\mathcal{G}(\alpha X) = \textstyle\sum_{\alpha_1+\alpha_2=\alpha} \mathcal{G}(\alpha_1 X_1) \boxtimes \mathcal{G}(\alpha_2 X_2), \tag{0.9}$$

*by replacing $Y_a$ with an open neighborhood of $X_a = f_a^{-1}(0)$ in $Y_a$ $(a = 1, 2)$ so that $\Sigma = \Sigma_1 \times \Sigma_2$ if necessary. Here we may assume $\alpha_1, \alpha_2 \in (0, \alpha)$.*

The formula (0.8) determines $\mathcal{J}(\alpha X)$ for any $\alpha \in \mathbf{Q}$, since $\mathcal{J}((\alpha + 1)X) = f\mathcal{J}(\alpha X)$ for $\alpha \geqslant 0$ and $\mathcal{J}(\alpha X) = \mathcal{O}_Y$ for $\alpha \leqslant 0$, see (5.1.2) below. The formula (0.9) for $\alpha = 1$ is more complicated (see Corollary (5.4) below), since it is closely related to the "irrationality" of the singularities of $X$ (see (4.2.5) below). Related to this, we have the following.

**Theorem 5.** *Let $X$, $Y$ be as in Theorem 1 with $X$ reduced, or $X$ be a reduced hypersurface in a smooth complex algebraic variety or a complex manifold $Y$ defined by $f\in\Gamma(Y,\mathcal{O}_Y)$. If $X$ has only Du Bois singularities, then*

$$M_0(X) := M_y(X)|_{y=0} = 0 \quad \textit{in} \;\; \mathbf{H}_\bullet(\Sigma). \tag{0.10}$$

*The converse holds if $\Sigma=\operatorname{Sing}X$ is a projective variety. More precisely, the converse holds if (0.10) holds for $(i_{\Sigma,\mathbf{P}^N})_*M_0(X)$ in $\mathbf{H}_\bullet(\mathbf{P}^N)$, where $\mathbf{P}^N$ is a projective space containing $\Sigma$.*

The first assertion of Theorem 5 is already known if $\mathbf{H}_\bullet(\Sigma)$ in (0.10) is replaced with $\mathbf{H}_\bullet(X)$, and $M_0$ with $(i_{\Sigma,X})_*M_0$ at least in the second case where $X$ is defined by a global function $f$, since we have $T^{\rm vir}_{y*}(X)|_{y=0}=td_*(\mathcal{O}_X)$, see [BrScYo, p. 6], [CaMaScSh, p. 2619], [Sch4, Corollary 2.3]. Its converse also holds if $X$ is projective. In the isolated singularity case, Theorem 5 is related to [St3, Theorem 3.12], [Is, Theorem 6.3] in the case where the smoothing is a base change of a smoothing with total space nonsingular, see (4.8) below.

Let $\mathrm{JC}(f)$ be the set of *jumping coefficients* of $f$, consisting of numbers $\alpha$ with $\mathcal{G}(\alpha X)\ne 0$ (see (5.1) below), and $\mathrm{lct}(f)$ be the *log canonical threshold* of $f$, which is by definition the minimal jumping coefficient. We have the following (which is an immediate consequence of the duality for nearby cycle functors [Sa4] together with [BuSa, Theorem 0.1]).

**Proposition 3.** *Let $X$ be a hypersurface in a smooth complex variety $Y$ (or a complex manifold $Y$) defined by $f\in\Gamma(Y,\mathcal{O}_Y)$. Let $\alpha\in(0,1)$. Then $\alpha\in\mathrm{JC}(f)$ if*

$$M^{\rm sp}_{\widetilde{y}}(X)|_{\widetilde{y}^\alpha}\ne 0 \quad \textit{in} \;\; \mathbf{H}_\bullet(\Sigma), \tag{0.11}$$

*where $M^{\rm sp}_{\widetilde{y}}(X)|_{\widetilde{y}^\alpha}$ is the coefficient of $\widetilde{y}^\alpha$ in $M^{\rm sp}_{\widetilde{y}}(X)\in\mathbf{H}_\bullet(\Sigma)\big[\widetilde{y}^{1/e}\big]$. The converse holds if $\Sigma=\operatorname{Sing}X$ is a projective variety. More precisely, if $\alpha\in\mathrm{JC}(f)\cap(0,1)$, then (0.11) holds for the image of $M^{\rm sp}_{\widetilde{y}}(X)|_{\widetilde{y}^\alpha}$ in $\mathbf{H}_\bullet(\mathbf{P}^N)$, where $\mathbf{P}^N$ is a projective space containing $\Sigma$.*

In the isolated singularity case, this is closely related to [Bu, Corollary in p. 258].

From Theorem 4 we can deduce, for instance, the following.

**Corollary 1.** *With the notation and the assumption of Theorem 4, we have*

$$\mathrm{JC}(f)\cap(0,1) = \big(\mathrm{JC}(f_1)+\mathrm{JC}(f_2)\big)\cap(0,1), \tag{0.12}$$

$$\mathrm{lct}(f) = \min\big\{1, \mathrm{lct}(f_1)+\mathrm{lct}(f_2)\big\}, \tag{0.13}$$

*by replacing $Y_a$ with an open neighborhood of $X_a$ $(a=1,2)$ so that $\Sigma=\Sigma_1\times\Sigma_2$ if necessary.*

Note that $\mathrm{JC}(f)$ is determined by (0.12) together with (5.1.5) below (since $1\in\mathrm{JC}(f)$ by looking at the smooth points of $X$). By the second equality of (4.5.2) below, we get

$$M_0(X) = \textstyle\bigoplus_{\alpha\in(0,1)} M^{\rm sp}_{\widetilde{y}}(X)|_{\widetilde{y}^\alpha} \quad \text{in} \;\; \mathbf{H}_\bullet(\Sigma). \tag{0.14}$$

So Proposition 3 may be viewed as a refinement of Theorem 5 modulo the assertion that we have $\mathrm{lct}(f)=1$ if and only if $X=f^{-1}(0)$ is reduced, and has only Du Bois singularities, see [Sa9, Theorem 0.5], [KoSc, Corollary 6.6], and (4.3.9) below.

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In Section 1 we review some basics of Hirzebruch classes, and prove Proposition 2. In Section 2 we prove Theorem 1 and Proposition 1 by using the short exact sequences associated with nearby and vanishing cycle functors of mixed Hodge modules. In Section 3 we prove Theorems 2 and 3 after giving a rather simple proof of the Thom-Sebastiani theorem for filtered $\mathcal{D}$-modules. In Section 4 some relations with rational and Du Bois singularities are explained. In Section 5 we give some applications to multiplier ideals in the hypersurface case.

## 1. Hirzebruch characteristic classes

In this section we review some basics of Hirzebruch classes, and prove Proposition 2.

**1.1. Hirzebruch classes.** For a complex algebraic variety $X$, we set in this paper

$$\mathbf{H}_k(X) := H_{2k}^{\mathrm{BM}}(X,\mathbf{Q}) \quad \text{or} \quad \mathrm{CH}_k(X)_{\mathbf{Q}}.$$

Let $\mathrm{MHM}(X)$ be the abelian category of mixed Hodge modules on $X$ (see [Sa3], [Sa5]). For $\mathcal{M}^\bullet \in D^b\mathrm{MHM}(X)$, we have the homology Hirzebruch characteristic class defined by

$$T_{y*}(\mathcal{M}^\bullet) := td_{(1+y)*}\big(\mathrm{DR}_y[\mathcal{M}^\bullet]\big) \in \mathbf{H}_\bullet(X)[y,y^{-1}]. \tag{1.1.1}$$

Here, setting $F^p = F_{-p}$ so that $\mathrm{Gr}_F^p = \mathrm{Gr}^F_{-p}$, we have

$$\mathrm{DR}_y[\mathcal{M}^\bullet] := \textstyle\sum_{i,p} (-1)^i \left[\mathcal{H}^i\mathrm{Gr}_F^p\mathrm{DR}(\mathcal{M}^\bullet)\right](-y)^p \in K_0(X)[y,y^{-1}], \tag{1.1.2}$$

with

$$td_{(1+y)*} : K_0(X)[y,y^{-1}] \to \mathbf{H}_\bullet(X)\left[y, \tfrac{1}{y(y+1)}\right] \tag{1.1.3}$$

defined by the composition of the scalar extension of the Todd class transformation

$$td_* : K_0(X) \to \mathbf{H}_\bullet(X),$$

(see [BaFuMa], where $td_*$ is denoted by $\tau$) with the multiplication by $(1+y)^{-k}$ on $\mathbf{H}_k(X)$ (see [BrScYo]). Note that $\mathcal{H}^i\mathrm{Gr}_F^p\mathrm{DR}(\mathcal{M}^\bullet) = 0$ for $|p| \gg 0$ (see for instance [Sa3, 2.2.10.5]). We have the last inclusion in (1.1.1), that is, $T_{y*}(\mathcal{M}^\bullet) \in \mathbf{H}_\bullet(X)[y,y^{-1}]$, by [Sch2, Proposition 5.21].

In this paper, we denote $a_X^*\mathbf{Q}_h \in D^b\mathrm{MHM}(X)$ by $\mathbf{Q}_{h,X}$ as in [MaSaSc1], where $\mathbf{Q}_h$ denotes the trivial $\mathbf{Q}$-Hodge structure of rank 1 and type $(0,0)$ (see [De1]), and $a_X : X \to pt$ is the structure morphism, see [Sa5]. The *homology Hirzebruch characteristic class* of $X$ is defined by

$$T_{y*}(X) := T_{y*}(\mathbf{Q}_{h,X}) \in \mathbf{H}_\bullet(X)[y] \big(\subset \mathbf{H}_\bullet(X)[y,y^{-1}]\big). \tag{1.1.4}$$

This inclusion is reduced to the $X$ smooth case by using a stratification of $X$ together with smooth partial compactifications of strata over $X$. Then it follows from the relation with the cohomology Hirzebruch class, see [BrScYo].

In the $X$ smooth case, we have

$$\mathrm{DR}_y[X] = \Lambda_y[T^*X], \tag{1.1.5}$$

with

$$\Lambda_y[V] := \textstyle\sum_{p\geqslant 0} [\Lambda^p V]\, y^p \quad \text{for a vector bundle } V. \tag{1.1.6}$$

In the case $X$ is a complete intersection in a smooth complex algebraic variety $Y$, the *virtual Hirzebruch characteristic class* $T_{y*}^{\rm vir}(X)$ can be defined like the virtual genus in [Hi] (see [MaSaSc2, Section 1.4]) by

$$T_{y*}^{\rm vir}(X) := td_{(1+y)*}{\rm DR}_y^{\rm vir}[X] \in \mathbf{H}_{\bullet}(X)[y], \tag{1.1.7}$$

where ${\rm DR}_y^{\rm vir}[X]$ is the image in $K_0(X)[[y]]$ of

$$\Lambda_y(T_{\rm vir}^*X) = \Lambda_y[T^*Y|_X]/\Lambda_y[N_{X/Y}^*] \in K^0(X)[[y]], \tag{1.1.8}$$

and belongs to $K_0(X)[y]$ (see for instance [MaSaSc1, Proposition 3.4]). Here $K^0(X)$, $K_0(X)$ are the Grothendieck group of locally free sheaves of finite length and that of coherent sheaves respectively. We denote by $T^*Y$ and $N_{X/Y}^*$ the cotangent and conormal bundles respectively. The virtual cotangent bundle is defined by

$$T_{\rm vir}^*X := [T^*Y|_X] - [N_{X/Y}^*] \in K^0(X). \tag{1.1.9}$$

Here $N_{X/Y}^*$ in the non-reduced case is defined by the locally free sheaf $\mathcal{I}_X/\mathcal{I}_X^2$ on $X$ with $\mathcal{I}_X \subset \mathcal{O}_Y$ the ideal of $X \subset Y$.

Note that the above definition of $T_{y*}^{\rm vir}(X)$ is compatible with the one in [CaMaScSh] by [MaSaSc1, Proposition 1.3.1].

**Remarks 1.2.** (i) Let $(M,F)$ be a filtered *left* $\mathcal{D}$-module on a smooth variety $X$ of dimension $d_X$. The filtration $F$ of the de Rham complex ${\rm DR}_X(M,F)$ is defined by

$${\rm DR}_X(M,F)^i = \Omega_X^{i+d_X} \otimes_{\mathcal{O}_X} (M, F[-i-d_X]) \qquad \big(i \in [-d_X, 0]\big), \tag{1.2.1}$$

where ${\rm DR}_X(M,F)^i$ denotes the $i$ th component of ${\rm DR}_X(M,F)$. Recall that $F_p = F^{-p}$ and $F[m]_p = F_{p-m}$ for $p,m \in \mathbf{Z}$.

This is compatible with the definition of ${\rm DR}_X$ for filtered *right* $\mathcal{D}_X$-modules as in [Sa3]; that is, we have the canonical isomorphism

$${\rm DR}_X(M,F) = {\rm DR}_X\big((\Omega_X^{d_X}, F) \otimes_{\mathcal{O}_X} (M,F)\big), \tag{1.2.2}$$

where $(\Omega_X^{d_X}, F) \otimes_{\mathcal{O}_X} (M,F)$ is the filtered right $\mathcal{D}_X$-module corresponding to a filtered $\mathcal{D}_X$-module $(M,F)$, and the filtration $F$ on $\Omega_X^{d_X}$ is shifted by $-d_X$ so that

$${\rm Gr}_p^F \Omega_X^{d_X} = 0 \quad (p \ne -d_X), \tag{1.2.3}$$

Under the direct image functor $i_*^{\mathcal{D}}$ for filtered $\mathcal{D}$-modules with $i : X \hookrightarrow Y$ a closed embedding of smooth varieties, the filtration $F$ of a filtered *left* $\mathcal{D}$-module $(M,F)$ is shifted by the codimension $r := \dim Y - \dim X$; more precisely

$$i_*^{\mathcal{D}} M = i_* M[\partial_1, \dots, \partial_r] \quad \text{with} \quad F_p(i_*^{\mathcal{D}} M) = \textstyle\sum_{\nu \in \mathbf{N}^r} i_*\big(F_{p-|\nu|-r} M \otimes \partial^{\nu}\big), \tag{1.2.4}$$

where $\partial^{\nu} := \prod_{j=1}^r \partial_j^{\nu_j}$ with $\partial_j := \partial/\partial y_j$ for local coordinates $y_i$ of $Y$ with $X = \bigcup_{i \leqslant r}\{y_i = 0\}$. This shift comes from (1.2.3) since there is no shift of filtration for filtered right $\mathcal{D}$-modules. (Globally we have to twist the right-hand side of (1.2.4) by $\omega_{X/Y}$.) Because of this shift of the filtration $F$, we have the canonical isomorphisms of $\mathcal{O}_Y$-modules

$$i_* \mathcal{H}^j {\rm Gr}_p^F {\rm DR}_X(M) = \mathcal{H}^j {\rm Gr}_p^F {\rm DR}_Y(i_*^{\mathcal{D}} M). \tag{1.2.5}$$

If $(M,F)$ is the underlying filtered left $\mathcal{D}_X$-module of a mixed Hodge module $\mathcal{M}$, then there is an equality in $K_0(X)[y, y^{-1}]$:

$$\textstyle\sum_{i,p} (-1)^i \big[\mathcal{H}^i {\rm Gr}_p^F {\rm DR}(M)\big](-y)^p = \sum_{i,p} (-1)^i \big[\Omega_X^{i+d_X} \otimes_{\mathcal{O}_X} {\rm Gr}_p^F M\big](-y)^{p-i-d_X}. \tag{1.2.6}$$

This is related to the right-hand side of (1.1.2), and follows directly from (1.2.1).

(ii) Let $Z$ be a locally closed smooth subvariety of a smooth variety $X$ with $i_Z : Z \hookrightarrow X$ the canonical inclusion. For $\mathcal{M}^\bullet \in D^b\mathrm{MHM}(X)$, set

$$\mathcal{M}_Z^j := H^j i_Z^* \mathcal{M}^\bullet \in \mathrm{MHM}(Z) \quad (j \in \mathbf{Z}).$$

(Here $H^j : D^b\mathrm{MHM}(Z) \to \mathrm{MHM}(Z)$ is the canonical cohomology functor.) Assume this mixed Hodge module is a variation of mixed Hodge structure $\mathbf{H}_Z^j$ on $Z$ for any $j \in \mathbf{Z}$; more precisely, its underlying $F$-filtered left $\mathcal{D}_Z$-module is a locally free $F$-filtered $\mathcal{O}_Z$-module which underlies $\mathbf{H}_Z^j$. (Note that there is a shift of the weight filtration $W$ by $d_Z$.) For $z \in Z$, we denote by $i_{z,Z} : \{z\} \hookrightarrow Z$ and $i_z : \{z\} \hookrightarrow X$ the canonical inclusions. We have $\mathcal{M}_z^j$, $\mathbf{H}_z^j$ by applying the above argument to the inclusion $i_z : \{z\} \hookrightarrow X$. Here $\mathcal{M}_z^j$ is naturally identified with $\mathbf{H}_z^j$ (including the weight filtration $W$) since $z$ is a point. The relations between these are given by

$$H^{-d_Z} i_{z,Z}^* \mathcal{M}_Z^j = i_{z,Z}^* \mathbf{H}_Z^j = \mathcal{M}_z^{j-d_Z} = \mathbf{H}_z^{j-d_Z}. \tag{1.2.7}$$

Here $H^k i_{z,Z}^* \mathcal{M}_Z^j = 0$ $(k \neq -d_Z)$, and $i_{z,Z}^* \mathbf{H}_Z^j$ is the restriction to $z$ as a variation of mixed Hodge structure. Note that there is *no shift* of the filtration $F$ in (1.2.7). (Consider, for instance, the case $\mathcal{M}^\bullet$ is the constant mixed Hodge module $\mathbf{Q}_{h,X}[d_X]$, where $\mathcal{M}_Z^{-r} = \mathbf{Q}_{h,Z}[d_Z]$ with $r = \mathrm{codim}_X Z$ and $\mathcal{M}_z^{-d_X} = \mathbf{Q}_h$.) This follows from the definition of the nearby cycle functors $\psi_{z_i}$ in [Sa3], [Sa5], where the $z_i$ are local coordinates of $Z$. The functor $i_{z,Z}^*$ can be given in this case by the iteration of the mapping cones of the canonical morphisms

$$\mathrm{can} : \psi_{z_i,1} \to \varphi_{z_i,1},$$

and the vanishing cycle functors $\varphi_{z_i,1}$ vanish for smooth mixed Hodge modules on $Z$. Here smooth means that their underlying $\mathbf{Q}$-complexes are local systems on $Z$ shifted by $d_Z$. (Note that the last property implies the shift of indices in (1.2.7).)

**1.3. Spectral Hirzebruch classes.** We denote by $\mathrm{MHM}(X, T_s)$ the abelian category of mixed Hodge modules $\mathcal{M}$ on a smooth variety $X$ (or more generally, on a variety embeddable into a smooth variety $X$) such that $\mathcal{M}$ is endowed with an action of $T_s$ of finite order. (For instance, $\mathcal{M} = \varphi_{f_{a,j}} \mathbf{Q}_{h,U_j}$ with $T_s$ the semisimple part of the monodromy in the notation of the introduction.) For $(\mathcal{M}, T_s) \in \mathrm{MHM}(X, T_s)$, let $(M, F)$ be the underlying filtered left $\mathcal{D}_X$-module. Since $T_s$ has a finite oder $e$, we have the canonical decomposition

$$(M, F) = \textstyle\sum_{\lambda \in \mu_e} (M_\lambda, F), \tag{1.3.1}$$

such that $T_s = \lambda$ on $M_\lambda \subset M$, where $\mu_e := \{\lambda \in \mathbf{C} \mid \lambda^e = 1\}$. We define the *spectral Hirzebruch class* by

$$T_{\widetilde{y}*}^{\mathrm{sp}}(\mathcal{M}, T_s) := td_{(1-\widetilde{y})*}\big(\mathrm{DR}_{\widetilde{y}}[\mathcal{M}, T_s]\big) \in \mathbf{H}_\bullet(X)\big[\widetilde{y}^{1/e}, \tfrac{1}{\widetilde{y}(\widetilde{y}-1)}\big], \tag{1.3.2}$$

with

$$\mathrm{DR}_{\widetilde{y}}[\mathcal{M}, T_s] := \textstyle\sum_{i,p,\lambda} (-1)^i \left[\mathcal{H}^i \mathrm{Gr}_F^p \mathrm{DR}(M_\lambda)\right] \widetilde{y}^{p+\ell(\lambda)} \quad \text{in} \quad K_0(X)[\widetilde{y}^{1/e}, \widetilde{y}^{-1/e}]. \tag{1.3.3}$$

Here

$$\ell(\lambda) \in [0, 1) \quad \text{with} \quad \exp(2\pi i \ell(\lambda)) = \lambda, \tag{1.3.4}$$

and

$$td_{(1-\widetilde{y})*} : K_0(X)[\widetilde{y}^{1/e}, \widetilde{y}^{-1/e}] \to \mathbf{H}_\bullet(X)\big[\widetilde{y}^{1/e}, \tfrac{1}{\widetilde{y}(\widetilde{y}-1)}\big] \tag{1.3.5}$$

is the scalar extension of the Todd class transformation $td_* : K_0(X) \to \mathbf{H}_\bullet(X)$ followed by the multiplication by $(1-\widetilde{y})^{-k}$ on $\mathbf{H}_k(X)$ as in (1.1) (where $\widetilde{y} = -y$). Actually the class belongs to $\mathbf{H}_\bullet(X)[\widetilde{y}^{1/e}, \widetilde{y}^{-1/e}]$ by generalizing [Sch2, Proposition 5.21], see Proposition (1.4) below.

The above definition can be generalized to the $X$ singular case by using locally defined closed embeddings into smooth varieties, where the independence of locally defined closed embeddings follows from the isomorphism (1.2.5) (at the level of $\mathcal{O}$-modules). We can further generalize this definition to the case of $\mathcal{M}^\bullet \in D^b\mathrm{MHM}(X)$ endowed with an action of $T_s$ of finite order by applying the above arguments to each cohomology module $H^i\mathcal{M}^\bullet$ ($i \in \mathbf{Z}$).

The above arguments imply the transformation

$$T_{\widetilde{y}*}^{\mathrm{sp}} : K_0^{\mathrm{mon}}(\mathrm{MHM}(X)) \to \textstyle\bigcup_{e\geqslant 1} \mathbf{H}_\bullet(X)\big[\widetilde{y}^{1/e}, \widetilde{y}^{-1/e}\big], \tag{1.3.6}$$

which is functorial for proper morphisms by using the compatibility of DR (or rather $\mathrm{DR}^{-1}$) with the direct images by proper morphisms (see [Sa3, Section 2.3.7]) together with the compatibility of $td_*$ with the pushforward by proper morphisms (see [BaFuMa]). Here the left-hand side is the Grothendieck group of mixed Hodge modules on $X$ endowed with a finite order automorphism, see also [CaMaScSh, Remark 1.3(4)].

**Proposition 1.4.** *In the notation of* (1.3.2)*, we have*

$$T_{\widetilde{y}*}^{\mathrm{sp}}(\mathcal{M}, T_s) \in \mathbf{H}_\bullet(X)[\widetilde{y}^{1/e}, \widetilde{y}^{-1/e}]. \tag{1.4.1}$$

*Proof.* Let $\mathcal{S}$ be a stratification of $X$ such that $H^i j_S^*\mathcal{M}$ are variations of mixed Hodge structures on any strata $S \in \mathcal{S}$ for any $i \in \mathbf{Z}$, where $j_S : S \hookrightarrow X$ is the canonical inclusion. By the same argument as in the proof of [MaSaSc1, Proposition 5.1.2], we have the equality

$$T_{\widetilde{y}*}^{\mathrm{sp}}(\mathcal{M}) = \textstyle\sum_{S,i} (-1)^i T_{\widetilde{y}*}^{\mathrm{sp}}\big((j_S)_! H^i (j_S)^*\mathcal{M}\big). \tag{1.4.2}$$

So the assertion is reduced to the case where $\mathcal{M} = j_!\mathcal{M}'$ with $\mathcal{M}' \in \mathrm{MHM}(X')$ a variation of mixed Hodge structure on an open subvariety $X'$ with $j : X' \hookrightarrow X$ the natural inclusion. We may further assume that $D := X \setminus X'$ is a divisor with simple normal crossings since the Todd class transformation $td_*$ and the de Rham functor DR commute with the pushforward or the direct image under a proper morphism (see [BaFuMa] for $td_*$).

Let $M^{>0}$ be the Deligne extension of the underlying $\mathcal{O}_{X'}$-module of $\mathcal{M}'$ such that the eigenvalues of the residues of the logarithmic connections are contained in $(0, 1]$. The action of $T_s$ is naturally extended to $M^{>0}$, and we have the canonical decomposition

$$M^{>0} = \textstyle\bigoplus_\lambda M_\lambda^{>0}.$$

It follows from [Sa5, Proposition 3.11] that each $M_\lambda^{>0}$ is identified with an $\mathcal{O}_X$-submodule of $M_\lambda$, and there is a canonical filtered quasi-isomorphism

$$\mathrm{DR}_{X\langle D\rangle}(M_\lambda^{>0}, F) \xrightarrow{\sim} \mathrm{DR}_X(M_\lambda, F), \tag{1.4.3}$$

where the left-hand side is the filtered logarithmic de Rham complex such that its $i$ th component is given by

$$\mathrm{DR}_{X\langle D\rangle}(M_\lambda^{>0}, F)^i = \Omega_X^{i+d_X}(\log D)\otimes_{\mathcal{O}_X}(M_\lambda^{>0}, F[-i-d_X]). \tag{1.4.4}$$

As in (1.2.6), we get by (1.4.3) the equality in $K_0(X)[\widetilde{y}^{1/e}, \widetilde{y}^{-1/e}]$:

$$\begin{aligned}&\textstyle\sum_{p,i,\lambda} (-1)^i \big[\mathcal{H}^i\mathrm{Gr}_p^F\mathrm{DR}_X(M_\lambda)\big]\, \widetilde{y}^{p+\ell(\lambda)}\\ &= \textstyle\sum_{p,i,\lambda} (-1)^i \big[\Omega_X^{i+d_X}(\log D)\otimes_{\mathcal{O}_X}\mathrm{Gr}_p^F M_\lambda^{>0}\big]\, \widetilde{y}^{p-i-d_X+\ell(\lambda)}.\end{aligned} \tag{1.4.5}$$

By [BaFuMa], the Todd class transformation $td_*$ satisfies the following property:

$$td_*(V\otimes \xi) = ch(V) \cap td_*(\xi) \quad \big(V \in K^0(X),\ \xi \in K_0(X)\big). \tag{1.4.6}$$

By using the *twisted* Chern character

$$ch^{(1-\widetilde{y})} : K^0(X) \ni V \mapsto \textstyle\sum_{k\geqslant 0} ch^k(V)(1-\widetilde{y})^k \in \mathbf{H}^\bullet(X)[\widetilde{y}], \tag{1.4.7}$$

with $ch^k$ the $k$ th component of the Chern character $ch$ (see [Sch2], [Yo1]), (1.4.6) is extended to

$$td_{(1-\widetilde{y})*}(V\otimes\widetilde{\xi}) = ch^{(1-\widetilde{y})}(V)\cap td_{(1-\widetilde{y})*}(\widetilde{\xi}) \quad \big(V\in K^0(X),\ \widetilde{\xi}\in K_0(X)[\widetilde{y}^{1/e},\widetilde{y}^{-1/e}]\big). \tag{1.4.8}$$

By applying this to the case $V=\mathrm{Gr}_p^F M_\lambda^{>0}$ in the right-hand side of (1.4.5), the assertion is now reduced to the case $\mathrm{Gr}_p^F M_\lambda^{>0}=\mathcal{O}_X$, and to the case $\mathcal{M}=j_*\mathbf{Q}_{h,X'}[d_X]$, as in the proof of [Sch2, Proposition 5.21].

The assertion is further reduced to the case $X'=X$ by using the weight filtration $W$ on the mixed Hodge module $j_*\mathbf{Q}_{h,X'}[d_X]$. In fact, if $D_i$ $(i=1,\dots,m)$ are the irreducible components of $D$, then

$$\mathrm{Gr}_k^W(j_*\mathbf{Q}_{h,X'}[d_X]) = \textstyle\bigoplus_{|I|=k-d_X}\mathbf{Q}_{h,D_I}(-|I|)[d_{D_I}] \quad (I\subset\{1,\dots,m\}), \tag{1.4.9}$$

where $D_I:=\bigcap_{i\in I}D_i$, and $|I|=\mathrm{codim}_X D_I$ so that $k=d_{D_I}+2|I|=d_X+|I|$.

In the case $\mathcal{M}=\mathbf{Q}_{h,X}[d_X]$, the assertion follows from the inclusion (1.1.4). This finishes the proof of Proposition (1.4).

**1.5. Proof of Proposition 2.** The assertion (0.4) follows from the construction in (1.3). By the argument in (1.4) together with (1.2.7) (where we have no shift of the filtration $F$), the remaining assertion (0.3) is reduced to the following:

$$\mathrm{Gr}_F^p H^j(F_{f,0},\mathbf{C}) = 0 \quad \text{for } p<0, \tag{1.5.1}$$

where $f$ is any holomorphic function $f$ on a complex manifold $X$, and $0\in f^{-1}(0)\subset X$. We denote by $F_{f,0}$ the Milnor fiber of $f$ around $0$ so that

$$H^j(F_{f,0},\mathbf{Q}) = H^j i_0^*\psi_f\mathbf{Q}_{h,X} \qquad (j\in\mathbf{Z}), \tag{1.5.2}$$

with $i_0:\{0\}\hookrightarrow X$ the natural inclusion. In fact, (0.3) is reduced to the inclusion (1.1.4) by using (1.4.8) together with the non-negativity of the codimension $|I|$ in (1.4.9).

For the proof of (1.5.1), we use an embedded resolution $\pi:\widetilde{X}\to X$ of $f^{-1}(0)$, where we may assume that $D:=\pi^{-1}(0)\subset\widetilde{X}$ is a divisor (by taking a point-center blow-up first). The latter is a union of irreducible components of $\pi^{-1}f^{-1}(0)$, and is also a divisor with normal crossings. Set $\pi_0:=\pi|_D:D\to\{0\}$, and $\widetilde{f}:=f\circ\pi$. We have the canonical isomorphisms

$$H^j i_0^*\psi_f\mathbf{Q}_{h,X} = H^j(\pi_0)_* i_D^*\psi_{\widetilde{f}}\mathbf{Q}_{h,\widetilde{X}} \qquad (j\in\mathbf{Z}), \tag{1.5.3}$$

since

$$\pi_*\psi_{\widetilde{f}}\mathbf{Q}_{h,\widetilde{X}} = \psi_f\pi_*\mathbf{Q}_{h,\widetilde{X}} = \psi_f\mathbf{Q}_{h,X} \quad \text{and} \quad (\pi_0)_*\circ i_D^* = i_0^*\circ\pi_*.$$

It is well-known that the variation of mixed Hodge structures $\mathbf{H}_Z^j$ in Remark (1.2)(ii) (applied to $\mathcal{M}^\bullet = i_D^*\psi_{\widetilde{f}}\mathbf{Q}_{h,\widetilde{X}}$ or $\psi_{\widetilde{f}}\mathbf{Q}_{h,\widetilde{X}}$) are direct sums of locally constant variations of Hodge structures of type $(k,k)$ with $k\in\mathbf{N}$, where $Z$ is a stratum of the canonical stratification associated with the divisor with normal crossings $\pi^{-1}f^{-1}(0)$, see [St1], [St2] (and also [Sa5, Proposition 3.5]). So the assertion (1.5.1) follows. This finishes the proof of Proposition 2.

We recall here the notion of topological filtration on the Grothendieck group (see [SGA6], [Fu]) which will be used in Section 4.

**1.6. Topological filtration.** Let $X$ be a complex algebraic variety, and $K_0(X)$ be the Grothendieck group of coherent sheaves on $X$. It has the topological filtration which is denoted in this paper by $G$ (in order to distinguish it from the Hodge filtration $F$), and such that $G_kK_0(X)$ is generated by the classes of coherent sheaves $\mathcal{F}$ with $\dim\mathrm{supp}\,\mathcal{F}\leqslant k$, see

[Fu, Examples 1.6.5 and 15.1.5], [SGA6]. It is known (see [Fu, Corollary 18.3.2]) that $td_*$ induces the isomorphisms

$$(1.6.1)\qquad \begin{aligned} (td_*)_{\mathbf{Q}} : K_0(X)_{\mathbf{Q}} &\xrightarrow{\sim} \textstyle\bigoplus_k \mathrm{CH}_k(X)_{\mathbf{Q}},\\ \mathrm{Gr}^G_k(td_*)_{\mathbf{Q}} : \mathrm{Gr}^G_k K_0(X)_{\mathbf{Q}} &\xrightarrow{\sim} \mathrm{CH}_k(X)_{\mathbf{Q}}. \end{aligned}$$

Moreover the inverse of the last isomorphism is given by $Z \mapsto [\mathcal{O}_Z]$ for irreducible reduced closed subvarieties $Z \subset X$ with dimension $k$. Here we set

$$\mathbf{H}_\bullet(X) := \textstyle\bigoplus_k \mathrm{CH}_k(X)_{\mathbf{Q}},$$

and use the filtration $G$ defined by

$$(1.6.2)\qquad G_k\mathbf{H}_\bullet(X) := \textstyle\bigoplus_{j\leqslant k} \mathbf{H}_j(X).$$

This definition is valid also in the case $\mathbf{H}_k(X) := H^{\mathrm{BM}}_{2k}(X, \mathbf{Q})$. Here

$$(1.6.3)\qquad \mathrm{Gr}^G_k(td_*)_{\mathbf{Q}} : \mathrm{Gr}^G_k K_0(X)_{\mathbf{Q}} = \mathrm{CH}_k(X)_{\mathbf{Q}} \to H^{\mathrm{BM}}_{2k}(X, \mathbf{Q})$$

is identified with the cycle class map.

If $X = \mathbf{P}^N$, then $\mathbf{H}_k(X) = \mathbf{Q}$ for $k \in [1, N]$. These are canonically generated by the classes of linear subspaces, and the $\mathrm{Gr}^G_k(td_*)_{\mathbf{Q}}$ are identified with the identity maps. For any irreducible reduced closed subvariety $Z \subset \mathbf{P}^N$ with $\dim Z = k$, we have the *positivity*:

$$(1.6.4)\qquad \mathrm{Gr}^G_k(td_*)_{\mathbf{Q}}[Z] = \deg Z > 0 \quad \text{in} \ \ \mathbf{H}_k(\mathbf{P}^N) = \mathbf{Q}.$$

Here $\deg Z$ is defined to be the intersection number of $Z$ with a sufficiently general linear subspace of the complementary dimension if $k > 0$ (and $\deg Z = 1$ if $k = 0$). Moreover, for any coherent sheaf $\mathcal{F}$ on $\mathbf{P}^N$ with $\dim \mathrm{supp}\, \mathcal{F} = k$, we have also the *positivity*:

$$(1.6.5)\qquad \mathrm{Gr}^G_k[\mathcal{F}] = \textstyle\sum_{i=1}^r m_i\, \mathrm{Gr}^G_k[Z_i] > 0 \quad \text{in} \ \ \mathrm{Gr}^G_k K_0(\mathbf{P}^N)_{\mathbf{Q}} = \mathrm{CH}_k(\mathbf{P}^N)_{\mathbf{Q}} = \mathbf{Q},$$

where the $Z_i$ are $k$-dimensional irreducible components of $\mathrm{supp}\mathcal{F}$, and $m_i \in \mathbf{Z}_{>0}$ are the multiplicity of $\mathcal{F}$ at the generic point of $Z_i$ ($i \in [1, r]$).

## 2. Proofs of Theorem 1 and Proposition 1

In this section we prove Theorem 1 and Proposition 1 by using the short exact sequences associated with nearby and vanishing cycle functors of mixed Hodge modules.

**2.1. Construction.** In the notation of the introduction, set $S := \mathbf{C}^r$ with coordinates $t_1, \dots, t_r$. Let $\mathcal{X} \subset Y \times S$ be the hypersurface such that

$$(2.1.1)\qquad \mathcal{X} \cap (Y \times \{a\}) = X_{a,r+1} \bigl(= s^{-1}_{a,r+1}(0) \subset Y\bigr) \quad \text{for any} \ \ a = (a_1, \dots, a_r) \in S,$$

where $s_{a,r+1}$ is as in the introduction. For $j \in [0, r]$, set

$$(2.1.2)\qquad S_j := \{t_k = 0 \ (k > j)\} \subset S, \quad \mathcal{X}_j := \mathcal{X} \times_S S_j \subset \mathcal{X}.$$

Note that $\dim S_j = j$, and the fiber of $\mathcal{X}_j \to S_j$ over $(a_1, \dots, a_j) \in S_j$ ($j \in [0, r]$) coincides with $X_{a,j+1}$ in the introduction. (Here $j$ is shifted by 1.) For $j \in [1, r]$, set

$$(2.1.3)\qquad S_{a,j} := \{t_k = a_k \ (k < j),\ t_k = 0 \ (k > j)\} \subset S_j, \quad Y_{a,j} := \mathcal{X} \times_S S_{a,j} \subset \mathcal{X}_j.$$

Then $Y_{a,j}$ is a one-parameter family containing $X_{a,j}$ over $t_j = 0$ and $X_{a,j+1}$ over $t_j = a_j$. By the definitions of $U_j$, $f_{a,j}$ in the introduction, there are natural isomorphisms

$$(2.1.4)\qquad U_j = Y_{a,j} \setminus (X'_j \times S_{a,j})\, (\subset Y \times S_{a,j}) \qquad (j \in [1, r]),$$

such that $f_{a,j}$ on the left-hand side is identified with $t_j$ on the right-hand side. In fact, we have on $X_{a,j+1} \setminus X'_j$

$$(s_{a,j}/s'^{m}_j)|_{U_j} = \big((s - a_1 s'^{m}_1 - \cdots - a_{j-1} s'^{m}_{j-1})/s'^{m}_j\big)|_{U_j} = a_j,$$

and the complex number $a_j$ is identified with the variable $t_j$ so that the disjoint union of $X_{a,j+1}$ with $a_j$ varying appropriately (including $a_j = 0$) is identified with $Y_{a,j}$. (Here $X_{a,j+1}$ for $a_j = 0$ is identified with $X_{a,j}$.)

By (2.1.4) we get the isomorphism

$$\varphi_{t_j}\mathbf{Q}_{h,Y_{a,j}} = (i_{\Sigma_j \setminus X'_j,\Sigma_j})_! \varphi_{f_{a,j}}\mathbf{Q}_{h,U_j} \qquad (j \in [1,r]), \tag{2.1.5}$$

since the following is shown in [MaSaSc2, Section 2.4]:

$$\varphi_{t_j}\mathbf{Q}_{h,Y_{a,j}}|_{\Sigma_j \cap X'_j} = 0. \tag{2.1.6}$$

For $j \in [1,r]$, we have the short exact sequences of mixed Hodge modules on $X_{a,j}$ :

$$0 \to \mathbf{Q}_{h,X_{a,j}}[n] \to \psi_{t_j}\mathbf{Q}_{h,Y_{a,j}}[n] \to \varphi_{t_j}\mathbf{Q}_{h,Y_{a,j}}[n] \to 0, \tag{2.1.7}$$

since the $a_j$ are sufficiently general.

**2.2. Proof of Theorems 1.** There are non-empty Zariski-open subsets $S^\circ_j \subset S_j$ $(j \in [1,r])$ such that the $D_j := S_j \setminus S^\circ_j$ are divisors on $S_j$ containing $S_{j-1}$ and satisfying

$$D_{j-1} \supset \overline{(D_j \setminus S_{j-1})} \cap S_{j-1} \quad (j \in [2,r]), \tag{2.2.1}$$

and moreover, by setting

$$\mathcal{X}^\circ_j := \mathcal{X} \times_S S^\circ_j \subset \mathcal{X}_j, \quad \mathcal{Y}^\circ_j := \mathcal{X} \times_S \big(S_j \setminus \overline{(D_j \setminus S_{j-1})}\big) \subset \mathcal{X}_j,$$

there are short exact sequences of mixed Hodge modules on $\mathcal{X}^\circ_{j-1}$ $(j \in [1,r])$ :

$$0 \to \mathbf{Q}_{h,\mathcal{X}^\circ_{j-1}}[d_{j-1}] \to \psi_{t_j}\mathbf{Q}_{h,\mathcal{Y}^\circ_j}[d_{j-1}]|_{\mathcal{X}^\circ_{j-1}} \to \varphi_{t_j}\mathbf{Q}_{h,\mathcal{Y}^\circ_j}[d_{j-1}]|_{\mathcal{X}^\circ_{j-1}} \to 0, \tag{2.2.2}$$

where $d_{j-1} := \dim \mathcal{X}_{j-1}\,(= n + j - 1)$. We may furthermore assume

$$\mathrm{Supp}\big(\varphi_{t_j}\mathbf{Q}_{h,\mathcal{Y}^\circ_j}|_{\mathcal{X}^\circ_{j-1}}\big) = \Sigma_j \times S^\circ_{j-1} \subset \mathcal{X}^\circ_{j-1}, \tag{2.2.3}$$

$$\varphi_{t_j}\mathbf{Q}_{h,\mathcal{Y}^\circ_j}|_{\Sigma_j \times S^\circ_{j-1}} \text{ is locally constant over } S^\circ_{j-1}, \tag{2.2.4}$$

by shrinking $S^\circ_{j-1}$ if necessary, where $\Sigma_j = \mathrm{Sing}\, X_{a,j} = \bigcap_{k<j} X'_k \cap \Sigma$ as in the introduction. For the proof of (2.2.4) we use the deformation to the normal bundle in (2.4) below together with a Thom-Sebastiani theorem in Theorem (3.2) as well as Remarks (2.5) below.

These imply by decreasing induction on $k \in [2, j-1]$ :

$$\psi_{t_k} \cdots \psi_{t_{j-1}}(\varphi_{t_j}\mathbf{Q}_{h,\mathcal{Y}^\circ_j}|_{\Sigma_j \times S^\circ_{j-1}})|_{\Sigma_j \times S^\circ_{k-1}} \text{ is locally constant over } S^\circ_{k-1}. \tag{2.2.5}$$

In the notation of (2.1) we have the isomorphisms

$$\psi_{t_j}\mathbf{Q}_{h,Y_{a,j}} = \psi_{t_j}\mathbf{Q}_{h,\mathcal{Y}^\circ_j}|_{X_{a,j}}, \quad \varphi_{t_j}\mathbf{Q}_{h,Y_{a,j}} = \varphi_{t_j}\mathbf{Q}_{h,\mathcal{Y}^\circ_j}|_{X_{a,j}}, \tag{2.2.6}$$

such that the restriction of the short exact sequence (2.2.2) to $X_{a,j}$ is identified with (2.1.7), since the $a_j$ are sufficiently general. (In fact, the $V$-filtration induces the $V$-filtration on the restriction to the transversal slice $Y_{a,j} \subset \mathcal{Y}^\circ_j$ passing through $X_{a,j} \subset \mathcal{X}^\circ_j$, see [DiMaSaTo, Theorem 1.1]. Moreover this restriction morphism induces a bistrict surjection for $(F,V)$, see [DiMaSaTo, Lemma 4.2]. So the assertion follows, since the weight filtration $W$ is given by the relative monodromy filtration.)

We then get (0.1) by applying the iterations of nearby cycle functors $\psi_{t_k}$ $(k < j)$ to (2.2.2) and using (2.1.5), (2.2.6). Here we also need [Sch3] (or [MaSaSc1, Proposition 3.3]) together with [Ve, Theorem 7.1] in order to show that the virtual Hirzebruch class $T_{y*}^{\mathrm{vir}}(X)$ can be obtained by applying the iteration of the nearby cycle functors $\psi_{t_j}$ $(j \leqslant r)$ to $\mathbf{Q}_{h,\mathcal{Y}_r^\circ}$. This finishes the proof of Theorem 1.

**2.3. Proof of Proposition 1.** The limit in (0.2) is defined by

$$\lim_{a_j \to 0} T_{y*}(X_{a,j+1}) := T_{y*}(\psi_{t_j}\mathbf{Q}_{h,Y_{a,j}}), \tag{2.3.1}$$

in the notation of (2.1). The assertion (0.2) then follows from (2.1.5), (2.1.7) and (2.2.4–6). This finishes the proof of Proposition 1.

We review here some basics of deformations to normal bundles which will be needed in the proofs of Theorems 2 and 3.

**2.4 Deformations to normal bundles.** In the notation of the introduction, set

$$x'_i := (s'_i/s'_j)|_{U_j} \quad (i<j).$$

Since $Z_j = \cap_{i<j}\{x'_i = 0\} \subset U_j = Y \setminus X'_j$, we have the decomposition

$$V_j := N_{Z_j/U_j} = Z_j \times \mathbf{C}^{j-1}, \tag{2.4.1}$$

where the left-hand side is the normal bundle of $Z_j$ in $U_j$. The total deformation space $\mathcal{U}_j$ of $U_j$ to the normal bundle $V_j$ can be defined by

$$\mathcal{U}_j := \mathrm{Spec}_{U_j}\big(\textstyle\bigoplus_{i\in\mathbf{Z}} \mathcal{I}_{Z_j}^i \otimes t^{-i}\big),$$

where $\mathcal{I}_{Z_j} \subset \mathcal{O}_{U_j}$ is the ideal of $Z_j$, and $\mathcal{I}_{Z_j}^i = \mathcal{O}_{U_j}$ for $i \leqslant 0$. We can identify $\mathcal{U}_j$ with a relative affine open subset of the blow-up of $U_j \times \mathbf{C}$ along $Z_j \times \{0\}$, on which the functions

$$x_i := x'_i/t \quad (i<j)$$

are well-defined, where $t$ is the coordinate of the second factor of $U_j \times \mathbf{C}$, that is, the parameter of deformation. Note that the normal bundle $V_j$ is contained in $\mathcal{U}_j$ as the fiber over $t=0$ (which is a relative affine open subset of the exceptional divisor of the blow-up), since

$$V_j = \mathrm{Spec}_{U_j}\big(\textstyle\bigoplus_{i\geqslant 0} \mathcal{I}_{Z_j}^i/\mathcal{I}_{Z_j}^{i+1} \otimes t^{-i}\big).$$

Set

$$z_i := x_i|_{V_j} \qquad (i<j).$$

These give the decomposition (2.4.1) by inducing coordinates of the second factor of (2.4.1).

Consider now the following function on $\mathcal{U}_j$ :

$$\widetilde{f}_{a,j} := \pi_j^*(s/{s'_j}^m)|_{U_j} - \textstyle\sum_{i<j} a_i\, x_i^m, \tag{2.4.2}$$

where $\pi_j : \mathcal{U}_j \to U_j$ is the canonical morphism. Restricting over $t=1$, we have

$$\widetilde{f}_{a,j}|_{t=1} = f_{a,j}, \tag{2.4.3}$$

On the other hand, restricting over $t=0$, we get

$$\widetilde{f}_{a,j}|_{V_j} = pr_1^*\big(f_{a,j}|_{Z_j}\big) - \textstyle\sum_{i<j} a_i\, z_i^m = pr_1^* f'_j - \textstyle\sum_{i<j} a_i\, z_i^m, \tag{2.4.4}$$

where $pr_1 : V_j \to Z_j$ is the canonical projection, and $f'_j$ is as in Theorem 2. To (2.4.4) we can apply a Thom-Sebastiani theorem (see Theorem (3.2) below).

**Remarks 2.5.** (i) The restriction of $\widetilde{f}_{a,j}$ to a sufficiently small neighborhood $\mathcal{U}'_j$ of $Z_j \times \mathbf{C}$ in $\mathcal{U}_j$ is a topologically locally trivial family parametrized by $t \in \mathbf{C}$, since the $s'_j$ are sufficiently general. The vanishing cycle functor along $\widetilde{f}_{a,j}$ for $\mathbf{Q}_{h,\mathcal{U}'_j}$ commutes with the restriction to

$t=c$ for any $c\in\mathbf{C}$. This follows from [DiMaSaTo] as in the proof of (2.2.6). (Here analytic mixed Hodge modules are used.)

(ii) Let $\mathcal{M}$ be a mixed Hodge module on $X\times\mathbf{C}$. Assume its underlying $\mathbf{Q}$-complex is isomorphic to the pull-back of a $\mathbf{Q}$-complex on $X$ by the first projection $p:X\times\mathbf{C}\to X$. Then $\mathcal{M}$ is isomorphic to the pull-back of a mixed Hodge module on $X$ by $p$ up to a shift of a complex. In fact, we have the isomorphism $p^*p_*\mathcal{M}\to\mathcal{M}$ in this case. (We apply this to $X=Z_j$ and $\mathcal{M}=\varphi_{\widetilde{f}_{a,j}}\mathbf{Q}_{h,\mathcal{U}_j}[d_{\mathcal{U}_j}-1]|_{Z_j}$.)

## 3. Application of Thom-Sebastiani theorem

In this section we prove Theorems 2 and 3 after giving a rather simple proof of the Thom-Sebastiani theorem for filtered $\mathcal{D}$-modules.

**3.1. Algebraic microlocalization.** Let $Y$ be a smooth complex algebraic variety (or a connected complex manifold) with $f$ a non-constant function on $Y$, that is $f\in\Gamma(Y,\mathcal{O}_Y)\setminus\mathbf{C}$. Let $i_f:Y\hookrightarrow Y\times\mathbf{C}$ be the graph embedding by $f$, and $t$ be the coordinate of the second factor of $Y\times\mathbf{C}$. Set

$$(\mathcal{B}_f,F):=(i_f)_*^{\mathcal{D}}(\mathcal{O}_Y,F)=(\mathcal{O}_Y[\partial_t],F),$$

where $(i_f)_*^{\mathcal{D}}$ is the direct image of filtered $\mathcal{D}$-modules, see (1.2.4). The last isomorphism is as filtered $\mathcal{O}_Y[\partial_t]$-modules, and the sheaf-theoretic direct image $(i_f)_*$ is omitted to simplify the notation. The actions of $t$ and $\partial_{y_i}$ with $y_i$ local coordinates of $Y$ are given by

$$\begin{aligned} t(g\,\partial_t^j)&=fg\,\partial_t^j-jg\,\partial_t^{j-1},\\ \partial_{y_i}(g\,\partial_t^j)&=(\partial_{y_i}g)\,\partial_t^j-(\partial_{y_i}f)g\,\partial_t^{j+1}\quad(g\in\mathcal{O}_Y), \end{aligned}\tag{3.1.1}$$

In this section, the Hodge filtration $F$ is indexed as in the case of *right* $\mathcal{D}$-modules (since there is a shift of filtration under the direct images by closed embeddings for left $\mathcal{D}$-modules, see (1.2.4)). So we have

$$\mathrm{Gr}_p^F\mathcal{B}_f=\begin{cases}\mathcal{O}_Y\partial_t^{p+d_Y} & \text{if}\quad p\geqslant -d_Y,\\ 0 & \text{otherwise.}\end{cases}\tag{3.1.2}$$

By (1.2.2) this does not cause a problem when we use the de Rham functor DR.

Let $\widetilde{\mathcal{B}}_f$ be the *algebraic microlocalization* of $\mathcal{B}_f$ (see [Sa7]), that is,

$$(\widetilde{\mathcal{B}}_f,F)=(\mathcal{O}_Y[\partial_t,\partial_t^{-1}],F)\quad\text{with}\quad \mathrm{Gr}_p^F\widetilde{\mathcal{B}}_f=\mathcal{O}_Y\partial_t^{p+d_Y}\quad(p\in\mathbf{Z}).\tag{3.1.3}$$

Let $V$ be the microlocal $V$-filtration on $\widetilde{\mathcal{B}}_f$ along $t=0$, see [Sa7]. This is obtained by modifying the $V$-filtration of Kashiwara [Ka2] and Malgrange [Mal] on $\mathcal{B}_f$. It is an exhaustive decreasing filtration indexed discretely by $\mathbf{Q}$ and satisfying the properties as below, and moreover it is uniquely determined by them.

(a) The $V^\alpha\widetilde{\mathcal{B}}_f$ are finitely generated over $\mathcal{D}_Y[\partial_t^{-1}]$ ($\alpha\in\mathbf{Q}$).

(b) $t(V^\alpha\widetilde{\mathcal{B}}_f)\subset V^{\alpha+1}\widetilde{\mathcal{B}}_f$, $\partial_t(V^\alpha\widetilde{\mathcal{B}}_f)=V^{\alpha-1}\widetilde{\mathcal{B}}_f$ ($\alpha\in\mathbf{Q}$).

(c) The action of $\partial_tt-\alpha$ on $\mathrm{Gr}_V^\alpha\widetilde{\mathcal{B}}_f$ is nilpotent ($\alpha\in\mathbf{Q}$).

The property (a) follows from the assertion that $\mathcal{D}_Y[s]f^s$ is locally finitely generated over $\mathcal{D}_Y$ (more precisely, it is subholonomic, see [Ka1]). In fact, the latter property implies that the $V^\alpha\mathcal{B}_f$ are also locally finitely generated over $\mathcal{D}_Y$. (Here it is also possible to use [Sa3, 3.2.1.2] together with Nakayama's lemma, since any element of $\mathcal{B}_f$ is annihilated by a sufficiently high power of $t-f$.)

By the construction in [Sa7] there are canonical isomorphisms

$$\mathrm{can} : \mathrm{Gr}_V^\alpha(\mathcal{B}_f, F) \xrightarrow{\sim} \mathrm{Gr}_V^\alpha(\widetilde{\mathcal{B}}_f, F) \qquad (\alpha < 1), \tag{3.1.4}$$

$$\partial_t^k : (\widetilde{\mathcal{B}}_f; F, V) \xrightarrow{\sim} (\widetilde{\mathcal{B}}_f; F[-k], V[-k]) \qquad (k \in \mathbf{Z}). \tag{3.1.5}$$

Note that the morphism can in (3.1.4) for $\alpha = 1$ is strictly surjective by [Sa3, Lemma 5.1.4 and Proposition 5.1.14]. In fact, by setting

$$X := f^{-1}(0) \subset Y,$$

the morphism can in (3.1.4) for $\alpha = 1$ is identified with the underlying morphism of filtered $\mathcal{D}_Y$-modules of the morphism can in the short exact sequence of mixed Hodge modules

$$0 \to \mathbf{Q}_{h,X}[d_X] \to \psi_{f,1}\mathbf{Q}_{h,Y}[d_X] \xrightarrow{\mathrm{can}} \varphi_{f,1}\mathbf{Q}_{h,Y}[d_X] \to 0. \tag{3.1.6}$$

By (3.1.4–5) we have the canonical isomorphisms for any $\alpha \in \mathbf{Q}$

$$\mathrm{DR}_Y(\mathrm{Gr}_V^\alpha \widetilde{\mathcal{B}}_f) = \varphi_{f,\mathbf{e}(-\alpha)}\mathbf{C}_Y[d_X], \tag{3.1.7}$$

where $\mathbf{e}(-\alpha) := \exp(-2\pi i\alpha)$. In fact, (3.1.7) is well-known for $\mathcal{B}_f$, instead of $\widetilde{\mathcal{B}}_f$, if $\alpha \in [0, 1)$ (but not $(0, 1]$). Then it holds for $\widetilde{\mathcal{B}}_f$ and for any $\alpha \in \mathbf{Q}$ by (3.1.4–5).

Consider the vanishing cycle mixed Hodge module

$$\varphi_f\mathbf{Q}_{h,Y}[d_X].$$

Its underlying filtered $\mathcal{D}$-module can be given by

$$\textstyle\bigoplus_{\alpha\in(-1,0]} \mathrm{Gr}_V^\alpha(\widetilde{\mathcal{B}}_f, F), \tag{3.1.8}$$

Here the filtration $F$ is *not* shifted, although it is shifted by 1 if we use $\mathrm{Gr}_V^\alpha \widetilde{\mathcal{B}}_f$ for $\alpha \in (0, 1]$ instead of $\alpha \in (-1, 0]$. If fact, if we use $\mathcal{B}_f$ instead of $\widetilde{\mathcal{B}}_f$ in (3.1.8), then we get the *nearby cycle functor* $\psi_f$ instead of the vanishing cycle functor $\varphi_f$, and the filtration $F$ is shifted by 1, where $\alpha \in (0, 1]$ (which corresponds to the so-called "lower extension"), see [Sa3, 5.1.3.3]. This implies a similar assertion for $\widetilde{\mathcal{B}}_f$, since the morphism can in (3.1.4) induces an isomorphism or a surjection for $\alpha \in (0, 1]$. Thus there is no shift of the filtration $F$ in (3.1.8) by using (3.1.5) for $j = 1$. (In this section, we index the filtration $F$ like *right* $\mathcal{D}$-modules as is explained before (3.1.2), and $V$ is indexed increasingly so that $V_\alpha = V^{-\alpha}$ and $\mathrm{Gr}_\alpha^V = \mathrm{Gr}_V^{-\alpha}$. In the case $\alpha = 0$, the above definition of the filtration $F$ on the vanishing cycle mixed Hodge module is compatible with the original definition of $\varphi$ in [Sa3, 5.1.3.3].)

In particular, we get for $\alpha \in (-1, 0]$

$$\mathrm{Gr}_p^F \mathrm{Gr}_V^\alpha \widetilde{\mathcal{B}}_f = 0 \quad (p < -d_X). \tag{3.1.9}$$

For $\alpha = 0$, this uses the strict surjectivity of (3.1.4) for $\alpha = 1$. (This is closely related to the strict negativity of the roots of $b$-functions, see [Ka1].)

We have a Thom-Sebastiani theorem as below. This is a special case of the assertion mentioned in [Sa7, Remark 4.5], and also follows from [Sa10] (see also [DeLo], [GeLoMe] for the motivic version, and [ScSt], [Va] for the isolated hypersurface singularity case). We give here a short proof for the convenience of the reader.

**Theorem 3.2.** *Let $Y_a$ be a smooth complex algebraic variety (or a connected complex manifold) with $f_a$ a non-constant function, that is, $f_a \in \Gamma(Y_a, \mathcal{O}_{Y_a}) \setminus \mathbf{C}$, for $a = 1, 2$. Set*

*$Y=Y_1\times Y_2$ with $f=f_1+f_2$. Then there are canonical isomorphisms of filtered $\mathcal{D}_Y$-modules for $\alpha\in(-1,0]$ :*

$$\begin{aligned}\mathrm{Gr}_V^{\alpha}(\widetilde{\mathcal{B}}_f,F)=\textstyle\bigoplus_{\alpha_1\in I(\alpha)}\mathrm{Gr}_V^{\alpha_1}(\widetilde{\mathcal{B}}_{f_1},F)\boxtimes\mathrm{Gr}_V^{\alpha-\alpha_1}(\widetilde{\mathcal{B}}_{f_2},F)\\ \oplus\textstyle\bigoplus_{\alpha_1\in J(\alpha)}\mathrm{Gr}_V^{\alpha_1}(\widetilde{\mathcal{B}}_{f_1},F)\boxtimes\mathrm{Gr}_V^{\alpha-1-\alpha_1}(\widetilde{\mathcal{B}}_{f_2},F[-1]),\end{aligned}\tag{3.2.1}$$

*by replacing $Y_a$ with an open neighborhood of $X_a:=f_a^{-1}(0)$ in $Y_a$ $(a=1,2)$ if necessary, where*

$$I(\alpha):=(-1,0]\cap[\alpha,\alpha+1)\qquad J(\alpha):=(-1,0]\cap[\alpha-1,\alpha).$$

**Note.** Setting $\alpha_2=\alpha-\alpha_1,\ \alpha'_2=\alpha-1-\alpha_1$, we have

$$\alpha_1\in I(\alpha)\iff\alpha_1,\,\alpha_2\in(-1,0],\qquad\alpha_1\in J(\alpha)\iff\alpha_1,\,\alpha'_2\in(-1,0].\tag{3.2.2}$$

*Proof.* Set $\Sigma_a=\mathrm{Sing}\,f_a^{-1}(0)$ $(a=1,2)$. By replacing $Y_a$ with an open neighborhood of $X_a$ $(a=1,2)$ if necessary, we may assume

$$\mathrm{Sing}\,f^{-1}(0)=\Sigma_1\times\Sigma_2.$$

In fact, $f^{-1}(0)$ is the inverse image of the anti-diagonal of $\mathbf{C}\times\mathbf{C}$ by $f_1\times f_2$.

By [Sa7, Section 4.1] we have the short exact sequence

$$0\to(\widetilde{\mathcal{B}}_{f_1}\boxtimes\widetilde{\mathcal{B}}_{f_2};F[1],V[1])\xrightarrow{\iota}(\widetilde{\mathcal{B}}_{f_1}\boxtimes\widetilde{\mathcal{B}}_{f_2};F,V)\xrightarrow{\eta}(\widetilde{\mathcal{B}}_f;F,V)\to0,\tag{3.2.3}$$

where $\iota$ is defined by

$$\partial_{t_1}\boxtimes id-id\boxtimes\partial_{t_2},$$

and $\eta$ by

$$\eta\big(g_1\partial_{t_1}^{i_1}\boxtimes g_2\partial_{t_2}^{i_1}\big):=g_1g_2\partial_t^{i_1+i_2}\quad\text{for}\ \ g_a\in\mathcal{O}_{Y_a}\ (a=1,2).$$

Note that

$$(\widetilde{\mathcal{B}}_{f_1}\boxtimes\widetilde{\mathcal{B}}_{f_2};F,V):=(\widetilde{\mathcal{B}}_{f_1};F,V)\boxtimes(\widetilde{\mathcal{B}}_{f_2};F,V),\tag{3.2.4}$$

where the external product $\boxtimes$ is taken as that of $\mathcal{O}_{Y_a}$-modules $(a=1,2)$.

By (3.2.4) we have the following filtered isomorphisms (see Remark (3.3)(i) below):

$$\mathrm{Gr}_V^{\alpha}(\widetilde{\mathcal{B}}_{f_1}\boxtimes\widetilde{\mathcal{B}}_{f_2},F)\xleftarrow{\ \sim\ }\textstyle\bigoplus_{\alpha_1\in\mathbf{Q}}\mathrm{Gr}_V^{\alpha_1}(\widetilde{\mathcal{B}}_{f_1},F)\boxtimes\mathrm{Gr}_V^{\alpha-\alpha_1}(\widetilde{\mathcal{B}}_{f_2},F)\qquad(\alpha\in\mathbf{Q}).\tag{3.2.5}$$

By the definition of $\iota$ and by using (3.1.5) for $j=1$, the filtered isomorphism (3.2.5) implies the filtered isomorphism

$$(\mathrm{Coker}\,\mathrm{Gr}_V^{\alpha}\iota,F)\cong\textstyle\bigoplus_{\alpha_1\in(-1,0]}\mathrm{Gr}_V^{\alpha_1}(\widetilde{\mathcal{B}}_{f_1},F)\boxtimes\mathrm{Gr}_V^{\alpha-\alpha_1}(\widetilde{\mathcal{B}}_{f_2},F)\qquad(\alpha\in\mathbf{Q}),\tag{3.2.6}$$

where the left-hand side is defined to be a quotient of $\mathrm{Gr}_V^{\alpha}(\widetilde{\mathcal{B}}_{f_1}\boxtimes\widetilde{\mathcal{B}}_{f_2},F)$.

We can verify that (3.2.3) induces an isomorphism of bi-filtered $\mathcal{D}_Y$-modules

$$(\mathrm{Coker}\,\iota;F,V)\xrightarrow{\ \sim\ }(\widetilde{\mathcal{B}}_f;F,V),\tag{3.2.7}$$

which is also compatible with the action of $t$, $\partial_t$. Here the action of $t$ and $\partial_t$ on $\mathrm{Coker}\,\iota$ is defined respectively by $t_1+t_2$ and either $\partial_{t_1}$ or $\partial_{t_2}$ (by the definition of $\iota$).

In fact, the compatibility of the isomorphism (3.2.7) with $F$ follows from the definition (3.1.3). The compatibility with the filtration $V$ is equivalent to that $\eta$ is strictly compatible with the filtration $V$. By the uniqueness of the microlocal filtration $V$ explained in (3.1), this is also equivalent to that the quotient filtration $V$ on $\mathrm{Coker}\,\iota$ satisfies the conditions of the microlocal $V$-filtration in (3.1). Here the finiteness condition (a) for $\widetilde{\mathcal{B}}_f$ follows from that for $\widetilde{\mathcal{B}}_{f_a}$. Condition (b) follows from the definition of the action of $t$, $\partial_t$ explained above.

Condition (c) is verified also by using the definition of the action of $t$, $\partial_t$ on the left-hand side (especially $t = t_1 + t_2$). Thus (3.2.7) follows.

Using (3.2.5), we can prove that $\iota$ is bistrictly injective. This implies that the short exact sequence (3.2.3) is bistrictly exact by using the theory of compatible filtrations in [Sa3, Section 1]. So the cokernel commutes with $\mathrm{Gr}_V^\alpha$ in a compatible way with the filtration $F$. (This does not necessarily hold if $\iota$ is not bistrictly compatible with $F, V$.)

The assertion (3.2.1) now follows from (3.2.6–7). In fact, (3.2.6) says that $\mathrm{Coker}\,\mathrm{Gr}_V^\alpha\iota$ for $\alpha \in (-1, 0]$ is given by the direct sum over the index set defined by the conditions:

$$\alpha_1 \in (-1, 0], \quad \alpha_1 + \alpha_2 = \alpha \in (-1, 0] \quad \text{with} \quad \alpha_2 := \alpha - \alpha_1,$$

where $\alpha_2 \in (-1, 1)$, and does not necessarily belong to $(-1, 0]$. However, the difference with the union of the index sets of the direct sums in (3.2.1) (see also (3.2.2)) can be recovered by using (3.1.5) for $a = 2$, $j = 1$, where we get the shift of $F$ by $-1$ in the last term of (3.2.1). Thus Theorem (3.2) follows.

**Remarks 3.3.** (i) The proof of (3.2.5) is not completely trivial, since we have to use the assertion that the filtrations $F$, $F^{(1)}$, $F^{(2)}$, $V$, $V_{(1)}$, $V_{(2)}$ on $\widetilde{\mathcal{B}}_{f_1} \boxtimes \widetilde{\mathcal{B}}_{f_2}$ form compatible filtrations in the sense of [Sa3, Section 1], where $F^{(a)}$ is induced by $F$ on $\widetilde{\mathcal{B}}_{f_a}$, and similarly for $V_{(a)}$ $(a = 1, 2)$. Note that $F$ is the convolution of $F^{(1)}$, $F^{(2)}$, and similarly for $V$, see Remark (ii) below for convolution. We can prove the compatibility of the above six filtrations by using [Sa3, Theorem 1.2.12]. In fact, the compatibility of the four filtrations $F^{(1)}$, $F^{(2)}$, $V_{(1)}$, $V_{(2)}$ follows from the definition, since the external product is an exact functor for both factors. Then we can apply Remark (ii) below, and (3.2.5) follows by taking $\mathrm{Gr}^{V_{(2)}}$. In fact, we have the canonical isomorphisms

$$\mathrm{Gr}_{V_{(1)}}^{\alpha_1}\mathrm{Gr}_{V_{(2)}}^{\alpha_2}(\widetilde{\mathcal{B}}_{f_1} \boxtimes \widetilde{\mathcal{B}}_{f_2}) = \mathrm{Gr}_V^{\alpha_1}\widetilde{\mathcal{B}}_{f_1} \boxtimes \mathrm{Gr}_V^{\alpha_2}\widetilde{\mathcal{B}}_{f_2},$$

which is compatible with the filtrations $F^{(1)}$, $F^{(2)}$.

In this case, however, there is an additional difficulty, since the filtration $V$ does not satisfy the condition $V^\alpha = 0$ for $\alpha \gg 0$ (and similarly for $V_{(1)}^\alpha$, $V_{(2)}^\alpha$). In order to avoid this problem, we restrict to $V_{(1)}^\beta$, and take the inductive limit for $\beta \to -\infty$. Note that the induced filtration $V_{(2)}$ on $\mathrm{Gr}_V^\alpha V_{(1)}^\beta$ satisfies the above property (since $V_{(2)}^\gamma \mathrm{Gr}_V^\alpha V_{(1)}^\beta = 0$ for $\gamma > \alpha - \beta$).

(ii) In general, if there are compatible $m$ filtrations $F_{(1)}, \dots, F_{(m)}$ of an object $M$ of an abelian category $\mathcal{A}$ where the inductive limit over a directed set is always an exact functor (for instance, the category of $\mathbf{C}$-vector spaces), then we can show by using [Sa3, Theorem 1.2.12] that the $m + 1$ filtrations $F_{(1,2)}, F_{(1)}, \dots, F_{(m)}$ also form compatible filtrations of $M$, where $F_{(1,2)}$ is the convolution of $F_{(1)}$ and $F_{(2)}$, that is,

$$F_{(1,2)}^p M = \textstyle\sum_{q\in\mathbf{Z}} F_{(1)}^q M \cap F_{(2)}^{p-q} M.$$

This assertion can be reduced to the finite sum case by using an inductive limit argument as above, and then to the finite filtration case (by replacing $F_{(1)}^p$, $F_{(2)}^q$ with 0 for $p, q \gg 0$). Here we have the canonical isomorphisms

$$\mathrm{Gr}_{F_{(1,2)}}^p M \xleftarrow{\sim} \textstyle\bigoplus_{q\in\mathbf{Z}} \mathrm{Gr}_{F_{(1)}}^q \mathrm{Gr}_{F_{(2)}}^{p-q} M,$$

which is compatible with the filtrations $F_{(i)}$ $(i > 2)$. This can be shown by using

$$\mathrm{Gr}_{F_{(2)}}^{p-q}\mathrm{Gr}_{F_{(1,2)}}^p M = \mathrm{Gr}_{F_{(1,2)}}^p \mathrm{Gr}_{F_{(2)}}^{p-q} M = \mathrm{Gr}_{F_{(1)}}^q \mathrm{Gr}_{F_{(2)}}^{p-q} M,$$

where we need the abelian category containing the exact category of $(m-2)$-filtered objects of $\mathcal{A}$ as in [Sa3, Sections 1.3.2–3].

(iii) Since the de Rham functor DR is compatible with the exterior product $\boxtimes$, we can deduce from Theorem (3.2) the following isomorphisms of complexes of $\mathcal{O}_Y$-modules for $\alpha\in(-1,0]$, $p\in\mathbf{Z}$:

$$\begin{aligned}&\mathrm{Gr}_F^p\mathrm{DR}_Y(\mathrm{Gr}_V^{\alpha}\widetilde{\mathcal{B}}_f)\\ &=\textstyle\bigoplus_{\alpha_1+\alpha_2=\alpha,\,p_1+p_2=p}\mathrm{Gr}_F^{p_1}\mathrm{DR}_{Y_1}(\mathrm{Gr}_V^{\alpha_1}\widetilde{\mathcal{B}}_{f_1})\boxtimes\mathrm{Gr}_F^{p_2}\mathrm{DR}_{Y_2}(\mathrm{Gr}_V^{\alpha_2}\widetilde{\mathcal{B}}_{f_2})\\ &\quad\oplus\textstyle\bigoplus_{\alpha_1+\alpha_2=\alpha-1,\,p_1+p_2+1=p}\mathrm{Gr}_F^{p_1}\mathrm{DR}_{Y_1}(\mathrm{Gr}_V^{\alpha_1}\widetilde{\mathcal{B}}_{f_1})\boxtimes\mathrm{Gr}_F^{p_2}\mathrm{DR}_{Y_2}(\mathrm{Gr}_V^{\alpha_2}\widetilde{\mathcal{B}}_{f_2}),\end{aligned}\tag{3.3.1}$$

where $\alpha_1,\alpha_2\in(-1,0]$, and $F^p=F_{-p}$. (Note that $\mathrm{Gr}_F^{p_2}=\mathrm{Gr}_{F[-1]}^{p_2'}$ with $p_2':=p_2+1$.)

Setting $\lambda_a=\exp(-2\pi i\alpha_a)$, we have $\alpha_a=-\ell(\lambda_a)$ $(a=1,2)$, and

$$\alpha_1+\alpha_2\leqslant-1\iff\ell(\lambda_1)+\ell(\lambda_2)\geqslant1.\tag{3.3.2}$$

If these equivalent conditions are satisfied, then (3.3.1) says that the index $p$ of the Hodge filtration $F$ increases by 1. This is very important for the proofs of Theorems 2 and 3.

(iv) In the notation of (3.1), assume $Y_2=\mathbf{C}$ with coordinate $x_2$, and $f_2=a\,x_2^m$ with $a\in\mathbf{C}^*$. Then the Milnor fiber $F_{f_2,0}$ consists of $m$ points, and

$$\widetilde{H}^0(F_{f_2,0},\mathbf{C})_\lambda=\begin{cases}\mathbf{C}&\text{if }\lambda^m=1\text{ and }\lambda\ne1,\\0&\text{otherwise,}\end{cases}\tag{3.3.3}$$

where $\widetilde{H}$ denotes the reduced cohomology. This implies

$$\mathrm{Gr}_F^{p_2}\mathrm{DR}_{Y_2}(\mathrm{Gr}_V^{\alpha_2}\widetilde{\mathcal{B}}_{f_2})\cong\begin{cases}\mathbf{C}&\text{if }p_2=0,\ \alpha_2\in\{\frac{1}{m},\dots,\frac{m-1}{m}\},\\0&\text{otherwise.}\end{cases}\tag{3.3.4}$$

Note that we have in this case

$$\varphi_f\mathbf{C}_Y=\varphi_{f_1}\mathbf{C}_{Y_a}\otimes_{\mathbf{C}}\widetilde{H}^0(F_{f_2,0},\mathbf{C})[-1].\tag{3.3.5}$$

In general the mixed Hodge modules are stable by $\psi[-1]$, $\varphi[-1]$, and Theorem (3.2) implies as is well-known (see [Mas], [Sch1, Corollary 1.3.4 in p. 72]):

$$\varphi_f\mathbf{C}_Y=\varphi_{f_1}\mathbf{C}_{Y_1}\boxtimes\varphi_{f_2}\mathbf{C}_{Y_2}[-1].\tag{3.3.6}$$

**3.4. Proofs of Theorems 2 and 3.** Theorem 2 follows from (2.4.3–4), (3.3.1–2), (3.3.4–6) and Remarks (2.5). Theorem 3 also follows by using (3.3.1–2) and (3.3.6) together with the compatibility of the Todd class transformation $td_*:K_0(X)\to\mathbf{H}_\bullet(X)$ with cross products (or Künneth maps), see [BaFuMa, Section III.3]. This finishes the proofs of Theorems 2 and 3.

**Remarks 3.5.** (i) By the proof of Theorem 3, the assertion holds at the level of Grothendieck groups, and we have the following equality in $K_0(\Sigma)\big[\widetilde{y}^{1/e}\big]$:

$$\mathrm{DR}_{\widetilde{y}}[\varphi_f\mathbf{Q}_{h,Y},T_s]=-\mathrm{DR}_{\widetilde{y}}[\varphi_{f_1}\mathbf{Q}_{h,Y_1},T_s]\boxtimes\mathrm{DR}_{\widetilde{y}}[\varphi_{f_2}\mathbf{Q}_{h,Y_2},T_s].\tag{3.5.1}$$

(ii) Theorem 3 does not necessarily hold if there are non-zero critical values $c_a$ of $f_a$ $(a=1,2)$ with $c_1+c_2=0$, since the last condition is equivalent to the condition $\Sigma\ne\Sigma_1\times\Sigma_2$. However, we can apply Theorem 3 with $f_a$ replaced by $f_a-c_a$ $(a=1,2)$ in the above case.

(iii) In Theorem 3, $X$ is never compact even if $X_1,X_2$ are compact, since $X$ is the inverse image of the anti-diagonal of $\mathbf{C}\times\mathbf{C}$ by $f_1\times f_2$. Extending the situation in Theorem 3, we may consider the case where $f_1$, $f_2$ are proper morphisms from smooth varieties to $\mathbf{P}^1$ and $X$ is defined by the inverse image of the anti-diagonal of $\mathbf{P}^1\times\mathbf{P}^1$. In this case $X$ is compact. We can apply Theorem 3 by choosing an appropriate local coordinate of $\mathbf{P}^1$ on a neighborhood of each critical value $c_a$ of $f_a$ $(a=1,2)$ with $(c_1,c_2)$ belonging to the anti-diagonal of $\mathbf{P}^1\times\mathbf{P}^1$.

## 4. Relation with rational and Du Bois singularities

In this section some relations with rational and Du Bois singularities are explained.

**4.1. Primitive decomposition of nearby cycles.** Let $Y$ be a smooth complex algebraic variety (or a connected complex manifold), and $f$ be a non-constant function on $Y$, that is, $f\in\Gamma(Y,\mathcal{O}_Y)\setminus\mathbf{C}$. Assume $X:=f^{-1}(0)\subset Y$ is *reduced* in this section.

With the notation of (3.1), we first show the short exact sequence

$$0\to\widetilde{\omega}_X\otimes_{\mathcal{O}_X}\omega_X^\vee\to F_{-d_X}\mathrm{Gr}_V^1\mathcal{B}_f\xrightarrow{\mathrm{can}}F_{-d_X}\mathrm{Gr}_V^1\widetilde{\mathcal{B}}_f\to 0. \tag{4.1.1}$$

Here

$$\widetilde{\omega}_X:=(\rho)_*\omega_{\widetilde{X}}\subset\omega_X,\quad \omega_X^\vee:=\mathcal{H}om_{\mathcal{O}_X}(\omega_X,\mathcal{O}_X),$$

with $\rho:\widetilde{X}\to X$ a resolution of singularities, and

$$\omega_X=\omega_Y\otimes_{\mathcal{O}_Y}\mathcal{O}_X,$$

since $X$ is globally defined by $f$, see [Sa6, Lemma 2.9].

For the proof of (4.1.1), we use the monodromy filtration $W$ on $\mathrm{Gr}_V^1\mathcal{B}_f$ shifted by $d_X$, which is uniquely characterized by the following conditions:

$$\begin{aligned}&N(W_i\,\mathrm{Gr}_V^1\mathcal{B}_f)\subset W_{i-2}\,\mathrm{Gr}_V^1\mathcal{B}_f\quad(i\in\mathbf{Z}),\\&N^i:\mathrm{Gr}^W_{d_X+i}\,\mathrm{Gr}_V^1\mathcal{B}_f\xrightarrow{\sim}\mathrm{Gr}^W_{d_X-i}\,\mathrm{Gr}_V^1\mathcal{B}_f\quad(i>0).\end{aligned}\tag{4.1.2}$$

The *primitive part* is defined by

$${}^P\mathrm{Gr}^W_{d_X+i}\,\mathrm{Gr}_V^1\mathcal{B}_f:=\mathrm{Ker}\,N^{i+1}\subset\mathrm{Gr}^W_{d_X+i}\,\mathrm{Gr}_V^1\mathcal{B}_f\quad(i\geqslant 0),\tag{4.1.3}$$

with ${}^P\mathrm{Gr}^W_{d_X+i}\,\mathrm{Gr}_V^1\mathcal{B}_f=0$ $(i<0)$. This implies the *primitive decomposition*

$$\mathrm{Gr}^W_j\,\mathrm{Gr}_V^1\mathcal{B}_f=\textstyle\bigoplus_{k\geqslant 0}N^k\big({}^P\mathrm{Gr}^W_{j+2k}\,\mathrm{Gr}_V^1\mathcal{B}_f\big)\quad(j\in\mathbf{Z}),\tag{4.1.4}$$

and the *co-primitive part* can be expressed by

$$\begin{aligned}N^i\big({}^P\mathrm{Gr}^W_{d_X+i}\,\mathrm{Gr}_V^1\mathcal{B}_f\big)&=\mathrm{Ker}\big(\mathrm{Gr}^W_{d_X-i}\,\mathrm{Gr}_V^1\mathcal{B}_f\xrightarrow{N}\mathrm{Gr}^W_{d_X-i-2}\,\mathrm{Gr}_V^1\mathcal{B}_f\big)\\&=\mathrm{Ker}\big(\mathrm{Gr}^W_{d_X-i}\,\mathrm{Gr}_V^1\mathcal{B}_f\xrightarrow{\mathrm{can}}\mathrm{Gr}^W_{d_X-i}\,\mathrm{Gr}_V^1\widetilde{\mathcal{B}}_f\big)\quad(i\geqslant 0).\end{aligned}\tag{4.1.5}$$

Indeed, the first isomorphism follows from the primitive decomposition (4.1.4). As for the last isomorphism, note that the canonical morphism

$$\mathrm{can}:\mathrm{Gr}_V^1\mathcal{B}_f\to\mathrm{Gr}_V^1\widetilde{\mathcal{B}}_f$$

is identified with the morphism

$$\mathrm{can}:\mathrm{Gr}_V^1\mathcal{B}_f\to\mathrm{Gr}_V^0\mathcal{B}_f,$$

which is defined by $-\mathrm{Gr}_V\partial_t$. Moreover, for the latter, we have

$$N=\mathrm{Var}\circ\mathrm{can},$$

where Var is defined by $\mathrm{Gr}_Vt$, and is injective, see [Sa3, 5.1.3.4]. So the last isomorphism of (4.1.5) also follows.

Returning to the proof of (4.1.1), we get by (3.1.9)

$$F_{-d_X}\big(N^i\big({}^P\mathrm{Gr}^W_{d_X+i}\,\mathrm{Gr}_V^1\mathcal{B}_f\big)\big)=F_{-d_X-i}\big({}^P\mathrm{Gr}^W_{d_X+i}\,\mathrm{Gr}_V^1\mathcal{B}_f\big)=0\quad(i>0),\tag{4.1.6}$$

and it follows from [Sa6, Proposition 2.7] that

$$F_{-d_X}\big({}^P\mathrm{Gr}^W_{d_X}\,\mathrm{Gr}^1_V\mathcal{B}_f\big)=\widetilde{\omega}_X\otimes_{\mathcal{O}_X}\omega_X^\vee, \tag{4.1.7}$$

since $\mathcal{B}_f$ is a left $\mathcal{D}$-module. So (4.1.1) follows.

We denote by $\mathrm{IC}_X\mathbf{Q}_h$ the mixed Hodge module of weight $d_X$ such that its underlying $\mathbf{Q}$-complex is the intersection complex $\mathrm{IC}_X\mathbf{Q}$. We have

$$\mathrm{IC}_X\mathbf{Q}_h=\mathrm{Gr}^W_{d_X}(\mathbf{Q}_{h,X}[d_X])={}^P\mathrm{Gr}^W_{d_X}\psi_{f,1}\mathbf{Q}_{h,Y}[d_X], \tag{4.1.8}$$

where the two isomorphisms follow from [Sa5, 4.5.9] and (3.1.6) together with the primitive decomposition as in (4.1.4).

Comparing (4.1.1) with (3.1.6), we see that the exactness of (4.1.1) is essentially equivalent to

$$F_{-d_X}(\mathbf{Q}_{h,X}[d_X])=F_{-d_X}(\mathrm{IC}_X\mathbf{Q}_h)=\widetilde{\omega}_X. \tag{4.1.9}$$

Here we set in general

$$F_{p_0}\mathcal{M}:=F_{p_0}M, \tag{4.1.10}$$

if $(M,F)$ is the underlying filtered *right* $\mathcal{D}$-module of a mixed Hodge module $\mathcal{M}$, where

$$p_0:=\min\big\{p\in\mathbf{Z}\mid \mathrm{Gr}^F_pM\neq 0\big\}.$$

These are independent of embeddings of algebraic varieties into smooth varieties as long as *right* $\mathcal{D}$-modules are used.

**4.2. Rational singularities.** With the notation and the assumption of (4.1), we have the following canonical isomorphism by (4.1.1):

$$(\omega_X/\widetilde{\omega}_X)\otimes_{\mathcal{O}_X}\omega_X^\vee=F_{-d_X}(\widetilde{\mathcal{B}}_f/V^{>1}\widetilde{\mathcal{B}}_f), \tag{4.2.1}$$

(see also [Sa6, Theorem 0.6]), since

$$F_{-d_X}(\mathcal{B}_f/V^{>1}\mathcal{B}_f)=\mathcal{O}_X, \tag{4.2.2}$$

where $V^{>\alpha}:=V^{\alpha+\varepsilon}$ $(0<\varepsilon\ll 1)$ for $\alpha\in\mathbf{Q}$ in general.

In fact, by (3.1.9), (3.1.4–5), and [Sa3, 3.2.1.2], we have

$$F_{-d_X}V^{>0}\mathcal{B}_f=F_{-d_X}\mathcal{B}_f=\mathcal{O}_Y,\quad F_{-d_X}V^{>1}\mathcal{B}_f=t(F_{-d_X}V^{>0}\mathcal{B}_f). \tag{4.2.3}$$

Thus (4.2.2) and (4.2.1) follow.

As a corollary of (4.2.1), we see that $X$ has only *rational singularities* if and only if

$$F_{-d_X}(\widetilde{\mathcal{B}}_f/V^{>1}\widetilde{\mathcal{B}}_f)=0,\quad\text{or equivalently}\quad F_{-d_X}(\varphi_f\mathbf{Q}_{h,Y}[d_X])=0, \tag{4.2.4}$$

under the notation (4.1.10), see also [Sa6, Theorem 0.6] (and [Sa1] in the isolated singularity case). Consider the classes

$$\begin{aligned}\big[(\omega_X/\widetilde{\omega}_X)\otimes_{\mathcal{O}_X}\omega_X^\vee\big]&=\big[F_{-d_X}(\widetilde{\mathcal{B}}_f/V^{>1}\widetilde{\mathcal{B}}_f)\big],\\ \big[\omega_X/\widetilde{\omega}_X\big]&=\big[F_{-d_X}(\varphi_f\mathbf{Q}_{h,Y}[d_X])\big]\quad\text{in }\ K_0(\Sigma).\end{aligned} \tag{4.2.5}$$

These may be called the *irrationality* of the singularities of $X$ at least in the $\Sigma$ projective case by the argument after (4.2.7) below. In the isolated singularity case, its dimension is called the geometric genus, see [Sa1].

If $X$ has only rational singularities, then the last condition in (4.2.4) implies

$$\mathrm{DR}_y\big[\varphi_f\mathbf{Q}_{h,Y}[d_X]\big]\big|_{y^{d_X}}=\big[F_{-d_X}(\varphi_f\mathbf{Q}_{h,Y}[d_X])\big]=0\quad\text{in }\ K_0(\Sigma), \tag{4.2.6}$$

(see also (1.2.6)), where $|_{y^{d_X}}$ means taking the coefficient of $y^{d_X}$. (Recall that *right* $\mathcal{D}$-modules are used in (4.1.10).) Moreover we have the following.

(4.2.7) The converse holds if $\Sigma$ is a *projective* variety.

In fact, if the singularities of $X$ are irrational, then we can show the non-vanishing of (4.2.6) in $K_0(\mathbf{P}^N)_{\mathbf{Q}}$ with $\mathbf{P}^N$ projective space containing $\Sigma$ by using the topological filtration in (1.6) together with the positivity (see (1.6.4–5)) of the image by the cycle class map (1.6.3) of the coherent sheaf

$$\mathcal{F} := F_{-d_X}(\varphi_f \mathbf{Q}_{h,Y}[d_X]).$$

**4.3. Du Bois singularities.** With the notation and the assumption of (4.1), let $\mathbf{D}_{h,X}$ be the dual of $\mathbf{Q}_{h,X}$. Since $\mathbf{Q}_{h,X}[d_X]$ is a mixed Hodge module, so is $\mathbf{D}_{h,X}[-d_X]$. Then we have the short exact sequence of mixed Hodge modules

$$0 \to \varphi_{f,1}\mathbf{Q}_{h,Y}(1)[d_X] \xrightarrow{\mathrm{Var}} \psi_{f,1}\mathbf{Q}_{h,Y}[d_X] \to \mathbf{D}_{h,X}(-d_X)[-d_X] \to 0, \tag{4.3.1}$$

which is the dual of (3.1.6) (up to a sign). In fact, Var is the dual of can in (3.1.6) up to a sign, see [Sa3, Section 5.2]. The underlying exact sequence of filtered $\mathcal{D}$-modules of (4.3.1) is identified with

$$0 \to (\mathrm{Im}\, N, F) \to (\mathrm{Gr}_V^1 \mathcal{B}_f, F) \to (\mathrm{Coker}\, N, F) \to 0, \tag{4.3.2}$$

and the primitive decomposition (4.1.4) implies

$$\mathrm{Gr}^W_{d_X+i}(\mathrm{Coker}\, N, F) = {}^P\mathrm{Gr}^W_{d_X+i}(\mathrm{Gr}_V^1 \mathcal{B}_f, F) \quad (i \geqslant 0), \tag{4.3.3}$$

since $\mathrm{Gr}^W$ commutes with taking the cokernel of $N$, see [Sa3, Proposition 5.1.14]. Setting

$$\widetilde{\omega}'_X := F_0\big(\mathbf{D}_{h,X}[-d_X]\big) = \mathrm{Gr}_F^0\big(\mathrm{DR}(\mathbf{D}_{h,X}[-d_X])\big),$$

we then get by (4.3.1–3) and (4.1.6)

$$\widetilde{\omega}'_X = F_{-d_X}\big(\mathbf{D}_{h,X}(-d_X)[-d_X]\big) = F_{-d_X}\big(\psi_{f,1}\mathbf{Q}_{h,Y}[d_X]\big). \tag{4.3.4}$$

Combined with (4.2.2), these imply

$$(\omega_X/\widetilde{\omega}'_X)\otimes_{\mathcal{O}_X}\omega_X^\vee = F_{-d_X}(\mathcal{B}_f/V^1\mathcal{B}_f) = F_{-d_X}(\widetilde{\mathcal{B}}_f/V^1\widetilde{\mathcal{B}}_f). \tag{4.3.5}$$

Since the dual functor $\mathbf{D}$ commute with DR (or rather $\mathrm{DR}^{-1}$, see [Sa3, Section 2.4.11]) and also with $\mathrm{Gr}_F^0$ by definition, we have by the definition of $\widetilde{\omega}'_X$ just before (4.3.4)

$$\mathbf{D}(\widetilde{\omega}'_X) = \mathrm{Gr}_F^0\mathrm{DR}(\mathbf{Q}_{h,X}[d_X]). \tag{4.3.6}$$

Here the left-hand side is the Grothendieck dual of the $\mathcal{O}_X$-module $\widetilde{\omega}'_X$, and we have

$$\mathbf{D}(\mathcal{F}) := \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \omega_X[d_X]) \quad \text{for} \quad \mathcal{F} \in D^b_{\mathrm{coh}}(\mathcal{O}_X).$$

It follows from (4.3.6) that $X$ has only *Du Bois singularities* (see [St3]) if and only if

$$\widetilde{\omega}'_X = \omega_X, \tag{4.3.7}$$

(since $\mathbf{D}(\omega_X) = \mathcal{O}_X[d_X]$). This condition is equivalent to the vanishing of the $\mathcal{O}_X$-modules in (4.3.5). Moreover the last condition is equivalent to

$$F_{-d_X}(\psi_{f,\neq 1}\mathbf{Q}_{h,Y}[d_X]) = F_{-d_X}(\varphi_{f,\neq 1}\mathbf{Q}_{h,Y}[d_X]) = 0. \tag{4.3.8}$$

This implies by using (5.1.6) below

(4.3.9) $X$ has only Du Bois singularities if and only if $\mathrm{lct}(f) = 1$,

where the *log canonical threshold* $\mathrm{lct}(f)$ is defined to be the minimal jumping coefficients as in (0.12) in the introduction.

**Remarks 4.4.** (i) The assertion (4.3.9) is equivalent to [Sa9, Theorem 0.5] where the statement is given in terms of the maximal root $-\alpha_f$ of the Bernstein-Sato polynomial $b_f(s)$. In fact, it is well-known that

$$\mathrm{lct}(f) = \alpha_f. \tag{4.4.1}$$

This follows, for instance, from [BuSa] combined with [Mal]. This well-known assertion together with some relevant references does not seem to be quoted in [KoSc], although the theorem in [Sa9] explained above is mentioned after [KoSc, Corollary 6.6], where it is shown that a reduced hypersurface $X \subset Y$ has only Du Bois singularities if and only if $(Y, X)$ is a log canonical pair, see also Remark (ii) below.

(ii) It is well-known (and is easy to show) that $(Y, X)$ is a log canonical pair with $X$ reduced if and only if $\mathrm{lct}(f) = 1$. In fact, let $\rho : (\widetilde{Y}, \widetilde{X}) \to (Y, X)$ be an embedded resolution. We have

$$\widetilde{X} = \rho^* X = \widetilde{X}' + \textstyle\sum_i m_i E_i, \quad \omega_{\widetilde{Y}} = (\rho^*\omega_Y)\big(\sum_i \nu_i E_i\big) \quad (m_i, \nu_i \in \mathbf{Z}_{>0}), \tag{4.4.2}$$

where the $E_i$ are the exceptional divisors of $\rho$, and $\widetilde{X}'$ is the proper transform of $X$. (The last equality is equivalent to that $\mathrm{div}\big(\mathrm{Jac}(\rho)\big) = \sum_i \nu_i E_i$, where $\mathrm{Jac}(\rho)$ is the Jacobian of $\rho$ with respect to some local coordinates of $\widetilde{Y}$, $Y$.) By (4.4.2) we then get

$$\omega_{\widetilde{Y}}(\widetilde{X}') = \big(\rho^*\omega_Y(X)\big)\big(\textstyle\sum_i (\nu_i - m_i)E_i\big). \tag{4.4.3}$$

Since $\mathrm{lct}(f) = \min \mathrm{JC}(f)$ by definition, the equality (4.4.3) implies

$$\begin{aligned} &(Y, X) \ \text{ is a log canonical pair } \iff \nu_i - m_i \geqslant -1 \ (\forall\, i) \\ &\iff (\nu_i + 1)/m_i \geqslant 1 \ (\forall\, i) \iff \mathrm{lct}(f) = 1, \end{aligned} \tag{4.4.4}$$

where the first equivalence is by the definition of canonical pairs together with (4.4.3), see [KoSc]. The last equivalence follows from the well-known assertion:

$$\mathrm{lct}(f) = \min\{(\nu_i + 1)/m_i\} \ \text{ if } \ \mathrm{lct}(f) < 1 \ \text{ or } \ \min\{(\nu_i + 1)/m_i\} < 1. \tag{4.4.5}$$

By the definition of $\mathrm{JC}(f)$ (see (5.1) below), the last assertion can be verified by calculating the integration of $\rho^*(|f|^{-2\alpha}\omega \wedge \overline{\omega})$ on $\widetilde{Y}$, where $\omega$ is a nowhere vanishing holomorphic form of degree $d_Y$ locally defined on $Y$, and $\alpha \in (0, 1]$. In fact, this can be reduced to a well-known assertion saying that we have for $\beta \in \mathbf{R}$, $c \in \mathbf{R}_{>0}$

$$\int_0^c r^\beta \mathrm{d}r < \infty \iff \beta > -1.$$

**4.5. Proof of Theorem 5.** We first show the second case where $X$ is globally defined by a function $f$ on $Y$. By the polarization on the nearby and vanishing cycle mixed Hodge modules (see [Sa3, Section 5.2], [Sa4]), we have the self-dualities

$$\begin{aligned} \mathbf{D}(\varphi_{f,\ne 1}\mathbf{Q}_{h,Y}[d_X]) &= (\varphi_{f,\ne 1}\mathbf{Q}_{h,Y}[d_X])(d_X), \\ \mathbf{D}(\varphi_{f,1}\mathbf{Q}_{h,Y}[d_X]) &= (\varphi_{f,1}\mathbf{Q}_{h,Y}[d_X])(d_X + 1), \end{aligned} \tag{4.5.1}$$

since $\psi_{f,\ne 1} = \varphi_{f,\ne 1}$, and $d_Y = d_X + 1$. (Note that the monodromy filtration is self-dual.) These imply in the notation of (4.1.10)

$$\begin{aligned} \mathbf{D}\big(\mathrm{Gr}_F^0 \mathrm{DR}(\varphi_{f,\ne 1}\mathbf{Q}_{h,Y}[d_X])\big) &= \mathrm{Gr}_F^{d_X} \mathrm{DR}(\varphi_{f,\ne 1}\mathbf{Q}_{h,Y}[d_X]) \\ &= F_{-d_X}(\varphi_{f,\ne 1}\mathbf{Q}_{h,Y}[d_X]), \\ \mathbf{D}\big(\mathrm{Gr}_F^0 \mathrm{DR}(\varphi_{f,1}\mathbf{Q}_{h,Y}[d_X])\big) &= \mathrm{Gr}_F^{d_X+1} \mathrm{DR}(\varphi_{f,1}\mathbf{Q}_{h,Y}[d_X]) \\ &= 0. \end{aligned} \tag{4.5.2}$$

Recall that *right* $\mathcal{D}$-modules are used in (4.1.10).

So the first assertion of Theorem 5 in the second case follows from the assertion concerning (4.3.8), since $\mathbf{D}^2 = id$ and

$$M_0(X) = td_*\big[\mathrm{Gr}_F^0\mathrm{DR}(\varphi_f\mathbf{Q}_{h,Y})\big]. \tag{4.5.3}$$

To show the converse, assume that $X$ is *not* Du Bois, that is,

$$\mathcal{F} := F_{-d_X}(\varphi_{f,\ne 1}\mathbf{Q}_{h,Y}[d_X]) \ne 0. \tag{4.5.4}$$

By (4.5.2–3) we have to show

$$td_*\big[\mathbf{D}(\mathcal{F})\big] \ne 0 \quad \text{in } \mathbf{H}_\bullet(\mathbf{P}^N), \tag{4.5.5}$$

where we can replace $\mathbf{H}_\bullet(\Sigma)$ with $\mathbf{H}_\bullet(\mathbf{P}^N)$ by the compatibility of $td_*$ with the pushforward by proper morphisms. Then the assertion follows by using the topological filtration on $K_0(\mathbf{P}^N)_{\mathbf{Q}}$ and $\mathbf{H}_\bullet(\mathbf{P}^N)$ in (1.6) together with the positivities in (1.6.4–5) (see also an argument after (4.2.7)). This finishes the proof of Theorem 5 in the second case.

For the proof in the first case, note that the support of $\mathcal{F}$ in (4.5.4) is independent of the choice of a local defining function of $X$ (where the ambiguity comes from the multiplication by a nowhere vanishing function). However, we have to take here the direct image $(i_{\Sigma_1\setminus X'_1,\Sigma'_1})_!$ of a mixed Hodge module. This can be calculated as in the proof of Proposition (1.4), and the latter shows that it is enough to take the *closure* of the support of the coherent sheaf which gives the non-Du Bois locus. This closure is independent of the choice of $s'_1$, and taking the direct image does not cause a problem as long as $s'_1$ is sufficiently general so that $X'_1 = {s'_1}^{-1}(0)$ does not contain this support. So the assertion follows. Here it is enough to consider the summand in (0.1) with $j = 1$ by using (0.13) together with Theorem 2. This finishes the proof of Theorem 5.

**Remark 4.6.** We do not know a priori the support of the coherent sheaf in the above argument, and there might be some problem about the genericity condition on $s'_1$ (that is, the condition that ${s'_1}^{-1}(0)$ does not contain the support). It may be better to argue as follows:

On a dense Zariski-open subset $U$ of the parameter space of $s'_1$, $X'_1 = {s'_1}^{-1}(0)$ intersects the strata of a Whitney stratification of $X$ transversally so that $M_0(X)$ can be defined. Moreover $M_0(X)$ in the graded pieces of the topological filtration in (1.6) is independent of the choice of $s'$, since it is given by the cycle map, see (1.6). (Here [DiMaSaTo] is also used.) There is another dense Zariski-open subset $U'$ of the parameter space of $s'_1$ such that $X'_1 = {s'_1}^{-1}(0)$ does not contain the non-Du Bois locus. We have $U \subset U'$, since the image of the cycle map would vanish if $s'_1 \notin U'$. So no problem occurs.

**4.7. Proof of Proposition 3.** The duality isomorphisms in (4.5.1) are compatible with the action of the semisimple part of the monodromy $T_s$, where the $\mathbf{e}(\alpha)$-eigenspace is the dual of the $\mathbf{e}(-\alpha)$-eigenspace, see also [Sa7, 2.4.3]. The argument is then essentially the same as in the proof of Theorem 5 by using the topological filtration in (1.6) together with [BuSa, Theorem 0.1] (see (5.1.6) below) which gives the relation with the jumping coefficients. This finishes the proof of Proposition 3.

**4.8. Isolated hypersurface singularity case.** Let $f : (Y,0) \to (\Delta,0)$ be a germ of a holomorphic function on a complex manifold $Y$ such that $X := f^{-1}(0)$ has an isolated singularity at 0, where $\Delta \subset \mathbf{C}$ is an open disk. Let $\mu_f$ be the Milnor number of $f$. As in [St2], the *spectrum*

$$\mathrm{Sp}(f) = \textstyle\sum_{i=1}^{\mu_f} t^{\alpha_{f,i}} \in \mathbf{Z}\big[t^{1/e}\big]$$

with $\alpha_{f,i} \leqslant \alpha_{f,i+1}$ $(i \in [1, \mu_f - 1])$ is defined by

$$\#\{i \mid \alpha_{f,i} = \alpha\} = \dim \mathrm{Gr}_F^p \widetilde{H}^{d_Y-1}(F_{f,0}, \mathbf{C})_{\mathbf{e}(-\alpha)} \qquad (p := [d_Y - \alpha],\ \alpha \in \mathbf{Q}), \tag{4.8.1}$$

where $F_{f,0}$ denotes the Milnor fiber of $f$, and $\widetilde{H}^k(F_{f,0}, \mathbf{Q})$ is identified with $\mathcal{H}^k\varphi_f\mathbf{Q}_{h,Y}$.

Set $Y' := Y \times_\Delta \Delta'$ with $\rho_m : (\Delta', 0) \to (\Delta, 0)$ a totally ramified $m$-fold covering. Let $\beta$ be the smallest positive rational number such that $\mathbf{e}(\beta)$ $(:= e^{2\pi i\beta})$ is an eigenvalue of the Milnor monodromy of $f$. Assume

$$\tfrac{1}{m} \leqslant \beta. \tag{4.8.2}$$

Then the following three conditions are equivalent to each other:

(a) $(X, 0)$ is a Du Bois singularity.

(b) $(Y', 0)$ is a rational singularity.

(c) $f : Y \to \Delta$ is a cohomologically insignificant smoothing.

Condition (c) means that $\mathrm{Gr}_F^0 \widetilde{H}^k(F_{f,0}, \mathbf{C}) = 0$ $(\forall\, k)$, see [St3]. (This condition is invariant by the base change of $\Delta$.)

Set $h = f - z^m$ on $Y \times \mathbf{C}$ with $z$ the coordinate of $\mathbf{C}$ so that $Y' = h^{-1}(0)$. Then the above three conditions are respectively equivalent to

(a)′ $\alpha_{f,1} \geqslant 1$.

(b)′ $\alpha_{h,1} > 1$.

(c)′ $\alpha_{f,\mu_f} \leqslant d_Y - 1$.

In fact, the first two equivalences follow from the arguments related to conditions (4.2.4), (4.3.8), and the last one from the above definition of spectrum, see (4.8.1). We have moreover the symmetry (see [St2]):

$$\alpha_{f,i} + \alpha_{f,j} = d_Y \quad \text{if} \quad i + j = \mu_f + 1, \tag{4.8.3}$$

together with the Thom-Sebastiani theorem as in [ScSt], [Va]:

$$\mathrm{Sp}(h) = \mathrm{Sp}(f)\,\mathrm{Sp}(g), \tag{4.8.4}$$

where $g := z^m$. Since $\mathrm{Sp}(g) = \sum_{k=1}^{m-1} t^{k/m}$ (see Remark (3.3)(iv)), we then get

$$\alpha_{h,1} = \alpha_{f,1} + \tfrac{1}{m}. \tag{4.8.5}$$

So the equivalences between (a), (b), (c) follow.

In the case $\rho_m$ is associated with a semi-stable reduction, the above equivalences are a special case of [St3, Theorem 3.12] combined with [Is, Theorem 6.3] where an arbitrary smoothing of a normal (or Cohen-Macaulay) isolated singularity is treated. We take a projective compactification of $f$ as in [Br] to apply [Is].

## 5. Applications to multiplier ideals

In this section we give some applications to multiplier ideals in the hypersurface case.

**5.1. Multiplier ideals.** Let $Y$ be a smooth complex algebraic variety (or a connected complex manifold), and $f$ be a non-constant function on $Y$, that is, $f \in \Gamma(Y, \mathcal{O}_Y) \setminus \mathbf{C}$. Let $\mathcal{J}(\alpha X) \subset \mathcal{O}_Y$ be the *multiplier ideal* of $X$ with coefficient $\alpha \in \mathbf{Q}$ (or $\mathbf{R}$ more generally). It can be defined by the local integrability of

$$|g|^2/|f|^{2\alpha} \quad \text{for} \quad g \in \mathcal{O}_Y, \tag{5.1.1}$$

see [Na], [La]. By definition, the $\mathcal{J}(\alpha X)$ form a decreasing sequence of ideal sheaves of $\mathcal{O}_Y$ indexed by $\mathbf{R}$ and satisfying

$$\mathcal{J}(\alpha X) = \mathcal{O}_Y \quad (\alpha \leqslant 0), \quad \mathcal{J}((\alpha+1)X) = f\mathcal{J}(\alpha X) \quad (\alpha \geqslant 0). \tag{5.1.2}$$

Multiplier ideals can be defined also by using an embedded resolution of $X$ (*loc. cit.*), and it implies

$$\mathcal{J}(\alpha X) = \mathcal{J}((\alpha+\varepsilon)X) \quad (0 < \forall\, \varepsilon \ll 1). \tag{5.1.3}$$

This means that $\mathcal{J}(\alpha X)$ is *right-continuous* for $\alpha$. More precisely, for any $\alpha' \in \mathbf{R}$, the argument using an embedded resolution shows

$$\big\{\alpha \in \mathbf{R} \mid \mathcal{J}(\alpha X) = \mathcal{J}(\alpha' X)\big\} = [\beta, \beta') \ \text{ or } \ (-\infty, \beta') \ \text{ for some } \ \beta, \beta' \in \mathbf{Q}.$$

We define the graded quotients $\mathcal{G}(\alpha X)$ by

$$\mathcal{G}(\alpha X) := \mathcal{J}((\alpha-\varepsilon)X)/\mathcal{J}(\alpha X) \quad (0 < \varepsilon \ll 1),$$

where the range of $\varepsilon$ may depend on $\alpha$ (this is the same in (5.1.3)). We then have

$$\mathrm{JC}(X) := \big\{\alpha \in \mathbf{R} \mid \mathcal{G}(\alpha X) \neq 0\big\} \subset \mathbf{Q}.$$

The members of $\mathrm{JC}(X)$ are called the *jumping coefficients* of $X$. We will restrict to rational numbers $\alpha$ when we consider $\mathcal{J}(\alpha X)$, $\mathcal{G}(\alpha X)$.

By (5.1.2) we get the isomorphisms

$$f : \mathcal{G}(\alpha X) \xrightarrow{\sim} \mathcal{G}((\alpha+1)X) \quad (\alpha > 0), \tag{5.1.4}$$

and

$$\mathrm{JC}(X) = \big(\mathrm{JC}(X) \cap (0,1]\big) + \mathbf{N}. \tag{5.1.5}$$

Consider the filtration $V$ on $\mathcal{O}_Y$ induced by the filtration $V$ on $\mathcal{B}_f$ via the inclusion

$$\mathcal{O}_Y = F_{-d_Y}\mathcal{B}_f \hookrightarrow \mathcal{B}_f.$$

By [BuSa, Theorem 0.1] we have

$$\begin{aligned} &\mathcal{J}(\alpha X) = V^\alpha\mathcal{O}_Y && \text{if } \alpha \notin \mathrm{JC}(X), \\ &\mathcal{G}(\alpha X) = \mathrm{Gr}_V^\alpha\mathcal{O}_Y = V^\alpha\mathcal{O}_Y/\mathcal{J}(\alpha X) && \text{if } \alpha \in \mathrm{JC}(X). \end{aligned} \tag{5.1.6}$$

This is related to the assertion that $\mathcal{J}(\alpha X)$ is *right-continuous* for $\alpha$ as is explained above, although $V^\alpha\mathcal{O}_Y$ is *left-continuous* for $\alpha$.

We now consider the *microlocal* $V$-filtration on $\mathcal{O}_Y$ which is denoted by $\widetilde{V}$, and is induced by the filtration $V$ on $\widetilde{\mathcal{B}}_f$ via the isomorphism

$$\mathcal{O}_Y = \mathrm{Gr}^F_{-d_Y}\widetilde{\mathcal{B}}_f.$$

Set

$$\widetilde{\mathrm{JC}}(X) := \big\{\alpha \in \mathbf{Q} \mid \mathrm{Gr}_{\widetilde{V}}^\alpha\mathcal{O}_Y \neq 0\big\}.$$

We have by (3.1.4)

$$\widetilde{\mathrm{JC}}(X) \subset (0, +\infty), \quad \widetilde{\mathrm{JC}}(X) \cap (0,1) = \mathrm{JC}(X) \cap (0,1). \tag{5.1.7}$$

However, the last equality does not necessarily hold if $(0,1)$ is replaced by $(0,1]$ (since $\widetilde{\mathrm{JC}}(X)$ does not necessarily contain 1), and (5.1.5) with $\mathrm{JC}(f)$ replaced by $\widetilde{\mathrm{JC}}(f)$ does not necessarily holds, see Example (5.6)(ii) below.

We have the *microlocal multiplier ideals* $\widetilde{\mathcal{J}}(\alpha X)$, and their graded quotients $\widetilde{\mathcal{G}}(\alpha X)$ such that $\widetilde{\mathcal{J}}(\alpha X)$ is right-continuous and

$$\begin{aligned}\widetilde{\mathcal{J}}(\alpha X) &= \widetilde{V}^{\alpha}\mathcal{O}_Y && \text{if } \alpha \notin \widetilde{\mathrm{JC}}(X),\\ \widetilde{\mathcal{G}}(\alpha X) &= \mathrm{Gr}^{\alpha}_{\widetilde{V}}\mathcal{O}_Y = \widetilde{V}^{\alpha}\mathcal{O}_Y/\widetilde{\mathcal{J}}(\alpha X) && \text{if } \alpha \in \widetilde{\mathrm{JC}}(X).\end{aligned} \tag{5.1.8}$$

As for the relation with the usual multiplier ideals, we have by (3.1.4), (4.1.1)

$$\mathcal{J}(\alpha X) = \widetilde{\mathcal{J}}(\alpha X), \quad \mathcal{G}(\alpha X) = \widetilde{\mathcal{G}}(\alpha X) \quad (\alpha < 1), \tag{5.1.9}$$

$$\widetilde{\mathcal{J}}(X)/\mathcal{J}(X) = \widetilde{\omega}_X \otimes_{\mathcal{O}_X} \omega_X^{\vee} \subset \mathcal{O}_X \quad (\alpha = 1), \tag{5.1.10}$$

$$0 \to \widetilde{\omega}_X \otimes_{\mathcal{O}_X} \omega_X^{\vee} \to \mathcal{G}(X) \to \widetilde{\mathcal{G}}(X) \to 0 \quad (\alpha = 1). \tag{5.1.11}$$

Here we assume $X$ reduced in (5.1.10–11). Note that we have by (5.1.2)

$$\mathcal{J}(X) = \mathcal{O}_Y(-X) = \mathcal{I}_X \quad (\alpha = 1), \tag{5.1.12}$$

where the last term is the ideal sheaf of $X$.

We have the Thom-Sebastiani type theorem for microlocal multiplier ideals as follows.

**Theorem 5.2.** *With the notation and the assumption of Theorem* (3.2)*, there are equalities for any* $\alpha \in \mathbf{Q}$ :

$$\widetilde{\mathcal{J}}(\alpha X) = \textstyle\sum_{\alpha_1+\alpha_2=\alpha} \widetilde{\mathcal{J}}(\alpha_1 X_1) \boxtimes \widetilde{\mathcal{J}}(\alpha_2 X_2) \quad \textit{in} \ \ \mathcal{O}_Y = \mathcal{O}_{Y_1} \boxtimes \mathcal{O}_{Y_2}. \tag{5.2.1}$$

*by replacing* $Y_a$ *with an open neighborhood of* $X_a = f_a^{-1}(0)$ *in* $Y_a$ $(a = 1, 2)$ *so that* $\Sigma = \Sigma_1 \times \Sigma_2$ *if necessary. Here we may assume* $\alpha_1, \alpha_2 \in (0, \alpha)$ *by the first equality in* (5.1.2) *together with* (5.1.9).

*Proof.* In (5.2.1) we may replace $\alpha_1 + \alpha_2 = \alpha$ by $\alpha_1 + \alpha_2 \geqslant \alpha$, and assume for $0 < \varepsilon \ll 1/m$

$$\alpha_a \in \widetilde{\mathrm{JC}}(X_a) - \varepsilon \ \ (a = 1, 2), \tag{5.2.2}$$

(since $\widetilde{\mathcal{J}}(\alpha X)$ is right-continuous), where $m$ is a positive integer such that $\widetilde{\mathrm{JC}}(X_a) \in \mathbf{Z}/m$. We now show that (5.2.1) is equivalent to the following.

$$\widetilde{V}^{\alpha}\mathcal{O}_Y = \textstyle\sum_{\alpha_1+\alpha_2=\alpha} \widetilde{V}^{\alpha_1}\mathcal{O}_{Y_1} \boxtimes \widetilde{V}^{\alpha_2}\mathcal{O}_{Y_2} \quad \text{in} \ \ \mathcal{O}_Y = \mathcal{O}_{Y_1} \boxtimes \mathcal{O}_{Y_2}. \tag{5.2.3}$$

We may replace $\alpha_1 + \alpha_2 = \alpha$ by $\alpha_1 + \alpha_2 \geqslant \alpha$ in (5.2.3), and assume

$$\alpha_a \in \widetilde{\mathrm{JC}}(X_a) \ \ (a = 1, 2), \tag{5.2.4}$$

since $\widetilde{V}^{\alpha}$ is left-continuous. However, we may also assume (5.2.2) with $0 < \varepsilon \ll 1/m$ instead of (5.2.4) by replacing $\alpha$ with $\alpha - 2\varepsilon$ if necessary. (Here $\varepsilon$ may depend on $\alpha$.) The equivalence between (5.2.1) and (5.2.3) then follows from (5.1.8).

We can show (5.2.3) by taking $\mathrm{Gr}^F_{-d_Y}$ of the isomorphism (3.2.7) and calculating $\mathrm{Gr}^F$ of $\iota$ in (3.2.7), since $\mathrm{Gr}^F_{p_a}(\widetilde{\mathcal{B}}_{f_a}, V)$ is essentially independent of $p_a$ by (3.1.5) $(a = 1, 2)$. This finishes the proof of Theorem (5.2).

**Corollary 5.3.** *With the notation and the assumption of Theorem* (3.2)*, there are canonical isomorphisms for any* $\alpha \in \mathbf{Q}$ :

$$\widetilde{\mathcal{G}}(\alpha X) = \textstyle\bigoplus_{\alpha_1+\alpha_2=\alpha} \widetilde{\mathcal{G}}(\alpha_1 X_1) \boxtimes \widetilde{\mathcal{G}}(\alpha_2 X_2), \tag{5.3.1}$$

*by replacing* $Y_a$ *with an open neighborhood of* $X_a = f_a^{-1}(0)$ *in* $Y_a$ $(a = 1, 2)$ *if necessary. Here we may assume* $\alpha_1, \alpha_2 \in (0, \alpha)$ *as in Theorem* (5.2).

It is also possible to deduce this from Theorem (3.2). Combining Corollary (5.3) with (5.1.9), (5.1.11), we get the following.

**Corollary 5.4.** *With the notation and the assumption of Theorem* (3.2)*, assume further $X$ reduced. We have the short exact sequence for $\alpha = 1$ :*

(5.4.1) $$0 \to \widetilde{\omega}_X \otimes_{\mathcal{O}_X} \omega_X^{\vee} \to \mathcal{G}(X) \to \textstyle\bigoplus_{\alpha_1+\alpha_2=1} \mathcal{G}(\alpha_1 X_1) \boxtimes \mathcal{G}(\alpha_2 X_2) \to 0.$$

*by replacing $Y_a$ with an open neighborhood of $X_a = f_a^{-1}(0)$ in $Y_a$ $(a = 1, 2)$ if necessary. Here we may assume $\alpha_1, \alpha_2 \in (0, 1)$ as in Theorem* (5.2)*.*

**5.5. Proof of Theorem 4.** The assertion follows from Theorem (5.2) and Corollary (5.3) together with (5.1.9).

**Examples 5.6.** (i) Let $Y = \mathbf{C}$ with coordinate $z$. Set $f = z^m$ for $m \geqslant 2$. Then

(5.6.1) $$\widetilde{V}^{i/m}\mathcal{O}_Y = \mathcal{O}_Y z^k \quad \text{if} \quad i = k + 1 + \big[k/(m-1)\big].$$

In fact, we have

(5.6.2) $$V^{i/m}\mathcal{O}_Y = \mathcal{O}_Y z^{i-1} \quad (i \in [1, m-1]),$$

where $V$ is the usual $V$-filtration on $\mathcal{O}_Y$, see (5.1). This is compatible with [Sa2], and can be proved by using the multiplier ideals together with (5.1.6).

We then get the inclusion $\supset$ in (5.6.1) by using (5.6.2) together with the definition of the action of $\partial_z$ in (3.1.1) and (3.1.5), since $\mathrm{Gr}^F \partial_z$ preserves the filtration $\widetilde{V}$ and $\partial_z f = m z^{m-1}$. So it is enough to show

(5.6.3) $$\dim \mathrm{Gr}_{\widetilde{V}}^{i/m} \mathcal{O}_Y = \begin{cases} 1 & \text{if } i \geqslant 1,\ i/m \notin \mathbf{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

This follows from (3.3.3) by using (3.1.5) and recalling the definition of the direct image of filtered $\mathcal{D}$-modules by the inclusion $\{0\} \hookrightarrow \mathbf{C}$.

(ii) Let $Y = \mathbf{C}^d$ with coordinates $z_1, \dots, z_d$. Set $f = \sum_{j=1}^d z_j^{m_j}$ for $m_j \geqslant 2$ $(j \in [1, d])$. Then Example (i) together with (5.2.3) implies

(5.6.4) $$\widetilde{V}^{\alpha}\mathcal{O}_Y = \textstyle\sum_{\nu} \mathcal{O}_Y z^{\nu},$$

where the summation is taken over $\nu = (\nu_1, \dots, \nu_d) \in \mathbf{N}^d$ satisfying

(5.6.5) $$\textstyle\sum_{j=1}^d \frac{1}{m_j}\big(\nu_j + 1 + \big[\frac{\nu_j}{m_j - 1}\big]\big) \geqslant \alpha.$$

In particular, we have

(5.6.6) $$\widetilde{V}^{\widetilde{\alpha}_f}\mathcal{O}_Y = \mathcal{O}_Y \ne \widetilde{V}^{>\widetilde{\alpha}_f}\mathcal{O}_Y \quad \text{with} \quad \widetilde{\alpha}_f := \textstyle\sum_{j=1}^d \frac{1}{m_j}, \quad \text{and} \quad \mathrm{lct}(f) = \min\{1, \widetilde{\alpha}_f\}.$$

By (5.6.4–5) we see that the microlocal $V$-filtration on $\mathcal{O}_{Y,0}$ has nothing to do with the filtration $V$ on the *microlocal* Gauss-Manin system as in [Sa2]. In fact, the latter coincides with the usual Gauss-Manin system (since the Milnor fiber is contractible), and the filtration $V$ on it is induced by the *usual* $V$-filtration on $\mathcal{B}_f$, see [Sa3, Proposition 3.4.8].

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