Meromorphic connections with logarithmic poles along a free divisor

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Abstract. We develop an approach to the study of meromorphic connections with logarithmic poles along a Saito free divisor D in a complex manifold S. In particular, some properties of Christoffel symbols of such a connection are established. We also compute the sets of all integrable homogeneous meromorphic connections on $\Omega^1_S(\log D)$ with logarithmic poles along the discriminant of the minimal versal deformation of an A_3-singularity and along Sato's hypersurface occurred in the context of the theory of prehomogeneous spaces.

Introduction

The notion of logarithmic connection originates in a work of P. Deligne [**De**], where he formulated and proved the theorem establishing a Riemann-Hilbert correspondence between monodromy groups and Fuchsian systems of integrable partial differential equations or flat connections on complex manifolds. He also gave a treatment of the theorem of Griffiths which states that Gauss-Manin or Picard-Fuchs systems of differential equations are regular singular. To be precise, let X be a nonsingular complex quasi-projective variety, and let $D \subset X$ be a divisor with *normal crossings* so that X is a nonsingular completion of $X' = X \setminus D$. Then there is a correspondence between representations of the fundamental group $\pi_1(X \setminus D)$ in $\mathrm{GL}_d(\mathbf{C})$ and logarithmic integrable connections $H = (\mathcal{H}, \nabla)$ on X with singularities on D. Here $\mathcal{H}$ is a torsion free coherent sheaf on X together with a map $\nabla\colon\ \mathcal{H} \to \Omega^1_X[\log D] \otimes \mathcal{H}$ satisfying Leibniz rule and the integrability condition, where $\Omega^1_X[\log D]$ is the sheaf of rational 1-forms on X with logarithmic singularities on D. In fact, Deligne proved that a non-singular connection $\nabla'\colon\ \mathcal{H} \to \Omega^1_{X'} \otimes \mathcal{H}$ on X' is *regular singular* if and only if it can be extended to a logarithmic connection ∇ on X. Other equivalent conditions under which the connection is regular singular are considered in [**AB**]. Remark also that some important properties of ∇ are discussed by N. Katz [**K**] in the case when the divisor D has normal crossings.

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Thus, the problem of classification of logarithmic connections arises naturally. Making use of the general theory of meromorphic differential equations, D.G. Babbitt and V.S. Varadarajan [**BV2**] investigated some local aspects of this problem. The approach of Deligne was further developed by H. Esnault and E. Viehweg ([**EV**], Appendix). Somewhat later, in 1990, C. Simpson [**Si**] constructed a moduli scheme for non-singular integrable connections on a projective variety and gave a number of applications of his results to the nonabelian Hodge theory, to the theory of Higgs bundles, to the theory of Gauss-Manin connexion, etc. However a moduli scheme for regular connections on a projective variety does not exist in general. This leads to the study of the moduli problem for *logarithmic* connections on a projective variety. Thus, in 1993, N. Nitsure [**N**] generalized the method of Simpson and investigated this problem for *semi-stable* logarithmic connections.

On the other side, in 1976, K. Saito [**S1**] calculated the Gauss-Manin connexion associated to the versal deformation of an A_3-singularity and described an elegant representation of the connexion, making use of the general theory of logarithmic differential forms. More precisely, he computed a set of integrable homogeneous meromorphic connections with logarithmic poles along the discriminant D of the minimal versal deformation of an A_3-singularity which contains the Gauss-Manin connexion. Furthermore, in 1988, K. Saito [**S3**] introduced a new concept of regular singular connection along an *arbitrary* reduced divisor. In particular, the corresponding meromorphic connection has *logarithmic* poles along the divisor but the converse is not true in contrast with the classical situation considered by Deligne and his followers.

In this paper we develop an approach to the study of the classification problem for logarithmic connections with poles along a Saito free divisor $D \subset S$ based on specific properties of such divisors. Note that the singular locus of a free divisor has a very complicated structure and its codimension is equal to one [**A2**]. In fact, the class of Saito free divisors contains some types of divisors with normal crossings as well as the discriminants of the minimal versal deformations of hypersurface singularities. Among other things we shall describe some useful properties of Christoffel symbols and compute explicitly the set of all integrable homogeneous connections on $\Omega^1_S(\log D)$ with logarithmic poles along the discriminant of the minimal versal deformation of an A_3-singularity and along a free divisor in $\mathbf{C}^3$ that occurs in the context of the theory of prehomogeneous spaces developed by M. Sato [**Se**].

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1. Free divisors

Let U be an open subset of $\mathbf{C}^m$, and let D be the hypersurface defined by an equation $h(z) = 0$, where $h(z) = h(z_1, \dots, z_m)$ is a holomorphic function in U, and $z_1, \dots, z_m$ is a system of coordinates. Suppose that D is *reduced,* that is, $h(z)$ has no multiple factors.

Definition 1.1. A meromorphic differential q-form ω, $q \geq 0$, on U with poles on D is called *logarithmic* along the divisor D if ω and its differential $d\omega$ have poles at worst of the first order. It means that $h\omega$ and $hd\omega$ are *holomorphic* on U.

REMARK 1.2. In fact, differential forms of such kind were first considered in 1972 in a work of J.-B. Poly [**Po**] who proved that the Leray residue is well defined for any (not necessarily d-closed) *semi-meromorphic* differential form ω, satisfying the conditions of Definition 1.1. Somewhat later K. Saito called such *meromorphic* differential forms *logarithmic* along a hypersurface [**S1, S2**].

Let S be an m-dimensional complex manifold, and let $\Omega^\bullet_S = \big(\Omega^q_S, d\big)_{q=0,1,\dots}$ be the de Rham complex of germs of holomorphic differential forms on S. Let D be a reduced hypersurface of S, and $x \in S$. We denote the $\mathcal{O}_{S,x}$-module of germs of logarithmic q-form at x and the corresponding *sheaf* of logarithmic differential q-form on S by $\Omega^q_{S,x}(\log D)$ and $\Omega^q_S(\log D)$, respectively. Thus, the *coherent* $\mathcal{O}_S$-module $\Omega^1_S(\log D)$ is a submodule of $\Omega^1_S(\star D)$, consisting of all the "differential forms with polar singularities along D." Obviously, the sheaves $\Omega^q_S(\log D)$ and Ω^q_S coincide outside the divisor D, for all $q \geq 0$. By definition,

$$\Omega^0_{S,x}(\log D) \cong \Omega^0_{S,x} \cong \mathcal{O}_{S,x}, \quad \Omega^m_{S,x}(\log D) \cong \frac{1}{h_x}\Omega^m_{S,x}.$$

Suppose that D is *reducible*, $h = h_1 \cdots h_k$, so that the irreducible components of the divisor D are defined by the zero sets of the functions $h_1, \dots, h_k$. Then the sheaf $\Omega^q_S(\log D)$ of meromorphic differential forms logarithmic along D contains the subsheaf $\Omega^q_S[\log D]$ of logarithmic differential forms *with respect* to D defined as follows (see [**Mo**]):

$$\Omega^q_S[\log D] = \bigwedge^q \left\langle \frac{dh_1}{h_1}, \dots, \frac{dh_k}{h_k}, dz_1, \dots, dz_m \right\rangle, \quad q \geq 1.$$

For example, if $\Omega^1_S(\log D)$ is generated by *closed* forms then $\Omega^1_S[\log D] = \Omega^1_S(\log D)$ and vice versa. It is equivalent to certain conditions on the reducible divisor D which are similar to normal crossings (see [**S2**], Theorem 2.9).

In what follows, when we consider the local situation the point x will be taken to be 0 for simplicity. We shall also assume that U is an open subset of $\mathbf{C}^m$ containing the origin. Following K.Saito [**S1**], we denote the $\mathcal{O}_S$-module of vector fields *logarithmic* along $D \subset S$ by $\mathrm{Der}_S(\log D)$. This module consists of germs of holomorphic vector fields $\eta \in \mathrm{Der}(\mathcal{O}_S)$ on S such that $\eta(h)$ belongs to the principal ideal $(h)\cdot\mathcal{O}_S$. In particular, the vector field η is tangent to D at its smooth points. The inner multiplication of vector fields and differential forms induces a natural pairing of $\mathcal{O}_S$-modules

$$\mathrm{Der}_S(\log D) \times \Omega^q_S(\log D) \longrightarrow \Omega^{q-1}_S(\log D). \tag{1.1}$$

Moreover, for $q = 1$ this $\mathcal{O}_S$-bilinear mapping is a *non-degenerate* (perfect) pairing so that $\mathrm{Der}_S(\log D)$ and $\Omega^1_S(\log D)$ become $\mathcal{O}_S$-dual.

The sheaves of logarithmic vector fields $\mathrm{Der}_S(\log D)$ and logarithmic differential forms $\Omega^q_S(\log D)$, $q = 0, 1, \dots, m$, are *coherent* sheaves of $\mathcal{O}_S$-modules. In fact, $\mathrm{Der}_S(\log D)$ is the kernel of the natural $\mathcal{O}_S$-morphism of coherent $\mathcal{O}_S$-modules

$$\Phi : \mathrm{Der}(\mathcal{O}_S) \to \mathcal{O}_D$$

defined by the rule $\Phi(\eta) = \eta(h)$. Furthermore, the direct sum $\oplus_{q=0}^m \Omega^q_S(\log D)$ is an $\mathcal{O}_S$-exterior algebra closed under the exterior de Rham differentiation d.

Suppose that $\Omega^1_S(\log D)$ is a locally *free* $\mathcal{O}_S$-module of rank m. In view of the duality (1.1) the module of germs of logarithmic vector fields $\mathrm{Der}_S(\log D)$ is free too and vice versa. In this case $\Omega^p_S(\log D) = \bigwedge^p \Omega^1_S(\log D)$, $p \geq 1$, and D is called the *free* or *Saito free* divisor [**Ca**]. The following statement is due to K.Saito [**S1**].

PROPOSITION 1.3. *The $\mathcal{O}_{S,0}$-module $\mathrm{Der}_{S,0}(\log D)$ is free if and only if there are m germs of logarithmic vector fields $V^1, \dots, V^m \in \mathrm{Der}_{S,0}(\log D)$ such that the determinant of the $m \times m$-matrix $\mathcal{V} = \|\,\upsilon_{ij}\|$ whose entries are the coefficients of V^i, $i = 1, \dots, m$, is equal to αh where α is a unit. In this case the set of vector fields $V^1, \dots, V^m$ forms a basis of the free module $\mathrm{Der}_{S,0}(\log D)$.*

For example, $\mathrm{Der}_S(\log D)$ as well as $\Omega^1_S(\log D)$ are locally free if D is a divisor with *strict* normal crossings. In this case $\Omega^1_S(\log D) = \Omega^1_S[\log D]$ and a defining equation of D can be written as $h = z_1 \cdots z_k = 0$, where $k \leq m$. The discriminant of the minimal versal deformation of an isolated hypersurface or complete intersection singularity is a Saito free divisor (see [**S1**]).

Let us denote the cofactor of the element υ_{ij} of the matrix $\mathcal{V}$ by Υ_{ij} and consider the following holomorphic differential 1-forms

$$\vartheta_i = \sum_{j=1}^m \Upsilon_{ij} dz_j, \quad i = 1, \dots, m.$$

Set $\Delta = \det(\mathcal{V})$. Then the elements

$$\omega_i = \frac{\vartheta_i}{\Delta}, \quad i = 1, \dots, m,$$

form a basis of the free $\mathcal{O}_S$-module $\Omega^1_S(\log D)$. It is not difficult to see that two systems of basis $V^1, \dots, V^m$ and $\omega_1, \dots, \omega_m$ are dual with respect to the pairing (1.1). In particular, one gets $\iota_{V^j}(\omega_i) = \delta^j_i$ for all $i, j = 1, \dots, m$, where the inner multiplication or, equivalently, the contraction along vector field V is denoted by ι_V. We mention also some simple properties of dual basis (see [**A1**] with minor changes of notations):

1) $df = \displaystyle\sum_{i=1}^m \frac{\partial f}{\partial z_i}\, dz_i = \sum_{i=1}^m V^i(f)\, \omega_i$, whenever $f \in \mathcal{O}_{S,0}$;

2) $\iota_{V^i}(\vartheta_i) = \det(\mathcal{V}) = \Delta = \alpha h$, $\alpha \in \mathcal{O}^*_{S,0}$, $\iota_{V^i}(\vartheta_j) = 0$, $i \neq j$;

3) given an arbitrary ordered p-collection $1 \leq i_1 < \ldots < i_p \leq m$ of the set of integers $\{1, \dots, m\}$, there is the following identity:

$$\iota_{V^{i_1}} \ldots \iota_{V^{i_p}}\Big(\frac{1}{\Delta} dz\Big) = (-1)^{i_1 + \ldots + i_p + p} \omega_1 \wedge \ldots \wedge \widehat{\omega}_{i_1} \wedge \ldots \wedge \widehat{\omega}_{i_p} \wedge \ldots \wedge \omega_m,$$

where $dz = dz_1 \wedge \ldots \wedge dz_m$, and the symbol $\widehat{\ }$ marks omitted elements. To prove 3), we note that if $\mathcal{V}'$ is the *reciprocal* matrix to $\mathcal{V}$, then $\det(\mathcal{V}') = \det(\mathcal{V})^{m-1}$, so that $\vartheta_1 \wedge \ldots \wedge \vartheta_m = \Delta^{m-1}(dz)$. It remains to apply contractions ι_{V^i} successively to both sides of this identity.

REMARK 1.4. Let $\{\omega_1, \dots, \omega_m\}$ and $\{V^1, \dots, V^m\}$ be systems of dual basis of the free $\mathcal{O}_S$-modules of logarithmic 1-forms $\Omega^1_S(\log D)$ and of logarithmic vector fields $\mathrm{Der}_S(\log D)$, respectively. Set $\mathcal{M} = \frac{1}{2}m(m-1)$. Then

$$\begin{aligned} \omega_i &= (-1)^{\mathcal{M}+i+1}\iota_{V^1}\ldots\widehat{\iota_{V^i}}\ldots\iota_{V^m}(\tfrac{1}{\Delta}dz), \\ \omega_i\wedge\omega_j &= (-1)^{\mathcal{M}+i+j+2}\iota_{V^1}\ldots\widehat{\iota_{V^i}}\ldots\widehat{\iota_{V^j}}\ldots\iota_{V^m}(\tfrac{1}{\Delta}dz), \end{aligned}$$

for all $i,j = 1,\ldots,m,\ i<j$, and so on. In the sequel for simplicity we shall sometimes omit constant factors of type $(-1)^{\mathcal{M}+1}$ or $(-1)^{\mathcal{M}+2}$ in formulas if it will not lead to confusion.

In conclusion of this section let us obtain a useful expression for the total differential of basis logarithmic differential forms. First remark that the subset $\{V^1,\ldots,V^m\}$ of $\mathrm{Der}_{S,0}(\log D)$ is an *involutive* system of vector fields with the commutation relations $[V^i,V^j] = \sum_{k=1}^m c_k^{ij}(z)V^k$, where $c_k^{ij}\in\mathcal{O}_{S,0}$ are the "structural constants" of the Lie algebra $\mathrm{Der}_{S,0}(\log D)$ with respect to this basis. We shall use the following formula [**A1**].

LEMMA 1.5. *Let $V^1,\ldots,V^q$ be a sequence of logarithmic differential forms, and let $2\le q\le m$. There is the following identity:*

$$d\iota_{V^1}\ldots\iota_{V^q} = \frac{1}{2(q-2)!}\sum_i\sigma(i_1,\ldots,i_q)\iota_{[V^{i_1},V^{i_2}]}\iota_{V^{i_3}}\ldots\iota_{V^{i_q}}$$

$$+\frac{(-1)^{q-1}}{(q-1)!}\sum_i\sigma(i_1,\ldots,i_q)\iota_{V^{i_1}}\ldots\iota_{V^{i_{q-1}}}d\iota_{V^{i_q}} + (-1)^{q-1}(q-1)\iota_{V^1}\ldots\iota_{V^q}d,$$

where the summations are over all subsets $(i_1,\ldots,i_q)\in\{1,\ldots,q\}$, and $\sigma(i_1,\ldots,i_q)$ denotes the sign of the substitution.

PROOF. Set $q=2$. Making use of the standard relation $[L_V,\iota_W] = \iota_{[V,W]}$, one obtains

$$d\iota_{V^1}\iota_{V^2} = \iota_{[V^1,V^2]} - \iota_{V^1}d\iota_{V^2} + \iota_{V^2}d\iota_{V^1} - \iota_{V^1}\iota_{V^2}d.$$

Then it is possible to use an induction since $d\iota_{V^1}\ldots\iota_{V^q} = (d\iota_{V^1}\ldots\iota_{V^{q-1}})\iota_{V^q}$. □

PROPOSITION 1.6. *Let D be a free divisor, let $\{\omega_1,\ldots,\omega_m\}$ and $\{V^1,\ldots,V^m\}$ be systems of dual basis of the free $\mathcal{O}_S$-modules of logarithmic differential forms $\Omega^1_S(\log D)$ and of logarithmic vector fields* $\mathrm{Der}_S(\log D)$, *respectively. Then*

$$(-1)^{(\mathcal{M}+j+1)}d\omega_j = \frac{1}{2(m-3)!}\sum_k\sum_i\sigma(i_1,\ldots,i_{m-1})c_k^{i_1i_2}\iota_{V^k}\iota_{V^{i_3}}\ldots\iota_{V^{i_{m-1}}}(\frac{1}{\Delta}dz)$$

$$+\frac{(-1)^{m-2}}{(m-2)!}\sum_i\sigma(i_1,\ldots,i_{m-1})\mathcal{L}_{V^{i_{m-1}}}\iota_{V^{i_1}}\ldots\iota_{V^{i_{m-2}}}(\frac{1}{\Delta}dz),$$

where the summations are over all subsets $(i_1,\ldots,i_{m-1})\in\{1,\ldots,\widehat{j},\ldots,m\}$, and

$$\mathcal{L}_V = \mathrm{Div}(V) - \frac{1}{\Delta}V(\Delta),\quad V = \sum_{k=1}^m v_k\frac{\partial}{\partial z_k},\quad \mathrm{Div}(V) = \sum_{k=1}^m\frac{\partial v_k}{\partial z_k}.$$

Proof. In fact, one can apply the differentiation d to the first identity of Remark 1.4

$$(-1)^{(\mathcal{M}+j+1)}d\omega_j = d\iota_{V^1}\dots\widehat{\iota_{V^j}}\dots\iota_{V^m}(\frac{1}{\Delta}dz),$$

and then use Lemma 1.5. $\square$

Corollary 1.7. *Under assumptions of Proposition* 1.6 *set* $m = 3$. *Then*

$$\begin{aligned}
d\omega_1 &= -c_1^{23}\omega_2\wedge\omega_3 + c_2^{23}\omega_1\wedge\omega_3 - c_3^{23}\omega_1\wedge\omega_2 + \mathcal{L}_{V^3}\omega_1\wedge\omega_3 - \mathcal{L}_{V^2}\omega_1\wedge\omega_2,\\
d\omega_2 &= c_1^{13}\omega_2\wedge\omega_3 - c_2^{13}\omega_1\wedge\omega_3 + c_3^{13}\omega_1\wedge\omega_2 - \mathcal{L}_{V^3}\omega_2\wedge\omega_3 + \mathcal{L}_{V^1}\omega_1\wedge\omega_2,\\
d\omega_3 &= -c_1^{12}\omega_2\wedge\omega_3 + c_2^{12}\omega_1\wedge\omega_3 - c_3^{12}\omega_1\wedge\omega_2 + \mathcal{L}_{V^2}\omega_2\wedge\omega_3 - \mathcal{L}_{V^1}\omega_1\wedge\omega_3.
\end{aligned}$$

Proof. It is a direct consequence of Proposition above. $\square$

2. Logarithmic connections

Suppose that S is an m-dimensional complex smooth manifold and $D \subset S$ is a *free* divisor. Let $\Omega_S(\log D) = \bigwedge^{\bullet}_{\mathcal{O}_S}\Omega^1_S(\log D)$ be the graded exterior $\mathbf{C}$-algebra.

Definition 2.1. Let $\mathcal{H}$ be an $\mathcal{O}_S$-module. A meromorphic connection ∇ on $\mathcal{H}$ with *logarithmic* poles along D is a $\mathbf{C}$-linear morphism

$$\nabla\colon\ \Omega_S(\log D)\otimes_{\mathcal{O}_S}\mathcal{H}\longrightarrow\Omega_S(\log D)\otimes_{\mathcal{O}_S}\mathcal{H} \tag{2.1}$$

with the following properties:

i) ∇ is homogeneous of degree 1, that is, $\nabla = \oplus_{p\in\mathbf{N}}\nabla^p$, where

$$\nabla^p\colon\ \Omega^p_S(\log D)\otimes_{\mathcal{O}_S}\mathcal{H}\longrightarrow\Omega^{p+1}_S(\log D)\otimes_{\mathcal{O}_S}\mathcal{H},\quad p\geq 0,$$

are $\mathbf{C}$-linear morphisms;

ii) for $\omega\in\Omega^p_S(\log D),\ u\in\mathcal{H}$, the Leibniz rule

$$\nabla^p(\omega\otimes u) = d\omega\otimes u + (-1)^p\omega\otimes\nabla^0(u)$$

holds. In particular, for $f\in\mathcal{O}_S,\ u\in\mathcal{H}$, we have $\nabla^0(f\cdot u) = df\otimes u + f\cdot\nabla^0(\omega)$. Usual extension arguments show that the operator ∇^0 uniquely determines ∇ and vice versa. The connection ∇ is called *integrable* if the composition map in

$$\mathcal{H}\xrightarrow{\nabla^0}\Omega^1_S(\log D)\otimes\mathcal{H}\xrightarrow{\nabla^1}\Omega^2_S(\log D)\otimes\mathcal{H}$$

is trivial. The subspace of *horizontal* sections relative to the connection ∇ is defined as follows:

$$\mathcal{H}^{\nabla} = \mathrm{Ker}\big(\nabla^0:\ \mathcal{H}\longrightarrow\Omega^1_S(\log D)\otimes\mathcal{H}\big).$$

Let $\mathcal{H} = \Omega^1_S(\log D)$, and consider a logarithmic connection on $\Omega^1_S(\log D)$ itself, that is,

$$\nabla\colon\ \Omega^1_S(\log D)\longrightarrow\Omega^1_S(\log D)\otimes_{\mathcal{O}_S}\Omega^1_S(\log D). \tag{2.2}$$

Let $\omega_1, \dots, \omega_m$ be a system of free generators of $\Omega^1_S(\log D)$. Then the connection ∇ can be described in the following way:

$$\nabla(\omega_i) = \sum_{j=1}^{m} \omega_i^j \otimes \omega_j, \quad \omega_i^j = \sum_{k=1}^{m} \Gamma_i^{jk} \omega_k.$$

By analogy with the theory of smooth or holomorphic connections the analytic functions Γ_i^{jk} are called the *Christoffel symbols* with respect to the basis $\omega_1, \dots, \omega_m$. The structural equation for the curvature of the connection ∇ implies that the condition of integrability can be expressed as $d\omega_i^j = \sum_{k=1}^{m} \omega_k^j \wedge \omega_i^k$. It is equivalent to the matrix identity $d\,\Omega = \Omega \wedge \Omega$, where Ω is the transposed matrix of $\| \omega_i^j \|$. We shall call Ω the *connection form* of ∇. It should be remarked that the trace of Ω is a closed form since $\operatorname{tr}(d\Omega) = 0$. The connection ∇ on $\Omega^1_S(\log D)$ is called *torsion free* if the composition map in

$$\Omega^1_S(\log D) \xrightarrow{\nabla^0} \Omega^1_S(\log D) \otimes \Omega^1_S(\log D) \xrightarrow{\wedge} \bigwedge^2 \Omega^1_S(\log D)$$

coincides with the exterior de Rham differentiation d. In fact, this equivalent to the condition $d\omega_i = \sum_{j=1}^{m} \omega_i^j \wedge \omega_j$.

PROPOSITION 2.2. *Let $V^1, \dots, V^m$ and $\omega_1, \dots, \omega_m$ be systems of dual basis of logarithmic vector fields and forms, and let $\omega \in \Omega^1_S(\log D)$ be a horizontal element. Set $\omega = f_1\omega_1 + \dots + f_m\omega_m$, $f_i \in \mathcal{O}_S$. Then the functions f_i, $i = 1, \dots, m$, satisfy the following system of differential equations:*

$$\begin{bmatrix} V^j(f_1) \\ \vdots \\ V^j(f_m) \end{bmatrix} = -\Gamma^j \cdot \begin{bmatrix} f_1 \\ \vdots \\ f_m \end{bmatrix}, \quad j = 1, \dots, m,$$

where $\Gamma^j = \| \Gamma_k^{ij} \|$ are matrices of order m, and $i, k = 1, \dots, m$.

PROOF. It is a direct consequence of standard arguments exploiting the techniques of duality. Among them we mention here only the following basic considerations. First remark that there is a canonical isomorphism of $\mathcal{O}_S$-modules

$$\operatorname{Hom}_{\mathcal{O}_S}\big(\Omega^1_S(\log D), \mathcal{O}_S\big) \longrightarrow \operatorname{Der}_S(\log D),$$

induced by the pairing (1.1) for $q = 1$ and given by the rule $\langle \omega, V \rangle = \iota_V(\omega)$. Further the map $\iota_V \circ d = V$ extends to an $\mathcal{O}_S$-bilinear map

$$\big(\Omega^1_S(\log D) \otimes \mathcal{H}\big) \times \operatorname{Der}_S(\log D) \longrightarrow \mathcal{H}$$

with $\langle \omega \otimes w, V \rangle = \langle \omega, V \rangle \cdot w$ for $w \in \mathcal{H}$. In particular, when $\omega = df$, $f \in \mathcal{O}_S$, then $\langle df \otimes w, V \rangle = V(f) \cdot w$. Hence, the connection (2.1) induces a connection

$$\widetilde{\nabla} \colon \operatorname{Der}_S(\log D) \times \mathcal{H} \longrightarrow \mathcal{H} \tag{2.3}$$

in the following way: $\widetilde{\nabla}_V(w) = \langle \nabla(w), V \rangle$. Of course, the Leibniz rule is then fulfilled:

$$\widetilde{\nabla}_V(fw) = \langle \nabla(fw), V \rangle = \langle df \otimes w + f\nabla(w), V \rangle = V(f) \cdot w + f \cdot \widetilde{\nabla}_V(w).$$

Moreover, one can pursue similar considerations and show that there is a one-to-one correspondence between the two sets of connections (2.1) and (2.3). The desirable statement follows from this construction by direct computations. □

The following two assertions were observed by K.Saito [**S1**] in a slightly different context. A hypersurface D is called *quasi-homogeneous* if it is locally defined by quasi-homogeneous polynomial germs $h = h(z_1, \dots, z_m)$ in the weighted indeterminates $z_i,\ i = 1, \dots, m$.

Corollary 2.3. *Let D be a quasi-homogeneous divisor. Assume the logarithmic connection* (2.2) *be torsion free. Then a logarithmic differential form $\omega \in \Omega^1_S(\log D)$ of non-zero degree is horizontal if and only if there is an analytic function $u \in \mathcal{O}_S$ such that*

i) $\omega = du = \sum_{i=1}^{m} V^i(u)\, \omega_i,$

ii) $V^k V^j(u) + \sum_{i=1}^{m} V^i(u) \Gamma_i^{jk} = 0, \quad j, k = 1, \dots, m.$

Proof. Let $E \in \mathrm{Der}_S(\log D)$ be the *Euler* vector field so that $E(h) = \gamma h$, where $\gamma \in \mathcal{O}_S^*$ is a unit. One can suppose that ω is homogeneous. Then $L_E(\omega) = E(\omega) = \deg(\omega)\omega$, where $L_E = \iota_E d + d\iota_E$ is the Lie derivative along E. Since $\nabla(\omega) = 0$, the torsion freeness implies $d\omega = 0$. Further, $L_E(\omega) = d\iota_E(\omega)$, since $\iota_E(d\omega) = 0$. Hence, $\deg(\omega)\omega = d\varphi$ with $\varphi = \iota_E(\omega) \in \mathcal{O}_{S,0}$ in virtue of the basic properties of dual logarithmic vector fields and forms. Since $\deg(\omega) \neq 0$, one can set $u := \varphi / \deg(\omega)$. This completes the proof of *i*). The second statement is an immediate consequence of Proposition 2.2. Alternatively, it can be deduced from the basic properties of dual basis; it suffices to apply the operator of the de Rham differentiation d to the equality $\omega = \sum_{j=1}^m V^j(u)\omega_j$, so that

$$d\omega = \sum_{j=1}^{m} dV^j(u)\omega_j + \sum_{j=1}^{m} V^j(u) d\omega_j.$$

Since $dV^j(u) = \sum_{j=1}^m V^k(V^j(u))\omega_k$, one gets

$$\begin{aligned} d^2 u = d\omega &= \sum_{j,k=1}^{m} V^k\big(V^j(u)\big)\omega_k \wedge \omega_j + \sum_{j=1}^{m} V^j(u) \sum_{i,k=1}^{m} \Gamma_j^{ik} \omega_k \wedge \omega_i = \\ &= \sum_{j,k=1}^{m} V^k\big(V^j(u)\big)\omega_k \wedge \omega_j + \sum_{i,k=1}^{m} \big(\sum_{j=1}^{m} V^j(u)\Gamma_j^{ik}\big)\omega_k \wedge \omega_i = \\ &= \sum_{j,k=1}^{m} V^k\big(V^j(u)\big)\omega_k \wedge \omega_j + \sum_{j,k=1}^{m} \big(\sum_{i=1}^{m} V^i(u)\Gamma_i^{jk}\big)\omega_k \wedge \omega_j = 0. \end{aligned}$$

This yields the desired system of equations *ii*). □

A free divisor D is called *Koszul free* [**CM**] if there is a basis $V^j \in \mathrm{Der}_S(\log D)$, $j = 1, \dots, m$, such that the sequence of *principal* symbols $\sigma(V^1), \dots, \sigma(V^m)$ of the differential operators $V^1, \dots, V^m$ is regular in the structure ring of the cotangent bundle of S.

COROLLARY 2.4. *Suppose that the divisor D is Koszul free. Then the system of Corollary* 2.3 *is a maximally overdetermined system of linear partial differential equations or, in other terms, a holonomic system.*

PROOF. The Koszul freeness means that the sequence $\sigma(V^1), \dots, \sigma(V^m)$ is regular in the graded ring $\mathcal{O}_S[\zeta_1, \dots, \zeta_m]$, where $\zeta_i = \partial/\partial z_i,\ i = 1, \dots, m$. Hence the principal symbols of the system in question generate the ideal

$$(\sigma(V^j)\sigma(V^k);\ j,k = 1, \dots, m) = \big(\sigma(V^1), \dots, \sigma(V^m)\big)^2$$

which defines the characteristic variety of the *smallest* possible dimension equal to m. In fact, this ideal defines the Lagrangian subvariety of the cotangent bundle. $\square$

3. Lie derivatives and Christoffel symbols

Now we establish some elementary properties of the Christoffel symbols of meromorphic connections with logarithmic poles along a Saito free divisor.

THEOREM 3.1. *Let D be a free divisor, let $\{\omega_1, \dots, \omega_m\}$ and $\{V^1, \dots, V^m\}$ be systems of dual basis of the free $\mathcal{O}_S$-modules of logarithmic* 1*-forms $\Omega^1_S(\log D)$ and of logarithmic vector fields $\mathrm{Der}_S(\log D)$, respectively. Then locally there is the following identity:*

$$L_{V^j}(\omega_i) = \Big(\mathrm{Div}(V^j) - \frac{1}{\Delta}V^j(\Delta)\Big)\omega_i + \Big(\sum_{\substack{k=1\\ k\neq i,j}}^{m} c^{jk}_k\Big)\omega_i - \Big(\sum_{\substack{k=1\\ k\neq i,j}}^{m} c^{jk}_i\omega_k\Big), \tag{3.1}$$

where L_{V^j} is the Lie derivative along the vector field $V^j = \sum_{k=1}^m v_{jk}\frac{\partial}{\partial z_k}$, the divergence of V^j is equal to $\mathrm{Div}(V^j) = \sum_{k=1}^m \frac{\partial v_{jk}}{\partial z_k}$ similarly to notations of Proposition 1.6, *and $[V^i, V^j] = \sum_{k=1}^m c^{ij}_k V^k$ for all $i, j = 1, \dots, m$.*

PROOF. In virtue of the first identity of Remark 1.4 one may set

$$\omega_i = (-1)^i \iota_{V^1} \dots \widehat{\iota_{V^i}} \dots \iota_{V^m}(\frac{1}{\Delta}dz), \quad i = 1, \dots, m,$$

up to sign $(-1)^{\mathcal{M}+1}$. First assume $i \neq 1$. Making use of the standard relation $[L_{V^i}, \iota_{V^j}] = \iota_{[V^i,V^j]}$, one gets

$$L_{V^j}(\omega_i) = L_{V^j}\big((-1)^i \iota_{V^1} \dots \widehat{\iota_{V^i}} \dots \iota_{V^m}(\frac{1}{\Delta}dz)\big) =$$

$$= (-1)^i\big(\iota_{V^1}L_{V^j}\iota_{V^2} \dots \widehat{\iota_{V^i}} \dots \iota_{V^m} + \iota_{[V^j,V^1]}\iota_{V^2} \dots \widehat{\iota_{V^i}} \dots \iota_{V^m}\big)(\frac{1}{\Delta}dz) =$$

$$= (-1)^i\big(\iota_{V^1}L_{V^j}\iota_{V^2} \dots \widehat{\iota_{V^i}} \dots \iota_{V^m} + \sum_{k=1}^m c^{j1}_k \iota_{V^k}\iota_{V^2} \dots \widehat{\iota_{V^i}} \dots \iota_{V^m}\big)(\frac{1}{\Delta}dz). \tag{3.2}$$

Because

$$\iota_{V^i}\iota_{V^j} + \iota_{V^j}\iota_{V^i} = 0, \quad \iota^2_{V^i} = \iota_{V^i}\iota_{V^i} = 0,$$

the latter summand is equal to

$$(-1)^i\big(c_1^{j1}\iota_{V^1}\iota_{V^2}\dots\widehat{\iota_{V^i}}\dots\iota_{V^m}+(-1)^{i-2}c_i^{j1}\widehat{\iota_{V^1}}\iota_{V^2}\dots\iota_{V^i}\dots\iota_{V^m}\big)(\frac{1}{\Delta}dz)=$$

$$=c_1^{j1}\omega_i+(-1)^{2i-1}c_i^{j1}\omega_1=c_1^{j1}\omega_i-c_i^{j1}\omega_1.$$

When $i=1$ one gets

$$L_{V^j}(\omega_1)=L_{V^j}\big(-\iota_{V^2}\dots\iota_{V^m}(\frac{1}{\Delta}dz)\big)=$$

$$=-\iota_{V^2}L_{V^j}\iota_{V^3}\dots\iota_{V^m}(\frac{1}{\Delta}dz)-\iota_{[V^j,V^2]}\iota_{V^3}\dots\iota_{V^m}(\frac{1}{\Delta}dz)=$$

$$=-\iota_{V^2}L_{V^j}\iota_{V^3}\dots\iota_{V^m}(\frac{1}{\Delta}dz)-\sum_{k=1}^{m}c_k^{j2}\iota_{V^k}\iota_{V^3}\dots\iota_{V^m}(\frac{1}{\Delta}dz)=$$

$$=-\iota_{V^2}L_{V^j}\iota_{V^3}\dots\iota_{V^m}(\frac{1}{\Delta}dz)-c_1^{j2}\omega_2+c_2^{j2}\omega_1.$$

Let us assume $i\neq 2$ and compute the former summand of the formula (3.2) :

$$(-1)^i\iota_{V^1}L_{V^j}\iota_{V^2}\dots\widehat{\iota_{V^i}}\dots\iota_{V^m}(\frac{1}{\Delta}dz)=$$

$$=(-1)^i\big(\iota_{V^1}\iota_{V^2}L_{V^j}\iota_{V^3}\dots\widehat{\iota_{V^i}}\dots\iota_{V^m}+\iota_{V^1}\iota_{[V^j,V^2]}\iota_{V^3}\dots\widehat{\iota_{V^i}}\dots\iota_{V^m}\big)(\frac{1}{\Delta}dz)=$$

$$=(-1)^i\big(\iota_{V^1}\iota_{V^2}L_{V^j}\iota_{V^3}\dots\widehat{\iota_{V^i}}\dots\iota_{V^m}+\sum_{k=1}^{m}c_k^{j2}\iota_{V^1}\iota_{V^k}\iota_{V^3}\dots\widehat{\iota_{V^i}}\dots\iota_{V^m}\big)(\frac{1}{\Delta}dz).$$

Similar reasoning shows that the latter summand is equal to

$$(-1)^i\big(c_2^{j2}\iota_{V^1}\iota_{V^2}\dots\widehat{\iota_{V^i}}\dots\iota_{V^m}+(-1)^{i-3}c_i^{j2}\iota_{V^1}\widehat{\iota_{V^2}}\dots\iota_{V^i}\dots\iota_{V^m}\big)(\frac{1}{\Delta}dz)=$$

$$=c_2^{j2}\omega_i+(-1)^{2i-3}c_i^{j2}\omega_2=c_2^{j2}\omega_i-c_i^{j2}\omega_2,$$

and so on. At the last step we get

$$L_{V^j}(\omega_i)=(-1)^i\iota_{V^1}\dots\widehat{\iota_{V^i}}\dots\iota_{V^m}L_{V^j}(\frac{1}{\Delta}dz)+\Big(\sum_{\substack{k=1\\k\neq i,j}}^{m}c_k^{jk}\Big)\omega_i-\Big(\sum_{\substack{k=1\\k\neq i,j}}^{m}c_i^{jk}\omega_k\Big).$$

On the other side, since $L_{V^j}=d\iota_{V^j}+\iota_{V^j}d$ one has

$$L_{V^j}(\frac{1}{\Delta}dz)=d\iota_{V^j}(\frac{1}{\Delta}dz)+\iota_{V^j}d(\frac{1}{\Delta}dz)=d\iota_{V^j}(\frac{1}{\Delta}dz)=$$

$$=d\Big(\sum_{k=1}^{m}(-1)^k\upsilon_{jk}\frac{dz_1\wedge\dots\wedge\widehat{dz_k}\wedge\dots\wedge dz_m}{\Delta}\Big)=$$

$$=\Big(\sum_{k=1}^{m}\frac{\partial\upsilon_{jk}}{\partial z_k}(\frac{1}{\Delta}dz)+V^j(\frac{1}{\Delta})dz\Big)=\Big(\mathrm{Div}(V^j)-\frac{1}{\Delta}V^j(\Delta)\Big)(\frac{1}{\Delta}dz).$$

As a result one obtains the desired identity. $\square$

Assume the hypersurface D be quasi-homogeneous. Let $h = h(z_1, \dots, z_m)$ be its defining polynomial germ. Set $w_i = \deg z_i$, $d = \deg(h)$, $d_j = \deg V^j = -\deg \omega_j$, $j = 1, \dots, m$, so that $d = \sum_{i=1}^m w_i + \sum_{j=1}^m d_j$. The latter relation follows from the identity $\omega_1 \wedge \ldots \wedge \omega_m = \Delta^{-1} dz_1 \wedge \ldots \wedge dz_m$ in view of the property 3) of systems of dual basis. Suppose $V^1(h) = \deg(h)h$, that is, $V^1 = E$ is the *Euler* vector field, and $0 = d_1 \leq d_2 \leq \ldots \leq d_m$.

Corollary 3.2. *Assume a connection of type* (2.2) *be torsion free. Then its Christoffel symbols satisfy relations* $\Gamma_i^{j1} - \Gamma_i^{1j} = d_j \delta_i^j$ *for all* $i, j = 1, \dots, m$.

Proof. Since $\iota_{V^k}(\omega_i) = \delta_i^k$, then $L_{V^k}(\omega_i) = d\iota_{V^k}(\omega_i) + \iota_{V^k}(d\omega_i) = \iota_{V^k}(d\omega_i)$. First applying the contraction ι_{V^k} to the relation $d\omega_i = \sum_{j=1}^m \omega_i^j \wedge \omega_j$ of torsion freeness, one gets

$$\iota_{V^k}(d\omega_i) = \sum_{j=1}^m \left(\iota_{V^k}(\omega_i^j) \wedge \omega_j - \omega_i^j \wedge \iota_{V^k}(\omega_j)\right) = \sum_{j=1}^m \left(\Gamma_i^{jk} - \Gamma_i^{kj}\right)\omega_j.$$

Set $k = 1$. Since $V^1 = w_1 z_1 \partial/\partial z_1 + \ldots + w_m z_m \partial/\partial z_m$, then there are the following identities

$$[V^1, V^j] = c_j^{1j} V^j = d_j V^j,\ c_i^{1j} = 0,\ i \neq j,$$

for all $i, j = 1, \dots, m$. As a result, Theorem 3.1 yields

$$L_{V^1}(\omega_i) = \left(\sum_{\kappa=1}^m w_\kappa + \sum_{\substack{\kappa=2 \\ \kappa \neq i}}^m d_\kappa\right)\omega_i = -d_i \omega_i,$$

and our statement as required. □

In fact, there are quite a few other useful relations between the Christoffel symbols produced in a similar manner.

4. Examples

In this section we compute the set of all integrable homogeneous logarithmic connections (2.2) in two special cases. The first was considered by K.Saito [**S1**] in a slightly different context (cf. also [**A4**]).

4.1. Swallowtail. Let $F(z, t) = z^4 + t_2 z^2 + t_3 z + t_4 = 0$ be the equation of the total space of the minimal versal deformation of an A_3-singularity. Then the discriminant set $D \subset S = \mathbf{C}^3$ is a Saito free divisor defined by the equation

$$h = 256t_4^3 - 128t_2^2 t_4^2 + 16t_2^4 t_4 + 144t_2 t_3^2 t_4 - 4t_2^3 t_3^2 - 27t_3^4 = 0, \tag{4.1}$$

where the polynomial h is quasi-homogeneous of degree 12 and weights of variables t_2, t_3 and t_4 are equal to 2, 3 and 4, respectively. The hypersurface D is often called the *swallowtail*. A basis of the module of logarithmic vector fields $\mathrm{Der}_S(\log D)$ can be described as follows:

$$V^1 = \tfrac{1}{6}\left(2t_2 \tfrac{\partial}{\partial t_2} + 3t_3 \tfrac{\partial}{\partial t_3} + 4t_4 \tfrac{\partial}{\partial t_4}\right),$$
$$V^2 = 3t_3 \tfrac{\partial}{\partial t_2} + (4t_4 - t_2^2)\tfrac{\partial}{\partial t_3} - \tfrac{1}{2} t_2 t_3 \tfrac{\partial}{\partial t_4},$$
$$V^3 = (t_2^2 - 12t_4)\tfrac{\partial}{\partial t_2} + 3t_2 t_3 \tfrac{\partial}{\partial t_3} + (\tfrac{9}{4} t_3^2 - 4t_2 t_4)\tfrac{\partial}{\partial t_4}.$$

The commutation relations are $[V^1,V^2]=\frac{1}{6}V^2$, $[V^1,V^3]=\frac{1}{3}V^3$, $[V^2,V^3]=t_2V^2$. The matrix $\mathcal{V}$ defined in Proposition 1.3 has the following representation:

$$\mathcal{V}=\begin{pmatrix}\frac{1}{3}t_2 & \frac{1}{2}t_3 & \frac{2}{3}t_4\\ 3t_3 & 4t_4-t_2^2 & -\frac{1}{2}t_2t_3\\ t_2^2-12t_4 & 3t_2t_3 & \frac{9}{4}t_3^2-4t_2t_4\end{pmatrix}.$$

The determinant of $\mathcal{V}$ is equal to $\frac{1}{8}h$. In our case the $\mathcal{O}_S$-module $\Omega^1_S(\log D)$ of logarithmic differential forms is free and there is the system of dual basis with respect to the vector fields V^1,V^2,V^3 defined by the minors of the second order of this matrix:

$$\omega_1=\frac{1}{2}\frac{dh}{h},\quad \omega_2=\frac{\widetilde{\vartheta}_2}{h},\quad \omega_3=\frac{2}{3}\frac{\widetilde{\vartheta}_3}{h},$$

where

$$\widetilde{\vartheta}_2=(32t_2t_3t_4-9t_3^3)d\,t_2+(6t_2t_3^2+64t_4^2-16t_2^2t_4)d\,t_3-(4t_2^2t_3+48t_3t_4)d\,t_4,$$
$$\widetilde{\vartheta}_3=(8t_2^2t_4-3t_2t_3^2-32t_4^2)d\,t_2+(2t_2^2t_3+24t_3t_4)d\,t_3+(16t_2t_4-4t_2^3-18t_3^2)d\,t_4,$$

so that one has $h=8\Delta$, $dh=12\vartheta_1$, $\widetilde{\vartheta}_2=8\vartheta_2$, $\widetilde{\vartheta}_3=12\vartheta_3$ in notations of Section 1. Making use of Corollary 1.7 and property 3) of Section 1, one obtains the following identities:

$$d\,\omega_1=0,\ d\,\omega_2=-\tfrac{1}{6}\omega_1\wedge\omega_2-t_2\omega_2\wedge\omega_3,\ d\,\omega_3=-\tfrac{1}{3}\omega_1\wedge\omega_3,$$
$$\omega_1\wedge\omega_2\wedge\omega_3=\frac{8\cdot dt_2\wedge dt_3\wedge dt_4}{h}.$$

By definition, $\iota_{V^j}(\omega_i)=\delta_i^j$ for all $i,j=1,2,3$. Because the weights of the variables t_2, t_3, t_4 induce the natural grading on the modules of logarithmic differential forms and vector fields, it is easy to see that the connection form of the differential operator (2.2) can be described by the following matrix:

$$\Omega=\begin{pmatrix}g_1\omega_1+at_2\omega_3 & m\omega_2 & n\omega_3\\ dt_2\omega_2+et_3\omega_3 & g_2\omega_1+kt_2\omega_3 & f\omega_2\\ ht_3\omega_2+(bt_2^2+ct_4)\omega_3 & it_2\omega_2+jt_3\omega_3 & g_3\omega_1+lt_2\omega_3\end{pmatrix},\tag{4.2}$$

where a, b, c, d, e, f, g_1, g_2, g_3, h, i, j, k, l, m, n are sixteen $\mathbf{C}$-valued parameters which can be considered as a coordinate system in $\mathbf{C}^{16}$. The matrix Ω satisfies the integrability condition $d\,\Omega=\Omega\wedge\Omega$ which implies the relations on the parameters. In order to compute these relations we first note (see Section 1) that

$$df=\sum_{i=1}^{3}V^i(f)\omega_i=\sum_{i=2}^{4}\frac{\partial f}{\partial t_i}\,dt_i,\quad \text{whenever } f\in\mathcal{O}_{\mathbf{C}^3}.$$

Hence, the columns of the matrix $\mathcal{V}$ define the relations between the total differentials of variables t_i and the basic logarithmic forms ω_i:

$$\begin{aligned}dt_2&=\tfrac{1}{3}t_2\omega_1+3t_3\omega_2+(t_2^2-12t_4)\omega_3,\\ dt_3&=\tfrac{1}{2}t_3\omega_1+(4t_4-t_2^2)\omega_2+3t_2t_3\omega_3,\\ dt_4&=\tfrac{2}{3}t_4\omega_1-\tfrac{1}{2}t_2t_3\omega_2+(\tfrac{9}{4}t_3^2-4t_2t_4)\omega_3.\end{aligned}$$

Performing further calculations, we obtain two relations

$$g_2 = (g_1 + \frac{1}{6}), \quad g_3 = (g_1 + \frac{1}{3}), \tag{4.3}$$

and the following system of 10 equations on 13 remaining parameters:

$$\left\{ \begin{array}{lcl} 3a - em + hn & = & 0 \\ m + mk - in - am & = & 0 \\ 2d + e - kd + bf + ad & = & 0 \\ 12d + 4e - cf & = & 0 \\ 3k + em - fj & = & 0 \\ f - fk + dn + fl & = & 0 \\ 3l - hn + fj & = & 0 \\ 2i + j - bm + ki - il & = & 0 \\ 12i + 4j + cm & = & 0 \\ 4h + \frac{1}{2}c - 6b + ei - dj - hl + ah & = & 0. \end{array} \right. \tag{4.4}$$

The equations of the system do not depend on the parameters g_i, $i = 1, 2, 3$. Note that the equations $(4.4)_1$, (4.4, $(4.4)_7$ imply the trace relation

$$a + k + l = \operatorname{tr}(d\Omega) = 0.$$

The equations $(4.4)_2$, $(4.4)_8$, $(4.4)_9$ are homogeneous with respect to the variables i, j, m. Two equations $(4.4)_2$ and $(4.4)_8^{\sim} := (4.4)_8 - \frac{1}{4}(4.4)_9$ form the system

$$\left\{ \begin{array}{lcl} -ni + (1 + k - a)m & = & 0 \\ 4(k - l - 1)i - (4b + c)m & = & 0. \end{array} \right.$$

Similar computations with the triple $(4.4)_3$, $(4.4)_4$, $(4.4)_6$ give us the system

$$\left\{ \begin{array}{lcl} nd + (1 - k + l)f & = & 0 \\ 4(a - k - 1)d + (4b + c)f & = & 0. \end{array} \right.$$

It is not difficult to see that the Jacobian determinants of both systems with respect to the variables i, m and d, f, respectively, are equal. To be precise,

$$n(4b + c) - 4(1 + k - a)(k - l - 1) = n(4b + c) - 4(1 - k + l)(a - k - 1).$$

Let us denote the corresponding polynomial by Π and suppose that $\Pi \neq 0$. This implies $i = m = d = f = 0$. In addition, the equations $(4.4)_8$ and $(4.4)_3$ yield $j = 0$ and $e = 0$, respectively, while the equation $(4.4)_5$ implies $k = 0$, that is, $a = -l$. As a result, the initial system of ten equations reduces to the system:

$$\left\{ \begin{array}{lcl} d = e = f = i = j = k = m = & & 0 \\ a + l & = & 0 \\ 3a + hn & = & 0 \\ 6b - \frac{1}{2}c - 4h - 2ah & = & 0. \end{array} \right. \tag{4.5}$$

The dimension of the space of its solutions in $\mathbf{C}^{13}$ is equal to 3. Let W_1 be the submanifold in $\mathbf{C}^{16}$ defined by the two relations (4.3) and the system (4.5). The dimension of W_1 is equal to 4; this manifold parameterizes a set of integrable logarithmic connections of type (2.2).

Assume $\Pi = 0$. Then one can replace two equations $(4.4)_3$ and $(4.4)_8$ by the equation $\Pi = 0$ and afterwards eliminate successively 6 parameters a, e, l, j, k, b.

Making use of the system of algebraic computations *"Macaulay"* [**BS**] or *"Singular"* [**GP**] one can determine the *minimal standard* basis of the obtained ideal. As a result the system (4.4) of ten equations reduces to the following one:

$$\begin{cases} cfm^2+4fim-8dm^2-\frac{4}{3}hmn+4in-4m & = & 0 \\ cf^2m+8f^2i-4dfm+\frac{4}{3}fhn+4dn+4f & = & 0 \\ f^2i^2+2dfim+d^2m^2+\frac{1}{6}cfin+\frac{2}{3}fhin & & \\ +\frac{1}{6}cdmn+\frac{2}{3}dhmn+\frac{5}{2}fi+\frac{5}{2}dm+\frac{1}{3}cn+\frac{4}{3}hn+1 & = & 0. \end{cases}$$

These 3 equations in 7 variables define a complete intersection which is a 4-dimensional variety. Together with the trace equation $a+k+l=0$ this system describes the other set of parameters in $\mathbf{C}^{13}$ corresponding to integrable logarithmic connections of type (2.2). Its dimension is equal to 9, while the dimension of the corresponding subvariety W_2 in the total parameter space $\mathbf{C}^{16}$ is equal to 10. Thus, the union $W_1\cup W_2\subset \mathbf{C}^{16}$ describes the set of all integrable homogeneous logarithmic connections on $\Omega^1_S(\log D)$ with poles along the swallowtail.

4.2. Sato's hypersurface. It is possible to prove [**Se**] that there are only *two* polynomials of degree 12 depending on 3 variables of weights $2,3,4,$ that define Saito free divisors and that are not equivalent under weighted coordinate transformations. One of them (4.1) defines the swallowtail, while the second polynomial

$$P=t_4^3-18t_2t_3^2t_4+18t_2^3t_3^2+27t_3^4-3t_2^4t_4+2t_2^6$$

has been found by M.Sato in the framework of the theory of prehomogeneous spaces [**Se**]. Let D be the corresponding hypersurface in $\mathbf{C}^3$. A basis of the module $\mathrm{Der}_S(\log D)$ can be described as follows:

$$\begin{aligned} V^1 &= 2t_2\tfrac{\partial}{\partial t_2}+3t_3\tfrac{\partial}{\partial t_3}+4t_4\tfrac{\partial}{\partial t_4}, \\ V^2 &= -\tfrac{3}{2}t_3\tfrac{\partial}{\partial t_2}+\tfrac{1}{4}(t_2^2-t_4)\tfrac{\partial}{\partial t_3}-3t_2t_3\tfrac{\partial}{\partial t_4}, \\ V^3 &= (t_2^2+t_4)\tfrac{\partial}{\partial t_2}+t_2t_3\tfrac{\partial}{\partial t_3}+(4t_2^3+6t_3^2)\tfrac{\partial}{\partial t_4}, \end{aligned}$$

with the commutation relations $[V^1,V^2]=V^2,\ [V^1,V^3]=2V^3,\ [V^2,V^3]=3t_2V^2$. The matrix $\mathcal{V}$ of Proposition 1.3 has the form:

$$\mathcal{V}=\begin{pmatrix} 2t_2 & 3t_3 & 4t_4 \\ -\frac{3}{2}t_3 & \frac{1}{4}(t_2^2-t_4) & -3t_2t_3 \\ t_2^2+t_4 & t_2t_3 & 4t_2^3+6t_3^2 \end{pmatrix},$$

and $\det(\mathcal{V})=P$. The system of dual basis of $\mathcal{O}_S$-module $\Omega^1_S(\log D)$ with respect to V^1,V^2,V^3 is defined by the minors of the second order of $\mathcal{V}$:

$$\omega_1=\frac{1}{12}\frac{dP}{P},\quad \omega_2=\frac{\vartheta_2}{P},\quad \omega_3=\frac{\vartheta_3}{P};$$

where

$$\vartheta_2=(4t_2t_3t_4-12t_2^3t_3-18t_3^3)d\,t_2+(8t_2^4+12t_2t_3^2-4t_2^2t_4-4t_4^2)d\,t_3+(t_2^2t_3+3t_3t_4)d\,t_4;$$

$$\vartheta_3=(t_4^2-9t_2t_3^2-t_2^2t_4)d\,t_2+(6t_2^2t_3-6t_3t_4)d\,t_3+\frac{1}{2}(t_2^3+9t_3^2-t_2t_4)d\,t_4.$$

In view of Corollary 1.7 and property 3) of Section 1 one obtains the following identities:

$$d\,\omega_1 = 0,\ d\,\omega_2 = -\omega_1 \wedge \omega_2 - 3t_2\omega_2 \wedge \omega_3,\ d\,\omega_3 = -2\omega_1 \wedge \omega_3,$$
$$\omega_1 \wedge \omega_2 \wedge \omega_3 = \frac{dt_2 \wedge dt_3 \wedge dt_4}{P}.$$

The connection form will be given by the matrix Ω of type (4.2) again. The integrability condition $d\,\Omega = \Omega \wedge \Omega$ gives us two relations

$$g_2 = (g_1 + 1), \quad g_3 = (g_1 + 2), \tag{4.6}$$

and the following system of 10 equations on 13 parameters:

$$\left\{\begin{array}{lcl}
-\frac{3}{2}a - em + hn & = & 0\\
3m + mk - in - am & = & 0\\
4d - \frac{1}{4}e - kd + bf + ad & = & 0\\
d + \frac{1}{4}e + cf & = & 0\\
-\frac{3}{2}k + em - fj & = & 0\\
3f - fk + dn + fl & = & 0\\
-\frac{3}{2}l - hn + fj & = & 0\\
4i - \frac{1}{4}j - bm + ki - il & = & 0\\
-i - \frac{1}{4}j + cm & = & 0\\
4h + 3c + 3b + ei - dj - hl + ah & = & 0.
\end{array}\right. \tag{4.7}$$

The equations of the system do not depend on the parameters g_i, $i = 1, 2, 3$. Again three equations $(4.7)_1$, $(4.7)_5$, $(4.7)_7$ imply the relation $a + k + l = \operatorname{tr}(d\Omega) = 0$. As in the previous example consider two triples of equations $(4.7)_2$, $(4.7)_8$, $(4.7)_9$, and $(4.7)_3$, $(4.7)_4$, $(4.7)_6$ which are homogeneous with respect to variables i, j, m, and d, e, f, respectively. Two equations $(4.7)_2$ and $(4.7)_{\tilde{8}} := (4.7)_8 - (4.7)_9$ form the system

$$\left\{\begin{array}{lcl}
-ni + (3 + k - a)m & = & 0\\
(5 + k - l)i - (b + c)m & = & 0
\end{array}\right.$$

with the Jacobian determinant $n(b+c) - (3+k-a)(5+k-l)$, while two equations $(4.7)_6$ and $(4.7)_{\tilde{3}} := (4.7)_3 + (4.7)_4$ lead to the system

$$\left\{\begin{array}{lcl}
nd + (3 - k + l)f & = & 0\\
(5 - k + a)d + (b + c)f & = & 0
\end{array}\right.$$

whose determinant is equal to $n(b + c) - (3 - k + l)(5 - k + a)$. Suppose that the first determinant does not vanish. Then $i = m = 0$ and the equation $(4.7)_9$ implies $j = 0$, while the equation $(4.7)_5$ implies $k = 0$, that is, $a = -l$. As a result, one deduces that both Jacobian determinants are equal and do not vanish. Hence

$d = f = 0$ and the system (4.7) reduces to the following system

$$\left\{\begin{array}{lcl} d = e = f = i = j = k = m = & 0 \\ a + l & = & 0 \\ 3a - 2hn & = & 0 \\ 3b + 3c + 4h + 2ah & = & 0. \end{array}\right. \tag{4.8}$$

Suppose that the second determinant does not vanish. The same considerations as before give us the system (4.8) again.

Finally let us suppose that both determinants vanish. Since they contain the common term $n(b+c)$, one gets the identity

$$(3 + k - a)(5 + k - l) = (3 - k + l)(5 - k + a).$$

This yields $k = 0$, that is, $a = -l$. One can successively eliminate the 4 parameters a, e, j, b, and reduce the system (4.7) to the following form:

$$\begin{cases} cfm - \frac{1}{2}fi + \frac{1}{2}dm & = 0 \\ fim + dm^2 + \frac{1}{2}hmn + \frac{3}{4}in - \frac{9}{4}m & = 0 \\ f^2i + dfm + \frac{1}{2}fhn - \frac{3}{4}dn - \frac{9}{4}f & = 0 \\ cf^2hm - \frac{3}{4}cf^2i - \frac{9}{4}cdfm + dfhm + \frac{1}{4}fh^2n - \frac{3}{2}d^2m - \frac{3}{8}dhn + \frac{3}{4}fh - \frac{45}{16}d & = 0 \\ cfhm^2 + \frac{3}{4}cfim - \frac{3}{4}cdm^2 + dhm^2 + \frac{1}{4}h^2mn + \frac{3}{2}dim + \frac{3}{8}hin + \frac{3}{4}hm + \frac{45}{16}i & = 0. \end{cases}$$

These 5 equations in 7 variables define a *non-complete* intersection. It is possible to analyze it explicitly in the following way.

First suppose $fm = 0$. If $f = 0$ then the first and third equations imply $dm = 0$ and $dn = 0$, respectively. The fourth equation gives $d = 0$. In this case the above system of equations reduces to the following system of two equations (corresponding to the second and fifth equations, respectively):

$$\left\{\begin{array}{lcl} \frac{1}{2}hmn + \frac{3}{4}in - \frac{9}{4}m & = & 0 \\ \frac{1}{4}h^2mn + \frac{3}{8}hin + \frac{3}{4}hm + \frac{45}{16}i & = & 0. \end{array}\right.$$

If $h = 0$, then $i = m = 0$. Otherwise, one can multiply the first equation by $-h/2$ and add it with the second. As a result one obtains $hm + \frac{3}{2}i = 0$. Now one can eliminate i from the first equation. This gives $m = 0$ and, consequently, $i = 0$ again. Thus the system (4.7) reduces to the system (4.8).

Now suppose $f \neq 0$. Hence $m = 0$. The above system of 5 equations reduces to the following one:

$$\left\{\begin{array}{lcl} \frac{1}{2}fhn - \frac{3}{4}dn - \frac{9}{4}f & = & 0 \\ \frac{1}{4}fh^2n - \frac{3}{8}dhn + \frac{3}{4}fh - \frac{45}{16}d & = & 0. \end{array}\right.$$

If $h = 0$, then $d = f = 0$. This is a contradiction. Otherwise, multiplying the first equation by $-h/2$ and adding it with the second one, we get $fh - \frac{3}{2}d = 0$. Eliminating d from the first equation, one obtains $f = 0$. This is a contradiction again.

Now let us suppose $fm \neq 0$. Multiplying the second and third equations by f and by $-m$, respectively, and then adding them, one gets $\frac{3}{4}fin + \frac{3}{4}dnm = 0$. This implies $n(fi+dm)=0$. If $n=0$, then the second equation gives $fim+dm^2-\frac{9}{4}m=0$, that is, $fi+dm-\frac{9}{4}=0$. In particular, $fi+dm\neq 0$. Multiplying the fourth and fifth equations by m and $-f$, respectively, and then adding them, one gets

$$(fi+dm)(cfm+dm+\tfrac{15}{8}) \;=\; 0.$$

This implies $cfm+dm+\frac{15}{8}=0$. However, after substitution $fi=\frac{9}{4}-dm$ in the first equation one gets $cfm+dm-\frac{9}{8}=0$. This is a contradiction.

It remains to consider the case $n\neq 0$. Then $fi+dm=0$. Since $m\neq 0$ the first equation gives $cf+d=0$. So from the above system of five equations one can eliminate the variable d in the following manner:

$$\begin{cases} cfm-fi & = \; 0\\ fim-cfm^2+\frac{1}{2}hmn+\frac{3}{4}in-\frac{9}{4}m & = \; 0\\ f^2i-cf^2m+\frac{1}{2}fhn+\frac{3}{4}cfn-\frac{9}{4}f & = \; 0\\ -\frac{3}{4}cf^2i+\frac{3}{4}c^2f^2m+\frac{1}{4}fh^2n+\frac{3}{8}cfhn+\frac{3}{4}fh+\frac{45}{16}cf & = \; 0\\ -\frac{3}{4}cfim+\frac{3}{4}c^2fm^2+\frac{1}{4}h^2mn+\frac{3}{8}hin+\frac{3}{4}hm+\frac{45}{16}i & = \; 0. \end{cases}$$

The first equation implies $cm-i=0$. One can eliminate the variable i from the remaining four equations:

$$\begin{cases} \frac{1}{2}hmn+\frac{3}{4}cmn-\frac{9}{4}m & = \; 0\\ \frac{1}{2}fhn+\frac{3}{4}cfn-\frac{9}{4}f & = \; 0\\ \frac{1}{4}fh^2n+\frac{3}{8}cfhn+\frac{3}{4}fh+\frac{45}{16}cf & = \; 0\\ \frac{1}{4}h^2mn+\frac{3}{8}chmn+\frac{3}{4}hm+\frac{45}{16}cm & = \; 0. \end{cases}$$

If $h=0$, then the third equation yields $cf=0$, that is, $c=0$, and, consequently, $d=i=0$. In such a case the first equation gives $m=0$. This is a contradiction.

Assume $h\neq 0$. Note that the first (or second) equation gives

$$\tfrac{1}{2}hn+\tfrac{3}{4}cn-\tfrac{9}{4} \;=\; 0,$$

while the third (or fourth) equation implies

$$\tfrac{1}{4}h^2n+\tfrac{3}{8}chn+\tfrac{3}{4}h+\tfrac{45}{16}c \;=\; 0.$$

Multiplying the former equation by $-h/2$ and adding it with the latter, one gets $h+\frac{3}{2}c=0$. Then one can eliminate c from the former equation. As a result an absurd equality $-9/4=0$ occurs.

Thus, in general, the initial system (4.7) of ten equations reduces to the system (4.8). Similarly to the example (4.1), it turns out that the dimension of the space of solutions of the system (4.7) containing in $\mathbf{C}^{13}$ is equal to 3. The dimension of the submanifold W_0 in the total parameter space $\mathbf{C}^{16}$ defined by the two relations (4.6) on g_i, $i=1,\,2,\,3$, and by this system is equal to 4. This manifold describes the set of all integrable homogeneous logarithmic connections on $\Omega^1_S(\log D)$ with poles along Sato's hypersurface.

5. Residue matrix, monodromy and moduli

In the framework of the theory of singularities there is a clear geometric interpretation of the connection (2.1) related with the notion of the Gauss-Manin connexion associated to deformations of singularities.

5.1. Gauss-Manin connexion. Let $f = (f_1, \dots , f_m)\colon X \to S$ be the minimal versal deformation of an isolated complete intersection singularity X_0, $\dim X_0 = n \geq 1$, and let $\mu = \mu(X_0)$ be the Milnor number of X_0. Then the discriminant $D \subset S$ of the deformation is a hypersurface with no multiple irreducible components. Set $S' = S \setminus D$. Then all fibers X_t, $t \in S'$, are smooth germs and the $\mathcal{O}_S$-module $\mathcal{H}^n(f_*\Omega^{\bullet}_{X/S})$ is free of rank $\mu = \dim H^n(X_s, \mathbf{C})$, $s \in S'$, while $\mathcal{H}^p(f_*\Omega^{\bullet}_{X/S}) = 0$ for all $0 < p < n$. Furthermore there is defined [**Gr**] the Gauss-Manin connexion

$$\nabla\colon\ \mathcal{H}^n(f_*\Omega^{\bullet}_{X/S}) \ \longrightarrow\ \Omega^1_S(\star D) \otimes \mathcal{H}^n(f_*\Omega^{\bullet}_{X/S}), \tag{5.1}$$

where $\Omega^1_S(\star D)$ denotes the sheaf of meromorphic differential 1-forms with poles along D. The differential operator ∇ is given by the rule

$$\nabla_{X/S}[\omega] = \sum_{i=1}^{m} \frac{dt_i}{h} \otimes h[\alpha_i].$$

In this formula $[\alpha_i]$ stands for the relative de Rham cohomology class of α_i in $f_*\Omega^n_{X/S}/d(f_*\Omega^{n-1}_{X/S})$, where α_i is determined from the following decomposition

$$d\omega = \sum_{i=1}^{m} \alpha_i \wedge df_i, \quad \alpha_i \in f_*\Omega^n_X,\ i = 1, \dots , m.$$

It is evident that the Gauss-Manin connexion has poles of the first order when h has no multiple factors. In other words, when D is reduced.

Due to ideas of K.Saito [**S1**], the connection (5.1) can be represented in a very elegant form. In fact, Saito's considerations imply that in the case where S is the minimal versal deformation of an isolated *complete intersection* singularity and $D \subset S$ is the discriminant, the connection (5.1) may be rewritten as a logarithmic connection of type (2.1) :

$$\nabla_{X/S}\colon \mathcal{H}^n(f_*\Omega^{\bullet}_{X/S}) \to \Omega^1_S(\log D) \otimes_{\mathcal{O}_S} \mathcal{H}^n(f_*\Omega^{\bullet}_{X/S}). \tag{5.2}$$

Remark 5.1. In the case when X_0 is a quasi-homogeneous germ the problems of computation of the connection (5.2) and the logarithmic connection (2.2) are closely related since the ranks of free $\mathcal{O}_S$-modules $\mathcal{H}^n = \mathcal{H}^n(f_*\Omega^{\bullet}_{X/S})$ and $\Omega^1_S(\log D)$ are equal. Moreover in this case one gets a method to compute the Gauss-Manin connexion. To be precise, first one has to compute all connections $\nabla_{X/S}$ of type (2.2) on $\Omega^1_S(\log D)$, where $D \subset S$ is the discriminant of the minimal versal deformation. Then it remains to select those which correspond to the required connection on $\mathcal{H}^n$. With minor changes this procedure can be adapted for a certain type of non-complete intersection singularities although two modules above have different ranks (see [**AM**]).

5.2. Local Picard-Fuchs systems. In fact, the Gauss-Manin connexion can be also described by a system of differential equations in the following way [**Pa**]. Under notations of the previous subsection let ω be a holomorphic form representing a class of the relative de Rham cohomology in $f^{-1}(B)$ where B is a small ball in S', and let $\gamma(t)$ be any continuous family of cycles over B. Then the integral

$$\mathcal{I}(t) = \int_{\gamma(t)} \omega$$

is a holomorphic function on B. Making use of the Stokes' formula one obtains

$$d\mathcal{I} = \sum \sigma_i dt_i, \quad \sigma_i(t) = \int_{\gamma(t)} \omega_i$$

where $d\omega = \sum df_i \wedge \omega_i$, and ω_i are holomorphic in $f^{-1}(B)$. Hence, the function $\mathcal{I}(t)$ is constant if and only if $\sigma_i \equiv 0$, for all i. In particular, there is defined a *non-singular* connection

$$\nabla_{X'/S'} : \mathcal{H}^n(f_*\Omega^\bullet_{X'/S'}) \to \mathcal{H}^n(f_*\Omega^\bullet_{X'/S'}) \otimes_{\mathcal{O}_{S'}} \Omega^1_{S'} \tag{5.3}$$

given as follows:

$$\nabla_{X'/S'}([\omega]) = \sum_{i=1}^{m} [\omega] \otimes dt_i,$$

where the cohomology class of the differential form ω is denoted by $[\omega]$. This differential operator is called the *local* Gauss-Manin connexion. Consider holomorphic forms ω_i, $i = 1, \dots, \mu$, on X' whose restrictions on every fiber X'_t of the map $f' \colon X' \to S'$ generate $H^n(X'_t, \mathbf{C})$. The corresponding sections w_i, $i = 1, \dots, m$, of the bundle $\mathcal{H}^n = \mathcal{H}^n(f_*\Omega^\bullet_{X'/S'})$ induce a basis. By construction, $d\,\omega_i = \sum df_j \wedge \omega_i^j$, where ω_i^j are holomorphic on X', and the corresponding holomorphic sections of the cohomological bundle can be written as $\sum_k \vartheta_i^{jk} w_k$, where ϑ_i^{jk} are holomorphic on S'. One then obtains $d\,\underline{w} = \Theta \underline{w}$, where $\underline{w} = (w_1, \dots, w_m)^{\mathrm{tr}}$ is a column-vector and $\Theta = \|\,\vartheta_i^k\,\|$ is the matrix with entries $\vartheta_i^k = \sum \vartheta_i^{jk} dt_j$. The integration of the sections w_i along any continuous family of cycles $\gamma_i(t)$ gives us $\mathcal{I}_i(t)$. Then the following system of differential equations describes the connection (5.3) :

$$d\,\mathbf{I} = \Theta \cdot \mathbf{I}, \tag{5.4}$$

where $\mathbf{I} = (\mathcal{I}_1, \dots, \mathcal{I}_m)^{\mathrm{tr}}$ is a vector-column of unknown functions, Θ is the *connection form* of the differential operator $\nabla_{X'/S'}$, and all entries of Θ are holomorphic 1-forms on S' satisfying the *integrability* condition $d\,\Theta = \Theta \wedge \Theta$. Such a system is called the *local Picard-Fuchs* system.

The above mentioned results of K. Saito [**S1**] imply that the connection (5.3) can be extended to the whole S as an integrable logarithmic connection (2.1) when D is a discriminant of the minimal versal deformation. In particular, all entries of the connection form can be expressed in terms of meromorphic forms with logarithmic poles along the divisor D.

In the case of an A_k-singularity, $k \geq 1$, there is a highly general method for computing the matrix Θ (see [**Pa**], [**Ma**]). Making use of elementary but tedious calculations, it is possible to produce from Θ another matrix with logarithmic entries corresponding to a *logarithmic* connection of type (2.1). Similarly to notations of Section 2 it will be denoted by Ω. For example, let us describe this matrix in the case of the *versal integrals* associated with an A_3-singularity. Making use of the presentation [**Pa**] for Θ, it is possible to verify that Ω has the following form (see [**A4**]):

$$\begin{pmatrix} (\frac{2}{3}q-\frac{1}{2})\omega_1+(\frac{3}{2}-4q)t_2\omega_3 & (4q-2)\omega_2 & (3-12q)\omega_3 \\ \frac{1}{2}t_2\omega_2+3(q-\frac{3}{4})t_3\omega_3 & (\frac{2}{3}q-\frac{1}{3})\omega_1+(2q-1)t_2\omega_3 & (4q-1)\omega_2 \\ -qt_3\omega_2+3t_4\omega_3 & (1-2q)t_2\omega_2+3(q-\frac{1}{2})t_3\omega_3 & (\frac{2}{3}q-\frac{1}{6})\omega_1+(2q-\frac{1}{2})t_2\omega_3 \end{pmatrix}$$

where q is a $\mathbf{C}$-valued parameter, and $\omega_1,\ \omega_2,\ \omega_3$ are the logarithmic differential forms defined in subsection 4.1.

5.3. The indicial polynomial and monodromy. It is known that the theory of local *Picard-Fuchs* system or maximally overdetermined systems of linear partial differential equations and the theory of ordinary differential equations are closely related. In fact, it is not difficult to define and compute explicitly the *characteristic* polynomial and monodromy of such a system at least in the case when its singular set D is an *irreducible* hypersurface. One can show that such a system is completely determined by its restriction to the generic family of smooth curves which cut D transversally (cf. [**AB**]). Then the restriction of the system to such a curve reduces to the differential equation in one variable with *regular* singularities of the following type

$$d\,\mathbf{I} = \left(\overline{\Theta}_{-1}\frac{dt}{t} + \sum_{s\geq 0}\overline{\Theta}_s t^s dt\right)\cdot\mathbf{I}, \tag{5.5}$$

that is,

$$\frac{d\,\mathbf{I}}{dt} = \left(\frac{\overline{\Theta}_{-1}}{t} + \sum_{s\geq 0}\overline{\Theta}_s t^s\right)\cdot\mathbf{I} = \overline{\Theta}\cdot\mathbf{I}.$$

Up to non-degenerate meromorphic coordinate transformations such a system is completely determined by the elementary divisors of the characteristic matrix $\|\,\lambda E-\overline{\Theta}_{-1}\,\|$ associated with the *residue matrix* $\overline{\Theta}_{-1}$ (see [**Ga**], Ch.XI, §10, [**BV1**]). By definition, the *indicial* or *characteristic* polynomial of the system (5.5) coincides with the characteristic polynomial $Q(\lambda) = \det(\lambda E - \overline{\Theta}_{-1})$ of the residue matrix where E is the identity matrix. For example, let us compute the indicial polynomial of Picard-Fuchs system for the versal integrals of an A_3-singularity. When $\varepsilon > 0$ is small enough then the following t-parameterization

$$t_2 = 0, \quad t_3 = \varepsilon^3, \quad t_4 = \sqrt[3]{2^{-8}}t,$$

defines the family of curves $\mathcal{C}_\varepsilon$ transversal to the discriminant D given by the polynomial h from (4.1). It is an easy exercise to verify that $\mathcal{C}_\varepsilon$ cuts D when $t = 3\varepsilon^4$. The restrictions of the polynomial h and the basis logarithmic forms to $\mathcal{C}_\varepsilon$ in a neighborhood of this point are the following:

$$\widetilde{h} = 3^3\varepsilon^8(t-3\varepsilon^4), \quad \widetilde{\omega_i} = \pi_i\frac{dt}{(t-3\varepsilon^4)}, \quad i = 1,2,3,$$

where

$$\pi_1 = 2^{-1},\ \pi_2 = -\sqrt[3]{2^{-4}}3^{-1}\varepsilon^{-1},\ \pi_3 = -\sqrt[3]{2^{-2}}3^{-2}\varepsilon^{-2}.$$

The restriction of the matrix written down at the end of subsection 5.2 gives us the residue matrix:

$$\overline{\Theta}_{-1} = \begin{pmatrix} (\frac{2}{3}q - \frac{1}{2})\pi_1 & (4q-2)\pi_2 & (3-12q)\pi_3 \\ 3(q-\frac{3}{4})\pi_3 & (\frac{2}{3}q - \frac{1}{3})\pi_1 & (4q-1)\pi_2 \\ -q\varepsilon^3\pi_2 + \sqrt[3]{2^{-8}}3^2\varepsilon^4\pi_3 & 3(q-\frac{1}{2})\varepsilon^3\pi_3 & (\frac{2}{3}q - \frac{1}{6})\pi_1 \end{pmatrix}.$$

Hence, $Q(\lambda) = \det(\lambda E - \overline{\Theta}_{-1}) = \lambda^2(\lambda - q + \frac{1}{2})$. In general it is not difficult to see that the degree of $Q(\lambda)$ is equal to m, the dimension of the ambient space of the system in question.

Let $\mathcal{L}$ be the field of formal Laurent series with complex coefficients containing only finitely many negative powers of t. In terms of the classical gauge adjoint action the connection or system associated with $\overline{\Theta}$ is regular (at $t=0$) if there is a constant matrix C and a *gauge* transformation $\Gamma \in \mathrm{GL}(m, \mathcal{L})$ such that

$$\Gamma\overline{\Theta}\Gamma^{-1} + \frac{d\Gamma}{dt}\Gamma^{-1} = \frac{1}{t}C.$$

Then the $\mathrm{GL}(m,\mathcal{L})$-conjugacy class of $\exp(2\pi i C)$ is uniquely determined; it is the local *monodromy* of $\overline{\Theta}$ at $t=0$. The classical results say [**Ga**] that $\exp(2\pi i C) = \exp(2\pi i \overline{\Theta}_{-1})$ if no two eigenvalues of $\overline{\Theta}_{-1}$ differ by an integer. In all cases the characteristic polynomials of the matrices $\exp(2\pi i C)$ and $\exp(2\pi i \overline{\Theta}_{-1})$ are equal. If there are pairs of eigenvalues of $\overline{\Theta}_{-1}$ differing by an integer (in such a situation one often says on the *resonance*), and if $n \geq 1$ is the maximum integer difference, then $\exp(2\pi i C)$ is completely determined by a knowledge of the Taylor coefficients $\overline{\Theta}_{-1+s}$, $s < n$, the bound being sharp [**BV1**].

5.4. Moduli. The formulation of the Riemann-Hilbert problem in terms of vector bundles suggests the question of studying the moduli of regular singular connections on punctured Riemann surfaces. The number of parameters needed to parameterize the corresponding Picard-Fuchs systems of differential equations with specified monodromy data was already of concern to Riemann. In the case of several variables one can pose the problem of classification of *local* Picard-Fuchs systems or logarithmic connections in a more rough sense. Thus, instead of conditions on monodromy data one can investigate the number of parameters or the parameter space which describe connections with specified type of *residue matrices* or indicial polynomials. In view of considerations in the previous subsection this question is very close to the classical setting. One can illustrate this idea by further analysis of the examples described in Section 4. Further calculations show that in the case of swallowtail for general values of parameters there are solutions of the following two types only:

Type I: $a=0,\ b=(2k+\frac{1}{2}),\ c=6,\ d=-\frac{4}{3}k,\ e=h=3k,\ f=-\frac{2}{3}k,$
$$g=\tfrac{1}{3}k-\tfrac{1}{6},\ i=(k+1),\ j=-\tfrac{9}{4}(2k+1),\ l=-k,\ m=n=(k-\tfrac{1}{2});$$

Type II: $a=0,\ b=-\frac{1}{4}(k+4)(k-2),\ c=3(3k^2+2k-4),$
$$d=-\tfrac{1}{3}k^{-1}(4k-2), e=h=-3(\tfrac{1}{2}k-1),\ f=-\tfrac{2}{3}k^{-1},\ g=\tfrac{1}{3}k-\tfrac{1}{6},$$
$$i=(k+1),\ j=-\tfrac{9}{4}k(k+2),\ l=-k,\ m=n=1;$$

where $g=g_1=g_2-\frac{1}{6}=g_3-\frac{1}{3}$ for the both types of solutions.

Similarly to the proof of Corollary 2.4, it is possible to verify that both types of solutions correspond to holonomic systems of differential equations of Picard-Fuchs type (cf. [**Pa**]). It is not difficult to compute that the systems of the first type have the indicial polynomial $Q(\lambda)=\lambda^2(\lambda-\frac{1}{2}k)$, while for the systems of the second type $Q(\lambda)=\lambda(\lambda-\frac{1}{4}k+\frac{1}{4})(\lambda-\frac{1}{4}k-\frac{1}{4})$. Here $k=3g+\frac{1}{2}$ is a $\mathbf{C}$-valued parameter, and the above polynomials are equal when $k=\pm1$. Hence the first and second types correspond to holonomic systems of *different* kinds and one obtains two families of integrable logarithmic connections (2.2). Both families are parameterized by k and they have two common members given by the values $k=\pm1$ (cf. [**S1**]).

Remark 5.2. It is possible to verify that the Picard-Fuchs system associated with the system of versal integrals of an A_3-singularity from subsection 5.2 can be considered as a system of Type I. In fact, under the substitution $q=\frac{1}{2}(k+1)$ the characteristic polynomial for this system reduces to $Q(\lambda)=\lambda^2(\lambda-\frac{1}{2}k)$.

At last, in notations of subsection 4.1 let us consider a hypersurface $N\subset W_1$ given by the equation $cn+4hn+3=0$. In fact, $N=W_1\cap W_2$, $\dim W_1=4$, $\dim W_2=10$, and the codimension of N in W_2 is equal to 7. The points of N correspond to systems whose restriction (5.5) to the generic curve has indicial polynomials with a root of multiplicity 3. To be precise, $Q(\lambda)=(\lambda-\frac{1}{2}(g+\frac{1}{6}))^3$. The points of the complement $W_1\setminus N$ correspond to the systems whose indicial polynomials have *different* roots.

Thus, up to gauge coordinate transformations the parameter space of integrable homogeneous logarithmic connections has the following decomposition. Two curves parameterize two families of integrable logarithmic connections (2.2) corresponding to the systems of Type I and Type II; these curves are met in two points given by the values $k=\pm1$. The third component occurs from the subset W_1; it contains a curve corresponding to the hypersurface N which intersects the first curve when $g=-\frac{1}{6}$, or $k=0$, etc.

Analogously to the above it is possible to verify that in the case of Sato's hypersurface the 4-dimensional submanifold $W_0\subset\mathbf{C}^{16}$ parameterizing all integrable homogeneous connections on $\Omega^1_S(\log D)$ contains a 3-dimensional hypersurface. Its points correspond to systems whose restriction (5.5) on the generic curve has indicial polynomials with a root of multiplicity 3. To be precise, this hypersurface is defined by the equation $36(g+1)^2-\frac{1}{4}g(g+2)-3cn-2hn=0$; the corresponding characteristic polynomial is equal to $Q(\lambda)=(\lambda-\frac{1}{12}(g+1))^3$, where $g=g_1$. The points of the complement of this hypersurface correspond to systems whose indicial polynomials have *different* roots.

6. Remarks and comments

The following arguments show that the systems of Type I correspond to the Gauss-Manin connexion associated with an A_3-singularity. In fact, the generic fiber X_t of the minimal versal deformation of an A_3-singularity

$$w^2 = z^4 + t_2 z^2 + t_3 z + t_4 \tag{6.1}$$

is isomorphic to a hyperelliptic curve of genus 1. Hence its first homology and cohomology groups with coefficients in $\mathbf{C}$ have the rank 2, and the following three versal (period) integrals

$$I_\ell(t) = \int_{\gamma(t)} \frac{z^\ell dz}{\sqrt{z^4 + t_2 z^2 + t_3 z + t_4}}, \qquad \ell = 0, 1, 2, \tag{6.2}$$

satisfy the differential equation of the *second* order, although the Milnor number of the special fiber X_0 is equal to 3. Here the contour of the integration $\gamma(t)$ is a homology cycle of the hyperelliptic curve. That is why the corresponding system of three differential equations has the characteristic polynomial with a multiple root (cf. [**S1**]).

In a more general context let us consider an arbitrary reduced divisor $D \subset S$, and set $S' = S \setminus D$. Let $j:\ S' \to S$ be the inclusion, and let $\mathcal{H} \to S'$ be a flat vector bundle with the sheaf of meromorphic sections $\mathcal{H}$. Then there is a unique extension of $\mathcal{H}$ to a coherent $\mathcal{O}_S(\star D)$-module with sections of moderate growth along D with respect to the multivalued flat sections [**De**]. In fact, two systems of Types I and II correspond two flat vector bundles on S' with different monodromies. Each has a sheaf $\mathcal{H}$ of holomorphic sections over S' and unique extension of $\mathcal{H}$ by the theory of Deligne. Moreover, in both cases there are extensions to *free* $\mathcal{O}_S$-modules with logarithmic connections. Such extensions are not at all unique (cf. [**He**], Ch. 3).

In fact, Deligne has defined the notion *regular singular* not for a single $\mathcal{O}_S$-coherent extension of $\mathcal{H}$, but for equivalence classes of such extensions with respect to a *meromorphic* equivalence relations, that is, for meromorphically equivalent class of coherent sheaves on S (cf. [**De**], [**S3**]). Moreover, he studied such extensions with logarithmic poles along D only in the normal crossings case. The corresponding systems of differential equations are considered up to meromorphic changes of independent variables, and from his point of view the notions of systems with regular singularities and of systems with logarithmic poles are equivalent. It was proved by reduction of the general case to the case of a divisor with normal crossings (see [**AB**]). However, it should be noted that K.Saito [**S3**] introduced a different concept of regular singularity. In his terminology a system with regular singularity along a divisor D has *logarithmic* poles along D. However, the converse is not true at least in the case of Picard-Fuchs systems which correspond to the Gauss-Manin connexion associated with a flat family of isolated singularities.

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