An index for gauge-invariant operators and the Dixmier-Douady invariant

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Abstract

Let $\mathcal{G} \to B$ be a bundle of compact Lie groups acting on a fiber bundle $Y \to B$. In this paper we introduce and study gauge-equivariant K-theory groups $K^i_{\mathcal{G}}(Y)$. These groups satisfy the usual properties of the equivariant K-theory groups, but also some new phenomena arise due to the topological non-triviality of the bundle $\mathcal{G} \to B$. As an application, we define a gauge-equivariant index for a family of elliptic operators $(P_b)_{b \in B}$ invariant with respect to the action of $\mathcal{G} \to B$, which, in this approach, is an element of $K^0_{\mathcal{G}}(B)$. We then give another definition of the gauge-equivariant index as an element of $K_0(C^*(\mathcal{G}))$, the K-theory group of the Banach algebra $C^*(\mathcal{G})$. We prove that $K_0(C^*(\mathcal{G})) \simeq K^0_{\mathcal{G}}(\mathcal{G})$ and that the two definitions of the gauge-equivariant index are equivalent. The algebra $C^*(\mathcal{G})$ is the algebra of continuous sections of a certain field of C^*-algebras with non-trivial Dixmier-Douady invariant. The gauge-equivariant K-theory groups are thus examples of twisted K-theory groups, which have recently proved themselves useful in the study of Ramond-Ramond fields.

Contents

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Introduction

Families of elliptic operators invariant with respect to a family of Lie groups appear in gauge theory and in the analysis of geometric operators on certain non-compact manifolds. They arise, for example, in the analysis of the Dirac operator on a compact S^1-manifold M, provided that we desingularize the action of S^1 by replacing the original metric g with $\phi^{-2}g$, where ϕ is the length of the infinitesimal generator of the S^1-action. The main result of [30] states that the kernel of the new Dirac operator on the open manifold $M \smallsetminus M^{S^1}$ is naturally isomorphic to the kernel of the original Dirac operator. A natural question to ask then is when is this new Dirac operator (on $M \smallsetminus M^{S^1}$) Fredholm. The answer [25, 26] is that, in general, the Fredholm property of elliptic geometric operators on $M \smallsetminus M^{S^1}$ is controlled by the invertibility of a certain family of operators invariant with respect to a family of solvable Lie groups. Operators invariant with respect to a family of Lie groups $\mathcal{G} \to B$ will be called *gauge-invariant operators* or *$\mathcal{G}$-invariant operators* in what follows.

Gauge-invariant operators were studied in [29], where an index theorem for these operators was also obtained in the case when $\mathcal{G} \to B$ is a family of solvable Lie groups. In this paper, we study the case of $\mathcal{G}$-invariant operators when $\mathcal{G}$ is a bundle of compact Lie groups. In a certain way, the general case of operators invariant with respect to a family of connected Lie groups can be reduced to the cases when the fibers are solvable or compact groups, by using the structure of connected Lie groups. There exist, however, quite significant differences between families of solvable Lie groups and families of compact Lie groups.

Here are two of the most important differences between families of compact Lie groups and families of solvable Lie groups. First, a family of compact Lie groups is always locally trivial. In particular, all fibers of a family of compact Lie groups are isomorphic over each connected component of the base, see Section 1. This need not be the case for families of nilpotent groups, for example. Second, the theory of families of compact Lie groups is more closely related to gauge theory and Physics than to Analysis on non-compact manifolds. This is mainly because bundles of compact Lie groups are one of the main objects of study in gauge theory (see, for example, [4, 21, 40, 14]), but there are also other reasons. For example, gauge-invariant operators are related to anomalies in physics and to the Aronov-Bohm effect [2, 1, 7, 8, 11, 27, 32]. Also, it is also possible that gauge-invariant operators are related to the second quantization in field theory [14, 19]. An interesting possibility is to study whether Ramond-Ramond charges [17, 18, 28, 41, 42] can be realized as gauge-equivariant indices.

From now on and throughout this paper, we shall assume that $\mathcal{G} \to B$ is a bundle of <u>*compact*</u> *Lie groups (except in Section 1 or when explicitly mentioned otherwise).* The relation between gauge-invariant operators and Physics provides further motivation for the study of gauge-invariant operators.

A first problem that we solve for gauge-invariant operators is to define their index. We actually give two definitions of this index. The first definition is geometric and the second one is algebraic. The advantage of the geometric definition is that it is closer to the classical definition of the index of a family of operators in the Atiyah and Singer paper on families [5]. The advantage of the algebraic definition of the index, however, is that it works in general, whereas the geometric definition requires a certain finite holonomy assumption on our bundle of Lie groups. This finite holonomy assumption is automatically satisfied if the typical fiber of our bundle of Lie groups has a center of dimension ≤ 1, but not in general.

For the geometric definition of the gauge-equivariant index, we first define groups $K^j_{\mathcal{G}}(Y)$ for any fiber bundle $\pi : Y \to B$ on which $\mathcal{G}$ acts smoothly. (We assume that the action of $\mathcal{G}$ preserves the fibers of $Y \to B$.) These groups are defined geometrically in terms of $\mathcal{G}$-equivariant vector bundles on Y and give rise to a contravariant functor in Y that has all the usual properties of equivariant K-theory: homotopy invariance, continuity, and Bott periodicity. These groups behave well when $\mathcal{G}$ has finite holonomy (or, more generally, representation theoretic finite holonomy, see Section 3 where these conditions were introduced). The geometric definition of the index associates an element of the group $K^0_{\mathcal{G}}(B)$ to any $\mathcal{G}$-invariant continuous family of pseudodifferential operators on Y. This is done as in the classical case by perturbing our family to a family consisting of operators whose range is closed (or, equivalently, such that the family of their kernels defines a vector bundle over the base). The difficulty is that we have to perform all these constructions equivariantly.

For the algebraic construction of the analytic index of a $\mathcal{G}$-equivariant family of elliptic operators, we consider $C^*(\mathcal{G})$, the C^*-algebra of $\mathcal{G}$, which (in our case) can be defined as the completion of the convolution algebra of $\mathcal{G}$ with respect to the action on each $L^2(\mathcal{G}_b)$. Then

$$K^j_{\mathcal{G}}(B) \cong K_j(C^*(\mathcal{G})) \qquad (1)$$

whenever $\mathcal{G}$ satisfies the finite holonomy assumption mentioned above (Theorem 5.2). Using some basic constructions in K-theory, we can then give a direct definition of the gauge-invariant index with values in $K_0(C^*(\mathcal{G}))$, which turns out to be equivalent to the geometric definition of the gauge-equivariant index, if we take into account the isomorphism of Equation (1) above. For this construction, we need no assumption on $\mathcal{G}$ (except that the fibers of $\mathcal{G} \to B$ are compact Lie groups), but it has the disadvantage that it is less elementary and certainly less explicit. We leave the problem of determining the gauge-equivariant index of a $\mathcal{G}$-invariant family of elliptic operators for another paper [31], since it requires several new techniques.

The algebra $C^*(\mathcal{G})$ turns out to decompose naturally as a direct sum of (algebras of continuous sections of) fields of finite dimensional algebras on finite coverings of B. Assume that the typical fiber G of $\mathcal{G} \to B$ is connected and denote by G' the derived group of G. The Dixmier-Douady invariants of these fields can be recovered from a unique class in $H^2(B, Z(G) \cap G')$ (see also [13, 15, 16, 36] for more on these invariants). The K-theory groups of these algebras are twisted K-theory groups, as the ones appearing in the study of Ramond-Ramond fields [10, 17, 18, 41, 42]. This suggests some possible connections between the structure of Ramond-Ramond fields and gauge-equivariant index theory. An interesting possibility would be that the Ramond-Ramond charges can be realized as gauge-equivariant indices.

Let us now briefly describe the contents of each section. In Section 1, we introduce bundles of Lie groups and define their actions on spaces. We also introduce the algebras of (families of) gauge-invariant pseudo-differential operators. This is the only section were we do not assume that the fibers of $\mathcal{G} \to B$ are compact. In Section 2, we discuss the holonomy of the representation spaces associated to bundles of Lie groups. For instance, we introduce the condition that a bundle of Lie groups have finite holonomy, which will play an important role in the study of gauge-equivariant K-theory. The gauge-equivariant K-theory groups $K^0_{\mathcal{G}}(Y)$ of a bundle $Y \to B$ on which $\mathcal{G}$ acts are introduced in Section 3. In that section, we establish several properties of these groups. If the bundle $\mathcal{G} \to B$ has finite holonomy, then the groups $K^0_{\mathcal{G}}$ behave like the usual equivariant K-theory groups, but they can behave very differently if this condition is not satisfied. For example, not every $\mathcal{G}$-equivariant vector bundle can be realized as a sub-bundle of a trivial bundle in general; this is possible, however, if $\mathcal{G} \to B$ has representation theoretic finite holonomy, which is actually the reason why we introduced this condition, in the first place. The gauge-equivariant K-theory groups appear naturally as the receptacles of the indices of gauge-equivariant families of elliptic operators. These indices are defined geometrically in Section 4 and analytically in Section 6. These two definition are shown to coincide using the structure of the K-theory of $C^*(G)$ and, especially, the isomorphism $K^0_{\mathcal{G}}(B) = K^0(C^*(\mathcal{G}))$. In the Appendix we recall several basic constructions in K-theory.

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1 Gauge-invariant pseudodifferential operators

We now describe the settings in which we shall work. If $f_i : Y_i \to B$, $i = 1, 2$, are two maps, we shall denote by

$$Y_1 \times_B Y_2 := \{(y_1, y_2) \in Y_1 \times Y_2,\ f_1(y_1) = f_2(y_2)\,\} \tag{2}$$

their fibered product.

Fiber bundles will figure prominently in this paper. Even if we are often interested primarily in smooth fiber bundles, we have found it more convenient to work in the framework of "longitudinally smooth" fiber bundles, whose definition is recalled below. This simplifies and makes more natural certain constructions in this paper, The reader can safely assume for most results that the bundles are smooth. In fact, we have included in "[ ]" the corresponding statements for smooth fiber bundles.

Definition 1.1. Let B be a locally compact topological space. A locally trivial fiber bundle $\pi : Y \to B$ with typical fiber F is called *longitudinally smooth* if, by definition, F is a smooth manifold and the structure group of this bundle reduces to $\mathrm{Diff}(F)$, the group of $\mathcal{C}^\infty$-diffeomorphisms of the fiber F.

In particular, every smooth bundle is longitudinally smooth. We now introduce bundles of Lie groups.

Definition 1.2. Let B be a compact Hausdorff [respectively, smooth manifold], and let G be a Lie group. We shall denote by $\mathrm{Aut}(G)$ the group of automorphisms of G. A *[smooth] bundle Lie groups $\mathcal{G}$ with typical fiber G over B* is, by definition, a [smooth] fiber bundle $\mathcal{G} \to B$ with typical fiber G and structure group $\mathrm{Aut}(G)$.

We do not assume in this section that $\mathcal{G} \to B$ has compact fibers.

We see thus that a [smooth] bundle of Lie groups with typical fiber G over B is a longitudinally smooth [respectively, smooth] fiber bundle over B with typical fiber G whose structure group reduces from $\mathrm{Diff}(G)$ to $\mathrm{Aut}(G)$.

Let $\mathcal{G} \to B$ and $\mathcal{G}' \to B'$ be bundles of Lie groups. A *morphism of bundles of Lie groups* (or simply, a *morphism*) $\gamma : \mathcal{G} \to \mathcal{G}'$ is a continuous map covering a continuous map $\gamma_1 : B \to B'$ such that the induced map $\mathcal{G}_b \to \mathcal{G}'_{\gamma_1(b)}$ is a group morphism for any $b \in B$.

Let $d : \mathcal{G} \to B$ be a bundle of Lie groups. We shall often let $\mathcal{G}$ act on spaces Y. Let $\pi : Y \to B$ be a fiber bundle. We say that $\mathcal{G}$ *acts continuously on* Y if each group $\mathcal{G}_b$ acts continuously on $Y_b := \pi^{-1}(b)$ and the induced map μ

$$\mathcal{G} \times_B Y := \{(g, y) \in \mathcal{G} \times Y, d(g) = \pi(y)\} \ni (g, y) \longrightarrow \mu(g, y) := gy \in Y,$$

continuous. We shall also say that Y is a *$\mathcal{G}$-fiber bundle.* If $\pi : Y \to B$ is longitudinally smooth bundle and $\mathcal{G}_b \times Y_b \to Y_b$ are smooth, then we shall say that $Y \to B$ is a *longitudinally smooth $\mathcal{G}$-fiber bundle.* If $\mathcal{G}$ is a smooth bundle of Lie groups, $Y \to B$ is a smooth fiber bundle and the map μ is smooth, we shall say that $Y \to B$ is a *smooth $\mathcal{G}$-fiber bundle.*

Assume for the rest of this section that the quotient $Y/\mathcal{G} := \cup Y_b/\mathcal{G}_b$ is compact. Then we shall denote by $\psi^m_{\mathrm{inv}}(Y)$ the space of continuous [respectively, smooth] families $D = (D_b)$, $b \in B$, of order m, classical pseudodifferential operators acting on the fibers of $Y \to B$ such that each D_b is invariant with respect to the action of the group $\mathcal{G}_b$. Unless mentioned otherwise, we assume that these operators act on half densities along each fiber. Then $\psi^\infty_{\mathrm{inv}}(Y) := \cup_{m \in \mathbb{Z}} \psi^m_{\mathrm{inv}}(Y)$ is an algebra, by classical results [5, 20]. Note also that $\psi^m_{\mathrm{inv}}(Y)$ also makes sense for m not an integer.

We now discuss the principal symbols of operators in $\psi^\infty_{\mathrm{inv}}(Y)$. Let

$$T_{vt}Y := \ker(\pi_* : TY \to TB)$$

be the bundle of *vertical* tangent vectors to Y, and let $T^*_{vt}Y$ be its dual. We fix compatible metrics on $T_{vt}Y$ and $T^*_{vt}Y$, and we define $S^*_{vt}Y$, the *cosphere bundle of the vertical tangent bundle* to Y, to be the set of vectors of length one of $T^*_{vt}Y$. Also, let

$$\sigma_m : \Psi^m(Y_b) \to \mathcal{C}^\infty(S^*Y_b)$$

be the usual principal symbol map, defined on the space of pseudodifferential operators of order m on Y_b and with values smooth functions on the unit cosphere bundle of Y_b. The definition of σ_m depends on the choice of a trivialization of the bundle of homogeneous functions of order m on $T^*_{vt}Y$, regarded as a bundle over $S^*_{vt}Y$. The principal symbols $\sigma_m(D_b)$ of an element (or family) $D = (D_b) \in \psi^m_{\mathrm{inv}}(Y)$ then gives rise to a smooth function on $\mathcal{C}^\infty(S^*_{vt}Y)$, which is invariant with respect to $\mathcal{G}$, and hence descends to a smooth function on $S^*_{vt}Y$. The resulting function,

$$\sigma_m(D) \in \mathcal{C}(S^*_{vt}Y))^{\mathcal{G}}, \tag{3}$$

will be referred to as the *principal symbol* of an element (or operator) in $\psi^m_{\mathrm{inv}}(Y)$. Note that, $\sigma_m(D) \in \mathcal{C}^\infty(S^*_{vt}Y)^{\mathcal{G}}$ when $\mathcal{G}$ and Y are smooth fiber bundles. As usual, an operator $D \in \psi^m_{\mathrm{inv}}(Y)$ is called *elliptic* if, and only if, its principal symbol is everywhere invertible.

In the particular case when $Y = \mathcal{G}$ and $\mathcal{G}$ is a smooth bundle, $\psi^{\infty}_{\mathrm{inv}}(\mathcal{G})$ identifies with convolution operators on each fiber $\mathcal{G}_b$ that have compactly supported kernels, are smooth outside the identity, and have only conormal singularities at the identity. In particular, $\psi^{-\infty}_{\mathrm{inv}}(\mathcal{G}) = \mathcal{C}^{\infty}_{\mathrm{c}}(\mathcal{G})$, with the fiberwise convolution product.

All constructions and definitions above extend to operators acting between sections of a $\mathcal{G}$-equivariant vector bundle $E \to Y$. Recall that a bundle $E \to Y$ is $\mathcal{G}$-equivariant if $\mathcal{G}$ acts on the total space of E, the projection $E \to Y$ is $\mathcal{G}$ equivariant, and the induced action $E_y \to E_{gy}$ between the fibers of $E \to Y$ is linear. The space of order m, classical, $\mathcal{G}$-invariant pseudodifferential operators acting on sections of E will be denoted by $\psi^m_{\mathrm{inv}}(Y; E)$. The same construction then generalizes also to $\mathcal{G}$-invariant operators acting between sections of two $\mathcal{G}$-equivariant vector bundles $E_i \to Y$, $i = 0, 1$. The resulting space of order m, $\mathcal{G}$-invariant pseudodifferential operators acting between sections of E_0 and E_1 will be denoted by $\psi^m_{\mathrm{inv}}(Y; E_0, E_1)$.

Let $T_{vt}Y$ denote the bundle of vertical tangent vectors to the fibration $\pi : Y \to B$. Assume that there are given a $\mathcal{G}$-invariant metric on $T_{vt}Y$ and a $\mathcal{G}$-equivariant bundle W of modules over the Clifford algebras of $T_{vt}Y$. Then a typical example of a family $D = (D_b) \in \psi^{\infty}_{\mathrm{inv}}(Y)$ is that of the family of Dirac operators D_b acting on the fibers Y_b of $Y \to B$. (Each D_b acts on sections of $W|_{Y_b}$, the restriction of the given Clifford module W to that fiber.)

Later on, we shall need the following structure theorem for smooth $\mathcal{G}$-fiber bundles Y (that is, when $\mathcal{G}$, Y, and the action of $\mathcal{G}$ on Y are smooth). Recall that $\pi : Y \to B$ is the structural projection map. Fix $b_0 \in B$ and let $G = \mathcal{G}_{b_0} := d^{-1}(b_0)$ and $F = Y_{b_0} := \pi^{-1}(b_0)$. Let $\mathrm{Aut}(G, F)$ be the group of automorphisms of the pair (G, F), that is, $\mathrm{Aut}(G, F)$ consists of pairs (α, β), where α is an automorphism of G and $\beta : F \to F$ is a diffeomorphism satisfying $\beta(gf) = \alpha(g)\beta(f)$. Note that $\mathrm{Aut}(G, F)$ acts on both G and F. Moreover, the map (action) $G \times F \to F$ is $\mathrm{Aut}(G, F)$-equivariant.

Theorem 1.3. *Let $\mathcal{G} \to B$ be a smooth bundle of Lie groups acting smoothly on the smooth fiber bundle $Y \to B$, whose typical fiber is denoted by F. Then there exists a principal* $\mathrm{Aut}(G, F)$*-bundle $\mathcal{Q} \to B$ such that*

$$\mathcal{G} \cong \mathcal{Q} \times_B G\,, \quad Y \cong \mathcal{Q} \times_B Y,$$

and the induced map $\mathcal{G} \times_B Y \to Y$ is obtained from the $\mathrm{Aut}(G, F)$*-equivariant map $G \times F \to F$.*

Proof. Let $\mathcal{Q}$ be the set of triples (b, α, β) where $b \in B$, $\alpha : G \to \mathcal{G}_b$ is a group isomorphism, and $\beta : F \to Y_b$ is an equivariant map, in the sense that $\beta(gf) = \alpha(g)\beta(f)$ (as in the definition of the group $\mathrm{Aut}(G, F)$). The projection map $p : \mathcal{Q} \to B$ is given by the projection onto the first coordinate. By definition, $p^{-1}(b_0) = \mathrm{Aut}(G, F)$, so in particular, this fiber is not empty.

It is clear that $\mathrm{Aut}(G, F)$ acts simply, transitively on each non-empty fiber $p^{-1}(b)$. To complete the proof, all we need then is to check that $\mathcal{Q}$ is locally trivial. To this end, we may assume that $\mathcal{G} = B \times G$. The action of $\mathcal{G}$ on Y reduces then to an action of G. Since G is compact, we can choose a G-invariant metric on Y. The Levi-Civita connection will give then rise to a G-equivariant diffeomorphism $Y_b \simeq F$ of the fibers Y_b, for b close to b_0. This proves the local triviality. $\square$

Similar ideas can be used to prove that a family of Lie groups with connected compact fibers is locally trivial, and hence that it defines a bundle of Lie groups. Let us first define the concept of a family of Lie groups, a concept that we consider only in the smooth category.

Definition 1.4. Let B be a smooth manifold. A *family of Lie groups* is a submersion$d : \mathcal{G} \to B$ such that each $\mathcal{G}_b := \pi^{-1}(b)$ is a Lie group and the induced map

$$\mathcal{G} \times_B \mathcal{G} := \{(g', g) \in \mathcal{G} \times \mathcal{G}, d(g') = d(g)\} \ni (g', g) \longrightarrow g'g^{-1} \in \mathcal{G} \tag{4}$$

is differentiable.

We remark that B embeds naturally in $\mathcal{G}$ as the space of units of the groups $\mathcal{G}_b$. If $\mathcal{G}$ is a smooth family of Lie groups, then B is a smooth submanifold of $\mathcal{G}$. We also remark that a bundle of Lie groups is a particular case of a continuous family groupoid (see [25, 26] for definitions) whose space of units identifies with B. Let us denote by $d, r : \mathcal{G} \to B$ the domain and, respectively, the range maps of this groupoid. Then $d = r$, for our groupoid. If we work in the differentiable category, this groupoid is a differentiable groupoid.

For the rest of this section and then throughout the paper, we shall assume that the fibers of the bundles (or families) of Lie groups that we consider are compact. The following result is probably already known.

Theorem 1.5. *Suppose that $\mathcal{G} \to B$ is a family of Lie groups, that B is connected, and that all groups $\mathcal{G}_b := d^{-1}(b)$ are compact and connected. Then $\mathcal{G} \to B$ is a locally trivial fiber bundle with structure group* $\mathrm{Aut}(G)$.

Proof. It is enough to prove that $\mathcal{G} \to B$ is locally trivial. Fix $b_0 \in B$, and let $t_0 \in \mathcal{G}_{b_0}$ be a topological generator of a maximal torus of $\mathcal{G}_{b_0}$ (recall that this means that the least closed subgroup of $\mathcal{G}_{b_0}$ containing t_0 is a maximal torus). Choose a smooth local section t of $\mathcal{G}$, such that $t(b_0) = t_0$. Let H_b be the set of elements of $\mathcal{G}_b$ commuting with $t(b)$.

Let $\mathfrak{g} = \cup \operatorname{Lie} \mathcal{G}_b$ be the union of the Lie algebras of the fibers of $\mathcal{G} \to B$. Then $\mathfrak{g}$ maps naturally to B and can be identified with the restriction to B of the vertical tangent bundle to $\mathcal{G} \to B$.

Assume that in a small neighborhood of b_0 all the groups $\mathcal{G}_b$ have maximal tori of the same dimension. Let H_b^0 be the connected component of H_b. Then H_b^0 contains a maximal torus of $\mathcal{G}_b$, has the same dimension as a maximal torus of $\mathcal{G}_{b_0}$, and hence H_b^0 is a maximal torus in $\mathcal{G}_b$, for any b in a small neighborhood of b_0. Then the root spaces associated to these maximal tori vary continuously with b and hence the root systems of all the groups $\mathcal{G}_b$ are isomorphic. This proves that the bundle $\mathfrak{g}$ of Lie algebras is locally constant, by Serre's theorem. Since $\pi_1(\mathcal{G}_b)$ is also constant, we also obtain that all the groups $\mathcal{G}_b$ are isomorphic.

Let us verify now the assumption that we made above, that is that in a small neighborhood of b_0 all the groups $\mathcal{G}_b$ have maximal tori of the same dimension.

The Lie algebra $\operatorname{Lie} H_b$ is the kernel of ad_{t_b}. Since the operators ad_{t_b} depend continuously on b (on any trivialization of $\mathfrak{g}$ as a vector bundle), the dimension of the kernels of ad_{t_b} is $\leq$ the dimension of H_{b_0} in a small neighborhood of b_0. Since H_b contains a maximal torus of $\mathcal{G}_b$, for any b, we obtain that the set of points b such that the dimension of a maximal torus of $\mathcal{G}_B$ is $\leq n$ is an open subset of B. We can approach then b_0 with a sequence of distinct points b_n such that $\mathcal{G}_{b_n}$ have maximal tori of the same dimension l, for any n.

Then the maximal tori of $H^0_{b_n}$ also have dimension l. Fix a metric on $\mathfrak{g}$ and let $X_1, \ldots, X_d \in \operatorname{Lie} \mathcal{G}_{b_0}$ be a basis of $\operatorname{Lie} \mathcal{G}_{b_0}$, which we extend to a basis of $\operatorname{Lie} H^0_{b_n}$, at least for n large. Let $\epsilon > 0$ be small, but fixed. If we denote by

$$g_{ijn} := \exp(\epsilon X_j(b_n)) \exp(\epsilon X_i(b_n)) \exp(-\epsilon X_j(b_n)) \exp(-\epsilon X_i(b_n))$$

then

$$dist(g_{ijn}, 1) \to 0 \tag{5}$$

as $n \to \infty$, for any i and j. Let $\delta > 0$ be arbitrary. This proves that for n large all the elements $\exp(\epsilon X_j(b_n))$ are in a δ neighborhood of a torus of dimension l. By letting $\delta \to 0$ and $n \to \infty$, we obtain that l coincides with the dimension of H_{b_0}. □

2 Finite holonomy conditions

From now on and throughout the paper all bundles of Lie groups that we shall consider will have compact fibers. Also, unless explicitly otherwise mentioned, B will be a compact space and $\mathcal{G} \to B$ will be a bundle of compact Lie groups.

We shall now take a closer look at the structure of bundles of Lie groups whose typical fiber is a compact Lie group G. Let $\operatorname{Aut}(G)$ be the group of automorphisms of G. By definition, there exists then a principal $\operatorname{Aut}(G)$-bundle $\mathcal{P} \to B$ such that $\mathcal{G} \cong \mathcal{P} \times_{\operatorname{Aut}(G)} G := (\mathcal{P} \times G)/\operatorname{Aut}(G)$.

Let $\widehat{\mathcal{G}}$ be the (disjoint) union of the sets $\widehat{\mathcal{G}}_b$ of equivalence classes of irreducible representations of the groups $\mathcal{G}_b$. Using the natural action of $\operatorname{Aut}(G)$ on $\widehat{G}$, we can naturally identify $\widehat{\mathcal{G}}$ with $\mathcal{P} \times_{\operatorname{Aut}(G)} \widehat{G}$ as fiber bundles over B.

Let $\operatorname{Aut}_0(G)$ be the connected component of the identity in $\operatorname{Aut}(G)$. The group $\operatorname{Aut}_0(G)$ will act trivially on the set $\widehat{G}$, because the later is discrete. Suppose,

$$H_R := \operatorname{Aut}(G)/\operatorname{Aut}_0(G), \qquad \mathcal{P}_0 := \mathcal{P}/\operatorname{Aut}_0(G).$$

Then $\mathcal{P}_0$ is an H_R-principal bundle and $\widehat{\mathcal{G}} \simeq \mathcal{P}_0 \times_{H_R} \widehat{G}$.

Assume now that B is a path-connected, locally simply-connected space and fix a point $b_0 \in B$. Then the bundle $\mathcal{P}_0$ is is classified by a morphism $\pi_1(B, b_0) \to H_R$ because the structure group H_R of this principal bundle is discrete.

The space $\widehat{\mathcal{G}}$ will be called *the representation space of* $\mathcal{G}$, the covering $\widehat{\mathcal{G}} \to B$ will be called *the representation covering associated to* $\mathcal{G}$. Fix arbitrary a base point of B. If B is path-connected and locally simply-connected, then the resulting morphism

$$\rho : \pi_1(B, b_0) \to H_R := \operatorname{Aut}(G)/\operatorname{Aut}_0(G), \tag{6}$$

will be called *the holonomy of the representation covering of* $\mathcal{G}$.

For our further reasoning, we shall sometimes need the following finite holonomy condition.

Definition 2.1. We say that $\mathcal{G}$ has *representation theoretic finite holonomy* if every $\sigma \in \widehat{\mathcal{G}}$ is contained in a compact-open subset of $\widehat{\mathcal{G}}$.

In the interesting cases, the above condition can be reformulated as follows.

Proposition 2.2. *Assume that B is path-connected and locally simply-connected. Then $\mathcal{G}$ has* representation theoretic finite holonomy *if, and only if, the following condition is satisfied: for any irreducible representation σ of G, the set $\pi_1(B, b_0)\sigma \subset \widehat{G}$ is finite.*

Proof. Since $\widehat{\mathcal{G}}$ is a covering of B, its compact-open subsets coincide with the finite unions of the connected components of $\widehat{\mathcal{G}}$ that are finite coverings of B (i.e. cover B finitely-many times to one). Let B_σ be the connected component of $\widehat{\mathcal{G}}$ containig a given point $\sigma \in \widehat{\mathcal{G}}$. The typical fiber of $B_\sigma \to B$ is $\pi_1(B, b_0)\sigma$. The result now follows. □

The above condition ensuring representation theoretic finite holonomy are difficult to check directly, so we shall also consider the following closely related condition.

Definition 2.3. Assume B is smooth and connected. We shall say that $\mathcal{G}$ has *finite holonomy* if the image $H_{\mathcal{G},b}$ of $\pi_1(B, b_0)$ in $H_R := \mathrm{Aut}(G)/\mathrm{Aut}_0(G)$ is finite.

These two "finite holonomy" conditions are related as follows.

Theorem 2.4. *Let $\mathcal{G} \to B$ be a bundle of compact Lie groups over a smooth connected manifold B. If $\mathcal{G}$ has finite holonomy then it has also representation theoretic finite holonomy. If the fibers of $\mathcal{G} \to B$ are connected, then the converse is also true.*

Proof. Let σ be an irreducible representation of $G = \mathcal{G}_{b_0}$. Then $\pi_1(B, b_0)\sigma = H_{\mathcal{G}}\sigma$ and hence $\pi_1(B, b_0)\sigma$ is a finite set.

To prove the converse, let us write $G = G_0'Z_0$, where $G_0' \subset G$ is the connected component of the subgroup G' generated by commutators and $Z_0 \subset G$ is the connected component of the center of G. This is possible because G is connected. Then $G_0' \cap Z_0 = A$ is a finite subgroup and $G \simeq (G_0' \times Z_0)/A$, where A is embedded diagonally in the two subgroups. Every automorphism of G maps each of G_0' and Z_0 to itself. This shows that the group of automorphisms of G identifies with the subgroup of automorphisms of $G_0' \times Z_0$ that map A to itself. Moreover, we have a canonical morphism $\mathrm{Aut}(G) \to \mathrm{Aut}(Z_0)$. Since $\mathrm{Aut}(G_0')/\mathrm{Aut}_0(G_0')$ is a finite group, it is enough then to show that the image of $\pi_1(B, b_0)$ in $\mathrm{Aut}(Z_0)$ is finite whenever $\mathcal{G}$ has representation theoretic finite holonomy.

The set $\widehat{S_0}$ of irreducible representations of Z_0 identifies with a lattice in the vector space $V = \mathrm{Lie}\, Z_0$. Choose a basis $v_1, \ldots, v_n \in \widehat{S_0}$ of V. By assumption, the sets $S_j = \pi_1(B, b_0)v_j$ are finite. The action of $\pi_1(B, b_0)$ then defines a group morphism from $\pi_1(B, b_0)$ to $\prod \mathrm{Aut}(S_j)$, the product of the permutation groups of the sets S_j. Let K be the kernel of this morphism, then the morphism $\pi_1(B, b_0) \to \mathrm{Aut}(Z_0)$ factors through K. This is enough to complete the proof of the fact that representation theoretic finite holonomy implies finite holonomy for a bundle of compact Lie groups $\mathcal{G}$ with connected fibers. □

Let us note the following two consequence of the above proof that will be useful in proving our results on the Dixmier-Douady invariants in Section 5. Denote by $G_{\mathrm{int}} \subset \mathrm{Aut}(G)$ the subgroup of inner automorphisms of G.

Proposition 2.5. *Suppose G is a compact, connected Lie group. Then $\mathrm{Aut}_0(G)$ consists of inner automorphisms. If the center of G has dimension ≤ 1, then $\mathrm{Aut}(G)/\mathrm{Aut}_0(G)$ is a finite group.*

Proof. We shall use the notation introduced in the proof of Theorem 2.4. First, since G_0' is semi-simple and connected, the connected component $\mathrm{Aut}_0(G_0')$ of $\mathrm{Aut}(G_0')$ consists of inner automorphisms. Since the image $G_{\mathrm{int}} \subset \mathrm{Aut}(G)$ of G acting by inner automorphisms is contained in $\mathrm{Aut}_0(G_0')$, we obtain that $G_{\mathrm{int}} = \mathrm{Aut}_0(G_0') = \mathrm{Aut}_0(G)$.

If $\dim Z_0 \leq 1$, the group of automorphisms of Z_0 is finite and this is enough to complete the proof. $\square$

Here is an example that the finite holonomy condition is not always satisfied by a bundle of compact Lie groups $\mathcal{G} \to B$. Let A be the matrix

$$A = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}.$$

This matrix induces an automorphism α of the compact torus $T = S^1 \times S^1$ by the formula

$$\alpha(z, w) = (z^3 w^2, z^4 w^3).$$

Let us consider on the unit circle S^1 a bundle of tori $\mathcal{G}_A$ with fiber T and holonomy A. This bundle can be realized as the quotient of $\mathbb{R} \times T$ by the equivalence relation $(t+n, z, w) \equiv (t, \alpha^n(z, w))$, $n \in \mathbb{Z}$. The morphism $\mathbb{Z} \simeq \pi_1(S^1) \to \mathrm{Aut}(T)$ sends then a generator of $\mathbb{Z}$ to the automorphism α. Clearly the range of this morphism is not finite. The only irreducible representation σ of T with the property that $\pi_1(S^1)\sigma$ is finite is the trivial representation.

3 Gauge equivariant K-theory

In this section, we define the gauge equivariant K-theory groups for spaces endowed with the action of a bundle of compact Lie groups $\mathcal{G}$. As we shall see, these are the right K-theory groups for our index calculations. We formulate everything for the case of (locally) compact Hausdorff topological spaces and continuous fiber bundles. However, one can easily extend the following results to the smooth setting (we usually include the necessary changes inside square brackets).

Let $d : \mathcal{G} \to B$ be a bundle of compact Lie groups over a compact space B and let $\pi : Y \to B$ be a fiber bundle. We assume, as in the previous section, that Y is a $\mathcal{G}$-fiber bundle (that is, that $\mathcal{G}$ acts on Y). Let E be a *finite-dimensional* vector bundle $E \to Y$ equipped with an action of $\mathcal{G}$. Such a vector bundle will also be called a *$\mathcal{G}$-equivariant vector bundle.* This implies, in particular, that $E_b := E|_{Y_b}$ is an ordinary $\mathcal{G}_b$-equivariant vector bundle over the $\mathcal{G}_b$-space Y_b, for any $b \in B$.

Definition 3.1. Let $E \to Y$ be a $\mathcal{G}$-equivariant vector bundle and let $E' \to Y'$ be a $\mathcal{G}'$-equivariant vector bundle, for two bundles of Lie groups $\mathcal{G} \to B$ and $\mathcal{G}' \to B'$. A *morphism* $(\gamma, \varphi) : (\mathcal{G}', E') \to (\mathcal{G}, E)$ is a pair of morphisms

$$(\gamma, \varphi), \quad \gamma : \mathcal{G}' \to \mathcal{G}, \quad \varphi : E' \to E,$$

assumed to satisfy

$$\varphi(ge) = \gamma(g)\varphi(e), \qquad e \in E'_b, \quad g \in \mathcal{G}'_b.$$

(A map φ with these properties will be called *γ-equivariant.*)

[All the above maps are assumed to be smooth when working in the smooth category.]

In particular, the maps (γ, φ) in the above definition give rise also to a map $B' \to B$ and to a γ-equivariant map $Y' \to Y$.

As usual, if $\psi : B' \to B$ is a continuous [respectively, smooth] map, we define the *inverse image* $(\psi^*\mathcal{G}, \psi^*E)$ of a $\mathcal{G}$-equivariant vector bundle $E \to Y$ by $\psi^*\mathcal{G} = \mathcal{G} \times_B B'$ and $\psi^*E = E \times_B B'$. A particular case of this definition is when ψ is an embedding, when it gives the definition of the restriction of a $\mathcal{G}$-equivariant vector bundle E to a closed subset $B' \subset B$ of the base of $\mathcal{G}$, yielding a $\mathcal{G}_{B'}$-equivariant vector bundle. Usually $\mathcal{G}$ will be fixed, however.

The set of isomorphism classes of equivariant vector bundles E on Y, as above, will be denoted by $\mathcal{E}_{\mathcal{G}}(Y)$. On this set we introduce a monoid operation, denoted "+," using the direct sum of vector bundles. This defines a monoid structure on the set $\mathcal{E}_{\mathcal{G}}(Y)$.

Definition 3.2. Let $\mathcal{G} \to B$ be a bundle of compact Lie groups acting on the fiber bundle $Y \to B$. Assume Y to be compact. The *$\mathcal{G}$-equivariant K-theory group $K^0_{\mathcal{G}}(Y)$* is defined as the group completion of the monoid $\mathcal{E}_{\mathcal{G}}(Y)$.

If $E \to Y$ is a $\mathcal{G}$-equivariant vector bundle on Y, we shall denote by $[E]$ its class in $K^0_{\mathcal{G}}(Y)$. Thus $K^0_{\mathcal{G}}(Y)$ consists of differences $[E] - [E_1]$.

The groups $K^0_{\mathcal{G}}(Y)$ will be called *gauge equivariant* K-theory groups, when we do not need to specify $\mathcal{G}$. If B is reduced to a point, then $\mathcal{G}$ is a Lie group, and the groups $K^0_{\mathcal{G}}(Y)$ reduce to the usual equivariant K-groups. More generally, this is true if $\mathcal{G} \simeq B \times G$ is a trivial bundle. The familiar functoriality properties of the usual equivariant K-theory groups extend to the gauge equivariant K-theory groups.

Theorem 3.3. *Assume that the bundle of Lie groups $\mathcal{G} \to B$ acts on a fiber bundle $Y \to B$ and that, similarly, $\mathcal{G}' \to B'$ acts on a fiber bundle $Y' \to B'$. Let $\gamma : \mathcal{G} \to \mathcal{G}'$ be a morphism of bundles of Lie groups and $f : Y \to Y'$ be a γ-equivariant map. Then we obtain a natural morphism*

$$(\gamma, f)^* : K^0_{\mathcal{G}'}(Y') \to K^0_{\mathcal{G}}(Y) \tag{7}$$

(denoted also $f^* : K^0_{\mathcal{G}}(Y) \to K^0_{\mathcal{G}}(Y')$ if γ is the identity morphism).

Proof. The pull-back operation preserves the direct sum of equivariant vector bundles, so it induces a morphism of monoids $\mathcal{E}_{\mathcal{G}'}(Y') \to \mathcal{E}_{\mathcal{G}}(Y)$. The morphism $(\gamma, f)^* : K^0_{\mathcal{G}'}(Y') \to K^0_{\mathcal{G}}(Y)$ is the group completion of this morphism. ☐

We now proceed to establish the main properties of the gauge equivariant K-theory groups. Most of them are similar to those of the usual equivariant K-theory groups, but there are also some striking differences.

First, let us observe that we can extend as usual the definition of the gauge-equivariant groups to non-compact $\mathcal{G}$-fiber bundles Y. Let Y be a $\mathcal{G}$-fiber bundle. We shall denote then by $Y^+ := Y \cup B$ the space obtained from Y by one-point compactifying each fiber.

Definition 3.4. Assume that the typical fiber of the $\mathcal{G}$-fiber bundle $Y \to B$ is a locally compact space. We define then

$$K_{\mathcal{G}}(Y) := \mathrm{Ker}\{K_{\mathcal{G}}(Y^+) \to K_{\mathcal{G}}(B)\}.$$

Note that the above groups are "compactly supported" K-groups. These are the only K-theory groups that we shall consider when working with non-compact manifolds.

We now discuss induction. Let $\mathcal{G} \subset \mathcal{G}'$ be a sub-bundle of the bundle of Lie groups $\mathcal{G}' \to B$. Also, let Y be a $\mathcal{G}$-fiber bundle and Y' be a $\mathcal{G}'$-bundle. The fibered product over $\mathcal{G}$, namely $\times_{\mathcal{G}}$, is defined as the quotient of $\times_B$, the fibered product over B, by the action of $\mathcal{G}$, its action on $\mathcal{G}'$ being by right translations, namely $\mathcal{G}' \times_{\mathcal{G}} Y := (\mathcal{G}' \times_B Y)/\mathcal{G}$. Then to any $\mathcal{G}$-equivariant vector bundle $E \to Y$ we can associate a $\mathcal{G}'$-equivariant vector bundle $E' = \iota(E)$ over $Y' := \mathcal{G}' \times_{\mathcal{G}} Y$ by the formula $E' := \mathcal{G}' \times_{\mathcal{G}} E$. This operation gives rise to a morphism

$$\iota : K^0_{\mathcal{G}}(Y) \to K^0_{\mathcal{G}'}(\mathcal{G}' \times_{\mathcal{G}} Y), \tag{8}$$

called the *induction* morphism.

Theorem 3.5. *The induction morphism $\iota : K^0_{\mathcal{G}}(Y) \to K^0_{\mathcal{G}'}(\mathcal{G} \times_{\mathcal{G}'} Y)$ is an isomorphism. The inverse is given by the restriction to $Y \subset Y'$ and $\mathcal{G} \subset \mathcal{G}'$.*

Proof. Note that Y identifies with the image of $B \times_B Y$ (same B) in $Y' := \mathcal{G}' \times_{\mathcal{G}} Y$. Moreover, the map $Y \to Y'$ is $\mathcal{G}$-equivariant. This shows that it makes sense to consider the restriction map $r : K^0_{\mathcal{G}'}(Y') \to K^0_{\mathcal{G}}(Y)$. It follows from the definition of $\iota(E)$, for a $\mathcal{G}$-equivariant bundle $E \to Y$, that $r \circ \iota$ is the identity.

The proof of the fact that $\iota \circ r$ is the identity follows from the fact that any $\mathcal{G}$-equivariant isomorphism $\beta : E|_Y \to F|_Y$ of two $\mathcal{G}'$-equivariant bundles extends uniquely to a $\mathcal{G}'$-equivariant isomorphism $E \to F$. We shall use this as follows. Let E be a $\mathcal{G}'$-equivariant bundle. Then take $F := \iota \circ r(E)$. By what we have proved, we know that $r(E) = E|_Y \simeq F|_Y = r(F)$ as $\mathcal{G}$-equivariant bundles. Then $E \simeq F$, as wanted. □

The groups $K^0_{\mathcal{G}}(Y)$ can be fairly small if the holonomy of $\mathcal{G}$ is "large." This is a new fenomenon, not encountered in the usual equivariant K-theory. Here is an example.

Proposition 3.6. *Let $\mathcal{G}_A$ be the bundle of two dimensional tori over the circle S^1 introduced at the end of Section 1, then $K^0_{\mathcal{G}_A}(S^1) \simeq K^0(S^1)$.*

If G were a compact group, the analogous statement would be that $K^0_G(point) \simeq K^0(point)$, which is clearly true only if G is trivial. This kind of pathologies are ruled out by considering only bundles of Lie groups with representation theoretic finite holonomy.

Let us denote by $C(\mathcal{G})$ the group of continuous sections of $\mathcal{G}$, that is, the group of continuous maps $\gamma : B \to \mathcal{G}$. The group $C(\mathcal{G})$ acts on $\Gamma(E)$ according to the rule $(\gamma s)(y) = \gamma(b)s((\gamma(b))^{-1}y)$, where $\gamma \in C(\mathcal{G})$, $s \in \Gamma(E)$, and $y \in Y_b$. We now define a continuous map $Av_{\mathcal{G}} : \Gamma(E) \to \Gamma^{\mathcal{G}}(E)$, where $\Gamma^{\mathcal{G}}(E)$ is the space of $C(\mathcal{G})$-invariant sections of E, by the formula

$$\big[Av_{\mathcal{G}}(s)\big](x) := \int_{\mathcal{G}_b} s(g^{-1}x)dg, \ \text{if } x \in Y_b,$$

the measure on $\mathcal{G}_b$ being normalized to have total mass one. In the following, by a $\mathcal{G}$-invariant section of a $C(\mathcal{G})$-module we shall understand a $C(\mathcal{G})$-invariant section.

Lemma 3.7. *Let X be a closed $\mathcal{G}$-invariant sub-bundle of a compact $\mathcal{G}$-fiber bundle Y, and let $E \to X$ be a $\mathcal{G}$-equivariant bundle. Let s' be a $\mathcal{G}$-invariant cross-section of the restriction $E|_X \to X$. Then s' can be extended to a $\mathcal{G}$-invariant cross-section of E.*

Proof. We proceed as in the classical case (cf. [3]). First extend the section s' to a section s_1 of E over the whole of Y, not necessarily equivariant. The desired extension is obtained by setting $s = Av_{\mathcal{G}}(s_1)$. □

We have the following analog of the corresponding classical result (see [3] or [23], for example).

Theorem 3.8. *Let E and F be $\mathcal{G}$-equivariant vector bundles over X, and $\alpha : E \to F$ a $\mathcal{G}$-equivariant morphism such that $\alpha_x : E_x \to F_x$ is an epimorphism for all points $x \in X$. Then there exists a $\mathcal{G}$-equivariant morphism $\beta : F \to E$ such that $\alpha\beta = \mathrm{Id}_F$.*

Proof. First, we can define a possibly not equivariant morphism of bundles $\widetilde{\beta} : F \to E$ such that $\alpha\widetilde{\beta} = \mathrm{Id}_F$ (see, e.g. [23, Theorem I.5.13]). Then, let us take $\beta := Av_{\mathcal{G}}(\widetilde{\beta})$, which we define by regarding $\widetilde{\beta}$ as an element of the $\mathcal{G}$-equivariant vector bundle $\mathrm{Hom}(F, E)$. Then

$$\alpha\beta = \alpha Av_{\mathcal{G}}(\widetilde{\beta}) = Av_{\mathcal{G}}(\alpha\widetilde{\beta}) = Av_{\mathcal{G}}(\mathrm{Id}) = \mathrm{Id}\,.$$

□

A $\mathcal{G}$-equivariant vector bundle $E \to Y$ on a $\mathcal{G}$-fiber bundle $Y \to B$, Y compact, is called *trivial* if, by definition, there exists a $\mathcal{G}$-equivariant vector bundle $E' \to B$ such that E is isomorphic to the pull-back of E' to Y. Thus $E \simeq Y \times_B E'$.

Theorem 3.9. *Assume that $\mathcal{G} \to B$ has representation theoretic finite holonomy. Let $Y \to B$ be a compact $\mathcal{G}$-fiber bundle and $E \to Y$ be a $\mathcal{G}$-equivariant vector bundle. Then there exists a $\mathcal{G}$-equivariant vector bundle $V \to B$ and a $\mathcal{G}$-equivariant vector bundle E' such that $Y \times_B V \cong E \bigoplus E'$.*

Proof. Let Y_b be the fiber of $Y \to B$ above some point $b \in B$. Let us recall how to embed $E_b := E|_{Y_b}$ into a trivial bundle, for each $b \in B$. Choose sections $s_1, \dots, s_n$ of $E|_{Y_b}$ such that they generate a finite dimensional $\mathcal{G}_b$-invariant subspace V_b and they generate $\Gamma(E_b)$ as a $C(Y_b)$-module. Then there exists a $\mathcal{G}_b$-equivariant map $C(Y_b) \times V_b \to \Gamma(E_b)$ that is surjective. The required embedding into a trivial bundle is then obtained by an application of Theorem 3.8 for B reduced to b.

A set of sections $s_1, \dots, s_n$ as above will be called a *generating set of sections.*

Let V_b be the representation space of $\mathcal{G}_b$ defined by these sections. Because $\mathcal{G}$ has representation theoretic finite holonomy, there exists a $\mathcal{G}$-equivariant vector bundle $W(b) \to B$ such that the fiber of this bundle at b is a representation of $\mathcal{G}_b$ containing V_b. There will exist then a $C(B)$-linear map $\Phi_b : \Gamma(W(b)) \to \Gamma(E)$ and sections $\xi_j \in \Gamma(W(b))$ such that $\Phi_b(\xi_j)|_{Y_b} = s_j$. By averaging with respect to $\mathcal{G}$ (using the map $Av_{\mathcal{G}}$), we can assume that Φ_b is $\mathcal{G}$-equivariant.

Then $\Phi(\xi_j)$ will define by restriction a set of generating sections of $E_{b'}$, for b' in a neighborhood U_b of b. Cover B with finitely many such neighborhoods U_{b_j}, and let $V \to B$ be the direct sum of all the bundles $W(b_j)$ and $\Phi := \oplus \Phi_{b_j}$.

Our construction then gives a $\mathcal{G}$-equivariant map

$$1 \otimes \Phi : C(Y) \otimes_{C(B)} \Gamma(W) \to \Gamma(E) \tag{9}$$

that is surjective by construction. We have thus constructed a surjective map $Y \times_B V \to E$. An application of Theorem 3.8 concludes the proof. □

Let us observe that the above result is not true for $\mathcal{G}_A$, the bundle of two dimensional tori over S^1 considered at the end of Section 1. Indeed, let $\mathcal{G} \subset \mathcal{G}_A$ be subset of all elements of order two of the fibers of $\mathcal{G}_A$. Then $\mathcal{G} \to S^1$ is a trivial bundle of finite groups: $\mathcal{G} = S^1 \times A$, with $A \simeq (\mathbb{Z}/2\mathbb{Z})^2$. Let

$$Y' := \mathcal{G}_A \times_{\mathcal{G}} S^1 = \mathcal{G}_A / \mathcal{G}.$$

We know by Theorem 3.5 that

$$K^0_{\mathcal{G}_A}(Y') \simeq K^0_A(S^1) \simeq R(A) \otimes K^0(S^1) = R(A),$$

the isomorphism being given by restriction to $S^1 \subset Y'$. On the other hand, if $E \to Y'$ were a sub-bundle of a trivial E' bundle over Y', then $E|_{S^1}$ would also be a $\mathcal{G}$-equivariant sub-bundle of the trivial $\mathcal{G}$-equivariant bundle $E'|_{S^1}$. If E'' is a $\mathcal{G}_A$-equivariant bundle over S^1, then the pull-back to Y' followed by the restriction to S^1 corresponds to restricting the action of $\mathcal{G}_A$ to an action of $\mathcal{G}$. Thus any bundle of the form $E'|_{S^1}$, with E' a trivial $\mathcal{G}_A$-bundle, will be trivial over S^1 and will have the trivial action of A, by Proposition 3.6. Any sub-bundle of E' will again have the trivial action of A. This shows that the $\mathcal{G}_A$-equivariant bundles over Y' that can be realized as sub-bundles of trivial bundles have a class in $K^0_{\mathcal{G}_A}(Y') \simeq R(A)$ corresponding to multiples of the trivial representation. This gives the following result.

Proposition 3.10. *Thus not every $\mathcal{G}_A$-equivariant bundle over Y' can be realized as a sub-bundle of a trivial bundle.*

We now check that the category of $\mathcal{G}$-equivariant vector bundles is a Banach category (see Definition 7.3). All the other properties of the gauge equivariant K-theory groups that we shall prove will turn out to be consequences of a some general theorems on Banach categories from [22, 23]. We shall obtain, in particular, that gauge equivariant K-theory has long exact sequences and satisfies Bott periodicity.

Proposition 3.11. *The category of $\mathcal{G}$-equivariant vector bundles over a $\mathcal{G}$-fiber bundle $Y \to B$, Y compact, is a Banach category.*

Proof. First, the set $\Gamma(E)$ of all continuous sections $s : Y \to E$ of a $\mathcal{G}$-equivariant vector bundle $E \to Y$ over a compact $\mathcal{G}$-fiber bundle $Y \to B$ becomes a Banach space when endowed with the "sup"-norm. Consider now two $\mathcal{G}$-equivariant vector bundles E and F over a compact $\mathcal{G}$-fiber bundle Y. The vector bundle $\mathrm{Hom}(E, F)$ will have a natural $\mathcal{G}$-action. As usual, we can identify $\Gamma^{\mathcal{G}}(\mathrm{Hom}(E,F))$ with the set of $\mathcal{G}$-equivariant morphisms $\varphi : E \to F$, which is hence a Banach space. The composition of morphisms $\mathrm{Hom}(E_1,E_2)\times\mathrm{Hom}(E_2,E_3)\to\mathrm{Hom}(E_1,E_3)$ is continuous because the category of vector bundles is a Banach category. The restriction to $\mathcal{G}$-equivariant morphisms will also be continuous. This checks all conditions of Definition 7.3 (see the Appendix, where Banach categories are discussed following [22, 23])), and hence the proof is now complete. □

This proposition allows now to establish several useful Lemmata.

Lemma 3.12. *Let $E,F\to X$ be $\mathcal{G}$-equivariant vector bundles and let $Y\subset X$ be a $\mathcal{G}$-equivariant sub-bundle over B. Let $\varphi' : E|_Y \to F|_Y$ be a morphism of the restrictions of the $\mathcal{G}$-fiber bundles E and F to Y. Then φ' can be extended to a $\mathcal{G}$-equivariant morphism $\varphi : E\to F$. If φ is an isomorphism, then there exists a $\mathcal{G}$-invariant open neighborhood U of Y such that*

$$\varphi|_U : E|_U \to F|_U$$

is an isomorphism. Any two such extensions are homotopic to each other in the class of $\mathcal{G}$-equivariant isomorphisms over some $\mathcal{G}$-invariant neighborhood of X.

Proof. We apply Lemma 3.7 to the bundle $\mathrm{Hom}(E,F)$ and use the fact that the set of isomorphisms forms an open subset of the set of all homomorphisms. □

If Y is a $\mathcal{G}$-fiber bundle over B and I is the unit interval, then we define the action of $\mathcal{G}_b$ on $Y_b\times I$ by the formula $g(y,t)=(gy,t)$. So we obtain a $\mathcal{G}$-fiber bundle $Y\times I$ over B. This is a particular case (for $Z=B\times I$ and the trivial action of $\mathcal{G}$ on Z) of the following construction. Suppose that Y and Z are $\mathcal{G}$-fiber bundles over B. Then the bundle $Y\times_B Z\to B$ can be equipped with the diagonal action of $\mathcal{G}$.

Lemma 3.13. *Let $\pi_Y : Y\to B$ and $\pi_X : X\to B$ be compact $\mathcal{G}$-fiber bundles and $f_t : Y\to X$ be a continuous homotopy of $\mathcal{G}$-equivariant mappings $(0\le t\le 1)$ satisfying $\pi_X\circ f_t = f_t\circ\pi_Y$. Suppose that E is a $\mathcal{G}$-equivariant vector bundle over X. Then $f_0^*E\cong f_1^*E$.*

Proof. We proceed as in [3], Lemma 1.4.3. Denote by I the unit interval $[0,1]$. Let $\pi : Y\times I\to Y$ be the first projection, and let $f : Y\times I\to X$, $f(y,t):=f_t(y)$ be the given homotopy. Let us consider the bundles f^*E and $\pi^*f_t^*E$ for some fixed t. Over $Y\times\{t\}$ we have a natural isomorphism (identification) of these bundles. By Lemma 3.12, since Y is compact, there exists a neighborhood U_t of t in I such that f^*E and $\pi^*f_t^*E$ are isomorphic over U_t. Hence the isomorphism class of $(f^*E)|_t = f_t^*E$ is a locally constant function of t. The connectedness of I completes then the proof. □

Corollary 3.14. *Let $Y\to B$ be a $\mathcal{G}$-fiber bundle and $E\to Y$ be a $\mathcal{G}$-equivariant vector bundle. Then there exists a $\mathcal{G}$-invariant Hermitian metric on E.*

Proof. Choose an arbitrary Hermitian metric on E, regarded as an element of the $\mathcal{G}$-equivariant vector bundle $E^*\otimes E^*$. Its average will then be a $\mathcal{G}$-invariant Hermitian metric. □

Another immediate consequence of Theorem 3.9 is the following important property.

Theorem 3.15. *Let $\rho^{Y,Y'} : \mathcal{E}_{\mathcal{G}}(Y) \to \mathcal{E}_{\mathcal{G}}(Y')$ be the restriction functor, where Y' is a closed $\mathcal{G}$-invariant sub-bundle of Y. Then $\rho^{Y,Y'}$ is a full quasi-surjective Banach functor in the sense of* 7.5 .

Proof. We shall use the concepts recalled in the Appendix. Since the restriction map

$$\rho_* : \Gamma^{\mathcal{G}}(\mathrm{Hom}(E,F)) \to \Gamma^{\mathcal{G}}(\mathrm{Hom}(E|_{Y'}, F|_{Y'}))$$

is a continuous linear map of Banach spaces (cf. the proof of Proposition 3.11), ρ is a Banach functor. By Lemma 3.12, ρ is full. Finally, it is quasi-surjective by Theorem 3.9, because $Y \times_B V$ extends $Y' \times_B V$. □

Definition 3.16. Following the general scheme presented in the Appendix (Section 7), we define the K-groups for a compact space B by setting

$$\begin{aligned} K^{p,q}_{\mathcal{G}}(Y) &= K^{p,q}(\mathcal{E}_{\mathcal{G}}(Y), \\ K^{n}_{\mathcal{G}}(Y) &= K^{n}(\mathcal{E}_{\mathcal{G}}(Y)), \\ K_{\mathcal{G}}(Y) &= K^{0,0}(\mathcal{E}_{\mathcal{G}}(Y)). \end{aligned}$$

By Theorem 3.15, setting $K^n_{\mathcal{G}}(Y,Y') := K^n(\rho^{Y,Y'})$ in the sense of Definition 7.12, we get (by [23, II.3.22] and [22, 2.3.1]) a long exact sequence

$$\begin{aligned} \dots \to K^{n-1}_{\mathcal{G}}(Y,Y') \to K^{n-1}_{\mathcal{G}}(Y) \to K^{n-1}_{\mathcal{G}_1}(Y') \\ \to K^{n}_{\mathcal{G}}(Y,Y') \to K^{n}_{\mathcal{G}}(Y) \to K^{n}_{\mathcal{G}}(Y') \to \dots \end{aligned} \tag{10}$$

Theorem 3.17. (Bott-Clifford periodicity) *We have a natural isomorphism*

$$K^n_{\mathcal{G}}(Y,Y') \cong K^{n-2}_{\mathcal{G}}(Y,Y').$$

Proof. The category of $\mathcal{G}$-equivariant vector bundles is a Banach category (see Definitions 7.3 and 7.4 of the Appendix) This and the periodicity of the Clifford algebras directly implies our result (see, for example, [23, §III.4]). □

Theorem 3.18. (Periodicity.) *Let $Y \to B$ be a compact $\mathcal{G}$-fiber bundle. We have a natural isomorphism*

$$K^1_{\mathcal{G}}(Y) \cong K_{\mathcal{G}}(Y \times D^1, Y \times S^0),$$

where (D^n, S^{n-1}) is a ball and its boundary.

Proof. This is proved for general (complex) Banach categories in [22, Theorem 2.3.3]. □

Theorem 3.19. *Suppose that $Y \to B$ is a closed invariant sub-bundle of a compact $\mathcal{G}$-fiber bundle $X \to B$. Then the projection $\kappa : X \to X/Y$ induces an isomorphism $\kappa^* : K_{\mathcal{G}}(X/Y, \{y\}) \to K_{\mathcal{G}}(X,Y)$.*

Proof. We shall follow the proof in [23, II.2.35].

Let us prove first that κ^* is surjective. Let $d(E,F,\alpha)$ be an element of $K_{\mathcal{G}}(X,Y)$. By adding the same bundle to E and F, we may assume, without loss of generality, that $F \cong X \times_B V$ (that is, that F is the pull-back from B of a vector bundle $V \to B$). We want to find a triple (E', F', α') defining an element of $K_{\mathcal{G}}(X/Y, \{y\})$, such that the triples $(\kappa^*(E'), \kappa^*(F'), \kappa_1^*(\alpha'))$ and (E,F,α) are isomorphic, where $\kappa_1 : Y \to \{y\}$ maps Y to a point. According to Lemma 3.12, there is a closed $\mathcal{G}$-invariant neighborhood N of Y and an isomorphism $\beta : E|_N \to F|_N$ such that $\beta|_Y = \alpha$. Let E' be the vector bundle over X/Y obtained by clutching the bundle $E|_{X\setminus Y}$ and the bundle $n/Y \times_B V$, using $\beta|_{X\setminus Y}$. One has $X \setminus Y = (X/Y) \setminus \{y\}$ and $N \setminus Y = (N/Y) \setminus \{y\}$. Let $F' = X/Y \times_B V$, and let $\alpha' : E'|_{\{y\}} \to F'|_{\{y\}}$ be the isomorphism induced by the above clutching.

Then we can define an isomorphism $f : E \to \kappa^*(E')$ by $f|_{X\setminus Y} = \mathrm{Id}$, with the identification $\kappa^* E'|_{X\setminus Y} = E'|_{X\setminus Y} = E|_{X\setminus Y}$, and $f|_N = \mathrm{Id}$ with the identification $\kappa^*(E')|_N = (X \times_B V)|_N$. It is now obvious that the diagram

$$\begin{tikzcd} E|_Y \arrow[r, "\alpha"] \arrow[d, "f|_Y"'] & F|_Y \arrow[d, equal] \\ \kappa^*(E')|_Y \arrow[r, "\kappa_1^*(\alpha')"] & \kappa^*(F')|_Y \end{tikzcd}$$

is commutative.

We now prove that κ^* is injective. Let $d(E', F', \alpha')$ be an element of $K_{\mathcal{G}}(X/Y, \{y\})$ such that

$$\kappa^*(d(E', F', \alpha')) = d(\kappa^*(E'), \kappa^*(F'), \kappa_1^*(\alpha')) = 0.$$

According to Proposition 7.7, there is a bundle T over X such that $\kappa_1^*(\alpha') \oplus \mathrm{Id}\,|_{T|_Y}$ can be extended by an isomorphism $\beta : \kappa^*(E') \oplus T \to \kappa^*(F') \oplus T$. As before we may assume that $T = X \times_B V$. Let $T' = X/Y \times_B V$. Let $\beta' : E' \oplus T' \to F' \oplus T'$ be the isomorphism which is equal to β over $X \setminus Y$, and to α' over $\{y\}$. Then β' is continuous and is an extension of $\alpha' \oplus \mathrm{Id}_{T|_{\{y\}}}$ over X/Y. By Proposition 7.7, $d(E', F', \alpha') = d(E' \oplus T', F' \oplus T', \alpha' \oplus \mathrm{Id}_{T'}) = 0$. ∎

Lemma 3.20. *Let $Y \to B$ be a compact $\mathcal{G}$-fiber bundle. Then $K_{\mathcal{G}}(Y) \cong K_{\mathcal{G}}(Y^+, B)$.*

Proof. It follows by writing the long exact sequence (Equation (10)) for the pair (Y^+, B). ∎

Using the above Lemma, we see that the long exact sequence extends of Equation (10) extends to non-compact $\mathcal{G}$-fiber bundles X and Y.

Definition 3.21. Assume $Y \to B$ is a possibly non-compact $\mathcal{G}$-fiber bundle. Let $Y^+ := Y \cup B$ be the fiberwise one-point compactification. Let

$$K_{\mathcal{G}}^{-1}(Y) = \mathrm{Ker}\{K_{\mathcal{G}}^{-1}(Y^+) \to K_{\mathcal{G}}^{-1}(B)\},$$

$$K_{\mathcal{G}}^{n}(Y, Y') = K_{\mathcal{G}}((Y \setminus Y') \times \mathbb{R}^n).$$

It is necessary to verify the compatibility condition $K_{\mathcal{G}}^{-1}(Y) \cong K_{\mathcal{G}}(Y \times \mathbb{R})$. For this purpose, let us consider the bundle $Z := Y \times \mathbb{R}_+$, where $\mathbb{R}_+ = [0, +\infty)$ (c.f., [23, Theorem II.4.8]). Then Z is fiberwise homeomorphic to $Y^+ \times [0,1] \setminus Y^+ \vee [0,1]$, where 1 is the base point of $[0,1]$. Hence Z^+ is fiberwise homeomorphic to the fiberwise quotient $Y_b^+ \times [0,1]/Y_b^+ \vee [0,1]$. Since $Y_b^+ \vee [0,1]$ is invariant, the identification is equivariant. Under this identification, let us define an equivariant fiberwise homotopy $r : Z^+ \times [0,1] \to Z^+$ by the formula $r([y,t],u) = [y, 1+(1-t)u]$, where $y \in Y^+$, $t, u \in [0,1]$, and $[\,,\,]$ means the class in the quotient space. When $u = 0$, the image is B. Hence $K_{\mathcal{G}}(Y \times \mathbb{R}_+) = K_{\mathcal{G}}^{-1}(Y \times \mathbb{R}_+) = 0$. The long exact sequence of the pair $(Y \times \mathbb{R}_+, Y)$

$$K_{\mathcal{G}}^{-1}(Y \times \mathbb{R}_+) \to K_{\mathcal{G}}^{-1}(Y) \to K_{\mathcal{G}}(Y \times \mathbb{R}) \to K_{\mathcal{G}}(Y \times \mathbb{R}_+),$$

proves immediately the required compatibility: $K_{\mathcal{G}}^{-1}(Y) \cong K_{\mathcal{G}}(Y \times \mathbb{R})$. In this proof we have also used the identifications

$$K_{\mathcal{G}}(Y \times \mathbb{R}_+, Y) \cong K_{\mathcal{G}}(Y \times \mathbb{R}_+ \setminus Y) \cong K_{\mathcal{G}}(Y \times \mathbb{R}),$$

in view of Theorem 3.19.

4 The analytic index: a geometric approach

For a family of elliptic operators invariant with respect to a bundle of compact Lie groups $\mathcal{G} \to B$, it is possible to extend the definition of the family index to obtain an index with values in the gauge-equivariant K-theory groups $K_{\mathcal{G}}^0(B)$ introduced in the previous section. In this section we provide an explicit geometric construction of this index when $\mathcal{G}$ has representation theoretic finite holonomy. The general case requires different methods and will be treated in Section 6. We continue to assume that B is compact.

Definition 4.1. A *locally trivial bundle of Hilbert spaces* over B is a fiber bundle $H \to B$ such that B can be covered with open sets U_α with the property that $H|_{U_\alpha} \simeq U_\alpha \times H_0$, for some fixed Hilbert space H_0 and the transition functions are continuous in norm.

Let $\mathcal{G} \to B$ be a bundle of Lie groups. A *locally trivial bundle of $\mathcal{G}$-Hilbert spaces* over B is a *locally trivial bundle of Hilbert spaces* $H \to B$ together with a fiber-preserving action of $\mathcal{G}$ on H that consists of continuous families in any trivialization of H.

It is known that every locally trivial bundle of *infinite dimensional* Hilbert spaces on a finite dimensional base is actually trivial, because the space of unitary operators of an infinite dimensional Hilbert space is contractable (Kuiper's theorem [24]). See also Dixmier's book [15].

We fix in this section a bundle of compact Lie groups $\mathcal{G} \to B$, with representation theoretic finite holonomy and with a B compact, path connected, and locally simply-connected topological space.

Lemma 4.2. *Assume that $\mathcal{G} \to B$ has representation theoretic finite holonomy (as above). Suppose that $H^0 \to B$ and $H^1 \to B$ are two locally trivial bundles of $\mathcal{G}$-Hilbert spaces. Suppose also that $F = (F_b : H_b^0 \to H_b^1)_{b \in B}$ is a family of $\mathcal{G}$-invariant Fredholm operators that is norm-continuous in any trivialization of $H^i \to B$. Then there exists a finite-dimensional $\mathcal{G}$-invariant vector sub-bundle* KER $\subset H^0$ *such that:*

1. $F_b : (\mathrm{KER}_b)^\perp \to F_b((\mathrm{KER}_b)^\perp)$ *is a* $\mathcal{G}_b$*-isomorphism for every* $b \in B$*;*
2. $\mathrm{COK} := \bigcup_{b\in B} (F_b(\mathrm{KER}_b))^\perp \subset H^1$ *is a finite-dimensional* $\mathcal{G}$*-invariant sub-bundle.*

Proof. For any irreducible representation σ of G, let us denote $S_\sigma = \pi_1(B, b_0)\sigma$. Our assumption that $\mathcal{G}$ has representation theoretic finite holonomy is equivalent to saying that all sets S_σ are finite (Proposition 2.2). These sets then partition $\widehat{G}$, the set of irreducible representations of G.

For each $b \in \mathcal{G}_b$ we obtain a subset $S_{\sigma,b} \subset \widehat{\mathcal{G}}_b$, defined as the fiber above b of the space $\mathcal{P} \times_{\mathrm{Aut}(G)} S_\sigma$. Let us identify G with one of the fibers $\mathcal{G}_{b_0}$ of $\mathcal{G} \to B$. In particular, this identifies S_σ with S_{σ,b_0}. Since B is path connected, the set $S_{\sigma,b}$ of irreducible representations of $\mathcal{G}_b$ is the fiber above b of the path connected component of S_σ.

For each $\sigma \in \widehat{G}$, we define then H^i_σ to be the union of the isotypical components corresponding to $S_{\sigma,b}$ in H^i_b. Then H^i_σ is again a bundle of Hilbert spaces. Moreover, F_b will map H^0_σ to H^1_σ, and the resulting map will be an isomorphism except maybe for finitely many irreducible representations σ.

We shall construct the bundle KER as a union of sub-bundles corresponding to each representation $\sigma \in \widehat{G}$, for those $\sigma \in \widehat{G}$ for which $F_b : H^0_\sigma \to H^1_\sigma$ is not already an isomorphism. We can thus fix σ in the following discussion.

If the bundle H^0_σ is consists of finite dimensional vector spaces, then H^1_σ consists also of finite dimensional vector spaces, and hence we can choose $\mathrm{KER} = H^0_\sigma$. Let us assume then that H^0_σ does not consist of finite dimensional vector spaces, then H^1_σ does not consist either of finite dimensional vector spaces. The triviality of the bundle H^0_σ implies then that we can choose an increasing sequence of finite dimensional sub-bundles $L_n \subset H^0_\sigma$ such that their union is dense in each fiber of H^0_σ. Using the fiberwise averaging with respect to $\mathcal{G}_b$, we see that we can assume each L_n to be invariant. We can then take the component of KER corresponding to σ to be L_n, for some large n. □

Definition 4.3. Let $F = (F_b : H^0_b \to H^1_b)_{b\in B}$ be a family of $\mathcal{G}$-invariant Fredholm operators that is norm-continuous in any trivialization of $H^i \to B$, as in Lemma 4.2. The element

$$\mathrm{ind}_{\mathcal{G}}(F) := [\mathrm{KER}] - [\mathrm{COK}] \in K_{\mathcal{G}}(B)$$

is called the *gauge-equivariant index* of the invariant Fredholm family F.

Lemma 4.4. *The gauge-equivariant index* $\mathrm{ind}_{\mathcal{G}} F$ *is well defined, that is, it depends only on* F *and not on the choice of the* $\mathcal{G}$*-equivariant sub-bundle* KER.

Proof. Let us consider two possible choices for the bundle KER of Lemma 4.2 that was used to define the gauge-equivariant index. Denote these two sub-bundles by KER_1 and KER_2 and identify them with the orthogonal the projections onto their ranges. Let P_1 and P_2 be these two projection. The proof of Lemma 4.2 shows that we can find a new sub-bundle KER with associated projection P such that $\|P_1 - P_1P\| < \epsilon$ and $\|P_2 - P_2P\| < \epsilon$, with ϵ as small as we want. It is enough to check then that both KER_1 and KER give rise to the same index.

But P_1 is close to the subprojection $\chi(PP_1P)$ of P obtained by applying the functional analytic calculus to PP_1P, where χ is locally constant, equal to 0 in a neighborhood of 0 and equal to 1 in a neighborhood of 1. Since close projections are homotopic and the index does not change under homotopies, we see that we can actually assume that $P_1 \leq P$ (that is, that P_1 is a subprojection of P). But then F is injective from the range of $P - P_1$, $(P - P_1)H^0$ to $F(P - P_1)H^1$. Moreover, $F(P - P_1)H$ is also a finite-dimensional vector bundle over B. Let COK_1 be the cokernel of F acting on P_1H^0. The above discussion shows that

$$[\mathrm{KER}] - [\mathrm{COK}] = [\mathrm{KER}_1] + [(P - P_1)H^0] - [\mathrm{COK}_1] - [F(P - P_1)H^1] = [\mathrm{KER}_1] - [\mathrm{COK}_1].$$

□

The gauge-equivariant index has the usual properties of the index of elliptic operators.

Lemma 4.5. *Let F and F' be two invariant families as in Lemma 4.2. Assume F consists of Fredholm operators. Then there exists $\epsilon > 0$ such that if $\|F - F'\| < \epsilon$, then F' consists also of Fredholm operators and has the same gauge-equivariant index. In particular, the gauge-equivariant index is homotopy invariant.*

Proof. The family F' is Fredholm by the usual Hilbert space argument which applies since our operators are norm continuous in any trivialization. Moreover, for F' sufficiently close to F, we can choose the same sub-bundle KER to define the gauge-equivariant index of F', while the corresponding COK and COK' will be isomorphic. See also [39]. □

Let us now consider a longitudinally smooth $\mathcal{G}$-fiber bundle $Y \to B$ with a $\mathcal{G}$-invariant complete metric on each fiber of Y. The invariant metrics give rise to Laplace-Beltrami operators Δ_b acting on functions. The Sobolev spaces $H^l(Y)$ are defined as the domains of Δ_b^l (this choice is classical, see [6, 33], for example). This definition extends right away to Sobolev spaces of sections of a $\mathcal{G}$-equivariant vector bundle E equipped with a $\mathcal{G}$-invariant hermitian metric to define a locally trivial bundle of $\mathcal{G}$-Hilbert spaces.

Assume that $F = (D_b)_{b\in B}$ is a $\mathcal{G}$-invariant family of order m operators acting between sections of some $\mathcal{G}$-equivariant vector bundles $E_0, E_1 \to Y$. The lth Sobolev spaces $H^l(Y; E_0)$ of sections of E_0 along the fibers of $Y \to B$ defines a locally trivial bundle of $\mathcal{G}$-Hilbert spaces H_0. Let H_1 be defined similarly using the $(l - m)$th Sobolev spaces of sections of E_1. Then $F : H_0 \to H_1$ is a Fredholm family as in the statement of Lemma 4.2. The homotopy invariance of the gauge-equivariant index shows that the gauge-equivariant index of the family $F = (D_b)_{b\in B}$ depends only on the principal symbol of this family. As in the non-equivariant case, the principal symbol of F gives rise to an element x of $K^0_{\mathcal{G}}(T^*Y_{vert})$, and the index depends only on x. Since every class $x \in K^0_{\mathcal{G}}(T^*Y_{vert})$ arises in this way, we obtain a well defined group morphism

$$a - \mathrm{ind} : K^0_{\mathcal{G}}(T^*Y_{vert}) \to K^0_{\mathcal{G}}(B), \tag{11}$$

which will be called the *analytic index morphism*. A more general definition of this morphism (without finite holonomy conditions will be obtained in Section 6.

The locally trivial bundle of $\mathcal{G}$-Hilbert spaces H_0 defined above will be called the bundle of Sobolev spaces of order l associated to $Y \to B$.

Definition 4.6. A locally trivial bundle of $\mathcal{G}$-Hilbert spaces $\pi : H \to B$ is called *saturated* if, for any $b \in B$ and any $\sigma \in \widehat{\mathcal{G}}_b$, the multiplicity of σ in the Hilbert space $H_b = \pi^{-1}(b)$ is either zero of infinite.

One has the following easy but useful statement.

Lemma 4.7. *Suppose that* $\dim Y > \dim \mathcal{G}$. *Than any bundle of Sobolev spaces* $H^s(Y; E)$ *associated to* $Y \to B$ *is saturated.*

Proof. Let G be the typical fiber of $\mathcal{G} \to B$ and fix $\sigma \in \widehat{G}$. Let $H \to B$ be a bundle of Sobolev spaces associated to $Y \to B$. Then the multiplicity of σ in the fiber H_b is a multiple of the dimension of $L^2(Y_b/\mathcal{G}_b)$. □

The following lemma explains why we are interested in saturated Hilbert bundles.

Lemma 4.8. *Suppose that* $H \to B$ *is a saturated locally trivial bundle of* $\mathcal{G}$*-Hilbert spaces. Let* $E \to B$ *be a* $\mathcal{G}$*-equivariant vector bundle. Assume that for any* $b \in B$ *and any* $\sigma \in \widehat{\mathcal{G}}_b$ *appearing in* E_b*, the multiplicity of* σ *in* H_b *is non-zero. Then* E *is* $\mathcal{G}$*-equivariantly isomorphic to a sub-bundle of* H*.*

Proof. Choose compact subsets $U_\alpha \subset B$ and trivializations

$$\mathcal{G}|_{U_\alpha} \simeq U_\alpha \times G, \quad E|_{U_\alpha} \simeq U_\alpha \times E_0, \quad \text{and } \mathrm{H}|_{\mathrm{U}_\alpha} \simeq \mathrm{U}_\alpha \times \mathrm{H}_0.$$

We can choose the sets U_α such that their interiors cover B.

We can choose then embeddings $J_\alpha : E|_{U_\alpha} \to H|_{U_\alpha}$ inductively such that J_α and J_β have orthogonal ranges above each $b \in U_\alpha \cap U_\beta$, if $\alpha \neq \beta$. Let ϕ_α^2 be a partition of unity subordinated to the interiors of U_α. Then $J = \sum \phi_\alpha J_\alpha$ is the desired embedding. □

For the proof of the next theorem we shall need the following lemma.

Lemma 4.9. *Let* $E_1, E_2 \to B$ *be two* $\mathcal{G}$*-equivariant vector bundles on a path-connected, locally simply-connected topological space* B *such that their classes in gauge equivariant theory coincide (that is,* $[E_1] = [E_2] \in K^0_{\mathcal{G}}(B)$*). Then there exists a* $\mathcal{G}$*-fiber bundle* $E \to B$ *such that*

1. $E_1 \oplus E \cong E_2 \oplus E$*;*
2. *if an irreducible representation* $\sigma \in \widehat{\mathcal{G}}_b$ *appears in* E_b*, then it appears also in* $(E_1)_b$ *and in* $(E_2)_b$*.*

Proof. The existence of E satisfying the first assumption follows from the definition of the group completion of a monoid.

To obtain the second property, we just decompose the vector bundles according to representations σ in the orbits of $\pi_1(B, b_0)$ on $\widehat{G}$. (See the discussion at the end of Section 1.) Then we notice that the dimension of the isotypical subspace $(E_b)_\sigma$ is the same for any $\sigma \in \widehat{\mathcal{G}}_b$ belonging to a fixed connected component of $\widehat{\mathcal{G}}$. □

In the proof of the following theorem, we shall use the completion of the algebra $\psi_{\text{inv}}^{-\infty}(Y;E)$ of order $-\infty$, invariant operators on Y and acting on square integrable sections of E. We shall denote by $C^*(Y;\mathcal{G},E)$ the resulting algebra. It has a natural norm

$$\|T\| = \sup_{b\in B} \|T_b\|_b, \tag{12}$$

where $\| \; \|_b$ is the norm of operators acting on the Hilbert space $L^2(Y_b;E_b)$, $E_b := E|_{Y_b}$. Since all operators in $\psi_{\text{inv}}^{-\infty}(Y;E)$ act as compact operators on $L^2(Y_b;E_b)$, a density and an averaging argument shows that $C^*(Y;\mathcal{G},E)$ identifies with the algebra of continuous families of $\mathcal{G}$-invariant, compact operators acting on the family of Hilbert spaces $L^*(Y_b;E_b)$. If E is a trivial vector bundle, we omit it from the notation. Also, if $Y = \mathcal{G}$, then we shall denote the resulting algebra $C^*(Y;\mathcal{G})$ simply by $C^*(\mathcal{G})$. Recall for the next theorem that the spaces $\psi_{\text{inv}}^m(Y;E,F)$ were defined in Section 1.

The following theorem shows that the $\mathcal{G}$-equivariant index identifies the obstruction to invertibility, as the usual (or Fredholm) index.

Theorem 4.10. *Suppose that* $\dim Y > \dim \mathcal{G}$ *and let* $D \in \psi_{\text{inv}}^m(Y;E,F)$ *be a* $\mathcal{G}$*-equivariant family of elliptic operators acting along the fibers of* $Y \to B$*. Then we can find* $R \in \psi_{\text{inv}}^{m-1}(Y;E,F)$ *such that*

$$D_b + R_b : H^s(Y_b;E_b) \to H^{s-m}(Y_b;F_b)$$

is invertible for all $b \in B$ *if, and only if,* $\operatorname{ind}_{\mathcal{G}}(D) = 0$*. Moreover, if* $\operatorname{ind}_{\mathcal{G}}(D) = 0$*, then we can choose the above* R *in* $\psi_{\text{inv}}^{-\infty}(Y;E,F)$*.*

Proof. Suppose that such a perturbation R exists. Then $\operatorname{ind}_{\mathcal{G}}(D+R) = 0$. Since the index depends only on the principal symbol of D, we obtain that $\operatorname{ind}_{\mathcal{G}}(D) = 0$.

Let $D \in \psi_{\text{inv}}^m(Y;E,F)$ and choose KER and COK as in Lemma 4.2. If $\operatorname{ind}_{\mathcal{G}}(D) = 0$, then, by Lemma 4.9, we can find a $\mathcal{G}$-equivariant vector bundle $E \to B$ such that $\text{KER} \oplus E \cong \text{COK} \oplus E$. We can choose this bundle E such that all irreducible representations $\sigma \in \widehat{\mathcal{G}}$ that appear in (some fiber of) E also appear in (some fiber of) the bundle of Sobolev spaces $H^s(Y;E)$. Lemma 4.8 then shows that we can identify E with a $\mathcal{G}$-equivariant sub-bundle the orthogonal complement of KER. Then, by replacing our original choice of KER with $\text{KER} \oplus E$ and then by replacing COK with the new cokernel space, we can assume that $\text{KER} \cong \text{COK}$.

Let $T : \text{KER} \to \text{COK}$ be a $\mathcal{G}$-equivariant isomorphism of these two $\mathcal{G}$-equivariant vector bundles. Then T is a $\mathcal{G}$-invariant family of compact operators acting on sections of $H^s(Y;E)$ with values sections of $H^{s-m}(Y;F)$. Let P be the orthogonal projection onto KER. Then $D' = D(1-P) + T$ is invertible. Moreover, $R' := D' - D$ is also a $\mathcal{G}$-invariant family of compact operators acting on sections of $H^s(Y;E)$ with values sections of $H^{s-m}(Y;F)$. Since $\psi_{\text{inv}}^{-\infty}(Y;E)$ is dense in $C^*(Y;\mathcal{G},E)$, we can find an operator $R \in \psi_{\text{inv}}^{-\infty}(Y;E)$ that is close enough to R' to ensure that $D + R$ is also invertible in all fibers. □

5 The structure of $\mathcal{G}$-equivariant K-groups

In this section we shall study the structure of the algebra $C^*(\mathcal{G})$ introduced in the previous section as the completion of $\psi_{\mathrm{inv}}^{-\infty}(\mathcal{G})$ in the norm $\| \, \|$ of Equation 12. We shall also relate the gauge-equivariant K-theory groups of $\mathcal{G}$ with the K-theory groups of $C^*(\mathcal{G})$.

For each $b \in B$, denote by μ_b the translation invariant measure on $\mathcal{G}_b$ whose total mass is one. Because the bundle $\mathcal{G} \to B$ is locally trivial, we know that the function $B \ni b \to \mu_b(f) \in \mathbb{C}$ is continuous, for any continuous function f on B. Let us denote by $C(B)$ the space of continuous functions on $\mathcal{G}$ with the fiberwise convolution product and the involution $f^*(g) = \overline{f(g^{-1})}$. The algebra $C(\mathcal{G})$ also acts on each $L^2(\mathcal{G})$ and we can check using the local trivialization of $\mathcal{G}$ that $C(\mathcal{G})$ is dense in $C^*(\mathcal{G})$.

The algebra that we have introduced above is usually denoted $C^*_r(\mathcal{G})$, whereas the notation $C^*(\mathcal{G})$ is reserved for the envelopping C^*-algebra of $C(\mathcal{G})$. It can be shown, but we shall not need this, that in our case $C^*(\mathcal{G}) = C^*_r(\mathcal{G})$, which justifies our notation. (See [26] for the definition of the envelopping C^*-algebra of a groupoid and for related concepts).

Recall that we have denoted by $\mathcal{P}$ a principal $\mathrm{Aut}(G)$-bundle that defines $\mathcal{G}$ in the sense that $\mathcal{G} = \mathcal{P} \times_{\mathrm{Aut}(G)} G$, for some fixed Lie group G. Also, recall that in this paper G was assumed to be compact beginning with Section 3.

If $\mathcal{G}$ (or, equivalently, $\mathcal{P}$) is trivial, then $C^*(\mathcal{G}) \simeq C(B, C^*(G))$, the algebra of continuous functions on B with values in $C^*(G)$. (As usual, we have denoted by $C^*(G)$ the norm completion of the convolution algebra of G actiong on $L^2(G)$.) This leads us to the following construction. Let $C^*_{\mathcal{G}} \to B$ be the locally trivial bundle with fibers $C^*(\mathcal{G}_b)$. It is the fiber bundle associated to $\mathcal{P}$ and its action on $C^*(G)$, that is, $C^*_{\mathcal{G}} \cong \mathcal{P} \times_{\mathrm{Aut}(G)} C^*(G)$ as bundles over B. The local triviality of this bundle allows us to talk about the space of continuous sections of this bundle, which is a complete normed algebra (even a C^*-algebra).

Lemma 5.1. *The algebra $C^*(\mathcal{G})$ identifies naturally with the algebra $\Gamma(C^*_{\mathcal{G}})$ of continuous sections of the bundle $C^*_{\mathcal{G}} \to B$.*

If $\mathcal{G}$ has representation theoretic finite holonomy, then there exist projections $p_n \in C^*(\mathcal{G})$, such that $p_n p_{n+1} = p_n$, $p_n x = x p_n$, for any $x \in C^*(\mathcal{G})$, and $\cup p_n C^*(\mathcal{G})$ is dense in $C^*(\mathcal{G})$.

Proof. The first statement follows from the fact that these two algebras are the completions of the same algebra $\psi_{\mathrm{inv}}^{-\infty}(\mathcal{G})$ with respect to the same norm.

Assume first that B is path-connected. We shall use the notation and the constructions introduced at the end of Section 1. In particular, $H_{\mathcal{G}}$ is the image of the holonomy morphism $\pi_1(B, b_0) \to H_R := \mathrm{Aut}(G)/\mathrm{Aut}_0(G)$. Let H be the inverse image of $H_{\mathcal{G}}$ in $\mathrm{Aut}(G)$. Then we can reduce the structure group of $\mathcal{P}$ to H. Choose sets $S_n \subset \widehat{G}$ such that $S_n \subset S_{n+1}$, each S_n is $\pi_1(B, b_0)$-invariant and $\cup S_n = \widehat{G}$. Also, let q_n be the central projection of $C^*(G)$ corresponding to S_n. By construction, q_n is invariant for H, and hence it gives rise to a section p_n of $C^*_{\mathcal{G}}$, which is the desired projection.

In general, we use an exhaustion of $\widehat{\mathcal{G}}$ by compact-open subsets. $\square$

Let A be a (possibly non-unital) algebra. By a *finitely-generated, projective module* over A we shall understand a left A-module of the form $A^N e$, where $e \in M_N(A)$ is a projection (that is, $e^2 = e$). All modules over non-commutative algebras used below will be assumed to be left-modules, unless otherwise mentioned.

Theorem 5.2. *Assume that the bundle of compact Lie groups $\mathcal{G} \to B$ has representation theoretic finite holonomy and that B is compact. Then there is a natural equivalence of categories between the category of locally trivial $\mathcal{G}$-equivariant vector bundles over B and the category of finitely-generated, projective modules over $C^*(\mathcal{G})$. In particular,*

$$K^*_{\mathcal{G}}(B) \cong K_*(C^*(\mathcal{G})).$$

Proof. If $E \to B$ is a $\mathcal{G}$-equivariant vector bundle, we can endow E with a $\mathcal{G}$-invariant metric (Corollary 3.14) and hence we obtain that $C(\mathcal{G})$ acts on $\Gamma(E)$. Using the local triviality of $\mathcal{G}$ and the metric on E, we see that we can extend this action of $C(\mathcal{G})$ to an action of $C^*(\mathcal{G})$. Thus $\Gamma(E)$ is a $C^*(\mathcal{G})$-module. We shall prove that if is projective and that $E \to \Gamma(E)$ is an equivalence of categories.

By looking at the representations of $\mathcal{G}$ that appear in the fibers of $E \to B$, we see that by choosing n large enough we can assume that p_n acts as the identity on $\Gamma(E)$. Then there exists a surjective map $(C^*(\mathcal{G})p_n)^N \to \Gamma(E)$ of $C^*(\mathcal{G})$-modules. By regarding $\cup_b (C^*(\mathcal{G}_b)p_n)^N$ as a $\mathcal{G}$-equivariant vector bundle, we see that E has a direct summand in it, which implies then that the $C^*(\mathcal{G})$-module $\Gamma(E)$ is a direct summand of $(C^*(\mathcal{G})p_n)^N$, and hence it is of the form $C^*(\mathcal{G})^N e$, for some projection $e \in M_N(p_n C^*(\mathcal{G})p_n)$.

Conversely, let us assume that M is a finitely-generated, projective $C^*(\mathcal{G})$-module, that is, that $M \cong C^*(\mathcal{G})^N e$, for some projection $e \in M_N(C^*(\mathcal{G}))$. Then the space of continuous sections of $\mathcal{G}$ will also act on M. Using the local triviality of $\mathcal{G} \to B$, we then see that there exists n such that $p_n e = e$. Then $M = (p_n C^*(\mathcal{G}))^N e$ is also a projective module over the unital algebra $p_n C^*(\mathcal{G})p_n = p_n C^*(\mathcal{G})$, which contains $C(B)$ in its center. Since $p_n C^*(\mathcal{G})$ is a projective $C(B)$-module, M will also be a projective $C(B)$-module, and hence it can be identified with the space of sections of a vector bundle $E \to B$. Then the action of $C^*(\mathcal{G})$ on E is fiberwise and, by restricting to $\mathcal{G}$, we obtain that E is also a $\mathcal{G}$-equivariant vector bundle. Thus $M \cong \Gamma(E)$ as $C^*(\mathcal{G})$-modules. □

We remark that the first part of the above theorem remains true even without the representation theoretic finite holonomy condition, but the proof has to be slightly modified. The second part, that is the isomorphism of the K-theory groups, is not true, in general, without the representation theoretic finite holonomy condition.

We now take a closer look at the structure of the algebra $C^*(\mathcal{G})$. Let us denote by $(\widehat{\mathcal{G}})_d$ the space of irreducible representations of dimension d of the groups $\mathcal{G}_b$. By the local triviality of $\mathcal{G} \to B$, $(\widehat{\mathcal{G}})_d$ is open and closed in $\widehat{\mathcal{G}}$ and is a covering space of B. Let f be a continuous function on $\mathcal{G}$. Then the function

$$(\widehat{\mathcal{G}})_d \ni \sigma \to f_{TR} := Tr(\sigma(f)) \in \mathbb{C}$$

is a continuous function on $(\widehat{\mathcal{G}})_d$ (this also follows from the local triviality of $\mathcal{G} \to B$). Moreover,

$$|Tr(\sigma(f - g))| \leq (\dim \sigma)\, \|f - g\|,$$

so if $f_n \in C(\mathcal{G})$ converges in $C^*(\mathcal{G})$, the functions $(f_n)_{TR}$ converge uniformly on each of the sets $(\widehat{\mathcal{G}})_d$, which shows that the definition of f_{TR} can be extended to $f \in C^*(\mathcal{G})$ by continuity, and the result will still be continuous on $\widehat{\mathcal{G}}$.

Let us denote by $PGL(d, \mathbb{C}) := GL(d, \mathbb{C})/Z(GL(d, \mathbb{C}))$ the group of automorphisms of the algebra $M_d(\mathbb{C})$. If $\mathcal{A} \to X$ is a bundle of algebras with structure group $PGL(d, \mathbb{C})$, then every fiber $\mathcal{A}_x \cong M_d(\mathbb{C})$ will have a unique C^*-norm denoted $\| \; \|_x$, for any $x \in X$. (Recall that a norm $\| \; \|$ on a $*$-algebra A is called a C^*-norm if it is a complete Banach algebra norm and $\|x^*x\| = \|x\|^2$ for any $x \in A$. A normed $*$-algebra is called a *C^*-algebra* if its norm is a C^*-norm.) We shall denote by $\Gamma_0(\mathcal{A})$ the space of continuous sections ξ of $\mathcal{A}$ such that for any $\epsilon > 0$, the set $\{x, \|\xi\|_x \geq \epsilon\}$ is a compact subset of $\widehat{\mathcal{G}}$. Then $\Gamma_0(\mathcal{A})$ is complete in the norm $\|\xi\| = \sup_x \|\xi(\sigma)\|_x$ and is a C^*-algebra.

Theorem 5.3. *Let $\mathcal{G} \to B$ be a bundle of compact Lie groups. Then there exists on each $(\widehat{\mathcal{G}})_d$ a locally trivial bundle of algebras $\mathcal{A}_d$ with fiber $M_d(\mathbb{C})$ and structure group $PGL(d, \mathbb{C}) := GL(d, \mathbb{C})/Z(GL(d, \mathbb{C}))$ such that the space $\Gamma_0(\mathcal{A}_d)$ identifies with a direct summand of $C^*(\mathcal{G})$ and*

$$C^*(\mathcal{G}) \cong \Gamma_0(\mathcal{A}_d).$$

In particular, $K_i(C^*(\mathcal{G})) \cong \oplus K_i(\Gamma_0(\mathcal{A}_d))$ and the primitive ideal spectrum of $C^*(\mathcal{G})$ is homeomorphic to $\widehat{\mathcal{G}}$, which in turn is homeomorphic to the disjoint union of the sets $(\widehat{\mathcal{G}})_d$.

Proof. The first part of the result follows from the fact that the pointwise trace $f \to f_{TR}$ defined above is continuous for any $f \in C^*(\mathcal{G})$. The second part is also a general property of continuous trace C^*-algebras. (See [15] and the references therein.) □

We thus obtain a description of $K_i(C^*(\mathcal{G}))$ in terms of twisted K-theory. (A *twisted K-theory group* of a space X is the K-theory of a bundle of matrix algebras or compact operators over X. See [10, 35, 18] and the references therein for more information on the subject.)

The above theorem has several consequences.

Proposition 5.4. *Suppose that all the fibers of $\mathcal{G} \to B$ are compact abelian Lie groups. Then*

$$K_*(C^*(\mathcal{G})) \cong \oplus K^*(\widehat{\mathcal{G}}).$$

Proof. If the fibers of $\mathcal{G} \to B$ are abelian groups, then $C^*(\mathcal{G})$ is also abelian and its primitive ideal spectrum is $\widehat{\mathcal{G}}$. □

We shall assume until the end of this section that B is a path-connected, locally simply-connected space. (Also, recall that beginning with Section 3, B is assumed to be compact.)

Let $\mathcal{P}$ be the principal $\mathrm{Aut}(G)$ bundle defining $\mathcal{G} \to B$. We shall identify G with one of the fibers of $\mathcal{G} \to B$. Fix $\sigma \in \widehat{G}$, we shall denote then by B_σ the connected component of $\widehat{\mathcal{G}}$ containing σ. Recall that $\pi_1(B, b_0)$ acts on $\widehat{G}$. Then $B_\sigma \to B$ is the covering space associated to the isotropy of σ in $\pi_1(B, b_0)$, more precisely, if $\tilde{B} \to B$ is a universal covering space of B, then the covering $B_\sigma \to B$ is equivalent to the covering $\tilde{B} \times_{\pi_1(B,b_0)} (\pi_1(B, b_0)\sigma) \to B$.

The space B_σ is contained in the space $(\widehat{\mathcal{G}})_d$, with $d = \dim \sigma$. We shall denote by $\mathcal{A}_\sigma$ the bundle of finite dimensional algebras obtained by restricting $\mathcal{A}_d$ to B_σ. We then have the following corollary.

Corollary 5.5. *Assume $\mathcal{G} \to B$ is a bundle of compact Lie groups over a path-connected, locally simply-connected space. Then we have a canonical isomorphism*

$$C^*(\mathcal{G}) \cong \oplus_\sigma \Gamma_0(\mathcal{A}_\sigma),$$

where σ ranges through a set of representatives of the orbits of $\pi_1(B, b_0)$ on $\widehat{G}$ and $\mathcal{A}_\sigma \to B_\sigma$ are bundles of algebras with fiber $M_d(\mathbb{C})$, $d = \dim \sigma$, obtained as the restriction of $\mathcal{A}$ to B_σ, as above.

For the following result, we shall need the following lemma.

Lemma 5.6. *Let $\mathcal{A} \to X$ be a locally trivial bundle of finite dimensional algebras with fiber $M_d(\mathbb{C})$. Then $K_i(\Gamma_0(\mathcal{A})) \otimes \mathbb{Q} \cong K^i(X) \otimes \mathbb{Q}$.*

Proof. Let $A = \Gamma_0(\mathcal{A})$. Endow each fiber of $\mathcal{A} \to X$ with the inner product $(T, T_1) := Tr(T_1^* T)$, with T and T_1 in the fiber. Then $\mathcal{A} \to X$ is a Hermitian vector bundle. Let $B = \Gamma_0(\mathrm{End}(\mathcal{A}))$, which will be then a C^*-algebra Morita equivalent to $C_0(X)$. In particular, $K_i(C_0(X)) \cong K_i(\Gamma_0(\mathrm{End}(\mathcal{A})))$, isomorphism that we shall denote by j in what follows. Note also that we have a natural morphism $j_0 : C_0(X) \to B$, which sends a function f to the operator of multiplication with that function f. Let $[\mathcal{A}]$ be the class in $K^0(X) = K_0(C_0(X))$ of the vector bundle $\mathcal{A} \to X$. Then, at the level of K_0-groups, we have $(j_0)_*(\xi) = j_*(\xi) \otimes [\mathcal{A}]$.

The center of A is isomorphic to $C_0(X)$, which gives rise to an algebra morphism $\phi : C_0(X) \to A$. Left multiplication with the elements of A gives rise to a second algebra morphism $\psi : A \to B$. The composite morphisms $(j \circ \psi_*) \circ \phi_* : K_0(C_0(X)) \to K_0(B) \simeq K_0(C_0(X))$ is then multiplication by $[\mathcal{A}]$.

Assume $\mathcal{A}$ is a trivial bundle of rank d^2. Then $(j \circ \psi_*) \circ \phi_* : K_0(C_0(X)) \to K_0(B) \simeq K_0(C_0(X))$ is multiplication by d^2. Similarly, $\phi_* \circ (j \circ \psi_*) : K_0(A) \to K_0(A)$ is also multiplication by d^2. This proves the rational isomorphism for trivial bundles $\mathcal{A}$.

In general, we use the fact that the maps ϕ_* and ψ_* are natural and a Meyer-Vietoris argument [9], to complete the proof. □

The above example is not true if we do not include rational coefficients, as implied by the following example due to M. Dadarlat.

Example 5.7. Let X be the mapping cone of the map $z \mapsto z^n$ of the circle. Then X is a two dimensional CW complex with

$$K_0(C(X)) = \mathbb{Z} \oplus \mathbb{Z}/n, \ K_1(C(X)) = 0.$$

Using the homotopy exact sequence

$$\mathbb{T} \to U(n) \to PU(n) \to B\mathbb{T} \to BU(n),$$

we see that any $C(X)$-linear automorphism of

$$A = C(X) \otimes M_n(\mathbb{C})$$

that is given by a map $X \to PU(n)$ is determined up to unitary equivalence by a line bundle E over X with $E \oplus \cdots \oplus E$ (n-times) trivial.

One also can verify that the map $\alpha_* : K_0(A) = K^0(X) \to K_0(A) = K^0(X)$ is induced by multiplication (in the ring $K^0(X)$) with the K_0-class $[E]$. Define the mapping cylinder M_α of α:

$$M_\alpha = \{f : [0,1] \to A : f(1) = \alpha(f(0))\}.$$

Then M_α is the C^*-algebra of sections of a bundle of $n \times n$-matrices with spectrum $Y = \mathbb{T} \times X$ and center $C(Y)$. The exact sequence

$$0 \to SA \to M_\alpha \to A \to 0$$

induces a six-term exact sequence

$$\begin{array}{ccccc} 0 = K_1(A) & \longrightarrow & K_0(M_\alpha) & \longrightarrow & K_0(A) = \mathbb{Z} \oplus \mathbb{Z}/n \\ {\scriptstyle 1-\alpha_*}\big\uparrow & & & & \big\downarrow{\scriptstyle 1-\alpha_*} \\ 0 = K_1(A) & \longleftarrow & K_1(M_\alpha) & \longleftarrow & K_0(A) = \mathbb{Z} \oplus \mathbb{Z}/n \end{array}$$

Choose E such that its class in $K^0(X) \cong \mathbb{Z} \oplus \mathbb{Z}/n$ is equal to $(1, -1)$ and set $n = 4$. Then $K_0(M_\alpha) = \mathbb{Z} \oplus \mathbb{Z}/2$ so that $K_0(M_\alpha) \neq K_0(C(Y)) = K_0(C(\mathbb{T} \times X)) = \mathbb{Z} \oplus \mathbb{Z}/4$. This completes the example of a bundle of algebras whose K-theory is not isomorphic to that of its center.

The decomposition of Corollary 5.5 leads to the following determination of the K-theory groups of the algebras $C^*(\mathcal{G})$, up to rational isomorphism.

Theorem 5.8. *Suppose B is a path-connected, locally simply-connected space and let $\mathcal{G} \to B$ be a bundle of compact Lie groups with representation theoretic finite holonomy. Then*

$$K_i(C^*(\mathcal{G})) \otimes \mathbb{Q} \simeq K^i(\widehat{\mathcal{G}}) \otimes \mathbb{Q} \simeq K^i(B) \otimes R(G)^{\pi_1(B,b_0)} \otimes \mathbb{Q}.$$

Proof. Let G be the typical fiber of $\mathcal{G} \to B$ and choose a set of representatives $S \subset \widehat{G}$ for the orbits of $\pi_1(B, b_0)$ on $\widehat{G}$. We shall use the results and the notation of Corollary 5.5.

We have then

$$K_i(C^*(\mathcal{G})) \otimes \mathbb{Q} \simeq \oplus_{\sigma \in S} K_i(\Gamma_0(\mathcal{A}_\sigma)) \otimes \mathbb{Q} \simeq \oplus_{\sigma \in S} K^i(B_\sigma) \otimes \mathbb{Q}.$$

But $K^i(B_\sigma) \otimes \mathbb{Q} \simeq K^i(B) \otimes \mathbb{Q}$, because $B_\sigma \to B$ is a finite covering. Thus $K_i(C^*(\mathcal{G})) \otimes \mathbb{Q} \simeq K^i(B) \otimes \mathbb{Q}^S$, where $\mathbb{Q}^{(S)}$ is the vector space with basis S. Moreover, $\mathbb{Q}^{(S)}$ identifies with $R(G)^{\pi_1(B,b_0)} \otimes \mathbb{Q}$, because $R(G) = \mathbb{Z}^{(\widehat{G})}$ (that is, the free abelian group with basis $\widehat{G}$). $\square$

Recall that the conditions of the above theorem are automatically satisfied if the typical fiber G of $\mathcal{G} \to B$ is a semi-simple Lie group or if G is the product of a semisimple Lie group by the one-dimensional torus S^1.

Let us now take a closer look at the algebras $A_\sigma = \Gamma_0(\mathcal{A}_\sigma)$ used above. Each of these algebras is the algebra of sections of the field $\mathcal{A}_\sigma$ of finite dimensional matrix algebras over B_σ. Let us denote the dimension of these fibers by d_σ^2, for any fixed σ, as above. In particular, $d_\sigma = \dim V_\sigma$. Then the bundle $\mathcal{A}_\sigma$ is a bundle with structure group

$$PGL(d_\sigma, \mathbb{C}) := GL(d_\sigma, \mathbb{C})/Z(GL(d_\sigma, \mathbb{C})) = SL(d_\sigma, \mathbb{C})/C_{d_\sigma} =: PSL(d_\sigma, \mathbb{C}), \qquad (13)$$

where by C_m we denote the cyclic group with m elements. The bundle $\mathcal{A}_\sigma$ is hence classified by a 1-cocycle in $H^1(B_\sigma, PSL(d_\sigma, \mathbb{C}))$. The connecting morphism

$$H^1(B_\sigma, PSL(d_\sigma, \mathbb{C})) \to H^2(B_\sigma, C_{d_\pi})$$

in non-abelian cohomology then gives rise to an element $\chi_\sigma \in H^2(B_\sigma, C_{d_\pi})$ (see [15, 16]). By definition, χ_σ is the Dixmier-Duady invariant of $\mathcal{A}_\sigma$ (see [15, 16]). We want to analyse these invariants for the fields of matrix algebras $\mathcal{A}_\sigma \to B_\sigma$ introduced above before the statement of Corollary 5.5.

Assume now that G, the typical fiber of $\mathcal{G}$, is connected. We shall use the notation used at the end of Section 1. In particular, $\mathcal{P}$ is the principal $\mathrm{Aut}(G)$-bundle defining $\mathcal{G}$ and

$$\pi_1(B, b_0) \to H_R = \mathrm{Aut}(G)/\,\mathrm{Aut}_0(G)$$

is the holonomy morphism defining the principal H_R-bundle $\mathcal{P}_0 := \mathcal{P}/\,\mathrm{Aut}_0(G)$. Recall that, in our case (G connected), $\mathcal{G} \to B$ has representation theoretic finite holonomy if, and only if, the connected components of $\widehat{\mathcal{G}}$ are compact. We then know by Theorem 2.4 that these conditions are also equivalent to the condition that the range $H_{\mathcal{G}}$ of the holonomy morphism be finite. We identify G with a fiber of $\mathcal{G} \to B$ and let B_σ be the connected component in $\widehat{\mathcal{G}}$ of $\sigma \in \widehat{G}$, as above. Let $\tilde{B} \to B$ be a universal covering space of B and let $B' := \tilde{B} \times_{\pi_1(B,b_0)} H_R$. Then for any σ we obtain a map

$$f_\sigma : B' \to B_\sigma = \tilde{B} \times_{\pi_1(B,b_0)} H_R\sigma. \qquad (14)$$

Let us lift $\mathcal{P}$ to an $\mathrm{Aut}(G)$-principal bundle on B'. Then, by construction, the resulting bundle reduces to a principal $\mathrm{Aut}_0(G) = G_{int} = G/Z(G)$ bundle classified by a one-cocycle in $H^1(B', G_{int})$ (we are using here Corollary 2.5). Let us denote by $\chi \in H^2(B', Z(G))$ the image of this cocycle under the connecting morphism in non-abelian cohomology associated to the exact sequence of groups

$$1 \to Z(G) \to G \to G/Z(G) \to 1.$$

Let G' be the derived group of G (G' is connected because we assumed G to be connected). Denote

$$Z' = G' \cap Z(G), \qquad (15)$$

a finite abelian group. Because G' maps onto $G/Z(G)$, the obstruction χ comes from a canonical element

$$\chi' \in H^2(B', Z'). \tag{16}$$

Let $\sigma : G \to GL(V_\sigma)$ be an irreducible representation of G and $d_\sigma = \dim V_\sigma$, as before. Then

$$\sigma(Z') \subset C_{d_\sigma} = Z(GL(\mathbb{C}, d_\sigma)) \cap SL(\mathbb{C}, d_\sigma),$$

which induces a morphism

$$\sigma_* : H^2(B', Z') \to H^2(B', C_{d_\sigma}) \simeq H^2(B', \mathbb{Z}/d_\sigma\mathbb{Z}).$$

From the above constructions, we obtain the following theorem.

Theorem 5.9. *Let $\mathcal{G} \to B$ be a bundle of compact, connected Lie groups on a compact, connected smooth manifold B. Let $f_\sigma : B' \to B$ and $Z' = G' \cap Z(G)$ be as before (Equations (14) and (15)). The obstructions $\chi' \in H^2(B', Z')$ of Equation (16) and the Dixmier-Douady invariant $\chi_\sigma \in H^2(B_\sigma, \mathbb{Z}/d_\sigma\mathbb{Z})$ are related by $f_\sigma^*(\chi_\sigma) = \sigma_*(\chi')$.*

Proof. This follows from the definitions of the obstructions χ' and χ_σ, from the fact that the morphism $G \to GL(V_\sigma)$ maps $Z' \subset C_{d_\sigma}$, and from the naturality of the boundary map in non-abelian cohomology. □

The Dixmier-Douady invariant was recently been shown to be relevant in the study of Ramond-Ramond fields (see [18, 10, 41, 42] and the references therein). On the other hand, the algebra $C^*(\mathcal{G})$ is naturally a direct sum of algebras with controlled Dixmier-Douady invariant. This suggests then the question whether Ramond-Ramond fields can be obtained as indices of operators invariant with respect to a bundle of compact Lie groups $\mathcal{G} \to B$. The Dixmier-Douady invariant of the resulting fields of algebras can be determined in terms of a unique obstruction defined in terms of $\mathcal{G}$, at least in the case when the holonomy map $\pi_1(B, b_0) \to \mathrm{Aut}(G)/\mathrm{Aut}_0(G)$ is trivial.

6 The analytic index: an algebraic approach

We shall use below $\widehat{\otimes}$, the minimal tensor product of C^*-algebras. Recall that the minimal tensor product of C^*-algebras is defined to be (isomorphic to) the completion of the image of $\pi_1 \otimes \pi_2$, the tensor product of two injective representations π_1 and π_2. The same definition applies to the tensor products $\widehat{\otimes}_C$ over a central subalgebra C [37].

Lemma 6.1. *Assume $\dim Y > \dim \mathcal{G}$. Also, let $\mathcal{K} \to B$ denote the locally trivial bundle of algebras whose fiber at $b \in B$ is the algebra $\mathcal{K}(Y_b)$ of compact operators on $L^2(Y_b)$. Define $C^*(\mathcal{G})\widehat{\otimes}\mathcal{K} = C^*(\mathcal{G})\widehat{\otimes}_{C(B)}\Gamma(\mathcal{K})$. Then $C^*(Y, \mathcal{G})$ is Morita-equivalent to a direct summand of of $C^*(\mathcal{G})\widehat{\otimes}\mathcal{K}$. Consequently, there is a natural map*

$$K_i(C^*(Y, \mathcal{G})) \to K_i(C^*(\mathcal{G})\widehat{\otimes}\mathcal{K}) \simeq K_i(C^*(\mathcal{G})).$$

Proof. We shall regard $C^*(\mathcal{G})\widehat{\otimes}\mathcal{K}$ as an algebra of operators acting on functions on $Y\times_B\mathcal{G}$. Let $\pi(g)$ be the action of some $g\in\mathcal{G}_b$ on $Y_b\times\mathcal{G}_b$. Denote by $p_b=\int_{\mathcal{G}_b}\pi(g)dg$, the integral being with respect to the normalized Haar measure. Then $p_b^2=p_b$ is a self-adjoint projection in the algebra $\mathcal{B}(L^2(Y_b))\otimes C^*(\mathcal{G}_b)$. Let $p=(p_b)$. Then pointwise multiplication by p_b defines a multiplier of $C^*(\mathcal{G})\widehat{\otimes}\mathcal{K}$ (that is, it maps $C^*(\mathcal{G})\widehat{\otimes}\mathcal{K}$ to itself by left or right multiplication).

By a standard argument [34], the algebra $p(C^*(\mathcal{G})\widehat{\otimes}\mathcal{K})p$ is isomorphic to $C^*(Y,\mathcal{G})$. It is then known that $p(C^*(\mathcal{G})\widehat{\otimes}\mathcal{K})p$ is Morita equivalent to $(C^*(\mathcal{G})\widehat{\otimes}\mathcal{K})p(C^*(\mathcal{G})\widehat{\otimes}\mathcal{K})$ [9]. This completes the proof. $\square$

We proceed now to define the index of an *elliptic, invariant family* of operators

$$D=(D_b)\in M_N(\psi^m_{\mathrm{inv}}(Y))=\psi^m_{\mathrm{inv}}(Y;\mathbb{C}^N),$$

without any holonomy assumption on $\mathcal{G}\to B$. We assume that Y is compact, for simplicity; otherwise, we need to use algebras with adjoint units. First, we observe that there exists an exact sequence

$$0\to C^*(Y,\mathcal{G})\to\mathcal{E}\to\mathcal{C}^\infty(S^*_{vt}Y)\to 0,\quad \mathcal{E}:=\psi^0_{\mathrm{inv}}(Y)+C^*(Y,\mathcal{G}). \tag{17}$$

The operator D (or, rather, the family of operators $D=(D_b)$) has an invertible principal symbol, and hence the family $T=(T_b)$,

$$T_b:=(1+D_b^*D_b)^{-1/2}D_b,$$

consists of elliptic, invariant operators, because the algebra of pseudodifferential operators on a compact manifold is closed under holomorphic functional calculus. Moreover, $T\in\mathcal{E}=\psi^0_{\mathrm{inv}}(Y)+C^*(Y,\mathcal{G})$. It's principal symbol is still invertible, and hence defines a class $[T]\in K_1(\mathcal{C}^\infty(S^*_{vt}Y))\simeq K^1(S^*_{vt}Y)$. Let

$$\partial: K_1^{alg}(S^*_{vt}Y))\to K_0^{alg}(C^*(Y,\mathcal{G}))\simeq K_0(C^*(Y,\mathcal{G})) \tag{18}$$

be the boundary map in the K-theory exact sequence

$$\begin{aligned}K_1^{alg}(C^*(Y,\mathcal{G}))\to K_1^{alg}(\mathcal{E})\to K_1^{alg}(S^*_{vt}Y))&\\ \xrightarrow{\partial} K_0^{alg}(C^*(Y,\mathcal{G}))\to K_0^{alg}(\mathcal{E})\to K_0^{alg}(S^*_{vt}Y))&\end{aligned}$$

associated to the exact sequence (17). We hence obtain a group morphism

$$\partial: K_1^{\mathrm{alg}}(\mathcal{C}^\infty(S^*_{vt}Y))\to K_0(C^*(Y,\mathcal{G})).$$

By combining ∂ with the canonical morphism $K_0(C^*(Y,\mathcal{G}))\to K_0(C^*(\mathcal{G}))$, we obtain our desired morphism,

$$\mathrm{ind}_a: K_1^{\mathrm{alg}}(\mathcal{C}^\infty(S^*_{vt}Y))\to K_0(C^*(\mathcal{G})), \tag{19}$$

which we shall call *the analytic index morphism.* The image of D under the composition of the above maps, that is $\mathrm{ind}_a([T])$, will be denoted $\mathrm{ind}_a(D)$ and will be called *the analytic index of D.* The analytic index morphism descends to a group morphism $K^1(S^*_{vt}Y) \to K_0(C^*(\mathcal{G}))$ denoted in the same way. For T acting between not necessarily equal vector bundles, we can proceed similarly by using bivariant K-theory [12, 38] to define a morphism

$$\mathrm{ind}_a : K^0(T^*_{vt}Y) \to K_0(C^*(\mathcal{G})). \tag{20}$$

We still need to prove that the two definitions of the analytic index coincide.

Theorem 6.2. *Let $\mathcal{G} \to B$ be a bundle of Lie groups with representation theoretic finite holonomy. Then the morphisms $a - \mathrm{ind}$ and ind_a defined in Equations* (11) *and* (20) *are equal:*

$$a - \mathrm{ind} = \mathrm{ind}_a : K^0(T^*_{vt}Y) \to K_0(C^*(\mathcal{G})).$$

Proof. Let $F = (D_b)$ be a family in $\psi^m_{\mathrm{inv}}(Y; E_0, E_1)$. Choose KER as in our first definition of the analytic index. Let P be the orthogonal projection onto the range of KER. By a small perturbation, we can arrange that $P \in \psi^{-\infty}_{\mathrm{inv}}(Y; \mathbb{C}^N)$. We can replace then F by $F(1 - P)$. A standard calculation (see [9] for example) then shows that $\partial[F] = [\mathrm{KER}] - [\mathrm{COK}]$. □

7 Appendix

Let us recall some general constructions of K-theory from [23] and [22]. First we need some definitions.

Definition 7.1. [23, I.6.7] An additive category $\mathcal{C}$ is called *pseudo-Abelian*, if for each object E from $\mathcal{C}$ and each morphism $p : E \to E$, satisfying to a condition $p^2 = p$ (i. e. an idempotent) there exists the kernel $\mathrm{Ker}\, p$. For an arbitrary additive category $\mathcal{C}$ there exists *associated pseudo-Abelian category* $\widetilde{\mathcal{C}}$ which is a solution of the corresponding universal problem and is defined as follows [23, I.6.10]. Objects of $\mathcal{C}$ are pairs (E, p), where $E \in \mathcal{O}b(\mathcal{C})$ and p is a projector in E. A morphism from (E, p) to (F, q) is such a morphism $f : E \to F$ in $\mathcal{C}$, that $f \circ p = q \circ f = f$.

Definition 7.2. [23, § II.1] Let M be an abelian monoid. On the product $M \times M$ we consider the equivalence relation

$$(m, n) \sim (m', n') \ \Leftrightarrow \ \exists\, p, q : \ (m, n) + (p, p) = (m', n') + (q, q).$$

Let $S(M)$ be the quotient of $M \times M$ by the above equivalence relation. Then $S(M)$ is a group, called the *group completion*. If we consider now an additive category $\mathcal{C}$ and denote by $\dot{E}$ the isomorphism class of an object E from $\mathcal{C}$, then the set $\Phi(\mathcal{C})$ of these classes is equipped with a structure of an Abelian monoid with respect to operation $\dot{E} + \dot{F} = (E \oplus F)^\bullet$. In this case the group $S(\Phi(\mathcal{C}))$ is denoted by $K(\mathcal{C})$ and is called *Grothendieck group* of the category $\mathcal{C}$.

Definition 7.3. [23, § II.2] A *Banach structure* on an additive category $\mathcal{C}$ is defined by a Banach space structure on all groups $\mathcal{C}(E,F)$, where E and F are arbitrary objects from $\mathcal{C}$ such that the composition of morphisms $\mathcal{C}(E,F)\times\mathcal{C}(F,G)\to\mathcal{C}(E,G)$ is bilinear and continuous. We also say that $\mathcal{C}$ is a *Banach category*.

Definition 7.4. [23, § I.6] Suppose that $\mathcal{C}$ is an additive category. The category $\mathcal{C}$ is called *pseudo-abelian,* if, for each object E of $\mathcal{C}$ and each morphism $p:E\to E$ satisfying the condition $p^2=p$, there exists the kernel of p.

Definition 7.5. [23, § II.2] Let $\mathcal{C}$ and $\mathcal{C}'$ be additive categories. An additive functor $\varphi:\mathcal{C}\to\mathcal{C}'$ is called *quasi-surjective* if each object of $\mathcal{C}'$ is a direct summand of an object of type $\varphi(E)$. A functor φ called *full* if, for any E, $F\in\mathcal{O}b(\mathcal{C})$, the map $\varphi(E,F):\mathcal{C}(E,F)\to\mathcal{C}'(\varphi(E),\varphi(F))$ is surjective. For Banach categories φ is called *Banach* if this map $\varphi(E,F)$ linear and continuous.

Definition 7.6. [23, II.2.13] Let $\varphi:\mathcal{C}\to\mathcal{C}'$ be a quasi-surjective Banach functor. We shall denote by $\Gamma(\varphi)$ the set consisting of triples of the form (E,F,α), where E and F are objects of the category $\mathcal{C}$ and $\alpha:\varphi(E)\to\varphi(F)$ is an isomorphism. The triples (E,F,α) and (E',F',α') are called *isomorphic*, if there are isomorphisms $f:E\to E'$ and $g:F\to F'$ such that the diagram

$$\begin{tikzcd}
\varphi(E) \arrow[r, "\alpha"] \arrow[d, "\varphi(f)"'] & \varphi(F) \arrow[d, "\varphi(g)"] \\
\varphi(E') \arrow[r, "\alpha'"] & \varphi(F')
\end{tikzcd}$$

commutes. A triple (E,F,α) is *elementary* if $E=F$ and isomorphism α is homotopic in the set of automorphisms of $\varphi(E)$ to the identical isomorphism $\mathrm{Id}_{\varphi(E)}$. We define *sum* of two triples (E,F,α) and (E',F',α') as

$$(E\oplus E',F\oplus F',\alpha\oplus\alpha').$$

The Grothendieck group $K(\varphi)$ of a functor φ is defined as quotient set of the monoid $\Gamma(\varphi)$ with respect to the following equivalence relation: $\sigma\sim\sigma'$ if and only if there exist elementary triples τ and τ', that the triple $\sigma+\tau$ is isomorphic to the triple $\sigma'+\tau'$. The operation of addition introduces on $K(\varphi)$ a structure of Abelian group. The class of a triple we shall denote by $d(E,F,\alpha)$.

Proposition 7.7. [23, II.2.28] Let $d(E,F,\alpha)$ be an element of $K(\varphi)$ where $\varphi:\mathcal{C}\to\mathcal{C}'$ is a full quasi-surjective Banach functor. Then $d(E,F,\alpha)=0$ if, and only if, there exists an object M of $\mathcal{C}$ and an isomorphism $\beta:E\oplus M\to F\oplus M$ such that $\varphi(\beta)=\alpha\oplus\mathrm{Id}_{\varphi(M)}$.

Definition 7.8. [23, II.3.3] Consider the set of pairs of the form (E,α), where E is an object of the category $\mathcal{C}$ and α is an automorphism of E. Two pairs (E,α) and (E',α') are called *isomorphic*, if there is an isomorphism $h:E\to E'$ in category $\mathcal{C}$ such that the diagram

$$\begin{tikzcd}
E \arrow[r, "h"] \arrow[d, "\alpha"'] & E' \arrow[d, "\alpha'"] \\
E \arrow[r, "h"] & E'
\end{tikzcd}$$

commutes. The direct sum defines the operation of *addition* of pairs. A pair (E, α) is called *elementary*, if the automorphism α is homotopic to Id_E in the set of automorphisms of E. Abelian group [23, II.3.4] $K^{-1}(\mathcal{C})$ is defined as a quotient set (with operation of addition) of the set of pairs $\{(E, \alpha)\}$ with respect to the following equivalence relation: $\sigma \sim \sigma'$ if and only if there are such elementary pairs τ and τ', that $\sigma + \tau$ is isomorphic to $\sigma' + \tau'$.

Definition 7.9. [23, II.4.1] Let $\mathcal{C}$ be a Banach category and $C^{p,q}$ be the Clifford algebra. We shall denote by $\mathcal{C}^{p,q}$ the category whose objects are pairs (E, ρ), where $E \in \mathcal{O}b(\mathcal{C})$ and $\rho : C^{p,q} \to \mathrm{End}(E)$ is a homomorphism of algebras. A morphism from a pair (E, ρ) to a pair (E', ρ') is a $\mathcal{C}$-morphism $f : E \to E'$ such that $f \circ \rho(\lambda) = \rho(\lambda) \circ f$ for each element $\lambda \in \mathcal{C}^{p,q}$.

Recall that there exist canonical morphisms $C^{p,q} \to C^{p,q+1}$.

Definition 7.10. [23, III.4.11] Let $\mathcal{C}$ be a pseudo-Abelian Banach category. *The group* $K^{p,q}(\mathcal{C})$ is defined as the Grothendieck group of the forgetful functor $\mathcal{C}^{p,q+1} \to \mathcal{C}^{p,q}$ (in the sense of the Definition 7.6).

The following statement can be easily obtained using the properties of Clifford algebras.

Theorem 7.11. [23, III.4.6, III.4.12] *The groups $K^{p,q}(\mathcal{C})$ depend only on the difference $p-q$. Besides, the groups $K^{0,0}(\mathcal{C})$ and $K^{0,1}(\mathcal{C})$ are canonically isomorphic to groups $K(\mathcal{C})$ and $K^{-1}(\mathcal{C})$.*

Definition 7.12. Now we can define $K^{p-q}(\mathcal{C}) = K^{p,q}(\mathcal{C})$ and similarly for K-groups of functors.

We need also another description of K-groups, which is equivalent [23, §§ III.4, III.5] to the initial.

Definition 7.13. [23, III.4.11, III.5.1] Let $\mathcal{C}$ be a pseudo-Abelian Banach category and let E be a $C^{p,q}$-module (an object of the category $\mathcal{C}^{p,q}$). A *grading* of E is an endomorphism η of E (considering as an object of $\mathcal{C}$) such that

1. $\eta^2 = 1$,
2. $\eta\rho(e_i) = -\rho(e_i)\eta$, where e_i are the generators of Clifford algebra and $\rho : C^{p,q} \to \mathrm{End}(E)$ is the homomorphism, determining the $C^{p,q}$-structure on E.

In other words, a grading of E is a $C^{p,q+1}$-structure on E, extending the initial $C^{p,q}$-structure (if we put $\rho(e_{p+q+1}) = \eta$).

Let us define the group $K^{p,q}(\mathcal{C})$ as the quotient group of the free Abelian group, generated by the triples (E, η_1, η_2), where E is a $C^{p,q}$-module and η_1, η_2 is a grading of E with respect to the subgroup, generated by relations

1. $(E, \eta_1, \eta_2) \oplus (F, \xi_1, \xi_2) = (E \oplus F, \eta_1 \oplus \xi_1, \eta_2 \oplus \xi_2)$,
2. $(E, \eta_1, \eta_2) = 0$, if η_1 is homotopic to η_2 in the set of gradations of E.

As usual, by $d(E, \eta_1, \eta_2) \in K^{p,q}(\mathcal{C})$ we shall denote the class of triple (E, η_1, η_2).

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