Twisted automorphisms of Hopf algebras

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Abstract

Twisted homomorphisms of bialgebras are bialgebra homomorphisms from the first into Drinfeld twistings of the second. They possess a composition operation extending composition of bialgebra homomorphisms. Gauge transformations of twists, compatible with adjacent homomorphisms, give rise to gauge transformation of twisted homomorphisms, which behave nicely with respect to compositions. Here we study (gauge classes of) twisted automorphisms of cocommutative Hopf algebras. After revising well-known relations between twists, twisted forms of bialgebras and R-matrices (for commutative bialgebras) we describe twisted automorphisms of universal enveloping algebras.

1 Introduction

The aim of the paper is to study "hidden" symmetries of bialgebras which manifest themselves in representation theory. It is very well-known that categories of representations (modules) of bialgebras are examples of so-called *monoidal categories*. It is less acknowledged that relations between bialgebras (*homomorphisms of bialgebras*) do not capture all relations (*monoidal functors*) between their representation categories. An algebraic notion which does the job (*bi-Galois (co)algebra*) is known only to specialists. They fully represent monoidal relations between representation categories but are sometimes not very easy to work with. For example, composition of monoidal functors corresponds to tensor product of Galois (co)algebras and in some situations is quite tricky to calculate explicitly. At the same time monoidal functors of interest could have some additional properties which put restrictions on the corresponding algebraic objects and allow one to have an alternative and perhaps simpler description.

Note that any representation category is equipped for free with a monoidal functor to the category of vector spaces, the functor forgetting the action of the bialgebra (*forgetful functor*). Here we deal with monoidal functors between representation categories which preserve (not necessarily monoidally) the forgetful functors. The corresponding algebraic notion is of *twisted homomorphism*. After defining them in the first section we examine algebraic counterparts of the composition (composition of twisted homomorphisms) and natural transformations of monoidal functors (*gauge transformations*).

The main object to study for us is the category of twisted automorphisms of a cocommutative Hopf algebra. We start by re-examining the actions of twisted automorphisms on twists (twisted homomorphisms from the ground field) and R-matrices. After that we treat the case of a universal enveloping algebra of a Lie algebra (over formal power series). It turns out that any twisted automorphism is a bialgebra automorphism together with an invariant twist (*separated* case). The gauge classes of invariant twists form an abelian group isomorphic to the invariant elements of the exterior square of the Lie algebra. The group of gauge classes of twisted automorphisms is a crossed product of the group of automorphisms of the Lie algebra and the group of invariant twists.

Throughout the paper k be a ground field, which is supposed to be algebraically closed of characteristic zero.

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2 Twisted homomorphisms of bialgebras and monoidal functors between categories of modules

2.1 Twisted homomorphisms of bialgebras

Here we recall the notions of twisted homomorphisms of bialgebras and their transformations and show how they enrich the category of bialgebras making it a 2-category.

A *twisted homomorphism* of bialgebras $(H,\Delta,\varepsilon) \to (H',\Delta',\varepsilon')$ is a pair (f,F) where $f: H \to H'$ is a homomorphism of algebras and F is an invertible element of $H'\otimes H'$ (f*-twist* or simply *twist*) such that

$$F\Delta'(f(x)) = (f\otimes f)\Delta(x)F, \quad \forall x \in H, \tag{1}$$

$$(F\otimes 1)(\Delta'\otimes I)(F) = (1\otimes F)(I\otimes\Delta')(F), \quad \text{2-cocycle condition}$$

$$(\varepsilon\otimes I)(F) = (I\otimes\varepsilon)(F) = 1, \quad \text{normalisation.}$$

For example, a homomorphism of bialgebras $f: H \to H'$ is a twisted homomorphism with the identity twist $(f,1)$. A twisted homomorphism is *separated* if the first component $f : H \to H'$ is a homomorphism of bialgebras. For a separated twisted homomorphism the condition (1) amounts to the invariance of the twist with respect to the sub-bialgebra $f(H) \subset H'$:

$$\Delta'(f(x))F = F\Delta'(f(x)), \quad \forall x \in H.$$

Note that the 2-cocycle and normalisation conditions for a twist F imply the coassociativity and counitality respectively of the *twisted* coproduct on H':

$$(\Delta')^F(x) = F\Delta'(x)F^{-1}.$$

The condition (1) says that $f : H \to (H')_F$ is a homomorphism of bialgebras, where ${H'}_F = (H', (\Delta')^F)$ is the twisted bialgebra. Note also that the condition on $F \in H' \otimes H'$ equivalent to the coassociativity of the twisted coproduct $(\Delta')^F$, is H'-invariance of

$$(\Delta' \otimes I)(F)^{-1}(F \otimes 1)^{-1}(1 \otimes F)(I \otimes \Delta')(F).$$

One could call such elements *quasi-twists.* Then a *quasi-twisted homomorphism* of bialgebras $(f, F) : H \to H'$ is a bialgebra homomorphisms $f : H \to {H'}_F$ for a quasi-twist F. Although, being a partial case of the general picture of twists for quasi-bialgebras (see [4]), quasi-twisted homomorphism are interesting in their own right. For instance, Drinfeld's rectification of Kohno's monodromy theorem establishes the existence of a quasi-twisted isomorphism between quantised and classical universal enveloping algebras $U_q(\mathfrak{g})[[h]] \to U(\mathfrak{g})[[h]]$ (see [4]). For another example see [2], where it was proved that finite groups have the same character tables if and only if there is a quasi-twisted isomorphism between their group algebras (over algebraically closed field of characteristic zero). Being much more versatile, quasi-twisted homomorphisms lack one important property valid for twisted homomorphisms. There is no composition operation defined for general quasi-twisted homomorphisms. It is related to the fact that (in contrast to twists) the image of a quasi-twist under a bialgebra homomorphism is not necessarily a quasi-twist.

For succesive twisted homomorphisms

$$(H, \Delta, \varepsilon) \xrightarrow{(f,F)} (H', \Delta', \varepsilon') \xrightarrow{(f',F')} (H'', \Delta'', \varepsilon'')$$

we define their *composition* as

$$(f', F') \circ (f, F) = (f'f, f'(F)F').$$

Here $f'(F) = (f' \otimes f')(F)$. It is not hard to verify that the result is a twisted homomorphism $(H, \Delta, \varepsilon) \to (H'', \Delta'', \varepsilon'')$ and that the composition is associative. See also section 3. Note that separated twisted homomorphisms are closed under composition.

By a *gauge transformation* $(f, F) \to (f', F')$ of twisted homomorphisms $(f, F), (f', F') : (H, \Delta, \varepsilon) \to (H', \Delta', \varepsilon')$ we will mean an element a of H' such that

$$af(x) = f'(x)a, \quad \forall x \in H, \tag{2}$$

$$F\Delta'(a) = (a \otimes a)F'. \tag{3}$$

We will depict it graphically as follows:

$$\begin{tikzcd}[column sep=huge]
(H,\Delta,\varepsilon) \arrow[r, bend left=40, "{(f,F)}"{name=U}] \arrow[r, bend right=40, "{(f',F')}"'{name=D}] & (H',\Delta',\varepsilon') \arrow[from=U, to=D, phantom, "\Downarrow a"]
\end{tikzcd}$$

Note that the condition (3) together with normalisation conditions for F and G implies $\varepsilon(a) = 1$. Note also that a twisted homomorphism gauge isomorphic to a separated twisted homomorphism is not necessarily separated.

For successive gauge transformations

$$\begin{tikzcd}[column sep=huge]
(H,\Delta,\varepsilon) \arrow[r, bend left=60, "{(f,F)}"{name=U}] \arrow[r, "{(g,G)}"'{name=M}] \arrow[r, bend right=60, "{(j,J)}"'{name=D}] & (H',\Delta',\varepsilon') \arrow[from=U, to=M, phantom, "\Downarrow a"] \arrow[from=M, to=D, phantom, "\Downarrow b"]
\end{tikzcd}$$

the composition $b.a : (f, F) \to (j, J)$ is simply the product ba in H'. Again it is quite straightforward to check that this is a transformation.

Note that if H' is a Hopf algebra, any gauge transformation between twisted homomorphisms into H' is invertible. This fact can be checked by pure algebraic computations. A sketch of a different proof will be given in section 3.

We can also define compositions of transformations and twisted homomorphisms in the following two situations:

$$\begin{tikzcd}[column sep=huge]
(H,\Delta,\varepsilon) \arrow[r, bend left=40, "{(f,F)}"{name=U}] \arrow[r, bend right=40, "{(f',F')}"'{name=D}] & (H',\Delta',\varepsilon') \arrow[from=U, to=D, phantom, "\Downarrow a"] \arrow[r, "{(g,G)}"] & (H'',\Delta'',\varepsilon'')
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=huge]
(H,\Delta,\varepsilon) \arrow[r, "{(f,F)}"] & (H',\Delta',\varepsilon') \arrow[r, bend left=40, "{(g,G)}"{name=U}] \arrow[r, bend right=40, "{(g',G')}"'{name=D}] & (H'',\Delta'',\varepsilon'') \arrow[from=U, to=D, phantom, "\Downarrow b"]
\end{tikzcd}$$

we define it to be $(g, G) \circ a = g(a)$ in the first case and $b \circ (f, F) = b$ in the second. The following properties intertwining compositions of twisted homomorphisms and gauge transformations are quite straightforward consequences of the definitions:

$$(a.b) \circ (f, F) = (a \circ (f, F)).(b \circ (f, F)),$$

$$(g,G)\circ(a.b) = ((g,G)\circ a).((g,G)\circ b),$$
$$(a\circ(f,F)).((g,G)\circ b) = ((g,G)\circ b).(a\circ(f,F)).$$

For an alternative explanation see section 3.

Note that the structures described above extend the category of bialgebras and twisted homomorphisms to a 2-category **Tw** with gauge transformations as 2-cells (2-morphisms).

Twisted homomorphisms have certain natural involutive symmetry.

Lemma 2.1.1. *If $(f,F): H \to H'$ is a twisted homomorphism, then $t(f,F) = (f,t(F)) : H^{co} \to (H')^{co}$ is a twisted homomorphism between bialgebras with opposite comultiplication.*

Proof. The condition (1) for $(f,t(F))$ is the transpose (the result of applying t) of the condition (1) for (f,F). The same with the normalisation condition for $t(F)$. The 2-cocycle equation for $t(F)$ is the 2-cocycle equation for F where the first and the last tensor factor have been interchanged:

$$(t(F)\otimes 1)(t\Delta'\otimes I)t(F) = (t_2t_1t_2)((1\otimes F)(I\otimes\Delta')(F)) =$$
$$(t_1t_2t_1)((F\otimes 1)(\Delta'\otimes I)(F)) = (1\otimes t(F))(I\otimes t\Delta')t(F).$$

□

Note that if we had a gauge transformation $a: (f,F)\to(g,G)$, then $a: t(f,F)\to t(g,G)$ is also a gauge transformation. Moreover, all compositions are compatible with t. In (2-)categorical language $t: \mathbf{Tw}\to\mathbf{Tw}$ is a 2-functor (2-isomorphism).

We call a twisted homomorphism $(f,F): H\to H'$ of cocommutative bialgebras *symmetric* if $t(f,F) = (f,F)$. Note that a twisted homomorphism gauge isomorphic to a symmetric twisted homomorphism is symmetric itself.

2.2 Twisted homomorphisms and Galois (co)algebras

It was observed by Drinfeld [3] that a twist F on a bialgebra H defines a new coalgebra structure on H:

$$\Delta_F(x) = F\Delta(x), \quad x\in H.$$

Coassociativity for Δ_F is equivalent to the 2-cocycle condition on F. The property $\Delta_F(xy) = \Delta_F(x)\Delta(y)$ guarantees that the right H-module structure on H is an H-module coalgebra structure on (H,Δ_F). Recall that a coalgebra (C,δ) is an *H-module coalgebra* if C is an H-module $a: C\otimes H\to C$ and

$$\delta(xy) = \delta(x)\Delta(y), \quad x\in C, y\in H.$$

An H-module coalgebra C is *Galois* if the composition

$$C\otimes H \xrightarrow{\delta\otimes I} C\otimes C\otimes H \xrightarrow{I\otimes a} C\otimes C$$

is an isomorphism. It is not hard to see that if H is a Hopf algebra (H, Δ_F) is a Galois H-module coalgebra. Moreover Galois H-module coalgebras of the form (H, Δ_F) are characterised by the so-called normal basis property. A Galois H-module coalgebra C has a *normal basis* if C is isomorphic to H as an H-module.

For a twisted homomorphism $(f, F) : H \to H'$, the H'-Galois coalgebra (H', Δ_F) comes equipped with the left H-action $H \otimes H' \to H'$ sending $x \otimes y$ to $f(x)y$. It follows from the definition of twisted homomorphism that this action preserves the coproduct Δ_F:

$$F\Delta'(f(x)y) = F\Delta'(f(x))\Delta'(y) = (f \otimes f)\Delta(x)F\Delta'(y).$$

Composition of twisted homomorphisms corresponds to the following operation on Galois coalgebras. Let C be a Galois H'-module coalgebra and C' be a Galois H''-module coalgebra with compatible left H'-action. Then the tensor product of H'-modules $C \otimes_{H'} C'$ is a Galois H''-coalgebra with respect to the coproduct $t_{23}(\delta \otimes \delta')$. Moreover a left H-action on C compatible with H'-module coalgebra structure will pass on to $C \otimes_{H'} C'$.

For successive twisted homomorphisms

$$(H, \Delta, \varepsilon) \xrightarrow{(f,H)} (H', \Delta', \varepsilon') \xrightarrow{(f',F')} (H'', \Delta'', \varepsilon'')$$

the tensor product $(H', \Delta_F) \otimes_{H'} (H'', \Delta_{F'})$ is isomorphic to H'' as an $H - H''$-bimodule via $x \otimes y \mapsto f'(x)y$. Moreover, the tensor product of comultiplications $\Delta_F \otimes \Delta_{F'}$ is carried out into the comultiplication $\Delta_{f(F)F'}$. Indeed, the tensor product of comultiplications followed by the isomorphism sends $x \otimes y$ into $(f' \otimes f')(F\Delta'(x))F'\Delta''(y)$ which can be rewritten as

$$(f' \otimes f')(F\Delta'(x))F'\Delta''(y) = (f' \otimes f')(F)(f' \otimes f')(\Delta'(x))F'\Delta''(y) =$$

$$(f' \otimes f')(F)F'\Delta''(f'(x))\Delta''(y) = (f' \otimes f')(F)F'\Delta''(f'(x)y).$$

Finally $(f' \otimes f')(F)F'\Delta''(f'(x)y)$ is the coproduct $\Delta_{f(F)F'}$ applied to $f'(x)y$.

A gauge transformation $a \in H'$ of twisted homomorphisms $(f, F), (f', F') : (H, \Delta, \varepsilon) \to (H', \Delta', \varepsilon')$ defines a homomorphism $x \mapsto ax$ of Galois H'-module coalgebras. Preservation of H'-module structure is obvious. Compatible H-module structure is preserved by the property (2) of gauge transformations, compatibility with coalgebra structures follows from the property (3):

$$F\Delta'(ax) = F\Delta'(a)\Delta'(x) = (a \otimes a)F'\Delta(x).$$

Define a bicategory **Gal** with objects being Hopf algebras, arrows from H to H' being Galois right H'-module coalgebras with compatible left H-action and composition given by tensor product, and homomorphisms of $H - H'$-bimodule coalgebras as 2-cells. The above construction defines a psedofunctor $\mathbf{Tw} \to \mathbf{Gal}$.

In the case of finite dimensional Hopf algebras we can replace Galois coalgebra with the more familiar notion of Galois algebra. Note that for a finite dimensional Hopf algebra H, a Galois H-module coalgebra must be finite dimensional (of the same dimension). Note also that the dual algebra to a Galois

H-coalgebra will be a Galois H-algebra. Recall that an *H-algebra* R is an algebra and an H-module in such way that the multiplication map μ is a map of H-modules:

$$\mu\Delta(x)(a \otimes b) = x\mu(a \otimes b) = x(ab), \quad x \in H, a, b \in R.$$

The *crossed product* $R * H$ of H with an H-algebra R is an algebra, which as a vector space is isomorphic to the tensor product $R \otimes H$. Denote elements of $R * H$ corresponding to tensors $a \otimes x$ by $a * x$. The product on $R * H$ is given by the rule:

$$(a * x)(b * y) = \sum_{(x)} ax_1 * x_2 y.$$

Here we use the so-called Swidler's notation for the coproduct $\Delta(x) = \sum_{(x)} x_1 \otimes x_2$. Following [10] we call an H-algebra R *Galois* if the homomorphism of algebras

$$A * H \to End(A), \quad a * x \mapsto (b \mapsto ax(a))$$

is an isomorphism.

For a finite dimensional H the dual of the Galois coalgebra (H, Δ_H) is the dual space H^* with the multiplication:

$$(l *_F l')(x) = (l \otimes l')\Delta_F(x) = (l \otimes l')(F\Delta(x)).$$

This can be seen as an ordinary multiplication on H^* twisted by F with respect to the (right) H-action on H^*:

$$l *_F l' = \sum l_1^F (l')_2^F.$$

Here $F = \sum F_1 \otimes F_2$ and $l^y(x) = l(xy)$.

3 Categorical interpretations

Most of the material included in this section is pretty standard (for example, see [6]; for more categorically oriented treatment see [13]).

3.1 Monoidal categories and functors

A *monoidal category* is a category $\mathcal{G}$ with a functor

$$\otimes : \mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{G} \qquad (X, Y) \mapsto X \otimes Y$$

(*tensor product*), a natural collection of isomorphisms (*associativity constraint*)

$$\varphi_{X,Y,Z} : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z \qquad \text{for any} \quad X, Y, Z \in \mathcal{G}$$

which satisfies the following *pentagon axiom*:

$$(X \otimes \varphi_{Y,Z,W})\varphi_{X,Y\otimes Z,W}(\varphi_{X,Y,Z} \otimes W) = \varphi_{X,Y,Z\otimes W}\varphi_{X\otimes Y,Z,W}$$

and an object 1 (*unit* object) together with natural isomorphisms

$$\rho_X : X \otimes 1 \to X \quad \lambda_X : 1 \otimes X \to X$$

such that $\lambda_1 = \rho_1$

$$\lambda_{X\otimes Y} = \lambda_X \otimes I : 1 \otimes X \otimes Y \to X \otimes Y, \quad \rho_X \otimes I = I \otimes \lambda_Y : X \otimes 1 \otimes Y \to X \otimes Y,$$

$$\rho_{X\otimes Y} = I \otimes \rho_Y : X \otimes Y \otimes 1 \to X \otimes Y$$

for any $X, Y \in \mathcal{G}$. Here I denote the identity morphism. It is known that the first two conditions follow from the rest. We formulate them for the sake of symmetry.

The celebrated MacLane coherence theorem [9] says that there is a unique isomorphism between any two bracket arrangements on the tensor products of objects $X_1, ..., X_n$, which is a composition of (tensor products of) the associativity constraints. This fact allows to omit brackets in tensor products. It is also easy to see that the unit object is unique up to an isomorphism. More precisely, any monoidal category is monoidally equivalent to a *strict* monoidal category (a monoidal category with identity associativity and unit object constraints) (see, for example, [7]).

A *monoidal functor* between monoidal categories $\mathcal{G}$ and $\mathcal{H}$ is a functor $F : \mathcal{G} \to \mathcal{H}$ with a natural collection of isomorphisms (the so-called *monoidal structure*)

$$F_{X,Y} : F(X \otimes Y) \to F(X) \otimes F(Y) \qquad \text{for any} \quad X, Y \in \mathcal{G}$$

satisfying the *coherence* axiom:

$$(I \otimes F_{Y,Z})F_{X,Y\otimes Z} = (F_{X,Y} \otimes I)F_{X\otimes Y,Z} \tag{4}$$

for any objects $X, Y, Z \in \mathcal{G}$.

A natural transformation $f : F \to G$ of monoidal functors F and G is *monoidal* if

$$G_{X,Y} f_{X\otimes Y} = (f_X \otimes f_Y)F_{X,Y}$$

for any $X, Y \in \mathcal{G}$.

Here we will be mostly interested in tensor categories and functors (linear over the ground field k). Recall that a monoidal category is *tensor* if it is k-linear abelian and the tensor product is k-linear and bi-exact. For a monoidal functor between tensor categories to be *tensor* we will ask it to be k-linear and left exact. Denote by **Tens** the 2-category of tensor categories, with tensor functors and monoidal natural transformations.

3.2 Categories of modules over bialgebras

Recall that the comultiplication in H can be used to define a structure of an H-module on the tensor product $M \otimes_k N$ of two H-modules:

$$h * (m \otimes n) = \Delta(h)(m \otimes n) \qquad h \in H, m \in M, n \in N.$$

The coassociativity axiom for the coproduct implies that the obvious associativity constraint for vector spaces

$$\varphi : L\otimes(M\otimes N)\to(L\otimes M)\otimes N \qquad \varphi(l\otimes(m\otimes n))=(l\otimes m)\otimes n$$

is H-linear. The counit defines an H-module structure on the ground field k and the counit axiom guarantees that this is a unit object. Thus the category $HMod$ of (left) modules over a bialgebra becomes monoidal.

For a homomorphism of algebras $f : H\to H'$ define by $f^* : H'Mod\to H$-Mod the *inverse image* functor, which turns an H'-module M into an H-module $f^*(M)$. Here as a vector space, $f^*(M)$ is the same as M but with a new module structure $x.m = f(x)m$ for $x\in H$ and $m\in M$. On the level of categories of modules, twisted homomorphisms and gauge transformations take the following meaning.

Proposition 3.2.1. *For a twisted homomorphism $(f,F) : H\to H'$ the inverse image functor $f^* : H'-Mod\to H-Mod$ becomes tensor, with the monoidal structure given by multiplication with the twist:*

$$F_{M,N} : f^*(M\otimes H)\to f^*(M)\otimes f^*(N), \quad m\otimes n\mapsto F(m\otimes n).$$

Compositions of twisted homomorphisms and corresponding functors are related as follows:

$$((f,F)\circ(g,G))^* = (g,G)^*\circ(f,F)^*.$$

A gauge transformation $a : (f,F)\to(g,G)$ defines a monoidal natural transformation $a : (f,F)^*\to(g,G)^*$:

$$a_M : f^*(M)\to g^*(M), \quad m\mapsto am.$$

Compositions of gauge transformations correspond to compositions of natural transformations.

Proof. It is straightforward to see that the condition (1) guarantees H-linearity of the monoidal constraint $F_{M,N}$ while the 2-cocycle condition for F is equivalent to the coherence axiom (4). Similarly, the condition (2) for a gauge transformation a says that a_M is a morphism of H-modules and the condition (3) is equivalent to the monoidality of a_M. $\square$

2-categorically the proposition 3.2.1 says that the inverse image functor construction defines a 2-functor **Tw** $\to$ **Tens** contravariant on morphisms and covariant on 2-cells. We can extend this to a bifunctor **Gal** $\to$ **Tens** as follows.

Proposition 3.2.2. *A Galois H'-module coalgebra C with compatible H-action defines a tensor functor $C\otimes_{H'} : H'Mod\to HMod$ with the monoidal structure*

$$\begin{tikzcd}
C\otimes_{H'}(M\otimes N) \arrow[d, "\delta\otimes I"] & C\otimes_{H'}M\otimes C\otimes_{H'}N \\
(C\otimes C)\otimes_{H'}(M\otimes N) \arrow[r] & (C\otimes C)\otimes_{H'\otimes H'}(M\otimes N) \arrow[u, "t_{23}"']
\end{tikzcd}$$

Homomorphisms of H-H'-bimodule coalgebras define monoidal natural transformations.

Proof. It follows from the Galois property that the above monoidal structure is an isomorphism. Coassociativity of the coproduct implies the coherence axiom. □

Obviously, composition of functors corresponds to the tensor product of coalgebras.

4 Twisted automorphisms of a Hopf algebra

In this section we will describe Cat-groups of twisted automorphisms along with some of their natural Cat-subgroups. One of the reasons (which will be particularly important in the last section) why Cat-groups are more natural objects to deal with rather than groups of (classes of) twisted automorphisms is the following. Suppose that we want to define an action of a group G on a bialgebra H by twisted homomorphisms. A homomorphism from the group G into the group $Out_{\mathbf{Tw}}(H)$ of classes of twisted homomorphism would just not have enough information. A homomorphism from G to the group $Aut_{\mathbf{Tw}}(H)$ of twisted automorphisms would do, but that would not capture all the cases. The right answer is a (monoidal) functor $G \to \mathcal{A}ut_{\mathbf{Tw}}(H)$ of Cat-groups (a map of crossed modules of groups).

4.1 Categorical groups, crossed modules and their maps

Recall that a *categorical group* is a monoidal category in which every arrow is invertible (monoidal groupoid) and for every object X there is an object X^* with an arrow $e_X : X^* \otimes X \to I$ (a dual object). A categorical group is *strict* (or a *Cat-group*) if it is strict as a monoidal category and e_X can be chosen to be an identity. In other words a Cat-group is a categorical group whose objects form a group (with the tensor product). A slight modification of Mac Lane's coherence theorem [9] says that any categorical group is monoidally equivalent to a strict one. Note that Cat-groups are group objects in the category of categories (thus the name).

For any object A of a 2-category $\mathbf{A}$ the category $\mathcal{A}ut_{\mathbf{A}}(A)$ of automorphisms of A (invertible 1-morphisms $A \to A$), with the composition as the tensor product and bijective natural transformations as morphisms, is a Cat-group.

It is well known (see [1] for the history) that Cat-groups are the same as crossed modules of Whitehead ([15]). Recall that a *crossed module* of groups is a pair of groups P, C with a (left) action of P on C (by group automorphisms):

$$P \times C \to C, \quad (p, c) \mapsto^p c$$

and a homomorphism of groups

$$P \stackrel{\partial}{\leftarrow} C$$

such that

$$\partial({}^{p}c) = p\partial(c)p^{-1},$$
$$^{\partial(c)}c' = cc'c^{-1}.$$

For a Cat-group $\mathcal{G}$ the corresponding crossed complex consists of the group of objects P, the group of morphisms $X \to I$ into the identity object with the product:

$$(X \xrightarrow{a} I).(Y \xrightarrow{b} I) = X \otimes Y \xrightarrow{a\otimes b} I \otimes I = I,$$

the action

$$^{Y}(X \xrightarrow{a} I) = Y \otimes X \otimes Y^{*} \xrightarrow{Y\otimes a\otimes Y^{*}} Y \otimes I \otimes Y^{*} = I,$$

and the homomorphism $\partial : C \to P$ sending $X \to I$ into X.

Relative simplicity and compactness of the notion of crossed module as a description for Cat-groups has its downside. A monoidal functor between Cat-groups will correspond to something less obvious (and less well-known) than a *homomorphism* of crossed modules (a pair of group homomorphisms preserving all the structures). That will describe only *strict* monoidal functors. To deal with general monoidal functors we need the following weaker relation. A *map* of crossed modules $(P, C) \to (E, N)$ is a triple (τ, ν, θ) where τ and ν are maps making the diagram commutative

$$\begin{tikzcd} P \arrow[d, "\tau"] & C \arrow[l, "\partial"'] \arrow[d, "\nu"] \\ E & N \arrow[l, "\partial"] \end{tikzcd}$$

and $\theta : P \times P \to N$ such that

$$\tau(pq) = \partial(\theta(p, q))\tau(p)\tau(q), \quad p, q \in P,$$

$$\nu(ab) = \theta(\partial(a), \partial(b))\nu(a)\nu(b), \quad a, b \in C,$$

$$\theta(p, qr)\,{}^{\tau(p)}\theta(q, r) = \theta(pq, r)\theta(p, q), \quad p, q, r \in P,$$

$$\theta(1, q) = 1 = \theta(p, 1), \quad p, q \in P,$$

$$\nu({}^{p}a) = \theta(p, \partial(a))\,{}^{\tau(p)}\nu(a).$$

Complete invariants of a categorical-group $\mathcal{G}$ with respect to monoidal equivalences are

$$\pi_0(\mathcal{G}),\ \pi_1(\mathcal{G}),\ \phi \in H^3(\pi_0(\mathcal{G}), \pi_1(\mathcal{G})),$$

where the first is the group of isomorphism classes of objects, the second is the abelian group ($\pi_0(\mathcal{G})$-module) $Aut_{\mathcal{G}}(I)$ of automorphisms of the unit object and the third is a cohomology class (the *associator*). In the crossed module setting

$$\pi_0 = coker(\partial),\ \pi_1 = ker(\partial).$$

Note that the image of ∂ is normal so the cokernal has sense. The class ϕ is defined as follows: choose a section $\sigma : coker(\partial) \to P$ and a map $a : coker(\partial) \times coker(\partial) \to C$ such that

$$\sigma(fg) = \partial(a(f,g))\sigma(f)\sigma(g), \quad f,g \in coker(\partial).$$

Then for any $f,g,h \in coker(\partial)$ the expression

$$a(f,gh)^{\sigma(f)}a(g,h)a(f,g)^{-1}a(fg,h)$$

is always in the kernel of ∂ and is a group 3-cocycle of $coker(\partial)$ with coefficients in $ker(\partial)$. The cohomology class ϕ does not depend on the choices made.

Analogously, complete invariants of a monoidal functor $F : \mathcal{G} \to \mathcal{F}$ between categorical groups with respect to monoidal isomorphisms are

$$\pi_0(F) : \pi_0(\mathcal{G}) \to \pi_0(\mathcal{F}),\ \pi_1(F) : \pi_1(\mathcal{G}) \to \pi_1(\mathcal{F}),\ \theta(F) : \pi_0(\mathcal{G})\times\pi_0(\mathcal{G}) \to \pi_1(\mathcal{F}),$$

where the first is the homomorphism of groups, the second is the homomorphism of $\pi_0(\mathcal{G})$-modules and the third is really a class in $C^2(\pi_0(\mathcal{G}),\pi_1(\mathcal{F}))/B^2(\pi_0(\mathcal{G}),\pi_1(\mathcal{F}))$ such that

$$d(\theta(F)) = \pi_1(F)_*(\phi(\mathcal{G})) - \pi_0(F)^*(\phi(\mathcal{F}))$$

Here

$$\pi_0(F)^* : C^*(\pi_0(\mathcal{F}),\pi_1(\mathcal{F})) \to C^*(\pi_0(\mathcal{G}),\pi_1(\mathcal{F})),$$
$$\pi_1(F)_* : C^*(\pi_0(\mathcal{G}),\pi_1(\mathcal{G})) \to C^*(\pi_0(\mathcal{G}),\pi_1(\mathcal{F}))$$

are the maps of cochain complexes induced by the group homomorphisms $\pi_0(F)$, $\pi_1(F)$.

4.2 Cat-groups of twisted automorphisms

As a part of the 2-category **Tw** the invertible twisted endomorphisms of a Hopf algebra H (twisted automorphisms) and gauge transformations between them form a Cat-group $\mathcal{A}ut_{\mathbf{Tw}}(H)$. The corresponding crossed module of groups has the form

$$Aut_{\mathbf{Tw}}(H) \xleftarrow{\partial} H^\times_\varepsilon.$$

Here $Aut_{\mathbf{Tw}}(H)$ is the group of twisted automorphisms of H with respect to the composition, $H^\times_\varepsilon$ is the group of invertible elements x of H such that $\varepsilon(x) = 1$, and ∂ sends x into the pair (an *inner* twisted automorphism) $({}^x(\ ), (x \otimes x)\Delta(x)^{-1})$ where the first component is the conjugation automorphism:

$${}^x(\ ) : H \to H, \quad {}^x(y) = xyx^{-1}.$$

The action of $Aut_{\mathbf{Tw}}(H)$ on $H^\times$ is given by the action of the first component $(f,F)(y) = f(y)$.

There are two important Cat-subgroups in $\mathcal{A}ut_{\mathbf{Tw}}(H)$. The first is the full Cat-subgroup $\mathcal{A}ut^1_{\mathbf{Tw}}(H)$ of twisted automorphisms with the identity as the first component. Its crossed module is

$$Aut^1_{\mathbf{Tw}}(H) \xleftarrow{\partial} (Z(H)_\varepsilon)^\times.$$

Here $Aut^1_{\mathbf{Tw}}(H)$ is the group of *invariant twists* on H (invertible elements of $H\otimes H$ commuting with the image $\Delta(H)$ and satisfying the 2-cocycle condition), $(Z(H)_\varepsilon)^\times$ is the group of invertible elements of the centre of counit 1: $\varepsilon(x)=1$. Again ∂ assigns to x the invariant twist $(x\otimes x)\Delta(x)^{-1}$. The action of $Aut^1_{\mathbf{Tw}}(H)$ on $(Z(H)_\varepsilon)^\times$ is trivial.

The second is the full Cat-subgroup $\mathcal{A}ut_{bialg}(H)$ of bialgebra automorphisms of H. Here the crossed module is

$$Aut_{bialg}(H) \overset{\partial}{\leftarrow} G(H),$$

where $Aut_{bialg}(H)$ is the group of automorphisms of H as a bialgebra, $G(H)=\{x\in H,\quad \Delta(x)=x\otimes x\}$ is the group of group-like elements of H and ∂ sends x into the conjugation automorphism. The action of $Aut_{bialg}(H)$ on $G(H)$ is obvious.

Note that the Cat-subgroup $\mathcal{A}ut^1_{\mathbf{Tw}}(H)$ is what might be called *normal*: the components of its crossed module are normal subgroups in the components of the crossed complex for $Aut_{\mathbf{Tw}}(H)$ and the action of $Aut_{\mathbf{Tw}}(H)$ on $H^\times$ preserve the subgroup $Z(H)^\times$.

The Cat-subgroup $\mathcal{A}ut_{bialg}(H)$ is not in general normal. In the next part we will characterise it as the stabiliser of a certain action. Recall that an *action* of a Cat-group (monoidal category) $\mathcal{G}$ on a category $\mathcal{A}$ is a monoidal functor $\mathcal{G}\to\mathcal{E}nd(\mathcal{A})$ into the category of endofunctors on $\mathcal{A}$. For an object $A\in\mathcal{A}$ the *stabiliser* $\mathcal{S}t_{\mathcal{G}}(A)$ is the category of pairs (G,g), where G is an object and $g:G(A)\to A$ is an isomorphism in $\mathcal{A}$. A morphism of pairs $(G,g)\to(F,f)$ is a morphism $x:G\to F$ in $\mathcal{G}$ such that the diagram

$$\begin{tikzcd} G(A) \arrow[rr, "x_A"] \arrow[dr, "g"'] & & F(A) \arrow[dl, "f"] \\ & A & \end{tikzcd}$$

commute.

4.3 Action on twists

Here we examine the (categorical) action of the Cat-group $\mathcal{A}ut_{\mathbf{Tw}}(H)$ on the category of twisted homomorphisms $\mathbf{Tw}(k,H)$ given by the composition.

Let us start with the category $\mathbf{Tw}(k,H)$. Any twisted homomorphism $k\to H$ must have the form (ι,F) where $\iota:k\to H$ is the unit inclusion and $F\in H^{\otimes 2}$ is an invertible element satisfying the 2-cocycle condition (a *twist*). A gauge transformation $(\iota,F)\to(\iota,F')$ is an invertible element $a\in H$ such that $F'(a\otimes a)=\Delta(a)F$. The category $\mathbf{Tw}(k,H)$ has a marked object $(\iota,1)$.

On the level of objects the action $Aut_{\mathbf{Tw}}(H)\times\mathbf{Tw}(k,H)\to\mathbf{Tw}(k,H)$ has the form

$$(f,F)\circ(\iota,F')\mapsto(\iota,{F'}^fF).$$

An object of the stabiliser $\mathcal{S}t_{Aut_{\mathbf{Tw}}(H)}(\iota,1)$ is a triple (f,F,a), where (f,F) is an object of $Aut_{\mathbf{Tw}}(H)$ and $a\in H$ is an invertible element such that $F=$

$(a \otimes a)\Delta(a)^{-1}$ (a transformation $(f,F) \circ (\iota,1) \to (\iota,1)$). The element a can be interpreted as a gauge transformation $(f,F) \to ({}^a(\ ) \circ f, 1)$. In particular, the composition ${}^a(\ ) \circ f$ of f with conjugation with a is an automorphism of bialgebras. The naturality of this construction implies the following result.

Proposition 4.3.1. *The inclusion $\mathcal{A}ut_{bialg}(H) \to \mathcal{S}t_{Aut_{\mathbf{Tw}}(H)}(\iota,1)$ is an equivalence of categories.*

Now we describe the stabilizer $\mathcal{S}t_{Aut^1_{\mathbf{Tw}}(H)}(\iota,1)$ in the Cat-subgroup of invariant twists. Its objects are pairs (F,a) where F is an invariant twist on H and $a \in H$ is an invertible element such that $F = (a \otimes a)\Delta(a)^{-1}$ (a transformation $(I,F) \circ (\iota,1) \to (\iota,1)$). Similarly, the element a can be seen as a gauge transformation $(I,F) \to ({}^a(\ ),1)$. In particular, the conjugation ${}^a(\ )$ is an automorphism of bialgebras. Thus we have the following.

Proposition 4.3.2. *The stabiliser $\mathcal{S}t_{Aut^1_{\mathbf{Tw}}(H)}(\iota,1)$ is equivalent to the Cat-group $\mathcal{A}ut^{inn}_{bialg}(H)$ with the crossed module:*

$$Aut^{inn}_{bialg}(H) \stackrel{\partial}{\leftarrow} G(H), \tag{5}$$

where $Aut^{inn}_{bialg}(H)$ is the group of bialgebra automorphisms of H which are inner as algebra automorphisms, i.e. the kernel of the homomorphism $Aut_{bialg}(H) \to Out_{alg}(H)$.

In particular, $\pi_0(\mathcal{S}t_{Aut^1_{\mathbf{Tw}}(H)}(\iota,1))$ is isomorphic to the kernel of $Out_{bialg}(H) \to Out_{alg}(H)$.

We finish this section with a description of the orbits of the Cat-group action of $\mathcal{A}ut_{\mathbf{Tw}}(H)$ on $\mathbf{Tw}(k,H)$ in terms of twisted forms of the bialgebra H. Recall that a twist $F \in H^{\otimes 2}$ allows us to define a new coproduct on H:

$$\Delta^F(x) = F^{-1}\Delta(x)F.$$

We call the bialgebra $Tf(F) = (H,\Delta^F)$ an *F-twisted form* of H (or just a *twisted form*). Note that a gauge transformation of twists $F'(a \otimes a) = \Delta(a)F$ defines a homomorphism of bialgebras $(\ )^a : (H,\Delta^{F'}) \to (H,\Delta^F)$. Indeed,

$$\Delta^F(a^{-1}xa) = F^{-1}\Delta(a)^{-1}\Delta(x)\Delta(a)F =$$

$$(a \otimes a)^{-1}(F')^{-1}\Delta(x)F'(a \otimes a) = (a \otimes a)^{-1}\Delta^{F'}(x)(a \otimes a).$$

Thus we have a functor $Tf : \mathbf{Tw}(k,H) \to Bialg$ into the category $Bialg$ of bialgebras and their automorphisms.

Proposition 4.3.3. *The functor $Tf : \mathbf{Tw}(k,H) \to Bialg$ is strictly constant on orbits of the action of $\mathcal{A}ut_{\mathbf{Tw}}(H)$ on $\mathbf{Tw}(k,H)$, i.e. isomorphisms of twisted forms $f : (H,\Delta^F) \to (H,\Delta^{F'})$ are in 1-1 correspondence with twisted automorphisms $(f,F'') : (H,\Delta) \to (H,\Delta)$ such that $(f,F'') \circ (\iota,F) = (\iota,F')$.*

Proof. If $(f, F''): (H,\Delta) \to (H,\Delta)$ is a twisted automorphism such that $F''(f\otimes f)(F) = F'$ then

$$(f\otimes f)\Delta^F(x) = (f\otimes f)(F)^{-1}(f\otimes f)\Delta(x)(f\otimes f)(F) =$$

$$(F')^{-1}F''(f\otimes f)\Delta(x)(F'')^{-1}F' = (F')^{-1}\Delta(f(x))F' = \Delta^{F'}(f(x))$$

so that f is an isomorphism of bialgebras $(H,\Delta^F) \to (H,\Delta^{F'})$. Conversely, for an isomorphism of bialgebras $f: (H,\Delta^F) \to (H,\Delta^{F'})$, the element $F'' = F'(f\otimes f)(F)^{-1}$ defines a structure of twisted isomorphism $(f,F''): (H,\Delta) \to (H,\Delta)$:

$$F''(f\otimes f)\Delta(x) = F'(f\otimes f)(F)^{-1}(f\otimes f)\Delta(x) =$$

$$F'(f\otimes f)(F^{-1}\Delta(x)) = (F')^{-1}F'\Delta(f(x))F'(f\otimes f)(F)^{-1}.$$

□

In particular, the orbits of the group action of $Aut_{\mathbf{Tw}}(H)$ on $Tw(k,H)$ are in 1-1 correspondence with isomorphism classes of twisted forms of H.

4.4 Action on triangular structures

Here we refine the action of twisted automorphisms on triangular structures to the level of categories.

Recall that a *triangular structure* on a bialgebra H is an invertible element $R \in H\otimes H$ (a *universal R-matrix*) satisfying

$$Rt\Delta(x) = \Delta(x)R \quad \forall x \in H, \tag{6}$$

along with *triangle equations*:

$$(I\otimes\Delta)(R) = R_{13}R_{12},$$

$$(\Delta\otimes I)(R) = R_{13}R_{23},$$

normalisation:

$$(\varepsilon\otimes I)(R) = (I\otimes\varepsilon)(R) = 1,$$

and *unitarity* condition:

$$R_{21} = R^{-1}.$$

Here $R_{21} = (12)R$ is the transposition of tensor factors of $R \in H\otimes H$, $R_{12} = R\otimes 1$, $R_{13} = (I\otimes(12))(R_{12})$ etc.

Note that any universal R-matrix on a cocommutative bialgebra H satisfies the 2-cocycle condition:

$$(R\otimes 1)(\Delta\otimes I)(R) = R_{12}R_{13}R_{23} = (I\otimes\Delta)(R)(1\otimes R).$$

Denote by $\mathcal{T}r(H)$ the set of triangular structures on the bialgebra H.
It was observed by Drinfeld (see also [11]) that the map

$$H^* \to H, \quad l \mapsto (l\otimes I)(R) \tag{7}$$

is a homomorphism of algebras and anti-homomorphism of coalgebras for any triangular structure R. In particular, its image is a (finite-dimensional) sub-bialgebra H_R in H, R belongs to $H_R^{\otimes 2}$ (the so-called *minimal triangular* sub-bialgebra), and the map (7) factors as follows

$$H^* \to H_R^* \simeq H_R \to H,$$

where the first surjection $H^* \to H_R^*$ is dual to the last inclusion $H_R \to H$ and the isomorphism $H_R^* \simeq H_R$ is self-dual. It is well known that the minimal triangular subalgebra for a cocommutative Hopf algebra possesses a quite simple description.

Proposition 4.4.1. *For a cocommutative Hopf algebra H the set $\mathcal{T}r(H)$ of triangular structures is isomorphic to the set of pairs (A, b), where A is a normal commutative cocommutative finite dimensional sub-bialgebra in H and $b : A^* \to A$ is an H-invariant isomorphism of bi-algebras.*

Proof. As a sub-biagebra of H the minimal triangular sub-bialgebra H_R must be cocommutative. It is commutative since H_R^* is a sub-bialgebra of H^*. Normality of H_R and H-invariance of $b : H_R^* \simeq H_R$ are equivalent to the condition (6) for R. ☐

For a twisted automorphism (f, F) and a triangular structure R on H define a *twisted* triangular structure:

$$R^{(f,F)} = F_{21}(f \otimes f)(R)F^{-1}.$$

It is straightforward to verify that the properties of the R-matrix are preserved. Moreover, gauge isomorphic twisted automorphisms act equally. Indeed, for $g(x) = af(x)a^{-1}$ and $G = \Delta(a)F(a \otimes a)^{-1}$,

$$R^{(g,G)} = \Delta(a)F_{21}(a \otimes a)^{-1}(a \otimes a)(f \otimes f)(R)(a \otimes a)^{-1}(a \otimes a)F^{-1}t\Delta(a)^{-1} =$$

$$\Delta(a)R^{(f,F)}t\Delta(a)^{-1} = R^{(f,F)}.$$

Thus an action of the group $Out_{\mathbf{Tw}}(H)$ on the set $\mathcal{T}r(H)$ is defined.

Recall that the *Drinfeld element* of a triangular structure R on a Hopf algebra H is $u = \mu(I \otimes S)(R)$, where $S : H \to H$ is the antipode and $\mu : H \otimes H \to H$ is the multiplication map. It was proven in [5] that u (also see [8]) is a group-like element: $\Delta(u) = u \otimes u$. Moreover, if H is cocommutative then u is central and of order 2. Thus we have a map $u : Tr(H) \to (G(H) \cap Z(H))_2$ from the set of triangular structures to the 2-torsion subgroup of group-like central elements. This map admits a section, which sends a group-like involution u to an R-matrix

$$R_u = \frac{1}{2}(1 \otimes 1 + 1 \otimes u + u \otimes 1 - u \otimes u).$$

Proposition 4.4.2. *For a cocommutative Hopf algebra the Drinfeld element map is constant on orbits of the action of $Out^1_{\mathbf{Tw}}(H)$ on $Tr(H)$.*

Proof. It is quite straightforward to check that the Drinfeld element is constant on orbits. Indeed, it was proved in [5] that the map $\mu(I\otimes S) : (H^{\otimes 2})^H \to Z(H)$ is a homomorphism of algebras. Thus

$$\mu(I\otimes S)(F_{21}RF^{-1}) = \mu(I\otimes S)(F_{21})\mu(I\otimes S)(R)\mu(I\otimes S)(F)^{-1}$$

equals $\mu(I\otimes S)(R)$ since $\mu(I\otimes S)(F) = 1$. $\square$

Categorically triangular structures correspond to symmetric structures on the category of modules. Recall that a monoidal category $\mathcal{G}$ is *symmetric* if it is equipped with a collection of isomorphisms $c_{X,Y} : X\otimes Y \to Y\otimes X$ natural in $X, Y \in \mathcal{G}$ and satisfying the following axioms:

$$c_{X,Y}c_{Y,X} = 1, \quad \text{symmetry},$$

hexagon axioms:

$$c_{X,Y\otimes Z} = \phi_{Y,Z,X}(Y\otimes c_{X,Z})\phi^{-1}_{Y,X,Z}(c_{X,Y}\otimes Z)\phi_{X,Y,Z},$$

$$c_{X\otimes Y,Z} = \phi^{-1}_{Z,X,Y}(c_{X,Z}\otimes Y)\phi_{X,Z,Y}(X\otimes c_{Y,Z})\phi^{-1}_{X,Y,Z}.$$

Note that the last condition is redundant and included here for the sake of symmetry.

Proposition 4.4.3. *A triangular structure R on a bialgebra H defines a symmetric structure:*

$$c_{M,N} : M\otimes N \to N\otimes N, \quad m\otimes n \mapsto R(n\otimes m)$$

on the category $H - Mod$

Proof. The condition (6) implies that $c_{M,N}$ is a morphism of H-modules:

$$c_{M,N}(\Delta(x)(m\otimes n)) = Rt\Delta(x)(n\otimes m) = \Delta(x)R(n\otimes m) = \Delta(x)c_{M,N}(m\otimes n).$$

Triangle equations are equivalent to hexagon axioms. Normalisation for R gives the conditions $c_{1,N} = I$, $c_{M,1} = I$. Unitarity for R implies symmetry for c. $\square$

Monoidal autoequivalences of a monoidal category act naturally on the set of symmetric structures of the category. For a monoidal autoequivalence F and a symmetry c define the new symmetry c^F by

$$\begin{tikzcd}
F(X)\otimes F(Y) \arrow[r, "c^F_{F(X),F(Y)}"] & F(Y)\otimes F(X) \\
F(X\otimes Y) \arrow[u, "F_{X,Y}"] \arrow[r, "F(c_{X,Y})"] & F(Y)\otimes F(X) \arrow[u, "F_{Y,X}"']
\end{tikzcd}$$

It is straightforward to see that this action corresponds to the action of twisted homomorphisms on R-matrices.

5 Twisted automorphisms of universal enveloping algebras

In this part we look at twisted automorphisms of universal enveloping algebras over formal power series $k[[h]]$.

5.1 Invariant twists of $U(\mathfrak{g})[[h]]$

Let $F \in U(\mathfrak{g})[[h]]^{\mathfrak{g}}$ be a twist. Expand it as a formal power series in h with coefficients in $U(\mathfrak{g})$:

$$F = \sum_{i=0}^{\infty} F_i h^i.$$

Since invertible elements of a universal enveloping algebra over a field k of characteristic zero are trivial (scalars), the constant term F_0 of F must be the identity. Let $X = F_l$ be the first non-zero coefficient. The degree l part of the 2-cocycle equation is the additive 2-cocycle condition for X:

$$1 \otimes X + (I \otimes \Delta)(X) = X \otimes 1 + (\Delta \otimes I)(X).$$

Following Drinfeld [4] consider the complex $(H^{\otimes *}, \partial)$ with the differential $\partial : H^{\otimes n} \to H^{\otimes n+1}$ defined by

$$\partial(X) = 1 \otimes X + \sum_{i=1}^{n} (-1)^i (I^{\otimes i-1} \otimes \Delta \otimes I^{\otimes n-i-1})(X) + (-1)^{n+1}(X \otimes 1). \quad (8)$$

The cohomology of this complex admits a simple description.

Proposition 5.1.1. *For a universal enveloping algebra $H = U(\mathfrak{g})$ the alternation map $Alt_n : H^{\otimes n} \to \Lambda^n H$ induces an isomorphism of the n-th cohomology of the complex (8) and $\Lambda^n \mathfrak{g}$.*

Proof. Sketch of the proof (for details see [4]):
By the Poincare-Birkhoff-Witt theorem, the universal enveloping algebra $U(\mathfrak{g})$ is isomorphic as a coalgebra to the symmetric algebra $S^*(\mathfrak{g})$. The complex (8) for $H = S^*(\mathfrak{g})$ breaks into graded pieces:

$$S^n(\mathfrak{g}) \to \oplus_{i_1+i_2=n} S^{i_1}(\mathfrak{g}) \otimes S^{i_2}(\mathfrak{g}) \to \ldots \to \oplus_{i_1+\ldots+i_s=n} \otimes_{j=1}^{s} S^{i_j}(\mathfrak{g}) \to \ldots \to (\mathfrak{g})^{\otimes n} \quad (9)$$

The degree n piece is isomorphic to the cochain complex of the simplicial n-cube tensored (over symmetric group) with $(\mathfrak{g})^{\otimes n}$. $\square$

In particular, for an additive 2-cocycle $X \in U(\mathfrak{g})^{\otimes 2}$ there is $a \in U(\mathfrak{g})$ such that

$$X = \overline{X} + a \otimes 1 + 1 \otimes a - \Delta(a), \quad \overline{X} = Alt_2(X) = \frac{1}{2}(X - X_{21}). \quad (10)$$

Note that both X and $\overline{X}$ are $\mathfrak{g}$-invariant which makes $\partial(a) = a \otimes 1 + 1 \otimes a - \Delta(a)$ $\mathfrak{g}$-invariant. The last implies that a can be chosen to be $\mathfrak{g}$-invariant (central). To see it we prove a slightly more general statement.

Lemma 5.1.2. *For a universal enveloping algebra $H = U(\mathfrak{g})$ the alternation map $Alt_n : (H^{\otimes n})^H \to (\Lambda^n H)^H$ induces an isomorphism of the n-th cohomology of the subcomplex of H-invariant elements of (8) and the space of $\mathfrak{g}$-invariant skewsymmetric tensors $(\Lambda^n \mathfrak{g})^{\mathfrak{g}}$.*

Proof. The coalgebra isomorphism between $U(\mathfrak{g})$ and $S^*(\mathfrak{g})$ is $\mathfrak{g}$-invariant. The isomorphism between the degree n piece (8) and the cochain complex of the simplicial n-cube tensored with $(\mathfrak{g})^{\otimes n}$ is natural in $\mathfrak{g}$ and in particular $\mathfrak{g}$-invariant. $\square$

For the central a satisfying (10) the exponent $exp(ah^l)$ defines a gauge transformation of invariant twists $F \to F'$ where $F' = 1 + \overline{X}h^l + ...$ Thus we can assume (up to a gauge transformation) that $X \in (\Lambda^2\mathfrak{g})^{\mathfrak{g}}$. Note that $\mathfrak{g}$-invariance of X implies

$$[1 \otimes X, (I \otimes \Delta)(X)] = [X \otimes 1, (\Delta \otimes I)(X)] = 0.$$

Hence the exponent $exp(Xh^l)$ is an invariant twist on $U(\mathfrak{g})[[h]]$:

$$(exp(Xh^l) \otimes 1)(\Delta \otimes I)(exp(Xh^l)) = exp((X \otimes 1 + (\Delta \otimes I)(X))h^l) =$$

$$exp((1 \otimes X + (I \otimes \Delta)(X))h^l) = (1 \otimes exp(Xh^l))(I \otimes \Delta)(exp(Xh^l)).$$

Writing F as $exp(Xh^l) \circ F'$ we will have at least the first l components of F' being zero. Iterating the argument we prove the following.

Proposition 5.1.3. *Any invariant twist on $U(\mathfrak{g})[[h]]$ is gauge isomorphic to a product $\prod_{i=1}^{\infty} exp(X_i h^i)$ where $X_i \in (\Lambda^2\mathfrak{g})^{\mathfrak{g}}$.*

Note that the components X_i are defined uniquely by the twist F. Thus the set $\pi_0(\mathcal{A}ut^1_{\mathbf{Tw}}(U(\mathfrak{g})[[h]]))$ of classes of invariant twists is isomorphic to $(\Lambda^2\mathfrak{g})^{\mathfrak{g}}[[h]]$. Another way to see it is to use the logarithmic map. Since, for an invariant twist F, the factors $1 \otimes F$, $(I \otimes \Delta)(F)$ and $F \otimes 1$, $(\Delta \otimes 1)(F)$ of the 2-cocycle equation pairwise commute, the logarithm $log(F)$ is an additive 2-cocycle. Hence F is gauge isomorphic to $exp(X)$, where $X = Alt_2(log(F)) \in (\Lambda^2\mathfrak{g})^{\mathfrak{g}}[[h]]$.

To examine the group structure on $\pi_0(\mathcal{A}ut^1_{\mathbf{Tw}}(U(\mathfrak{g})[[h]]))$ we will use the Baker-Campbell-Hausdorff formula:

$$exp(X)exp(Y) = exp(X + Y)exp(A(X, Y)),$$

where $A(X, Y)$ is an element of the completion (with respect to the natural grading) of the free Lie algebra on X, Y. Note that

$$A(X, Y) = \frac{1}{2}[X, Y] + \text{higher terms}.$$

Now for $X, Y \in (\Lambda^2\mathfrak{g})^{\mathfrak{g}}[[h]]$ the commutator $[X, Y]$ is an additive $\mathfrak{g}$-invariant 2-cocycle and is symmetric. Thus there is a central $a(X, Y) \in Z(U(\mathfrak{g}))[[h]]$ such that

$$[X, Y] = a(X, Y) \otimes 1 + 1 \otimes a(X, Y) - \Delta(a(X, Y)). \tag{11}$$

Note also that any $Z \in (\Lambda^2\mathfrak{g})^{\mathfrak{g}}[[h]]$ must commute with $[X, Y]$. Indeed, the first commutator in

$$[[X, Y], Z] = [a(X, Y) \otimes 1 + 1 \otimes a(X, Y), Z] - [\Delta(a(X, Y)), Z]$$

is zero by centrality of $a(X, Y)$ while the second vanishes because of $\mathfrak{g}$-invariance of Z. In particular, the higher terms in $A(X, Y)$ are all zero and $A(X, Y) = \partial(\frac{1}{2}a(X, Y))$. Thus the exponent $exp(\frac{1}{2}a(X, Y))$ is a gauge transformation between the invariant twists $exp(X)exp(Y)$ and $exp(X + Y)$, which proves the following statement.

Theorem 5.1.4. *The group $\pi_0(\mathcal{A}ut^1_{\mathbf{Tw}}(U(\mathfrak{g})[[h]]))$ of classes of invariant twists is isomorphic to the addititve group $(\Lambda^2\mathfrak{g})^{\mathfrak{g}}[[h]]$.*

Moreover, we can calculate the associator class

$$\phi \in H^3(\pi_0(\mathcal{A}ut^1_{\mathbf{Tw}}(U(\mathfrak{g})[[h]])), \pi_1(\mathcal{A}ut^1_{\mathbf{Tw}}(U(\mathfrak{g})[[h]])))$$

of the Cat-group $\mathcal{A}ut^1_{\mathbf{Tw}}(U(\mathfrak{g})[[h]])$ of invariant twists. For our choice of gauge transformation between the invariant twists $exp(X)exp(Y)$ and $exp(X + Y)$ the logarithm of the associator on $exp(X), exp(Y), exp(Z)$ (as an element of $Z(\mathfrak{g})[[h]] \subset Z(U(\mathfrak{g}))[[h]]$) equals

$$\frac{1}{2}(a(X, Y) + a(X + Y, Z) - a(Y, Z) - a(X, Y + Z)). \tag{12}$$

Since both $\pi_i(\mathcal{A}ut^1_{\mathbf{Tw}}(U(\mathfrak{g})[[h]])), i = 0, 1$ are divisible abelian groups (vector spaces over k) with trivial action of the first on the second, the cohomology group

$$H^3(\pi_0(\mathcal{A}ut^1_{\mathbf{Tw}}(U(\mathfrak{g})[[h]])), \pi_1(\mathcal{A}ut^1_{\mathbf{Tw}}(U(\mathfrak{g})[[h]]))) \simeq H^3((\Lambda^2\mathfrak{g})^{\mathfrak{g}}[[h]], Z(\mathfrak{g})[[h]])$$

is isomorphic to the group $Hom(\Lambda^3((\Lambda^2\mathfrak{g})^{\mathfrak{g}}[[h]]), Z(\mathfrak{g})[[h]])$ of skew-symmetric maps via the map which takes a group 3-cocycle into its alternation. Clearly, the alternation of the (logarithm of the) associator (12) is zero. Thus the class ϕ is trivial.

Remark 5.1.5. *Tangent Cat-Lie algebra of $\mathcal{A}ut^1_{\mathbf{Tw}}(U(\mathfrak{g}))$.*

We can formalise the ground ring dependence of $\mathcal{A}ut^1_{\mathbf{Tw}}(U(\mathfrak{g}))$ in the form of a Cat-group valued pseudo-functor

$$k \mapsto \mathcal{A}ut^1_{\mathbf{Tw}}(U_k(\mathfrak{g}))$$

on the category of Artinian local commutative algebras. This point of view allows us to define in a standard way the tangent Cat-Lie algebra (crossed module of Lie algebras) for $\mathcal{A}ut^1_{\mathbf{Tw}}(U(\mathfrak{g}))$:

$$Z^2 \xleftarrow{\partial} C^1.$$

Here Z^2 is the Lie algebra (with respect to the commutator in $U(\mathfrak{g})^{\otimes 2}$) of 2-cocycles of the subcomplex of $\mathfrak{g}$-invariants of (8) and C^1 is the abelian Lie algebra of 1-cochains of the same subcomplex. The action of Z^2 on C^1 is trivial and the commutator $[X, \partial(a)]$ is zero for any $X \in Z^2$ and $a \in C^1$ (thus fulfilling the axioms of a crossed module of Lie algebras). Writing $[X,Y] = [Alt_2(X), Alt_2(Y)]$ as $\partial(a(X,Y))$ for $a(X,Y) = a(Alt_2(X), Alt_2(Y))$ as before we can see that $a(X,[Y,Z]) = 0$ so the Jacobiator of the crossed module of Lie algebras is trivial.

5.2 Separation for twisted automorphisms

Here we examine twisted automorphisms $(f,F) : H \to H$ where $H = U(\mathfrak{g})[[h]]$. The constant term (with respect to h) of F must be the identity. Thus by condition (1) the constant term of f must be an automorphism of the bialgebra H, hence must be induced by an automorphism of the Lie algebra $\mathfrak{g}$. These allow us to assume without loss of generality (up to an automorphism of $\mathfrak{g}$) that

$$f = I + \sum_{i=1}^{\infty} f_i h^i, \quad F = 1 + \sum_{i=1}^{\infty} F_i h^i.$$

Let $X = F_l$ be the first non-zero coefficient. As before, the degree l part of the 2-cocycle equation is the additive 2-cocycle condition for X:

$$1 \otimes X + (I \otimes \Delta)(X) = X \otimes 1 + (\Delta \otimes I)(X).$$

As before we write

$$X = \overline{X} + a \otimes 1 + 1 \otimes a - \Delta(a)$$

for $\overline{X} = Alt_2(X)$ and some $a \in U(\mathfrak{g})$. The exponent $exp(ah^l)$ defines a gauge transformation of twisted automorphisms $(f,F) \to (f',F')$ where $F' = 1 + \overline{X}h^l + ...$ Hence we can assume (up to a gauge transformation) that $X \in \Lambda^2\mathfrak{g}$. Now the left hand side of the degree l part of the condition (1), namely,

$$\sum_{i+j=l} (f_i \otimes f_j)\Delta(x) - \Delta(f_l(x)) = [X, \Delta(x)]$$

is symmetric while the right hand side is anti-symmetric. That means both sides are zero. In particular, $X \in \Lambda^2\mathfrak{g}$ is $\mathfrak{g}$-invariant and $exp(Xh^l)$ is a twist on $U(\mathfrak{g})[[h]]$. Writing (f,F) as $(1, exp(Xh^l)) \circ (f,F')$ we will have at least the first l components of F' being zero. Proceeding like that (using induction by the number of first successive zero components in F) we prove the following.

Proposition 5.2.1. *Any twisted automorphism of a universal enveloping algebra $U(\mathfrak{g})[[h]]$ is gauge isomorphic to a separated twisted automorphism, i.e. a twisted automorphism of the form $(1,F) \circ (f,1)$ where F is an invariant twist on $U(\mathfrak{g})[[h]]$ and f is a bialgebra automorphism of $U(\mathfrak{g})[[h]]$.*

5.3 The Cat-group $\mathcal{A}ut_{\mathbf{Tw}}(U(\mathfrak{g})[[h]])$

Since any symmetric invariant twist is isomorphic to a trivial one, the group

$$Aut_{\mathbf{Tw}}(U(\mathfrak{g})[[h]]) = \pi_0(\mathcal{A}ut_{\mathbf{Tw}}(U(\mathfrak{g})[[h]])) \simeq Out_{bialg}(U(\mathfrak{g})[[h]]) \ltimes (\Lambda^2\mathfrak{g})^{\mathfrak{g}}[[h]]$$

is the crossed product of the group of outer bialgebra automorphisms of $U(\mathfrak{g})[[h]]$ and the group of gauge classes of invariant twists. Any bialgebra automorphism of a universal enveloping algebra is induced by a Lie algebra automorphism. Thus the group $Out_{bialg}(U(\mathfrak{g})[[h]])$ in its turn is the crossed product

$$Aut(\mathfrak{g}) \ltimes (1 + hOutDer(\mathfrak{g})[[h]])$$

of the group of automorphisms of the Lie algebra $\mathfrak{g}$ and the exponent of the Lie algebra $hOutDer(\mathfrak{g})[[h]]$ of outer derivations of $\mathfrak{g}[[h]]$ of degree ≥ 1. The action of the subgroup $Aut(\mathfrak{g})$ on the group of gauge classes of invariant twists $(\Lambda^2\mathfrak{g})^{\mathfrak{g}}[[h]]$ is

$$f(F) = (f \otimes f)(F).$$

The action of the degree ≥ 1 part $(1+hOutDer(\mathfrak{g})[[h]])$ is induced by the action of the Lie algebra of derivations $Der(\mathfrak{g})$ on the space $(\Lambda^2(\mathfrak{g}))^{\mathfrak{g}}$:

$$dX = (d \otimes I + I \otimes d)(X).$$

It is straightforward to see that inner derivations act trivially (this is equivalent to $\mathfrak{g}$-invariance). To see that $\mathfrak{g}$-invariance is preserved by this action we need to verify that $d_x d(X) = 0$ for any $x \in \mathfrak{g}$. Here $d_x(y) = [x, y]$ is the inner derivation corresponding to x. Since $d_x d = dd_x + d_{d(x)}$ we have $d_x dX = dd_x(X) + d_{d(x)}X = 0$.

Note that $\pi_1(\mathcal{A}ut_{\mathbf{Tw}}(U(\mathfrak{g})[[h]])) = \pi_1(\mathcal{A}ut^1_{\mathbf{Tw}}(U(\mathfrak{g})[[h]])) = Z(\mathfrak{g})[[h]]$.

Proposition 5.3.1. *The associator*

$$\phi \in H^3(Aut_{\mathbf{Tw}}(U(\mathfrak{g})[[h]]), Z(U(\mathfrak{g})[[h]]))$$

of the Cat-group $\mathcal{A}ut_{\mathbf{Tw}}(U(\mathfrak{g})[[h]])$ is the image of the associator

$$\psi \in H^3(Out_{bialg}(U(\mathfrak{g})[[h]]), Z(U(\mathfrak{g})[[h]]))$$

of the Cat-group $\mathcal{A}ut_{bialg}(U(\mathfrak{g})[[h]])$ under the homomorphism of groups $Out_{bialg}(U(\mathfrak{g})[[h]]) \to Aut_{\mathbf{Tw}}(U(\mathfrak{g})[[h]])$.

Proof. We need to check that the associator is trivial if at least one of the arguments belongs to the subgroup $(\Lambda^2\mathfrak{g})^{\mathfrak{g}}[[h]]$. We have seen in section 5.1 that it is trivial if all three arguments are from $(\Lambda^2\mathfrak{g})^{\mathfrak{g}}[[h]]$. In section 5.5 we will construct the solution $a(X,Y)$ of (11) such that $a(g(X), g(Y)) = a(X,Y)$ for any automorphism $g \in Aut(\mathfrak{g})$ (and $a(dX,Y) + a(X,dY) = 0$ for any derivation $d \in Der(\mathfrak{g})$) thus covering the case when two of the arguments of the associator belong to $(\Lambda^2\mathfrak{g})^{\mathfrak{g}}[[h]]$. Finally, the fact $Out(\mathfrak{g})$ and $OutDer(\mathfrak{g})$ act on $(\Lambda^2\mathfrak{g})^{\mathfrak{g}}$ guarantees that the associator is trivial when one of the arguments belongs to the subgroup $(\Lambda^2\mathfrak{g})^{\mathfrak{g}}[[h]]$. $\square$

5.4 Twists on $U(\mathfrak{g})[[h]]$

Recall that $\mathbf{Tw}(k[[h]], U(\mathfrak{g})[[h]])$ is a complicated notation for the groupoid of twists on $U(\mathfrak{g})[[h]]$.

Writing the twist as a formal power series $F = \sum_{i=0}^{\infty} f_i h^i$, we get, for the degree n part of the 2-cocycle equation:

$$\sum_{i+j=n,\ i,j>n} ((1 \otimes f_i)(I \otimes \Delta)(f_j) - (f_i \otimes 1)(\Delta \otimes I)(f_j)) = 0.$$

The degree 1 part is simply $d(f_1) = 0$. Thus f_1 is an additive 2-cocycle and $r = Alt_2(f_1)$ belongs to $\Lambda^2(\mathfrak{g})$. Up to a gauge isomorphism we can assume that $f_1 = r$. The degree 2 part reads as

$$(1 \otimes f_1)(I \otimes \Delta)(f_1) - (f_1 \otimes 1)(\Delta \otimes I)(f_1) = d(f_2). \tag{13}$$

The left hand side is in general a 2-cocycle (this can be checked directly). Thus the alternation of the left hand side is an element of $\Lambda^3(\mathfrak{g})$. This element can be written explicitly if we assume as before that $f_1 = r$ is bi-primitive and skew-symmetric. Indeed, for such a choice, the left hand side of (13) has the form

$$r_{23}(r_{13} + r_{12}) - r_{12}(r_{13} + r_{23}).$$

Note that

$$Alt_3(r_{23}r_{13}) = \frac{1}{6}(r_{23}r_{13} + r_{31}r_{21} + r_{12}r_{32} - r_{13}r_{23} - r_{32}r_{12} - r_{21}r_{31}),$$

which for a skew-symmetric r equals

$$\frac{1}{6}([r_{23}, r_{13}] + [r_{23}, r_{12}] + [r_{13}, r_{12}]).$$

Note also that

$$Alt_3(r_{23}r_{13}) = Alt_3(r_{23}r_{12}) = -Alt_3(r_{12}r_{13}) = -Alt_3(r_{12}r_{23}).$$

Thus

$$Alt_3(r_{23}(r_{13} + r_{12}) - r_{12}(r_{13} + r_{23})) =$$
$$\frac{4}{6}([r_{23}, r_{13}] + [r_{23}, r_{12}] + [r_{13}, r_{12}]).$$

Together with (13) it means that

$$[r_{23}, r_{13}] + [r_{23}, r_{12}] + [r_{13}, r_{12}] = 0. \tag{14}$$

This equation is known as the *classical Yang-Baxter* equation (*CYBE*). Denote by $CYB(\mathfrak{g})$ the set of solutions (in $\Lambda^2(\mathfrak{g})$) of the classical Yang-Baxter equation. Thus we have the following.

Proposition 5.4.1. *The above defines a map*

$$\pi_0(\mathbf{Tw}(k[[h]], U(\mathfrak{g})[[h]])) \to CYB(\mathfrak{g}) \tag{15}$$

from the set of gauge isomorphism classes to the set of solutions to CYBE.

A section to the map (15) was constructed in [3] (see also [6]). In particular, the map (15) is surjective.

The map (15) can be extended to a map of groupoids in the following way. For a gauge automorphism $a \in U(\mathfrak{g})[[h]]$ of a twist $F \in \mathbf{Tw}(k[[h]], U(\mathfrak{g})[[h]])$ the condition $\Delta(a)F = F(a \otimes a)$ expanded in h gives

$$\sum_{i+j=n} \Delta(a_i)f_j = \sum_{i+j=n} f_j(a_i \otimes 1 + 1 \otimes a_i).$$

Since $a_0 = f_0 = 1$, the degree 1 part is $\Delta(x) + r = r + x \otimes 1 + 1 \otimes x$, where $x = a_1$ and $r = f_1$. Thus x belongs to $\mathfrak{g}$. After cancellation the skew-symmetric degree 2 part reads $\Delta(x)r = r(x \otimes 1 + 1 \otimes x)$ or

$$[r, x \otimes 1 + 1 \otimes x] = 0. \tag{16}$$

Define the *centraliser* $C_{\mathfrak{g}}(r)$ of r as the Lie subalgebra in $\mathfrak{g}$ of those x which satisfy (16). Denote by $\mathcal{CYB}(\mathfrak{g})$ a disconnected groupoid with the set of (classes of) objects $CYB(\mathfrak{g})$ and with the abelian automorphism groups $Aut_{\mathcal{CYB}(\mathfrak{g})}(r) = C_{\mathfrak{g}}(r)$ where $r \in \mathcal{CYB}(\mathfrak{g})$. Then the map (15) lifts to a functor of groupoids

$$\mathbf{Tw}(k[[h]], U(\mathfrak{g})[[h]]) \to \mathcal{CYB}(\mathfrak{g}).$$

The action of the cat-group $\mathcal{A}ut^1_{\mathbf{Tw}}(U(\mathfrak{g})[[h]])$ on $\mathbf{Tw}(k[[h]], U(\mathfrak{g})[[h]])$ corresponds to the action of $(\Lambda^2(\mathfrak{g}))^{\mathfrak{g}}$ on $CYB(\mathfrak{g})$ given by addition: for $X \in (\Lambda^2(\mathfrak{g}))^{\mathfrak{g}}$ and $r \in CYB(\mathfrak{g})$ the sum $X + r$ belongs to $CYB(\mathfrak{g})$. The action of $Aut(\mathfrak{g}[[h]])$ on $CYB(\mathfrak{g})$ boils down to the group action of $Aut(\mathfrak{g})$.

Remark 5.4.2.

It seems that the construction of [3] (as well as [6]) can be extended to a bijection $CYB_h(\mathfrak{g}) \to \pi_0(\mathbf{Tw}(k[[h]], U(\mathfrak{g})[[h]]))$. Here we understand $CYB_h(\mathfrak{g})$ as the set of solutions to CYBE in $\Lambda^2(\mathfrak{g})[[h]]$. $\mathfrak{g}$-equivariance of the construction from [3] would imply that the map $\mathcal{CYB}_h(\mathfrak{g}) \to \mathbf{Tw}(k[[h]], U(\mathfrak{g})[[h]])$ is a functor if we define automorphism groups in $\mathcal{CYB}_h(\mathfrak{g})$ to be exponents of Lie algebras $C_{\mathfrak{g}}(r)[[h]]$.

5.5 Geometric description

For $X \in \Lambda^2(\mathfrak{g})$ define its *support subspace* to be $\mathfrak{a}(X) = \{(l \otimes I)(X), l \in \mathfrak{g}^*\} \subset \mathfrak{g}$. Note that X belongs to $\Lambda^2\mathfrak{a}(X)$ and defines a linear isomorphism

$$\mathfrak{a}(X)^* \to \mathfrak{a}(X),\ l \mapsto (l\otimes)(X).$$

Thus X is the *Casimir* element for the symplectic form $b = b(X)$ on $\mathfrak{a}(X)$:

$$b((l \otimes I)(X), (l' \otimes I)(X)) = (l \otimes l')(X)$$

$$(I \otimes b)(X \otimes I) = (b \otimes I)(I \otimes X) = I : \mathfrak{a}(X) \to \mathfrak{a}(X).$$

The following geometric characterisation of solutions to CYBE was obtained by Drinfeld [3].

Proposition 5.5.1. *The support* $\mathfrak{s} = \{(I \otimes l)(r),\ l \in \mathfrak{g}^*\}$ *of a solution* $r \in \Lambda^2(\mathfrak{g})$ *to CYBE is a Lie subalgebra of* $\mathfrak{g}$*. The symplectic form* b *on* $\mathfrak{s}$ *is a Lie 2-cocycle.*

Proof. Applying $(I \otimes l \otimes m)$ to CYBE we get

$$[(I \otimes l)(r), (I \otimes m)(r)] + (I \otimes l')(r) + (I \otimes m')(r) = 0,$$

where $l'(x) = l([x, (I \otimes m)(r)])$ and $m'(x) = m([x, (l \otimes I)(r)]$. Thus $\mathfrak{s} \subset \mathfrak{g}$ is a Lie subalgebra.

To see that the form b is a 2-cocycle note that

$$b([(I \otimes l)(r), (I \otimes m)(r)], (I \otimes n)(r)) = (n \otimes m \otimes l)([r_{12}, r_{23}] - [r_{13}, r_{23}).$$

Thus the alternation (in l, m, n) of the left hand side is $(n \otimes m \otimes l)$ applied to a multiple of CYBE. ∎

In terms of $\mathfrak{s}$ and b the centraliser $C_{\mathfrak{g}}(r)$ is the stabiliser $St_{C_{\mathfrak{g}}(\mathfrak{s})}(b)$ of the form b in the centraliser $C_{\mathfrak{g}}(\mathfrak{s})$ of the Lie subalgebra $\mathfrak{s}$ in $\mathfrak{g}$.

For $\mathfrak{g}$-invariant $X \in \Lambda^2(\mathfrak{g})$, its support $\mathfrak{a}(X)$ is an ideal in $\mathfrak{g}$ and the form $b(X)$ is $\mathfrak{g}$-invariant.

Lemma 5.5.2. *For any* $\mathfrak{g}$*-invariant* $X \in \Lambda^2(\mathfrak{g})^{\mathfrak{g}}$,

$$[X_{13}, X_{23}] = 0. \tag{17}$$

Proof. Since X_{23} is $ad(\mathfrak{g})$-invariant $[X_{13}, X_{23}] = -[X_{12}, X_{23}]$ which equals $[X_{23}, X_{12}]$. Again, by $ad(\mathfrak{g})$-invariance of X_{12}, we have $[X_{23}, X_{12}] = -[X_{13}, X_{12}]$ which coincides with $[X_{13}, X_{21}]$ since $X_{12} = -X_{21}$. Now, by $ad(\mathfrak{g})$-invariance of X_{13}, $[X_{13}, X_{21}] = -[X_{13}, X_{23}]$ which finally implies

$$[X_{13}, X_{23}] = -[X_{13}, X_{23}].$$ ∎

It follows from the relation (17) that $\mathfrak{a}(X)$ is an abelian ideal for any $X \in \Lambda^2(\mathfrak{g})^{\mathfrak{g}}$:

$$[(l \otimes I)(X), (l' \otimes I)(X)] = [(l \otimes l' \otimes I)([X_{13}, X_{23}]) = 0.$$

Thus we assign an abelian ideal with an $\mathfrak{g}$-invariant symplectic form to any element of $\Lambda^2(\mathfrak{g})^{\mathfrak{g}}$. Conversely, for such a pair $(\mathfrak{a}, b)$ the Casimir element X_b of b obviously belongs to $\Lambda^2(\mathfrak{g})^{\mathfrak{g}}$. Thus we have the following.

Proposition 5.5.3. *The support construction establishes a bijection between* $\Lambda^2(\mathfrak{g})^{\mathfrak{g}}$ *and the set of pairs* $(\mathfrak{a}, b)$*, where* $\mathfrak{a} \subset \mathfrak{g}$ *is an abelian ideal and* b *is a* $\mathfrak{g}$*-invariant symplectic form of* $\mathfrak{a}$

For a Lie algebra $\mathfrak{g}$ denote by $nil(\mathfrak{g})$ its *nilradical*, i.e. the maximal nilpotent ideal (= sum of all nilpotent ideals).

Corollary 5.5.4. *For any Lie algebra* $\mathfrak{g}$*,*

$$\Lambda^2(\mathfrak{g})^{\mathfrak{g}} = \Lambda^2(nil(\mathfrak{g}))^{\mathfrak{g}}.$$

In particular $\Lambda^2(\mathfrak{g})^{\mathfrak{g}} = 0$ *if* $nil(\mathfrak{g}) = 0$*.*

Proof. Any abelian ideal is obviously nilpotent. Thus it must be contained in $nil(\mathfrak{g})$. □

Now we describe vector space structure of $\Lambda^2(\mathfrak{g})$ in terms of pairs $(\mathfrak{a}, b)$. Obviously, multiplication by scalars is given by the following rule:

$$c(\mathfrak{a}, b) = (\mathfrak{a}, cb), \text{ for } c \in k \setminus \{0\}.$$

The geometric presentation for the addition is more involved.

For $X_1, X_2 \in \Lambda^2(\mathfrak{g})$, the direct sum of their supports $\mathfrak{a}_1 \oplus \mathfrak{a}_2$ is equipped with the symplectic form $b_1 \oplus b_2$. Denote by

$$(\mathfrak{a}_1 \cap \mathfrak{a}_2)^{\perp} = \{(u_1, u_2) \in \mathfrak{a}_1 \oplus \mathfrak{a}_2 : b_1(u_1, x) = b_2(u_2, x)\ \forall x \in \mathfrak{a}_1 \cap \mathfrak{a}_2\}$$

the orthogonal complement of the anti-diagonal image of $\mathfrak{a}_1 \cap \mathfrak{a}_2$ in $\mathfrak{a}_1 \oplus \mathfrak{a}_2$. Denote by K the intersection $(\mathfrak{a}_1 \cap \mathfrak{a}_2) \cap (\mathfrak{a}_1 \cap \mathfrak{a}_2)^{\perp}$ which coincides with the kernel $ker(b_1|_{\mathfrak{a}_1\cap\mathfrak{a}_2} - b_2|_{\mathfrak{a}_1\cap\mathfrak{a}_2})$ of the difference of the symplectic forms b_i restricted to $\mathfrak{a}_1 \cap \mathfrak{a}_2$. The kernel of the surjection $\mathfrak{a}_1 \oplus \mathfrak{a}_2 \to \mathfrak{a}_1 + \mathfrak{a}_2 \subset \mathfrak{g}$ coincides with the anti-diagonal image of $\mathfrak{a}_1 \cap \mathfrak{a}_2$ and the short exact sequence

$$\mathfrak{a}_1 \cap \mathfrak{a}_2 \to \mathfrak{a}_1 \oplus \mathfrak{a}_2 \to \mathfrak{a}_1 + \mathfrak{a}_2$$

extends to a commutative diagram with short exact rows and columns:

$$\begin{tikzcd}
K \arrow[r] \arrow[d] & \mathfrak{a}_1 \cap \mathfrak{a}_2 \arrow[r] \arrow[d] & ((\mathfrak{a}_1 \cap \mathfrak{a}_2)/K)^* \arrow[d] \\
(\mathfrak{a}_1 \cap \mathfrak{a}_2)^{\perp} \arrow[r] \arrow[d] & \mathfrak{a}_1 \oplus \mathfrak{a}_2 \arrow[r] \arrow[d] & (\mathfrak{a}_1 \cap \mathfrak{a}_2)^* \arrow[d] \\
\mathfrak{a} \arrow[r] & \mathfrak{a}_1 + \mathfrak{a}_2 \arrow[r] & K^*
\end{tikzcd} \tag{18}$$

We can use the bottom row to define a subspace $\mathfrak{a} \subset \mathfrak{g}$ as the kernel of the map $\mathfrak{a}_1 + \mathfrak{a}_2 \to K^*$ induced by the map $\mathfrak{a}_1 \oplus \mathfrak{a}_2 \to (\mathfrak{a}_1 \cap \mathfrak{a}_2)^*$:

$$(u_1, u_2) \mapsto (x \mapsto b_1(u_1, x) - b_2(u_2, x))\ x \in \mathfrak{a}_1 \cap \mathfrak{a}_2\}.$$

Then the left column allows us to define a symplectic form on $\mathfrak{a}$. Indeed, the kernel of the restriction of $b_1 \oplus b_2$ to $(\mathfrak{a}_1 \cap \mathfrak{a}_2)^{\perp}$ is K. Thus $b_1 \oplus b_2$ induces a non-degenerate skew-symmetric bilinear form b on $(\mathfrak{a}_1 \cap \mathfrak{a}_2)^{\perp}/K = \mathfrak{a}$.

Proposition 5.5.5. *The subspace* $\mathfrak{a} \subset \mathfrak{g}$ *is the support of the sum* $X_1 + X_2 \in \Lambda^2(\mathfrak{g})$*. The Casimir element of the symplectic form* b *coincides with* $X_1 + X_2$*.*

Proof. First we need to verify that the support of $X_1 + X_2$ lies in $\mathfrak{a}$. For that purpose we need a more explicit description of the map $\mathfrak{a}_1 + \mathfrak{a}_2 \to K^*$. For $(l_1 \otimes I)(X_1) + (l_2 \otimes I)(X_2) \in \mathfrak{a}_1 + \mathfrak{a}_2$ it gives a linear function $K \to k$: if we write an element $x \in K$ as $x = (m_1 \otimes I)(X_1) = (m_2 \otimes I)(X_2)$ the value of this function on x is

$$(l_1 \otimes m_1)(X_1) - (l_2 \otimes m_2)(X_2).$$

Note that this expression is zero for any $l_i \in \mathfrak{g}^*$ such that $(l_1 \otimes I)(X_1) = (l_2 \otimes I)(X_2)$ so it is well defined on $\mathfrak{a}_1 + \mathfrak{a}_2$ (does not depend on the presentation $(l_1 \otimes I)(X_1) + (l_2 \otimes I)(X_2)$). The function associated to an element $(l \otimes I)(X_1) + (l \otimes I)(X_2)$ of the support of $X_1 + X_2$ is clearly zero. Thus the support of $X_1 + X_2$ belongs to $\mathfrak{a}$.

Now it suffices to check that $(b \otimes I)(I \otimes (X_1 + X_2))$ is the identity on $\mathfrak{a}$, or that $(b \otimes l)(I \otimes (X_1 + X_2)) = l$ for any linear function l on $\mathfrak{a}$. The bilinear form b on $\mathfrak{a}$ assigns to $(l_1 \otimes I)(X_1) + (l_2 \otimes I)(X_2), (m_1 \otimes I)(X_1) + (m_2 \otimes I)(X_2) \in \mathfrak{a}$ the number

$$(l_1 \otimes m_1)(X_1) - (l_2 \otimes m_2)(X_2).$$

In particular, for $x = (l_1 \otimes I)(X_1) + (l_2 \otimes I)(X_2) \in \mathfrak{a}$,

$$(b \otimes l)(x \otimes (X_1 + X_2)) = b(x, (I \otimes l)(X_1) + (I \otimes l)(X_2)) =$$

$$(l_1 \otimes l)(X_1) + (l_2 \otimes l)(X_2) = l(x). \qquad \square$$

In particular, if both X_1, X_2 are $\mathfrak{g}$-invariant we get a geometric description of the addition on $\Lambda^2(\mathfrak{g})^{\mathfrak{g}}$.

Remark 5.5.6.

According to proposition 5.5.3, the support subspace $\mathfrak{a}$ of $X_1 + X_2$ must be an abelian ideal. The fact that $\mathfrak{a}$ is an ideal follows from the $\mathfrak{g}$-equivariance of the construction for $\mathfrak{a}$, while the abelian property can be checked by direct computation. First note that the commutant $[\mathfrak{a}_1, \mathfrak{a}_2]$ lies in K. As a consequence, the map $\mathfrak{a}_1 + \mathfrak{a}_2 \to K^*$ is a homomorphism of Lie algebras (with abelian structure on K^*). Thus $\mathfrak{a}$ is a Lie subalgebra. To see that the commutant $[\mathfrak{a}, \mathfrak{a}]$ is zero, it is enough to check that $b_1([u_1 + u_2, v_1 + v_2], y) = 0$ for any $u_1 + u_2, v_1 + v_2 \in \mathfrak{a}$ and any $y \in \mathfrak{a}_1$. Writing

$$[u_1 + u_2, v_1 + v_2] = [u_1, v_2] + [u_2, v_1] = [u_1, v_2] - [v_1, u_2],$$

we need to verify that $b_1([u_1, v_2], y) = b_1([v_1, u_2], y)$. Indeed, by $\mathfrak{g}$-invariance of b_i and the defining relations for $u_1 + u_2, v_1 + v_2$ (together with $[\mathfrak{a}_1, \mathfrak{a}_2] \subset K$), we have the chain of equalities:

$$b_1([u_1, v_2], y) = -b_1(u_1, [y, v_2]) = -b_2(u_2, [y, v_2]) =$$

$$b_2([y, u_2], v_2) = b_1([y, u_2], v_1) = -b_1(y, [v_1, u_2]) = b_1([v_1, u_2], y).$$

Note also that the arrows of the diagram (18) are homomorphisms of vector spaces and not of Lie algebras. To turn it into a diagram of Lie algebras it suffices to introduce an appropriate Lie algebra structure on the direct sum $\mathfrak{a}_1 \oplus \mathfrak{a}_2$. Denote by $\mathfrak{a}_1 \bowtie \mathfrak{a}_2$ the Lie algebra of pairs (x_1, x_2), $x_i \in \mathfrak{a}_i$ with the bracket:

$$[(x_1, x_2), (y_1, y_2)] = \frac{1}{2}([x_2, y_1] + [x_1, y_2], [x_2, y_1] + [x_1, y_2]).$$

Then the maps of the diagram

$$\begin{tikzcd}
K \arrow[r] \arrow[d] & \mathfrak{a}_1 \cap \mathfrak{a}_2 \arrow[r] \arrow[d] & ((\mathfrak{a}_1 \cap \mathfrak{a}_2)/K)^* \arrow[d] \\
(\mathfrak{a}_1 \cap \mathfrak{a}_2)^\perp \arrow[r] \arrow[d] & \mathfrak{a}_1 \bowtie \mathfrak{a}_2 \arrow[r] \arrow[d] & (\mathfrak{a}_1 \cap \mathfrak{a}_2)^* \arrow[d] \\
\mathfrak{a} \arrow[r] & \mathfrak{a}_1 + \mathfrak{a}_2 \arrow[r] & K^*
\end{tikzcd} \tag{19}$$

become homomorphisms of Lie algebras (one should think of objects in the right column as abelian Lie algebras).

When X_1 is $\mathfrak{g}$-invariant and X_2 is a solution to CYBE the sum $X_1 + X_2$ is also a solution to the CYBE. In this case the support space $\mathfrak{a}$ is still a Lie algebra. Again the diagram (18) can be made into a diagram of Lie algebras. One needs to think of $\mathfrak{a}_1 \bowtie \mathfrak{a}_2$ as the Lie algebra with the bracket:

$$[(x_1, x_2), (y_1, y_2)] = \frac{1}{2}([x_2, y_1] + [x_1, y_2], [x_2, y_1] + [x_1, y_2] + 2[x_2, y_2]).$$

In other words, the diagram (19) defines an action of the set of abelian ideals with invariant symplectic forms (which coincides with the vector space $(\Lambda^2\mathfrak{g})^\mathfrak{g}$) on the set of subalgebras with non-degenerate 2-cocycles (the set $CYB(\mathfrak{g})$).

Now we give a geometric description of the map

$$(\Lambda^2\mathfrak{g})^\mathfrak{g} \otimes (\Lambda^2\mathfrak{g})^\mathfrak{g} \to (S^3\mathfrak{g})^\mathfrak{g}, \tag{20}$$

which for X_1, X_2 gives a coboundary a for the commutator

$$[X_1, X_2] = a \otimes 1 + 1 \otimes a - \Delta(a).$$

Here we identify $(S^*\mathfrak{g})^\mathfrak{g}$ with the centre $Z(U(\mathfrak{g})) = U(\mathfrak{g})^\mathfrak{g}$ of the universal enveloping algebra by means of the Kirillov-Duflo isomorphism.

For two abelian ideals $\mathfrak{a}_1, \mathfrak{a}_2 \subset \mathfrak{g}$ denote by $\mathfrak{b} = [\mathfrak{a}_1, \mathfrak{a}_2]$ their commutant. Note that orthogonal complements $\mathfrak{b}^\perp_{b_i}$ with respect to invariant symplectic forms b_i on $\mathfrak{a}_i$ have the following commutation property:

$$[\mathfrak{b}^\perp_{b_1}, \mathfrak{a}_2] = [\mathfrak{a}_1, \mathfrak{b}^\perp_{b_2}] = 0. \tag{21}$$

Indeed, for $x \in \mathfrak{b}_{b_1}^{\perp}, y \in \mathfrak{a}_2$,

$$b_1([x,y],z) = -b_1(x,[z,y]) = 0, \quad \forall z \in \mathfrak{a}_1.$$

Hence $[x,y] = 0$. Similarly for $[\mathfrak{a}_1, \mathfrak{b}_{b_2}^{\perp}]$. Now define a map $\mathfrak{b}^* \otimes \mathfrak{b}^* \to \mathfrak{b}$ by sending $l_1 \otimes l_2$ into $[x_1, x_2]$, where $x_i \in \mathfrak{a}_i$ are defined by

$$b_i(x_i, u) = l_i(u), \quad \forall u \in \mathfrak{b}.$$

The elements x_i are defined up to $\mathfrak{b}_{b_i}^{\perp}$ so in view of the relations (21) the commutator $[x_1, x_2]$ is well defined. Obviously, the map $\mathfrak{b}^* \otimes \mathfrak{b}^* \to \mathfrak{b}$ corresponds to a 3-vector $c \in \mathfrak{b}^{\otimes 3}$, which is $\mathfrak{g}$-invariant by the construction. To see that this 3-vector is symmetric, think of it as a map

$$(\mathfrak{b}^*)^{\otimes 3} \to k, \quad l_1 \otimes l_2 \otimes l_3 \mapsto l_3([x_1, x_2]).$$

Find $y_i \in \mathfrak{a}_i$ such that

$$b_i(y_i, u) = l_3(u), \quad \forall u \in \mathfrak{b}.$$

Then

$$l_3([x_1,x_2]) = b_1(y_1,[x_1,x_2]) = -b_1([y_1,x_2],x_1) = l_1([y_1,x_2]),$$

which in terms of the 3-vector c means that $c_{13} = c$. Similarly

$$l_3([x_1,x_2]) = b_2(y_2,[x_1,x_2]) = -b_2([x_1,y_2],x_2) = l_2([x_1,y_2]),$$

which means that $c_{23} = c$.

Lemma 5.5.7. *The element a corresponding to the invariant $c \in S^3(\mathfrak{g})^{\mathfrak{g}}$ via the isomorphism $S^*(\mathfrak{g})^{\mathfrak{g}} \to Z(U(\mathfrak{g}))$ satisfies the equation*

$$[X_1, X_2] = a \otimes 1 + 1 \otimes a - \Delta(a).$$

Proof. Since $\mathfrak{b} = [\mathfrak{a}_1, \mathfrak{a}_2]$ is isotropic with respect to the forms b_i we can choose subspaces $L_i \subset \mathfrak{a}_i$, which are Lagrangian with respect to b_i respectively and contain $\mathfrak{b}$. Write $\mathfrak{a}_i = L_i \oplus L_i^*$. A choice of bases $L_1 =< e_i >,\ L_2 =< f_j >$ allows us to write

$$X_1 = \sum_i e_i \wedge e^i, \quad X_2 = \sum_j f_j \wedge f^j.$$

Here e^i (f^j) is the dual basis in L_1^* (respectively L_2^*). By the choice, $[L_1, L_2] = 0$ so

$$[X_1, X_2] = \sum_{i,j} [e_i \wedge e^i, f_j \wedge f^j] = \sum_{i,j} e_i e_j \odot [e^i, f^j].$$

Here $\odot$ is the symmetric product $x \odot y = x \otimes y + y \otimes x$. Since the commutator pairing $[\ ,\ ] : L_1^* \otimes L_2^* \to \mathfrak{b}$ factors through $c : \mathfrak{b}^* \otimes \mathfrak{b}^* \to \mathfrak{b}$ we can write

$$a = \sum_{i,j} e_i f_j [e^i, f^j] = \sum_{s,t} x_s x_t c(x^s, x^t),$$

where x_s is a basis of $\mathfrak{b}$. Finally note that the element c belongs to $S^*(\mathfrak{b})^{\mathfrak{g}} = U(\mathfrak{b})^{\mathfrak{g}} \subset Z(U(\mathfrak{g}))$ so we do not need to worry about the particular choice of isomorphism $S^*(\mathfrak{g})^{\mathfrak{g}} \to Z(U(\mathfrak{g}))$. □

Remark 5.5.8.

It follows from the invariance of the construction of c that $c(g(X_1), g(X_2)) = c(X_1, X_2)$ for an automorphism g of the Lie algebra $\mathfrak{g}$.

We can use a symmetric 3-vector $c \in \mathfrak{b}^{\otimes 3}$ on a vector space $\mathfrak{b}$ to define a structure of (meta-abelian) Lie algebra $\mathfrak{g}(\mathfrak{b}, c)$ on the vector space $\mathfrak{b}^* \oplus \mathfrak{b}^* \oplus \mathfrak{b}$:

$$[(l_1, m_1, x_1), (l_2, m_2, x_2)] = (0, 0, (l_1 \otimes m_2 \otimes I - l_2 \otimes m_1 \otimes I)(c)).$$

The subspaces

$$\mathfrak{a}_1 = \mathfrak{b}^* \oplus 0 \oplus \mathfrak{b}, \quad \mathfrak{a}_2 = 0 \oplus \mathfrak{b}^* \oplus \mathfrak{b}$$

are abelian ideals in $\mathfrak{g}(\mathfrak{b}, c)$. The symplectic forms b_i on $\mathfrak{a}_i$

$$b_1((l_1, 0, x_1), (l_2, 0, x_2)) = l_1(x_2) - l_2(x_1),$$

$$b_2((0, m_1, x_1), (0, m_2, x_2)) = m_1(x_2) - m_2(x_1)$$

are $\mathfrak{g}(\mathfrak{b}, c)$-invariant. Indeed, by $c_{13} = c$

$$b_1([(0, m, x), (l_1, 0, x_1)], (l_2, 0, x_2)) = b_1(-(0, 0, (l_1 \otimes m \otimes I)(c)), (l_2, 0, x_2)) =$$

$= (l_1 \otimes m \otimes l_2)(c)$, which coincides with

$$b_1((l_1, 0, x_1), [(l_2, 0, x_2), (0, m, x)]) = b_1((l_1, 0, x_1), (0, 0, (l_2 \otimes m \otimes I)(c))) = (l_2 \otimes m \otimes l_1)(c).$$

Similarly, by $c_{23} = c$

$$b_2([(0, m_1, x_1), (l, 0, x)], (0, m_2, x_2)) = b_1(-(0, 0, (l \otimes m_1 \otimes I)(c)), (0, m_2, x_2)) =$$

$= (l \otimes m_1 \otimes m_2)(c)$, which coincides with

$$b_2((0, m_1, x_1), [(l, 0, x), (0, m_2, x_2)]) = b_2((0, m_1, x_1), (0, 0, (l \otimes m_2 \otimes I)(c))) =$$

$= (l \otimes m_2 \otimes m_1)(c)$. Clearly, the 3-vector corresponding to the pair $(\mathfrak{a}_i, b_i)$ via (20) is c.

The above construction is universal in the following sense. For a pair $(\mathfrak{a}_i, b_i)$ of abelian ideals with symplectic invariant forms, the sum $\mathfrak{a}_1 + \mathfrak{g}_2$ (which is a meta-abelian ideal) maps on to $\mathfrak{g}(\mathfrak{b}, c)$, where $\mathfrak{b} = [\mathfrak{a}_1, \mathfrak{a}_2]$ and c is the 3-vector defined by (20). Note that $\mathfrak{b}$ is isotropic in both $\mathfrak{a}_i$ so the maps $\mathfrak{a}_i \to \mathfrak{b}^*$ defined by the forms b_i factor through $\mathfrak{a}_i/\mathfrak{b}$. Thus we have a diagram

$$\begin{tikzcd}
\mathfrak{b} \arrow[r] \arrow[d] & \mathfrak{a}_1 + \mathfrak{a}_2 \arrow[r] \arrow[d] & \mathfrak{a}_1/\mathfrak{b} \oplus \mathfrak{a}_2/\mathfrak{b} \arrow[d] \\
\mathfrak{b} \arrow[r] & \mathfrak{g}(\mathfrak{b}, c) \arrow[r] & \mathfrak{b}^* \oplus \mathfrak{b}^*
\end{tikzcd}$$

The middle vertical map is a homomorphism of Lie algebras since the right vertical map is compatible with commutator pairings $\mathfrak{a}_1/\mathfrak{b} \otimes \mathfrak{a}_2 \to \mathfrak{b}$ (induced from $\mathfrak{a}_1 + \mathfrak{g}_2$) and $\mathfrak{b}^* \otimes \mathfrak{b}^* \to \mathfrak{b}$ (in $\mathfrak{g}(\mathfrak{b}, c)$).

Example 5.5.9. *Heisenberg algebra*

Let (V, b) be a symplectic vector space and $\mathfrak{g} = \mathcal{H}(V, b)$ its Heisenberg Lie algebra with the central generator c. Then the map $v \mapsto v \wedge c = v \otimes c - c \otimes v$ is an isomorphism $V \to (\Lambda^2\mathfrak{g})^{\mathfrak{g}}$. Clearly $v \wedge c$ is $\mathfrak{g}$-invariant:

$$[u \otimes 1 + 1 \otimes u, v \wedge c] = [u, v] \wedge c = b(u, v)c \wedge c = 0.$$

To see that there are no other $\mathfrak{g}$-invariant elements in $\Lambda^2\mathfrak{g}$ write $\mathfrak{g}$ as $V \oplus < c >$. Then $\Lambda^2\mathfrak{g} = \Lambda^2 V \oplus V$ with the $\mathfrak{g}$-action on the first component $V \otimes \Lambda^2 V \to V$ being induced by the bilinear form b and hence having no invariants.

The commutator (in $U(\mathfrak{g})^{\otimes 2}$) of two elements from $(\Lambda^2\mathfrak{g})^{\mathfrak{g}}$

$$[v \wedge c, u \wedge c] = b(v, u)(c \otimes c^2 + c^2 \otimes c)$$

is a coboundary $a \otimes 1 + 1 \otimes a - \Delta(a)$. For example, we can choose $a : \lambda^2(\Lambda^2\mathfrak{g}) \to U(\mathfrak{g})$ to be

$$a(v \wedge c, u \wedge c) = \frac{b(u, v)}{3}c^3.$$

The subalgebras in $\mathcal{H}(V, b)$ break into two classes depending on whether or not they contain the centre. A subalgebra of the first type has the form $U \oplus < c >$ for an arbitrary subspace $U \subset V$, while the second type is simply an *isotropic* subspace U of V (isotropic means that the restriction of the symplectic form b to U is zero). A non-degenerate 2-cocycle on a subalgebra of the second type is a symplectic form. For a subalgebra of the first type $U \oplus < c >$ with a non-degenerate 2-cocycle β denote by U' the orthogonal complement (with respect to β) of c in U. Clearly the restriction of β on U' is non-degenerate. Moreover, the 2-cocycle condition implies that U' is isotropic with respect to b. This corresponds to the fact that any $X \in CYB(\mathfrak{g})$ for $\mathfrak{g} = \mathcal{H}(V, b)$ can be uniquely written as $Y + v \wedge c$ for $Y \in CYB(\mathfrak{g})$ supported on a subalgebra of the second type. In particular, $(\Lambda^2\mathfrak{g})^{\mathfrak{g}}$ acts transitively on $CYB(\mathfrak{g})$ with orbits corresponding to subalgebras of the second type with symplectic forms on them.

References

[1] R. Brown, C.B. Spencer, *G*-groupoids, crossed modules and the fundamental groupoid of a topological group. Nederl. Akad. Wetensch. Proc., Ser. A, 38 (1976), no. 4, 296–302.

[2] A. Davydov, Finite groups with the same character tables, Drinfel'd algebras and Galois algebras. Algebra (Moscow, 1998), 99–111, de Gruyter, Berlin, 2000.

[3] V. G. Drinfel'd, Constant quasiclassical solutions of the Yang-Baxter quantum equation. (Russian) Dokl. Akad. Nauk SSSR 273 (1983), no. 3, 531–535.

[4] V. G. Drinfel'd, Quasi-Hopf algebras. (Russian) Algebra i Analiz 1 (1989), no. 6, 114–148; translation in Leningrad Math. J. 1 (1990), no. 6, 1419–1457.

[5] V. G. Drinfel'd, Almost cocommutative Hopf algebras. (Russian) Algebra i Analiz 1 (1989), no. 2, 30–46; translation in Leningrad Math. J. 1 (1990), no. 2, 321–342

[6] P. Etingof, O. Schiffmann, Lectures on quantum groups. Lectures in Mathematical Physics. International Press, Boston, MA, 1998. 239 pp.

[7] A. Joyal, R. Street, Braided tensor categories. Adv. Math. 102 (1993), no. 1, 20–78.

[8] V. Liubashenko, Super-analisis and solutions of triangle equation. Ph. D. thesis, 1986.

[9] S. MacLane, Categories for the Working Mathematician, Springer-Verlag, 1988.

[10] S. Montgomery, Hopf algebras and their actions on rings. CBMS Regional Conference Series in Mathematics, 82. American Mathematical Society, Providence, RI, 1993. 238 pp

[11] Radford D., Minimal quasitriangular Hopf algebras, J Algebra, 157 (1993), n 2, 285-315.

[12] P. Schauenburg, Hopf bi-Galois extensions. Comm. Algebra 24 (1996), no. 12, 3797–3825.

[13] R. Street, Quantum groups. A path to current algebra. Cambridge University Press, 2007, 141 pp.

[14] K.-H.Ulbrich, On Galois H-dimodule algebras, Math. J. Okayama Univ., 31 (1989), 79-84.

[15] J. H. C. Whitehead, Combinatorial homotopy. I,II. Bull. Amer. Math. Soc. 55, (1949), 213–245, 453–496.