ON THE SOLUTIONS OF THE HOMOGENEOUS COMPLEX MONGE-AMPÈRE EQUATION.

by

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1. Introduction

In [5] Stoll showed that if M is a non compact, connected complex manifold of dimension m and $\tau : M \to [0,+\infty)$ is a C^∞ exhaustion function such that $dd^c\tau > 0$ on M and $(dd^c\log\tau)^m \equiv 0$ on $M - \tau^{-1}(0)$, then there exists a biholomorphic map $\Phi : \mathbb{C}^m \to M$ such that for all $Z \in \mathbb{C}^m$ we have $\tau\circ\Phi(Z) = \|Z\|^2$. Stoll's proof was simplified in various ways (see [2],[7]) and Stoll himself was able to give a version of this theorem on complex spaces ([6]). The next step has been to study the case when the exhaustion verifies the same assumtions on $M - \tau^{-1}(0)$ and some weaker ones on $\tau^{-1}(0)$. This was carried out in several directions by Burns [2], Wong [8] and by the author in [4].

Stoll's theorem and the successive results can be viewed from different angles. On one hand they allow one to characterize complex manifolds which carry a strictly plurisubharmonic exhaustion with some additional properties - most notheworthy its logarithm satisfies the complex homogeneous Monge-Ampère equation. On the other hand theese results give a classification, up to biholomorphic maps, for certain kind of solution of the complex homogeneous Monge-Ampère equation. This is of particular interest since, at the moment, there is no satisfactory understanding of this equation from the PDE point of view.

It is easy to give examples of solutions which are not classified by the known theory. If $H : \mathbb{C}^m \to \mathbb{R}$ is a positive homogeneous polynomial of bidegree (p,p) (i.e. such that $H(\lambda Z) = |\lambda|^{2p}H(Z)$ for all $\lambda \in \mathbb{C}$ and $Z \in \mathbb{C}^m$) with the property that $d\bar{d}^cH > 0$ on $\mathbb{C}^m - \{0\}$, then $\mathrm{rank}_{\mathbb{C}} dd^c\log H = m-1$ and therefore $(dd^c\log H)^m \equiv 0$ on $\mathbb{C}^m - \{0\}$. Clearly up to linear isomorphisms (and in fact biholomorphisms) the only positive homogeneous polynomial of bidegree $(1,1)$ is $\| \|^2$. For $p > 1$ it is known that there are many non equivalent such polynomials.

In light of Stoll's theorem it is natural to ask whether it is possible to characterize the solutions of the complex homogeneous Monge-Ampère equation which pull back via a biholomorphic map to $\log H$ where H is a homogeneous polynomial of bidegree (p,p), $p > 1$. Related to this problem there is also a question of Burns ([2]) who asks whether a positive homogeneous polynomial H on $\mathbb{C}^m$ (i.e. such that for some positive integer n one has $H(tZ) = t^n H(Z)$ for all $t \in \mathbb{R}$ and $Z \in \mathbb{C}^m$) such that $dd^c H > 0$ and $(dd^c \log H)^m \equiv 0$ on $\mathbb{C}^m - \{0\}$ has to be necessarily of bidegree (p,p) for some p.

In this paper we address theese problems. Firstly, we give a positive answer to Burns' question (Theorem 3.4). Then we take over the general case and we are able to give a characterization (Theorem 4.3) which coincides with Stoll's result when $p = 1$.

The main difference with Stoll's theorem is that, when $p > 1$, one has that the exhaustion is not anylonger strictly plurisubharmonic on its zero set since its order of vanishing is too high. This difficulty is overcome by taking suitable roots and using the classification of non smooth exhaustions of Monge-Ampère type given in [4] where we characterized the strictly plurisubharmonic Finsler metrics over $\mathbb{C}^m$. Some of the results have alternative proofs. We chose those which we felt were more elementary and made the paper as selfcontained as possible.

A word about notations. Upper indices will denote components and lower ones derivatives. Summation conventions are also used unless they may cause confusion. Finally we denote $d = \partial + \bar{\partial}$ and $d^c = \frac{i}{4\pi}(\bar{\partial} - \partial)$ so that $dd^c = \frac{i}{2\pi}\partial\bar{\partial}$.

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2. Preliminaries

a) The Monge-Ampère foliation.

Let M be a connected complex manifold of dimension m and $\tau : M \to (0,R)$ be a C^∞ proper function such that

$$(2.1)\qquad dd^c\tau > 0$$

$$(2.2)\qquad (dd^c\log\tau)^m \equiv 0.$$

Since

$$(2.3)\qquad dd^c\tau = (dd^c\log\tau + d\log\tau \wedge d^c\log\tau),$$

by taking exterior powers, we get using (2.1) and (2.2)

$$(2.4)\qquad 0 < (dd^c\tau)^m = \tau^m(dd^c\log\tau)^{m-1} \wedge d\log\tau \wedge d^c\log\tau.$$

From (2.4) one has immediately $d\tau = \tau d\log\tau \neq 0$ and $(dd^c\log\tau)^{m-1} \neq 0$. Thus, because of (2.2), $\mathrm{rank}_{\mathbb{C}} dd^c\log\tau = m-1$. Using the equality

$$(2.5)\qquad \tau^2 dd^c\log\tau = \tau dd^c\tau - d\tau \wedge d^c\tau,$$

taking exterior powers and recalling (2.2), one has

$$(2.6)\qquad \tau(dd^c\tau)^m = (dd^c\tau)^{m-1} \wedge d\tau \wedge d^c\tau$$

so that, if (2.1) is satisfied, it follows that (2.2) is equivalent to the local equation

$$(2.7) \qquad \tau = \tau_{\bar{\nu}}\tau^{\bar{\nu}\mu}\tau_{\mu}$$

where $(\tau^{\bar{\nu}\mu}) = (\tau_{\mu\bar{\nu}})^{-1}$.

As $dd^c\log\tau$ has rank $m-1$ and it is closed, a rank 1, integrable distribution is defined on M by

$$(2.8) \quad \Sigma = \mathrm{Ann}\, dd^c\log\tau = \{V \in T^{1,0}(M) \mid dd^c\log\tau(V,\bar{V}) = 0\}.$$

The maximal integral manifold of Σ are Riemann surfaces and define the so called <u>Monge-Ampère</u> <u>foliation</u> associated to τ. By construction the leaves of the Monge-Ampère foliation are exactly the one dimensional complex submanifold of M along which $\log\tau$ is harmonic (for more details see [1] or [7] for example).

If X is the <u>complex</u> <u>gradient</u> of τ, i.e. the vector field dual with respect to the Kähler metric $dd^c\tau > 0$ to the form $\partial\tau$, then in local coordinates

$$(2.9) \qquad X = X^{\mu}\frac{\partial}{\partial z^{\mu}} = \tau^{\bar{\nu}\mu}\tau_{\bar{\nu}}\,\frac{\partial}{\partial z^{\mu}}.$$

From (2.7) and (2.9) it follows that $X(\tau) = \tau$ and thus, as $d\tau \neq 0$, we have $X \neq 0$ on M. Again from (2.7) and (2.9), a simple calculation shows that $dd^c\log\tau(X,\bar{X}) = 0$ and therefore Σ is a trivial subbundle of $T^{1,0}(M)$ generated by X. The leaves of the Monge-Ampère foliation are then just the integral (complex) curves of X. In particular one should note that the Monge-Ampère foliation is holomorphic (i.e. Σ is a holomorphic subbundle of $T^{1,0}(M)$) if and only if X is holomorphic. This is quite an exceptional occurrence although X is always holomorphic when restricted to one leaf (see [5], Proposition 3.5 for example).

It should be noted that when (2.1) and (2.2) are satisfied, then we have also

(2.10) $\quad dd^c \log\tau \geq 0.$

For a proof see for instance [8], Remark 2 in Section 5. We shall also need two simple lemmata. The first one is a trivial consequence of the definition of complex gradient and therefore we state it without proof.

Lemma 2.1. For $j = 1,2$, let M_j be a connected complex manifold of dimension m and $\tau_j : M_j \to (0,R)$ be a C^∞ proper function satisying (2.1) and (2.2). If $\Phi : M_1 \to M_2$ is a biholomorphic map such that $\tau_1 = \tau_2 \circ \Phi$ and X_j is the complex gradient of τ_j, then $X_2 = \Phi_* X_1$ and X_1 is holomorphic if and only if X_2 is so.

Lemma 2.2. Let M be a connected complex manifold of dimension m and $\tau : M \to (0,R)$ be a C^∞ proper function satisfying (2.1) and (2.2). If p is a positive integer and $\sigma = \tau^{1/p}$, then also σ satisfies (2.1) and (2.2).

Proof. Since τ satisfies (2.1) and (2.2), we have $d\tau \neq 0$ and $\mathrm{rank}_{\mathbb{C}} dd^c \log\tau = m - 1$ on M. Thus

(2.11) $\quad d\sigma = \frac{1}{p}\tau^{(1-p)/p}\, d\tau \neq 0 \qquad d\log\sigma = \sigma^{-1} d\sigma \neq 0$

(2.12) $\quad \mathrm{rank}_{\mathbb{C}} dd^c \log\sigma = \mathrm{rank}_{\mathbb{C}} dd^c \log\tau = m - 1.$

In particular $(dd^c \log\tau)^m \equiv 0$ on M. Moreover we have $dd^c \log\sigma = \frac{1}{p} dd^c \log\tau \geq 0$. From the formula

(2.13) $\quad dd^c\sigma = \sigma(dd^c \log\sigma + d\log\sigma \wedge d^c \log\sigma),$

one obtains $dd^c\sigma \geq dd^c \log\sigma \geq 0$ on M. Taking exterior powers of the right and left handside of (2.13) and using (2.11) and

(2.12), we have

$$0 \leq (dd^c\sigma)^m = \sigma^m(dd^c\log\sigma)^{m-1} \wedge d\log\sigma \wedge d^c\log\sigma \neq 0.$$

Hence $(dd^c\sigma)^m > 0$ and therefore $dd^c\sigma > 0$.

q.e.d.

b) Manifolds of circular type.

In [4] we introduced the following notion. Let M be a non compact, connected complex manifold of dimension m and $\tau : M \to [0,R)$ be an exhaustion function. Define $M_* = \{x \in M \mid \tau(x) > 0\}$. We say that the pair (M,τ) is a <u>manifold of circular type</u> if τ has the following properties:

(2.14) $\tau \in C^0(M) \cap C^\infty(M_*)$;

(2.15) $dd^c\tau > 0$ on M_*;

(2.16) $(dd^c\log\tau)^m \equiv 0$ on M_*.

Moreover there exists $p \in \tau^{-1}(0)$ and a coordinate neighborhood U of p such that:

(2.17) if $\|\ \|$ denotes the euclidean norm, then there exist constants $C, K > 0$ such that $C\|Z\|^2 \leq \tau(Z) \leq K\|Z\|^2$ for all $Z \in U$.

(2.18) there exists $\varepsilon > 0$ so that $tZ \in U$ if $|t| < \varepsilon$ and $\|Z\| < 2$ and such that the function $h : (-\varepsilon,\varepsilon)\times(\mathbb{B}(2)-\{0\}) \to \mathbb{R}_+$ defined by $h(t,Z) = \tau(tZ)$ is of class C^∞.

It turns out that, if the other assumtions are satisfied, (2.18) is equivalent to

(2.18') if $\pi : \tilde{M} \to M$ is the blow up of M at p, then $\tau \circ \pi \in C^{\infty}(\tilde{M})$.

Clearly this assumption is nicer to state, but for the porpouse of this paper it is important that the main results of [4] can be obtained assuming only (2.18). It should be noted that in [4] we assumed for simplicity also $dd^c \log \tau \geq 0$ on M_*. This, as we noted before, follows from (2.15) and (2.17).

The main result of [4], which we shall need later, can be stated as follows:

<u>Theorem 2.3.</u> Let (M,τ) be a manifold of circular type. If $\sup\tau=+\infty$, then there exists a biholomorphic map $\Phi : \mathbb{C}^m \to M$ such that $\sigma = \tau\circ\Phi$ is a strictly plurisubharmonic exhaustion of $\mathbb{C}^m$ with the property that $\sigma(\lambda Z) = |\lambda|^2\sigma(Z)$ for all $Z \in \mathbb{C}^m$ and $\lambda \in \mathbb{C}$. If $\sup\tau < +\infty$ and the Monge-Ampère foliation associated to τ is holomorphic, then there exists a strictly pseudoconvex, complete circular domain $G \subset\subset \mathbb{C}^m$ and a biholomorphic map $\Psi : G \to M$ such $\tau\circ\Psi$ is the Minkowski functional squared of G.

3. Homogeneous polynomials on $\mathbb{C}^m$

A polynomial $H : \mathbb{C}^m \to \mathbb{C}$ is said to be homogeneous of degree n if for all $t \in \mathbb{R}$ and $Z \in \mathbb{C}^m$

$$(3.1) \qquad H(tZ) = t^n H(Z).$$

In fact any C^∞ function $H : \mathbb{C}^m \to \mathbb{C}$ which satisfies (3.1) is a homogeneous polynomial of degree n. We also say that a polynomial $H : \mathbb{C}^m \to \mathbb{C}$ is homogeneous of bidegree (p,q) if for all $\lambda \in \mathbb{C}$ and $Z \in \mathbb{C}^m$

$$(3.2) \qquad H(\lambda Z) = \lambda^q \bar{\lambda}^p H(Z).$$

Again any C^∞ function satisfying (3.2) is a homogeneous polynomial of bidegree (p,q).

Given a homogeneous polynomial $H : \mathbb{C}^m \to \mathbb{C}$ of degree n, there exists a unique decomposition

$$(3.3) \qquad H = \sum_{p+q=n} H^{p,q}$$

where $H^{p,q}$ is a homogeneous polynomial of bidegree (p,q).

If $H : \mathbb{C}^m \to \mathbb{C}$ is a homogeneous polynomial of bidegree (p,q), then one checks immediately that H_μ (resp. $H_{\bar{\mu}}$) is a homogeneous polynomial of bidegree $(p-1,q)$ if $p \geq 1$ (resp. $(p,q-1)$ if $q \geq 1$). Moreover, for every $Z \in \mathbb{C}^m$ one has the formulas

$$(3.4) \qquad \begin{aligned} H_\mu(Z) z^\mu &= pH(Z) \quad \text{if } p \geq 1, \\ H_{\bar{\mu}}(Z) \bar{z}^\mu &= qH(Z) \quad \text{if } q \geq 1. \end{aligned}$$

Finally, we shall say that a homogeneous polynomial H is positive if it is real valued and $H(Z) > 0$ for all $Z \in \mathbb{C}^m$.

We need a number of properties of homogeneous polynomials. We group them together in the following

Lemma 3.1. (i) If H is a homogeneous polynomial on $\mathbb{C}^m$ of bidegree (p,p) with $dd^cH > 0$ on $\mathbb{C}^m - \{0\}$, then H is positive.
(ii) If H is a positive homogeneous polynomial on $\mathbb{C}^m$ of degree n, then $n = 2p$ is even and $H^{p,p}$ is positive.
(iii) If H is a homogeneous polynomial on $\mathbb{C}^m$ of degree n and with the property that $dd^cH > 0$ on $\mathbb{C}^m - \{0\}$, then $n = 2p$ is even and $dd^cH^{p,p} > 0$ on $\mathbb{C}^m - \{0\}$.

Proof. (i): By hypothesis and using (3.4), we have for $Z \in \mathbb{C}^m - \{0\}$

$$0 < H_{\mu\bar{\nu}}(Z) z^\mu \bar{z}^\nu = pH_\mu(Z) z^\mu = p^2 H(Z).$$

(ii): Let $Z \in \mathbb{C}^m - \{0\}$ and $t \in \mathbb{R}$. Then

$$0 < H(e^{it}Z) = \sum_{r+s=n} H^{r,s}(e^{it}Z) = \sum_{r+s=n} e^{it(r-s)} H^{r,s}(Z).$$

If n were odd, then

$$0 < \int_0^{2\pi} H(e^{it}Z)\,dt = \sum_{r+s=n} H^{r,s}(Z) \int_0^{2\pi} e^{it(r-s)}\,dt = 0.$$

Thus n must be even: $n = 2p$. Moreover

$$0 < \int_0^{2\pi} H(e^{it}Z)\,dt = \sum_{r+s=n} H^{r,s}(Z) \int_0^{2\pi} e^{it(r-s)}\,dt = 2\pi H^{p,p}(Z).$$

(iii): Under the hypothesis, given any $Z, W \in \mathbb{C}^m - \{0\}$, we have

$$0 < H_{\mu\bar{\nu}}(Z) w^\mu \bar{w}^\nu = \sum_{r+s=n} H^{r,s}_{\mu\bar{\nu}}(Z) w^\mu \bar{w}^\nu = H_W(Z).$$

For any $W \in \mathbb{C}^m - \{0\}$, H_W is a positive homogeneous polynomial of degree $n-2$ whose component of bidegree $(r-1, s-1)$ is given by

$H^{r,s}_{\mu\bar{\nu}} w^{\mu}\bar{w}^{\nu}$. Thus (ii) implies that $n = 2p$ for some p and that for every $Z \in \mathbb{C}^m - \{0\}$ we have $H^{p,p}_{\mu\bar{\nu}}(Z) w^{\mu}\bar{w}^{\nu} > 0$. Since $W \in \mathbb{C}^m - \{0\}$ was arbitrary, we are done.

q.e.d.

It is known that homogeneous polynomials of bidegree (p,p) give rise to solutions of the homogeneous Monge-Ampère equation of the kind described in Section 2. For the reader's convenience we give here a precise statement.

Proposition 3.2. Let $H : \mathbb{C}^m \to \mathbb{R}$ be a homogeneous polynomial of bidegree (p,p) such that $dd^c H > 0$ on $\mathbb{C}^m - \{0\}$. Then H is an exhaustion of $\mathbb{C}^m$ such that on $\mathbb{C}^m - \{0\}$ we have $H > 0$ and $(dd^c \log H)^m \equiv 0$. Moreover the complex gradient X of H is given by $X(Z) = p^{-1} z^{\mu} \frac{\partial}{\partial z^{\mu}}$ so that the Monge-Ampère foliation associated to H is holomophic and its leaves are complex lines through the origin.

Proof. From Lemma 3.1 it follows that H is positive and therefore that it is an exhaustion of $\mathbb{C}^m$. Let $u = \log H$. Then given $\lambda \in \mathbb{C}$ and $Z \in \mathbb{C}^m - \{0\}$, one has

$$(3.5) \qquad u(\lambda Z) = p \log|\lambda|^2 + u(Z).$$

Differentiating both sides of (3.5) with respect to λ and $\bar{\lambda}$ and taking $\lambda = 1$, one obtains

$$(3.6) \qquad u_{\mu\bar{\nu}}(Z) z^{\mu}\bar{z}^{\nu} = 0.$$

Equation (3.6) implies that $\operatorname{rank}_{\mathbb{C}} dd^c u < m$ and therefore that $(dd^c u)^m \equiv 0$ on $\mathbb{C}^m - \{0\}$. In order to compute the complex gradient of H it is enough to observe that, using (3.4), we have $H_{\bar{\nu}}(Z) = p^{-1} H_{\mu\bar{\nu}}(Z) z^{\mu}$. The rest of the statement then follows immediately.

q.e.d.

The aim of this section is to show that among the homogeneous polynomials only those of bidegree (p,p) have the propeties listed in Proposition 3.2. We need the following preliminary observation.

Lemma 3.3. Let H be a positive homogeneous polynomial on $\mathbb{C}^m$ of degree n with $dd^c H > 0$ and $(dd^c \log H)^m \equiv 0$ on $\mathbb{C}^m - \{0\}$. Then $n = 2p$ and, if $\tilde{H} = H^{1/p}$, the pair $(\mathbb{C}^m, \tilde{H})$ is a manifold of circular type.

Proof. Because of Lemma 3.1, we have $n = 2p$ for some positive integer p. As H is positive, the function $\tilde{H} = H^{1/p}$ is a well defined exhaustion of $\mathbb{C}^m$, of class C^∞ on $\mathbb{C}^m - \{0\}$. Since for any $t \in \mathbb{R}$ and $Z \in \mathbb{C}^m$ one has $\tilde{H}(tZ) = t^2 \tilde{H}(Z)$, it is easy to verify that $\tilde{H}$ fulfill the conditions (2.17) and (2.18). Lemma 2.2 shows that $\tilde{H}$ satisfies also (2.15) and (2.16) and therefore the claim is proved.

q.e.d.

Theorem 3.4. Let H be a positive homogeneous polynomial on $\mathbb{C}^m$ of degree n such that $dd^c H > 0$ and $(dd^c \log H)^m \equiv 0$ on $\mathbb{C}^m - \{0\}$. Then $n = 2p$ is even and H is homogeneous of bidegree (p,p).

Proof. Because of Lemma 3.3 and Theorem 2.3, we know that $n = 2p$ and that, if $\tilde{H} = H^{1/p}$, there exists an automorphism Φ of $\mathbb{C}^m$ such that for every $Z \in \mathbb{C}^m$ and $\lambda \in \mathbb{C}$ we have $\tilde{H} \circ \Phi(\lambda Z) = |\lambda|^2 \tilde{H} \circ \Phi(Z)$. Thus if we denote $\hat{H} = H \circ \Phi$, we have that $\hat{H} \in C^\infty(\mathbb{C}^m)$

and satisfies $\hat{H}(\lambda Z) = |\lambda|^{2p}\hat{H}(Z)$ for all $\lambda \in \mathbb{C}$ and $Z \in \mathbb{C}^m$. Hence $\hat{H}$ is a homogeneous polynomial of bidegree (p,p) such that $dd^c\hat{H} > 0$. In particular, as we saw in Proposition 3.2, its complex gradient $\hat{X}$ is holomorphic. Then Lemma 2.1 implies that $\hat{X} = \Phi_* X$ and that X is holomorphic.

Since H is homogeneous of degree $2p$, we have for all $t \in \mathbb{R}$ and $Z \in \mathbb{C}^m$

$$H_\mu(tZ) = t^{2p-1}H_\mu(Z) \qquad H_{\bar{\nu}}(tZ) = t^{2p-1}H_{\bar{\nu}}(Z)$$

$$H_{\mu\bar{\nu}}(tZ) = t^{2p-2}H_{\mu\bar{\nu}}(Z) \qquad H^{\bar{\nu}\mu}(tZ) = t^{2-2p}H^{\bar{\nu}\mu}(Z).$$

Recalling the definition (2.5) of X, we have

$$X(tZ) = t^{2-2p}H^{\bar{\nu}\mu}(Z)t^{2p-1}H_{\bar{\nu}}(Z) = tX(Z).$$

Since X is holomorphic, it must be of the form

$$X(Z) = X^\mu(Z)\frac{\partial}{\partial z^\mu} = a^\mu_\nu z^\nu \frac{\partial}{\partial z^\mu} \tag{3.7}$$

where $A = (a^\mu_\nu) \in GL(m,\mathbb{C})$.

Let $H = \sum_{r+s=2p} H^{r,s}$ be the decomposition of H in homogeneous polynomials of bidegree (r,s). Because of Lemma 3.1 (iii) and Proposition 3.2, we know that

$$(H^{p,p})^{\bar{\nu}\mu}(Z)H^{p,p}_{\bar{\nu}}(Z) = \frac{1}{p}z^\mu. \tag{3.8}$$

on the other hand

$$\sum_{\substack{r+s=2p\\ r,s\geq 1}} X^\mu H^{r,s}_{\mu\bar{\nu}} = X^\mu H_{\mu\bar{\nu}} = H_{\bar{\nu}} = \sum_{\substack{r+s=2p\\ s\geq 1}} H^{r,s}_{\bar{\nu}} .$$

Since by (3.7) each X^μ is homogeneous of bidegree $(1,0)$, comparing the degrees of the two ends of this equality, we can conclude

(3.9) $$H^{0,2p}_{\bar\nu} = 0,$$

(3.10) $$X^\mu H^{r,s}_{\mu\bar\nu} = H^{r,s}_{\bar\nu} \quad \text{if} \quad r,s \geq 1 \quad \text{and} \quad r+s = 2p.$$

Thus, using (3.8), we compute

(3.11) $$\begin{aligned} X^\alpha(Z) &= (H^{p,p})^{\bar\nu\alpha}(Z) H^{p,p}_{\mu\bar\nu}(Z) X^\alpha(Z) \\ &= (H^{p,p})^{\bar\nu\alpha}(Z) H^{p,p}_{\bar\nu}(Z) = \frac{1}{p} z^\alpha . \end{aligned}$$

Now, using (3.4) and (3.9), we have

$$\overline{H^{2p,0}(Z)} = H^{0,2p}(Z) = \frac{1}{2p} H^{0,2p}_{\bar\nu}(Z)\bar z^\nu = 0.$$

From (3.10), using (3.4) and (3.11), we compute

$$H^{r,s}_{\bar\nu}(Z) = X^\mu(Z) H^{r,s}_{\mu\bar\nu}(Z) = \frac{1}{p} H^{r,s}_{\mu\bar\nu}(Z) z^\mu = \frac{r}{p} H^{r,s}_{\bar\nu}(Z)$$

and

$$sH^{r,s}(Z) = H^{r,s}_{\bar\nu}(Z)\bar z^\nu = \frac{r}{p} H^{r,s}_{\bar\nu}(Z)\bar z^\nu = \frac{rs}{p} H^{r,s}(Z).$$

Hence, if $r,s \geq 1$ and $r+s = 2p$, then either $r = s = p$ or $H^{r,s} = 0$. In conclusion $H = H^{p,p}$ and the proof is complete.

q.e.d.

<u>Remark 1.</u> Let H be a homogeneous polynomial on $\mathbb{C}^m$ of bidegree (p,p) such that $dd^cH > 0$ on $\mathbb{C}^m - \{0\}$. If $\Phi \in \mathrm{Aut}(\mathbb{C}^m)$ and $\hat H = H\circ\Phi$ is a homogeneous polynomial of bidegree (q,q), then it follows that Φ is linear and $p = q$. This is easy to see since, if $\hat D = \{Z \in \mathbb{C}^m \mid \hat H(Z) < 1\}$ and $D = \{Z \in \mathbb{C}^m \mid H(Z) < 1\}$, then both $\hat D$ and D are complete circular domains and Φ restricted to $\hat D$ is a biholomorphic map onto D which fixes the origin. By a classical theorem of Cartan, then Φ is necessarily linear. Theorem 3.4 shows that if $\hat H = H\circ\Phi$ is just a homogeneous polynomial

of degree n, then the same conclusion holds i.e. Φ is linear and $n = 2p$. This is immediate from the theorem since if H satisfies (2.1) and (2.2) on $\mathbb{C}^m - \{0\}$ then so does $\hat{H}$.

Remark 2. Up to linear isomorphisms (and therefore up to biholomorphisms) there exists only one strictly plurisubharmonic homogeneous polynomial of bidegree (1,1): $\|\ \|^2$. This is not the case for higher bidegree. This is fairly obvious but it may be convenient to give an explicit example. Let $H^\alpha : \mathbb{C}^2 \to \mathbb{R}$ be defined by $H^\alpha(z,w) = |z|^4 + |w|^4 + \alpha|z|^2|w|^2$ with $\alpha > 0$. Clearly $dd^cH^\alpha > 0$ for all $\alpha > 0$ and $H^2(z,w) = \|(z,w)\|^4$. We claim that if $\alpha \neq 2$ then H^α cannot be equivalent to H^2 up to complex linear isomorphisms. Assume that there exists $A \in GL(2,\mathbb{C})$ such that $\rho = H^2 \circ A = H^\alpha$ for some $\alpha \neq 2$. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then we must have

$$1 = H^\alpha(1,0) = \rho(1,0) = |a|^2 + |c|^2.$$

Also one computes

$$H^\alpha_{z\bar{z}}(z,w) = 4|z|^2 + \alpha|w|^2 \qquad H^\alpha_{w\bar{w}}(z,w) = \alpha|z|^2 + 4|w|^2$$

$$\rho_{z\bar{z}}(z,w) = 4|az + bw|^2 + 2|cz + dw|^2$$

$$\rho_{w\bar{w}}(z,w) = 2|az + bw|^2 + 4|cz + dw|^2.$$

Thus

$$4 = H^\alpha_{z\bar{z}}(1,0) = \rho_{z\bar{z}}(1,0) = 4|a|^2 + 2|c|^2 = 2|a|^2 + 2$$

$$\alpha = H^\alpha_{w\bar{w}}(1,0) = \rho_{w\bar{w}}(1,0) = 2|a|^2 + 4|c|^2 = 2|c|^2 + 2$$

and therefore

$$2 \neq \alpha = 2|a|^2 + 2|c|^2 = 2$$

which is impossible.

Remark 3. Let $\tau : \mathbb{C}^m \to [0,+\infty)$ be an exhaustion such that $\tau \in C^0(\mathbb{C}^m) \cap C^\infty(\mathbb{C}^m - \{0\})$ and satisfying (2.1) and (2.2). Moreover let us assume that τ is homogeneous of degree n: $\tau(tZ) = t^n\tau(Z)$ for all $t \in \mathbb{R}$ and $Z \in \mathbb{C}^m$. If τ were of class C^∞ also at the origin then it would be a homogeneous polynomial and Theorem 3.4 would apply. In general if $\sigma = \tau^{2/n}$, then using the homogeneity of τ and Lemma 2.2, one shows as in Lemma 3.3 that $(\mathbb{C}^m, \sigma)$ is a manifold of circular type. Thus there exists $\Phi \in \mathrm{Aut}(\mathbb{C}^m)$ so that if $\rho = \sigma \circ \Phi$, then ρ is homogeneous of bidegree (1,1). In particular ρ can be written as $\rho(Z) = \|Z\|^2 g(Z)$ where g is a bounded positive function, which is constant on $L \cap \mathbb{C}^m - \{0\}$ for each complex line L through the origin (of course in general g is not continuous at the origin). Using the results of Section 4 of [4], it can be shown that $\sigma(Z) = \rho(Z) + O(\|Z\|^3)$. Therefore $\tau(Z) = \|Z\|^n h(Z) + O(\|Z\|^{n+1})$ for all $Z \in \mathbb{C}^m$ and where $h = g^{n/2}$. But then, since τ is homogeneous of degree n, we can conclude that τ must be expressed by $\tau(Z) = \|Z\|^n h(Z)$. In other words any such exhaustion τ is the product of a power of the norm times the pull back from $\mathbb{P}_{n-1}$ of a suitable smooth function.

4. The general characterization

We shall now turn to the problem of characterizing in general the solutions of the homogeneous complex Monge-Ampère equation which, up to biholomorphic maps, are the logarithm of a homogeneous polynomial of bidegree (p,p). We start by fixing some terminology.

Let M be a complex manifold of dimension m and $f : M \to \mathbb{R}$ be a function of class C^∞. We say that f <u>vanishes of order</u> n <u>at</u> $a \in M$ if f and all its derivatives of order $s < n$ vanish at a and some derivative of order s of f is nonzero at a. If f vanishes of order n at $a \in M$, then the <u>leading Hessian</u> $H_f : T_a(M) \to \mathbb{R}$ <u>of</u> f <u>at</u> a is defined by

$$(4.1) \qquad H_f(X) = \sum_{p+q=n} \frac{\partial^n f(a)}{\partial z^{\mu_1}..\partial z^{\mu_p}\partial \bar z^{\nu_1}..\partial \bar z^{\nu_q}} X^{\mu_1}\ldots X^{\mu_p}\bar X^{\nu_1}\ldots \bar X^{\nu_q}$$

where $X = X^\nu \frac{\partial}{\partial z^\mu} + \bar X^\nu \frac{\partial}{\partial \bar z^\nu} \in T_a(M)$. The form H_f decomposes into the sum of its components of bidegree (p,q): $H_f = \sum_{p+q=n} H_f^{p,q}$ where each $H_f^{p,q}$ is defined by

$$(4.2) \qquad H_f^{p,q}(X) = \frac{\partial^n f(a)}{\partial z^{\mu_1}..\partial z^{\mu_p}\partial \bar z^{\nu_1}..\partial \bar z^{\nu_q}} X^{\mu_1}\ldots X^{\mu_p}\bar X^{\nu_1}\ldots \bar X^{\nu_q}.$$

If $n = 2p$ is even, we say that $H_f^{p,p}$ is the <u>leading Levi form</u> of f.

If we identify $T_a(M)$ with $\mathbb{C}^m$, then H_f is a homogeneous polinomial of degree n on $\mathbb{C}^m$ and the decomposition of H_f given by the $H_f^{p,q}$ is exactly the decomposition of H_f into the sum of homogeneous polynomials of bidegree (p,q).

We shall say that H_f is <u>positive</u> if $H_f(X) > 0$ for all $X \in T_a(M) - \{0\}$ i.e. if H_f is positive as a homogeneous polynomial.

We shall consider the following situation. Let M be a connected, non compact complex manifold of dimension m and $\tau : M \to [0,+\infty)$ be an exhaustion function of class C^∞ which satisfies the following assumptions:

(4.3) $dd^c\tau > 0$ on $M_* = \{x \in M \mid \tau(x) > 0\}$;

(4.4) $(dd^c\log\tau)^m \equiv 0$ M_*;

(4.5) τ vanishes of order n at a point $a \in \tau^{-1}(0)$ and if H is the leading Hessian of τ at a, then H is positive and $dd^cH > 0$ on $T_a(M) - \{0\}$.

As a first consequence, we have the following:

<u>Proposition 4.1.</u> The zero set of τ consists of exactly one point 0_M which we call the <u>center</u> of M.

<u>Proof.</u> It is known that the hypothesis (4.3) and (4.4) imply that $\tau^{-1}(0)$ is non empty and connected (see [4], Theorem 2.5 for example). Thus the the conclusion follows from (4.5) since $a = 0_M$ is a strict minimum for τ and therefore an isolated point of $\tau^{-1}(0)$.

q.e.d.

Before going into our classification, it is of interest to give a more precise description of τ near the center 0_M.

<u>Proposition 4.2.</u> The order of vanishing of τ at the center 0_M is even: $n = 2p$. Moreover the leading Hessian of τ at 0_M coincides with the leading Levi form.

Proof. Let H be the leading Hessian of τ at 0_M. Here we shall freely identify $T_{0_M}(M) = \mathbb{C}^m$. Since H is positive, Lemma 3.1 (ii) shows that the degree of H and therefore the order of vanishing of τ at 0_M is even $n = 2p$. Since by hypothesis (4.5) we have $dd^cH > 0$ on $\mathbb{C}^m - \{0\}$, the claim will follow from Theorem 3.4 if we can show $(dd^c \log\tau)^m \equiv 0$ on $\mathbb{C}^m - \{0\}$.

Let U be a small enough coordinate neighborhood centered at 0_M and let $Z \in \mathbb{C}^m - \{0\}$. For $t \in \mathbb{R}$ such that $tZ \in U$ we have

$$\tau(tZ) = t^nH(Z) + 0(t^{n+1})$$

$$\tau_\mu(tZ) = t^{n-1}H_\mu(Z) + 0(t^n) \qquad \tau_{\bar\nu}(tZ) = t^{n-1}H_{\bar\nu}(Z) + 0(t^n)$$

$$\tau_{\mu\bar\nu}(tZ) = t^{n-2}H_{\mu\bar\nu}(Z) + 0(t^{n-1})$$

$$\tau^{\bar\nu\mu}(tZ) = t^{2-n}H^{\bar\nu\mu}(Z) + 0(t^{3-n})$$

Also recall that, as we noted in Section 2, whenever (2.1) is satisfied, locally the Monge-Ampère equation (2.2) is equivalent to the equation (2.7). Thus for small $t \neq 0$ we compute

$$\begin{aligned} t^nH(Z) + 0(t^{n+1}) = \tau(tZ) &= \tau_{\bar\nu}(tZ)\tau^{\bar\nu\mu}(tZ)\tau_\mu(tZ) \\ &= t^nH_{\bar\nu}(Z)H^{\bar\nu\mu}(Z)H_\mu(Z) + 0(t^{n+1}). \end{aligned}$$

Dividing the first and the last term of the equality by t^n and takig limit as $t \to 0$, we obtain

$$H(Z) = H_{\bar\nu}(Z)H^{\bar\nu\mu}(Z)H_\mu(Z)$$

which, as observed above, is equivalent to $(dd^c\log H(Z))^m = 0$.

q.e.d.

<u>Remark 1.</u> According to Stoll [5], a <u>strictly parabolic manifold of infinite radius</u> is a pair (M,τ) such that M is a connected, non compact complex manifold of dimension m and $\tau : M \to [0,+\infty)$ is an exaustion of class C^∞ satisfying (4.3), (4.4) and such that $dd^c\tau > 0$ also on $\tau^{-1}(0)$. Then it can be shown (see [5], Proposition 2.2) that τ vanishes of order 2 at every point of $\tau^{-1}(0)$ and that (4.5) is verified. Conversely if τ satisfies (4.3), (4.4) and (4.5) with order of vanishing $n = 2$, then (M,τ) is strictly parabolic of infinite radius. This is immediate from Proposition 4.1 and 4.2. In fact $\tau^{-1}(0) = \{0_M\}$ and near 0_M we have $\tau(Z) = H(Z) + 0(\|Z\|^3)$ where H is a homogeneous polynomial of bidegree (1,1) which is strictly plurisubharmonic outside the origin. Thus $H(Z) = \|A(Z)\|^2$ for some $A \in GL(m,\mathbb{C})$ and therefore $dd^c\tau(0_M) > 0$. In conclusion when the order of vanishing is 2 our class of manifolds coincides with Stoll's strictly parabolic manifolds. Therefore, as we shall see in a moment, our classification theorem extends Stoll's result ([5]) to exhaustions with higher order of vanishing.

We can now state and prove our main result.

<u>Theorem 4.3.</u> Let M be a connected, non compact complex manifold of dimension m and $\tau : M \to [0,+\infty)$ a C^∞ exhaustion satisfying (4.3), (4.4) and (4.5). Then the order of vanishing n of τ at its zero set is even and there exists a biholomorphin map $\Phi : \mathbb{C}^m \to M$ such that $\tau\circ\Phi$ is a homogeneous polynomial of bidegree (p,p) where $p = \frac{1}{2}n$.

<u>Proof.</u> We know already that $n = 2p$ for some positive integer p

and that $\tau^{-1}(0) = \{0_M\}$. We want to use Theorem 2.3. To this end we shall show that if $\sigma = \tau^{1/p}$, then the pair (M,σ) is a manifold of circular type. Clearly $\sigma \in C^0(M) \cap C^\infty(M_*)$ and from Lemma 2.2 it follows that σ satisfies also (2.15) and (2.16). We need to show only that (2.17) and (2.17) hold for σ.

If U is a small enough coordinate neighborhood centered at 0_M, then τ has the following expansion on U:

$$\tau(Z) = H(Z) + R(Z)$$

where H is a positive homogeneous polynomial of bidegree (p,p) and R is a C^∞ function such that $|R(Z)| < A\|Z\|^{2p+1}$ for some $A > 0$. There exist $m, M > 0$ so that $2m\|Z\|^{2p} \le H(Z) \le M\|Z\|^{2p}$. Since for $\|Z\|$ small enough we have $A\|Z\|^{2p+1} < m\|Z\|^{2p}$, we can conclude for Z in a neighborhood of 0_M:

$$\begin{aligned} m^{1/p}\|Z\|^2 &\le (H(Z) - |R(Z)|)^{1/p} \le \sigma(Z) \\ &\le (H(Z) + |R(Z)|)^{1/p} \le (M + m)^{1/p}\|Z\|^2 \end{aligned}$$

so that σ satisfies (2.17). Let $\varepsilon > 0$ so that if $|t| < \varepsilon$ and $\|Z\| < 2$ we have $tZ \in U$ and define $W = (-\varepsilon,\varepsilon) \times (\mathbb{B}(2)-\{0\})$. Define $h : W \to \mathbb{R}_+$ by $h(t,Z) = \sigma(tZ)$. We need to show that h is of class C^∞. For $(t,Z) \in W$, $t \ne 0$, we have

$$0 < \tau(tZ) = t^{2p}H(Z) + R(tZ) \tag{4.6}$$

with $\lim\limits_{t\to 0} t^{-2p}R(tZ) = 0$. There exists a function $T : W \to \mathbb{R}$ of class C^∞ such that $R(tZ) = t^{2p+1}T(t,Z)$. From (4.6) and since $H(Z) > 0$ if $Z \ne 0$, we get

$$H(Z) + tT(t,Z) > 0$$

for any $(t,Z) \in W$. But then

$$h(t,Z) = \sigma(tZ) = t^2\big(H(Z) + tT(t,Z)\big)^2$$

and therefore h is of class C^∞.

Since $\sup\sigma = +\infty$, by Theorem 2.3, there exists a biholomorphic map $\Phi : \mathbb{C}^m \to M$ such that we have $\sigma(\Phi(\lambda Z)) = |\lambda|^2\sigma(\Phi(Z))$ for all $\lambda \in \mathbb{C}$ and $Z \in \mathbb{C}^m$. But then we have also $\tau\circ\Phi(\lambda Z) = |\lambda|^{2p}\tau\circ\Phi(Z)$ and, since $\tau\circ\Phi$ is of class C^∞, the claim follows.

q.e.d.

<u>Remark 2.</u> We always assumed τ to be of class C^∞. It should be underlined that less is needed. In fact Theorem 2.3 can be proved assuming that the exhaustion is of class C^5 on M_* since this much is needed in [2] to prove Theorem 3.1. Thus our Theorem 4.3 holds even assuming that τ is of class C^k where $k = \max\{5, n\}$. At any rate Theorem 4.3 can be viewed also as a regularity result for degenerate Monge-Ampère equations. In fact it implies that τ is real analytic on M.

<u>Remark 3.</u> If M is as above and $\tau : M \to [0,+\infty)$ is an exhaustion satisfying (4.3) and (4.4) but only continuous on $\tau^{-1}(0)$, we can still classify M and τ if we assume instead of (4.5) the following:

(4.5') There exists $a \in \tau^{-1}(0)$ sush that

(i) with respect to coordinate centered at a we have $C\|Z\|^n \le \tau(Z) \le K\|Z\|^n$ for some $C, K > 0$ and positive integer n;

(ii) the function h definite for $|t| < \varepsilon$ and $0 < \|Z\| < \varepsilon'$, for some $\varepsilon, \varepsilon' > 0$, by $h(t,Z) = (\tau(tZ))^{2/n}$ is of class C^∞.

Theese assumptions, which for τ smooth on $\tau^{-1}(0)$ are equivalent to those of Theorem 4.3, allow one to apply Theorem 2.3 to the exhaustion $\sigma = \tau^{2/n}$. Then, in the same way as in Remark 3 of Section 3, it can be shown that there exists a biholomorphic map $\Phi : \mathbb{C}^m \to M$ such that $\tau\circ\Phi(Z) = \|Z\|^n g(Z)$ where g is a bounded function on $\mathbb{C}^m$ constant on each punctured complex line through the origin.

5. Final remarks

A number of questions arise naturally in this context and yet cannot be answered.

Firstly, our characterization of homogeneous polynomials (Theorem 4.3) applies only to unbounded exhaustions while Stoll's theorem ([5]) classifies also bounded ones. In fact if M is a non compact, connected complex manifold of dimension m which carries a C^∞, strictly plurisubharmonic exhaustion τ such that $\sup\tau = 1$ and $(dd^c\log\tau)^m \equiv 0$ on $\{\tau > 0\}$, then M is biholomorphic to the ball in $\mathbb{C}^m$ and τ pulls back to $\|\ \|^2$. We conjecture that the some kind of theorem should hold for homogeneous polynomials of bidegree (p,p). More explicitly if M and τ are as in Theorem 4.3 but $\sup\tau = 1$, there should be a a homogeneous polynomial H of bidegree (p,p) so that, if $G = \{Z \in \mathbb{C}^m \mid H(Z) < 1\}$, then there exists a biholomorphic map $\Phi : \mathbb{C}^m \to M$ with $\tau\circ\Phi = H$.

The difficulty in proving this result, at least with our method, is that, in order to apply Theorem 2.3, one needs to show that the Monge-Ampère foliation associated to τ is holomorphic. In the case of unbounded exhaustion, this follows from a theorem of Burns [2]. When the exhaustion is bounded instead, the Monge-Ampère foliation is generically not holomorphic (cfr.[3]). However we feel that under the above assumptions the foliation is indeed holomorphic and that therefore our conjecture holds true.

A second kind of problems concernes ways to improve our characterization. Namely we need a relatively strong assumption about the behavior of τ at its zero set. It is quite interesting to investigate whether they can be relaxed. A step in this direction would be to prove Theorem 3.4 assuming that the homogeneous polynomial H is just nonnegative. It should be observed that the nature of the theorem changes considerably. In fact, if H is only nonnegative, then $H^{-1}(0)$ is a priori a noncompact set and thus H is not anylonger a proper function. In this case the associated Monge-Ampère exhaustion could be very wild. An entirely different approach may be needed to solve this problem and even a counterexample may be almost as interesting as a positive result.

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