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Geometry of Noncommutative 'Spaces' and Schemes

by

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Introduction

1. Antecedents. We begin with a few relevant historical observations.

1.1. Serre's Proj and Gabriel's spectrum. The most important early sources of noncommutative algebraic geometry are the description by Serre of the category of coherent sheaves on a projective variety [S] and the introduction by Gabriel of the *injective spectrum* of a locally noetherian Grothendieck category. Gabriel assigned, in a canonical way, to every locally noetherian Grothendieck category a locally ringed space, whose underlying topological space is the injective spectrum – the set of isomorphism classes of indecomposable injective objects endowed with Zariski topology. He proved that this assignment reconstructs any noetherian scheme uniquely up to isomorphism [Gab, Chapter 6].

Note that the work of Serre appeared several years prior to scheme theory and the Gabriel's work around the same time as the first two volumes of EGA.

1.2. First attempts to define noncommutative schemes. There were attempts (which started around the end of the sixties and continued to be visible for more than a decade) to initiate noncommutative scheme theory based on a rather straightforward extension of the Gabriel spectrum to the category of left modules over an arbitrary associative unital ring R (its points are those isomorphism classes of indecomposable injective objects $[E]$ for which the quotient category by the left orthogonal to E has simple objects) endowed with *Zariski* topology and a *structure* sheaf of associative rings determined by the ring R. Schemes were defined as ringed spaces which are locally affine (see [Go1], [Go2] and references therein). If R is a commutative ring, then there is a natural embedding of the prime spectrum of R into the above defined spectrum of the category of R-modules, which is an isomorphism if the ring R is noetherian (the case considered by Gabriel), but, not in general. So, the restriction of this concept of a noncommutative scheme to the commutative case recovers only locally noetherian schemes, which is already an indication of a certain inadequacy of the spectrum used here. Nevertheless, even under noetherian hypothesis, this theory did not go beyond the above quoted definition of a scheme (given in the last section of [Go2]). The declared goal – the creation of a noncommutative version of local algebra, was never achieved.

Other movements towards noncommutative algebraic geometry (initiated around the mid-seventies) were based on the prime spectrum of rings endowed with Zariski topology and a structure sheaf of associative rings whose construction required noetherian hypothesis. This produced a version of an *affine noetherian* scheme. General noetherian schemes were defined as locally affine ringed spaces [VOV]. One can show that this version of noncommutative schemes can be obtained from the previous one by considering only left noetherian rings and taking a much coarser version of Zariski topology than the one used in [Go2]. Note that the prime spectrum of most of noncommutative algebras of interest is rather poor (e.g. it is trivial in the case of Weyl algebras over fields of zero characteristic).

1.3. Supporting motivations. There was a certain outside interest in the quest of noncommutative algebraic geometry already at that time, i.e. in the middle of seventies (see the introduction to [Dix]), which was mostly due to the algebraization of representation theory (initiated by works of Kirillov, Gelfand and Kirillov, Dixmier, and his school) and a promise of new insights and possible applications to representation theory of algebraic groups, enveloping algebras of Lie algebras, and some other algebras of interest.

1.4. D-modules and D-schemes. Then, starting from 1980, Beilinson and Bernstein developed a compromise-type noncommutative algebraic geometry – the theory of *D-schemes* (which are usual commutative schemes equipped with a subsheaf of the sheaf of (twisted) differential operators) in order to study representation theory of reductive algebraic groups. This important development led to a break-through in representation theory and distracted the curiosity of most working mathematicians from attempts to construct noncommutative scheme theory based on Gabriel's injective spectrum, or on the prime spectrum of associative rings.

1.5. The Cohn's spectrum. There was another approach to noncommutative local algebra, due to P. Cohn, which is based on the notion of the *universal localization.* Technically, the main difference between Cohn's approach and the other approaches mentioned above is that instead of dealing with abelian categories of modules over a ring, Cohn's theory operates with the *exact* category of projective modules of finite type (Cohn's original formulations use only matrix rings over a given associative unital ring).

It is worth mentioning that Cohn's philosophy serves as a base for works of Gelfand and Retakh and their collaborators on birational noncommutative algebra. Recently, Cohn's universal localization found applications in topology (see [Loc]).

1.6. Imposing naive geometric spaces. The above mentioned approaches to noncommutative algebraic geometry insisted on a naive generalization of the standard pattern of commutative scheme theory: noncommutative versions of schemes were sought as *geometric spaces*, and the latter were understood as topological spaces endowed with a *structure* sheaf of associative rings. This holds for D-schemes of Beilinson and Bernstein and for much more recent Kapranov's version of formal noncommutative geometry [Ka], because, by nature, D-schemes, as well as Kapranov's formal NC schemes, are quasi-coherent sheaves of associative algebras on commutative schemes. But, an arbitrary left noetherian associative algebra is not isomorphic to the algebra of global sections of the corresponding structure sheaf on Gabriel's or Cohn's (or any other) spectrum. It is therefore not surprising that imposing ringed spaces as the framework for noncommutative algebraic geometry and trying to literally mimic the pattern of commutative algebra and algebraic geometry, led to considerable difficulties already on a very basic level.

1.7. Pseudo-geometry versus geometry. The discovery of quantum groups triggered a flow of new examples supplied mostly by mathematical physics and attributed to noncommutative geometry, reviving some stagnating areas (e.g. Hopf algebras) and involving a big number of mathematicians and theoretical physicists fascinated by the geometric flavor of this suddenly wide open field of research. This rise of the interest in noncommutative algebraic geometry was marked by the transition from attempts to build its foundations relying on naive generalizations of geometric spaces to the opposite extreme – viewing noncommutative algebraic geometry as *pseudo-geometry*, that is geometry in which spaces are replaced by something else. The transition was greatly influenced by Connes' approach to noncommutative differential geometry. On a more advanced stage, its roots can be found in the pseudo-geometric development of Grothendieck's algebraic geometry between the end of the fifties and the beginning of the seventies – going from the category of geometric (that is locally ringed) spaces to the category Esp of *spaces* which are sheaves of sets on the **fpqc** presite of affine schemes, then expanding to toposes, algebraic spaces and stacks. Note that in commutative algebraic geometry all these notions and points of view coexisted and complemented each other.

1.8. Points from commutative algebraic geometry. The abandon of the geometric point of view was due not so much to the limitations of Gabriel's injective spectrum and shortcomings in the attempts of using it, but, mostly to the fact that the Gabriel's spectrum was known to and appreciated by only a few algebraists, while the dominating paradigm of a $\mathfrak{point}$ came from commutative algebraic geometry: points of a commutative scheme are equivalence classes of geometric points, i.e. morphisms from spectra of fields.

A naive noncommutative generalization of this notion is obtained by replacing fields by skew fields. Thus, the *naive points* of an affine 'space' corresponding to an associative unital ring R are morphisms from R to skew fields, and the equivalence classes of morphisms from R to skew fields are in natural bijective correspondence with *completely prime* two-sided ideals of the ring R (i.e. ideals $\mathfrak{p}$ such that the set $R-\mathfrak{p}$ is closed under multiplication). Noncommutative rings usually have very few completely prime two-sided ideals (enveloping algebras of finite-dimensional solvable Lie algebras being among rare worthy exceptions). One consequence of this other transplantation of a commutative paradigm into noncommutative setting, was a widely adopted opinion that noncommutative algebraic geometry is essentially a *geometry without points*. Such a viewpoint reduces noncommutative algebraic geometry to the condition of a poor relative of its commutative predecessor: one cannot count on a noncommutative version of local algebra, in particular, one cannot count on a local study of spaces and morphisms of spaces, which constitute at least a half of the content of commutative algebraic geometry. Fortunately, this opinion is wrong.

2. 'Spaces' of noncommutative algebraic geometry.

One of the benefits of the pseudo-geometric viewpoint in noncommutative algebraic geometry is a considerable increase of its range. Roughly, the picture is as follows.

2.1. Spaces and algebras. The duality between compact topological spaces and commutative unital C^*-algebras is a fundamental fact of functional analysis discovered by I.M. Gelfand in the late thirties. A. Connes extended formally this duality to the noncommutative setting identifying 'noncommutative spaces' with noncommutative C^*-algebras. This eventually led to the creation of noncommutative differential geometry [C1], [C2]. Following the Connes' example, V. Drinfeld [Dr] defined the category of noncommutative affine schemes (he called them 'quantum spaces') in a similar way, as the category dual to the category of unital associative algebras, forcing to the noncommutative case the duality

$$[\text{algebras} \;\leftrightarrow\; \text{affine schemes}]$$

of commutative algebraic geometry.

2.2. Noncommutative Proj. Noncommutative projective spaces were introduced (by Manin's suggestion) via a formal extension of the Serre's description of the category of quasi-coherent sheaves on a projective variety [S]: the category of quasi-coherent sheaves on the projective spectrum of an associative graded algebra R is the quotient category of the category of graded R-modules by the subcategory of locally finite ones (this approach was further developed in [V1], [V2], [A2], [AZ], [OW], and in a number of other works).

Thus, a noncommutative projective space X is represented by a category, C_X, which is regarded as the category of quasi-coherent sheaves on X. This viewpoint is well adapted to the affine case: for any associative ring R, the category of quasi-coherent sheaves on the corresponding affine scheme is identified with the category $R-mod$ of left R-modules.

2.3. 'Spaces' represented by abelian categories. From the prospective of the above mentioned developments, a point of view which looked plausible at the end of eighties (and was later, after appearance of [R1] and [R], adopted by most mathematicians working in the area) is that 'spaces' of noncommutative algebraic geometry are represented by abelian categories (thought as their categories of quasi-coherent or coherent sheaves). If X and Y are 'spaces' represented by abelian categories, respectively C_X and C_Y, then morphisms from X to Y are isomorphism classes of additive functors $C_Y \longrightarrow C_X$ called *inverse image* functors of the morphism they represent.

2.4. 'Spaces' represented by triangulated categories. Another viewpoint motivated in the first place by representation theory of reductive algebraic groups, and later (around 1993) by problems of mathematical physics (– homological mirror symmetry) is to consider 'spaces' represented by (enhanced) triangulated categories, which sometimes can be thought as derived categories of the categories of (quasi-)coherent sheaves on these 'spaces'.

2.5. 'Spaces' represented by A-infinity categories. At the end of nineties, working on deformation theory, M. Kontsevich expanded geometric flavor by considering 'spaces' represented by A-infinity categories.

2.6. 'Spaces' defined by presheaves of sets on the category of noncommutative affine schemes. The category $\mathbf{Aff}_k$ of affine noncommutative k-schemes is the category opposite to the category of associative unital k-algebras. Some of the important examples of noncommutative 'spaces', such as noncommutative Grassmannians, flag varieties and many others [KR1], [KR2], [KR3], are defined in two steps. The first step is a construction of a presheaf of sets on $\mathbf{Aff}_k$ (i.e. a functor from the category of unital associative k-algebras to the category of sets). In commutative algebraic geometry, the second step is taking the associated sheaf with respect to an appropriate (**fpqc** or Zariski) topology on $\mathbf{Aff}_k$. In noncommutative geometry, we assign, instead, to every presheaf of sets on $\mathbf{Aff}_k$ a fibred category whose fibers are categories of modules over k-algebras and define the category of quasi-coherent sheaves on this presheaf as the category opposite to the category of cartesian sections of this fibred category [KR4]. The category of quasi-coherent presheaves *represents* the 'space' corresponding to the presheaf of sets.

2.7. Commutative 'spaces' which "live" in symmetric monoidal categories. After the formalism of Tannakian categories appeared at the end of the sixties-beginning of the seventies [Sa], [DeM], and *'super'-mathematics* approximately at the same time, the idea of mathematics (or at least algebra and geometry), which uses general symmetric monoidal categories, instead of the symmetric monoidal category of vector spaces, became familiar. In [De], Deligne presented a sketch of a fragment of commutative projective geometry in symmetric monoidal k-linear abelian categories as a part of his proof of the characterization of rigid monoidal abelian categories having a fiber functor.

Manin defined the (category of coherent sheaves on the) **Proj** of a commutative $\mathbb{Z}_+$-graded algebra in a symmetric monoidal abelian category endowed with a fiber functor [M1] using, once again, the Serre's description of the category of coherent sheaves on a projective variety as its definition.

2.8. Quantized enveloping algebras and algebraic geometry in braided monoidal categories. While working (in 1995) on a quantum analog of Beilinson-Bernstein localization construction, it was discovered that 'spaces' of noncommutative algebraic geometry could be something different from just abelian or Grothendieck categories. In this particular situation, the natural action of the quantized enveloping algebra of a semisimple Lie algebra on its quantum base affine space becomes *differential* only if the whole picture is put into the monoidal category of $\mathbb{Z}^n$-graded modules endowed with a braiding determined by the Cartan matrix of the Lie algebra (see [LR2], [LR3], [LR4]).

This list (which is far from being complete) shows that the range of objects – *spaces* and *morphisms of spaces*, of noncommutative algebraic geometry is considerably larger than the range of objects of commutative algebraic geometry.

3. Pseudo-geometric start.

The pseudo-geometric noncommutative landscape sketched above is a natural point of departure, by the simple reason that it includes most examples of interest. Instead of trying to impose, from the very beginning, general notions of 'spaces' and morphisms of 'spaces', which absorb all the known cases, we approach these notions by studying algebraic geometry in certain key pseudo-geometric settings, which are simple enough to not to get lost and, at the same time, sufficient to obtain a rich theory and to see what one should expect or look for in more sophisticated pseudo-geometries.

3.1. 'Spaces' represented by categories. In the very first, in a sense the simplest, setting of this kind, 'spaces' are represented by svelte (– equivalent to small) categories and morphisms of 'spaces' $X \longrightarrow Y$ are isomorphism classes of (*inverse image*) functors $C_Y \longrightarrow C_X$ between the corresponding categories. This defines the category $|Cat|^o$ of 'spaces'. A morphism of 'spaces' is called *continuous* if its inverse image functor has a right adjoint (called a direct image functor), and it is called *flat* if, in addition, the inverse image functor is left exact (i.e. preserves finite limits). A continuous morphism is called *affine* if its direct image functor is conservative (i.e. it reflects isomorphisms) and has a right adjoint. These notions (introduced in [R]) unveil unexpectedly rich algebraic geometry, more precisely, *geometries*, living inside of $|Cat|^o$. They appear as follows.

3.2. Continuous monads. Fix a 'space' S such that the category C_S has cokernels of pairs of arrows. We consider the category $\mathfrak{End}(C_S)$ of *continuous* (i.e. having a right adjoint) endofunctors of C_S. It is a monoidal category with respect to the composition of functors whose unit object is the identical functor. The monoids in this category are called *continuous monads* on C_S. In other words, continuous monads on C_S are pairs (F, μ), where F is a continuous functor $C_S \longrightarrow C_S$ and μ is a functor morphism $F^2 \longrightarrow F$ such that $\mu \circ F\mu = \mu \circ \mu F$ and $\mu \circ F\eta = id_F = \mu \circ \eta F$ for a (unique) morphism $Id_{C_S} \xrightarrow{\eta} F$ called the unit of the monad (F, μ). A monad morphism $(F, \mu) \longrightarrow (F', \mu')$ is given by a functor morphism $F \xrightarrow{\varphi} F'$ such that $\varphi \circ \mu = \mu' \circ \varphi F' \circ F\varphi$ and $\varphi \circ \eta$ is the unit of the monad (F', μ'). This defines the category $\mathfrak{Mon}_{\mathfrak{c}}(S)$ of continuous monads on C_S.

If $C_S = \mathbb{Z} - mod$, then the category $\mathfrak{Mon}_{\mathfrak{c}}(S)$ is naturally equivalent to the category *Rings* of associative unital rings. If C_S is the category of quasi-coherent sheaves on a scheme $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, then $\mathfrak{Mon}_{\mathfrak{c}}(S)$ is equivalent to the category of quasi-coherent sheaves $\mathcal{A}$ of rings on $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ endowed with a morphism $\mathcal{O}_{\mathcal{X}} \longrightarrow \mathcal{A}$ of sheaves of rings. In particular, the sheaf of rings of (twisted) differential operators can be regarded as a monad on C_S.

If C_S is the category of sets, then the category $\mathfrak{Mon}_\mathfrak{c}(S)$ is equivalent to the category of monoids in the usual sense.

3.3. Relative affine 'spaces'. Given a 'space' S, we define the category Aff_S of *affine S-spaces* as the full subcategory of $|Cat|^o/S$ whose objects are pairs $(X, X \xrightarrow{f} S)$ with f an affine morphism.

3.4. Theorem. *The category Aff_S is anti-equivalent to the category $\mathfrak{Ass}_S$ whose objects are continuous monads on the category C_S and morphisms are conjugacy classes of monad morphisms.*

If $C_S = \mathbb{Z} - mod$, then the category $\mathfrak{Ass}_S$ is equivalent to the category whose objects are associative unital rings and morphisms are conjugacy classes of ring morphisms. If C_S is the category **Sets**, then $\mathfrak{Ass}_S$ is equivalent to the category whose objects are monoids and morphisms are conjugacy classes of monoid morphisms. This shows that the choice of the *base 'space'* S influences drastically the rest of the story.

3.5. Locally affine relative 'spaces'. *Locally affine* S-'spaces' are defined in an obvious way, once a notion of a cover (a *quasi-pretopology*) is fixed. We introduce several canonical quasi-pretopologies on the category $|Cat|^o$. Their common feature is the following: if a set of morphisms to X is a cover, then the set of their inverse image functors is conservative and all inverse image functors are exact in a certain mild way. If, in addition, morphisms of covers are continuous, X has a finite affine cover, and the category C_S has finite limits, then this requirement suffices to recover the object X from the covering data uniquely up to isomorphism (i.e. the category C_X is recovered uniquely up to equivalence) via 'flat descent'.

3.6. 'Spaces' determined by presheaves of sets on $\mathbf{Aff}_k$. By definition, the category $\mathbf{Aff}_k$ of noncommutative affine k-schemes is the category opposite to the category Alg_k of associative unital k-algebras; so that presheaves of sets on $\mathbf{Aff}_k$ are functors from Alg_k to **Sets**. The presheaves of sets on $\mathbf{Aff}_k$ appeared in our work with Maxim Kontsevich, for the first time in order to introduce noncommutative projective spaces. It was an attempt to imitate the standard commutative approach realizing schemes (and more general spaces) as sheaves of sets on the category of affine schemes endowed with an appropriate Grothendieck pretopology. It turned out that it is not clear a priori what an *appropriate pretopology* in the noncommutative case is: Zariski pretopology is irrelevant, because the noncommutative projective 'space' is not a scheme – it does not have an affine Zariski cover. Flat affine covers seemed to be as a natural choice, but, they do not form a pretopology – invariance under the base change fails. Similar story with Grassmannians and other analogs of commutative constructions. The elucidation of this problem is as follows. Consider the fibred category $\widetilde{\mathbf{Aff}}_k$ with the base $\mathbf{Aff}_k$ whose fibers are categories of left modules over corresponding algebras. For every presheaf of sets X on $\mathbf{Aff}_k$, we have the fibred category $\widetilde{\mathbf{Aff}}_k/X$ induced by $\widetilde{\mathbf{Aff}}_k$ along the forgetful functor $\mathbf{Aff}_k/X \longrightarrow \mathbf{Aff}_k$. The category $Qcoh(X)$ of quasi-coherent sheaves on the presheaf X is defined as the category opposite to the category of cartesian sections of the fibred category $\widetilde{\mathbf{Aff}}_k$. For a pretopology τ on $\mathbf{Aff}_k/X$, we define the subcategory $Qcoh(X,\tau)$ of *quasi-coherent sheaves* on $(\mathbf{Aff}_k/X,\tau)$.

3.7. Theorem. *(a) A pretopology τ on $\mathbf{Aff}_k$ is subcanonical (– all representable presheaves of sets are sheaves) iff $Qcoh(X,\tau) = Qcoh(X)$ for any presheaf of sets on $\mathbf{Aff}_k$ (in other words, 'descent' pretopologies on $\mathbf{Aff}_k$ are precisely subcanonical pretopologies).*

In this case, $Qcoh(X) = Qcoh(X,\tau) \hookrightarrow Qcoh(X^\tau) = Qcoh(X^\tau,\tau)$, where X^τ is the sheaf on $(\mathbf{Aff}_k,\tau)$ associated with the presheaf X and $\hookrightarrow$ is a natural full embedding.

(b) If a pretopology τ is of effective descent, then the embedding $Qcoh(X) \hookrightarrow Qcoh(X^\tau)$ is a category equivalence.

This theorem says that, roughly speaking, the category $Qcoh(X)$ of quasi-coherent presheaves knows itself which pretopologies to choose. It also indicates where one should look for a correct noncommutative version of the category Esp (of sheaves of sets on the **fpqc** site of commutative affine schemes): this should be the category $NEsp_\tau$ of sheaves of sets on the presite $(\mathbf{Aff}_k,\tau)$, where τ is a pretopology of *effective descent.* From the minimalistic point of view, the best choice would be the (finest) pretopology of effective descent. But, there is a more important consideration. The main role of a pretopology is that it is used for gluing new 'spaces' (so that the preference given in commutative algebraic geometry to **fpqc** pretopology on the category of affine schemes shows the readiness to consider more general locally affine spaces than schemes).

The pretopology that seems to be the most relevant for Grassmannians (in particular, for noncommutative projective 'spaces') and a number of other smooth noncommutative spaces constructed in [KR5] is the *smooth* topology introduced in [KR2].

The theorem is quite useful on a pragmatical level. Namely, if $\mathfrak{X}$ is a sheaf of sets on $(\mathbf{Aff}_k,\tau)$ for an appropriate pretopology of effective descent and X is a presheaf of sets on Aff_k such that its associated sheaf is isomorphic to $\mathfrak{X}$, and $\mathfrak{R} \underset{\mathfrak{p}_2}{\overset{\mathfrak{p}_1}{\rightrightarrows}} \mathfrak{U} \xrightarrow{\pi} X$ is an exact sequence of presheaves with $\mathfrak{R}$ and $\mathfrak{U}$ representable, then the category $Qcoh(X)$ (hence the category $Qcoh(\mathfrak{X})$) is constructively described (uniquely up to equivalence) via the pair $\mathcal{A} \underset{p_2}{\overset{p_1}{\rightrightarrows}} \mathcal{R}$ of k-algebra morphisms representing $\mathfrak{R} \underset{\mathfrak{p}_2}{\overset{\mathfrak{p}_1}{\rightrightarrows}} \mathfrak{U}$. This consideration is used to describe the categories of quasi-coherent sheaves on noncommutative 'spaces'.

3.8. Noncommutative stacks. There is one more important observation in connection with this theorem: categories which appear in noncommutative algebraic geometry are categories of quasi-coherent sheaves on noncommutative stacks.

4. From pseudo-geometry to geometry.

4.0. Spectra. The general expectation is that pseudo-geometric 'spaces' should have canonical spectral theories, and a choice of a spectral theory implies a *geometric realization* of 'spaces', which associates with every 'space' a stack whose base is a topological space – the *spectrum* of the 'space', and fibers at points are *local* 'spaces' – their spectrum has only one closed point, which belongs to the closure of any other point.

An important evidence for this thesis is the spectral theory of 'spaces' represented by svelte abelian categories, which was started in the middle of the eighties. Below follows a brief outline of some of its basic notions and facts. A detailed exposition is in Chapter II of this monograph.

4.1. Topologizing subcategories and the spectrum Spec$(-)$. A full subcategory of an abelian category C_X is called *topologizing* if it is closed under finite coproducts and subquotients. For an object M, we denote by $[M]$ the smallest topologizing subcategory of C_X containing M. One can show that objects of $[M]$ are subquotients of finite coproducts of copies of M. The *spectrum* $\mathbf{Spec}(X)$ of the 'space' X consists of all nonzero $[M]$ such that $[L] = [M]$ for any nonzero subobject L of M. We endow $\mathbf{Spec}(X)$ with the preorder $\supseteq$ which is called (with a good reason) the *specialization* preorder.

If M is a simple object, then the objects of $[M]$ are isomorphic to finite direct sums of copies of M and $[M]$ is a minimal element of $(\mathbf{Spec}(X), \supseteq)$. If C_X is the category of modules over a commutative unital ring R, then the map $p \longmapsto [R/p]$ is an isomorphism between the prime spectrum of R with specialization preorder and $(\mathbf{Spec}(X), \supseteq)$.

4.2. Local 'spaces'. An abelian category C_Y (and the 'space' Y) is called *local* if it has the smallest nonzero topologizing subcategory. It follows that this subcategory coincides with $[M]$ for any of its nonzero objects M; so that it is the smallest element of $\mathbf{Spec}(Y)$. If a local category has a simple object, M, then this smallest category coincides with $[M]$. In particular, all simple objects of C_Y (if any) are isomorphic one to another. The category of modules over a commutative ring is local iff the ring is local.

4.3. Serre subcategories and Spec$^-(-)$. A topologizing subcategory of an abelian category C_X is called *thick* if it is closed under extensions. For any subcategory $\mathcal{T}$ of C_X, let $\mathcal{T}^-$ denote the full subcategory of C_X whose objects are characterized by the following property: their subquotients have nonzero subobjects from $\mathcal{T}$. One can show that $(\mathcal{T}^-)^- = \mathcal{T}^-$ and the subcategory $\mathcal{T}^-$ is thick. We call a subcategory $\mathcal{T}$ of C_X a *Serre* subcategory if $\mathcal{T} = \mathcal{T}^-$.

Let $\mathbf{Spec}^-(X)$ denote the set of all Serre subcategories $\mathcal{P}$ such that the quotient category $C_X/\mathcal{P}$ is local. One can show that if C_X is a locally noetherian Grothendieck category (more generally, a Grothendieck category with a Gabriel-Krull dimension), then $\mathbf{Spec}^-(X)$ is isomorphic to the Gabriel spectrum of C_X.

Let $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$ denote the set of all Serre subcategories of C_X such that the intersection $\mathcal{P}^*$ of all topologizing subcategories properly containing $\mathcal{P}$ is not equal to $\mathcal{P}$.

4.4. Theorem. *(a)* $\mathbf{Spec}_{\mathfrak{t}}^{1,1} \subseteq \mathbf{Spec}^-(X)$.

(b) The map which assigns to a topologizing subcategory $\mathcal{Q}$ the union $\widehat{\mathcal{Q}}$ of all topologizing subcategories which do not contain $\mathcal{Q}$ is a bijection $\mathbf{Spec}(X) \xrightarrow{\sim} \mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$.

(c) Let $\mathcal{T}$ be a Serre subcategory of C_X and $C_X \xrightarrow{\mathfrak{q}^*} C_X/\mathcal{T}$ the localization functor.

(c1) If $\mathcal{T} \not\supseteq [M] \in$ $\mathbf{Spec}(X)$*, then $[\mathfrak{q}^*(M)] \in$* $\mathbf{Spec}(X/\mathcal{T})$.

(c2) The map $\mathcal{P} \longmapsto {\mathfrak{q}^*}^{-1}(\mathcal{P})$ is a bijection from $\mathbf{Spec}^-(X/\mathcal{T})$ *onto the subset* $\{\mathcal{P} \in \mathbf{Spec}^-(X) \mid \mathcal{T} \subseteq \mathcal{P}\}$.

(d) Let $\{\mathcal{T}_i \mid i \in J\}$ be a finite set of Serre subcategories such that $\bigcap_{i \in J} \mathcal{T}_i = 0$. Then

(d1) $\mathbf{Spec}^-(X) = \bigcup_{i \in J} \mathbf{Spec}^-(X/\mathcal{T}_i)$.

(d2) An element $\mathcal{P}$ of $\mathbf{Spec}^-(X)$ *belongs to* $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$ *iff* $\mathcal{P}/\mathcal{T}_i \in \mathbf{Spec}_{\mathfrak{t}}^{1,1}(X/\mathcal{T}_i)$ *whenever $\mathcal{T}_i \subseteq \mathcal{P}$.*

The assertion (c) can be extracted from [R, Ch.III]. The last assertion, the most important one, states that an element of $\mathbf{Spec}^-(X)$ belongs to $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$ (that is it corresponds to an element of $\mathbf{Spec}(X)$) iff this element belongs to $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$ *locally.*

4.5. The geometric center of a 'space' and the reconstruction of commutative schemes. Recall that the *center* of the category C_Y is the (commutative) ring of endomorphisms of its identical functor. If C_Y is a category of left modules over an associative unital ring R, then the center of C_Y is naturally isomorphic to the center of R.

We endow the spectrum $\mathbf{Spec}(X)$ with *Zariski* topology (which we do not describe here). The map $\widetilde{\mathcal{O}}_X$ which assigns to every open subset W of $\mathbf{Spec}(X)$ the center of the quotient category $C_X/\mathcal{S}_W$, where $\mathcal{S}_W = \bigcap_{\mathcal{Q} \in W} \widehat{\mathcal{Q}}$, is a presheaf on $\mathbf{Spec}(X)$. We denote by $\mathcal{O}_X$ its associated sheaf. One can show that the stalk of the sheaf $\mathcal{O}_X$ at a point $\mathcal{Q}$ of the spectrum is isomorphic to the center of the local category $C_X/\widehat{\mathcal{Q}}$, and the center of a local category is a local commutative ring. The locally ringed space $(\mathbf{Spec}(X), \mathcal{O}_X)$ is called the *geometric center* (or *Zariski geometric center*) of the 'space' X.

One of the consequences of the theorem above is the following reconstruction theorem:

4.6. Theorem. *If C_X is the category of quasi-coherent sheaves on a commutative quasi-compact quasi-separated scheme, then the geometric center* $(\mathbf{Spec}(X), \mathcal{O}_X)$ *of the 'space' X is naturally isomorphic to the scheme. So that any quasi-separated quasi-compact commutative scheme is canonically reconstructed, uniquely up to isomorphism, from its category of quasi-coherent sheaves.*

In the case of a noetherian scheme, this theorem recovers Gabriel's reconstruction theorem [Gab], because it is easy to show that if C_X is the category of modules over a commutative noetherian ring, then the injective spectrum of C_X is naturally isomorphic to the spectrum $\mathbf{Spec}(X)$.

4.7. Geometric realization of a noncommutative scheme. Let C_X be an abelian category with enough objects of finite type. We have a contravariant pseudo-functor from the category of the Zariski open sets of the spectrum $\mathbf{Spec}(X)$ to Cat which assigns to each open set $\mathcal{U}$ of $\mathbf{Spec}(X)$ the quotient category $C_X/\mathcal{S}_{\mathcal{U}}$, where $\mathcal{S}_{\mathcal{U}} = \bigcap\limits_{\mathcal{Q}\in\mathcal{U}} \widehat{\mathcal{Q}}$, and to each embedding $\mathcal{U} \hookrightarrow \mathcal{V}$ the corresponding localization functor. To this pseudo-functor, there corresponds (by a standard formalism) a fibred category over the Zariski topology of $\mathbf{Spec}(X)$. The associated stack, $\mathfrak{F}^{\mathfrak{z}}_X$, is a stack of local categories: its stalk at each point $\mathcal{Q}$ of $\mathbf{Spec}(X)$ is equivalent to the local category $C_X/\widehat{\mathcal{Q}}$.

We regard the stack $\mathfrak{F}^{\mathfrak{z}}_X$ as a *geometric realization* of the abelian category C_X.

If X is a (noncommutative) scheme, then the stack $\mathfrak{F}^{\mathfrak{z}}_X$ is *locally affine.*

4.8. Note: the geometric center of noncommutative schemes. Taking the center of each fiber of the stack $\mathfrak{F}^{\mathfrak{z}}_X$, we recover the presheaf of commutative rings $\widetilde{\mathcal{O}}_X$, hence the geometric center of the 'space' X.

Note that the stalks at points of a noncommutative scheme are local abelian categories, which only in exceptional cases are equivalent to categories of modules over rings. This explains why imposing that noncommutative schemes should be ringed topological spaces did not work.

4.9. The spectrum $\mathbf{Spec}^0_{\mathfrak{c}}(X)$. If C_X is the category of quasi-coherent sheaves on a non-quasi-compact scheme, like, for instance, the flag variety of a Kac-Moody Lie algebra, or a noncommutative scheme which does not have a finite affine cover (say, the quantum flag variety of a Kac-Moody Lie algebra, or the corresponding quantum D-scheme), then the spectrum $\mathbf{Spec}(X)$ is insufficient. It should be replaced by the spectrum $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ whose elements are coreflective topologizing subcategories of C_X of the form $[M]_{\mathfrak{c}}$ (i.e. generated by the object M) such that if L is a nonzero subobject of M, then $[L]_{\mathfrak{c}} = [M]_{\mathfrak{c}}$.

There is a natural map $\mathbf{Spec}(X) \longrightarrow \mathbf{Spec}^0_{\mathfrak{c}}(X)$ which assigns to every $\mathcal{Q} \in \mathbf{Spec}(X)$ the smallest coreflective subcategory $[\mathcal{Q}]_{\mathfrak{c}}$ containing $\mathcal{Q}$. If the category C_X has enough objects of finite type, this canonical map is a bijection.

4.10. Theorem. *Let $\{\mathcal{T}_i \mid i \in J\}$ be a set of coreflective thick subcategories of an abelian category C_X such that $\bigcap\limits_{i\in J} \mathcal{T}_i = 0$; and let $C_X \xrightarrow{u_i^*} C_X/\mathcal{T}_i$ be the localization functor. The following conditions on a nonzero coreflective topologizing subcategory $\mathcal{Q}$ of C_X are equivalent:*

(a) $\mathcal{Q} \in \mathbf{Spec}^0_{\mathfrak{c}}(X)$,
(b) $[u_i^*(\mathcal{Q})]_{\mathfrak{c}} \in \mathbf{Spec}^0_{\mathfrak{c}}(X/\mathcal{T}_i)$ *for every* $i \in J$ *such that* $\mathcal{Q} \not\subseteq \mathcal{T}_i$.

One of the consequences of this theorem is the following reconstruction theorem.

4.11. Theorem. *Let* C_X *be the category of quasi-coherent sheaves on a commutative scheme* $\mathbf{X} = (\mathcal{X}, \mathcal{O})$. *Suppose that there is an affine cover* $\{\mathcal{U}_i \hookrightarrow \mathcal{X} \mid i \in J\}$ *of the scheme* $\mathbf{X}$ *such that all immersions* $\mathcal{U}_i \hookrightarrow \mathcal{X}$, $i \in J$, *have a direct image functor (say, the scheme* $\mathbf{X}$ *is quasi-separated). Then the geometric center* $(\mathbf{Spec}^0_{\mathfrak{c}}(X), \mathcal{O}_X)$ *is isomorphic to the scheme* $\mathbf{X}$.

If $\mathbf{X} = (\mathcal{X}, \mathcal{O})$ is a quasi-compact quasi-separated scheme, then the category C_X of quasi-coherent sheaves on $\mathbf{X}$ has enough objects of finite type, hence the spectrum $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ coincides with $\mathbf{Spec}(X)$. Thus, the reconstruction theorem for quasi-compact quasi-separated schemes is a special case of the theorem above.

5. Noncommutative local algebra and representation theory.

A classical problem of representation theory is the construction of (interesting classes of) irreducible representations. From the point of view of noncommutative algebraic geometry, this problem is a part of a more natural and more general problem of constructing objects representing elements of an appropriate spectrum. Likewise, in commutative algebraic geometry, the set of maximal ideals of a ring is replaced by its prime spectrum.

On the experimental level, the work on the realizations of points of the spectrum started at the end of nineteen eighties with constructing realizations of the spectrum of several 'small' algebras which appear in representation theory and mathematical physics, like the first Weyl and Heisenberg algebras and their quantum analogs, (classical and quantized) enveloping algebra of sl(2), quantum algebra of functions on SL(2). Some of the computations are gathered in Chapters II and IV of the monograph [R]. These examples, however, are of a special nature – they belong to the class of so called 'hyperbolic' algebras or rank 1 [R, Ch.II] (or 'hyperbolic monads' of rank 1 in [R, Ch.IV]) which is particularly convenient for spectral computations. Algebras of skew differential operators is the only other class of algebras whose spectrum was effectively computed "by hands"[R8].

5.1. Associated points. Let M be an object of the category C_X. An element $\mathcal{Q}$ of $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ is called an *associated point of* M *in* $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ if M has a nonzero subobject L such that $\mathcal{Q} = [L]_{\mathfrak{c}}$ and L is right orthogonal to $\widehat{\mathcal{Q}}$). We denote the set of associated points of M in $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ by $\mathfrak{Ass}_{\mathfrak{c}}(M)$.

Associated points have properties analogous to the known properties of associated points of modules over commutative rings.

5.2. Induction problem. Let X and Y be 'spaces' represented by abelian categories, resp. C_X and C_Y, $X \xrightarrow{\mathfrak{f}} Y$ a continuous morphism of 'spaces', $\mathcal{Q}$ a point of the spectrum of X. The *induction problem* is to find representatives M of the spectrum of $\mathfrak{X}$ such that $\mathcal{Q}$ is an associated point of $\mathfrak{f}_*(M)$.

If C_X and C_Y are categories of quasi-coherent sheaves on commutative schemes, respectively $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ and $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$, and $\mathfrak{f}^*$ is an inverse image functor of a scheme morphism $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \xrightarrow{(\varphi,\xi)} (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$, then the induction problem is the problem of the construction of the correspondence $x \mapsto \varphi^{-1}(x)$ inverse to the map of the underlying topological spaces.

It turns out that all previously obtained realizations of spectral points (in particular those of [R, Ch.II and Ch.IV]) are specializations of an *induction construction*, which gives a solution of the induction problem in the case when $\mathfrak{f}$ is a locally affine morphism and the pair $(\mathfrak{f}, \mathcal{Q})$ satisfies certain additional conditions.

In the case of an affine morphism $X \xrightarrow{\mathfrak{f}} Y$, the induction construction is as follows. There is a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "\mathfrak{f}_{\mathcal{Q}}"] \arrow[dr, "\mathfrak{f}"'] & & X_{\mathcal{Q}} \arrow[dl, "\phi_{\mathcal{Q}}"] \\ & Y & \end{tikzcd}$$

of affine morphisms, where $X_{\mathcal{Q}} \xrightarrow{\phi_{\mathcal{Q}}} Y$ is the so called *stabilizator* of the point $\mathcal{Q}$ (defined in Chapter III). Let $\mathfrak{L}_{\mathcal{Q}}$ denote the composition of the functor $\mathfrak{f}_{\mathcal{Q}}^*$ and the functor which assigns to every object of the category C_X the quotient of this object by its $\mathfrak{f}_*^{-1}(\widehat{\mathcal{Q}})$-torsion, where ${}^{\mathfrak{c}}\widehat{\mathcal{Q}}$ is the Serre subcategory of C_X corresponding to $\mathcal{Q}$ (generated by all objects N such that $\mathcal{Q} \not\subseteq [N]_{\mathfrak{c}}$).

Let $Spec_{\mathfrak{c}}^0(X)$ denote the family of *representatives* of elements of the $\mathbf{Spec}_{\mathfrak{c}}^0(X)$, i.e. objects M such that $[M]_{\mathfrak{c}} = \mathcal{Q} \in \mathbf{Spec}_{\mathfrak{c}}^0(X)$ and M is ${}^{\mathfrak{c}}\widehat{\mathcal{Q}}$-torsion free. Let $Spec_{\mathfrak{c}}^{\mathcal{Q}}(X_{\mathcal{Q}})$ denote the family of all objects of $Spec_{\mathfrak{c}}^0(X_{\mathcal{P}})$ such that $\mathcal{Q}$ is an associated point of their image in C_X. If the inverse image functor $\mathfrak{f}_{\mathcal{Q}}^*$ is exact and faithful and certain 'ampleness' conditions are satisfied, then the functor $\mathfrak{L}_{\mathcal{Q}}$ transforms every object of $Spec_{\mathfrak{c}}^{\mathcal{Q}}(X_{\mathcal{Q}})$ into an object of the spectrum of the 'space' X. Moreover, every object of the spectrum of X whose image in $C_{X_{\mathcal{Q}}}$ has an associated point which belongs to $Spec_{\mathfrak{c}}^{\mathcal{Q}}(X_{\mathcal{Q}})$ is equivalent to the image of this associated point by the functor $\mathfrak{L}_{\mathcal{Q}}$. The functor $\mathfrak{L}_{\mathcal{Q}}$ maps simple objects from $Spec_{\mathfrak{c}}^{\mathcal{Q}}(X_{\mathcal{Q}})$ to simple objects of C_X

The induction construction is purely noncommutative: if the morphism $X \xrightarrow{\mathfrak{f}} Y$ corresponds to a morphism of commutative schemes, then $X \xrightarrow{\mathfrak{f}_{\mathcal{Q}}} X_{\mathcal{Q}}$ is an isomorphism, i.e. the construction is trivial. The best results are achieved when the stabilizer is trivial, that is $X_{\mathcal{Q}} \xrightarrow{\phi_{\mathcal{Q}}} Y$ is an isomorphism. In general, the 'size' of the stabilizer measures noncommutativity (or commutativity) of the pair $(\mathfrak{f}, \mathcal{Q})$.

6. Geometry of t-'spaces'.

These are 'spaces' represented by svelte triangulated Karoubian categories. We denote by $\mathcal{CT}_{\mathfrak{X}}$ the triangulated category corresponding to the t-'space' $\mathfrak{X}$. Morphisms from $\mathfrak{X}$ to $\mathfrak{Y}$ are isomorphism classes of triangle functors $\mathcal{CT}_{\mathfrak{Y}} \longrightarrow \mathcal{CT}_{\mathfrak{X}}$. The pseudo-geometric picture of t-'spaces' is more convenient and graceful than in the case of 'spaces' represented by abelian categories. The key fact is an analogue of the Beck's theorem for triangle functors and resulted from this analogue triangular replacement of *flat* descent.

6.1. The spectra of a t-'space'. We start with the spectrum $\mathbf{Spec}^1_{\mathfrak{L}}(\mathfrak{X})$ of a t-'space' $\mathfrak{X}$ related with exact localizations. It consists of all thick triangulated subcategories $\mathcal{P}$ of the triangulated category $\mathfrak{TC}_{\mathfrak{X}}$ such that the intersection $\mathcal{P}^*$ of all thick triangulated subcategories properly containing $\mathcal{P}$ does not coincide with $\mathcal{P}$. This spectrum is decomposed into the disjoint union of two parts. One part, $\mathbf{Spec}^{1,0}_{\mathfrak{L}}(\mathfrak{X})$, consists of points $\mathcal{P}$, which are *fat* in the sense that the right orthogonal complement to $\mathcal{P}$ inside of the subcategory $\mathcal{P}^*$ is zero; i.e. $\mathcal{P}$ contains a set of generators of the subcategory $\mathcal{P}^*$. In the case when the triangulated category has infinite products or coproducts, $\mathcal{P}$ generates the whole triangulated category: its right orthogonal complement, $\mathcal{P}^\perp$, is zero. The complementary part $\mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X})$, which consists of non-fat points, is the object of our study. We observe that the spectrum $\mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X})$ has a natural counterpart $\mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X})$, which might be regarded as the triangulated version of the spectrum $\mathbf{Spec}(X)$. The map, which assigns to a thick subcategory $\mathcal{P}$ the intersection $\mathcal{P}_* = \mathcal{P}^* \cap \mathcal{P}^\perp$ is a bijection from $\mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X})$ onto $\mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X})$. There are natural notions of *supports* of objects which are used, among other things, to define topologies on the spectra, in particular analogs of the Zariski topology. These spectra have simple local properties, similar to those of the spectrum $\mathbf{Spec}^-(-)$. Namely, if $\{\mathcal{T}_i \mid i \in J\}$ is a finite family of thick triangulated subcategories of the triangulated category $\mathcal{CT}_{\mathfrak{X}}$ such that $\bigcap\limits_{i\in J} \mathcal{T}_i = 0$, then $\mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X}) = \bigcup\limits_{i\in J} \mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X}/\mathcal{T}_i)$.

Moreover, these spectra have much better functorial properties than the spectra of 'spaces' represented by abelian categories. Explicitly, this means that the triangulated analog of the *induction construction* outlined above – a spectral version of cohomological induction, works without additional "ampleness" conditions on the pair $(\mathfrak{f}, \mathcal{Q})$, unlike its abelian prototype.

7. A sequence of events. Different parts of this story moved in different directions dictated mostly by immediate needs of several concrete examples and problems.

The first serious progress was due to the discovery (in the middle of eighties) of the spectrum $\mathbf{Spec}(-)$ of 'spaces' represented by svelte abelian categories [R, Ch.III) and its applications to representations of algebras of mathematical physics (see [R], Chapters II, IV). At that stage, there was a harmony between pseudo-geometry and geometry: only 'spaces' represented by abelian (or even Grothendieck) categories were considered and the spectrum $\mathbf{Spec}(X)$ (endowed with a version of Zariski topology) was regarded as the *underlying topological* space of X. The monograph [R] marked the end of that period.

Then, in the middle of the nineties, appeared the work on D-modules on noncommutative 'spaces' with the goal to obtain a quantized version of the Beilinson-Bernstein localization construction [LR1], [LR2]. The contemplation of the quantum analogues of the base affine space, flag variety and the *quantum D-'spaces'* determined by the action of quantized enveloping algebra on this 'spaces' (see [LR2]) led to the notion of a *noncommutative scheme* [R2], [R3]. This notion is purely categorical and does not use any abelian (or even additivity) hypothesis. The work on noncommutative projective spaces [KR1] and noncommutative Grassmannians provoked the study of 'spaces' defined by presheaves of sets on the category of noncommutative affine schemes and (formally) smooth 'spaces' [KR2], considerably extended the use of flat descent turning it into a tool for describing the categories of quasi-coherent sheaves on noncommutative 'spaces' [KR3]. This eventually triggered (actually, required) the introduction of noncommutative stacks [KR4].

As for the geometric part, it continued to develop, for quite a while, only as the spectral theory of ('spaces' represented by) abelian categories of [R, Ch.III], [R4], without any reaction to all these pseudo-geometric developments. It seemed at the time that the spectrum $\mathbf{Spec}(-)$ is an exceptional notion. The absence of a spectral theory of 'spaces' represented by triangulated categories had been of a particular nuisance, considering the role triangulated categories play in representation theory and started to play in mathematical physics. Then spectra of 'spaces' represented by arbitrary svelte categories – *spectra related with localizations*, were discovered. They were easily adjusted to more sophisticated settings; in particular, a satisfactory spectral theory of 'spaces' represented by svelte triangulated categories was, finally, found [R6].

Moreover, these several spectra provided enough experimental material to figure out a general pattern producing spectra – *spectral cuisine* [R5]. Thanks to this work, geometry (i.e. spectral theory) almost caught up with pseudo-geometry, at least potentially. Still, the most important spectrum, $\mathbf{Spec}(-)$, continued to resist generalizations. Its straightforward version for 'spaces' represented by exact categories (appeared in [R, Ch.5]) does not inherit important properties with respect to localizations (explained in Chapter II of this book).

Note that, from the algebraic point of view, exact categories are much more robust than abelian categories. For instance, unlike abelian categories, they are stable under transition to the categories of filtered objects. They also contain the categories of projective modules over associative rings and, more generally, categories of vector bundles on ringed spaces (in particular, on schemes), which was the first reason for introducing them in homological algebra and K-theory. Important categories of functional analysis (starting from the category of Banach spaces) are exact. Therefore, studying 'spaces' represented by exact categories instead of abelian categories should create a bridge towards noncommutative geometries related to analysis, in particular, to noncommutative differential geometry.

An attempt to understand cycles and K-theory of noncommutative 'spaces' and schemes led to a new homological algebra and, as a byproduct, to an important expansion of pseudo-geometry: 'spaces' represented by *right* (or *left*) *exact* categories. Right exact categories are categories endowed with a Grothendieck pretopology whose covers are strict epimorphisms. In particular, *exact* categories are a special case of right exact categories. Finally, an adequate extension of the spectrum $\mathbf{Spec}(-)$ to 'spaces' represented by svelte right exact categories was found [R12]. Considering that every category has a canonical (the finest) right exact structure, this last development establishes (at least temporarily) a harmony between pseudo-geometric and geometric parts of noncommutative algebraic geometry and opens entirely new prospectives.

8. Texts. The above described evolution (which occurred since the appearance of the monograph [R]) formed a whole cycle of development: first new, important pseudo-geometric notions and poorly understood pseudo-geometric examples and constructions emerged during the second half of the nineties; then new notions, facts and insights discovered during the first several years of the twenty first century allowed to fill up gaps between different, seemingly unrelated pieces of noncommutative pseudo-geometry and permitted to find the missing geometric parts of the story. The present state of the subject looks, therefore, appropriate for organizing material scattered among the papers ([KR1]-[KR5], [LR1], [LR2], [R2]-[R8]) and unpublished notes into a coherent exposition of (certain chapters of) foundations of noncommutative algebraic geometry. The manuscripts

Noncommutative 'Spaces' and Stacks

Geometry of Noncommutative 'Spaces' and Schemes

Homological Algebra of Noncommutative 'Spaces' I

are the parts of the treatise written so far. The first one is mostly based on the papers [KR1]-[KR5] and the notes of courses on noncommutative algebraic geometry and algebra given at Kansas State University. It might be called "basics of noncommutative pseudo-geometry". Due to the role of pseudo-geometry, results of this manuscript are used, directly or indirectly, in the rest of the treatise. Therefore, we give a brief outline of its content.

We start with 'spaces' represented by svelte categories and morphisms of 'spaces' represented by (their inverse image) functors, and develop the basic theory of locally affine 'spaces' and schemes with a stress on flat descent (used as a tool for describing categories of quasi-coherent sheaves on noncommutative 'spaces') and the noncommutative analogs of smooth and étale morphisms etc.. Then we study 'spaces' determined by presheaves of sets on the category $\mathbf{Aff}_k$ of noncommutative affine schemes over a commutative ring k. We introduce and study Grassmannians, generalized Grassmannians – the non-commutative analog of Quot schemes, and (generalized) flag varieties. We associate with presheaves of sets on $\mathbf{Aff}_k$ ringed categories, which give rise to the categories of quasi-coherent (pre)sheaves of modules. Finally, there are two more aspects that we take into consideration: *constructions of important 'spaces'* and *geometries, which "live" in different monoidal categories*. We combine the two aspects together. Namely, the constructions of 'spaces' are made inside of geometries living in monoidal categories.

There is no need to explain here details of the third manuscript, but, of course, we sketch the organization of the second one – the present volume.

10. Content. In order to make the exposition self-contained, the necessary pseudo-geometric preliminaries are summarized (without proofs) in the first chapter.

Chapter II describes the spectral theory of 'spaces' represented by abelian categories.

Chapter III studies the functorial properties of spectra and gives some of its applications to representation theory of (quantized) enveloping algebras.

Chapter IV is dedicated to the geometry of 'spaces' represented by triangulated categories. It follows the pattern of the first three chapters. We start with a description of continuous triangle morphisms, which is a triangulated version of Beck's theorem. The purpose of the chapter is to present a triangulated version of the main facts of Chapters II and III – the relevant spectra and their functorial properties.

The reader who is interested only in geometry of abelian and triangulated categories and their applications to algebraic representation theory, can ignore the rest of the book.

In Chapter V, "Spectra related with localizations", we start to fill up the most obvious aesthetical gap between the fact that main pseudo-geometric notions (like schemes, for example) are defined for 'spaces' represented by arbitrary svelte categories, while there was no geometric (i.e. spectral) counterpart. Part of material is taken from [R6]. The spectra we introduced here are directly related to exact localizations. In a sense, we obtain natural extensions of Gabriel's spectrum for 'spaces' represented by arbitrary svelte categories. Our main spectrum, $\mathbf{Spec}(-)$, remains out of reach in this approach.

It turns out that the spectrum $\mathbf{Spec}(-)$ can be recovered if we take into consideration a structure of a *right exact* category. This is done in Chapter VI, "Geometry of right exact 'spaces"', dedicated to spectral theory of 'spaces'represented by svelte right exact categories. In particular, we extend to right exact 'spaces' the spectrum $\mathbf{Spec}(-)$ and establish the analogues of the main facts of Chapter II.

Chapter VII is called, for a good reason, "Spectral cuisine for the working mathematicians". It is based on [R5] (enriched with some more recent observations) and describes a general machinery, which produces spectra. All spectra appeared so far here and in [R] can be obtained using this machine, as well as new spectra.

In order to make the exposition "user friendly", all spectral constructions and facts coming from the "Spectral cuisine" which appear in other Chapters are explained independently in each case. So that "users" might omit reading Chapter VII.

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Chapter I
Locally Affine 'Spaces' and Schemes.

In Section 1, we review the first notions of noncommutative algebraic geometry – preliminaries on 'spaces' represented by categories, morphisms represented by their inverse image functors. We recall the notions of continuous, flat and affine morphisms and illustrate them with a couple of examples. In Section 2, we remind Beck's theorem characterizing monadic morphisms and apply it to the study of affine relative schemes. In Section 3, we introduce the notions of a *weakly locally affine* morphism and a *weak scheme* over a 'space'. Section 4 is dedicated to *flat descent* which is one of the main tools of noncommutative algebraic geometry. In Section 5, we sketch several examples of noncommutative schemes and more general locally affine spaces, which are among illustrations and/or motivations of constructions of this work. The whole chapter can be regarded as a review of the few facts of the noncommutative algebraic (or categoric pseudo-)geometry which are used in the rest of the work. There are practically no proofs. They can be found in the first Chapter of [KR7] and in [KR3].

1. Noncommutative 'spaces' represented by categories and morphisms between them. Continuous, affine and locally affine morphisms.

1.1. Categories and 'spaces'. As usual, Cat, or $Cat_{\mathfrak{U}}$, denotes the bicategory of categories, which belong to a fixed universum $\mathfrak{U}$. We call objects of Cat^{op} 'spaces'. For any 'space' X, the corresponding category C_X is regarded as the category of quasi-coherent sheaves on X. For any $\mathfrak{U}$-category $\mathcal{A}$, we denote by $|\mathcal{A}|$ the corresponding object of Cat^{op} (the underlying 'space') defined by $C_{|\mathcal{A}|} = \mathcal{A}$.

We denote by $|Cat|^o$ the category having same objects as Cat^{op}. Morphisms from X to Y are isomorphism classes of functors $C_Y \longrightarrow C_X$. For a morphism $X \xrightarrow{f} Y$, we denote by f^* any functor $C_Y \longrightarrow C_X$ representing f and call it an *inverse image functor of the morphism* f. We shall write $f = [F]$ to indicate that f is a morphism having an inverse image functor F. The composition of morphisms $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ is defined by $g \circ f = [f^* \circ g^*]$.

1.2. Localizations and conservative morphisms. Let Y be an object of $|Cat|^o$ and Σ a class of arrows of the category C_Y. We denote by $\Sigma^{-1}Y$ the object of $|Cat|^o$ such that the corresponding category coincides with (the standard realization of) the quotient of the category C_Y by Σ (cf. [GZ, 1.1]): $C_{\Sigma^{-1}Y} = \Sigma^{-1}C_Y$. The canonical *localization functor* $C_Y \xrightarrow{p_\Sigma^*} \Sigma^{-1}C_Y$ is regarded as an inverse image functor of a morphism, $\Sigma^{-1}Y \xrightarrow{p_\Sigma} Y$.

For any morphism $X \xrightarrow{f} Y$ in $|Cat|^o$, we denote by Σ_{f^*} the family of all arrows s of the category C_Y such that $f^*(s)$ is invertible (notice that Σ_{f^*} does not depend on the choice of an inverse image functor f^*). Thanks to the universal property of localizations, f^* is represented as the composition of the localization functor $p_f^* = p^*_{\Sigma_{f^*}} : C_Y \longrightarrow \Sigma_{f^*}^{-1} C_Y$ and a uniquely determined functor $\Sigma_{f^*}^{-1} C_Y \xrightarrow{f_{\mathfrak{c}}^*} C_X$. In other words, $f = p_f \circ f_{\mathfrak{c}}$ for a uniquely determined morphism $X \xrightarrow{f_{\mathfrak{c}}} \Sigma_{f^*}^{-1} Y$.

A morphism $X \xrightarrow{f} Y$ is called *conservative* if Σ_{f^*} consists of isomorphisms, or, equivalently, p_f is an isomorphism.

A morphism $X \xrightarrow{f} Y$ is called a *localization* if $f_{\mathfrak{c}}$ is an isomorphism, i.e. the functor $f_{\mathfrak{c}}^*$ is an equivalence of categories.

Thus, $f = p_f \circ f_{\mathfrak{c}}$ is a unique decomposition of a morphism f into a localization and a conservative morphism.

1.3. Continuous, flat, and affine morphisms. A morphism is called *continuous* if its inverse image functor has a right adjoint (called a direct image functor), and *flat* if, in addition, the inverse image functor is left exact (i.e. preserves finite limits). A continuous morphism is called *affine* if its direct image functor is conservative (i.e. it reflects isomorphisms) and has a right adjoint.

1.4. Categoric spectrum of a unital ring. For an associative unital ring R, we define the *categoric spectrum* of R as the object $\mathbf{Sp}(R)$ of $|Cat|^o$ represented by the category $R-mod$ of left R-modules; i.e. $C_{\mathbf{Sp}(R)} = R-mod$.

Let $R \xrightarrow{\phi} S$ be a unital ring morphism and $R-mod \xrightarrow{\bar{\phi}^*} S-mod$ the functor $S \otimes_R -$. The canonical right adjoint to $\bar{\phi}^*$ is the pull-back functor $\bar{\phi}_*$ along the ring morphism ϕ. A right adjoint to $\bar{\phi}_*$ is given by

$$S - mod \xrightarrow{\bar{\phi}^!} R - mod, \quad L \longmapsto Hom_R(\phi_*(S), L).$$

The map

$$\left(R \xrightarrow{\phi} S\right) \longmapsto \left(\mathbf{Sp}(S) \xrightarrow{\bar{\phi}} \mathbf{Sp}(R)\right)$$

is a functor

$$Rings^{op} \xrightarrow{\mathbf{Sp}} |Cat|^o$$

which takes values in the subcategory of $|Cat|^o$ formed by affine morphisms.

The image $\mathbf{Sp}(R) \xrightarrow{\bar{\phi}} \mathbf{Sp}(T)$ of a ring morphism $T \xrightarrow{\phi} R$ is flat (resp. faithful) iff ϕ turns R into a flat (resp. faithful) right T-module.

1.4.1. Continuous, flat, and affine morphisms from $\mathbf{Sp}(S)$ to $\mathbf{Sp}(R)$. Let R and S be associative unital rings. A morphism $\mathbf{Sp}(S) \xrightarrow{f} \mathbf{Sp}(R)$ with an inverse image functor f^* is continuous iff

$$f^* \simeq \mathcal{M}\otimes_R \ : L \longmapsto \mathcal{M} \otimes_R L \tag{1}$$

for an (S,R)-bimodule $\mathcal{M}$ defined uniquely up to isomorphism. The functor

$$f_* = Hom_S(\mathcal{M}, -) : N \longmapsto Hom_S(\mathcal{M}, N) \tag{2}$$

is a direct image of f.

By definition, the morphism f is conservative iff $\mathcal{M}$ is *faithful* as a right R-module, i.e. the functor $\mathcal{M} \otimes_R -$ is faithful.

The direct image functor (2) is conservative iff $\mathcal{M}$ is a generator in the category of left S-modules, i.e. for any nonzero S-module N, there exists a nonzero S-module morphism $\mathcal{M} \longrightarrow N$.

The morphism f is flat iff $\mathcal{M}$ is flat as a right R-module.

The functor (2) has a right adjoint, $f^!$, iff f_* is isomorphic to the tensoring (over S) by a bimodule. This happens iff $\mathcal{M}$ is a projective S-module of finite type. The latter is equivalent to the condition: the natural functor morphism $\mathcal{M}^*_S \otimes_S - \longrightarrow Hom_S(\mathcal{M}, -)$ is an isomorphism. Here $\mathcal{M}^*_S = Hom_S(M, S)$. In this case, $f^! \simeq Hom_R(\mathcal{M}^*_S, -)$.

1.5. Example. Let $\mathcal{G}$ be a monoid and R a $\mathcal{G}$-graded unital ring. We define the 'space' $\mathbf{Sp}_{\mathcal{G}}(R)$ by taking as $C_{\mathbf{Sp}_{\mathcal{G}}(R)}$ the category $gr_{\mathcal{G}}R - mod$ of left $\mathcal{G}$-graded R-modules. There is a natural functor $gr_{\mathcal{G}}R - mod \xrightarrow{\phi_*} R_0 - mod$, which assigns to each graded R-module its zero component ('zero' is the unit element of the monoid $\mathcal{G}$). The functor ϕ_* has a left adjoint, ϕ^*, which maps every R_0-module M to the graded R-module $R \otimes_{R_0} M$. The adjunction arrow $Id_{R_0 - mod} \longrightarrow \phi_*\phi^*$ is an isomorphism. This means that the functor ϕ^* is fully faithful, or, equivalently, the functor ϕ_* is a localization.

The functors ϕ_* and ϕ^* are regarded as respectively a direct and an inverse image functor of a morphism $\mathbf{Sp}_{\mathcal{G}}(R) \xrightarrow{\phi} \mathbf{Sp}(R_0)$. It follows from the above that the morphism ϕ is affine iff ϕ is an isomorphism (i.e. ϕ^* is an equivalence of categories).

In fact, if ϕ is affine, the functor ϕ_* should be conservative. Since ϕ_* is a localization, this means, precisely, that ϕ_* is an equivalence of categories.

1.6. The cone of a non-unital ring. Let R_0 be a unital associative ring, and let R_+ be an associative ring, non-unital in general, in the category of R_0-bimodules; i.e. R_+ is endowed with an R_0-bimodule morphism $R_+ \otimes_{R_0} R_+ \xrightarrow{m} R_+$ satisfying the associativity condition. Let $R = R_0 \oplus R_+$ denote the augmented ring described by this data. Let $\mathcal{T}_{R_+}$

denote the full subcategory of the category $R - mod$ whose objects are all R-modules annihilated by R_+. Let $\mathcal{T}^-_{R_+}$ be the Serre subcategory (that is a full subcategory closed by taking subquotients, extensions, and arbitrary direct sums) of the category $R - mod$ spanned by $\mathcal{T}_{R_+}$.

We define the 'space' *cone of* R_+ by taking as $C_{\mathbf{Cone}(R_+)}$ the quotient category $R - mod/\mathcal{T}^-_{R_+}$. The localization functor $R - mod \xrightarrow{u^*} R - mod/\mathcal{T}^-_{R_+}$ is an inverse image functor of a morphism of 'spaces' $\mathbf{Cone}(R_+) \xrightarrow{u} \mathbf{Sp}(R)$. The functor u^* has a (necessarily fully faithful) right adjoint, i.e. the morphism u is continuous. If R_+ is a unital ring, then u is an isomorphism (see C3.2.1). The composition of the morphism u with the canonical affine morphism $\mathbf{Sp}(R) \longrightarrow \mathbf{Sp}(R_0)$ is a continuous morphism $\mathbf{Cone}(R_+) \longrightarrow \mathbf{Sp}(R_0)$. Its direct image functor is (regarded as) the *global sections functor*.

1.7. The graded version: $\mathbf{Proj}_{\mathcal{G}}$. Let $\mathcal{G}$ be a monoid and $R = R_0 \oplus R_+$ a $\mathcal{G}$-graded ring with zero component R_0. Then we have the category $gr_{\mathcal{G}} R - mod$ of $\mathcal{G}$-graded R-modules and its full subcategory $gr_{\mathcal{G}} \mathcal{T}_{R_+} = \mathcal{T}_{R_+} \cap gr_{\mathcal{G}} R - mod$ whose objects are graded modules annihilated by the ideal R_+. We define the 'space' $\mathbf{Proj}_{\mathcal{G}}(R)$ by setting

$$C_{\mathbf{Proj}_{\mathcal{G}}(R)} = gr_{\mathcal{G}} R - mod/gr_{\mathcal{G}} \mathcal{T}^-_{R_+}.$$

Here $gr_{\mathcal{G}} \mathcal{T}^-_{R_+}$ is the Serre subcategory of the category $gr_{\mathcal{G}} R - mod$ spanned by $gr_{\mathcal{G}} \mathcal{T}_{R_+}$. One can show that $gr_{\mathcal{G}} \mathcal{T}^-_{R_+} = gr_{\mathcal{G}} R - mod \cap \mathcal{T}^-_{R_+}$. Therefore, we have a canonical projection

$$\mathbf{Cone}(R_+) \xrightarrow{\mathfrak{p}} \mathbf{Proj}_{\mathcal{G}}(R).$$

The localization functor $gr_{\mathcal{G}} R - mod \longrightarrow C_{\mathbf{Proj}_{\mathcal{G}}(R_+)}$ is an inverse image functor of a continuous morphism $\mathbf{Proj}_{\mathcal{G}}(R) \xrightarrow{\mathfrak{v}} \mathbf{Sp}_{\mathcal{G}}(R)$. The composition $\mathbf{Proj}_{\mathcal{G}}(R) \xrightarrow{\mathfrak{v}} \mathbf{Sp}(R_0)$ of the morphism $\mathfrak{v}$ with the canonical morphism $\mathbf{Sp}_{\mathcal{G}}(R) \xrightarrow{\phi} \mathbf{Sp}(R_0)$ defines $\mathbf{Proj}_{\mathcal{G}}(R)$ as a 'space' over $\mathbf{Sp}(R_0)$. Its direct image functor is called the *global sections functor*.

1.7.1. Example: cone and Proj of a $\mathbb{Z}_+$-graded ring. Let $R = \oplus_{n \geq 0} R_n$ be a $\mathbb{Z}_+$-graded ring, $R_+ = \oplus_{n \geq 1} R_n$ its 'irrelevant' ideal. Thus, we have the *cone* of R_+, $\mathbf{Cone}(R_+)$, and $\mathbf{Proj}(R) = \mathbf{Proj}_{\mathbb{Z}}(R)$, and a canonical morphism $\mathbf{Cone}(R_+) \longrightarrow \mathbf{Proj}(R)$.

2. Beck's Theorem and affine morphisms.

2.1. Beck's Theorem. Let $X \xrightarrow{f} Y$ be a continuous morphism in with inverse image functor f^*, direct image functor f_*, and adjunction morphisms

$$Id_{C_Y} \xrightarrow{\eta_f} f_* f^* \quad \text{and} \quad f^* f_* \xrightarrow{\epsilon_f} Id_{C_X}.$$

Let $\mathcal{F}_f$ denote the monad (F_f, μ_f) on Y, where $F_f = f_*f^*$ and $\mu_f = f_*\epsilon_f f^*$.

We denote by $\mathcal{F}_f - mod$, or by $(\mathcal{F}_f/Y) - mod$ the category of $\mathcal{F}_f$-modules. Its objects are pairs (M, ξ), where $M \in ObC_Y$ and ξ is a morphism $F_f(M) \longrightarrow M$ such that the diagram

$$\begin{array}{ccc} F_f^2(M) & \xrightarrow{\mu_f(M)} & F_f(M) \\ {\scriptstyle F_f(\xi)} \downarrow & & \downarrow {\scriptstyle \xi} \\ F_f(M) & \xrightarrow{\xi} & M \end{array}$$

commutes and $\xi \circ \eta_f(M) = id_M$. Morphisms from (M, ξ) to $(\widetilde{M}, \widetilde{\xi})$ are given by morphisms $M \xrightarrow{g} \widetilde{M}$ of the category C_Y such that the diagram

$$\begin{array}{ccc} F_f(M) & \xrightarrow{F_f(g)} & F_f(\widetilde{M}) \\ {\scriptstyle \xi} \downarrow & & \downarrow {\scriptstyle \widetilde{\xi}} \\ M & \xrightarrow{g} & \widetilde{M} \end{array}$$

commutes. The composition is defined in a standard way.

We denote by $\mathbf{Sp}(\mathcal{F}_f/Y)$ the 'space' represented by the category of $\mathcal{F}_f$-modules and call it the *categoric spectrum* of the monad $\mathcal{F}_f$.

There is a commutative diagram

$$\begin{array}{ccc} C_X & \xrightarrow{\bar{f}_*} & (\mathcal{F}_f/Y) - mod \\ & {\scriptstyle f_*} \searrow \quad \swarrow {\scriptstyle \mathfrak{f}_*} & \\ & C_Y & \end{array} \qquad (3)$$

Here $\bar{f}_*$ is the canonical functor

$$C_X \longrightarrow (\mathcal{F}_f/Y) - mod, \quad M \longmapsto (f_*(M), f_*\epsilon_f(M)),$$

and $\mathfrak{f}^*$ is the forgetful functor $(\mathcal{F}_f/Y) - mod \longrightarrow C_Y$.

The following assertion is one of the versions of Beck's theorem.

2.1.1. Theorem. *Let $X \xrightarrow{f} Y$ be a continuous morphism.*

(a) If the category C_Y has cokernels of reflexive pairs of arrows, then the functor $\bar{f}_*$ has a left adjoint, $\bar{f}^*$; hence $\bar{f}_*$ is a direct image functor of a continuous morphism $\bar{X} \xrightarrow{f} \mathbf{Sp}(\mathcal{F}_f/Y)$.

(b) If, in addition, the functor f_* preserves cokernels of reflexive pairs, then the adjunction arrow $\bar{f}^*\bar{f}_* \longrightarrow Id_{C_X}$ is an isomorphism, i.e. $\bar{f}_*$ is a localization.

(c) If, in addition to (a) and (b), the functor f_* is conservative, then $\bar{f}_*$ is a category equivalence.

Proof. See [MLM], IV.4.2, or [ML], VI.7. ■

2.1.2. Corollary. *Let $X \xrightarrow{f} Y$ be an affine morphism (cf. 1.3). If the category C_Y has cokernels of reflexive pairs of arrows (e.g. C_Y is an abelian category), then the canonical morphism $X \xrightarrow{\mathfrak{f}} \mathbf{Sp}(\mathcal{F}_f/Y)$ is an isomorphism.*

2.1.3. Monadic morphisms. A continuous morphism $X \xrightarrow{f} Y$ is called *monadic* if the functor

$$C_X \xrightarrow{\widetilde{f_*}} \mathcal{F}_f - mod, \quad M \longmapsto (f_*(M), f_*\epsilon_f(M)),$$

is an equivalence of categories.

2.2. Continuous monads and affine morphisms. A functor F is called *continuous* if it has a right adjoint. A monad $\mathcal{F} = (F, \mu)$ on a 'space' Y (i.e. on the category C_Y) is called *continuous* if the functor F is continuous.

2.2.1. Proposition. *A monad $\mathcal{F} = (F, \mu)$ on Y is continuous iff the canonical morphism $\mathbf{Sp}(\mathcal{F}/Y) \xrightarrow{\hat{f}} Y$ is affine.*

Proof. A proof in the case of a continuous monad can be found in [KR2, 6.2], or in [R3, 4.4.1] (see also [R4, 2.2]). ■

2.2.2. Corollary. *Suppose that the category C_Y has cokernels of reflexive pairs of arrows. A continuous morphism $X \xrightarrow{f} Y$ is affine iff its direct image functor $C_X \xrightarrow{f_*} C_Y$ is the composition of a category equivalence*

$$C_X \longrightarrow (\mathcal{F}_f/Y) - mod$$

for a continuous monad $\mathcal{F}_f$ on Y and the forgetful functor $(\mathcal{F}_f/Y) - mod \longrightarrow C_Y$. The monad $\mathcal{F}_f$ is determined by f uniquely up to isomorphism.

Proof. The conditions of the Beck's theorem are fulfilled if f is affine, hence f_* is the composition of an equivalence $C_X \longrightarrow (\mathcal{F}_f/Y) - mod$ for a monad $\mathcal{F}_f = (f_*f^*, \mu_f)$ in C_Y and the forgetful functor $(\mathcal{F}_f/Y) - mod \longrightarrow C_Y$ (see (1)). The functor $F_f = f_*f^*$ has a right adjoint $f_*f^!$, where $f^!$ is a right adjoint to f_*. The rest follows from 2.2.1. ■

2.3. The category of affine schemes over a 'space' and the category of monads on this 'space'.

2.3.1. Proposition. *Let*

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ f \searrow & & \swarrow g \\ & S & \end{array}$$

be a commutative diagram in $|Cat|^o$. *Suppose* C_Z *has cokernels of reflexive pairs of arrows. If* f *and* g *are affine, then* h *is affine.*

Let Aff_S denote the full subcategory of the category $|Cat|^o/S$ of 'spaces' over S whose objects are pairs $(X, X \xrightarrow{f} S)$, where f is an affine morphism. On the other hand, we have the category $\mathfrak{Mon}_\mathfrak{c}(S)$ of continuous monads on the 'space' S (i.e. on the category C_S) and the functor

$$\mathfrak{Mon}_\mathfrak{c}(S)^{op} \longrightarrow Aff_S \qquad (1)$$

which assigns to every continuous monad $\mathcal{F}$ the object $(\mathbf{Sp}(\mathcal{F}/S, \mathfrak{f})$, where $\mathbf{Sp}(\mathcal{F}/S)$ is the 'space' represented by the category $\mathcal{F}-mod$ and the morphism $\mathfrak{f}$ has the forgetful functor $\mathcal{F}-mod \longrightarrow C_S$ as a direct image functor. It follows from 2.3.1 and 2.2.2 that this functor is essentially full (that is its image is equivalent to the category Aff_S).

For every endofunctor $C_S \xrightarrow{G} C_S$, let $|G|$ denote the set $Hom(Id_{C_S}, G)$ of *elements* of G. If $\mathcal{F} = (F, \mu)$ is a monad, then the set of elements of F has a natural monoid structure; we denote this monoid by $|\mathcal{F}|$. And we denote by $|\mathcal{F}|^*$ the group of the invertible elements of the monoid $|\mathcal{F}|$. We say that two monad morphisms $\mathcal{F} \underset{\psi}{\overset{\phi}{\rightrightarrows}} \mathcal{G}$ are *conjugate* to each other of $\phi = t \cdot \psi \cdot t^{-1}$ for some $t \in |\mathcal{G}|^*$.

Let $\mathfrak{Mon}_\mathfrak{c}^\mathfrak{r}(S)$ denote the category whose objects are continuous monads on C_S and morphisms are *conjugacy classes* of morphisms of monads.

2.3.2. Proposition *The functor (1) induces an equivalence between the category* $\mathfrak{Mon}_\mathfrak{c}^\mathfrak{r}(S)$ *and the category* Aff_S *of affine schemes over* S.

2.3.3. Example. Let $S = \mathbf{Sp}(R)$ for an associative ring R. Then the category $\mathfrak{Mon}_\mathfrak{c}(S)$ of monads on $C_S = R - mod$ is naturally equivalent to the category $R\backslash Rings$ of associative rings over R. The conjugacy classes of monad morphisms correspond to conjugacy classes of ring morphisms. Let $\mathfrak{Ass}$ denote the category whose objects are associative rings and morphisms the conjugacy classes of ring morphisms.

One deduces from 2.3.2 the following assertion:

2.3.3.1. Proposition. *The category* Aff_S *of affine schemes over* $S = \mathbf{Sp}(R)$ *is naturally equivalent to the category* $(R\backslash\mathfrak{Ass})^{op}$.

3. Noncommutative weakly locally affine 'spaces' and schemes.

3.1. Weak covers. We call a family $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ of morphisms of 'spaces' a *weak cover* if

– all inverse image functors u_i^* are *exact* (i.e. the functors u_i^* preserve finite limits and colimits),

– the family $\{u_i^* \mid i \in J\}$ is *conservative* (i.e. if $u_i^*(s)$ is an isomorphism for all $i \in J$, then s is an isomorphism).

3.2. Weakly locally affine morphisms of 'spaces'. We call a morphism $X \xrightarrow{f} S$ of 'spaces' *locally affine* if there exists a cover $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ of the 'space' X such that all the compositions $f \circ u_i$ are affine.

3.2.1. Semi-separated covers and semi-separated locally affine 'spaces'. A cover $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ is called *semi-separated* if each of the morphisms u_i is affine.

A locally affine 'space' with a semi-separated affine cover is called *semi-separated.*

3.3. Weak schemes over S. *Weak schemes* over a 'space' S are locally affine morphisms $X \longrightarrow S$, which have an affine cover $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ formed by *localizatios.* The latter means that each inverse image functor u_i^* is the composition of the localization functor $C_X \longrightarrow \Sigma_{u_i^*}^{-1} C_X$, where $\Sigma_{u_i^*} = \{s \in HomC_X \mid u_i^*(s) \text{ is invertible}\}$, and an equivalence of categories $\Sigma_{u_i^*}^{-1} C_X \longrightarrow C_{U_i}$.

3.4. Schemes. A weak scheme $X \longrightarrow S$ with an affine cover $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ is a *scheme* if for every $i \in J$, the multiplicative system $\Sigma_{u_i^*}$ is finitely generated.

4. Descent: "covers", comonads, and gluing.

4.1. Comonads associated with "covers". Let $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ be a family of continuous morphisms and $\mathfrak{u}$ the corresponding morphism $\mathcal{U} = \coprod_{i \in J} U_i \xrightarrow{\mathfrak{u}} X$ with the inverse image functor

$$C_X \xrightarrow{\mathfrak{u}^*} \prod_{i \in J} C_{U_i} = C_{\mathcal{U}}, \quad M \longmapsto (u_i^*(M) | i \in J).$$

It follows that the family of inverse image functors $\{C_X \xrightarrow{u_i^*} C_{U_i} \mid i \in J\}$ is conservative iff the functor $\mathfrak{u}^*$ is conservative.

Suppose that the category C_X has products of $|J|$ objects. Then the morphism $\mathcal{U} = \coprod_{i \in J} U_i \xrightarrow{\mathfrak{u}} X$ is continuous: its direct image functor assigns to every object $(L_i | i \in J)$ of the category $C_{\mathcal{U}} = \prod_{i \in J} C_{U_i}$ the product $\prod_{i \in J} u_{i*}(L_i)$.

The adjunction morphism $Id_{C_X} \xrightarrow{\eta_{\mathfrak{u}}} \mathfrak{u}_*\mathfrak{u}^*$ assigns to each object M of C_X the morphism $M \longrightarrow \prod_{i\in J} u_{i*}u_i^*(M)$ determined by adjunction arrows $Id_{C_X} \xrightarrow{\eta_{u_i}} u_{i*}u_i^*$.

The adjunction morphism $\mathfrak{u}^*\mathfrak{u}_* \xrightarrow{\epsilon_{\mathfrak{u}}} Id_{C_{\mathcal{U}}}$ assigns to each object $\mathcal{L} = (L_i|i \in J)$ of $C_{\mathcal{U}}$ the morphism $(\epsilon_{\mathfrak{u},i}(\mathcal{L})|i \in J)$, where

$$u_i^*(\prod_{j\in J} u_{j*}(L_j)) \xrightarrow{\epsilon_{\mathfrak{u},i}(\mathcal{L})} L_i$$

is the composition of the image

$$u_i^*(\prod_{j\in J} u_{j*}(L_j)) \xrightarrow{u_i^*(p_i)} u_i^*u_{i*}(L_i)$$

of the image of the projection p_i and the adjunction arrow $u_i^*u_{i*}(L_i) \xrightarrow{\epsilon_{u_i}(L_i)} L_i$.

4.2. Beck's theorem and gluing. Suppose that for each $i \in J$, the category C_{U_i} has kernels of coreflexive pairs of arrows and the functor u_i^* preserves them. Then the inverse and direct image functors of the morphism $\mathfrak{u}$ satisfy the conditions of Beck's theorem, hence the category C_X is equivalent to the category of comodules over the comonad $\mathcal{G}_{\mathfrak{u}} = (G_{\mathfrak{u}}, \delta_{\mathfrak{u}}) = (\mathfrak{u}^*\mathfrak{u}_*, \mathfrak{u}^*\eta_{\mathfrak{u}}\mathfrak{u}_*)$ associated with the choice of inverse and direct image functors of $\mathfrak{u}$ together with an adjunction morphism $Id_{C_X} \xrightarrow{\eta_{\mathfrak{u}}} \mathfrak{u}_*\mathfrak{u}^*$.

Recall that $\mathcal{G}_{\mathfrak{u}}$-comodule is a pair $(\mathcal{L}, \zeta)$, where $\mathcal{L}$ is an object of $C_{\mathcal{U}}$ and ζ a morphism $\mathcal{L} \longrightarrow G_{\mathfrak{u}}(\mathcal{L})$ such that $\epsilon_{\mathfrak{u}}(\mathcal{L}) \circ \zeta = id_{\mathcal{L}}$ and $G_{\mathfrak{u}}(\zeta) \circ \zeta = \delta_{\mathfrak{u}}(\mathcal{L}) \circ \zeta$. Beck's theorem says that if the category $C_{\mathcal{U}}$ has kernels of coreflexive pairs of arrows and the functor $\mathfrak{u}^*$ preserves and reflects them, then the functor $C_X \xrightarrow{\widetilde{\mathfrak{u}^*}} (\mathcal{U}\backslash\mathcal{G}_{\mathfrak{u}}) - comod$ which assigns to each object M of C_X the $\mathcal{G}_{\mathfrak{u}}$-comodule $(\mathfrak{u}^*(M), \delta_{\mathfrak{u}}(M))$ is an equivalence of categories.

In terms of our local data – the "cover" $\{U_i \xrightarrow{u_i} X \mid i \in J\}$, a $\mathcal{G}_{\mathfrak{u}}$-comodule $(\mathcal{L}, \zeta)$ is the data $(L_i, \zeta_i|i \in J)$, where $(L_i|i \in J) = \mathcal{L}$ and ζ_i is a morphism

$$L_i \longrightarrow u_i^*\mathfrak{u}_*(\mathcal{L}) = u_i^*(\prod_{j\in J} u_{j*}(L_j))$$

which equalizes the pair of arrows

$$u_i^*\mathfrak{u}_*(\mathcal{L}) = u_i^*(\prod_j u_{j*}(L_j)) \underset{u_i^*(u_{j*}\zeta_j)}{\overset{u_i^*\eta_{\mathfrak{u}}\mathfrak{u}_*(\mathcal{L})}{\rightrightarrows}} u_i^*(\prod_m u_{m*}u_m^*(\prod_j u_{j*}(L_j))) = u_i^*\mathfrak{u}_*\mathfrak{u}^*\mathfrak{u}_*(\mathcal{L})$$

and such that $\epsilon_{\mathfrak{u},i}(\mathcal{L}) \circ \zeta_i = id_{L_i}$, $i \in J$.

The exactness of the diagram

$$\mathcal{L} \xrightarrow{\ \zeta\ } G_{\mathfrak{u}}(\mathcal{L}) \overset{\delta_{\mathfrak{u}}(\mathcal{L})}{\underset{G_{\mathfrak{u}}(\zeta)}{\rightrightarrows}} G^2_{\mathfrak{u}}(\mathcal{L})$$

is equivalent to the exactness of the diagram

$$L_i \xrightarrow{\ \zeta_i\ } u_i^*(\prod_{j\in J} u_{j*}(L_j)) \overset{u_i^*\eta_{\mathfrak{u}}\mathfrak{u}_*(\mathcal{L})}{\underset{u_i^*(u_{j*}\zeta_j)}{\rightrightarrows}} u_i^*(\prod_{m\in J} u_{m*}u_m^*(\prod_{j\in J} u_{j*}(L_j))) \tag{1}$$

for every $i \in J$. If the functors u_k^* preserve products of J objects (or just the products involved into (1)), then the diagram (1) is isomorphic to the diagram

$$L_i \xrightarrow{\ \zeta_i\ } \prod_{j\in J} u_i^* u_{j*}(L_j) \overset{u_i^*\eta_{\mathfrak{u}}\mathfrak{u}_*(\mathcal{L})}{\underset{u_i^*(u_{j*}\zeta_j)}{\rightrightarrows}} \prod_{j,m\in J} u_i^* u_{m*} u_m^* u_{j*}(L_j) \tag{2}$$

4.3. Remark. The exactness of the diagram (1) might be viewed as a sort of sheaf property. This interpretation looks more plausible (or less stretched) when the diagram (1) is isomorphic to the diagram (2), because $u_i^*u_{j*}(L_j)$ can be regarded as the section of L_j over the 'intersection' of U_i and U_j and $u_i^*u_{m*}u_m^*u_{j*}(L_j)$ as the section of L_j over the intersection of the elements U_j, U_m, and U_i of the "cover".

4.4. The condition of the continuity of the comonad associated with a "cover". Suppose that each direct image functor $C_{U_i} \xrightarrow{u_{i*}} C_X$, $i \in J$, has a right adjoint, $u_i^!$; and let $\mathfrak{u}^!$ denote the functor $C_X \longrightarrow C_{\mathcal{U}} = \prod_{i\in J} C_{U_i}$ which maps every object M to $(u_i^!(M)|i \in J)$. If the category C_X has coproducts of $|J|$ objects, then the functor $\mathfrak{u}^!$ has a left adjoint, which maps every object $(L_i|i \in J)$ of $C_{\mathcal{U}}$ to the coproduct $\coprod_{i\in J} u_{i*}(L_i)$.

Therefore, if the canonical morphism $\coprod_{i\in J} u_{i*}(L_i) \longrightarrow \prod_{i\in J} u_{i*}(L_i)$ is an isomorphism for every object $(L_i|i \in J)$ of the category $C_{\mathcal{U}}$, then (and only then) the functor $\mathfrak{u}^!$ is a right adjoint to the functor $\mathfrak{u}_*$.

In particular, $\mathfrak{u}^!$ is a right adjoint to $\mathfrak{u}_*$, if the category C_X is additive and J is finite.

4.5. Note. If, in addition, the functors u_{i*} are conservative for all $i \in J$, then the functor $\mathfrak{u}_*$ is conservative, and the category $C_{\mathcal{U}}$ is equivalent to the category of modules over the continuous monad $\mathcal{F}_{\mathfrak{u}} = (F_{\mathfrak{u}}, \mu_{\mathfrak{h}})$, where $F_{\mathfrak{u}} = \mathfrak{u}_*\mathfrak{u}^*$ and $\mu_{\mathfrak{u}} = \mathfrak{u}_*\epsilon_{\mathfrak{u}}\mathfrak{u}^*$ for an adjunction morphism $\mathfrak{u}^*\mathfrak{u}_* \xrightarrow{\epsilon_{\mathfrak{u}}} Id_{C_{\mathcal{U}}}$.

5. Some motivating examples.

5.1. The base affine 'space' and the flag variety of a reductive Lie algebra from the point of view of noncommutative algebraic geometry. Let $\mathfrak{g}$ be a reductive Lie algebra over $\mathbb{C}$ and $U(\mathfrak{g})$ the enveloping algebra of $\mathfrak{g}$. Let $\mathcal{G}$ be the group of integral weights of $\mathfrak{g}$ and $\mathcal{G}_+$ the semigroup of nonnegative integral weights. Let $R = \bigoplus_{\lambda\in\mathcal{G}_+} R_\lambda$, where R_λ is the vector space of the (canonical) irreducible finite dimensional representation with the highest weight λ. The module R is a $\mathcal{G}_+$-graded algebra with the multiplication determined by the projections $R_\lambda \otimes R_\nu \longrightarrow R_{\lambda+\nu}$, for all $\lambda, \nu \in \mathcal{G}_+$. By construction, the algebra R carries a $\mathfrak{g}$-module structure such that the multiplication is a $\mathfrak{g}$-module morphism. It is well known that the algebra R is isomorphic (both as an algebra and a $\mathfrak{g}$-module) to the algebra of regular functions on the *base affine space* of $\mathfrak{g}$. Recall that the *base affine* space of $\mathfrak{g}$ (which is, actually, not affine, but, a quasi-affine variety) is, by definition, the quotient G/U, where G is a connected simply connected algebraic group with the Lie algebra $\mathfrak{g}$, and U is its maximal unipotent subgroup.

The category $C_{\mathbf{Cone}(R)}$ is equivalent to the category of quasi-coherent sheaves on the base affine space Y of the Lie algebra $\mathfrak{g}$. The category $C_{\mathbf{Proj}_{\mathcal{G}}(R)}$ is equivalent to the category of quasi-coherent sheaves on the flag variety of $\mathfrak{g}$.

5.2. The quantized base affine 'space' and quantized flag variety of a semisimple Lie algebra. Let now $\mathfrak{g}$ be a semisimple Lie algebra over a field k of zero characteristic, and let $U_q(\mathfrak{g})$ be the quantized enveloping algebra of $\mathfrak{g}$. Define the $\mathcal{G}$-graded algebra $R = \bigoplus_{\lambda\in\mathcal{G}_+} R_\lambda$ the same way as above. This time, however, the algebra R is not commutative. Following the classical example (and identifying spaces with categories of quasi-coherent sheaves on them), we call $\mathbf{Cone}(R)$ *the quantum base affine 'space'* and $\mathbf{Proj}_{\mathcal{G}}(R)$ the *quantum flag variety* of the Lie algebra $\mathfrak{g}$.

5.2.1. Canonical affine covers of the base affine 'space' and the flag variety. Let W be the Weyl group of the Lie algebra $\mathfrak{g}$. Fix a $w \in W$. For any $\lambda \in \mathcal{G}_+$, choose a nonzero w-extremal vector $e^\lambda_{w\lambda}$ generating the one dimensional vector subspace of R_λ formed by the vectors of the weight $w\lambda$. Set $S_w = \{k^*e^\lambda_{w\lambda} | \lambda \in \mathcal{G}_+\}$. It follows from the Weyl character formula that $e^\lambda_{w\lambda}e^\mu_{w\mu} \in k^*e^{\lambda+\mu}_{w(\lambda+\mu)}$. Hence S_w is a multiplicative set. It was proved by Joseph [Jo] that S_w is a left and right Ore subset in R. The Ore sets $\{S_w | w \in W\}$ determine a conservative family of affine localizations

$$\mathbf{Sp}(S_w^{-1}R) \xrightarrow{\widetilde{\mathfrak{u}}_w} \mathbf{Cone}(R), \quad w \in W, \tag{4}$$

of the quantum base affine 'space' and a conservative family of affine localizations

$$\mathbf{Sp}_{\mathcal{G}}(S_w^{-1}R) \longrightarrow \mathbf{Proj}_{\mathcal{G}}(R), \quad w \in W, \tag{5}$$

of the quantum flag variety. We claim that the category $gr_{\mathcal{G}} S_w^{-1}R - mod$ of $\mathcal{G}$-graded $S_w^{-1}R$-modules is naturally equivalent to the category $(S_w^{-1}R)_0 - mod$.

In fact, by 1.5, it suffices to verify that the canonical functor

$$gr_{\mathcal{G}} S_w^{-1}R - mod \longrightarrow (S_w^{-1}R)_0 - mod$$

which assigns to every graded $S_w^{-1}R$-module its zero component is faithful; i.e. the zero component of every nonzero $\mathcal{G}$-graded $S_w^{-1}R$-module is nonzero. This is, really, the case, because if z is a nonzero element of the λ-component of a $\mathcal{G}$-graded $S_w^{-1}R$-module, then $(e_{w\lambda}^{\lambda})^{-1}z$ is a nonzero element of the zero component of this module.

Thus, we obtain an affine cover

$$\mathbf{Sp}((S_w^{-1}R)_0) \xrightarrow{u_w} \mathbf{Proj}_{\mathcal{G}}(R), \quad w \in W, \tag{6}$$

of the quantum flag variety $\mathbf{Proj}_{\mathcal{G}}(R)$ of the Lie algebra $\mathfrak{g}$.

The covers (4) and (6) are scheme structures on respectively quantum base affine 'space' and quantum flat variety. One can check that all morphisms of (4) and (6) are affine, i.e. the covers (4) and (5) are semi-separated.

5.3. Noncommutative Grassmannians. Fix an associative unital k-algebra R. Let $R\backslash Alg_k$ be the category of associative *k-algebras over R* (i.e. pairs $(S, R \to S)$, where S is a k-algebra and $R \to S$ a k-algebra morphism). We call them for convenience R-rings. We denote by R^e the k-algebra $R \otimes_k R^o$. Here R^o is the algebra opposite to R.

5.3.1. The functor $Gr_{M,V}$. Let M, V be left R-modules. Consider the functor

$$R\backslash Alg_k \xrightarrow{Gr_{M,V}} \mathbf{Sets},$$

which assigns to any R-ring $(S, R \xrightarrow{s} S)$ the set of isomorphism classes of epimorphisms $s^*(M) \longrightarrow s^*(V)$ (here $s^*(M) = S \otimes_R M$) and to any R-ring morphism

$$(S, R \xrightarrow{s} S) \xrightarrow{\phi} (T, R \xrightarrow{t} T)$$

the map $Gr_{M,V}(S, s) \xrightarrow{Gr_{M,V}(\phi)} Gr_{M,V}(T, t)$ induced by the inverse image functor

$$S - mod \xrightarrow{\phi^*} T - mod, \quad \mathcal{N} \longmapsto T \otimes_S \mathcal{N}.$$

5.3.2. The functor $G_{M,V}$. Denote by $G_{M,V}$ the functor $R\backslash Alg_k \longrightarrow$ **Sets**, which assigns to any R-ring $(S, R \stackrel{s}{\to} S)$ the set of pairs of morphisms $s^*(V) \stackrel{v}{\to} s^*(M) \stackrel{u}{\to} s^*(V)$ such that $u \circ v = id_{s^*(V)}$ and acts naturally on morphisms of R-rings. Suppose that V is a projective R-module. Then $s^*(V)$ is a projective S-module for any R-ring $(S, R \stackrel{s}{\to} S)$, so that the map

$$\pi_{M,V}(S,s) : G_{M,V}(S,s) \longrightarrow Gr_{M,V}(S,s), \quad (v,u) \longmapsto [u],$$

is surjective. Thus, we have is a functor epimorphism.

$$G_{M,V} \xrightarrow{\pi_{M,V}} Gr_{M,V}. \tag{1}$$

5.3.3. Relations. Denote by $\mathfrak{R}_{M,V}$ the "functor of relations" $G_{M,V} \times_{Gr_{M,V}} G_{M,V}$. By definition, $\mathfrak{R}_{M,V}$ is a subfunctor of $G_{M,V} \times G_{M,V}$, which assigns to each R-ring, $(S, R \stackrel{s}{\to} S)$, the set of all 4-tuples $(u_1, v_1; u_2, v_2) \in G_{M,V} \times G_{M,V}$ such that the epimorphisms u_1, u_2 are equivalent. The latter means that there exists an isomorphism $s^*(V) \xrightarrow{\varphi} s^*(V)$ such that $u_2 = \varphi \circ u_1$, or, equivalently, $\varphi^{-1} \circ u_2 = u_1$. Since $u_i \circ v_i = id$, $i = 1, 2$, these equalities imply that $\varphi = u_2 \circ v_1$ and $\varphi^{-1} = u_1 \circ v_2$. Thus, $\mathfrak{R}_{M,V}(S,s)$ is a subset of all $(u_1, v_1; u_2, v_2) \in G_{M,V}(S,s) \times G_{M,V}(S,s)$ satisfying the following relations:

$$u_2 = (u_2 \circ v_1) \circ u_1, \quad u_1 = (u_1 \circ v_2) \circ u_2 \tag{2}$$

in addition to the relations describing $G_{M,V}(S,s) \times G_{M,V}(S,s)$:

$$u_1 \circ v_1 = id_{S \otimes_R V} = u_2 \circ v_2 \tag{3}$$

Denote by p_1, p_2 the canonical projections $\mathfrak{R}_{M,V} \rightrightarrows G_{M,V}$. It follows from the surjectivity of $G_{M,V} \longrightarrow Gr_{M,V}$ that the diagram

$$\mathfrak{R}_{M,V} \underset{p_2}{\overset{p_1}{\rightrightarrows}} G_{M,V} \xrightarrow{\pi_{M,V}} Gr_{M,V} \tag{4}$$

is exact.

5.3.4. Proposition. *If both M and V are projective modules of a finite type, then the functors $G_{M,V}$ and $\mathfrak{R}_{M,V}$ are corepresentable.*

Proof. See [KR2, 10.4.3]. ■

5.3.5. Quasi-coherent presheaves on presheaves of sets. Consider the category $\mathbf{Aff}_k$ of *affine k-schemes* which we identify with the category of representable functors on the category Alg_k of k-algebras, and the fibred category with the base $\mathbf{Aff}_k$ whose fibers are categories of left modules over corresponding algebras. Let X be a presheaf of sets on $\mathbf{Aff}_k$. Then we have a fibred category $\widetilde{\mathbf{Aff}}_k/X$ with the base $\mathbf{Aff}_k/X$ induced by the forgetful functor $\mathbf{Aff}_k/X \longrightarrow \mathbf{Aff}_k$. The category $Qcoh(X)$ of *quasi-coherent presheaves* on X is the opposite to the category of *cartesian sections* of $\widetilde{\mathbf{Aff}}_k/X$.

5.3.6. Quasi-coherent presheaves on $Gr_{M,V}$. Suppose that M and V are projective modules of a finite type, hence the functors $G_{M,V}$ and $\mathfrak{R}_{M,V}$ are corepresentable by R-rings resp. $(\mathfrak{G}_{M,V}, R \to \mathfrak{G}_{M,V})$ and $(\mathcal{R}_{M,V}, R \to \mathcal{R}_{M,V})$. Then the category $Qcoh(G_{M,V})$ (resp. $Qcoh(\mathfrak{R}_{M,V})$) is equivalent to $\mathfrak{G}_{M,V} - mod$ (resp. $\mathcal{R}_{M,V} - mod$), and the category $Qcoh(Gr_{M,V})$ of quasi-coherent presheaves on $Gr_{M,V}$ is equivalent to the kernel of the diagram

$$Qcoh(G_{M,V}) \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} Qcoh(\mathfrak{R}_{M,V}) \tag{5}$$

This means that, after identifying categories of quasi-coherent presheaves in (5) with corresponding categories of modules, quasi-coherent presheaves on $Gr_{M,V}$ can be realized as pairs (L, ϕ), where L is a $\mathfrak{G}_{M,V}$-module and ϕ is an isomorphism $p_1^*(L) \xrightarrow{\sim} p_2^*(L)$. Morphisms $(L, \phi) \longrightarrow (N, \psi)$ are given by morphisms $L \xrightarrow{g} N$ such that the diagram

$$\begin{array}{ccc} p_1^*(L) & \xrightarrow{p_1^*(g)} & p_1^*(N) \\ \phi \big\downarrow \wr & & \wr \big\downarrow \psi \\ p_2^*(L) & \xrightarrow{p_2^*(g)} & p_2^*(N) \end{array}$$

commutes. The functor

$$Qcoh(Gr_{M,V}) \xrightarrow{\pi^*_{M,V}} Qcoh(G_{M,V}), \quad (L, \phi) \longmapsto L,$$

is an inverse image functor of the projection $G_{M,V} \xrightarrow{\pi_{M,V}} Gr_{M,V}$ (see 5.3.3(4)).

One can show that the functor $\pi^*_{M,V}$ is an inverse image functor of a faithfully flat affine morphism $\bar{\pi}_{M,V}$ from an affine 'space' $\mathbf{Sp}(\mathcal{G}_{\mathcal{M},\mathcal{V}})$ (where $\mathcal{G}_{\mathcal{M},\mathcal{V}}$ is a ring representing the functor $G_{\mathcal{M},\mathcal{V}}$) to the 'space' $\mathfrak{Grass}_{\mathcal{M},\mathcal{V}}$ represented by the category $Qcoh(Gr_{M,V})$ of quasi-coherent sheaves on $Gr_{M,V}$. In our terminology, this means that $\bar{\pi}_{M,V}$ is an affine semi-separated cover of $\mathfrak{Grass}_{\mathcal{M},\mathcal{V}}$.

5.3.7. Quasi-coherent sheaves on presheaves of sets. Let X be a presheaf of sets on $\mathbf{Aff}_k$ Given a *(pre)topology* τ on $\mathbf{Aff}_k/X$, we define the subcategory $Qcoh(X,\tau)$ of *quasi-coherent sheaves* on (X,τ) [KR4].

5.3.7.1. Theorem ([KR4]). *(a) A topology τ on $\mathbf{Aff}_k$ is subcanonical (i.e. all representable presheaves are sheaves) iff $Qcoh(X) = Qcoh(X,\tau)$ for every presheaf of sets X on $\mathbf{Aff}_k$ (in other words, 'descent' topologies on $\mathbf{Aff}_k$ are precisely subcanonical topologies). In this case, $Qcoh(X) = Qcoh(X,\tau) \hookrightarrow Qcoh(X^\tau) = Qcoh(X^\tau,\tau)$, where X^τ is the sheaf associated to X and $\hookrightarrow$ is a natural full embedding.*

(b) If τ is a topology of effective descent [KR4] (e.g. the **fpqc** *or smooth topology [KR2]), then the categories $Qcoh(X,\tau)$ and $Qcoh(X^\tau)$ are naturally equivalent.*

This theorem says, roughly speaking, that the category $Qcoh(X)$ of quasi-coherent presheaves knows, which topologies to choose. A topology that seems to be the most plausible for Grassmannians, in particular, for $N\mathbb{P}_k^n$, is the *smooth* topology introduced in [KR2]. It is of effective descent, and the category of quasi-coherent sheaves on $N\mathbb{P}_k^n$ defined in [KR1] is naturally equivalent to the category of quasi-coherent sheaves of the projective space defined via smooth topology on $\mathbf{Aff}_k$.

Chapter II
The Spectra of 'Spaces' Represented by Abelian Categories.

Spectral theory of abelian categories was started by P. Gabriel in early sixties [Gab] with the introduction of the spectrum of a locally noetherian Grothendieck category. The elements of the Gabriel's spectrum are isomorphism classes of indecomposable injectives. If R is a commutative noetherian ring, then the Gabriel's spectrum of the category of R-modules is naturally isomorphic to the prime spectrum of the ring. More generally, the Gabriel's spectrum of the category of quasi-coherent sheaves on a noetherian scheme is isomorphic to the underlying space of the scheme [Gab, Ch. VI, Theorem 1]. The Gabriel's spectrum does not recover, in general, the prime spectrum of a non-noetherian commutative ring, which prevented the extension of this remarkable theorem to non-noetherian schemes. The central character of this chapter is the spectrum $\mathbf{Spec}(-)$, which possesses the desired property: if C_X is the category of modules over an arbitrary commutative unital ring R, then $\mathbf{Spec}(X)$ is naturally isomorphic to the prime spectrum of R. For an arbitrary abelian category C_X, isomorphism classes of simple objects of C_X correspond to closed points of $\mathbf{Spec}(X)$. The main purpose is establishing local properties of the spectrum $\mathbf{Spec}(X)$, which are needed to study the underlying topological spaces of non-affine noncommutative schemes, and are crucial for reconstruction problems. These local properties are also used in computations of the spectra and applications of noncommutative local algebra and algebraic geometry to representation theory (see Ch. III).

Section 1 contains the necessary preliminaries on topologizing, thick, and Serre subcategories. In Section 2, the spectrum $\mathbf{Spec}(X)$ is introduced. In Section 3 is dedicated to local 'spaces', the spectrum $\mathbf{Spec}^-(X)$ (whose points are Serre subcategories such that the quotient 'space' is local) and the counterpart $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$ of the spectrum $\mathbf{Spec}(X)$ – the image of a natural embedding of $\mathbf{Spec}(X)$ into $\mathbf{Spec}^-(X)$. Section 4 is dedicated to the pretopology of Serre localizations and the resulting local property of the spectrum $\mathbf{Spec}^-(X)$. In Section 5, we discuss analogous facts for the pretopology of exact localizations and the related spectrum. In Section 6, we discuss shortly spectra related with localizations of abelian categories which are used in Section 7 in formulation of its main result: the local property of the spectrum $\mathbf{Spec}(X)$ in terms of its counterpart $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$.

In Section 8, we introduce the *geometric center* of a 'space' related to the spectrum $\mathbf{Spec}(X)$ and use the results of Section 7 to show that if C_X is the category of quasi-coherent sheaves on a quasi-compact quasi-separated scheme, then the geometric center of X is isomorphic to the scheme. Section 9 is dedicated to the spectra related to the preorder of reflective topologizing categories and their local properties. We introduce a pair spectra – $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ and $\mathbf{Spec}^1_{\mathfrak{c}}(X)$, together with a canonical map between them, which turns out to be an isomorphism. For an arbitrary abelian category C_X, there is a natural embedding $\mathbf{Spec}(X) \hookrightarrow \mathbf{Spec}^0_{\mathfrak{c}}(X)$. If C_X has enough objects of finite type (for instance, C_X is the category of quasi-coherent sheaves on a quasi-compact (noncommutative) scheme), then $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ and $\mathbf{Spec}(X)$ coincide. The main fact of the section is the local property of the spectrum $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ (or, ruther, of its counterpart $\mathbf{Spec}^1_{\mathfrak{c}}(X)$) with respect to infinite covers, which we formulate and prove in 9.6. If C_X is the category of quasi-coherent sheaves on a scheme $\mathbf{X}$, then $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ endowed with the Zariski topology (which is defined in terms of topologizing subcategories) is naturally isomorphic to the underlying topological space of the scheme $\mathbf{X}$, under the condition that $\mathbf{X}$ admits an affine cover $\{U_i \hookrightarrow \mathbf{X} \mid i \in J\}$ such that each immersion $U_i \hookrightarrow \mathbf{X}$ has a direct image functor. The latter condition holds if the scheme is quasi-separated.

In Section C1, we relate topologies on spectra with some natural 'topological' structures on the monoid of topologizing subcategories. Besides, this appendix contains facts (mostly borrowed from [R4]), which are used in the main body of the paper, especially in Sections 9 and 9. Section C2 contains some observations on supports of objects and the Krull filtrations. In Section C3, we apply the results of Section 9 to compare closed points of the spectrum $\mathbf{Spec}^-(X)$ and $\mathbf{Spec}(X)$. Closed points of $\mathbf{Spec}(X)$ play a special role due to their significance for representation theory and algebraic geometry. The spectrum is usually easier to compute than $\mathbf{Spec}(X)$ due to its better functorial properties. We show that, although $\mathbf{Spec}^-(X)$ is, usually, considerably larger than $\mathbf{Spec}(X)$, their closed points are in natural bijective correspondence in many (if not all) cases of interest.

1. Topologizing, thick, and Serre subcategories.

1.1. Topologizing subcategories. A full subcategory $\mathbb{T}$ of an abelian category C_X is called *topologizing* if it is closed under finite coproducts and subquotients.

A subcategory $\mathbb{S}$ of C_X is called *coreflective* if the inclusion functor $\mathbb{S} \hookrightarrow C_X$ has a right adjoint; that is every object of C_X has a biggest subobject, which belongs to $\mathbb{S}$.

We denote by $\mathfrak{T}(X)$ the preorder (with respect to $\subseteq$) of topologizing subcategories and by $\mathfrak{T}_{\mathfrak{c}}(X)$ the preorder of coreflective topologizing subcategories of C_X.

1.1.1. The Gabriel product and infinitesimal neighborhoods of topologizing categories. The *Gabriel product*, $\mathbb{S} \bullet \mathbb{T}$, of the pair of subcategories $\mathbb{S}$, $\mathbb{T}$ of C_X is the full subcategory of C_X spanned by all objects M such that there exists an exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

with $L \in Ob\mathbb{T}$ and $N \in Ob\mathbb{S}$. It follows that $0 \bullet \mathbb{T} = \mathbb{T} = \mathbb{T} \bullet 0$ for any strictly full subcategory $\mathbb{T}$. The Gabriel product of two topologizing subcategories is a topologizing subcategory, and its restriction to topologizing categories is associative; i.e. $(\mathfrak{T}(X), \bullet)$ is a monoid. Similarly, the Gabriel product of coreflective topologizing subcategories is a coreflective topologizing subcategory, hence $\mathfrak{T}_{\mathfrak{c}}(X)$ is a submonoid of $(\mathfrak{T}(X), \bullet)$.

The n^{th} *infinitesimal neighborhood*, $\mathbb{T}^{(n+1)}$, of a subcategory $\mathbb{T}$ is defined by $\mathbb{T}^{(0)} = 0$ and $\mathbb{T}^{(n+1)} = \mathbb{T}^{(n)} \bullet \mathbb{T}$ for $n \geq 0$.

1.2. The preorder $\succ$ and topologizing subcategories. For any two objects, M and N, of an abelian category C_X, we write $M \succ N$ if N is a subquotient of a finite coproduct of copies of M. For any object M of the category C_X, we denote by $[M]$ the full subcategory of C_X whose objects are all $L \in ObC_X$ such that $M \succ L$. It follows that $M \succ N \Leftrightarrow [N] \subseteq [M]$. In particular, M and N are equivalent with respect to $\succ$ (i.e. $M \succ N \succ M$) iff $[M] = [N]$. Thus, the preorder $\big(\{[M] \mid M \in ObC_X\}, \supseteq\big)$ is a canonical realization of the quotient of $(ObC_X, \succ)$ by the equivalence relation associated with $\succ$.

1.2.1. Lemma. *(a) For any object M of C_X, the subcategory $[M]$ is the smallest topologizing subcategory containing M.*

(b) The smallest topologizing subcategory spanned by a family of objects $\mathcal{S}$ coincides with $\bigcup_{N \in \mathcal{S}_\Sigma} [N]$, where $\mathcal{S}_\Sigma$ denotes the family of all finite coproducts of objects of $\mathcal{S}$.

Proof. (a) Since $\succ$ is a transitive relation, the subcategory $[M]$ is closed with respect to taking subquotients. If $M \succ M_i$, $i = 1, 2$, then $M \succ M \oplus M \succ M_1 \oplus M_2$, which shows that $[M]$ is closed under finite coproducts, hence it is topologizing. Clearly, any topologizing subcategory containing M contains the subcategory $[M]$.

(b) The union $\bigcup_{N \in \mathcal{S}_\Sigma} [N]$ is contained in every topologizing subcategory containing the family $\mathcal{S}$. It is closed under taking subquotients, because each $[N]$ has this property. It is closed under finite coproducts, because if $N_1, N_2 \in \mathcal{S}_\Sigma$ and $N_i \succ M_i$, $i = 1, 2$, then $N_1 \oplus N_2 \succ M_1 \oplus M_2$. ■

For any subcategory (or a class of objects) $\mathcal{S}$, we denote by $[\mathcal{S}]$ (resp. by $[\mathcal{S}]_{\mathfrak{c}}$) the smallest topologizing resp. coreflective topologizing) subcategory containing $\mathcal{S}$.

1.2.2. Proposition. *Suppose that C_X is an abelian category with small coproducts. Then a topologizing subcategory of the category C_X is coreflective iff it is closed under small coproducts. The smallest coreflective topologizing subcategory containing a set of objects $\mathcal{S}$ coincides with $\bigcup_{N \in \widetilde{\mathcal{S}}} [N]$, where $\widetilde{\mathcal{S}}$ is the family of all small coproducts of objects of $\mathcal{S}$.*

Suppose that the category C_X satisfies (AB4), i.e. it has infinite coproducts and the coproduct of a set of monomorphisms is a monomorphism. Then, for any object M of C_X, the smallest coreflective topologizing subcategory $[M]_{\mathfrak{c}}$ spanned by M is generated by subquotients of coproducts of sets of copies of M.

Proof. The argument is similar to that of 1.2.1 and left to the reader as an exercise. ∎

1.3. Thick subcategories. A topologizing subcategory $\mathbb{T}$ of the category C_X is called *thick* if $\mathbb{T} \bullet \mathbb{T} = \mathbb{T}$; in other words, $\mathbb{T}$ is thick iff it is closed under extensions.

We denote by $\mathfrak{Th}(X)$ the preorder of thick subcategories of C_X. For a thick subcategory $\mathcal{T}$ of C_X, we denote by $X/\mathcal{T}$ the *quotient 'space'* defined by $C_{X/\mathcal{T}} = C_X/\mathcal{T}$.

1.4. Serre subcategories. We recall the notion of a Serre subcategory of an abelian category as it is defined in [R, III.2.3.2]. For a subcategory $\mathbb{T}$ of C_X, let $\mathbb{T}^-$ denote the full subcategory of C_X generated by all objects L of C_X such that any nonzero subquotient of L has a nonzero subobject which belongs to $\mathbb{T}$.

1.4.1. Proposition. *Let $\mathbb{T}$ be a subcategory of C_X. Then*

(a) The subcategory $\mathbb{T}^-$ is thick.

(b) $(\mathbb{T}^-)^- = \mathbb{T}^-$.

(c) $\mathbb{T} \subseteq \mathbb{T}^-$ iff any subquotient of an object of $\mathbb{T}$ is isomorphic to an object of $\mathbb{T}$.

Proof. See [R, III.2.3.2.1]. ∎

1.4.2. Remark. It follows from 1.4.1 and the (definition of $\mathbb{T}^-$) that, for any subcategory $\mathbb{T}$ of an abelian category C_X, the associated Serre subcategory $\mathbb{T}^-$ is the largest topologizing (or the largest thick) subcategory of C_X such that every its nonzero object has a nonzero subobject from $\mathbb{T}$.

1.4.3. Definition. A subcategory $\mathbb{T}$ of an abelian category C_X is called a *Serre subcategory* if $\mathbb{T}^- = \mathbb{T}$. We denote by $\mathfrak{Se}(X)$ the preorder (with respect to $\subseteq$) of all Serre subcategories of C_X.

The following characterization of Serre subcategories turns to be quite useful.

1.4.4. Proposition. *Let $\mathbb{T}$ be a subcategory of an abelian category C_X closed under taking subquotients. The following conditions are equivalent:*

(a) $\mathbb{T}$ is a Serre subcategory.

(b) If $\mathbb{S}$ is a subcategory of the category C_X, which is closed under subquotients and is not contained in $\mathbb{T}$, then $\mathbb{S} \bigcap \mathbb{T}^\perp \neq 0$.

Proof. Recall that $\mathbb{T}^\perp$ is the full subcategory of the category C_X generated by all objects L of C_X such that $C_X(L, M) = 0$ for all $M \in Ob\mathbb{T}$.

$(a) \Rightarrow (b)$. Let $\mathbb{T}$ be a subcategory of C_X closed under taking quotients. By the definition of $\mathbb{T}^-$, an object M does not belong to $\mathbb{T}^-$ iff it has a nonzero subquotient, L, which does not have a nonzero subobject from $\mathbb{T}$. Since $\mathbb{T}$ is closed under taking quotients, the latter means precisely that $Hom(N, L) = 0$ for every $N \in Ob\mathbb{T}$, i.e. $L \in Ob\mathbb{T}^\perp$. Thus, M does not belong to $\mathbb{T}^-$ iff it has a nonzero subquotient which belongs to $\mathbb{T}^\perp$.

$(b) \Rightarrow (a)$. By the condition (b), if an object M does not belong to $\mathbb{T}$, then it has a nonzero subquotient, which belongs to $\mathbb{T}^{\perp}$. But, by the observation above, this means that the object M does not belong to $\mathbb{T}^{-}$. So that $\mathbb{T}^{-} \subseteq \mathbb{T}$. The inverse inclusion holds, because $\mathbb{T}$ is closed under taking subquotients (see 1.4.1(c)). ■

1.4.5. The property (sup). Recall that X (or the corresponding category C_X) has the property (sup) if for any ascending chain, Ω, of subobjects of an object M, the supremum of Ω exists, and for any subobject L of M, the natural morphism

$$sup(N \cap L \mid N \in \Omega) \longrightarrow (sup\Omega) \cap L$$

is an isomorphism.

1.4.6. Coreflective thick subcategories and Serre subcategories. Recall that a full subcategory $\mathcal{T}$ of a category C_X is called *coreflective* if the inclusion functor $\mathcal{T} \hookrightarrow C_X$ has a right adjoint. In other words, each object of C_X has the largest subobject, which belongs to $\mathcal{T}$.

1.4.6.1. Lemma. *Any coreflective thick subcategory is a Serre subcategory. If C_X has the property (sup), then any Serre subcategory of C_X is coreflective.*

Proof. See [R, III.2.4.4]. ■

2. The spectrum Spec(X). We denote by $Spec(X)$ the family of all nonzero objects M of the category C_X such that $L \succ M$ for any nonzero subobject L of M.

The spectrum $\mathbf{Spec}(X)$ of the 'space' X is the family of topologizing subcategories $\{[M] \mid M \in Spec(X)\}$ endowed with the *specialization* preorder $\supseteq$.

Let $\tau^{\succ}$ denote the topology on $\mathbf{Spec}(X)$ associated with the specialization preorder: the closure of $W \subseteq \mathbf{Spec}(X)$ consists of all $[M]$ such that $[M] \subseteq [M']$ for some $[M'] \in W$.

2.1. Proposition. *(a) Every simple object of the category C_X belongs to $Spec(X)$. The inclusion $Simple(X) \hookrightarrow Spec(X)$ induces an embedding of the set of the isomorphism classes of simple objects of C_X into the set of closed points of $(\mathbf{Spec}(X), \tau^{\succ})$.*

(b) If every nonzero object of C_X has a simple subquotient, then each closed point of $(Spec(X), \tau^{\succ})$ is of the form $[M]$ for some simple object M of the category C_X.

Proof. (a) If M is a simple object, then $Ob[M]$ consists of all objects isomorphic to coproducts of finite number of copies of M. In particular, if M and N are simple objects, then $[M] \subseteq [N]$ iff $M \simeq N$.

(b) If L is a subquotient of M, then $[L] \subseteq [M]$. If $[M]$ is a closed point of $\mathbf{Spec}(X)$, this implies the equality $[M] = [L]$. ■

Notice that the notion of a simple object of an abelian category is selfdual, i.e. $Simple(X) = Simple(X^o)$, where X^o is the dual 'space' defined by $C_{X^o} = C_X^{op}$. In particular, the map $M \longmapsto [M]$ induces an embedding of isomorphism classes of simple objects of C_X into the intersection $\mathbf{Spec}(X) \bigcap \mathbf{Spec}(X^o)$.

2.1.1. Proposition. *If the category C_X has enough objects of finite type, then the set of closed points of* $\mathbf{Spec}(X)$ *coincides with* $\mathbf{Spec}(X) \bigcap \mathbf{Spec}(X^o)$.

Proof. Since every nonzero object of C_X has a nonzero subobject of finite type, $\mathbf{Spec}(X)$ consists of $[M]$ such that M is of finite type and belongs to $Spec(X)$. On the other hand, if M is of finite type and $[M]$ belongs to $\mathbf{Spec}(X^o)$, then $[M] = [M_1]$, where M_1 is a simple quotient of M. Hence the assertion. ■

2.2. Supports of objects. For any object M of the category C_X, the support of M is defined by $Supp(M) = \{\mathcal{Q} \in \mathbf{Spec}(X) \mid \mathcal{Q} \subseteq [M]\}$. This notion enjoys the usual properties:

2.2.1. Proposition. *(a) If* $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$, *is a short exact sequence, then*

$$Supp(M) = Supp(M') \bigcup Supp(M'').$$

(b) Suppose the category C_X has the property (sup). Then

(b1) If M is the supremum of a filtered system $\{M_i \mid i \in J\}$ of its subobjects, then

$$Supp(M) = \bigcup_{i \in J} Supp(M_i).$$

(b2) As a consequence of (a) and (b1), we have

$$Supp(\bigoplus_{i \in J} M_i) = \bigcup_{i \in J} Supp(M_i).$$

Proof. (a) Since $[M'] \subseteq [M] \supseteq [M'']$, we have the inclusion

$$Supp(M') \bigcup Supp(M'') \subseteq Supp(M).$$

In order to show the inverse inclusion, notice that for any object L of the subcategory $[M]$, there exists an exact sequence $0 \longrightarrow L' \longrightarrow L \longrightarrow L'' \longrightarrow 0$ such that L' is an object of $[M']$ and L'' belongs to $[M'']$. This follows from the fact that L is a subquotient of a coproduct $M^{\oplus n}$ of n copies of M the related commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & {M'}^{\oplus n} & \longrightarrow & M^{\oplus n} & \longrightarrow & {M''}^{\oplus n} & \longrightarrow & 0 \\
 & & \uparrow & cart & \uparrow & & \uparrow & & \\
0 & \longrightarrow & K' & \longrightarrow & K & \longrightarrow & K'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L' & \longrightarrow & L & \longrightarrow & L'' & \longrightarrow & 0
\end{array} \qquad (1)$$

whose rows are exact sequences, the upper vertical arrows are monomorphisms, the lower ones epimorphisms, and the left upper square is cartesian.

Now if $[L] \in \mathbf{Spec}(X)$ and the object L' in the diagram (1) is nonzero, then, by the definition of the spectrum, $[L'] = [L]$, hence $[L] \in Supp(M')$. If $L' = 0$, then the arrow $L \longrightarrow L''$ is an isomorphism, in particular, $[L] = [L''] \in Supp(M'')$.

(b1) The inclusion $Supp(M) \supseteq \bigcup_{i \in J} Supp(M_i)$ is obvious. It follows from the property (sup) that if an object L is a nonzero subquotient of $M^{\oplus n}$ for some n, then it contains a nonzero subobject, L', which is a subquotient of M_i for some $i \in J$. If $[L] \in \mathbf{Spec}(X)$, this implies that $[L] = [L'] \in Supp(M_i)$.

(b2) If J is finite, the assertion follows from (a). If J is infinite, it is a consequence of (a) and (b1). ■

2.3. Topologies on $\mathbf{Spec}(X)$. Let Ξ be a class of objects of C_X closed under finite coproducts. For any set E of objects of Ξ, let $\mathcal{V}(E)$ denote the intersection $\bigcap_{M \in E} Supp(M)$. Then, for any family $\{E_i \mid i \in \mathfrak{I}\}$ of such sets, we have, evidently,

$$\mathcal{V}(\bigcup_{i \in J} E_i) = \bigcap_{i \in J} \mathcal{V}(E_i).$$

It follows from the equality $Supp(M \oplus N) = Supp(M) \bigcup Supp(N)$ (see 2.2.1(a)) that $\mathcal{V}(E \oplus \widetilde{E}) = \mathcal{V}(E) \bigcup \mathcal{V}(\widetilde{E})$. Here $E \oplus \widetilde{E} \stackrel{\mathrm{def}}{=} \{M \oplus N \mid M \in E,\ N \in \widetilde{E}\}$.

This shows that the subsets $\mathcal{V}(E)$ of $\mathbf{Spec}(X)$, where E runs through subsets of Ξ, are all closed sets of a topology, τ_Ξ, on $\mathbf{Spec}(X)$.

2.4. Zariski topology on the spectrum. Notice that the class $\Xi_{\mathfrak{f}}(X)$ of objects of finite type is closed under finite coproducts, hence defines a topology on $\mathbf{Spec}(X)$, which we denote by $\tau_{\mathfrak{z}}$.

2.4.1. Example. Let R be a commutative unital ring and C_X the category $R-mod$ of R-modules. Then $\mathbf{Spec}(X)$ is isomorphic to the prime spectrum $Spec(R)$ of the ring R and the topology $\tau_{\mathfrak{z}}$ corresponds to the Zariski topology on $Spec(R)$.

2.4.2. Zariski topology. If the category C_X has enough objects of finite type, we shall call the topology $\tau_{\mathfrak{z}}$ on $Spec(X)$ the *Zariski* topology.

3. Local 'spaces' and $\mathbf{Spec}^{-}(-)$.

3.1. Local 'spaces'. A 'space' X and the representing it abelian category C_X are called *local* if C_X has the smallest nonzero topologizing subcategory, $C_{X_{\mathfrak{t}}}$.

It follows that $C_{X_{\mathfrak{t}}}$ is the only closed point of $\mathbf{Spec}(X)$.

3.1.1. Proposition. *Let X be local, and let the category C_X have simple objects. Then all simple objects of C_X are isomorphic to each other, and every nonzero object of $C_{X_{\mathfrak{t}}}$ is a finite coproduct of copies of a simple object.*

Proof. In fact, if M is a simple object in C_X, then $[M]$ is a closed point of $\mathbf{Spec}(X)$. If X is local, this closed point is unique. Therefore, objects of $C_{X_{\mathfrak{t}}}$ are finite coproducts of copies of M (see the argument of 2.1). ■

3.1.2. The residue 'space' of a local 'space'. Let X be local 'space' and $C_{X_{\mathfrak{t}}}$ the smallest non-trivial topologizing subcategory of the category C_X. We regard the inclusion functor $C_{X_{\mathfrak{t}}} \hookrightarrow C_X$ as an inverse image functor of a morphism of 'spaces' $X \longrightarrow X_{\mathfrak{t}}$ and call $X_{\mathfrak{t}}$ the *residue* 'space' of X.

3.1.3. The residue skew field of a local 'space'. Suppose that X is a local 'space' such that the category C_X has a simple object, M. We denote by k_X the ring $C_X(M,M)^o$ opposite to the ring of endomorphisms of the object M. Since M is simple, k_X is a skew field, which we call the *residue skew field* of the local 'space' X. It follows from 3.1.1 that the residue skew field of X (if any) is defined uniquely up to isomorphism.

It follows that the *residue* category $C_{X_{\mathfrak{t}}}$ of the 'space' X is naturally equivalent to the category of finitely dimensional k_X-vector spaces.

3.2. $\mathbf{Spec}^-(X)$. By definition, $\mathbf{Spec}^-(X)$ is formed by all Serre subcategories $\mathcal{P}$ of C_X such that $X/\mathcal{P}$ is a local 'space'. It is endowed with the preorder $\supseteq$.

We define the support of an object M of C_X in $\mathbf{Spec}^-(X)$ as the set $Supp^-(M)$ of all $\mathcal{P} \in \mathbf{Spec}^-(X)$, which do not contain M, or, equivalently, the localization of M at $\mathcal{P}$ is nonzero. We leave as an exercise proving the analogue of 2.2.1 for $Supp^-(-)$.

We introduce the *Zariski* topology, $\tau_{\mathfrak{z}}^-$, on $\mathbf{Spec}^-(X)$ the same way as the topology on $\mathbf{Spec}(X)$: its closed sets are the intersections of $Supp^-(M)$, where M is an arbitrary object of finite type.

3.2.1. Indecomposable injectives and $\mathbf{Spec}^-(-)$. If C_X is a Grothendieck category with Gabriel-Krull dimension (say, C_X is locally noetherian), then the elements of $\mathbf{Spec}^-(X)$ are in bijective correspondence with the set of isomorphism classes of indecomposable injectives of the category C_X. The bijective correspondence is given by the map which assigns to every indecomposable injective E of C_X its *left orthogonal* – the full subcategory ${}^{\perp}E$ generated by all objects M of C_X such that $C_X(M,E) = 0$.

In other words, $\mathbf{Spec}^-(X)$ is isomorphic to the Gabriel spectrum of the category C_X.

An advantage of $\mathbf{Spec}^-(X)$ is that it makes sense for all abelian categories, even those, which do not have indecomposable injective objects at all. For instance, if C_X is the category of coherent sheaves on a noetherian scheme, then its Gabriel spectrum is empty, while $\mathbf{Spec}^-(X)$ coincides with $\mathbf{Spec}(X)$ and is homeomorphic to the the underlying topological space of the scheme.

3.3. $\mathbf{Spec}(X)$, $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$, and $\mathbf{Spec}^{-}(X)$. For any subcategory $\mathcal{P}$ of C_X, we denote by $\mathcal{P}^{\mathfrak{t}}$ the intersection of all topologizing subcategories of C_X properly containing $\mathcal{P}$. Let $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$ denote the set of all Serre subcategories $\mathcal{P}$ of C_X such that $\mathcal{P}^{\mathfrak{t}} \neq \mathcal{P}$.

3.3.1. Proposition. $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$ *consists of all topologizing subcategories $\mathcal{P}$ such that $\mathcal{P}_{\mathfrak{t}} \overset{\mathrm{def}}{=} \mathcal{P}^{\mathfrak{t}} \bigcap \mathcal{P}^{\perp}$ is nonzero.*

Proof. If $\mathcal{P} \in \mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$, i.e. $\mathcal{P}$ is a Serre subcategory of C_X, which is properly contained in $\mathcal{P}^{\mathfrak{t}}$, then it follows from 1.4.4 that $\mathcal{P}_{\mathfrak{t}} \neq 0$.

Suppose now that $\mathcal{P}$ is a topologizing subcategory of C_X such that $\mathcal{P}_{\mathfrak{t}} \neq 0$. We claim that then $\mathcal{P}$ is a Serre subcategory, i.e. $\mathcal{P} = \mathcal{P}^{-}$.

In fact, let $\mathcal{S}$ be a topologizing subcategory of C_X, which is not contained in $\mathcal{P}$. Then $\mathcal{P} \bullet \mathcal{S}$ contains $\mathcal{P}^{\mathfrak{t}}$ properly and $(\mathcal{P} \bullet \mathcal{S}) \bigcap \mathcal{P}^{\perp} \subseteq \mathcal{S}$. In particular, $\mathcal{P}_{\mathfrak{t}} \subseteq \mathcal{S}$. Since $\mathcal{P}_{\mathfrak{t}} \neq 0$, this implies that $\mathcal{S}$ is not contained in $\mathcal{P}^{-}$. This (and 1.4.2) shows that $\mathcal{P} = \mathcal{P}^{-}$. ∎

For any subcategory $\mathcal{Q}$ of the category C_X, we denote by $\widehat{\mathcal{Q}}$ the union of all topologizing subcategories of C_X, which do not contain $\mathcal{Q}$. It is easy to see, that for a pair $\mathcal{Q}_1$, $\mathcal{Q}_2$ topologizing subcategories, $\mathcal{Q}_1 \subseteq \mathcal{Q}_2$ iff $\widehat{\mathcal{Q}}_1 \subseteq \widehat{\mathcal{Q}}_2$.

If $\mathcal{Q}$ has one object, L, then the subcategory $\widehat{\mathcal{Q}}$ is the union of all topologizing subcategories of C_X which do not contain L. We shall write $\langle L \rangle$ instead of $\widehat{[L]}$.

3.3.2. Proposition. *(a)* $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X) \subseteq \mathbf{Spec}^{-}(X)$.

(b) For any $\mathcal{Q} \in \mathbf{Spec}(X)$, the subcategory $\widehat{\mathcal{Q}}$ is an element of $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$ and the map

$$\mathbf{Spec}(X) \longrightarrow \mathbf{Spec}_{\mathfrak{t}}^{1,1}(X), \quad \mathcal{Q} \longmapsto \widehat{\mathcal{Q}},$$

is an isomorphism of preorders.

Proof. (a) If $\mathcal{P} \in \mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$, then $\mathcal{P}^{\mathfrak{t}}/\mathcal{P}$ is naturally identified with the smallest nonzero topologizing subcategory of $C_X/\mathcal{P}$.

(b1) If $\mathcal{Q} \in \mathbf{Spec}(X)$, then $\widehat{\mathcal{Q}}$ is a Serre subcategory.

In fact, suppose that $\widehat{\mathcal{Q}} \neq \widehat{\mathcal{Q}}^{-}$, and let M be an object of $\widehat{\mathcal{Q}}^{-}$, which does not belong to its subcategory $\widehat{\mathcal{Q}}$. The latter means that $\mathcal{Q} \subseteq [M]$. Let $\mathcal{Q} = [L]$ for some $L \in Spec(X)$. The inclusion $\mathcal{Q} \subseteq [M]$ means that L is a subquotient of a coproduct of a finite number, $M^{\oplus n}$, of copies of M. Since $M^{\oplus n}$ is an object of $\widehat{\mathcal{Q}}^{-}$, the object L has a nonzero subobject N, which belongs to $\widehat{\mathcal{Q}}$; i.e. $\mathcal{Q} \not\subseteq [N]$. But, since $L \in Spec(X)$, the subcategories $[N]$ and $[L] = \mathcal{Q}$ coincide. Contradiction.

(b2) It follows from the definition of $\widehat{\mathcal{Q}}$ that, for any subcategory $\mathcal{Q}$, the subcategory $\widehat{\mathcal{Q}}^{\mathfrak{t}}$ coincides with the intersection of all topologizing subcategories of C_X containing $\widehat{\mathcal{Q}} \bigcup \mathcal{Q}$. In particular, $\widehat{\mathcal{Q}}$ belongs to $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$ whenever $\widehat{\mathcal{Q}}$ is a Serre subcategory. Together with (b1), this shows that the assignment $\mathcal{Q} \longmapsto \widehat{\mathcal{Q}}$ induces a map $\mathbf{Spec}(X) \longrightarrow \mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$.

(b3) Let $\mathcal{P} \in \mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$. It follows from 3.3.1 that $\mathcal{P}_{\mathfrak{t}} \neq 0$. Moreover, by the argument of 3.3.1, if $\mathbb{T}$ is a topologizing subcategory of C_X such that $\mathbb{T} \nsubseteq \mathcal{P}$, then $\mathcal{P}_{\mathfrak{t}} = \mathcal{P}^{\mathfrak{t}} \bigcap \mathcal{P}^{\perp} \subseteq \mathbb{T}$.

(b4) Let $\mathcal{P} \in \mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$. Every nonzero object of $\mathcal{P}_{\mathfrak{t}} = \mathcal{P}^{\mathfrak{t}} \bigcap \mathcal{P}^{\perp}$ belongs to $Spec(X)$.

Let L be a nonzero object of $\mathcal{P}_{\mathfrak{t}}$ and L_1 a nonzero subobject of L. Then $[L_1] \subseteq [L]$. If $[L_1] \nsubseteq [L]$, then it follows from (b3) above that $[L_1] \subseteq \mathcal{P}$, or, equivalently, $L_1 \in Ob\mathcal{P}$. This contradicts to the assumption that the object L is $\mathcal{P}$-torsion free.

(b5) Let $\mathcal{P} \in \mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$. Then $\mathcal{P} = \langle L \rangle$ for any nonzero object of $\mathcal{P}_{\mathfrak{t}} = \mathcal{P}^{\mathfrak{t}} \bigcap \mathcal{P}^{\perp}$.

Let L be a nonzero object of $\mathcal{P}_{\mathfrak{t}}$. Since L does not belong to the Serre subcategory $\langle L \rangle$, by (b3), we have the inclusion $\langle L \rangle \subseteq \mathcal{P}$. On the other hand, if $\langle L \rangle \subsetneqq \mathcal{P}$, then $L \in Ob\mathcal{P}$ which is not the case. Therefore $\mathcal{P} = \langle L \rangle$.

(b6) The topologizing subcategory $[\mathcal{P}_{\mathfrak{t}}]$ coincides with the subcategory $[L]$ for any nonzero object L of $\mathcal{P}_{\mathfrak{t}}$.

Clearly $[L] \subseteq [\mathcal{P}_{\mathfrak{t}}]$ for any $L \in Ob\mathcal{P}_{\mathfrak{t}}$. By (b3), if $\mathcal{P}_{\mathfrak{t}} \nsubseteq [L]$, then $[L] \subseteq \mathcal{P}$, hence $L = 0$.

Since, by (c), every nonzero object of $\mathcal{P}_{\mathfrak{t}}$ belongs to $Spec(X)$, this shows that $[\mathcal{P}_{\mathfrak{t}}]$ is an element of $\mathbf{Spec}(X)$.

(b7) It follows from the argument above that the map

$$\mathbf{Spec}(X) \longrightarrow \mathbf{Spec}_{\mathfrak{t}}^{1,1}(X), \quad \mathcal{Q} \longmapsto \widehat{\mathcal{Q}},$$

is inverse to the map $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X) \longrightarrow \mathbf{Spec}(X)$ which assigns to every $\mathcal{P}$ the topologizing subcategory $[\mathcal{P}_{\mathfrak{t}}]$. ■

3.3.3. The difference between $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$ and $\mathbf{Spec}^{-}(X)$. If $C_X = R - mod$, where R is a commutative noetherian ring, then the map, which assigns to each prime ideal p of R the isomorphism class of the injective hull of R/p is an isomorphism between the Gabriel spectrum of C_X (hence $\mathbf{Spec}^{-}(X)$) and the prime spectrum of the ring R [Gab]. In this case, $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X) = \mathbf{Spec}^{-}(X)$, i.e. the map $\mathcal{Q} \longmapsto \widehat{\mathcal{Q}}$ is an isomorphism between $\mathbf{Spec}(X)$ and $\mathbf{Spec}^{-}(X)$.

If R is a non-noetherian commutative ring, $\mathbf{Spec}^{-}(X)$ might be much bigger than the prime spectrum of R, while $\mathbf{Spec}(X)$ is naturally isomorphic to $Spec(R)$: the isomorphism is given by the map, which assigns to a prime ideal p the topologizing subcategory $[R/p]$; the inverse map assigns to $\mathcal{Q} = [M]$ the annihilator of the module M.

4. The pretopology of Serre localizations and the related spectrum.

4.1. Lemma. *Let C_X be an abelian category. For any finite set $\{T_i \mid i \in J\}$ of topologizing subcategories of C_X, we have the equality* $\big(\bigcap_{i\in J} T_i\big)^{-} = \bigcap_{i\in J} T_i^{-}$.

Proof. Clearly $\big(\bigcap\limits_{i\in J} T_i\big)^- \subseteq \bigcap\limits_{i\in J} T_i^-$. We need to prove the inverse inclusion.

Let $J = \{1, 2, ..., n\}$. Let M be a nonzero object of $\bigcap\limits_{i\in J} T_i^-$. And let L be any nonzero subquotient of the object M. Since M is a nonzero object of T_1^-, the object L has a nonzero subobject, L_1, which belongs to T_1. Since M is a nonzero object of T_2^- and L_1 is a nonzero subquotient of M, the object L_1 has a nonzero subobject, L_2, which belongs to T_2. Since T_1 contains all subobjects of its objects, $L_2 \in T_1 \bigcap T_2$. Continuing this way, we obtain a descending chain of nonzero subobjects of L, $L_n \to L_{n-1} \to ... \to L_1 \to L$, such that $L_i \in Ob \bigcap\limits_{1\le j\le i} T_j$. This shows that $M \in \big(\bigcap\limits_{i\in J} T_i\big)^-$. ■

4.2. The Gabriel multiplication. The *Gabriel product* of two subcategories $\mathbb{T}$ and $\mathbb{S}$ of an abelian category C_X is the full subcategory $\mathbb{T}\bullet\mathbb{S}$ of C_X generated by all $M \in ObC_X$ for, which there exists an exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

with $M' \in Ob\mathbb{S}$ and $M'' \in Ob\mathbb{T}$. If $\mathbb{T}$ and $\mathbb{S}$ are topologizing subcategories, then such is $\mathbb{T} \bullet \mathbb{S}$. This multiplication is associative and has an identity element – the subcategory **0**. Note that a topologizing subcategory $\mathbb{T}$ of C_X is *thick* iff $\mathbb{T} \bullet \mathbb{T} = \mathbb{T}$.

By Lemma III.6.2.1 in [R], if $\mathbb{T}$, $\mathbb{S}$ are coreflective subcategories of C_X, their Gabriel product $\mathbb{T} \bullet \mathbb{S}$ is coreflective too.

4.2.1. Lemma. *Let C_X be an abelian category. For any finite set $\{S,\ T_i \mid i \in J\}$ of topologizing subcategories of the category C_X, the following equalities hold:*

$$\big(\bigcap_{i\in J} T_i\big) \bullet S = \bigcap_{i\in J} T_i \bullet S \quad \textit{and} \quad S \bullet \big(\bigcap_{i\in J} T_i\big) = \bigcap_{i\in J} S \bullet T_i.$$

Proof. (a) The inclusions $\big(\bigcap\limits_{i\in J} T_i\big) \bullet S \subseteq \bigcap\limits_{i\in J} T_i \bullet S$ and $S \bullet \big(\bigcap\limits_{i\in J} T_i\big) \subseteq \bigcap\limits_{i\in J} S \bullet T_i$ are evident. We need to prove the inverse inclusions.

(b) Let $M \in Ob \bigcap\limits_{i\in J} T_i \bullet S$; i.e. for any $i \in J$, there is a monomorphism $f_i : M_i \to M$ such that $M_i \in ObT_i$ and $Cok(f_i) \in ObS$. This gives an exact sequence

$$0 \longrightarrow \bigcap_{i\in J} M_i \longrightarrow M \longrightarrow \prod_{i\in J} Cok(f_i)$$

Clearly $\bigcap_{i\in J} M_i \in Ob \bigcap_{i\in J} T_i$. Since J is finite, $\prod_{i\in J} Cok(f_i) \in ObS$, hence $M \in Ob\big(\bigcap_{i\in J} T_i\big) \bullet S$.

(c) Suppose $S \bullet M \in Ob \bigcap_{i\in J} T_i$; i.e. for any $i \in J$, there is a monomorphism $M_i \xrightarrow{f_i} M$ such that $M_i \in ObS$ and $Cok(f_i) \in ObT_i$. Since J is finite, $sup(M_i|i \in J) \in ObS$, and $M/sup(M_i|i \in J) \in Ob \bigcap_{i\in J} T_i$, hence $M \in \bigcap_{i\in J} S \bullet T_i$. ■

For any pair S and T of Serre subcategories of the category C_X, the symbol $S \vee T$ denotes the minimal Serre subcategory of C_X containing S and T. It follows that $S \vee T = (S \bullet T)^-$.

4.3. Proposition. *Let C_X be an abelian category. For any finite set $\{T_i|i \in J\}$ of Serre subcategories of the category C_X, the equality* $\big(\bigcap_{i\in J} T_i\big) \vee S = \bigcap_{i\in J}(T_i \vee S)$ *holds.*

Proof. By Lemma 4.1, $\bigcap_{i\in J}(T_i \vee S) = \bigcap_{i\in J}(T_i \bullet S)^- = \big(\bigcap_{i\in J}(T_i \bullet S)\big)^-$. By Lemma 4.2.1, $\big(\bigcap_{i\in J}(T_i \bullet S)\big)^- = \big((\bigcap_{i\in J} T_i) \bullet S\big)^- = \big(\bigcap_{i\in J} T_i\big) \vee S$. ■

4.4. The pretopology of Serre localizations. We define the *quasi-pretopology of Serre localizations*, $\tau_{\mathfrak{L}_\mathfrak{s}}$, on the category $|\mathfrak{L}_\mathfrak{s}\mathfrak{Ab}|^o$ by taking as covers all families of morphisms $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ such that the corresponding family of inverse image functors is conservative. We define by $\tau^{\mathfrak{f}}_{\mathfrak{L}_\mathfrak{s}}$ the quasi-pretopology on $|\mathfrak{L}_\mathfrak{s}\mathfrak{Ab}|^o$ obtained by taking all covers of $\tau_{\mathfrak{L}_s}$ containing a finite subcover.

It follows from Proposition 4.3 that $\tau^{\mathfrak{f}}_{\mathfrak{L}_s}$ is a Grothendieck pretopology. We call it the *pretopology of Serre localizations.*

4.5. The local property of the spectrum $\mathbf{Spec}^-(-)$.

4.5.1. Proposition. *Let $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ be a cover in the pretopology $\tau^{\mathfrak{f}}_{\mathfrak{L}_s}$. Then* $\mathbf{Spec}^-(X) = \bigcup_{i\in J} \mathbf{Spec}^-(U_i)$.

Proof. The inclusion $\bigcup_{i\in J} \mathbf{Spec}^-(U_i) \subseteq \mathbf{Spec}^-(X)$ follows from the functoriality of the S-spectrum (see 3.4.1). We need to check the inverse inclusion.

Let $\mathbf{x} \stackrel{q}{\to} X$ be any point of $\mathbf{Spec}^-(X)$. Consider cartesian squares

$$\begin{array}{ccccc} U_i^{\mathbf{x}} & \xrightarrow{q_i} & U_i & & \\ u_i^{\mathbf{x}} \downarrow & & \downarrow u_i & & \\ \mathbf{x} & \xrightarrow{q} & X & & i \in J \end{array} \qquad (1)$$

Since $\tau^{\mathfrak{f}}_{\mathfrak{L}_s}$ is a pretopology, the pull-back $\{U_i^{\mathbf{x}} \stackrel{u_i^{\mathbf{x}}}{\to} \mathbf{x} \mid i \in J\}$ of the cover $\{U_i \stackrel{u_i}{\to} X \mid i \in J\}$ is a cover. Let P be a quasi-final object of the local category $C_{\mathbf{x}}$. Since (by the definition of a cover) the set of inverse image functors $\{C_{\mathbf{x}} \xrightarrow{u_i^{\mathbf{x}*}} C_{U_i^{\mathbf{x}}} \mid i \in J\}$ is conservative, $u_j^{\mathbf{x}*}(P) \neq 0$ for some $j \in J$. Since P is the point of the spectrum of $\mathbf{x}$, the object $u_j^{\mathbf{x}*}(P)$ belongs to the spectrum, hence to the flat spectrum, of $U_j^{\mathbf{x}}$. By functoriality of the S-spectrum (cf. 3.4.1), the morphism $U_j^{\mathbf{x}} \xrightarrow{q_j} U_j$ induces a map $\mathbf{Spec}^-(U_j)^{\mathbf{x}} \longrightarrow \mathbf{Spec}^-(U_j)$ which sends the point $\langle u_j^{\mathbf{x}*}(P)\rangle$ to a point $\mathbf{P}_j$ of $\mathbf{Spec}^-(U_j)$. It follows from the commutativity of (1) that the image of $\mathbf{P}_j$ by $U_j \xrightarrow{u_j} X$ coincides with $\mathbf{x} \stackrel{q}{\to} X$. ■

5. The complete spectrum and the pretopology of exact localizations. Fix a svelte abelian category C_X. The *complete spectrum* of the 'space' X is the preorder (with respect to the inclusion) $\mathbf{Spec}^1(X)$ of all thick subcategories $\mathcal{P}$ of C_X such that $X/\mathcal{P}$ is a local 'space'. Thus, $\mathbf{Spec}^-(X)$ is the intersection of $\mathbf{Spec}^1(X)$ and the preorder $\mathfrak{Se}(X)$ of Serre subcategories of C_X.

It is immediate that $\mathbf{Spec}^1(X)$ is functorial with respect to exact localizations: any morphism $U \xrightarrow{u} X$ whose inverse image functor, u^*, is an exact localization induces an embedding $\mathbf{Spec}^1(U) \longrightarrow \mathbf{Spec}^1(X)$, which identifies $\mathbf{Spec}^1(U)$ with the subset of all $\mathcal{P} \in \mathbf{Spec}^1(X)$ containing $Ker(u^*)$.

For any pair S and T of thick subcategories of the category C_X, the symbol $S \sqcup T$ denotes the smallest thick subcategory of C_X containing S and T.

5.1. Proposition. *Let C_X be an abelian category. Let $\{T_i | i \in J\}$ be a finite family of thick subcategories of the category C_X. Then $\big(\bigcap_{i\in J} T_i\big) \sqcup S = \bigcap_{i\in J}(T_i \sqcup S)$ for any thick subcategory S.*

Proof. The inclusion $\big(\bigcap_{i\in J} T_i\big) \sqcup S \subseteq \bigcap_{i\in J}(T_i \sqcup S)$ is evident. We need to prove the inverse inclusion.

(a) Let $\mathcal{T}$ be a topologizing subcategory of C_X and $\mathcal{T}^\infty$ the smallest thick subcategory spanned by $\mathcal{T}$. Objects of $\mathcal{T}^\infty$ are $M \in ObC_X$ having a filtration

$$0 = M_0 \hookrightarrow M_1 \hookrightarrow \ldots \hookrightarrow M_n = M$$

such that $M_i/M_{i-1} \in Ob\mathcal{T}$ for every $1 \le i \le n$.

In fact, by an obvious induction argument, every object M of C_X possessing such a filtration belongs to the subcategory $\mathcal{T}^\infty$.

On the other hand, due to the fact that the subcategory $\mathcal{T}$ is closed under taking subquotients, the full subcategory of the category C_X generated by objects M having a filtration as above form a thick subcategory of C_X.

Indeed, a filtration $0 = M_0 \hookrightarrow M_1 \hookrightarrow \ldots \hookrightarrow M_n = M$ with subsequent quotients from $\mathcal{T}$ induces a filtration with the same property on every subobject and every quotient object of M. Conversely, let $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ be an exact sequence such that the objects M' and M'' have increasing finite filtrations with subsequent quotients from $\mathcal{T}$. Then the pull-back to M the filtration on M'' combined with the filtration on its subobject M' produces a desired filtration on M.

(b) Let $\{\mathcal{T}_i \mid i \in J\}$ be a finite set of topologizing subcategories of C_X. Then

$$\big(\bigcap_{i\in J}\mathcal{T}_i\big)^\infty = \bigcap_{i\in J}\mathcal{T}_i^\infty.$$

The inclusion $\big(\bigcap_{i\in J}\mathcal{T}_i\big)^\infty \subseteq \bigcap_{i\in J}\mathcal{T}_i^\infty$ is obvious. We need to prove the inverse inclusion.

Ordering elements of J, we assume that $J = \{1, \ldots, m\}$. Let M be an object of $\bigcap_{i=1}^m \mathcal{T}_i^\infty$. Let $0 = M_0 \hookrightarrow M_1 \hookrightarrow \ldots \hookrightarrow M_n = M$ be a filtration such that M_i/M_{i-1} is an object of $\mathcal{T}_1$, hence it is an object of $\bigcap_{i=2}^m \big(\mathcal{T}_1 \cap \mathcal{T}_i^\infty\big)$ for every $1 \le i \le n$. In particular, M_i/M_{i-1} is an object of $\mathcal{T}_1 \cap \mathcal{T}_2^\infty$ for all $1 \le i \le n$. Taking a filtration of each M_i/M_{i-1} with respect to $\mathcal{T}_2^\infty$ and pulling it back to a filtration on M_i, we obtain a finite increasing filtration of M such that its consecutive quotients belong to $\mathcal{T}_1 \cap \mathcal{T}_2$; etc..

(c) It follows from (a), (b) and 4.2.1 that

$$\bigcap_{i\in J}(\mathcal{T}_i \sqcup \mathcal{S}) = \bigcap_{i\in J}(\mathcal{S} \bullet \mathcal{T}_i)^\infty = \big(\bigcap_{i\in J}\mathcal{S} \bullet \mathcal{T}_i\big)^\infty = \big(\mathcal{S} \bullet (\bigcap_{i\in J}\mathcal{T}_i)\big)^\infty = \mathcal{S} \sqcup \big(\bigcap_{i\in J}\mathcal{T}_i\big),$$

hence the assertion. ∎

5.2. The pretopology of exact localizations. Let $|\mathfrak{L}_\mathfrak{e}\mathfrak{Ab}|^o$ be a category whose objects are 'spaces' X with abelian category C_X and morphisms are exact localizations. We define the *quasi-pretopology of exact localizations*, $\tau_{\mathfrak{L}_\mathfrak{e}}$, on the category $|\mathfrak{L}_\mathfrak{e}\mathfrak{Ab}|^o$ by taking as covers all families of morphisms $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ such that the corresponding family of inverse image functors is conservative. We define by $\tau^{\mathfrak{f}}_{\mathfrak{L}_{\mathfrak{e}}}$ the quasi-pretopology on $|\mathfrak{L}_{\mathfrak{e}}\mathfrak{Ab}|^o$ obtained by taking all covers of $\tau_{\mathfrak{L}_e}$ containing a finite subcover.

It follows from Proposition 5.1 that $\tau^{\mathfrak{f}}_{\mathfrak{L}_e}$ is a Grothendieck pretopology. We call it the *pretopology of exact localizations*, or simply the *pretopology of localizations.*

The following assertion is refered to as the *local property* of the complete spectrum.

5.3. Proposition. *Let* $\{U_i \overset{u_i}{\to} X \mid i \in J\}$ *be a cover in the pretopology* $\tau^{\mathfrak{f}}_{\mathfrak{L}_e}$. *Then*

$$\mathbf{Spec}^1(X) = \bigcup_{i\in J} \mathbf{Spec}^1(U_i)$$

Proof. The argument is similar to that of 4.5.1. Details are left to the reader. ∎

5.4. Proposition. *Let* $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ *be a finite conservative set of exact localizations. Then* $\mathbf{Spec}^-(X) \subseteq \bigcup_{i\in J} \mathbf{Spec}^-(U_i)$. *Here* $\mathbf{Spec}^-(X)$ *and* $\mathbf{Spec}^-(U_i)$, $i \in J$, *are realized as subsets of the complete spectrum* $\mathbf{Spec}^1(X)$.

Proof. Let T_i denote the kernel of the localization functor $C_X \xrightarrow{u_i^*} C_{U_i}$, $i \in J$. The condition that $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ is conservative means that $\bigcap_{i\in J} T_i = 0$. By 4.1, $\bigcap_{i\in J} T_i^- = \big(\bigcap_{i\in J} T_i\big)^- = 0$, or, equivalently, $\{\widetilde{U}_i \xrightarrow{\widetilde{u}_i} X \mid i \in J\}$ is a finite cover of X by Serre localizations. Here $\widetilde{U}_i$ denote the 'space' $X/|T_i^-|$ (i.e. $C_{\widetilde{U}_i} = C_X/T_i^-$). Since $T_i \subseteq T_i^-$, the localization $\widetilde{U}_i \longrightarrow X$ factors through a localization $\widetilde{U}_i \longrightarrow U_i$ with the kernel T_i^-/T_i. The latter is a Serre subcategory of the quotient category C_X/T_i. Therefore, $\mathbf{Spec}^-(\widetilde{U}_i) \subseteq \mathbf{Spec}^-(U_i)$. By 4.5.1, $\mathbf{Spec}^-(X) = \bigcup_{i\in J} \mathbf{Spec}^-(\widetilde{U}_i) \subseteq \bigcup_{i\in J} \mathbf{Spec}^-(U_i)$. ∎

6. Spectra related with localizations. Let C_X be a svelte abelian category. For any subcategory $\mathcal{S}$ of C_X, we denote by $\mathcal{S}^{\mathfrak{th}}$ the intersection of all thick subcategories of C_X properly containing $\mathcal{S}$, and by $\mathcal{S}_{\mathfrak{th}}$ the intersection $\mathcal{S}^{\mathfrak{th}} \bigcap \mathcal{S}^{\perp}$.

We denote by $\mathbf{Spec}^1_{\mathfrak{Th}}(X)$ the preorder of all thick subcategories $\mathcal{P}$ such that the intersection $\mathcal{P}^{\mathfrak{th}}$ of all thick subcategories properly containing $\mathcal{P}$ is not equal to $\mathcal{P}$.

We denote by $\mathbf{Spec}^1_{\mathfrak{Se}}(X)$ the intersection $\mathbf{Spec}^1_{\mathfrak{Th}}(X) \bigcap \mathfrak{Se}(X)$ of $\mathbf{Spec}^1_{\mathfrak{Th}}(X)$ with the preorder $\mathfrak{Se}(X)$ of Serre subcategories of the category C_X.

6.1. Proposition. *The following conditions on a topologizing subcategory* $\mathcal{T}$ *of* C_X *are equivalent:*

(a) $\mathcal{T} \in \mathbf{Spec}^1_{\mathfrak{Se}}(X)$,
(b) $\mathcal{T}_{\mathfrak{th}} \neq 0$.
In particular, $\mathbf{Spec}^1_{\mathfrak{Se}}(X) = \{\mathcal{P} \in \mathbf{Spec}^1_{\mathfrak{Th}}(X) \mid \mathcal{P}_{\mathfrak{th}} \neq 0\}$.

Proof. The fact follows from 1.4.4 and the argument is similar to that of 3.3.1. ■

6.2. Elementary properties. There are obvious inclusions

$$\mathbf{Spec}^1_{\mathfrak{Th}}(X) \supseteq \mathbf{Spec}^1_{\mathfrak{Se}}(X) \supseteq \mathbf{Spec}^-(X) \subseteq \mathbf{Spec}^1(X) \subseteq \mathbf{Spec}^1_{\mathfrak{Th}}(X).$$

6.3. Proposition. *Let* $\{\mathcal{T}_i \mid i \in J\}$ *be a finite set of thick subcategories of an abelian category* C_X *such that* $\bigcap_{i\in J} \mathcal{T}_i = 0$. *Then*

$$\mathbf{Spec}^1_{\mathfrak{Th}}(X) = \bigcup_{i\in J} \mathbf{Spec}^1_{\mathfrak{Th}}(U_i) \qquad (1)$$

and

$$\mathbf{Spec}^1_{\mathfrak{Se}}(X) \subseteq \bigcup_{i\in J} \mathbf{Spec}^1_{\mathfrak{Se}}(U_i). \qquad (2)$$

If $\{\mathcal{T}_i \mid i \in J\}$ *are Serre subcategories, then*

$$\mathbf{Spec}^1_{\mathfrak{Se}}(X) = \bigcup_{i\in J} \mathbf{Spec}^1_{\mathfrak{Se}}(U_i). \qquad (3)$$

Proof. The arguments establishing the equalities (1) and (3) are similar to the proof of 4.5.1. The proof of the inclusion (2) follows the argument of 5.4. ■

7. Local properties of $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$ and $\mathbf{Spec}(X)$.

7.1. Proposition. *Let* $\{\mathcal{T}_i \mid i \in J\}$ *be a finite set of thick subcategories of the category* C_X *such that* $\bigcap_{i\in J} \mathcal{T}_i = 0$. *The following conditions on a thick subcategory* $\mathcal{P}$ *of* C_X *are equivalent:*

(a) $\mathcal{P} \in \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$,
(b) $\mathcal{P} \in \mathbf{Spec}^1_{\mathfrak{Se}}(X)$ *and* $\mathcal{P}/\mathcal{T}_i \in \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X/\mathcal{T}_i)$ *for every* $i \in J$ *such that* $\mathcal{T}_i \subseteq \mathcal{P}$.

If the category C_X *has the property (sup), then the conditions (a) and (b) are equivalent to the condition*

(c) $\mathcal{P} \in \mathbf{Spec}^1_{\mathfrak{Th}}(X)$ *and* $\mathcal{P}/\mathcal{T}_i \in \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X/\mathcal{T}_i)$ *for every* $i \in J$ *such that* $\mathcal{T}_i \subseteq \mathcal{P}$.

Proof. $(a) \Rightarrow (b)$. Let $\mathcal{P} \in \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$, and let $\mathcal{T}$ be a thick subcategory of C_X contained in $\mathcal{P}$. Then $(\mathcal{T} \bullet \mathcal{P}^{\mathfrak{t}} \bullet \mathcal{T})/\mathcal{T}$ is the smallest topologizing subcategory of $C_X/\mathcal{T}$ properly containing $\mathcal{P}/\mathcal{T}$, and the localization functor $C_X \xrightarrow{u^*} C_X/\mathcal{T}$ maps nonzero objects of $\mathcal{P}_{\mathfrak{t}} = \mathcal{P}^{\mathfrak{t}} \cap \mathcal{P}^{\perp}$ to nonzero objects of $(\mathcal{P}/\mathcal{T})_{\mathfrak{t}} = (\mathcal{P}/\mathcal{T})^{\mathfrak{t}} \bigcap (\mathcal{P}/\mathcal{T})^{\perp}$.

$(b) \Rightarrow (a)$. Let u_i^* denote the localization functor $C_X \longrightarrow C_X/\mathcal{T}_i$. Set $J_{\mathcal{P}} = \{j \in J \mid \mathcal{T}_j \subseteq \mathcal{P}\}$. For every $i \in J_{\mathcal{P}}$, we denote by $\widetilde{\mathcal{Q}}_i$ the intersection $u_i^{*^{-1}}((\mathcal{P}/\mathcal{T}_i)^{\mathfrak{t}}) \bigcap \mathcal{P}^{\perp}$ and by $\mathcal{Q}_i$ the topologizing subcategory $[\widetilde{\mathcal{Q}}_i]$ spanned by $\widetilde{\mathcal{Q}}_i$. By assumption, $\widetilde{\mathcal{Q}}_i \neq 0$ for each $i \in J_{\mathcal{P}}$, hence $\mathcal{Q}_i \nsubseteq \mathcal{P}$. The latter implies that, for every $j \in J_{\mathcal{P}}$, the topologizing subcategory $[u_j^*(\mathcal{Q}_i \bullet \mathcal{P})]$ contains $(\mathcal{P}/\mathcal{T}_j)^{\mathfrak{t}}$, or, equivalently, $u_j^{*^{-1}}((\mathcal{P}/\mathcal{T}_i)^{\mathfrak{t}}) \subseteq \mathcal{T}_j \bullet \mathcal{Q}_i \bullet \mathcal{P}$. Therefore,

$$\widetilde{\mathcal{Q}}_j = u_j^{*^{-1}}((\mathcal{P}/\mathcal{T}_j)^{\mathfrak{t}}) \bigcap \mathcal{P}^{\perp} \subseteq (\mathcal{T}_j \bullet \mathcal{Q}_i \bullet \mathcal{P}) \bigcap \mathcal{P}^{\perp} = (\mathcal{T}_j \bullet \mathcal{Q}_i) \bigcap \mathcal{P}^{\perp} \subseteq \mathcal{T}_j \bullet \mathcal{Q}_i$$

which implies the inclusion $\mathcal{Q}_j \subseteq \mathcal{T}_j \bullet \mathcal{Q}_i$ for every $(i,j) \in J_{\mathcal{P}} \times J_{\mathcal{P}}$, hence

$$\mathcal{Q}_j \subseteq \bigcap_{i \in J_{\mathcal{P}}} (\mathcal{T}_j \bullet \mathcal{Q}_i) = \mathcal{T}_j \bullet \big(\bigcap_{i \in J_{\mathcal{P}}} \mathcal{Q}_i \big).$$

Here the equality is due to the finiteness of $J_{\mathcal{P}}$.

It follows from the inclusion $\mathcal{Q}_j \subseteq \mathcal{T}_j \bullet \big(\bigcap_{i \in J_{\mathcal{P}}} \mathcal{Q}_i \big)$ that $\bigcap_{i \in J_{\mathcal{P}}} \mathcal{Q}_i \neq 0$, because otherwise $\mathcal{Q}_j \subseteq \mathcal{T}_j \bullet 0 = \mathcal{T}_i$, which is impossible, since $\mathcal{T}_j \subseteq \mathcal{P}$ and $\mathcal{Q}_j \nsubseteq \mathcal{P}$.

There are two cases: $J = J_{\mathcal{P}}$ and $J \neq J_{\mathcal{P}}$. Consider each of them.

(i) Let $J_{\mathcal{P}} = J$. We set $\mathcal{Q} = \bigcap_{i \in J_{\mathcal{P}}} \mathcal{Q}_i$ and claim that $\mathcal{Q}$ is an element of $\mathbf{Spec}(X)$ corresponding to $\mathcal{P}$, that is $\mathcal{P} = \langle \mathcal{Q} \rangle$.

In fact, let $\mathcal{S}$ is a topologizing subcategory of C_X, which is not contained in $\mathcal{P}$. Then $\mathcal{P}$ is properly contained in $\mathcal{S} \bullet \mathcal{P}$ and, therefore, $u_i^{*^{-1}}((\mathcal{P}/\mathcal{T}_i)^{\mathfrak{t}}) \subseteq \mathcal{T}_i \bullet \mathcal{S} \bullet \mathcal{P}$ for each $i \in J$. This implies that $u_i^{*^{-1}}((\mathcal{P}/\mathcal{T}_i)^{\mathfrak{t}}) \bigcap \mathcal{P}^{\perp} \subseteq \mathcal{T}_i \bullet \mathcal{S} \bullet \mathcal{P} \bigcap \mathcal{P}^{\perp} \subseteq \mathcal{T}_i \bullet \mathcal{S}$. Therefore,

$$\widetilde{\mathcal{Q}} = \bigcap_{i \in J} u_i^{*^{-1}}((\mathcal{P}/\mathcal{T}_i)^{\mathfrak{t}}) \bigcap \mathcal{P}^{\perp} \subseteq \bigcap_{i \in J} (\mathcal{T}_i \bullet \mathcal{S}) = \big(\bigcap_{i \in J} \mathcal{T}_i \big) \bullet \mathcal{S} = 0 \bullet \mathcal{S} = \mathcal{S},$$

which implies that $\mathcal{Q} = [\widetilde{\mathcal{Q}}] \subseteq \mathcal{S}$.

(ii) Consider now the second case: $J_{\mathcal{P}} \neq J$, i.e. $J^{\mathcal{P}} = J - J_{\mathcal{P}}$ is non-empty. This case can be reduced to the first case as follows.

1) Set $C_{\mathcal{V}_{\mathcal{P}}} = \bigcap_{j \in J^{\mathcal{P}}} \mathcal{T}_j$. Notice that $C_{\mathcal{V}_{\mathcal{P}}} \nsubseteq \mathcal{P}$.

In fact, if $i \in J^{\mathcal{P}} = J - J_{\mathcal{P}}$, then $\mathcal{T}_i \nsubseteq \mathcal{P}$. Therefore, for every $j \in J_{\mathcal{P}}$, the topologizing subcategory $[u_j^*(\mathcal{T}_i \bullet \mathcal{P})]$ contains $(\mathcal{P}/\mathcal{T}_j)^{\mathfrak{t}}$, or, equivalently, $u_j^{*^{-1}}((\mathcal{P}/\mathcal{T}_j)^{\mathfrak{t}}) \subseteq \mathcal{T}_j \bullet \mathcal{T}_i \bullet \mathcal{P}$, which implies that $\widetilde{\mathcal{Q}}_j = u_j^{*^{-1}}((\mathcal{P}/\mathcal{T}_j)^{\mathfrak{t}}) \bigcap \mathcal{P}^\perp \subseteq (\mathcal{T}_j \bullet \mathcal{T}_i) \bigcap \mathcal{P}^\perp$. Thanks to the finiteness of $J^{\mathcal{P}}$, we obtain:

$$\widetilde{\mathcal{Q}}_j \subseteq \Big(\bigcap_{i \in J^{\mathcal{P}}} (\mathcal{T}_j \bullet \mathcal{T}_i)\Big) \bigcap \mathcal{P}^\perp = \Big(\mathcal{T}_j \bullet \big(\bigcap_{i \in J^{\mathcal{P}}} \mathcal{T}_i\big)\Big) \bigcap \mathcal{P}^\perp. \tag{4}$$

The inclusion $\bigcap_{i \in J^{\mathcal{P}}} \mathcal{T}_i \subseteq \mathcal{P}$ implies (together with (4)) that $\widetilde{\mathcal{Q}}_j \subseteq \mathcal{T}_j \subseteq \mathcal{P}$, which is impossible. So that $\bigcap_{i \in J^{\mathcal{P}}} \mathcal{T}_i \nsubseteq \mathcal{P}$.

2) Since $C_{\mathcal{V}_{\mathcal{P}}} \nsubseteq \mathcal{P}$, the intersection $\mathcal{P}_0 = C_{\mathcal{V}_{\mathcal{P}}} \bigcap \mathcal{P}$ is an element of $\mathbf{Spec}^1_{\mathfrak{S}\mathfrak{e}}(\mathcal{V}_{\mathcal{P}})$. Notice that $\{\mathcal{T}_i \cap C_{\mathcal{V}_{\mathcal{P}}} = \widetilde{\mathcal{T}}_i \mid i \in J_{\mathcal{P}}\}$ is a cocover of $\mathcal{V}_{\mathcal{P}}$, i.e. $\bigcap_{j \in J_{\mathcal{P}}} \widetilde{\mathcal{T}}_j = 0$. It remains to notice that $\mathcal{P}_0/\widetilde{\mathcal{T}}_j \in \mathbf{Spec}^{1,1}_{\mathfrak{t}}(\mathcal{V}_{\mathcal{P}}/\widetilde{\mathcal{T}}_j)$ for each $j \in J_{\mathcal{P}}$.

In fact, the localization functor $C_X \longrightarrow C_X/\mathcal{T}_j$ induces an equivalence of $C_{\mathcal{V}_{\mathcal{P}}}/\widetilde{\mathcal{T}}_j$ and the topologizing subcategory $(\mathcal{T}_j \bullet C_{\mathcal{V}_{\mathcal{P}}} \bullet \mathcal{T}_j)/\mathcal{T}_j$ of $C_X/\mathcal{T}_j$. The subcategory $\mathcal{P}_0/\widetilde{\mathcal{T}}_j$ of $C_{\mathcal{V}_{\mathcal{P}}}/\widetilde{\mathcal{T}}_j$ is the preimage of the intersection of the $\mathcal{P}/\mathcal{T}_j \in \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X/\mathcal{T}_j)$ with the topologizing subcategory $(\mathcal{T}_j \bullet C_{\mathcal{V}_{\mathcal{P}}} \bullet \mathcal{T}_j)/\mathcal{T}_j$, hence it belongs to $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(\mathcal{V}_{\mathcal{P}}/\widetilde{\mathcal{T}}_j)$.

3) Thus, the 'space' $\mathcal{V}_{\mathcal{P}}$, the cocover $\{\widetilde{\mathcal{T}}_i \mid i \in J_{\mathcal{P}}\}$, and the point $\mathcal{P}_0 = \mathcal{P} \bigcap C_{\mathcal{V}_{\mathcal{P}}}$ of the spectrum $\mathbf{Spec}^1_{\mathfrak{S}\mathfrak{e}}(\mathcal{V}_{\mathcal{P}})$ satisfy the conditions (b) with all $\widetilde{\mathcal{T}}_i$ being subcategories of $\mathcal{P}_0$. By 2) above, $\mathcal{P}_0$ belongs to the spectrum $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(\mathcal{V}_{\mathcal{P}})$, and $\mathcal{P}_0 = \langle \widetilde{\mathcal{Q}}_0 \rangle_{\mathcal{V}_{\mathcal{P}}} = \langle \mathcal{Q}_0 \rangle_{\mathcal{V}_{\mathcal{P}}}$, where $\mathcal{Q}_0$ is the smallest topologizing subcategory of ($C_{\mathcal{V}_{\mathcal{P}}}$, hence) C_X containing $\widetilde{\mathcal{Q}}_0$. Therefore, $\mathcal{Q}_0$ is a point of the spectrum $\mathbf{Spec}^0_{\mathfrak{t}}(X)$ and $\langle \mathcal{Q}_0 \rangle_X = \mathcal{P}$.

Obviously, $(b) \Rightarrow (c)$ without additional conditions on the category C_X. Suppose now that C_X has the property (sup).

$(c) \Rightarrow (b)$. It follows from (c) that $\mathcal{P} \in \mathbf{Spec}^1_{\mathfrak{s}}(X)$, i.e. $\mathcal{P}$ is a Serre subcategory.

In fact, by C1.4.1, the equality $\bigcap_{i \in J} \mathcal{T}_i = 0$ implies that $\bigcap_{i \in J} \mathcal{T}_i^- = 0$. In other words, $\{\mathcal{T}_i^- \mid i \in J\}$ is a finite cocover, which implies, by the local property of $\mathbf{Spec}^1_{\mathfrak{T}\mathfrak{h}}(X)$, that $\mathcal{T}_i^- \subseteq \mathcal{P}$ for some $i \in J$. Notice that if $\mathcal{T}_i^- \subseteq \mathcal{P}$, then $\mathcal{P}/\mathcal{T}_i^-$ belongs to $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X/\mathcal{T}_i^-)$. This follows from the fact that $\mathcal{P}/\mathcal{T}_i$ is, by the condition (c), an element of $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X/\mathcal{T}_i)$, and the spectrum $\mathbf{Spec}^{1,1}_{\mathfrak{t}}$ is functorial with respect to localizations (see the proof of $(a) \Rightarrow (b)$ above), in particular, with respect to $C_X/\mathcal{T}_i \longrightarrow C_X/\mathcal{T}_i^-$.

Therefore, $\mathcal{P}/\mathcal{T}_i^-$ is a Serre subcategory of the quotient category $C_X/\mathcal{T}_i^-$. Thanks to the property (sup), the Serre subcategory $\mathcal{T}_i^-$ is coreflective. By the argument 9.5(b1), this implies that $\mathcal{P}$ is a Serre subcategory of C_X. ■

7.2. Note. Proposition 7.1 is a stronger statement than [R4, 6.3] in all respects. The equivalence $(a) \Leftrightarrow (b)$ is essentially the assertion of [R4, 6.3], but the argument presented here is valid for arbitrary abelian categories, while the proof of [R4, 6.3] used the property (sup). The equivalence (a) and (b) to (c) (when C_X has the property (sup)) is a new observation (which could of be made in [R4]).

7.3. Proposition. *Let $\{\mathcal{T}_i \mid i \in J\}$ be a finite set of thick subcategories of an abelian category C_X such that $\bigcap_{i\in J} \mathcal{T}_i = 0$; and let u_i^* be the localization functor $C_X \longrightarrow C_X/\mathcal{T}_i$. The following conditions on a nonzero coreflective topologizing subcategory $\mathcal{Q}$ of C_X are equivalent:*

(a) $\mathcal{Q} \in \mathbf{Spec}(X)$,

(b) $[u_i^*(\mathcal{Q})] \in \mathbf{Spec}(X/\mathcal{T}_i)$ for every $i \in J$ such that $\mathcal{Q} \nsubseteq \mathcal{T}_i$.

Here $[u_i^*(\mathcal{Q})]$ denote the topologizing subcategory spanned by $u_i^*(\mathcal{Q})$.

Proof. The assertion follows from 7.1. ■

7.3.1. Note. The condition (b) of 7.3 can be reformulated as follows:

(b') For any $i \in J$, either $u_i^*(\mathcal{Q}) = 0$, or $[u_i^*(\mathcal{Q})] \in \mathbf{Spec}(X/\mathcal{T}_i)$.

7.4. Proposition. *Let C_X be an abelian category. Let $\mathfrak{U} = \{U_i \xrightarrow{u_i} X \mid i \in J\}$ be a finite cover of the 'space' X such that all morphisms $U_{ij} = U_i \cap U_j \xrightarrow{u_{ij}} U_i$ are continuous. Let $\mathcal{P}_i \in \mathbf{Spec}_{\mathfrak{t}}^{1,1}(U_i)$, and let L_i be an object of $Spec(U_i)$ such that $\mathcal{P}_i = \langle L_i \rangle$.*

The following conditions are equivalent:

(a) $\mathcal{P} = u_i^{*^{-1}}(\mathcal{P}_i)$ belongs to $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$; i.e. $\mathcal{P} = \langle M \rangle$ for some object M of $Spec(X)$;

(b) for any $j \in J$ such that $u_{ij}^*(L_i) \neq 0$, the object $u_{ji*}u_{ij}^*(L_i)$ of C_{U_j} has an associated point; i.e. it has a subobject L_j, which belongs to $Spec(U_j)$;

(c) $\mathcal{P}/Ker(u_j^*) = \mathcal{P}_j$ belongs to $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(U_j)$ for all j such that $Ker(u_j^*) \subseteq \mathcal{P}$.

Proof. (a)⇒(c) follows from 3.2(ii) and the functoriality of **Spec** (hence $\mathbf{Spec}_{\mathfrak{t}}^{1,1}$) with respect to localizations.

(c)⇒(a) follows from 7.1.

(b)⇒(c). Suppose that $Ker(u_j^*) \subseteq \mathcal{P}$, or, equivalently, $u_{ij}^*(L_i) \neq 0$. Then $\mathcal{P}_j = \mathcal{P}/Ker(u_j^*)$ is a point of $\mathbf{Spec}^1(U_j)$. Let $U_i \xleftarrow{u_{ij}} U_i \cap U_j = U_{ij} \xrightarrow{u_{ji}} U_j$ be the canonical embeddings. Since $L_i \in Spec(U_i)$ and $u_{ij}^*(L_i) \neq 0$, it follows that $u_{ij}^*(L_i) \in Spec(U_{ij})$.

Let L_j be a nonzero subobject of $u_{ji*}u_{ij}^*(L_i)$, and $L_j \in Spec(U_j)$. Then $u_{ji}^*(L_j)$ is a nonzero subobject of $u_{ij}^*(L_i)$. Therefore, since $u_{ij}^*(L_i)$ belongs to $Spec(U_{ij})$, the objects $u_{ji}^*(L_j)$ and $u_{ij}^*(L_i)$ are equivalent. Notice that, it follows from $\mathcal{P}_i = \langle L_i \rangle$ that $\mathcal{P}_i/Ker(u_{ij}^*) = \langle u_{ij}^*(L_i) \rangle$. But, $\mathcal{P}_i/Ker(u_{ij}^*) = \mathcal{P}_j/Ker(u_{ji}^*) = \mathcal{P}/Ker(u_{ji}^*u_j^*)$ and, by the argument above, $\langle u_{ij}^*(L_i) \rangle = \langle u_{ji}^*(L_j) \rangle$. Together with the fact that L_j is an object of $Spec(U_j)$, this shows that $\mathcal{P}_j = \langle L_j \rangle$.

(a)⇒(b). Suppose that $\mathcal{P} = {u_i^*}^{-1}(\mathcal{P}_i)$ belongs to $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$; i.e. $\mathcal{P} = \langle M \rangle$ for some object M of *Spec*(X). Let $\widetilde{L}_i$ be a $\mathcal{P}$-torsion free object of C_X such that $u_i^*(\widetilde{L}_i) \simeq L_i$. The relation $u_i^*(M) \succ L_i$ means that there exists a diagram $M^{\oplus n} \xleftarrow{\mathfrak{j}} K \xrightarrow{\mathfrak{e}} L_1 \xrightarrow{g} \widetilde{L}_i$ in, which $\mathfrak{e}$ is an epimorphism, the arrows $\mathfrak{j}$ and g are nonzero monomorphisms; in particular, $M \succ L_1$. Notice that $L_1 \succ M$, i.e. M and L_1 are equivalent. In fact, $u_i^*(L_1)$ is a nonzero subobject of L_i. Since the latter belongs to *Spec*(U_i), they are equivalent. Therefore, $u_i^*(L_1)$ is equivalent to $u_i^*(M)$. The relation $u_i^*(L_1) \succ u_i^*(M)$ is expressed by a diagram $L_1^{\oplus m} \xleftarrow{\mathfrak{j}'} \widetilde{K} \xrightarrow{\mathfrak{e}'} M_1 \xrightarrow{h} M$ in which $\mathfrak{e}'$ is an epimorphism and $\mathfrak{j}'$ and h are nonzero monomorphisms. Since $M \in$ *Spec*(X), M_1 is equivalent to M, hence the relation $L_1 \succ M_1$ which is explicit in the diagram above, implies that $L_1 \succ M$. Thus $L_1 \in$ *ObSpec*(X).

By the functoriality of *Spec* with respect to exact localizations, $u_j^*(L_1) = L_j$ belongs to *Spec*(U_j). Since L_1 is $\mathcal{P}$-torsion free, the adjunction arrow $L_j = u_j^*(L_1) \longrightarrow u_{ji*}u_{ji}^*u_j^*(L_1)$ is a monomorphism. On the other hand,

$$u_{ji*}u_{ji}^*u_j^*(L_1) \simeq u_{ji*}u_{ij}^*u_i^*(L_1) \longrightarrow u_{ji*}u_{ij}^*u_i^*(\widetilde{L}_i) \simeq u_{ji*}u_{ij}^*(L_i), \tag{5}$$

where the arrow in the middle is the image of the monomorphism $L_1 \longrightarrow \widetilde{L}_i$. Since all functors in the diagram (5) are left exact, this arrow is a monomorphism. Altogether gives the desired monomorphism $L_j \longrightarrow u_{ji*}u_{ij}^*(L_i)$. ∎

7.4.1. Proposition. *Let C_X be an abelian category and $\mathfrak{U} = \{U_i \xrightarrow{u_i} X \mid i \in J\}$ a finite set of continuous morphisms such that $\{C_X \xrightarrow{u_i^*} C_{U_i} \mid i \in J\}$ is a conservative family of exact localizations.*

(a) The morphisms $U_{ij} = U_i \cap U_j \xrightarrow{u_{ij}} U_i$ are continuous for all $i, j \in J$.

(b) Let L_i be an object of Spec(U_i); i.e. $[L_i]_{\mathfrak{c}} \in \mathbf{Spec}(U_i)$ and L_i is $\langle L_i \rangle$-torsion free. The following conditions are equivalent:

(i) $L_i \simeq u_i^*(L)$ for some $L \in Spec(X)$;

(ii) for any $j \in J$ such that $u_{ij}^*(L_i) \neq 0$, the object $u_{ji*}u_{ij}^*(L_i)$ of C_{U_j} has an associated point; i.e. it has a subobject L_{ij} which belongs to Spec(U_j).

Proof. The assertion follows from 7.4. ∎

7.5. Example. Let C_X be the category of quasi-coherent sheaves on a quasi-compact quasi-separated scheme $\mathbf{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. Let $\{\mathcal{U}_i \hookrightarrow \mathcal{X} \mid i \in J\}$ be an affine cover and C_{U_i} the category of quasi-coherent sheaves on $(\mathcal{U}_i, \mathcal{O}_{\mathcal{U}_i})$. Then all morphisms $U_i \cap U_j \longrightarrow U_i$ are continuous and the equivalent conditions (a), (b), (c) hold for every point $\mathcal{P}_i \in \mathbf{Spec}^{1,1}_{\mathfrak{t}}(U_i)$. This reflects the fact that $\mathbf{Spec}(U_i)$ is naturally identified with $\mathcal{U}_i$ and is an open subset of the spectrum $\mathbf{Spec}(X) \simeq \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$. It follows from 7.1 that

$\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X) = \bigcup_{i \in J} \mathbf{Spec}_{\mathfrak{t}}^{1,1}(U_i)$. So, Proposition 7.4 becomes trivial in the case of commutative schemes. It is non-trivial and meaningful in the case of noncommutative schemes, even in the case of D-schemes.

7.6. Example: simple holonomic D-modules. Let C_X be the category of holonomic D-modules on a smooth quasi-compact scheme $\mathbf{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. Let $\{\mathcal{U}_i \hookrightarrow \mathcal{X} \mid i \in J\}$ be an affine cover of $\mathbf{X}$, and let C_{U_i} be the category of holonomic D-modules on the affine subscheme $(\mathcal{U}_i, \mathcal{O}_{\mathcal{U}_i})$. Then all morphisms $U_i \cap U_j \longrightarrow U_i$ are continuous and the equivalent conditions (a), (b), (c) hold for every simple object L_i of C_{U_i}. The latter is due to the fact that direct and inverse image functors of open immersions preserve holonomicity. Thanks to the fact that all holonomic D-modules are of finite length, the 'space' X (i.e. the category C_X) has the Gabriel-Krull dimension zero, hence elements of $\mathbf{Spec}(X)$ are in a bijective correspondence with isomorphism classes of holonomic simple objects. Therefore, it follows from 7.4 and 7.1 that $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X) = \bigcup_{i \in J} \mathbf{Spec}_{\mathfrak{t}}^{1,1}(U_i)$. Thus, the problem of the description of simple holonomic modules on a smooth quasi-compact scheme is local: it can be reduced to the affine case.

Consider, for instance, the cover of the flag variety G/B of a reductive algebraic connected group G over $\mathbb{C}$ (or any other algebraically closed field of zero characteristic) by translations U_w, $w \in W$, of the big Schubert cell (here, as usual, W denotes the Weyl group of G). Then for any $w \in W$, the category C_{U_w} is equivalent to the category $A_n - mod$ of left modules over the Weyl algebra A_n. So the problem of a classification of holonomic D-modules on G/B is reduced to the problem of classification of holonomic D-modules on the affine n-dimensional space $\mathbb{A}^n$, that is holonomic A_n-modules.

7.7. Proposition. *Let C_X have the property (sup), and let $\{\mathcal{T}_i \mid i \in J\}$ be a finite set of Serre subcategories of the category C_X such that $\bigcap_{i \in J} \mathcal{T}_i = 0$. Suppose that $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X/\mathcal{T}_i) = \mathbf{Spec}^-(X/\mathcal{T}_i)$ for all $i \in J$. Then $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X) = \mathbf{Spec}^-(X)$; i.e. the map*

$$\mathbf{Spec}(X) \longrightarrow \mathbf{Spec}^-(X), \quad \mathcal{P} \longmapsto \langle \mathcal{P} \rangle,$$

is an isomorphism.

Proof. By 7.1, $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$ coincides with

$$\{\mathcal{P} \in \mathbf{Spec}^-(X) \mid \mathcal{P}/\mathcal{T}_i \in \mathbf{Spec}_{\mathfrak{t}}^{1,1}(X/\mathcal{T}_i) \text{ if } \mathcal{T}_i \subseteq \mathcal{P}\}.$$

By 9.5, $\mathbf{Spec}^-(X) = \bigcup_{i \in J} \mathbf{Spec}^-(X/\mathcal{T}_i)$. In particular,

$$\mathbf{Spec}^-(X) = \{\mathcal{P} \in \mathbf{Spec}^-(X) \mid \mathcal{P}/\mathcal{T}_i \in \mathbf{Spec}^-(X/\mathcal{T}_i) \text{ if } \mathcal{T}_i \subseteq \mathcal{P}\}.$$

Hence the assertion. ■

7.8. Example. Let C_X be the category of quasi-coherent sheaves on a smooth quasi-compact scheme $\mathcal{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ of dimension n. Let $C_{\mathfrak{A}}$ be the category of D-modules on $\mathcal{X}$ and $C_{\mathfrak{A}} \xrightarrow{u_*} C_X$ the pull-back functor corresponding to the embedding of the structure sheaf $\mathcal{O}_{\mathcal{X}}$ into the sheaf $\mathcal{D}_{\mathcal{X}}$ of differential operators on $\mathcal{X}$. Let $\mathfrak{U} = \{U_i \xrightarrow{u_i} \mathcal{X} \mid i \in J\}$ be an affine finite cover of $\mathcal{X}$ such that each U_i is isomorphic to the affine space $\mathbb{A}^n$. Then $\mathbf{Spec}^-(\mathfrak{A}) = \bigcup_{i\in J} \mathbf{Spec}^-(|A_n - mod|)$, where A_n is the n-th Weyl algebra.

7.8.1. The case of a curve. Suppose $n = 1$, i.e. $\mathcal{X}$ is a curve. Then $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$ and $\mathbf{Spec}^-(X)$ coincide.

In fact, the equality holds when C_X is the category of left modules over the first Weyl algebra A_1. This follows from the fact that A_1 has Gabriel-Krull dimension one, hence $\mathbf{Spec}^-(X)$ consists of closed points and one generic point.

In the general case, the equality follows from this and 7.7.

7.8.2. Corollary. *Let $C_{\mathfrak{A}}$ be the category $U(sl_2) - mod_0$ of $U(sl_2)$-modules with the trivial central character. Then* $\mathbf{Spec}^-(\mathfrak{A}) = \mathbf{Spec}(\mathfrak{A})$.

Proof. The fact is true if the base field is of positive characteristic, because then $U(sl_2)$ is finite-dimensional over its center.

Suppose that the base field is of characteristic zero. The category $C_{\mathfrak{A}} = U(sl_2) - mod_0$ is equivalent to the category $D(\mathbb{P}^1)$ of D-modules on the one-dimensional projective space. The assertion follows from 7.7. ■

8. Reconstruction of quasi-compact schemes.

8.1. Geometric center of a 'space'. Let C_X be an abelian category. Fix a topology τ on $\mathbf{Spec}(X)$. The map $\widetilde{\mathcal{O}}_{X,\tau}$ which assigns to every open subset W of $\mathbf{Spec}(X)$ the center of the quotient category $C_X/\mathcal{S}_W$, where $\mathcal{S}_W = \bigcap_{\mathcal{Q}\in W} \widehat{\mathcal{Q}}$, is a presheaf on $(\mathbf{Spec}(X), \tau)$. Recall that the *center* of the category C_Y is the (commutative) ring of endomorphisms of its identical functor. If C_Y is a category of left modules over a ring R, then the center of C_Y is naturally isomorphic to the center of R.

We denote by $\mathcal{O}_{X,\tau}$ the associated sheaf. The ringed space $((\mathbf{Spec}(X), \tau), \mathcal{O}_{X,\tau})$ is called the *geometric center* of the 'space' X. If τ is the Zariski topology, then we write simply $(\mathbf{Spec}(X), \mathcal{O}_X)$ and call this ringed space the *Zariski geometric center* of X. Recall that open sets in Zariski topology are sets of the form $U(\mathbb{T}) = \{\mathcal{Q} \in \mathbf{Spec}(X) \mid \mathcal{Q} \not\subseteq \mathbb{T}\}$, where $\mathbb{T}$ is an arbitrary bireflective topologizing subcategory of C_X. Recall that 'bireflective' means that the inclusion functor $\mathbb{T} \hookrightarrow C_X$ has right and left adjoints.

8.2. Commutative schemes which can be reconstructed from their categories of quasi-coherent or coherent sheaves. Let $\mathbf{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a ringed topological space and $\mathbf{U} = (\mathcal{U}, \mathcal{O}_{\mathcal{U}}) \xrightarrow{\mathfrak{j}} (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ an open immersion. Then the morphism $\mathfrak{j}$ has an exact inverse image functor $\mathfrak{j}^*$ and a fully faithful direct image functor $\mathfrak{j}_*$. This implies that $Ker(\mathfrak{j}^*)$ is a Serre subcategory of the category $\mathcal{O}_X - \mathcal{M}od$ of sheaves of $\mathcal{O}_X$-modules and the unique functor

$$\mathcal{O}_X - \mathcal{M}od/Ker(\mathfrak{j}^*) \longrightarrow \mathcal{O}_{\mathcal{U}} - \mathcal{M}od$$

induced by $\mathfrak{j}^*$ is an equivalence of categories [Gab, III.5].

Suppose now that $\mathbf{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a scheme and $Qcoh_{\mathbf{X}}$ the category of quasi-coherent sheaves on $\mathbf{X}$. The inverse image functor $\mathfrak{j}^*$ of the immersion $\mathfrak{j}$ maps quasi-coherent sheaves to quasi-coherent sheaves. Let u^* denote the functor $Qcoh_{\mathbf{X}} \longrightarrow Qcoh_{\mathbf{U}}$ induced by $\mathfrak{j}^*$. The functor u^*, being the composition of the exact full embedding of $Qcoh_{\mathbf{X}}$ into $\mathcal{O}_X - \mathcal{M}od$ and the exact functor $\mathfrak{j}^*$, is exact; hence it is represented as the composition of an exact localization $Qcoh_{\mathbf{X}} \longrightarrow Qcoh_{\mathbf{X}}/Ker(u^*)$ and a uniquely defined exact functor $Qcoh_{\mathbf{X}}/Ker(u^*) \longrightarrow Qcoh_{\mathbf{U}}$. If the direct image functor $\mathfrak{j}_*$ of the immersion $\mathfrak{j}$ maps quasi-coherent sheaves to quasi-coherent sheaves, then it induces a fully faithful functor $Qcoh_{\mathbf{U}} \xrightarrow{u_*} Qcoh_{\mathbf{X}}$ which is right adjoint to u^*. In particular, the canonical functor $Qcoh_{\mathbf{X}}/Ker(u^*) \longrightarrow Qcoh_{\mathbf{U}}$ is an equivalence of categories.

The reconstruction of a scheme $\mathbf{X}$ from the category $Qcoh_{\mathbf{X}}$ of quasi-coherent sheaves on $\mathbf{X}$ is based on the existence of an affine cover $\{\mathbf{U}_i \xrightarrow{u_i} \mathbf{X} \mid i \in J\}$ such that the canonical functors $Qcoh_{\mathbf{X}}/Ker(u_i^*) \longrightarrow Qcoh_{\mathbf{U}_i}$, $i \in J$, are category equivalences. It follows from the discussion above (or from [GZ, I.2.5.2]) that this is guaranteed if the inverse image functor $Qcoh_{\mathbf{X}} \xrightarrow{u_i} Qcoh_{\mathbf{U}_i}$ has a fully faithful right adjoint.

8.2.1. Proposition. *Let $\mathbf{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a quasi-compact scheme such that there exists an affine cover $\{\mathbf{U}_i \xrightarrow{u_i} \mathbf{X} \mid i \in J\}$ such that the canonical functors*

$$Qcoh_{\mathbf{X}}/Ker(u_i^*) \longrightarrow Qcoh_{\mathbf{U}_i}, \quad i \in J,$$

are category equivalences. Then

(a) The scheme $\mathbf{X}$ is isomorphic to the Zariski geometric center $((\mathbf{Spec}(X), \tau_{\mathfrak{z}}), \mathcal{O}_X)$ of the 'space' X, where $C_X = Qcoh_{\mathbf{X}}$, $\tau_{\mathfrak{z}}$ is the Zariski topology on X and $\mathcal{O}_X$ is the sheaf of commutative rings defined in 7.4.

(b) For every open immersion $\mathbf{U} \xrightarrow{u} \mathbf{X}$ such that $Qcoh_{\mathbf{X}}/Ker(u^*) \longrightarrow Qcoh_{\mathbf{U}}$ is a category equivalence, $Ker(u^*)$ is a Serre subcategory of $Qcoh_{\mathbf{X}}$. In particular, $Ker(u_i^*)$ is a Serre subcategory for all $i \in J$.

Proof. (a) Set $C_{U_i} = Qcoh_{\mathbf{U}_i}$ and $\mathcal{T}_i = Ker(u_i^*)$. Since $\mathbf{X}$ is quasi-compact, we can and will assume that J is finite. The condition that $\{\mathbf{U}_i \xrightarrow{u_i} \mathbf{X} \mid i \in J\}$ is a cover means precisely that $\bigcap_{i \in J} \mathcal{T}_i = 0$.

(a1) Let x be a point of the underlying space $\mathcal{X}$ of the scheme $\mathbf{X}$. Let $\mathfrak{I}_{\bar{x}}$ be the defining ideal of the closure $\bar{x}$ of the point x and $\mathcal{M}_{\bar{x}}$ the quotient sheaf $\mathcal{O}/\mathfrak{I}_{\bar{x}}$. Set $J_x = \{i \in J \mid \mathcal{M}_{\bar{x}} \notin Ob\mathcal{T}_i\}$. We claim that $\mathcal{Q}_x = [\mathcal{M}_{\bar{x}}]$ is an element of $\mathbf{Spec}(X)$.

For every $i \in J_x$, the object $u_i^*(\mathcal{M}_{\bar{x}})$ of the category C_{U_i} belongs to $Spec(U_i)$ and $Spec(U_i)$, because C_{U_i} is (equivalent to) the category of modules over a ring. Therefore, $[u_i^*(\mathcal{Q}_x)] = [u_i^*(\mathcal{M}_{\bar{x}})]$ is an element of $\mathbf{Spec}(U_i)$. By 7.1, $\mathcal{Q}_x \in \mathbf{Spec}(X)$.

(a2) Conversely, let $\mathcal{Q}$ be an element of $\mathbf{Spec}(X)$. Let $\mathcal{Q} \not\subseteq \mathcal{T}_i$, or, equivalently, $\mathcal{T}_i \subseteq \widehat{\mathcal{Q}}$. By the functoriality of $\mathbf{Spec}(X)$ under exact localizations, $[u_i^*(\mathcal{Q})]$ is an element of $\mathbf{Spec}(U_i)$. Since U_i is affine, $\mathbf{Spec}(U_i)$ is in bijective correspondence with the underlying space $\mathcal{U}_i$ of the subscheme $\mathbf{U}_i = (\mathcal{U}_i, \mathcal{O}_{\mathcal{U}_i})$; in particular, to the element $[u_i^*(\mathcal{Q})]$ there corresponds a point x of $\mathcal{U}_i$ which we identify with its image in $\mathcal{X}$. Notice that the point x does not depend on the choice of $i \in J_{\widehat{\mathcal{Q}}} = \{j \in J \mid \mathcal{T}_j \subseteq \widehat{\mathcal{Q}}\}$. This gives a map $\mathbf{Spec}(X) \longrightarrow \mathcal{X}$ which is inverse to the map $\mathcal{X} \longrightarrow \mathbf{Spec}(X)$ constructed in (a1) above. These maps are homeomorphisms in the case if the cover consists of one element, i.e. the scheme is affine. The general case follows from the commutative diagrams

$$\begin{array}{ccc} \mathbf{Spec}(U_i) & \xrightarrow{\ \sim\ } & \mathcal{U}_i \\ \big\downarrow & & \big\downarrow \\ \mathbf{Spec}(X) & \longrightarrow & \mathcal{X}, \qquad i \in J. \end{array} \tag{5}$$

in which vertical arrows are open immersions and the upper horizontal arrow is a homeomorphism; hence the lower horizontal arrow is a homeomorphism.

(a3) The diagrams (5) extend to the commutative diagrams of ringed spaces

$$\begin{array}{ccc} (\mathbf{Spec}(U_i), \mathcal{O}_{U_i}) & \xrightarrow{\ \sim\ } & (\mathcal{U}_i, \mathcal{O}_{\mathcal{U}_i}) \\ \big\downarrow & & \big\downarrow \\ (\mathbf{Spec}(X), \mathcal{O}_X) & \longrightarrow & (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \qquad i \in J. \end{array} \tag{6}$$

in which $\mathbf{Spec}^0_{\mathfrak{c}}(U_i), \mathcal{O}_{U_i})$ and $(\mathbf{Spec}^0_{\mathfrak{c}}(X), \mathcal{O}_X)$ are Zariski geometric centra of resp. U_i and X, vertical arrows are open immersions and upper horizontal arrow is an isomorphism. Therefore the lower horizontal arrow is an isomorphism.

(b) Let x be a point of $\mathcal{U}_i$, which we identify with its image in the underlying space $\mathcal{X}$ of the scheme $\mathbf{X}$. Let $\mathcal{Q}_x$ denote the corresponding element of $\mathbf{Spec}(X)$. It follows that $\mathcal{Q}_x \not\subseteq \mathcal{T}_i$, or, equivalently, $\mathcal{T}_i \subseteq \widehat{\mathcal{Q}}_x$. Thus, $\mathcal{T}_i \subseteq \bigcap_{x \in \mathcal{U}_i} \widehat{\mathcal{Q}}_x$. We claim that $\mathcal{T}_i = \bigcap_{x \in \mathcal{U}_i} \widehat{\mathcal{Q}}_x$. In fact, if M is an object of $C_X - \mathcal{T}_i$, then $u_i^*(M) \neq 0$. Since the category C_{U_i} is equivalent to the category of modules over a ring, every nonzero object of C_{U_i} has a non-empty support. In particular, there is a point $x \in \mathcal{U}_i$ which belongs to the support of $u_i^*(M)$. The latter means that $u_i^*(M) \notin Ob(\widehat{\mathcal{Q}}_x/\mathcal{T}_i)$, or, what is the same, $M \notin Ob\widehat{\mathcal{Q}}_x$.

Since each $\widehat{\mathcal{Q}}_x$ is a Serre subcategory and the intersection of any family of Serre subcategories is a Serre subcategory, $\mathcal{T}_i$ is a Serre subcategory. ∎

8.3. Remarks. (i) A comment to the assertion 8.2.1(b): if C_Y is a Grothendieck category and $\mathcal{T}$ is a Serre subcategory than the localization functor $C_Y \longrightarrow C_Y/\mathcal{T}$ has a right adjoint.

(ii) The quasi-compactness of the scheme in 8.2.1 is an essential requirement. If the scheme is not quasi-compact, the spectrum $\mathbf{Spec}(X)$ might be not sufficiently big to reconstruct the underlying space. This was observed by O. Gabber who produced an example of a scheme which is not isomorphic to the ringed space $(\mathbf{Spec}(X), \mathcal{O}_X)$ associated with its category of quasi-coherent sheaves.

9. The spectra related to coreflective topologizing subcategories.

For a svelte abelian category C_X, we denote by $\mathfrak{T}_\mathfrak{c}(X)$ (resp. by $\mathfrak{Th}_\mathfrak{c}(X)$) the preorder of all coreflective topologizing (resp. thick) subcategories of C_X.

9.1. The spectra $\mathbf{Spec}_\mathfrak{c}^0(X)$ and $\mathbf{Spec}_\mathfrak{c}^1(X)$. Elements of the spectrum $\mathbf{Spec}_\mathfrak{c}^1(X)$ are coreflective thick subcategories $\mathcal{P}$ such that the intersection $\mathcal{P}^\mathfrak{c}$ of all coreflective topologizing subcategories properly containing $\mathcal{P}$ contains $\mathcal{P}$ properly too. The spectrum $\mathbf{Spec}_\mathfrak{c}^0(X)$ is formed by coreflective topologizing subcategories $\mathcal{Q}$ of C_X such that the union ${}^\mathfrak{c}\widehat{\mathcal{Q}}$ of all coreflective subcategories of C_X which do not contain $\mathcal{Q}$ is a coreflective thick subcategory. The canonical injective morphism $\mathbf{Spec}_\mathfrak{c}^0(X) \longrightarrow \mathbf{Spec}_\mathfrak{c}^1(X)$ maps $\mathcal{Q}$ to ${}^\mathfrak{c}\widehat{\mathcal{Q}}$. It follows from the definition of ${}^\mathfrak{c}\widehat{\mathcal{Q}}$ that every coreflective topologizing subcategory properly containing ${}^\mathfrak{c}\widehat{\mathcal{Q}}$ contains $\mathcal{Q}$, hence the smallest coreflective subcategory $[{}^\mathfrak{c}\widehat{\mathcal{Q}}, \mathcal{Q}]_\mathfrak{c}$ containing $\mathcal{Q}$ and ${}^\mathfrak{c}\widehat{\mathcal{Q}}$ coincides with ${}^\mathfrak{c}\widehat{\mathcal{Q}}^\mathfrak{c}$. The injectivity of the map $\mathcal{Q} \longmapsto {}^\mathfrak{c}\widehat{\mathcal{Q}}$ is a consequence of the following fact which is going to be used more than once.

9.1.1. Lemma. *If $\mathcal{Q}_1, \mathcal{Q}_2$ are elements of $\mathcal{T}_\mathfrak{c}(X)$, then $\mathcal{Q}_1 \subseteq \mathcal{Q}_2$ iff ${}^\mathfrak{c}\widehat{\mathcal{Q}}_1 \subseteq {}^\mathfrak{c}\widehat{\mathcal{Q}}_2$.*

Proof. The argument is the same as in 4.1. ∎

9.1.2. Proposition. *Let C_X be an abelian category with the property (sup).*
(a) The canonical morphism

$$\mathbf{Spec}_\mathfrak{c}^0(X) \longrightarrow \mathbf{Spec}_\mathfrak{c}^1(X), \quad \mathcal{Q} \longmapsto {}^\mathfrak{c}\widehat{\mathcal{Q}}, \tag{1}$$

is an isomorphism.

(b) There are natural injective morphisms

$$\mathbf{Spec}(X) \longrightarrow \mathbf{Spec}^0_{\mathfrak{c}}(X) \quad \text{and} \quad \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X) \longrightarrow \mathbf{Spec}^1_{\mathfrak{c}}(X) \tag{2}$$

such that the diagram

$$\begin{array}{ccc} \mathbf{Spec}(X) & \longrightarrow & \mathbf{Spec}^0_{\mathfrak{c}}(X) \\ \wr\downarrow & & \downarrow\wr \\ \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X) & \longrightarrow & \mathbf{Spec}^1_{\mathfrak{c}}(X) \end{array}$$

commutes.

(c) If C_X has enough objects of finite type, then the morphisms (2) are isomorphisms.

Proof. (a) For every $\mathcal{P} \in \mathbf{Spec}^1_{\mathfrak{c}}(X)$, the intersection $\mathcal{P}^{\mathfrak{c}} \cap \mathcal{P}^{\perp}$ is nonzero, because $\mathcal{P}$ is a Serre subcategory. The claim is that the coreflective topologizing subcategory $[\mathcal{P}_*]_{\mathfrak{c}}$ spanned by the subcategory $\mathcal{P}_* = \mathcal{P}^{\mathfrak{c}} \cap \mathcal{P}^{\perp}$ belongs to the spectrum $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ and ${}^{\mathfrak{c}}\widehat{[\mathcal{P}_*]_{\mathfrak{c}}} = \mathcal{P}$. The map

$$\mathbf{Spec}^1_{\mathfrak{c}}(X) \longrightarrow \mathbf{Spec}^0_{\mathfrak{c}}(X), \quad \mathcal{P} \longmapsto {}^{\mathfrak{c}}\widehat{[\mathcal{P}_*]_{\mathfrak{c}}}, \tag{3}$$

is inverse to the map (1) above.

(a1) Notice that $\langle [\mathcal{P}_*]_{\mathfrak{c}} \rangle = \langle \mathcal{P}_* \rangle$ because a coreflective topologizing subcategory does not contain $[\mathcal{P}_*]_{\mathfrak{c}}$ iff it does not contain $\mathcal{P}_*$. Therefore, our claim is that $\langle \mathcal{P}_* \rangle = \mathcal{P}$.

(a2) If $\mathcal{T}$ is a coreflective topologizing subcategory of C_X which is not contained in $\mathcal{P}$, then $\mathcal{P}_* = \mathcal{P}^{\mathfrak{c}} \cap \mathcal{P}^{\perp} \subseteq \mathcal{T}$.

In fact, if $\mathcal{T} \nsubseteq \mathcal{P}$, then the coreflective topologizing subcategory $\mathcal{T} \bullet \mathcal{P}$ contains $\mathcal{P}$ properly, hence it contains $\mathcal{P}^{\mathfrak{c}}$. Notice that every $\mathcal{P}$-torsion free object of $\mathcal{T} \bullet \mathcal{P}$ belongs to $\mathcal{T}$. In particular, $\mathcal{P}_* \subseteq \mathcal{T}$.

(a3) It follows from (a1) that if $\mathcal{T} \in \mathfrak{T}_{\mathfrak{c}}(X)$ is such that $\mathcal{P}_* \nsubseteq \mathcal{T}$, then $\mathcal{T} \subseteq \mathcal{P}$. This means that $\langle \mathcal{P}_* \rangle \subseteq \mathcal{P}$. On the other hand, $\mathcal{P}_* \nsubseteq \mathcal{P}$ and $\mathcal{P}$ is a Serre subcategory; in particular, it is coreflective and topologizing; hence the inverse inclusion, $\mathcal{P} \subseteq \langle \mathcal{P}_* \rangle$.

(a4) Since the map (1) is injective and has a right inverse, $\mathcal{P} \longmapsto [\mathcal{P}_*]_{\mathfrak{c}}$, it is bijective.

(b) Thanks to the property (sup), a thick subcategory of the category C_X is coreflective iff it is a Serre subcategory. In particular, since elements of $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$ are Serre subcategories, $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X) \subseteq \mathfrak{Th}_{\mathfrak{c}}(X)$. A Serre subcategory $\mathcal{P}$ belongs to $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$ iff the intersection $\mathcal{P}^{\mathfrak{t}}$ of topologizing subcategories properly containing $\mathcal{P}$ contains $\mathcal{P}$ properly. Therefore, $\mathcal{P}^{\mathfrak{c}}$ contains $\mathcal{P}$ properly. The map $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X) \longrightarrow \mathbf{Spec}^1_{\mathfrak{c}}(X)$ is the inclusion.

Let $\mathcal{Q} \in \mathbf{Spec}(X)$, and let $[\mathcal{Q}]_{\mathfrak{c}}$ be the smallest coreflective topologizing subcategory of C_X containing $\mathcal{Q}$. Clearly $[\mathcal{Q}]_{\mathfrak{c}} \nsubseteq \widehat{\mathcal{Q}}$, hence $\widehat{\mathcal{Q}} \subseteq \langle [\mathcal{Q}]_{\mathfrak{c}} \rangle$. On the other hand, if $\mathcal{T}$ is a coreflective topologizing subcategory of C_X such that $[\mathcal{Q}]_{\mathfrak{c}} \nsubseteq \mathcal{T}$, then $\mathcal{Q} \nsubseteq \mathcal{T}$, or, equivalently, $\mathcal{T} \subseteq \widehat{\mathcal{Q}}$. This shows the inverse inclusion, $\langle[\mathcal{Q}]_{\mathfrak{c}}\rangle \subseteq \widehat{\mathcal{Q}}$. The equality $\langle[\mathcal{Q}]_{\mathfrak{c}}\rangle = \widehat{\mathcal{Q}}$, together with the fact that $\widehat{\mathcal{Q}}$ is a Serre subcategory, shows that $[\mathcal{Q}]_{\mathfrak{c}} \in \mathbf{Spec}^0_{\mathfrak{c}}(X)$ for every $\mathcal{Q} \in \mathbf{Spec}(X)$. The map $\mathbf{Spec}(X) \longrightarrow \mathbf{Spec}^0_{\mathfrak{c}}(X)$ assigns to every element $\mathcal{Q}$ of $\mathbf{Spec}(X)$ the coreflective topologizing subcategory $[\mathcal{Q}]_{\mathfrak{c}}$ spanned by $\mathcal{Q}$.

(c) Let $\mathcal{Q}$ be an object of $\mathbf{Spec}^0_{\mathfrak{c}}(X)$. One can see that ${}^{\mathfrak{c}}\widehat{\mathcal{Q}} = \langle M\rangle$ for every object M of $\mathcal{Q} - {}^{\mathfrak{c}}\widehat{\mathcal{Q}}$: the inclusion $\langle M\rangle \subseteq^{\mathfrak{c}} \widehat{\mathcal{Q}}$ is due to the fact that $M \in Ob\mathcal{Q}$ and the inverse inclusion holds because $M \notin Ob^{\mathfrak{c}}\widehat{\mathcal{Q}}$. This implies that $\mathcal{Q} = [M]_{\mathfrak{c}}$ for any object M of $\mathcal{Q} - {}^{\mathfrak{c}}\widehat{\mathcal{Q}}$. In particular, $\mathcal{Q} = [M]_{\mathfrak{c}}$ for any nonzero object M of $\mathcal{Q} \cap {}^{\mathfrak{c}}\widehat{\mathcal{Q}}^{\perp}$.

Suppose that the category C_X has enough objects of finite type, i.e. every nonzero object of C_X has a nonzero subobject of finite type. In particular, any nonzero object of the subcategory $\mathcal{Q} \cap {}^{\mathfrak{c}}\widehat{\mathcal{Q}}^{\perp}$ has a nonzero subobject L. Since L belongs to $\mathcal{Q} \cap {}^{\mathfrak{c}}\widehat{\mathcal{Q}}^{\perp}$ and is nonzero, $[L]_{\mathfrak{c}} = \mathcal{Q}$. We claim that L is an object of $Spec(X)$, which implies that the topologizing subcategory $[L]$ generated by L belongs to $\mathbf{Spec}(X)$.

In fact, let N be a nonzero subobject of L. Then $[N]_{\mathfrak{c}} = \mathcal{Q} = [L]_{\mathfrak{c}}$. In particular, L is an object of the coreflective topologizing subcategory of C_X spanned by N. Objects of the subcategory $[N]_{\mathfrak{c}}$ are precisely objects of the category C_X which are supremums of their subobjects from $[N]$. In particular, L is a supremum of its subobjects from $[N]$. Since subobjects of L which belong to the topologizing subcategory $[N]$ form a filtered system and L is of finite type, it follows that L is isomorphic to one of its subobjects from $[N]$, i.e. $L \in Ob[N]$. This proves that L belongs to $Spec(X)$. ∎

9.2. The spectra $\mathbf{Spec}^i_{\mathfrak{c}}(X)$ and $\mathbf{Spec}^i_{\mathfrak{Se}}(X)$. Recall that $\mathbf{Spec}^i_{\mathfrak{Se}}(X)$, $i = 0, 1$, are the spectra of the preorder $\mathfrak{Se}(X)$ of Serre subcategories of C_X (see 8.7): points of $\mathbf{Spec}^1_{\mathfrak{Se}}(X)$ are Serre subcategories $\mathcal{P}$ of C_X such that the intersection $\mathcal{P}^{\mathfrak{s}}$ of all Serre subcategories of C_X properly containing $\mathcal{P}$ does not coincide with $\mathcal{P}$; and $\mathbf{Spec}^0_{\mathfrak{Se}}(X)$ is formed by Serre subcategories $\mathcal{Q}$ such that the union ${}^{\mathfrak{c}}\widehat{\mathcal{Q}}_{\mathfrak{s}}$ of Serre subcategories not containing $\mathcal{Q}$ is a Serre subcategory.

9.2.1. Proposition. *There are natural injective morphisms*

$$\mathbf{Spec}^i_{\mathfrak{c}}(X) \longrightarrow \mathbf{Spec}^i_{\mathfrak{Se}}(X) \quad i = 0, 1,$$

such that the diagram

$$\begin{array}{ccc} \mathbf{Spec}^0_{\mathfrak{c}}(X) & \longrightarrow & \mathbf{Spec}^0_{\mathfrak{Se}}(X) \\ \wr\downarrow & & \downarrow \\ \mathbf{Spec}^1_{\mathfrak{c}}(X) & \longrightarrow & \mathbf{Spec}^1_{\mathfrak{Se}}(X) \end{array} \qquad (1)$$

commutes.

Proof. The spectrum $\mathbf{Spec}^1_{\mathfrak{c}}(X)$ is contained in the spectrum $\mathbf{Spec}^1_{\mathfrak{Se}}(X)$, because if $\mathcal{P}^{\mathfrak{c}} \neq \mathcal{P}$, then $(\mathcal{P}^{\mathfrak{c}})^-$ is the smallest Serre subcategory properly containing $\mathcal{P}$, hence

$\mathcal{P}$ belongs to $\mathbf{Spec}^1_{\mathfrak{S}\mathfrak{e}}(X)$. The map $\mathbf{Spec}^0_{\mathfrak{c}}(X) \longrightarrow \mathbf{Spec}^0_{\mathfrak{S}\mathfrak{e}}(X)$ assigns to every $\mathcal{Q} \in \mathbf{Spec}^0_{\mathfrak{c}}(X)$ the Serre subcategory $\mathcal{Q}^-$ spanned by $\mathcal{Q}$ (cf. 1.5). ■

9.2.2. Extended spectra. Extended spectra are obtained via adjoining to the original spectra a *marked point*. In the case of $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ and $\mathbf{Spec}^0_{\mathfrak{S}\mathfrak{e}}(X)$, this marked point might be realized as the zero subcategory. In the case of the spectra $\mathbf{Spec}^1_{\mathfrak{c}}(X)$ and $\mathbf{Spec}^1_{\mathfrak{S}\mathfrak{e}}(X)$, the marked point is realized as the empty subcategory which reflects the equalities $\langle 0\rangle = \emptyset = \langle 0\rangle_{\mathfrak{s}}$.

Morphisms between the original spectra determine morphisms between the corresponding extended spectra mapping marked points to marked points. In particular, the commutative diagram (1) extends to the commutative diagram

$$\begin{array}{ccc} \mathbf{Spec}^0_{\mathfrak{c}}(X)_\star & \longrightarrow & \mathbf{Spec}^0_{\mathfrak{S}\mathfrak{e}}(X)_\star \\ \wr\downarrow & & \downarrow \\ \mathbf{Spec}^1_{\mathfrak{c}}(X)_\star & \longrightarrow & \mathbf{Spec}^1_{\mathfrak{S}\mathfrak{e}}(X)_\star \end{array} \qquad (1_\star)$$

9.2.2.1 Proposition. *There are natural maps*

$$\mathbf{Spec}^i_{\mathfrak{S}\mathfrak{e}}(X)_\star \longrightarrow \mathbf{Spec}^i_{\mathfrak{c}}(X)_\star, \quad i = 0, 1,$$

such that the diagram

$$\begin{array}{ccccc} \mathbf{Spec}^0_{\mathfrak{c}}(X)_\star & \longrightarrow & \mathbf{Spec}^0_{\mathfrak{S}\mathfrak{e}}(X)_\star & \longrightarrow & \mathbf{Spec}^0_{\mathfrak{c}}(X)_\star \\ \wr\downarrow & & \downarrow & & \downarrow\wr \\ \mathbf{Spec}^1_{\mathfrak{c}}(X)_\star & \longrightarrow & \mathbf{Spec}^1_{\mathfrak{S}\mathfrak{e}}(X)_\star & \longrightarrow & \mathbf{Spec}^1_{\mathfrak{c}}(X)_\star \end{array} \qquad (2)$$

commutes and the compositions of its horizontal arrows are identical morphisms.

Proof. The map $\mathbf{Spec}^1_{\mathfrak{S}\mathfrak{e}}(X)_\star \longrightarrow \mathbf{Spec}^1_{\mathfrak{c}}(X)_\star$ assigns to each $\mathcal{P} \in \mathbf{Spec}^1_{\mathfrak{S}\mathfrak{e}}(X)$ the Serre subcategory $\langle \mathcal{P}^{\mathfrak{c}} \cap \mathcal{P}^{\perp}\rangle$ if $\mathcal{P}^{\mathfrak{c}} \neq \mathcal{P}$ (i.e. if $\mathcal{P} \in \mathbf{Spec}^1_{\mathfrak{c}}(X)$) and the marked point, $\langle 0\rangle = \emptyset$, if $\mathcal{P}^{\mathfrak{c}} = \mathcal{P}$. The map $\mathbf{Spec}^1_{\mathfrak{S}\mathfrak{e}}(X)_\star \longrightarrow \mathbf{Spec}^1_{\mathfrak{c}}(X)_\star$ is uniquely defined by the commutativity of the right square in the diagram (2). It follows from the (argument of) 9.1.2 that the composition of the lower horizontal arrows in (2) is the identical map. The similar fact for the upper horizontal arrows is a consequence of this and the commutativity of the diagram (2). ■

9.3. Functorial properties of $\mathbf{Spec}^1_{\mathfrak{c}}(X)$ and $\mathbf{Spec}^0_{\mathfrak{c}}(X)$. For any topologizing subcategory $\mathbb{T}$ of the category C_X, we set

$$\begin{aligned} &U^1_{\mathfrak{c}}(\mathbb{T}) = \{\mathcal{P} \in \mathbf{Spec}^1_{\mathfrak{c}}(X) \mid \mathbb{T} \subseteq \mathcal{P}\} \\ &V^1_{\mathfrak{c}}(\mathbb{T}) = \mathbf{Spec}^1_{\mathfrak{c}}(X) - U^1_{\mathfrak{c}}(\mathbb{T}) = \{\mathcal{P} \in \mathbf{Spec}^1_{\mathfrak{c}}(X) \mid \mathbb{T} \nsubseteq \mathcal{P}\}. \\ &U^0_{\mathfrak{c}}(\mathbb{T}) = \{\mathcal{Q} \in \mathbf{Spec}^0_{\mathfrak{c}}(X) \mid \mathcal{Q} \nsubseteq [\mathbb{T}]_{\mathfrak{c}}\} \quad \text{and} \\ &V^0_{\mathfrak{c}}(\mathbb{T}) = \mathbf{Spec}^0_{\mathfrak{c}}(X) - U^0_{\mathfrak{c}}(\mathbb{T}) = \{\mathcal{Q} \in \mathbf{Spec}^0_{\mathfrak{c}}(X) \mid \mathcal{Q} \subseteq [\mathbb{T}]_{\mathfrak{c}}\} \end{aligned} \qquad (1)$$

9.3.1. Proposition. *Let $\mathbb{T}$ be a topologizing subcategory of the category C_X.*

(a) The isomorphism

$$\mathbf{Spec}^0_{\mathfrak{c}}(X) \xrightarrow{\sim} \mathbf{Spec}^1_{\mathfrak{c}}(X), \quad \mathcal{Q} \longmapsto {}^{\mathfrak{c}}\widehat{\mathcal{Q}},$$

(cf. 9.1.2) induces isomorphisms

$$U^0_{\mathfrak{c}}(\mathbb{T}) \xrightarrow{\sim} U^1_{\mathfrak{c}}(\mathbb{T}) \quad \textit{and} \quad V^0_{\mathfrak{c}}(\mathbb{T}) \xrightarrow{\sim} V^1_{\mathfrak{c}}(\mathbb{T}). \tag{2}$$

(b) There are equalities $V^i_{\mathfrak{c}}(\mathbb{T}) = V^i_{\mathfrak{c}}(\mathbb{T}^-)$ and $U^i_{\mathfrak{c}}(\mathbb{T}) = U^i_{\mathfrak{c}}(\mathbb{T}^-)$, $i = 0, 1$.

(c) For every $\mathcal{P} \in V^1_{\mathfrak{c}}(\mathbb{T})$, the intersection $\mathcal{P} \cap \mathbb{T}$ is an element of $\mathbf{Spec}^1_{\mathfrak{c}}(|\mathbb{T}|)$, where $C_{|\mathbb{T}|} = \mathbb{T}$, and the map

$$V^1_{\mathfrak{c}}(\mathbb{T}) \longrightarrow \mathbf{Spec}^1_{\mathfrak{c}}(|\mathbb{T}|), \quad \mathcal{P} \longmapsto \mathcal{P} \cap \mathbb{T}, \tag{3}$$

is an isomorphism. The inverse map is given by $\widetilde{\mathcal{P}} \longmapsto \widetilde{\mathcal{P}}_+$ (see 7.1).

Similarly, the map $\mathcal{Q} \longmapsto \mathcal{Q} \cap \mathbb{T}$ induces an isomorphism $V^0_{\mathfrak{c}}(\mathbb{T}) \longrightarrow \mathbf{Spec}^0_{\mathfrak{c}}(|\mathbb{T}|)$.

(c^{bis}) If $\mathbb{T}$ is coreflective, then the inverse isomorphism, $\mathbf{Spec}^0_{\mathfrak{c}}(|\mathbb{T}|) \longrightarrow V^0_{\mathfrak{c}}(\mathbb{T})$, is given by the identical map.

(d) The maps $\mathcal{P} \longmapsto \mathcal{P}/\mathbb{T}^-$ and $\mathcal{Q} \longmapsto (\mathbb{T}^- \bullet \mathcal{Q} \bullet \mathbb{T}^-)/\mathbb{T}^-$ define injective morphisms resp.

$$U^1_{\mathfrak{c}}(\mathbb{T}) \longrightarrow \mathbf{Spec}^1_{\mathfrak{c}}(X/\mathbb{T}^-) \quad \textit{and} \quad U^0_{\mathfrak{c}}(\mathbb{T}) \longrightarrow \mathbf{Spec}^0_{\mathfrak{c}}(X/\mathbb{T}^-) \tag{4}$$

such that the diagram

$$\begin{tikzcd} U^0_{\mathfrak{c}}(\mathbb{T}) \arrow[r] \arrow[d, "\wr"'] & \mathbf{Spec}^0_{\mathfrak{c}}(X/\mathbb{T}^-) \arrow[d, "\wr"] \\ U^1_{\mathfrak{c}}(\mathbb{T}) \arrow[r] & \mathbf{Spec}^1_{\mathfrak{c}}(X/\mathbb{T}^-) \end{tikzcd} \tag{5}$$

commutes.

Proof. (a) Let $\mathcal{Q} \in U^0_{\mathfrak{c}}(\mathbb{T})$, i.e. $\mathcal{Q} \in \mathbf{Spec}^0_{\mathfrak{c}}(X)$ and $\mathcal{Q} \not\subseteq [\mathcal{T}]_{\mathfrak{c}}$. This means precisely that ${}^{\mathfrak{c}}\widehat{\mathcal{Q}} \in \mathbf{Spec}^0_{\mathfrak{c}}(X)$ and $\mathcal{T} \subseteq^{\mathfrak{c}} \widehat{\mathcal{Q}}$, i.e. $\mathcal{Q}$ is an element of $U^0_{\mathfrak{c}}(\mathbb{T})$ iff ${}^{\mathfrak{c}}\widehat{\mathcal{Q}}$ is an element of $U^1_{\mathfrak{c}}(\mathbb{T})$. The isomorphism $V^0_{\mathfrak{c}}(\mathbb{T}) \xrightarrow{\sim} V^1_{\mathfrak{c}}(\mathbb{T})$ follows from this and the isomorphism $\mathbf{Spec}^0_{\mathfrak{c}}(X) \xrightarrow{\sim} \mathbf{Spec}^1_{\mathfrak{c}}(X)$.

(b) The equalities $V^1_{\mathfrak{c}}(\mathbb{T}) = V^1_{\mathfrak{c}}(\mathbb{T}^-)$ and $U^1_{\mathfrak{c}}(\mathbb{T}) = U^1_{\mathfrak{c}}(\mathbb{T}^-)$ follow from an observation that elements of $\mathbf{Spec}^1_{\mathfrak{c}}(X)$ are Serre subcategories, and if $\mathcal{P}$ is a Serre subcategory, then $\mathbb{T} \subseteq \mathcal{P}$ iff $\mathbb{T}^- \subseteq \mathcal{P}$. The other two equalities follow from these isomorphisms (2) above.

(c) Let $\mathcal{P} \in V^1_{\mathfrak{c}}(\mathbb{T})$, i.e. $\mathcal{P} \in \mathbf{Spec}^1_{\mathfrak{c}}(X)$ and $\mathbb{T} \not\subseteq \mathcal{P}$. The latter implies that $[\mathbb{T}]_{\mathfrak{c}} \bullet \mathcal{P}$ is a coreflective topologizing subcategory of C_X properly containing $\mathcal{P}$. Therefore, it contains $\mathcal{P}^{\mathfrak{c}}$, and we have:

$$\mathcal{P}^{\mathfrak{c}} \cap \mathcal{P}^{\perp} \subseteq ([\mathbb{T}]_{\mathfrak{c}} \bullet \mathcal{P}) \cap \mathcal{P}^{\perp} = [\mathbb{T}]_{\mathfrak{c}} \cap \mathcal{P}^{\perp} \subseteq [\mathbb{T}]_{\mathfrak{c}}.$$

In particular, the intersection $\widetilde{\mathcal{Q}}_{\mathbb{T}} = \mathbb{T} \cap \mathcal{P}^{\mathfrak{c}} \cap \mathcal{P}^{\perp}$ is nonzero. This implies that $\langle \widetilde{\mathcal{Q}}_{\mathbb{T}} \rangle = \mathcal{P}$ (see the argument 9.1.2(c)). Notice that if $\mathcal{S}$ is a coreflective topologizing subcategory of $\mathbb{T}$, then $\mathcal{S}$ coincides with the intersection of $\mathbb{T}$ with the smallest coreflective topologizing subcategory of C_X containing $\mathcal{S}$. Therefore, the union $\langle \widetilde{\mathcal{Q}}_{\mathbb{T}} \rangle_{\mathbb{T}}$ of coreflective topologizing subctegories of $\mathbb{T}$ which do not contain $\widetilde{\mathcal{Q}}_{\mathbb{T}}$ coincides with the intersection $\langle \widetilde{\mathcal{Q}}_{\mathbb{T}} \rangle \cap \mathbb{T}$. Thus, $\langle \widetilde{\mathcal{Q}}_{\mathbb{T}} \rangle_{\mathbb{T}} = \mathcal{P} \cap \mathbb{T}$; in particular, $\mathcal{P} \cap \mathbb{T} \in \mathbf{Spec}^1_{\mathfrak{c}}(|\mathbb{T}|)$ and the corresponding element of $\mathbf{Spec}^0_{\mathfrak{c}}(|\mathbb{T}|)$ is $[\mathcal{P}^{\mathfrak{c}} \cap \mathcal{P}^{\perp}]_{\mathfrak{c}} \cap \mathbb{T}$. In other words, it is obtained from $\mathcal{P}$ by applying the composition of the isomorphism $\mathbf{Spec}^1_{\mathfrak{c}}(X) \xrightarrow{\sim} \mathbf{Spec}^0_{\mathfrak{c}}(X)$ and the intersection with $\mathbb{T}$, i.e. the diagram

$$\begin{array}{ccc} V^0_{\mathfrak{c}}(\mathbb{T}) & \longrightarrow & \mathbf{Spec}^0_{\mathfrak{c}}(|\mathbb{T}|) \\ \wr\downarrow & & \downarrow\wr \\ V^1_{\mathfrak{c}}(\mathbb{T}) & \longrightarrow & \mathbf{Spec}^1_{\mathfrak{c}}(|\mathbb{T}|) \end{array} \qquad (6)$$

whose horizontal arrows are given by $\mathcal{P} \longmapsto \mathcal{P} \cap \mathbb{T}$, commutes. It follows from the argument above that the map $V^0_{\mathfrak{c}}(\mathbb{T}) \longrightarrow \mathbf{Spec}^0_{\mathfrak{c}}(|\mathbb{T}|)$, $\mathcal{Q} \longmapsto \mathcal{Q} \cap \mathbb{T}$, is an isomorphism with the inverse map which assigns to every element $\mathcal{Q}'$ of $\mathbf{Spec}^0_{\mathfrak{c}}(|\mathbb{T}|)$ the coreflective topologizing subcategory $[\mathcal{Q}']_{\mathfrak{c}}$ in C_X spanned by $\mathcal{Q}'$. Therefore the lower horizontal arrow in (6) is an isomorphism too.

(c^{bis}) If $\mathbb{T}$ is coreflective, then every coreflective topologizing subcategory of $\mathbb{T}$ is a coreflective topologizing subcategory of C_X, hence the isomorphism

$$\mathbf{Spec}^0_{\mathfrak{c}}(|\mathbb{T}|) \xrightarrow{\sim} V^0_{\mathfrak{c}}(\mathbb{T}), \quad \mathcal{Q}' \longmapsto [\mathcal{Q}']_{\mathfrak{c}},$$

discussed above becomes an identical map.

(d) Let q^* denote the localization functor $C_X \longrightarrow C_X/\mathbb{T}^-$.

If $\mathcal{P}$ belongs to $U^1_{\mathfrak{c}}(\mathbb{T})$, i.e. $\mathbb{T} \subseteq \mathcal{P} \subsetneq \mathcal{P}^{\mathfrak{c}}$, then

$$(\mathcal{P}/\mathbb{T}^-)^{\mathfrak{c}} = [q^*(\mathcal{P}^{\mathfrak{c}})]_{\mathfrak{c}} = (\mathbb{T}^- \bullet \mathcal{P}^{\mathfrak{c}} \bullet \mathbb{T}^-)/\mathbb{T}^- \supsetneq \mathcal{P}/\mathbb{T}^-.$$

This shows that $\mathcal{P}/\mathbb{T}^-$ is an element of $\mathbf{Spec}^1_{\mathfrak{c}}(X/\mathbb{T}^-)$.

If $\mathcal{Q}$ is the image of the element $\mathcal{P}$ in $U^0_{\mathfrak{c}}(\mathbb{T})$ (see the assertion (a) above), then the coreflective subcategory $[q^*(\mathcal{Q})]_{\mathfrak{c}} = (\mathbb{T}^- \bullet \mathcal{Q} \bullet \mathbb{T}^-)/\mathbb{T}^-$ is the image of $\mathcal{P}/\mathbb{T}^-$ in $\mathbf{Spec}^0_{\mathfrak{c}}(X/\mathbb{T}^-)$. The commutativity of the diagram (5) follows from the definition of its arrows. ∎

9.4. The local property of the spectrum $\mathbf{Spec}^1_{\mathfrak{c}}(X)$.

9.4.1. Proposition. *Let $\{\mathcal{T}_i \mid i \in J\}$ be a set of Serre subcategories of the category C_X such that $\bigcap_{i \in J} \mathcal{T}_i = 0$; and let u^*_i denote the localization functor $C_X \longrightarrow C_X/\mathcal{T}_i$.*

1) The following conditions on $\mathcal{P} \in \mathbf{Spec}^1_{\mathfrak{S}\mathfrak{c}}(X)$ are equivalent:

(a) $\mathcal{P} \in \mathbf{Spec}^1_{\mathfrak{c}}(X)$,

(b) $\bigcap_{i \in J_{\mathcal{P}}} u_i^{*^{-1}}((\mathcal{P}/\mathcal{T}_i)^{\mathfrak{c}}) \nsubseteq \mathcal{P}$, where $J_{\mathcal{P}} = \{j \in J \mid \mathcal{T}_j \subseteq \mathcal{P}\}$, and if $J^{\mathcal{P}} = J - J_{\mathcal{P}} \neq \emptyset$, then $\bigcap_{j \in J^{\mathcal{P}}} \mathcal{T}_j \nsubseteq \mathcal{P}$.

2) The conditions (a) and (b) imply the condition

(c) $\mathcal{P}/\mathcal{T}_i \in \mathbf{Spec}^1_{\mathfrak{c}}(X/\mathcal{T}_i)$ for each $i \in J_{\mathcal{P}}$.

If J is finite, then the conditions (a) and (b) are equivalent to the condition (c).

Proof. 1) Since $\mathcal{P}$ is a Serre subcategory, the condition $\bigcap_{i \in J} \mathcal{T}_i^- = 0$ implies that $J_{\mathcal{P}} = \{i \in J \mid \mathcal{T}_i \subseteq \mathcal{P}\}$ is not empty.

In fact, if $\mathcal{T}_i \nsubseteq \mathcal{P}$ for all $i \in J$, then $\mathcal{T}_i \bullet \mathcal{P} \supseteq \mathcal{P}^{\mathfrak{c}}$, hence $\mathcal{P}^{\mathfrak{c}} \subseteq \bigcap_{i \in J}(\mathcal{T}_i \bullet \mathcal{P})$. But, by C1.2.1, $\bigcap_{i \in J}(\mathcal{T}_i \bullet \mathcal{P}) = \big(\bigcap_{i \in J} \mathcal{T}_i\big) \bullet \mathcal{P} = 0 \bullet \mathcal{P} = \mathcal{P}$, hence $\mathcal{P}^{\mathfrak{c}} = \mathcal{P}$, which contradicts to the assumption that $\mathcal{P} \in \mathbf{Spec}^1_{\mathfrak{c}}(X)$, i.e. $\mathcal{P} \subsetneq \mathcal{P}^{\mathfrak{c}}$.

2) Notice that if $\mathcal{S}$ is a Serre subcategory, and $\mathbb{T}$ a subcategory of C_X closed under taking subquotients, then $\mathbb{T} \nsubseteq \mathcal{S}$ iff $\mathbb{T} \cap \mathcal{S}^{\perp} \neq 0$, because an object M of C_X does not belong to $\mathcal{S}$ iff it has a nonzero $\mathcal{S}$-torsion free subquotient.

In particular, the condition (b) above can be written as follows:

(b) $\bigcap_{i \in J_{\mathcal{P}}} u_i^{*^{-1}}((\mathcal{P}/\mathcal{T}_i)^{\mathfrak{c}}) \bigcap \mathcal{P}^{\perp} \neq 0$, and $\big(\bigcap_{j \in J^{\mathcal{P}}} \mathcal{T}_j\big) \bigcap \mathcal{P}^{\perp} \neq 0$, if $J^{\mathcal{P}} = J - J_{\mathcal{P}} \neq \emptyset$.

$(a) \Rightarrow (b)$. Let $\mathcal{P} \in \mathbf{Spec}^1_{\mathfrak{c}}(X)$, i.e. $\mathcal{P} \neq \mathcal{P}^{\mathfrak{c}}$.

If $i \in J_{\mathcal{P}}$, that is $\mathcal{T}_i \subseteq \mathcal{P}$, then $u_i^{*^{-1}}((\mathcal{P}/\mathcal{T}_i)^{\mathfrak{c}}) = \mathcal{T}_i \bullet \mathcal{P}^{\mathfrak{c}} \bullet \mathcal{T}_i$.

Since $\mathcal{T}_i \subseteq \mathcal{P}$, the intersection $(\mathcal{T}_i \bullet \mathcal{P}^{\mathfrak{c}} \bullet \mathcal{T}_i) \bigcap \mathcal{P}^{\perp}$ coincides with $(\mathcal{T}_i \bullet \mathcal{P}^{\mathfrak{c}}) \bigcap \mathcal{P}^{\perp}$. Therefore

$$\begin{gathered}\bigcap_{i \in J_{\mathcal{P}}} u_i^{*^{-1}}((\mathcal{P}/\mathcal{T}_i)^{\mathfrak{c}}) \bigcap \mathcal{P}^{\perp} = \bigcap_{i \in J_{\mathcal{P}}} ((\mathcal{T}_i \bullet \mathcal{P}^{\mathfrak{c}}) \bigcap \mathcal{P}^{\perp}) = \\ \big(\bigcap_{i \in J_{\mathcal{P}}} (\mathcal{T}_i \bullet \mathcal{P}^{\mathfrak{c}})\big) \bigcap \mathcal{P}^{\perp} = \big(\big(\bigcap_{i \in J_{\mathcal{P}}} \mathcal{T}_i\big) \bullet \mathcal{P}^{\mathfrak{c}}\big) \bigcap \mathcal{P}^{\perp} \supseteq \mathcal{P}^{\mathfrak{c}} \bigcap \mathcal{P}^{\perp} \neq 0.\end{gathered} \tag{1}$$

Here we used the equality $\bigcap_{i \in J_{\mathcal{P}}} (\mathcal{T}_i \bullet \mathcal{P}^{\mathfrak{c}}) = \big(\bigcap_{i \in J_{\mathcal{P}}} \mathcal{T}_i\big) \bullet \mathcal{P}^{\mathfrak{c}}$ which is due, by C1.2.1, to the fact that the subcategory $\mathcal{P}^{\mathfrak{c}}$ is coreflective.

Notice that if $J_{\mathcal{P}} = J$, that is $\mathcal{T}_i \subseteq \mathcal{P}$ for all $j \in J$, then $\bigcap_{i \in J_{\mathcal{P}}} \mathcal{T}_i = 0$, hence the last inclusion in (1) can be replaced by the equality, i.e. the intersection $\bigcap_{i \in J_{\mathcal{P}}} u_i^{*^{-1}}((\mathcal{P}/\mathcal{T}_i)^{\mathfrak{c}}) \bigcap \mathcal{P}^{\perp}$ coincides with $\mathcal{P}^{\mathfrak{c}} \cap \mathcal{P}^{\perp}$.

Suppose that $J^{\mathcal{P}} = J - J_{\mathcal{P}}$ is non-empty. If $j \in J^{\mathcal{P}}$, that is $\mathcal{T}_j$ is not contained in $\mathcal{P}$, then $\mathcal{T}_j \bullet \mathcal{P}$ is a coreflective topologizing subcategory properly containing both $\mathcal{T}_j$ and $\mathcal{P}$, hence properly containing $\mathcal{P}$. Therefore $\mathcal{P}^{\mathfrak{c}} \subseteq \mathcal{T}_j \bullet \mathcal{P}$ for all $j \in J^{\mathcal{P}}$, or $\mathcal{P}^{\mathfrak{c}} \subseteq \bigcap_{j \in J^{\mathcal{P}}} (\mathcal{T}_j \bullet \mathcal{P})$.

Since $\mathcal{P}$ is a coreflective subcategory, $\bigcap_{j \in J^{\mathcal{P}}} (\mathcal{T}_j \bullet \mathcal{P}) = \big(\bigcap_{j \in J^{\mathcal{P}}} \mathcal{T}_j \big) \bullet \mathcal{P}$ (see C1.2.1). Thus, $\mathcal{P}^{\mathfrak{c}} \subseteq \big(\bigcap_{j \in J^{\mathcal{P}}} \mathcal{T}_j \big) \bullet \mathcal{P}$, which implies (actually, is equivalent to) that $\bigcap_{j \in J^{\mathcal{P}}} \mathcal{T}_j \nsubseteq \mathcal{P}$.

$(b) \Rightarrow (a)$. There are two cases: $J_{\mathcal{P}} = J$ and $J_{\mathcal{P}} \neq J$.

(i) We start with the first case; i.e. we assume that $\mathcal{T}_i \subseteq \mathcal{P}$ for all $i \in J$. Set $\widetilde{\mathcal{Q}} = \bigcap_{i \in J} u_i^{*^{-1}}((\mathcal{P}/\mathcal{T}_i)^{\mathfrak{c}}) \bigcap \mathcal{P}^{\perp}$ and $\mathcal{Q} = [\widetilde{\mathcal{Q}}]_{\mathfrak{c}}$ – the smallest coreflective topologizing subcategory of C_X containing $\widetilde{\mathcal{Q}}$. We claim that $\mathcal{Q}$ belongs to $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ and ${}^{\mathfrak{c}}\widehat{\mathcal{Q}} = \mathcal{P}$. Since, by condition (b), $\mathcal{Q} \nsubseteq \mathcal{P}$, it suffices to show that ${}^{\mathfrak{c}}\widehat{\mathcal{Q}} = \mathcal{P}$.

Let $\mathcal{S}$ be a coreflective topologizing subcategory of C_X which is not contained in $\mathcal{P}$. Then $\mathcal{P}$ is properly contained in $\mathcal{S} \bullet \mathcal{P}$ and, therefore, $u_i^{*^{-1}}((\mathcal{P}/\mathcal{T}_i)^{\mathfrak{c}}) \subseteq \mathcal{T}_i \bullet \mathcal{S} \bullet \mathcal{P}$ for each $i \in J$. This implies that $u_i^{*^{-1}}((\mathcal{P}/\mathcal{T}_i)^{\mathfrak{c}}) \bigcap \mathcal{P}^{\perp} \subseteq \mathcal{T}_i \bullet \mathcal{S} \bullet \mathcal{P} \bigcap \mathcal{P}^{\perp} \subseteq \mathcal{T}_i \bullet \mathcal{S}$. Therefore,

$$\widetilde{\mathcal{Q}} = \bigcap_{i \in J} u_i^{*^{-1}}((\mathcal{P}/\mathcal{T}_i)^{\mathfrak{c}}) \bigcap \mathcal{P}^{\perp} \subseteq \bigcap_{i \in J} (\mathcal{T}_i \bullet \mathcal{S}) = \big(\bigcap_{i \in J} \mathcal{T}_i \big) \bullet \mathcal{S} = 0 \bullet \mathcal{S} = \mathcal{S},$$

so that $\mathcal{Q} = [\widetilde{\mathcal{Q}}]_{\mathfrak{c}} \subseteq \mathcal{S}$.

(ii) Consider now the second case: $J_{\mathcal{P}} \neq J$, i.e. $J^{\mathcal{P}} = J - J_{\mathcal{P}}$ is non-empty. This case can be reduced to the first case as follows.

Set $C_{\mathcal{V}_{\mathcal{P}}} = \bigcap_{j \in J^{\mathcal{P}}} \mathcal{T}_j$. Since $\bigcap_{j \in J^{\mathcal{P}}} \mathcal{T}_j \nsubseteq \mathcal{P}$, the intersection $\mathcal{P}_{\mathfrak{t}} = C_{\mathcal{V}_{\mathcal{P}}} \bigcap \mathcal{P}$ is an element of $\mathbf{Spec}^1_{\mathfrak{S}\mathfrak{c}}(\mathcal{V}_{\mathcal{P}})$. Notice that $\{\mathcal{T}_i \cap C_{\mathcal{V}_{\mathcal{P}}} = \widetilde{\mathcal{T}}_i \mid i \in J_{\mathcal{P}}\}$ is a cocover of $\mathcal{V}_{\mathcal{P}}$, i.e. $\bigcap_{j \in J_{\mathcal{P}}} \widetilde{\mathcal{T}}_j = 0$. It remains to show that the condition $\widetilde{\mathcal{Q}} = \bigcap_{i \in J} u_i^{*^{-1}}((\mathcal{P}/\mathcal{T}_i)^{\mathfrak{c}}) \bigcap \mathcal{P}^{\perp} \neq 0$ implies the analogous condition for the object $\mathcal{P}_{\mathfrak{t}} = C_{\mathcal{V}_{\mathcal{P}}} \bigcap \mathcal{P}$ of $\mathbf{Spec}^1_{\mathfrak{S}\mathfrak{c}}(\mathcal{V}_{\mathcal{P}})$ and the cover $\{\widetilde{\mathcal{T}}_i \mid i \in J_{\mathcal{P}}\}$;

that is

$$\widetilde{\mathcal{Q}}_0 = \bigcap_{i\in J} \widetilde{u}_i^{*^{-1}}((\mathcal{P}_{\mathfrak{t}}/\widetilde{\mathcal{T}}_i)^{\mathfrak{c}}) \bigcap \mathcal{P}_{\mathfrak{t}}^{\perp} \neq 0.$$

In fact, let $\widetilde{u}_i^*$ denote the localization functor $C_{\mathcal{V}_{\mathcal{P}}} \longrightarrow C_{\mathcal{V}_{\mathcal{P}}}/\widetilde{\mathcal{T}}_i$. Then

$$\widetilde{u}_i^{*^{-1}}((\mathcal{P}_{\mathfrak{t}}/\widetilde{\mathcal{T}}_i)^{\mathfrak{c}}) = u_i^{*^{-1}}((\mathcal{P}/\mathcal{T}_i)^{\mathfrak{c}}) \bigcap C_{\mathcal{V}_{\mathcal{P}}}, \quad \text{and}$$
$$\widetilde{u}_i^{*^{-1}}((\mathcal{P}_{\mathfrak{t}}/\widetilde{\mathcal{T}}_i)^{\mathfrak{c}}) \bigcap \mathcal{P}_{\mathfrak{t}}^{\perp} = u_i^{*^{-1}}((\mathcal{P}/\mathcal{T}_i)^{\mathfrak{c}}) \bigcap C_{\mathcal{V}_{\mathcal{P}}} \bigcap \mathcal{P}^{\perp}.$$

Therefore,

$$\bigcap_{i\in J} \widetilde{u}_i^{*^{-1}}((\mathcal{P}_{\mathfrak{t}}/\widetilde{\mathcal{T}}_i)^{\mathfrak{c}}) \bigcap \mathcal{P}_{\mathfrak{t}}^{\perp} = \bigcap_{i\in J} u_i^{*^{-1}}((\mathcal{P}/\mathcal{T}_i)^{\mathfrak{c}}) \bigcap \mathcal{P}^{\perp} \bigcap C_{\mathcal{V}_{\mathcal{P}}} = \widetilde{\mathcal{Q}} \bigcap C_{\mathcal{V}_{\mathcal{P}}}. \tag{2}$$

On the other hand, for every $i \in J_{\mathcal{P}}$, there is an inclusion $\mathcal{T}_i \bullet C_{\mathcal{V}_{\mathcal{P}}} \bullet \mathcal{P} \supseteq u_i^{*^{-1}}((\mathcal{P}/\mathcal{T}_i)^{\mathfrak{c}})$, because $C_{\mathcal{V}_{\mathcal{P}}} \nsubseteq \mathcal{P}$, which implies the inclusion $u_i^{*^{-1}}((\mathcal{P}/\mathcal{T}_i)^{\mathfrak{c}}) \bigcap \mathcal{P}^{\perp} \subseteq \mathcal{T}_i \bullet C_{\mathcal{V}_{\mathcal{P}}}$. Taking the intersection, we obtain the inclusion

$$\widetilde{\mathcal{Q}} \subseteq \bigcap_{i\in J_{\mathcal{P}}} (\mathcal{T}_i \bullet C_{\mathcal{V}_{\mathcal{P}}}) = \Big(\bigcap_{i\in J_{\mathcal{P}}} \mathcal{T}_i\Big) \bullet C_{\mathcal{V}_{\mathcal{P}}}. \tag{3}$$

Notice that $\widetilde{\mathcal{Q}}$ is a full subcategory of C_X closed under taking subobjects. In particular, the equality $\widetilde{\mathcal{Q}} \bigcap C_{\mathcal{V}_{\mathcal{P}}} = 0$ means precisely that every object of $\widetilde{\mathcal{Q}}$ is $C_{\mathcal{V}_{\mathcal{P}}}$-torsion free. The latter fact together with the inclusion $\widetilde{\mathcal{Q}} \subseteq \Big(\bigcap\limits_{i\in J_{\mathcal{P}}} \mathcal{T}_i\Big) \bullet C_{\mathcal{V}_{\mathcal{P}}}$ (see (3)) implies that $\widetilde{\mathcal{Q}} \subseteq \bigcap\limits_{i\in J_{\mathcal{P}}} \mathcal{T}_i \subseteq \mathcal{P}$, which contradicts to the fact that $\widetilde{\mathcal{Q}}$ is a nonzero subcategory of $\mathcal{P}^{\perp}$.

This contradiction shows that $\widetilde{\mathcal{Q}} \bigcap C_{\mathcal{V}_{\mathcal{P}}} \neq 0$, hence, by (2),

$$\widetilde{\mathcal{Q}}_0 = \bigcap_{i\in J} \widetilde{u}_i^{*^{-1}}((\mathcal{P}_{\mathfrak{t}}/\widetilde{\mathcal{T}}_i)^{\mathfrak{c}}) \bigcap \mathcal{P}_{\mathfrak{t}}^{\perp} \neq 0.$$

(iii) Thus, the 'space' $\mathcal{V}_{\mathcal{P}}$, the cocover $\{\widetilde{\mathcal{T}}_i \mid i \in J_{\mathcal{P}}\}$, and the point $\mathcal{P}_{\mathfrak{t}} = \mathcal{P} \bigcap C_{\mathcal{V}_{\mathcal{P}}}$ of the spectrum $\mathbf{Spec}^1_{\mathfrak{S}\mathfrak{c}}(\mathcal{V}_{\mathcal{P}})$ satisfy the conditions (b) with all $\widetilde{\mathcal{T}}_i$ being subcategories of $\mathcal{P}_{\mathfrak{t}}$. By (i) above, $\mathcal{P}_{\mathfrak{t}}$ belongs to the spectrum $\mathbf{Spec}^1_{\mathfrak{c}}(\mathcal{V}_{\mathcal{P}})$, and $\mathcal{P}_{\mathfrak{t}} = \langle \widetilde{\mathcal{Q}}_0 \rangle_{\mathcal{V}_{\mathcal{P}}} = \langle \mathcal{Q}_0 \rangle_{\mathcal{V}_{\mathcal{P}}}$, where $\mathcal{Q}_0$ is the smallest coreflective topologizing subcategory of $C_{\mathcal{V}_{\mathcal{P}}}$ containing $\widetilde{\mathcal{Q}}_0$. Therefore, $[\mathcal{Q}_0]_{\mathfrak{c}}$ is a point of the spectrum $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ and $\langle \mathcal{Q}_0 \rangle_X = \mathcal{P}$.

$(b) \Rightarrow (c)$. The condition $\bigcap_{i\in J_{\mathcal{P}}} u_i^{*^{-1}}((\mathcal{P}/\mathcal{T}_i)^{\mathfrak{c}})\bigcap \mathcal{P}^{\perp} \neq 0$ implies that the intersection $u_i^{*^{-1}}((\mathcal{P}/\mathcal{T}_i)^{\mathfrak{c}})\bigcap \mathcal{P}^{\perp}$ is nonzero for every $i \in J_{\mathcal{P}}$. In particular, $(\mathcal{P}/\mathcal{T}_i)^{\mathfrak{c}}$ does not coincide with $\mathcal{P}/\mathcal{T}_i$, which means that $\mathcal{P}/\mathcal{T}$ belongs to the spectrum $\mathbf{Spec}^1_{\mathfrak{c}}(X/\mathcal{T}_i)$.

$(c) \Rightarrow (b)$ (when J is finite). For every $i \in J_{\mathcal{P}}$, let $\widetilde{\mathcal{Q}}_i$ denote the intersection $u_i^{*^{-1}}((\mathcal{P}/\mathcal{T}_i)^{\mathfrak{c}})\bigcap \mathcal{P}^{\perp}$ and $\mathcal{Q}_i = [\widetilde{\mathcal{Q}}_i]_{\mathfrak{c}}$ – the smallest coreflective topologizing subcategory of C_X containing $\widetilde{\mathcal{Q}}_i$. By assumption, $\widetilde{\mathcal{Q}}_i \neq 0$ for each $i \in J_{\mathcal{P}}$, hence $\mathcal{Q}_i \nsubseteq \mathcal{P}$. The latter implies that, for every $j \in J_{\mathcal{P}}$, the coreflective topologizing subcategory spanned by $u_j^*(\mathcal{Q}_i \bullet \mathcal{P})$ contains $(\mathcal{P}/\mathcal{T}_j)^{\mathfrak{c}}$, or, equivalently, $u_j^{*^{-1}}((\mathcal{P}/\mathcal{T}_i)^{\mathfrak{c}}) \subseteq \mathcal{T}_j \bullet \mathcal{Q}_i \bullet \mathcal{P}$. Therefore,

$$\widetilde{\mathcal{Q}}_j = u_j^{*^{-1}}((\mathcal{P}/\mathcal{T}_j)^{\mathfrak{c}})\bigcap \mathcal{P}^{\perp} \subseteq (\mathcal{T}_j \bullet \mathcal{Q}_i \bullet \mathcal{P})\bigcap \mathcal{P}^{\perp} = (\mathcal{T}_j \bullet \mathcal{Q}_i)\bigcap \mathcal{P}^{\perp} \subseteq \mathcal{T}_j \bullet \mathcal{Q}_i$$

which implies the inclusion $\mathcal{Q}_j \subseteq \mathcal{T}_j \bullet \mathcal{Q}_i$ for every $(i,j) \in J_{\mathcal{P}} \times J_{\mathcal{P}}$, hence the inclusion

$$\mathcal{Q}_j \subseteq \bigcap_{i\in J_{\mathcal{P}}} (\mathcal{T}_j \bullet \mathcal{Q}_i) = \mathcal{T}_j \bullet \Big(\bigcap_{i\in J_{\mathcal{P}}} \mathcal{Q}_i\Big).$$

Here the equality is due to the finiteness of $J_{\mathcal{P}}$.

It follows from the inclusion $\mathcal{Q}_j \subseteq \mathcal{T}_j \bullet \big(\bigcap_{i\in J_{\mathcal{P}}} \mathcal{Q}_i\big)$ that $\bigcap_{i\in J_{\mathcal{P}}} \mathcal{Q}_i \neq 0$, because otherwise $\mathcal{Q}_j \subseteq \mathcal{T}_j \bullet 0 = \mathcal{T}_i$, which is impossible, since $\mathcal{T}_j \subseteq \mathcal{P}$ and $\mathcal{Q}_j \nsubseteq \mathcal{P}$.

If $J = J_{\mathcal{P}}$, the condition (b) is fulfilled. If $J \neq J_{\mathcal{P}}$, we need to check that $\bigcap_{i\in J^{\mathcal{P}}} \mathcal{T}_i \nsubseteq \mathcal{P}$.

In fact, if $i \in J^{\mathcal{P}} = J - J_{\mathcal{P}}$, then $\mathcal{T}_i \nsubseteq \mathcal{P}$. Therefore, for every $j \in J_{\mathcal{P}}$, the coreflective subcategory spanned by $u_j^*(\mathcal{T}_i \bullet \mathcal{P})$ contains $(\mathcal{P}/\mathcal{T}_j)^{\mathfrak{c}}$, or, equivalently, $u_j^{*^{-1}}((\mathcal{P}/\mathcal{T}_j)^{\mathfrak{c}}) \subseteq \mathcal{T}_j \bullet \mathcal{T}_i \bullet \mathcal{P}$, which implies that $\widetilde{\mathcal{Q}}_j = u_j^{*^{-1}}((\mathcal{P}/\mathcal{T}_j)^{\mathfrak{c}})\bigcap \mathcal{P}^{\perp} \subseteq (\mathcal{T}_j \bullet \mathcal{T}_i)\bigcap \mathcal{P}^{\perp}$. Taking the intersection and using the finiteness of $J^{\mathcal{P}}$, we obtain:

$$\widetilde{\mathcal{Q}}_j \subseteq \Big(\bigcap_{i\in J^{\mathcal{P}}} (\mathcal{T}_j \bullet \mathcal{T}_i)\Big)\bigcap \mathcal{P}^{\perp} = \Big(\mathcal{T}_j \bullet \Big(\bigcap_{i\in J^{\mathcal{P}}} \mathcal{T}_i\Big)\Big)\bigcap \mathcal{P}^{\perp}. \tag{4}$$

The inclusion $\bigcap_{i\in J^{\mathcal{P}}} \mathcal{T}_i \subseteq \mathcal{P}$ implies (together with (4)) that $\widetilde{\mathcal{Q}}_j \subseteq \mathcal{T}_j \subseteq \mathcal{P}$, which is impossible. So that $\bigcap_{i\in J^{\mathcal{P}}} \mathcal{T}_i \nsubseteq \mathcal{P}$. ■

9.4.2. Note. The reader had, probably, noticed that some parts of the proof of 9.4.1 are similar to some parts of the argument of 7.1. If one considers only the case of finite cocovers, one can follow the argument of 7.1 which is considerably shorter than the proof above.

9.4.3. Proposition. *Let $\{\mathcal{T}_i \mid i \in J\}$ be a set of Serre subcategories of the category C_X such that $\bigcap_{i\in J} \mathcal{T}_i = 0$; and let u_i^* denote the localization functor $C_X \longrightarrow C_X/\mathcal{T}_i$.*

The following conditions on a nonzero coreflective topologizing subcategory $\mathcal{Q}$ of C_X are equivalent:

(a) $\mathcal{Q} \in \mathbf{Spec}^0_{\mathfrak{c}}(X)$,

(b) $[u_i^*(\mathcal{Q})]_{\mathfrak{c}} \in \mathbf{Spec}^0_{\mathfrak{c}}(X/\mathcal{T}_i)$ for every $i \in J$ such that $\mathcal{Q} \nsubseteq \mathcal{T}_i$.

Proof. The implication $(a) \Rightarrow (b)$ follows from 9.3.1(d).

$(b) \Rightarrow (a)$. Let $J_{\mathfrak{c}\widehat{\mathcal{Q}}}$ denote the set of all $i \in J$ such that $\mathcal{Q} \nsubseteq \mathcal{T}_i$, or, equivalently, $\mathcal{T}_i \subseteq^{\mathfrak{c}} \widehat{\mathcal{Q}}$. Notice that $J_{\mathfrak{c}\widehat{\mathcal{Q}}}$ is non-empty, because $\mathcal{Q}$ is nonzero and $\bigcap_{i\in J} \mathcal{T}_i = 0$.

Fix an $i \in J_{\mathfrak{c}\widehat{\mathcal{Q}}}$ and set $\mathcal{P}_i = u_i^{*^{-1}}(\langle u_i^*(\mathcal{Q})\rangle)$. It follows from the formula for $\mathcal{P}_i$ that $\mathcal{Q} \nsubseteq \mathcal{P}_i$. Notice that $\mathcal{P}_i = \mathcal{P}_j$ for every $j \in J_{\mathfrak{c}\widehat{\mathcal{Q}}}$.

In fact, replacing C_X by $C_{X'} = C_X/(\mathcal{P}_i \cap \mathcal{P}_j)$ and $\mathcal{P}_k$ by $\mathcal{P}'_k = \mathcal{P}_k/(\mathcal{P}_i \cap \mathcal{P}_j)$, $k = i, j$, we can obtain that $\mathcal{P}'_i \cap \mathcal{P}'_j = 0$. It follows from 9.3.1 that the condition (b) survives this operation. By 7.1, the image $\mathcal{Q}'$ of $\mathcal{Q}$ in $C_{X'}$ belongs to $\mathbf{Spec}^0_{\mathfrak{c}}(X')$. Therefore, $\mathcal{P}'_i = \mathcal{P}'_j$, which implies that $\mathcal{P}_i = \mathcal{P}_j$.

So, we write $\mathcal{P}$ instead of $\mathcal{P}_i$. For every $i \in J_{\mathfrak{c}\widehat{\mathcal{Q}}}$, the subcategory $(\mathcal{P}/\mathcal{T}_i)^{\mathfrak{c}}$ contains $u_i^*(\mathcal{Q})$, hence its preimage, $u_i^{*^{-1}}((\mathcal{P}/\mathcal{T}_i)^{\mathfrak{c}})$, contains $\mathcal{Q}$. Since $\bigcap_{i\in J_{\mathfrak{c}\widehat{\mathcal{Q}}}} u_i^{*^{-1}}((\mathcal{P}/\mathcal{T}_i)^{\mathfrak{c}})$ contains $\mathcal{Q}$, it is not contained in $\mathcal{P}$. Similarly, if $J^{\mathfrak{c}\widehat{\mathcal{Q}}} = J - J_{\mathfrak{c}\widehat{\mathcal{Q}}}$ is non-empty, then $\mathcal{Q} \subseteq \bigcap_{i\in J^{\mathfrak{c}\widehat{\mathcal{Q}}}} \mathcal{T}_j$, hence $\mathcal{Q} \subseteq \bigcap_{i\in J^{\mathfrak{c}\widehat{\mathcal{Q}}}} \mathcal{T}_j \nsubseteq \mathcal{P}$. Thus, $\mathcal{P}$ satisfies the condition (b) of 9.4.1. Therefore, by 9.4.1, $\mathcal{P} \in \mathbf{Spec}^1_{\mathfrak{c}}(X)$. It remains to show that $\mathcal{Q}$ is an element of $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ corresponding to $\mathcal{P}$.

It follows from the argument of 9.4.1 that the element of $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ corresponding to $\mathcal{P}$ is the coreflective topologizing subcategory $[\mathcal{Q}_{\mathcal{P}}]_{\mathfrak{c}}$ generated by $\mathcal{Q}_{\mathcal{P}} = \mathcal{Q} \cap \mathcal{P}^{\perp}$. In particular, $[\mathcal{Q}_{\mathcal{P}}]_{\mathfrak{c}} \subseteq \mathcal{Q}$. Let M be an object of the subcategory $\mathcal{Q}$. Since $[\mathcal{Q}_{\mathcal{P}}]_{\mathfrak{c}}$ is a coreflective subcategory of C_X, the object M has the biggest subobject $M_{\mathcal{P}} \hookrightarrow M$ with $M_{\mathcal{P}} \in Ob[\mathcal{Q}_{\mathcal{P}}]_{\mathfrak{c}}$. Consider the corresponding exact sequence

$$0 \longrightarrow M_{\mathcal{P}} \xrightarrow{\mathfrak{j}} M \longrightarrow N \longrightarrow 0. \qquad (5)$$

For every $i \in J_{\mathfrak{c}_{\widehat{\mathcal{Q}}}}$, the morphism $u_i^*(\mathfrak{j})$ is an isomorphism, because the images of $[\mathcal{Q}_{\mathcal{P}}]_{\mathfrak{c}}$ and $\mathcal{Q}$ in $C_X/\mathcal{T}_i$ coincide. This means that the object N in (5) belongs to $\mathcal{T}_i = Ker(u_i^*)$ for each $i \in J_{\mathfrak{c}_{\widehat{\mathcal{Q}}}}$. On the other hand, $\mathcal{Q} \subseteq \mathcal{T}_\nu$ for every $\nu \in J - J_{\mathfrak{c}_{\widehat{\mathcal{Q}}}}$ (by definition of $J_{\mathfrak{c}_{\widehat{\mathcal{Q}}}}$); in particular, $N \in T_\nu$ for all $\nu \in J - J_{\mathfrak{c}_{\widehat{\mathcal{Q}}}}$. Thus, N is an object of $\bigcap\limits_{i \in J} \mathcal{T}_i = 0$, i.e. $N = 0$, or, equivalently, the arrow $M_{\mathcal{P}} \xrightarrow{\mathfrak{j}} M$ in (5) is an isomorphism. This proves the inverse inclusion, $[\mathcal{Q}_{\mathcal{P}}]_{\mathfrak{c}} \supseteq \mathcal{Q}$. ■

9.4.4. Support. For every object M of C_X, we define the *support* of M in $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ by $Supp^0_{\mathfrak{c}}(M) = \{\mathcal{Q} \in \mathbf{Spec}^0_{\mathfrak{c}}(X) \mid \mathcal{Q} \subseteq [M]_{\mathfrak{c}}\}$.

9.5. Some consequences.

9.5.1. Proposition. *Let $\{\mathcal{T}_i \mid i \in J\}$ be a finite set of Serre subcategories of an abelian category C_X such that $\bigcap\limits_{i \in J} \mathcal{T}_i = 0$ and $\mathbf{Spec}^1_{\mathfrak{c}}(X/\mathcal{T}_i) = \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X/\mathcal{T}_i)$ for every $i \in J$. Then $\mathbf{Spec}^1_{\mathfrak{c}}(X) = \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$. In particular, the canonical map*

$$\mathbf{Spec}(X) \longrightarrow \mathbf{Spec}^0_{\mathfrak{c}}(X), \quad \mathcal{Q} \longmapsto [\mathcal{Q}]_{\mathfrak{c}},$$

is an isomorphism.

Proof. The inclusion $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X) \subseteq \mathbf{Spec}^1_{\mathfrak{c}}(X)$ holds by 9.1.2(b). Let $\mathcal{P} \in \mathbf{Spec}^1_{\mathfrak{c}}(X)$. Then, by the implication $(a) \Rightarrow (c)$ in 9.4.1, $\mathcal{P}/\mathcal{T}_i \in \mathbf{Spec}^1_{\mathfrak{c}}(X/\mathcal{T}_i)$ for every $i \in J$ such that $\mathcal{T}_i \subseteq \mathcal{P}$. By hypothesis, $\mathbf{Spec}^1_{\mathfrak{c}}(X/\mathcal{T}_i) = \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X/\mathcal{T}_i)$ for all $i \in J$. Therefore, by the implication $(b) \Rightarrow (a)$ in 7.1, $\mathcal{P}$ belongs to $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$. ■

9.5.2. Corollary. *Let $\{\mathcal{T}_i \mid i \in J\}$ be a finite set of Serre subcategories of an abelian category C_X such that $\bigcap\limits_{i \in J} \mathcal{T}_i = 0$ and for every $i \in J$, the quotient category $C_X/\mathcal{T}_i$ has enough objects of finite type. Then $\mathbf{Spec}^1_{\mathfrak{c}}(X) = \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$.*

In particular, the canonical map $\mathbf{Spec}(X) \longrightarrow \mathbf{Spec}^0_{\mathfrak{c}}(X)$ is an isomorphism.

Proof. Since each quotient category $C_X/\mathcal{T}_i$ has enough objects of finite type, it follows from 9.1.2(c) that $\mathbf{Spec}^1_{\mathfrak{c}}(X/\mathcal{T}_i) = \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X/\mathcal{T}_i)$ for all $i \in J$. The assertion follows now from 9.5.1. ■

9.5.3. Affine and quasi-affine cocovers. A morphism $X \xrightarrow{f} Y$ is called *affine* if it has a conservative (i.e. reflecting isomorphisms) direct image functor, $C_X \xrightarrow{f_*} C_Y$ (– a right adjoint to f^*) which has, in turn, a right adjoint. We call a 'space' X *affine over a ring* R, if there is an affine morphism $X \longrightarrow \mathbf{Sp}(R)$, where $C_{\mathbf{Sp}(R)} = R - mod$. A 'space' X is called *affine* if it is affine over $\mathbb{Z}$. By [R4, 9.3.3], X is affine iff the category C_X has a projective generator of finite type. By a well-known theorem of Gabriel and Mitchell, the latter condition means precisely that the category C_X is equivalent to the category of modules over an associative ring.

We call set $\{T_i \mid i \in J\}$ of thick subcategories of the category C_X an *affine cocover* of the 'space' X if $\bigcap_{i\in J} \mathcal{T}_i = 0$ and $X/\mathcal{T}_i$ is affine for every $i \in J$.

9.5.4. Proposition. *Let a finite set $\{T_i \mid i \in J\}$ of Serre subcategories of C_X be a cocover of X (that is $\bigcap_{i\in J} \mathcal{T}_i = 0$) such that every quotient category $C_X/\mathcal{T}_i$ has a family of generators of finite type. Then* $\mathbf{Spec}^1_{\mathfrak{c}}(X) = \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$.

In particular, $\mathbf{Spec}^1_{\mathfrak{c}}(X) = \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$, *if $\{T_i \mid i \in J\}$ is an affine cocover of X.*

Proof. In fact, quotients of an object of finite type is an object of finite type. Therefore, if a category C_Y has a family of generators of finite type, then every nonzero object of C_Y has a subobject of finite type. The assertion follows now from 9.5.3. ∎

9.5.5. Remark. If C_X is a Grothendieck category and $\mathcal{S}$ is a Serre subcategory of C_X, then the localization functor $C_X \longrightarrow C_X/\mathcal{S}$ has a right adjoint, hence $C_X/\mathcal{S}$ is Grothendieck category (see [BD, Ch.6]). In particular, all $C_X/\mathcal{T}_i$ are Grothendieck categories. One can regard Grothendieck categories with a generator of finite type as a noncommutative version of a quasi-affine scheme.

Recall that quasi-affine commutative schemes are, by definition, quasi-compact open subschemes of affine commutative schemes.

9.6. Geometric centers of a 'space' X associated with $\mathbf{Spec}^0_{\mathfrak{c}}(X)$.

9.6.1. The geometric center associated with a topology on $\mathbf{Spec}^0_{\mathfrak{c}}(X)$. Let τ be a topology on $\mathbf{Spec}^0_{\mathfrak{c}}(X)$. For every open subset $\mathcal{U}$, let $\widetilde{\mathcal{O}}_X(\mathcal{U})$ denote the *center* of the quotient category $C_X/\langle\mathcal{U}\rangle$, where $\langle\mathcal{U}\rangle = \bigcap_{\mathcal{Q}\in\mathcal{U}} {}^{\mathfrak{c}}\widehat{\mathcal{Q}}$. Recall that the center of the category C_Y is the (commutative) ring of the endomorphisms of the identical functor $C_Y \longrightarrow C_Y$.

The correspondence $\mathcal{U} \longmapsto \widetilde{\mathcal{O}}_X(\mathcal{U})$ is a presheaf of commutative rings on the topological space $(\mathbf{Spec}^0_{\mathfrak{c}}(X), \tau)$. We denote by $\mathcal{O}_X$ the associated sheaf. The ringed space $((\mathbf{Spec}^0_{\mathfrak{c}}(X), \tau), \mathcal{O}_X)$ is called the *geometric center* of X associated with $(\mathbf{Spec}^0_{\mathfrak{c}}(X), \tau)$.

9.6.2. Zariski geometric center. A topologizing subcategory $\mathbb{T}$ of C_X is called a *Zariski topologizing subcategory* if it is bireflective, i.e. the inclusion functor $\mathbb{T} \hookrightarrow C_X$ has right and left adjoints. Subsets $U^0_{\mathfrak{c}}(\mathbb{T}) = \{\mathcal{Q} \in \mathbf{Spec}^0_{\mathfrak{c}}(X) \mid \ \mathcal{Q} \nsubseteq \mathbb{T}\}$, where $\mathbb{T}$ runs through the preorder $\mathfrak{T}_{\mathfrak{z}}(X)$ of Zariski topologizing subcategories, are open sets of the Zariski topology on $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ (cf. C1.2.4.1). We call the corresponding geometric center the *Zariski geometric center* of X.

9.6.3. The topologies $\tau_{\mathfrak{c}}$ and $\tau^{\mathfrak{c}}$ on $\mathbf{Spec}^0_{\mathfrak{c}}(X)$. The Zariski topology might be trivial or too coarse in the noncommutative case. For instance, it is trivial if $C_X = R-mod$, where the ring R is simple (i.e. it does not have nonzero proper two-sided ideals), like, for instance, any Weyl algebra. Following C1.7, we introduce other topologies on $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ by fixing a set Ξ of coreflective topologizing subcategories of C_X and then taking subsets $U^0_{\mathfrak{c}}(\mathbb{T}) = \{\mathcal{Q} \in \mathbf{Spec}^0_{\mathfrak{c}}(X) \mid \ \mathcal{Q} \not\subseteq \mathbb{T}\}, \ \mathbb{T} \in \Xi$, as a base of open sets of a topology τ_Ξ.

Taking $\Xi = \mathbf{Spec}^0_{\mathfrak{c}}(X)$, we obtain the topology $\tau_{\mathfrak{c}}$ (compare with C1.7.2). Thus the sets $V^0_{\mathfrak{c}}(\mathcal{Q}) = \{\mathcal{Q}' \in \mathbf{Spec}^0_{\mathfrak{c}}(X) \mid \mathcal{Q}' \subseteq \mathcal{Q}\}, \ \mathcal{Q} \in \mathbf{Spec}^0_{\mathfrak{c}}(X)$, form a base of closed subsets of this topology.

The topology $\tau^{\mathfrak{c}}$ is determined by Ξ consisting of the subcategories $[M]_{\mathfrak{c}}$ spanned by objects which are *locally* of finite type. Here 'locally of finite type' means that the localization of M at every point $\mathcal{Q}$ of $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ (i.e. its image in the quotient category $C_X/^{\mathfrak{c}}\widehat{\mathcal{Q}}$) is an object of finite type. It seems that $\tau^{\mathfrak{c}}$ is an appropriate version of Zariski topology for the spectrum $\mathbf{Spec}^0_{\mathfrak{c}}(X)$. If C_X has enough objects of locally finite type, then the topology $\tau^{\mathfrak{c}}$ is finer than the topology $\tau_{\mathfrak{c}}$.

9.7. Proposition. *Let C_X be the category of quasi-coherent sheaves on a scheme $\mathbf{X} = (\mathcal{X}, \mathcal{O})$. Suppose that there is an affine cover $\{\mathcal{U}_i \hookrightarrow \mathcal{X} \mid i \in J\}$ of the scheme $\mathbf{X}$ such that all immersions $\mathcal{U}_i \hookrightarrow \mathcal{X}, \ i \in J$, have a direct image functor. Then the geometric center $(\mathbf{Spec}^0_{\mathfrak{c}}(X), \mathcal{O}_X)$ is isomorphic to the scheme $\mathbf{X}$.*

Proof. The argument follows the lines of the proof of 8.2.1.

(a) The underlying space $\mathcal{X}$ of the scheme $\mathbf{X}$ is isomorphic to $\mathbf{Spec}^0_{\mathfrak{c}}(X)$.

Let $\{\mathcal{U}_i \hookrightarrow \mathcal{X} \mid i \in J\}$ be an affine cover of the scheme $\mathbf{X}$. For each $i \in J$, we denote by C_{U_i} the category of quasi-coherent sheaves on the affine scheme $(\mathcal{U}_i, \mathcal{O}_{\mathcal{U}_i})$ and by $\mathcal{T}_i$ the kernel of the inverse image functor $C_X \xrightarrow{u_i^*} C_{U_i}$. This inverse image functor uniquely determines the equivalence of the quotient category $C_X/\mathcal{T}_i$ and C_{U_i}. The fact that $\{\mathcal{U}_i \hookrightarrow \mathcal{X} \mid i \in J\}$ is a cover means precisely that $\bigcap_{i \in J} \mathcal{T}_i = 0$. The existence of a direct image functor, $C_{U_i} \xrightarrow{u_{i*}} C_X$, of the embedding $\mathcal{U}_i \hookrightarrow \mathcal{X}$ implies that the subcategory $\mathcal{T}_i$ is coreflective: a right adjoint to the inclusion functor $\mathcal{T}_i \longrightarrow C_X$ assigns to every object M of C_X the kernel of the adjunction morphism $M \longrightarrow u_{i*}u_i^*(M)$.

(a1) Let x be a point of the underlying space $\mathcal{X}$ of the scheme $\mathbf{X}$. Let $\mathcal{I}_{\bar{x}}$ be the defining ideal of the closure $\bar{x}$ of the point x and $\mathcal{M}_{\bar{x}}$ the quotient sheaf $\mathcal{O}/\mathcal{I}_{\bar{x}}$. Set $J_x = \{i \in J \mid \mathcal{M}_{\bar{x}} \notin Ob\mathcal{T}_i\}$. We claim that $\mathcal{Q}_x = [\mathcal{M}_{\bar{x}}]_{\mathfrak{c}}$ is an element of $\mathbf{Spec}^0_{\mathfrak{c}}(X)$.

For every $i \in J_x$, the object $u_i^*(\mathcal{M}_{\bar{x}})$ of the category C_{U_i} belongs to $Spec(U_i)$ and $Spec(U_i) = Spec^0_{\mathfrak{c}}(U_i)$, because C_{U_i} is (equivalent to) the category of modules over a ring. Therefore, $[u_i^*(\mathcal{Q}_x)]_{\mathfrak{c}} = [u_i^*(\mathcal{M}_{\bar{x}})]_{\mathfrak{c}}$ is an element of $\mathbf{Spec}^0_{\mathfrak{c}}(U_i)$. By 9.4.3, $\mathcal{Q}_x \in \mathbf{Spec}^0_{\mathfrak{c}}(X)$.

(a2) Conversely, let $\mathcal{Q}$ be an element of $\mathbf{Spec}^0_{\mathfrak{c}}(X)$. Let $\mathcal{Q} \not\subseteq \mathcal{T}_i$, or, equivalently,

$\mathcal{T}_i \subseteq^{\mathfrak{c}} \widehat{\mathcal{Q}}$. By 9.4.3, $[u_i^*(\mathcal{Q})]_{\mathfrak{c}}$ is an element of $\mathbf{Spec}_{\mathfrak{c}}^0(U_i)$. Since U_i is affine, $\mathbf{Spec}_{\mathfrak{c}}^0(U_i)$ is in bijective correspondence with the underlying space $\mathcal{U}_i$ of the subscheme $(\mathcal{U}_i, \mathcal{O}_{\mathcal{U}_i})$; in particular, to the element $[u_i^*(\mathcal{Q})]_{\mathfrak{c}}$ there corresponds a point x of $\mathcal{U}_i$ which we identify with its image in $\mathcal{X}$. Notice that the point x does not depend on the choice of $i \in J_{\mathfrak{c}\widehat{\mathcal{Q}}} = \{j \in J \mid \mathcal{T}_j \subseteq^{\mathfrak{c}} \widehat{\mathcal{Q}}\}$. This gives a map $\mathbf{Spec}_{\mathfrak{c}}^0(X) \longrightarrow \mathcal{X}$ which is inverse to the map $\mathcal{X} \longrightarrow \mathbf{Spec}_{\mathfrak{c}}^0(X)$ constructed in (a1) above. These maps are homeomorphisms in the case if the cover consists of one element, i.e. the scheme is affine. The general case follows from the commutative diagrams

$$\begin{tikzcd} \mathbf{Spec}_{\mathfrak{c}}^0(U_i) \arrow[r, "\sim"] \arrow[d] & \mathcal{U}_i \arrow[d] \\ \mathbf{Spec}_{\mathfrak{c}}^0(X) \arrow[r] & \mathcal{X}, \end{tikzcd} \qquad i \in J. \tag{6}$$

in which vertical arrows are open immersions and the upper horizontal arrow is a homeomorphism; hence the lower horizontal arrow is a homeomorphism.

(b) The diagrams (6) extend to the commutative diagrams of ringed spaces

$$\begin{tikzcd} (\mathbf{Spec}_{\mathfrak{c}}^0(U_i), \mathcal{O}_{U_i}) \arrow[r, "\sim"] \arrow[d] & (\mathcal{U}_i, \mathcal{O}_{\mathcal{U}_i}) \arrow[d] \\ (\mathbf{Spec}_{\mathfrak{c}}^0(X), \mathcal{O}_X) \arrow[r] & (\mathcal{X}, \mathcal{O}) \end{tikzcd} \qquad i \in J. \tag{7}$$

in which $\mathbf{Spec}_{\mathfrak{c}}^0(U_i), \mathcal{O}_{U_i})$ and $(\mathbf{Spec}_{\mathfrak{c}}^0(X), \mathcal{O}_X)$ are Zariski geometric centra of resp. U_i and X, vertical arrows are open immersions and upper horizontal arrow is an isomorphism. Therefore the lower horizontal arrow is an isomorphism. ■

Complementary facts.

C1. The noncommutative cosite of topologizing subcategories and topologies on spectra.

C1.1. The noncommutative finite cosite of topologizing subcategories. We regard topologizing subcategories of an abelian category C_X as 'closed sets' of a 'finite topology' $\tau_X^{\mathfrak{f}}$ defined as follows. We call a set of inclusions $\{\mathbb{T} \hookrightarrow \mathbb{T}_i \mid i \in J\}$ of topologizing subcategories a *cocover* if there exists a finite subset J_0 of J such that $\bigcap\limits_{i \in J_0} \mathbb{T}_i = \mathbb{T}$.

Two of the three standard properties of cocovers follow immediately:

(a) $\mathbb{T} \xrightarrow{id} \mathbb{T}$ is a cocover;

(b) the composition of cocovers is a cocover: if $\{\mathbb{T} \hookrightarrow \mathbb{T}_i \mid i \in J\}$ is a cocover and $\{\mathbb{T}_i \hookrightarrow \mathbb{T}_{ij} \mid j \in J_i\}$ is a cocover for every $i \in J$, then $\{\mathbb{T} \hookrightarrow \mathbb{T}_{ij} \mid i \in J,\ j \in J_i\}$ is a cocover.

The third standard property – the invariance under the base change, acquires the following form:

(c) If $\{\mathbb{T} \hookrightarrow \mathbb{T}_i \mid i \in J\}$ is a cocover, then, for any $\mathbb{S} \in \mathfrak{T}(X)$, both $\{\mathbb{T}\bullet\mathbb{S} \hookrightarrow \mathbb{T}_i\bullet\mathbb{S} \mid i \in J\}$ and $\{\mathbb{S} \bullet \mathbb{T} \hookrightarrow \mathbb{S} \bullet \mathbb{T}_i \mid i \in J\}$ are cocovers.

The property (c) follows from [R4, 4.2.1]. Its proof is also contained in the argument of C1.2.3(b) below.

We call the triple $(\mathfrak{T}(X), \bullet; \tau_X^{\mathfrak{f}})$ the *noncommutative finite cosite* of topologizing subcategories of C_X.

C1.1.1. Note. One can define a finer topological structure on $\mathfrak{T}(X)$ by taking as cocovers all sets of inclusions $\{\mathbb{T} \hookrightarrow \mathbb{T}_i \mid i \in J\}$ such that $\bigcap\limits_{i \in J} \mathbb{T}_i = \mathbb{T}$. The family τ_X of such cocovers satisfies the conditions (a) and (b) above, but fails, in general, the invariance with respect to a base change. The situation improves when one considers instead of all topologizing subcategories coreflective or reflective topologizing categories. This is made precise below.

C1.2. Coreflective and reflective topologizing categories. Let $\mathfrak{T}_{\mathfrak{c}}(X)$ (resp. $\mathfrak{T}^{\mathfrak{c}}(X)$) denote the preorder of all coreflective (resp. reflective) topologizing subcategories of the category C_X. Recall that a subcategory $\mathcal{B}$ of C_X is called *coreflective* (resp. *reflective*) if the inclusion functor $\mathcal{B} \hookrightarrow C_X$ has a right (resp. left) adjoint. By [R, III.6.2.1], both $\mathfrak{T}_{\mathfrak{c}}(X)$ and $\mathfrak{T}^{\mathfrak{c}}(X)$ are monoidal subcategories of the monoidal category (preorder) $(\mathfrak{T}(X), \bullet)$. We shall use the same notation – $\tau_X^{\mathfrak{f}}$, for the restrictions of the topological structure $\tau_X^{\mathfrak{f}}$ defined in C1.1 to $\mathfrak{T}_{\mathfrak{c}}(X)$ and to $\mathfrak{T}^{\mathfrak{c}}(X)$.

Notice that $(\mathfrak{T}_{\mathfrak{c}}(X), \bullet; \tau_X^{\mathfrak{f}})$ is naturally anti-isomorphic to $(\mathfrak{T}^{\mathfrak{c}}(X^o), \bullet; \tau_{X^o}^{\mathfrak{f}})$.

The term 'anti-isomorphic' refers to the monoidal structure $(\mathbb{S}\bullet\mathbb{T})^o = \mathbb{T}^o\bullet\mathbb{S}^o$, where $\mathbb{S}^o$ denotes the subcategory of $C_{X^o} = C_X^{op}$ corresponding to the subcategory $\mathbb{S}$ of C_X.

C1.2.1. Lemma. *Suppose that C_X is an abelian category with supremums of sets of subobjects (for instance, if C_X has infinite coproducts). Then*

(a) The intersection of any set of reflective topologizing subcategories is a reflective topologizing subcategory.

(b) $\big(\bigcap_{i\in J}\mathbb{T}_i\big)\bullet\mathbb{S} = \bigcap_{i\in J}(\mathbb{T}_i\bullet\mathbb{S})$ for any set $\{\mathbb{T}_i \mid i\in J\}$ of topologizing subcategories and any coreflective subcategory $\mathbb{S}$.

Proof. (a) See [R, III.6.2.2].

(b) The inclusion $\big(\bigcap_{i\in J}\mathbb{T}_i\big)\bullet\mathbb{S} \subseteq \bigcap_{i\in J}(\mathbb{T}_i\bullet\mathbb{S})$ is obvious. On the other hand, let M be an object of the subcategory $\bigcap_{i\in J}(\mathbb{T}_i\bullet\mathbb{S})$; that is, for every $i\in J$, there exists an exact sequence $0\longrightarrow M_i\longrightarrow M\longrightarrow L_i\longrightarrow 0$ such that $M_i\in Ob\mathbb{S}$ and $L_i\in Ob\mathbb{T}_i$. Since $\mathbb{S}$ is a coreflective subcategory, the supremum M_J of the set of subobjects $\{M_i \mid i\in J\}$ is an object of $\mathbb{S}$. The canonical epimorphism $M\longrightarrow M/M_J$ factors through $M\longrightarrow L_i$ for each $i\in J$. Therefore, the object M/M_J, being a quotient of the object L_i, belongs to the subcategory $\mathbb{T}_i$ for each $i\in J$, hence it belongs to $\bigcap_{i\in J}\mathbb{T}_i$. ■

C1.2.2. Corollary. *Let C_X be an abelian category with infinums of sets of quotient objects (i.e. the dual category $C_{X^o} = C_X^{op}$ has maximums of sets of subobjects). Then*

(a) The intersection of any set of coreflective topologizing subcategories is a coreflective topologizing subcategory.

(b) $\mathbb{S}\bullet\big(\bigcap_{i\in J}\mathbb{T}_i\big) = \bigcap_{i\in J}(\mathbb{S}\bullet\mathbb{T}_i)$ for any set $\{\mathbb{T}_i \mid i\in J\}$ of topologizing subcategories and any reflective subcategory $\mathbb{S}$.

Proof. The assertion is dual to the assertion of C1.2.1. ■

C1.2.3. Corollary. *Let C_X be an abelian category with supremums of sets of subobjects and infinums of sets of quotient objects. Then*

(a) The intersection of any set of reflective (resp. coreflective) topologizing subcategories is a reflective (resp. coreflective) topologizing subcategory.

(b) If $\mathbb{S}$ is a reflective and $\mathbb{U}$ a coreflective topologizing subcategory of C_X, then

$$\mathbb{S}\bullet\Big(\bigcap_{i\in J}\mathbb{T}_i\Big) = \bigcap_{i\in J}(\mathbb{S}\bullet\mathbb{T}_i) \quad \textit{and} \quad \Big(\bigcap_{i\in J}\mathbb{T}_i\Big)\bullet\mathbb{U} = \bigcap_{i\in J}(\mathbb{T}_i\bullet\mathbb{U})$$

for any set $\{\mathbb{T}_i \mid i\in J\}$ of topologizing subcategories.

C1.2.3.1. Note. The conditions on C_X in C1.2.3 hold if the category C_X has infinite products and coproducts. In particular, they hold for any Grothendieck category, or the category of quasi-coherent sheaves on an arbitrary scheme.

C1.2.4. Interpretations. Let τ_X denote the family of cocovers on $\mathfrak{T}(X)$ in the sense of C1.1.1; that is $\{\mathbb{T} \hookrightarrow \mathbb{T}_i \mid i \in J\}$ is a cocover iff the intersection of all $\mathbb{T}_i$ coincides with $\mathbb{T}$. Corollary C1.2.3 can be spelled as follows:

If the category C_X has supremums of sets of subobjects and infinums of sets of quotient objects, then τ_X induces the structure of a *right cosite* on the monoid $(\mathfrak{T}_{\mathfrak{c}}(X), \bullet)$ of coreflective topologizing subcategories and the structure of a *left cosite* on the monoid $(\mathfrak{T}^{\mathfrak{c}}(X), \bullet)$ of reflective topologizing subcategories.

C1.2.4.1. Zariski cosite. We denote by $\mathfrak{T}_{\mathfrak{z}}(X)$ the intersection $\mathfrak{T}_{\mathfrak{c}}(X) \bigcap \mathfrak{T}^{\mathfrak{c}}(X)$. Objects of $\mathfrak{T}_{\mathfrak{z}}(X)$ – *bireflective* topologizing subcategories of C_X, are interpreted as Zariski closed subspaces. By this reason, we shall call them sometimes *Zariski* topologizing subcategories. Under conditions of C1.2.3 (i.e. if C_X and C_{X^o} have supremums of sets of subobjects), $(\mathfrak{T}_{\mathfrak{z}}(X), \bullet; \tau_X)$ is a *two-sided cosite.* The latter means that for any set $\{\mathbb{S},\ \mathbb{T}_i \mid i \in J\}$ of Zariski topologizing subcategories, the intersection $\bigcap\limits_{i \in J} \mathbb{T}_i$ is a Zariski topologizing subcategory and

$$\mathbb{S} \bullet \big(\bigcap_{i \in J} \mathbb{T}_i\big) = \bigcap_{i \in J} (\mathbb{S} \bullet \mathbb{T}_i), \qquad \big(\bigcap_{i \in J} \mathbb{T}_i\big) \bullet \mathbb{S} = \bigcap_{i \in J} (\mathbb{T}_i \bullet \mathbb{S}). \tag{1}$$

We call $(\mathfrak{T}_{\mathfrak{z}}(X), \bullet; \tau_X)$ *the noncommutative Zariski finite cosite of the 'space' X*. One of the reasons for these interpretaions comes from the following example.

C1.2.5. Example. Let $C_X = R - mod$ for an associative ring R. For every two-sided ideal α in R, let $\mathbb{T}_\alpha$ denote the full subcategory of $R - mod$ whose objects are modules annihilated by the ideal α. By [R, III.6.4.1], the map $\alpha \longmapsto \mathbb{T}_\alpha$ is an isomorphism of the preorder $(I(R), \supseteq)$ of two-sided ideals of the ring R onto $(\mathfrak{T}^{\mathfrak{c}}(X), \subseteq)$. Moreover, $\mathbb{T}_\alpha \bullet \mathbb{T}_\beta = \mathbb{T}_{\alpha\beta}$ for any pair of two-sided ideals $\alpha,\ \beta$. This means that the map $\alpha \longmapsto \mathbb{T}_\alpha$ is an isomorphism of monoidal categories (preorders), where the monoidal structure on $I(R)$ is the multiplication of ideals.

It follows from this description that every reflective topologizing subcategory of $C_X = R - mod$ is coreflective, that is $\mathfrak{T}^{\mathfrak{c}}(X) = \mathfrak{T}_{\mathfrak{z}}(X)$.

One can see that $\bigcap\limits_{i \in J} \mathbb{T}_{\alpha_i} = \mathbb{T}_{\alpha_J}$, where $\alpha_J = \sup(\alpha_i | i \in J)$. Thus, the cotopology τ_X on $\mathfrak{T}^{\mathfrak{c}}(X) = \mathfrak{T}_{\mathfrak{z}}(X)$ induces a (noncommutative) Zariski topology on $I(R)$: the set of inclusions of two-sided ideals $\{\alpha_i \hookrightarrow \alpha \mid i \in J\}$ is a *cover* if $\alpha = \sup(\alpha_i | i \in J_0)$.

The invariance with respect to base change in $I(R)$ is expressed by the equalities

$$\beta \sup(\alpha_i|\ i \in J) = \sup(\beta\alpha_i|\ i \in J) \quad \text{and} \quad \sup(\alpha_i|\ i \in J)\beta = \sup(\alpha_i\beta|\ i \in J)$$

for any set of two-sided ideals $\{\beta,\ \alpha_i \mid i \in J\}$. One can deduce directly from these equalities the base change invariance on $\mathfrak{T}_{\mathfrak{z}}(X)$ (in the case when $C_X = R - mod$). In fact, we have

$$\bigcap_{i\in J}(\mathbb{T}_{\alpha_i} \bullet \mathbb{T}_\beta) = \bigcap_{i\in J} \mathbb{T}_{\alpha_i\beta} = \mathbb{T}_{\sup(\alpha_i\beta|\ i\in J)} = \mathbb{T}_{\sup(\alpha_i|\ i\in J)\beta} =$$
$$\mathbb{T}_{\sup(\alpha_i|\ i\in J)} \bullet \mathbb{T}_\beta = \Big(\bigcap_{i\in J} \mathbb{T}_{\alpha_i}\Big) \bullet \mathbb{T}_\beta.$$

Similar calculation shows that $\bigcap_{i\in J}(\mathbb{T}_\beta \bullet \mathbb{T}_{\alpha_i}) = \mathbb{T}_\beta \bullet \Big(\bigcap_{i\in J} \mathbb{T}_{\alpha_i}\Big)$.

C1.2.6. Example: reflective topologizing subcategories of the category of quasi-coherent sheaves on a scheme. Let C_X be the category $Qcoh_{\mathbf{X}}$ of quasi-coherent sheaves on a scheme $\mathbf{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. Then elements of $\mathfrak{T}^{\mathfrak{c}}(X)$ are in one to one correspondence with quasi-coherent ideals of the structure sheaf $\mathcal{O}_{\mathcal{X}}$, or, equivalently, with closed subschemes of the scheme $\mathbf{X}$.

C1.3. Example: coreflective topologizing subcategories of an affine 'space'. Let C_X be the category $R - mod$ of left modules over an associative ring R. We denote by $I_\ell(R)$ the set of left ideals of R.

Recall that a set $\mathfrak{F}$ of left ideals of R is called a *topologizing filter* if it is closed under finite intersections, contains with every left ideal $\mathfrak{m}$ left ideals $(\mathfrak{m} : r) = \{a \in R \mid ar \in \mathfrak{m}\}$ for all $r \in R$ and all left ideals containing $\mathfrak{m}$.

Topologizing filters of left ideals form a monoidal category (a preorder) with respect to the *Gabriel multiplication* defined as follows.

$$\mathfrak{F} \circ \mathfrak{G} = \bigcup_{\mathfrak{m}\in\mathfrak{G}} \mathfrak{F} \circ \{\mathfrak{m}\}, \quad \text{where} \quad \mathfrak{F} \circ \{\mathfrak{m}\} = \{\mathfrak{n} \in I_\ell(R) \mid (\mathfrak{n} : r) \in \mathfrak{F} \text{ for all } r \in \mathfrak{m}\}.$$

There is a natural bijective correspondence between topologizing filters of left ideals and coreflective topologizing subcategories of the category $R - mod$. Namely, to each coreflective topologizing subcategory $\mathbb{T}$ of $R - mod$, there corresponds the filter $\mathfrak{F}_{\mathbb{T}}$ formed by annihilators of elements of modules from $\mathbb{T}$. The inverse map assigns to each topologizing filter $\mathfrak{F}$ the full subcategory $\mathbb{T}_{\mathfrak{F}}$ whose objects are all R-modules M such that each element of M is annihilated by some left ideal from $\mathfrak{F}$. These maps are mutually inverse isomorphisms between the monoidal preorder $(\mathfrak{T}\mathfrak{F}_\ell(R), \circ)$ of topologizing filters of left ideals of the ring R and the monoidal preorder $(\mathfrak{T}_{\mathfrak{c}}(X), \bullet)$ of coreflective topologizing subcategories of $C_X = R - mod$.

To every left ideal $\mathfrak{m}$ in R, one can assign the smallest topologizing filter $[\mathfrak{m}]$ containing $\mathfrak{m}$. It is easy to see that $[\mathfrak{m}]$ consists of all left ideals $\mathfrak{n}$ which contain $(\mathfrak{m} : x) = \{r \in R \mid rx \subset \mathfrak{m}\}$ for some finite set x of elements of R. The corresponding coreflective topologizing subcategory is formed by all R-modules M such that every element of M is annihilated by the left ideal $(\mathfrak{m} : x)$ for some finite set x of elements of R.

Notice that if $\mathfrak{m}$ is a two-sided ideal, then $\mathfrak{m} \subseteq (\mathfrak{m} : x)$ for any $x \subset R$. In this case the filter $[\mathfrak{m}]$ consists of all left ideals of R containing $\mathfrak{m}$ and the corresponding topologizing subcategory coincides with the subcategory $\mathbb{T}_{\mathfrak{m}}$ whose objects are modules annihilated by $\mathfrak{m}$ (see C1.2.1). If α and β are two-sided ideals, then $[\alpha] \circ [\beta] = [\alpha\beta]$. This shows that the map $I(R) \longrightarrow \mathfrak{T}\mathfrak{F}_\ell(R),\ \alpha \longmapsto [\alpha]$ is an embedding of monoidal preorders.

Thus, we have a commutative diagram of morphisms of monoidal preorders

$$\begin{array}{ccc} (I(R), \cdot) & \longrightarrow & (\mathfrak{T}\mathfrak{F}_\ell(R), \circ) \\ \wr\downarrow & & \downarrow\wr \\ (\mathfrak{T}^{\mathfrak{c}}(X), \bullet) & \longrightarrow & (\mathfrak{T}_{\mathfrak{c}}(X), \bullet) \end{array}$$

where the lower horizontal arrow is the inclusion functor and $C_X = R - mod$.

C1.4. The cosites of thick and Serre subcategories. Consider the preorder $\mathfrak{Th}(X)$ of thick subcategories and the preorder $\mathfrak{Se}(X)$ of Serre subcategories of the category C_X together with cocovers induced from $\mathfrak{T}(X)$.

C1.4.1. Proposition. *(a) $\mathfrak{Th}(X)$ and $\mathfrak{Se}(X)$ are Grothendieck precosites.*

(b) The map $\mathfrak{T}(X) \longrightarrow \mathfrak{Th}(X)$ which assigns to every topologizing subcategory $\mathbb{T}$ the thick subcategory $\mathbb{T}^\infty$ generated by $\mathbb{T}$ is a morphism from the noncommutative precosite $(\mathfrak{T}(X), \bullet)$ of topologizing subcategories to the Grothendick precosite $\mathfrak{Th}(X)$ of thick subcategories of C_X.

(c) The map $\mathfrak{T}(X) \longrightarrow \mathfrak{Se}(X),\ \mathbb{T} \longmapsto \mathbb{T}^-$, is a morphism from the noncommutative precosite $(\mathfrak{T}(X), \bullet)$ to the Grothendick precosite $\mathfrak{Se}(X)$ of Serre subcategories.

Proof. For any pair of topologizing subcategories $\mathbb{S}$ and $\mathbb{T}$, the thick subcategory $(\mathbb{S}\bullet\mathbb{T})^\infty$ generated by $\mathbb{S}\bullet\mathbb{T}$ coincides with the coproduct $\mathbb{S}^\infty \sqcup \mathbb{T}^\infty$ of the thick subcategories $\mathbb{S}^\infty$ and $\mathbb{T}^\infty$ generated respectively by $\mathbb{S}$ and $\mathbb{T}$.

Similarly, $(\mathbb{S} \bullet \mathbb{T})^-$ coincides with the coproduct $\mathbb{S}^- \vee \mathbb{T}^-$ of Serre subcategories generated respectively by $\mathbb{S}$ and $\mathbb{T}$.

In other words, the maps

$$\mathfrak{T}(X) \longrightarrow \mathfrak{Th}(X),\ \mathbb{T} \longmapsto \mathbb{T}^\infty, \quad \text{and} \quad \mathfrak{T}(X) \longrightarrow \mathfrak{Se}(X),\ \mathbb{T} \longmapsto \mathbb{T}^-, \qquad (1)$$

are morphisms of monoidal preorders resp.

$$(\mathfrak{T}(X),\bullet) \longrightarrow (\mathfrak{Th}(X),\sqcup) \quad \text{and} \quad (\mathfrak{T}(X),\bullet) \longrightarrow (\mathfrak{Se}(X),\vee).$$

It remains to verify that the maps (1) transfer cocovers to cocovers, or, equivalently, for any finite set $\{\mathbb{T}_i \mid i \in J\}$ of topologizing subcategories of C_X, there are equalities

$$\big(\bigcap_{i\in J}\mathbb{T}_i\big)^\infty = \bigcap_{i\in J}\mathbb{T}_i^\infty \quad \text{and} \quad \big(\bigcap_{i\in J}\mathbb{T}_i\big)^- = \bigcap_{i\in J}\mathbb{T}_i^-.$$

The first equality is proven in [R4, 4.6.1] and the second equality is the assertion [R4, 4.1].

Altogether proves (b) and (c). The assertion (a) is a consequence of (b) and (c).

In fact, for any finite set $\{\mathbb{S},\ \mathbb{T}_i \mid i \in J\}$ of thick subcategories, we have by (b)

$$\bigcap_{i\in J}(\mathcal{T}_i \sqcup \mathcal{S}) = \bigcap_{i\in J}(\mathcal{S}\bullet\mathcal{T}_i)^\infty = \big(\bigcap_{i\in J}\mathcal{S}\bullet\mathcal{T}_i\big)^\infty = \big(\mathcal{S}\bullet(\bigcap_{i\in J}\mathcal{T}_i)\big)^\infty = \mathcal{S}\sqcup\big(\bigcap_{i\in J}\mathcal{T}_i\big),$$

Similarly, if $\{\mathbb{S},\ \mathbb{T}_i \mid i \in J\}$ are Serre subcategories, then it follows from (c) that

$$\bigcap_{i\in J}(\mathcal{T}_i \vee \mathcal{S}) = \bigcap_{i\in J}(\mathcal{S}\bullet\mathcal{T}_i)^- = \big(\bigcap_{i\in J}\mathcal{S}\bullet\mathcal{T}_i\big)^- = \big(\mathcal{S}\bullet(\bigcap_{i\in J}\mathcal{T}_i)\big)^- = \mathcal{S}\vee\big(\bigcap_{i\in J}\mathcal{T}_i\big),$$

hence the assertion. ∎

C1.5. Monoidal subcategories of $(\mathfrak{T}(X),\bullet)$ and topologies on spectra. Any full monoidal subcategory $\mathfrak{G}$ of $(\mathfrak{T}(X),\bullet)$ closed under arbitrary intersections defines a topology $\tau_{\mathfrak{G}}$ on $\mathbf{Spec}^0_{\mathfrak{t}}(X)$ (hence on $\mathbf{Spec}(X)$) by taking $V^0_{\mathfrak{t}}(\mathbb{T}) = \{\mathcal{P}\in\mathbf{Spec}^0_{\mathfrak{t}}(X) \mid \mathcal{P}\subseteq\mathbb{T}\}$ (resp. $V(\mathbb{T}) = \{\mathcal{P}\in\mathbf{Spec}(X) \mid \mathcal{P}\subseteq\mathbb{T}\}$), $\mathbb{T}\in\mathfrak{G}$, as the set of closed subsets.

The map $\mathfrak{G}\longmapsto\tau_{\mathfrak{G}}$ is a surjective map from the family of full monoidal subcategories of $(\mathfrak{T}(X),\bullet)$ closed under arbitrary intersections onto the set of topologies on $\mathbf{Spec}^0_{\mathfrak{t}}(X)$ which are coarser than the topology $\tau^0_{\mathfrak{t}}$ corresponding to $\mathfrak{T}(X)$.

C1.6. Coarse Zariski topology. Suppose that the category C_X has supremums of sets of subobjects (for instance, C_X has infinite coproducts). Then, by [R, III.6.2.2], the intersection of any set of reflective topologizing subcategories is a reflective topologizing subcategory. Taking as $\mathfrak{G}$ the subcategory $\mathfrak{T}^{\mathfrak{c}}(X)$ of reflective topologizing subcategories, we obtain the *coarse Zariski* topology on $\mathbf{Spec}^0_{\mathfrak{t}}(X)$ which we denote by $\tau^0_{\mathfrak{z}}$. Its restriction to $\mathbf{Spec}(X)$ will be denoted by $\tau_{\mathfrak{z}}$.

C1.6.1. Proposition. *Suppose C_X has the property (sup) and a generator of finite type. Then the topological space* $(\mathbf{Spec}(X),\tau_{\mathfrak{z}})$ *is quasi-compact.*

Proof. See [R, III.6.5.2.1]. ■

C1.6.2. Example. Example C1.2.6 shows that if C_X is the category of quasi-coherent sheaves on a (commutative) scheme **X**, then the elements of $\mathfrak{T}^{\mathfrak{c}}(X)$ are categories of quasi-coherent sheaves on closed subschemes of **X**. Suppose that the scheme **X** is quasi-compact and quasi-separated (more generally, quasi-compact and the embeddings of every point has a direct image functor). Then $\mathbf{Spec}(X)$ is the set of points of the underlying space of the scheme **X** and closed sets of the Zariski topology on $\mathbf{Spec}(X)$ are spectra of closed subschemes. So that the Zariski topology on $\mathbf{Spec}(X)$ coincides with the Zariski topology in the conventional sense.

C1.6.3. Example: Zariski topology on an affine noncommutative scheme. Let $C_X = R - mod$ for an associative unital ring R. It follows from C1.6.1 that the topological space $(\mathbf{Spec}(X), \tau_{\mathfrak{z}})$ is quasi-compact.

This fact is a special case of a much stronger assertion: the open subset $\mathcal{U}$ of the space $(\mathbf{Spec}(X), \tau_{\mathfrak{z}})$ is quasi-compact iff $\mathcal{U} = U(\mathbb{T}_\alpha) = \mathbf{Spec}(X) - V(\mathbb{T}_\alpha)$ for a finitely generated two-sided ideal α of the ring R (cf. C1.2.1).

Two different proofs of this theorem can be found in [R]: I.5.6 and III.6.5.3.1. One of its consequences is that quasi-compact open sets form a base of the Zariski topology on $\mathbf{Spec}(X)$. In fact, every two-sided ideal α is the supremum of a set $\{\alpha_i \mid i \in J\}$ of its two-sided subideals, so that $U(\mathbb{T}_\alpha) = U(sup(\mathbb{T}_{\alpha_i} \mid i \in J)) = \bigcup_{i \in J} U(\mathbb{T}_{\alpha_i})$ (see C1.2.1).

C1.6.4. Note. Unlike the commutative case, the Zariski topology is trivial or too coarse in many important examples of noncommutative affine schemes. It follows from the previous discussion that if $C_X = R - mod$, then the Zariski topology on $\mathbf{Spec}(X)$ is trivial iff R is a simple ring (i.e. it does not have non-trivial two-sided ideals). In particular, the Zariski topology on $\mathbf{Spec}(X)$ is trivial if C_X is the category of D-modules on the affine space $\mathbb{A}^n$, because the algebra A_n of differential operators on $\mathbb{A}^n$ is simple. It is not sufficiently rich in the case when C_X is the category of representations of a semisimple Lie algebra over a field of characteristic zero.

C1.7. Some other canonical topologies. A way to define a topology on $\mathbf{Spec}^0_{\mathfrak{t}}(X)$ (and on $\mathbf{Spec}(X)$) is to single out a class of topologizing subcategories, Ξ, of C_X, take the smallest monoidal subcategory $\mathfrak{G}_\Xi$ of $(\mathfrak{T}(X), \bullet)$ which contains Ξ and is closed under arbitrary intersections (which are products in $(\mathfrak{T}(X), \subseteq)$) and obtain this way the topology $\tau^0_{\mathfrak{G}_\Xi}$. This is the same as taking the smallest topology on $\mathbf{Spec}^0_{\mathfrak{t}}(X)$ for which the sets $V^0_{\mathfrak{t}}(\mathbb{T})$, $\mathbb{T} \in \Xi$, are closed.

C1.7.1. The topology τ^*. For instance, taking as Ξ the class of all topologizing subcategories $[M]$, where M is an object of finite type, we obtain a topology τ^* on $\mathbf{Spec}(X)$ which in the case when C_X is the category of modules over a commutative ring (more generally, the category of quasi-coherent sheaves on a quasi-compact quasi-separated scheme; see C1.2.6 above) coincides with the Zariski topology. It is drastically different in most of noncommutative cases. For any simple ring R (in particular, for any Weyl algebra A_n), the Zariski topology is trivial, while the topology τ^* separates distinct points of the spectrum in Kolmogorov's sense, i.e. $(\mathbf{Spec}(X), \tau^*)$ is a Kolmogorov's space.

C1.7.2. The topology $\tau_{\mathfrak{s}}$. We take as Ξ the set $\mathbf{Spec}(X)$ and denote the corresponding topology on $\mathbf{Spec}(X)$ by $\tau_{\mathfrak{s}}$. This means that finite unions of sets $V(\mathcal{P})$ form a base of the closed sets of the topology $\tau_{\mathfrak{s}}$.

Notice that if the category C_X has enough objects of finite type (i.e. every nonzero object of C_X has a nonzero subobject of finite type), then the topology $\tau_{\mathfrak{s}}$ is coarser than the topology τ^*. In fact, in this case every element $\mathcal{P}$ of $\mathbf{Spec}(X)$ is of the form $[M]$ for some object M of finite type.

C1.8. Functorialities.

C1.8.1. Proposition. *Let C_X and C_Y be abelian categories, and let $X \xrightarrow{f} Y$ be a continuous morphism such that adjunction arrows $f^*f_* \xrightarrow{\epsilon_f} Id_{C_X}$ and $Id_{C_Y} \xrightarrow{\eta_f} f_*f^*$ are monomorphisms. Then the map $\mathbb{T} \longmapsto [f^{*^{-1}}(\mathbb{T})]$ defines a morphism of monoids $(\mathfrak{T}(X), \bullet) \xrightarrow{\mathfrak{T}(f)} (\mathfrak{T}(Y), \bullet)$.*

Proof. (a) Let $X \xrightarrow{f} Y$ be a morphism such that f^* is *semi-exact*; i.e. f^* maps any exact sequence $M' \longrightarrow M \longrightarrow M''$ to an exact sequence (for instance, f^* is right, or left exact). Then $f^{*^{-1}}(\mathbb{T}) \bullet f^{*^{-1}}(\mathbb{S}) \subseteq f^{*^{-1}}(\mathbb{T} \bullet \mathbb{S})$ for any pair $\mathbb{T}$, $\mathbb{S}$ of subcategories of C_X. In particular, $[f^{*^{-1}}(\mathbb{T})] \bullet [f^{*^{-1}}(\mathbb{S})] \subseteq [f^{*^{-1}}(\mathbb{T} \bullet \mathbb{S})]$.

In fact, if $M' \longrightarrow M \longrightarrow M''$ is an exact sequence with $f^*(M') \in Ob\mathbb{S}$ and $f^*(M'') \in Ob\mathbb{T}$, then $f^*(M) \in Ob\mathbb{T} \bullet \mathbb{S}$, because the sequence $f^*(M') \longrightarrow f^*(M) \longrightarrow f^*(M'')$ is exact, due to the semi-exactness of the functor f^*.

(a1) Notice that the inverse image functor of a continuous morphism is right exact, hence semi-exact.

(b) In order to prove the inverse inclusion, $[f^{*^{-1}}(\mathbb{T})] \bullet [f^{*^{-1}}(\mathbb{S})] \supseteq [f^{*^{-1}}(\mathbb{T} \bullet \mathbb{S})]$, it suffices to show that $[f^{*^{-1}}(\mathbb{T})] \bullet [f^{*^{-1}}(\mathbb{S})] \supseteq f^{*^{-1}}(\mathbb{T} \bullet \mathbb{S})$.

Let $f^*(M) \in Ob\mathbb{T} \bullet \mathbb{S}$; i.e. there is an exact sequence $L' \longrightarrow f^*(M) \longrightarrow L''$ with $L' \in Ob\mathbb{S}$ and $L'' \in Ob\mathbb{T}$. Consider the commutative diagram

$$\begin{array}{ccccc}
f^*f_*(L') & \longrightarrow & f^*f_*f^*(M) & \longrightarrow & f^*f_*(L') \\
{\scriptstyle \epsilon_f(L')} \big\downarrow & & {\scriptstyle \epsilon_f f^*(M)} \big\downarrow & & \big\downarrow {\scriptstyle \epsilon_f(L'')} \\
L' & \longrightarrow & f^*(M) & \longrightarrow & L''
\end{array}$$

Since ϵ_f is a monomorphism and $\epsilon_f f^*(M)$ is a strict epimorphism (coretraction), $\epsilon_f f^*(M)$ is an isomorphism. The monomorphness of $\epsilon_f(L')$ and $\epsilon_f(L'')$ imply that $f^*f_*(L') \in Ob\mathbb{S}$

and $f^*f_*(L'') \in Ob\mathbb{T}$. Thus, we have an exact sequence $f_*(L') \longrightarrow f_*f^*(M) \longrightarrow f_*(L'')$ with $f_*(L') \in Obf^{*^{-1}}(\mathbb{S})$ and $f_*(L'') \in Obf^{*^{-1}}(\mathbb{T})$, hence $f_*f^*(M) \in Obf^{*^{-1}}(\mathbb{T}) \bullet f^{*^{-1}}(\mathbb{S})$. If the adjunction morphism $M \longrightarrow f_*f^*(M)$ is a monoarrow, the object M belongs to the subcategory $[f^{*^{-1}}(\mathbb{T}) \bullet f^{*^{-1}}(\mathbb{S})] = [f^{*^{-1}}(\mathbb{T})] \bullet [f^{*^{-1}}(\mathbb{S})]$. ∎

C1.8.2. Note. The conditions of C1.8.1 hold if C_Y is a coreflective full subcategory of C_X and f^* is the inclusion functor $C_Y \hookrightarrow C_X$. In this case, the adjunction arrow $Id_{C_Y} \xrightarrow{\eta_f} f_*f^*$ is an isomorphism, and the second adjunction arrow, $f^*f_* \xrightarrow{\epsilon_f} Id_{C_X}$, is a monomorphism.

C2. Supports and specializations. Krull filtrations.

C2.1. Support in Spec(X). Let M be an object of an abelian category C_X. The *support* of M in $\mathbf{Spec}(X)$ is the set $Supp(M)$ of all $[P] \in \mathbf{Spec}(X)$ such that $M \succ P$, or, equivalently, $[P] \subseteq [M]$.

C2.2. Supports in Spec$^1(X)$ and in Spec$^-(X)$. The *support* of an object M of C_X in $\mathbf{Spec}^1(X)$ is the set $Supp^1(M)$ of all $\mathcal{P} \in \mathbf{Spec}^1(X)$ such that $M \notin Ob\mathcal{P}$.

The support of M in the S-spectrum is the set

$$Supp^-(M) = Supp^1(M) \bigcap \mathbf{Spec}^-(X) = Supp^1(M) \bigcap \mathfrak{Se}(X).$$

C2.3. Lemma. *Let M be an object of C_X.*

(a) The following conditions are equivalent:

(a1) $\mathcal{P} \in Supp^1(M)$;

(a2) $M \succ L$ for some nonzero object L of $\mathcal{P}^{\circledast} - \mathcal{P}$.

(b) The following conditions are equivalent:

(b1) $\mathcal{P} \in Supp^-(M)$;

(b2) $M \succ L$ for some nonzero object L of $\mathcal{P}_{\circledast} = \mathcal{P}^{\circledast} \cap \mathcal{P}^{\perp}$.

Proof. Let $C_X \xrightarrow{q^*_{\mathcal{P}}} C_{X/\mathcal{P}}$ be the localization functor at $\mathcal{P} \in \mathbf{Spec}^1(X)$.

(a1)⇒(a2). The condition $\mathcal{P} \in Supp^-(M)$ means precisely that $q^*_{\mathcal{P}}(M) \neq 0$. On the other hand, $q^*_{\mathcal{P}}(L')$ is a quasi-final object of $C_{X/\mathcal{P}}$ for every nonzero object L' of $\mathcal{P}^{\circledast} - \mathcal{P}$. Therefore, $q^*_{\mathcal{P}}(M) \succ q^*_{\mathcal{P}}(L')$. The latter means that there exists a diagram

$$M^{\oplus n} \xleftarrow{\mathfrak{j}'} K' \xrightarrow{\mathfrak{e}''} L'' \xrightarrow{\mathfrak{s}} L' \qquad (9)$$

such that $Ker(\mathfrak{j}')$ and $Cok(\mathfrak{s})$ are objects of $\mathcal{P}$, $\mathfrak{e}''$ is an epimorphism and $\mathfrak{s}$ is a monomorphism. Replacing K' by $K = K/Ker(\mathfrak{j}')$ and L' by the cokernel of the composition

$Ker(\mathfrak{j}') \longrightarrow K' \xrightarrow{\mathfrak{e}''} L''$, we obtain the diagram $M^{\oplus n} \xleftarrow{\mathfrak{j}} K \xrightarrow{\mathfrak{e}} L$ in which $\mathfrak{j}$ is a monomorphism and $\mathfrak{e}$ is an epimorphism; i.e. $M \succ L$. Since $q^*_{\mathcal{P}}(L)$ is isomorphic to $q^*_{\mathcal{P}}(L')$ and L' is an object of $\mathcal{P}^{\circledast} - \mathcal{P}$, the object L belongs to $\mathcal{P}^{\circledast} - \mathcal{P}$ too.

(a2)⇒(a1) & (b2)⇒(b1). If $M \succ L$ and $L \notin Ob\mathcal{P}$, then $M \notin Ob\mathcal{P}$, i.e. $\mathcal{P} \in Supp^1(X)$.

(b1)⇒(b2). If $\mathcal{P} \in Supp^-(M)$ and L' is a nonzero object of $\mathcal{P}^{\circledast} \cap \mathcal{P}^{\perp}$, then $q^*_{\mathcal{P}}(M) \succ q^*_{\mathcal{P}}(L')$ which is expressed by the diagram (9). Since this time L' is a $\mathcal{P}$-torsion object, the composition of $Ker(\mathfrak{j}') \longrightarrow K' \xrightarrow{\mathfrak{e}''} L''$ is zero. Therefore, replacing K' by $K = K'/Ker(\mathfrak{j}')$, we obtain a diagram $M^{\oplus n} \xleftarrow{\mathfrak{j}} K \xrightarrow{\mathfrak{e}''} L'' \hookrightarrow L'$ in which $\mathfrak{j}$ is a monomorphism and $\mathfrak{e}''$ is an epimorphism. So that $M \succ L''$, where L'' is a subobject of an object of $\mathcal{P}^{\circledast} \cap \mathcal{P}^{\perp}$, hence L'' belongs to $\mathcal{P}^{\circledast} \cap \mathcal{P}^{\perp}$. ■

C2.4. Proposition. *Let $\mathcal{P}_1$, $\mathcal{P}_2$ be elements of* $\mathbf{Spec}^-(X)$. *Then the following conditions are equivalent:*

(a) $\mathcal{P}_2 \subseteq \mathcal{P}_1$;

(b) for every nonzero object M_1 of $\mathcal{P}_1^{\circledast} \cap \mathcal{P}_1^{\perp}$, there exists a nonzero object M_2 of $\mathcal{P}_2^{\circledast} \cap \mathcal{P}_2^{\perp}$ such that $M_1 \succ M_2$.

(c) There exists a nonzero object M_1 of $\mathcal{P}_1^{\circledast} \cap \mathcal{P}_1^{\perp}$ with the following property: for any nonzero subobject L_1 of M_1, there is an object M_2 of $\mathcal{P}_2^{\circledast} \cap \mathcal{P}_2^{\perp}$ such that $L_1 \succ M_2$.

Proof. (a)⇒(b). If $\mathcal{P}_2 \subseteq \mathcal{P}_1$ and M_1 is a nonzero object of $\mathcal{P}_1^{\circledast} \cap \mathcal{P}_1^{\perp}$), then $\mathcal{P}_2 \in Supp^-(M_1)$. By C2.3(b), there exists an object M_2 of $\mathcal{P}_2^{\circledast} \cap \mathcal{P}_2^{\perp}$ such that $M_1 \succ M_2$.

(b)⇒(a). Suppose that $\mathcal{P}_2 \nsubseteq \mathcal{P}_1$; and let N be an object of $\mathcal{P}_2 - \mathcal{P}_1$. In particular, $\mathcal{P}_1 \in Supp^-(N)$. By C2.3(b), there exists a nonzero object M_1 of $\mathcal{P}_1^{\circledast} \cap \mathcal{P}_1^{\perp}$ such that $N \succ M_1$. By condition (b), $M_1 \succ M_2$ for some nonzero object M_2 of $\mathcal{P}_2^{\circledast} \cap \mathcal{P}_2^{\perp}$ which implies that $N \succ M_2$. The latter is impossible, because $N \in Ob\mathcal{P}_2$ and $M_2 \notin Ob\mathcal{P}_2$. Therefore $\mathcal{P}_2 \subseteq \mathcal{P}_1$.

Obviously, (b)⇒(c).

(c)⇒(a). Replacing X by $X/(\mathcal{P}_1 \cap \mathcal{P}_2)$ and the objects M_1 and M_2 by their images in $C_{X/(\mathcal{P}_1 \cap \mathcal{P}_2)}$, we can assume that $\mathcal{P}_1 \cap \mathcal{P}_2 = 0$. Suppose that $\mathcal{P}_2 \neq 0$. Then $\mathcal{P}_2$ is a local subcategory. If $\mathcal{P}_1 = 0$, then C_X is local too, and nonzero objects of $\mathcal{P}_1^{\circledast} \cap \mathcal{P}_1^{\perp} = 0^{\mathfrak{t}}$ are precisely quasi-final objects of C_X. Since $\mathcal{P}_2 \neq 0$, it contains $0^{\mathfrak{t}}$; in particular, $M_1 \in Ob\mathcal{P}_2$. This contradicts to the condition (c) according to which $M_1 \succ M_2$ for some $M_2 \in \mathcal{P}_2^{\circledast} \cap \mathcal{P}_2^{\perp}$.

Suppose now that both $\mathcal{P}_1$ and $\mathcal{P}_2$ are nonzero, hence both of them are local. There exists a quasi-final object L_2 of $\mathcal{P}_2$ and a monomorphism $L_2 \longrightarrow M_1$ such that $M_1/L_2 \in Ob\mathcal{P}_1$. By condition (c), there exists a nonzero object M_2 of $\mathcal{P}_2^{\circledast} \cap \mathcal{P}_2^{\perp}$ such that $L_2 \succ M_2$. Since $L_2 \in \mathcal{P}_2$, we run into a contradiction again. Altogether shows that $\mathcal{P}_2 = 0$. ■

C2.5. The Krull filtration of $\mathbf{Spec}^-(X)$ and the associated filtration of X. Fix an abelian category C_X. For every cardinal α, we define a subset $\mathfrak{S}^-_{\alpha}(X)$ of $\mathbf{Spec}^-(X)$ as follows.

$\mathfrak{S}_0^-(X) = \emptyset$;

if α is not a limit cardinal, then $\mathfrak{S}_\alpha^-(X)$ consists of all $\mathcal{P} \in \mathbf{Spec}^-(X)$ such that any $\mathcal{P}' \in \mathbf{Spec}^-(X)$ properly contained in $\mathcal{P}$ belongs to $\mathfrak{S}_{\alpha-1}^-(X)$;

if α is a limit cardinal, then $\mathfrak{S}_\alpha^-(X) = \bigcup\limits_{\beta<\alpha} \mathfrak{S}_\beta^-(X)$.

It follows from this definition (borrowed from [R, VI.6.3]) that $\mathfrak{S}_1^-(X)$ consists of all closed points of $\mathbf{Spec}^-(X)$.

We denote by $\mathfrak{S}_\omega^-(X)$ the union of all $\mathfrak{S}_\alpha^-(X)$. The filtration $\{\mathfrak{S}_\alpha^-(X)\}$ determines a filtration

$$C_{X_0} \hookrightarrow C_{X_1} \hookrightarrow \dots C_{X_\alpha} \hookrightarrow \dots \tag{5}$$

of the category C_X (or the 'space' X) by taking as C_{X_α} the full subcategory of C_X generated by objects M such that $Supp^-(M) \subseteq \mathfrak{S}_\alpha^-(X)$. Recall that $Supp^-(M) = \{\mathcal{P} \in \mathbf{Spec}^-(X) \mid M \notin Ob\mathcal{P}\}$. In particular, C_{X_ω} is the full subcategory of C_X generated by all $M \in ObC_X$ such that $Supp^-(M) \subseteq \mathfrak{S}_\omega^-(X)$.

It follows from the general properties of supports that C_{X_α} is a Serre subcategory of C_X and $\mathbf{Spec}^-(X_\alpha)$ is naturally identified with $\mathfrak{S}_\alpha^-(X)$; in particular, $\mathbf{Spec}^-(X_\omega)$ is identified with $\mathfrak{S}_\omega^-(X)$.

C2.6. The Krull dimension. For every element $\mathcal{P}$ of $\mathbf{Spec}^-(X_\omega)$, there is the biggest cardinal, $\mathfrak{ht}^-(\mathcal{P})$, among all the cardinals α such that $\mathcal{P} \notin \mathfrak{S}_\alpha^-(X)$. The cardinal $\mathfrak{ht}^-(\mathcal{P})$ is called the hight of $\mathcal{P}$ ([R, VI.6.3]).

The *Krull dimension* of X is the supremum of all $\mathfrak{ht}^-(\mathcal{P})$, where $\mathcal{P}$ runs through $\mathbf{Spec}^-(X_\omega)$ (in [R] it is called the *flat* dimension).

An object M of C_X is said to have a Krull dimension if it belongs to the subcategory C_{X_ω}. Finally, the 'space' X (or the category C_X) has a Krull dimension if $X = X_\omega$ (that is $C_X = C_{X_\omega}$) and every nonzero object of C_X has a nonempty support, i.e. $C_{X_0} = \mathbb{O}$.

C2.7. The Krull dimension and the Gabriel-Krull dimension. We recall the notion of the *Gabriel filtration* of an abelian category as it is defined in [R, 6.6]. Let C_X be an abelian category. The *Gabriel filtration of* X assigns to every cardinal α a Serre subcategory $C_{X_\alpha^-}$ of C_X which is constructed as follows:

Set $C_{X_0^-} = \mathbb{O}$.

If α is not a limit cardinal, then $C_{X_\alpha^-}$ is the smallest Serre subcategory of C_X containing all objects M such that the localization $q_{\alpha-1}^*(M)$ of M at $C_{X_{\alpha-1}^-}$ has a finite length.

If β is a limit cardinal, then $C_{X_\beta^-}$ is the smallest Serre subcategory containing all subcategories $C_{X_\alpha^-}$ for $\alpha < \beta$.

Let $C_{X_\omega^-}$ denote the smallest Serre subcategory containing all the subcategories $C_{X_\alpha^-}$. It follows that the quotient category C_{X/X_ω^-} has no simple objects.

An object M is said to have the *Gabriel-Krull dimension* β, if β is the smallest cardinal such that M belongs to $C_{X_\beta^-}$.

The 'space' X has a Gabriel-Krull dimension if $X = X_\omega^-$.

Every locally noetherian abelian category (e.g. the category of quasi-coherent sheaves on a noetherian scheme, or the category of left modules over a left noetherian associative algebra) has a Gabriel-Krull dimension.

It is argued in [R, VI.6] that if X has a Gabriel-Krull dimension, then the filtration (5) coincides with the Gabriel filtration of the category C_X. In particular, X has a Krull dimension: $X = X_\omega = X_\omega^-$. Thus, the Krull dimension is an extension of the Gabriel-Krull dimension to a wider class of 'spaces'.

C2.8. A description of $\mathbf{Spec}_\rtimes(X_\omega)$. The filtration $\{\mathfrak{S}_\alpha^-(X)\}$ of $\mathbf{Spec}^-(X)$ induces, via the isomorphism $\mathbf{Spec}_\rtimes(X) \xrightarrow{\sim} \mathbf{Spec}^-(X)$ (defined in 6.4), a filtration $\{\mathfrak{S}_\alpha^\rtimes(X)\}$ of the spectrum $\mathbf{Spec}_\rtimes(X)$. We call it the *Krull* filtration of $\mathbf{Spec}_\rtimes(X)$.

C2.8.1. Proposition. *The spectrum $\mathbf{Spec}_\rtimes(X_\omega)$ of X_ω is naturally isomorphic to $\bigcup\limits_\alpha \mathbf{Spec}_\rtimes(X_\alpha/X_{\alpha-1})$, and $\mathbf{Spec}^-(X_\omega)$ is isomorphic to $\bigcup\limits_\alpha \mathbf{Spec}^-(X_\alpha/X_{\alpha-1})$, where α runs through non-limit cardinals. These isomorphisms are compatible with the isomorphisms $\mathbf{Spec}_\rtimes(X_\omega) \xrightarrow{\sim} \mathbf{Spec}^-(X_\omega)$ and $\mathbf{Spec}_\rtimes(X_\alpha/X_{\alpha-1}) \xrightarrow{\sim} \mathbf{Spec}^-(X_\alpha/X_{\alpha-1})$.*

Proof. More precisely,

$$\mathbf{Spec}_\rtimes(X_\omega) = \bigcup_\alpha (\mathfrak{S}_\alpha^\rtimes(X) - \mathfrak{S}_{\alpha-1}^\rtimes(X)),$$

where α runs though non-limit cardinals, and for every non-limit cardinal α, there is a natural isomorphism

$$\mathfrak{S}_\alpha^\rtimes(X) - \mathfrak{S}_{\alpha-1}^\rtimes(X) \xrightarrow{\sim} \mathbf{Spec}_\rtimes(X_\alpha/X_{\alpha-1}). \tag{6}$$

The isomorphism (6) is given by the map $\mathfrak{S}_\alpha^\rtimes(X) \longrightarrow \mathfrak{T}(X/X_{\alpha-1})$ which assigns to every element $\mathcal{P}_\circledast$ of $\mathfrak{S}_\alpha^\rtimes(X)$ the smallest topologizing subcategory $[q_{\alpha-1}^*(\mathcal{P}_\circledast)]$ of $C_{X/X_{\alpha-1}}$ spanned by the image of $\mathcal{P}_\circledast$.

Let $\mathcal{P} \in \mathbf{Spec}^-(X_\omega)$, i.e. $\mathcal{P} \in \mathfrak{S}_\alpha(X)$ for some α. Consider all cardinals β such that $C_{X_\beta} \subseteq \mathcal{P}$. Since $\mathcal{P}$ is a Serre subcategory, the smallest Serre subcategory spanned by all C_{X_β} coincides with $C_{X_{\alpha-1}}$ for a non-limit cardinal α. The image $\mathcal{P}_\circledast = \mathcal{P}^\circledast \cap \mathcal{P}^\perp$ of $\mathcal{P}$ in $\mathbf{Spec}_\rtimes(X)$ is an element of $\mathfrak{S}_\alpha^\rtimes(X) - \mathfrak{S}_{\alpha-1}^\rtimes(X)$. ■

C2.9. The Krull filtrations and equivalences of categories.

C2.9.1. Proposition. *Let C_X and C_Y be abelian categories. Any category equivalence $C_X \xrightarrow{\Theta} C_Y$ induces equivalences $C_{X_\alpha} \xrightarrow{\Theta_\alpha} C_{Y_\alpha}$ for all cardinals α. In particular, Θ induces a category equivalence $C_{X_\omega} \xrightarrow{\Theta_\omega} C_{Y_\omega}$*

Proof. The argument is by (transfinite) induction. The assertion is, obviously, true for $\alpha = 0$. It is also true for $\alpha = 1$: if $\mathcal{P}$ is a closed point of $\mathbf{Spec}^-(X)$, then $[\Theta(\mathcal{P})]$ is a closed point of $\mathbf{Spec}^-(Y)$.

Suppose now that Θ induces equivalences $C_{X_\alpha} \xrightarrow{\Theta_\alpha} C_{Y_\alpha}$ for all cardinals $\alpha < \beta$. We claim that then it induces a category equivalence $C_{X_\beta} \xrightarrow{\Theta_\beta} C_{Y_\beta}$.

(a) If β is a limit cardinal, then it follows from the definition of the filtration (cf. C2.5), that $C_{X_\beta} = \big(\bigcup\limits_{\alpha<\beta} C_{X_\alpha}\big)^-$. By the induction hypothesis, Θ induces a category equivalence $\bigcup\limits_{\alpha<\beta} C_{X_\alpha} \longrightarrow \bigcup\limits_{\alpha<\beta} C_{Y_\alpha}$. It is easy to show that if Θ induces an equivalence between a subcategory $\mathbb{T}$ of C_X and a subcategory $\mathbb{S}$ of C_Y, then Θ induces an equivalence $\mathbb{T}^- \longrightarrow \mathbb{S}^-$. In particular, Θ induces a category equivalence from $C_{X_\beta} = \big(\bigcup\limits_{\alpha<\beta} C_{X_\alpha}\big)^-$ to $C_{Y_\beta} = \big(\bigcup\limits_{\alpha<\beta} C_{Y_\alpha}\big)^-$.

(b) Suppose now that β is not a limit cardinal. By the induction hypothesis, Θ induces a category equivalence $C_{X_{\beta-1}} \longrightarrow C_{Y_{\beta-1}}$; hence Θ induces an equivalence between quotient categories $C_{X/X_{\beta-1}} \xrightarrow{\widehat{\Theta}_{\beta-1}} C_{Y/Y_{\beta-1}}$. The equivalence $\widehat{\Theta}_{\beta-1}$ induces an equivalence $C_{(X/X_{\beta-1})_1} \longrightarrow C_{(Y/Y_{\beta-1})_1}$. Notice that C_{X_β} is the preimage of $C_{(X/X_{\beta-1})_1}$ in C_X. Similarly for C_{Y_β}. Therefore Θ induces a functor $C_{X_\beta} \xrightarrow{\Theta_\beta} C_{Y_\beta}$ and its quasi-inverse, Θ^*, induces a functor $C_{Y_\beta} \xrightarrow{\Theta^*_\beta} C_{X_\beta}$. Since Θ is an equivalence, Θ_β is an equivalence with a quasi-inverse Θ^*_β. ■

C2.9.2. Proposition. *Any category equivalence $C_X \xrightarrow{\Theta} C_Y$ induces isomorphisms*

$$\mathfrak{S}^-_\alpha(X) \xrightarrow{\sim} \mathfrak{S}^-_\alpha(Y) \quad \textit{and} \quad \mathfrak{S}^\rtimes_\alpha(X) \xrightarrow{\sim} \mathfrak{S}^\rtimes_\alpha(Y)$$

for all cardinals α. In particular, Θ induces an isomorphisms

$$\mathfrak{S}^-_\omega(X) \xrightarrow{\sim} \mathfrak{S}^-_\omega(Y) \quad \textit{and} \quad \mathfrak{S}^\rtimes_\omega(X) \xrightarrow{\sim} \mathfrak{S}^\rtimes_\omega(Y).$$

Proof. The assertion follows from C2.9.1 and from the fact that the natural isomorphisms
$$\mathfrak{S}^-_\alpha(X) \simeq \mathbf{Spec}^-(X_\alpha), \quad \mathfrak{S}^{\rtimes}_\alpha(X) \simeq \mathbf{Spec}_{\rtimes}(X_\alpha)$$
are compatible with category equivalences for all cardinals α. In particular, we have commutative diagrams

$$\begin{array}{ccc} \mathfrak{S}^-_\omega(X) & \xrightarrow{\sim} & \mathfrak{S}^-_\omega(Y) \\ \wr\downarrow & & \downarrow\wr \\ \mathbf{Spec}^-(X_\omega) & \xrightarrow{\sim} & \mathbf{Spec}^-(Y_\omega) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathfrak{S}^{\rtimes}_\omega(X) & \xrightarrow{\sim} & \mathfrak{S}^{\rtimes}_\omega(Y) \\ \wr\downarrow & & \downarrow\wr \\ \mathbf{Spec}_{\rtimes}(X_\omega) & \xrightarrow{\sim} & \mathbf{Spec}_{\rtimes}(Y_\omega) \end{array}$$

Details are left to the reader. ■

C2.9.3. Corollary. *Let C_X be an abelian category and $C_X \xrightarrow{\Theta} C_X$ an autoequivalence.*

(a) If $\mathcal{P} \in \mathbf{Spec}^-(X_\omega)$, then $\Theta(\mathcal{P}) \subseteq \mathcal{P} \Leftrightarrow [\Theta(\mathcal{P})] = \mathcal{P} \Leftrightarrow \mathcal{P} \subseteq [\Theta(\mathcal{P})]$.

(b) If $\mathcal{P}_\circledast \in \mathbf{Spec}_{\rtimes}(X_\omega)$, then $\Theta(\mathcal{P}_\circledast) \subseteq \mathcal{P}_\circledast \Leftrightarrow [\Theta(\mathcal{P}_\circledast)) = \mathcal{P}_\circledast \Leftrightarrow \mathcal{P}_\circledast \subseteq [\Theta(\mathcal{P}_\circledast))$.

(a) If $\mathcal{P} \in \mathbf{Spec}(X_\omega)$, then $\Theta(\mathcal{P}) \subseteq \mathcal{P} \Leftrightarrow [\Theta(\mathcal{P})] = \mathcal{P} \Leftrightarrow \mathcal{P} \subseteq [\Theta(\mathcal{P})]$.

Here $[\Theta(\mathcal{P})]$ and $[\Theta(\mathcal{P}_\circledast))$ coincide with strictly full subcategories of C_X generated by resp. $\Theta(\mathcal{P})$ and $\Theta(\mathcal{P}_\circledast)$.

Proof. (a) (i) Let $\Theta(\mathcal{P}) \subseteq \mathcal{P}$. If $\mathcal{P} \nsubseteq [\Theta(\mathcal{P})]$, then $\mathfrak{ht}^-([\Theta(\mathcal{P})]) < \mathfrak{ht}^-(\mathcal{P})$. By C2.9.1, this implies that $\mathfrak{ht}^-([\Theta^*\Theta(\mathcal{P})]) \leq \mathfrak{ht}^-([\Theta(\mathcal{P})]) < \mathfrak{ht}^-(\mathcal{P})$. But, since Θ^* is a quasi-inverse to Θ, $[\Theta^*\Theta(\mathcal{P})] = \mathcal{P}$. Therefore $\mathcal{P} = [\Theta(\mathcal{P})]$.

(ii) The implication $\mathcal{P} \subseteq [\Theta(\mathcal{P})] \Rightarrow [\Theta(\mathcal{P})] = \mathcal{P}$ follows from (i), because the inclusion $\mathcal{P} \subseteq [\Theta(\mathcal{P})]$ is equivalent to the inclusion $\Theta^*(\mathcal{P}) \subseteq \mathcal{P}$.

(b) The assertion (b) follows from (a) and the observation that the isomorphism $\mathbf{Spec}^-(X) \xrightarrow{\sim} \mathbf{Spec}_{\rtimes}(X)$ (cf. 6.4) is compatible with the actions of auto-equivalences on resp. $\mathbf{Spec}^-(X)$ and $\mathbf{Spec}_{\rtimes}(X)$.

(c) The assertion (c) follows from (b) and an observation that the canonical embedding
$$\mathbf{Spec}(X) \longrightarrow \mathbf{Spec}_{\rtimes}(X), \quad \mathcal{P} \longmapsto \mathcal{P} \cap \langle \mathcal{P} \rangle^\perp$$
is compatible with the actions of auto-equivalences on resp. $\mathbf{Spec}(X)$ and $\mathbf{Spec}_{\rtimes}(X)$. Details are left to the reader. ■

C3. Local properties of spectra and closed points.

C3.1. Closed points of spectra and Gabriel-Krull dimension. If X has a Gabriel-Krull dimension, then the set $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)_1$ of the closed points of $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$

coincides with the set $\mathbf{Spec}^-(X)_1$ of the closed points of $\mathbf{Spec}^-(X)$. Since in this case $\mathbf{Spec}^-(X) = \mathbf{Spec}^1_{\mathfrak{S}\mathfrak{e}}(X)$, the spectra $\mathbf{Spec}^1_{\mathfrak{S}\mathfrak{e}}(X)$, $\mathbf{Spec}^-(X)$, and $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$ have the same sets of closed points.

C3.2. Lemma. *Suppose that every nonzero object of C_X has a non-empty support in $\mathbf{Spec}(X)$ (for instance, C_X has enough objects of finite type; cf. C2.1). Then for any closed point $\mathcal{P}$ in $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$ and for any thick subcategory $\mathbb{T}$ of C_X such that $\mathbb{T} \subseteq \mathcal{P}$, the subcategory $\mathcal{P}/\mathbb{T}$ of $C_X/\mathbb{T}$ is a closed point in $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X/\mathbb{T})$.*

Proof. By 3.2(ii), $\mathcal{P} = \widehat{\mathcal{Q}}$ for a uniquely determined by this equality element $\mathcal{Q}$ of $\mathbf{Spec}(X)$, which is a closed point in $\mathbf{Spec}(X)$, since $\mathcal{P}$ is a closed point in $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$. We claim that image $[q^*(\mathcal{Q})]$ of $\mathcal{Q}$ is $\mathcal{T}(X/\mathbb{T})$ is a closed point of $\mathbf{Spec}(X/\mathbb{T})$.

In fact, let $\mathcal{Q}'$ be a nonzero topologizing subcategory of $C_X/\mathbb{T}$ contained in $[q^*(\mathcal{Q})]$. This means that the preimage $\mathcal{Q}'' = {q^*}^{-1}(\mathcal{Q}')$ of $\mathcal{Q}'$ in C_X is a topologizing subcategory of C_X which does not contain $\mathbb{T}$ and is contained in ${q^*}^{-1}([q^*(\mathcal{Q})])$. By 5.1.1, ${q^*}^{-1}([q^*(\mathcal{Q})]) = \mathbb{T} \bullet \mathcal{Q} \bullet \mathbb{T}$. Any object M of the subcategory $\mathbb{T} \bullet \mathcal{Q} \bullet \mathbb{T}$ can be described by the diagram

$$\begin{array}{ccccccccc}
 & & M_1' & \longleftarrow & 0 & & & & \\
 & & \big\downarrow & & & & & & \\
0 & \longrightarrow & M_1 & \longrightarrow & M & \longrightarrow & M_2 & \longrightarrow & 0 \\
 & & \big\downarrow & & & & & & \\
 & & M_1'' & \longrightarrow & 0 & & & &
\end{array} \qquad (1)$$

which incorporates two short exact sequences such that the objects M_1' and M_2, belong to $\mathbb{T}$, and $M_1'' \in Ob\mathcal{Q}$. One can see from this description that M is an object of $\mathbb{T} \bullet \mathcal{Q} \bullet \mathbb{T} - \mathbb{T}$ iff M_1'' is an object of $\mathcal{Q} - \mathbb{T}$. It follows from the diagram (1) that $M_1'' \in Ob[M]$. Since $\mathbb{T} \not\supseteq \mathcal{Q}'' \subseteq \mathbb{T} \bullet \mathcal{Q} \bullet \mathbb{T}$, the topologizing subcategory $\mathcal{Q}'' \cap \mathcal{Q}$ is not contained in $\mathbb{T}$. In particular, it is nonzero. Let M be a nonzero element of $\mathcal{Q}'' \cap \mathcal{Q}$. By hypothesis, $Supp(M)$ is non-empty; i.e. there exists an element $\widetilde{\mathcal{Q}}$ of $\mathbf{Spec}(X)$ such that $\widetilde{\mathcal{Q}} \subseteq [M]$. Thus, we have inclusions $\widetilde{\mathcal{Q}} \subseteq [M] \subseteq \mathcal{Q}'' \cap \mathcal{Q} \subseteq \mathcal{Q}$. Since $\mathcal{Q}$ is a closed point of $\mathbf{Spec}(X)$, the inclusion $\widetilde{\mathcal{Q}} \subseteq \mathcal{Q}$ implies that $\widetilde{\mathcal{Q}} = \mathcal{Q}$. Therefore the inclusions above can be replaced by equalities. In particular, $\mathcal{Q}'' \cap \mathcal{Q} = \mathcal{Q}$, that is $\mathcal{Q} \subseteq \mathcal{Q}''$ which means that $\mathcal{Q}'$ coincides with $[q^*(\mathcal{Q})] = (\mathbb{T} \bullet \mathcal{Q} \bullet \mathbb{T})/\mathbb{T}$. ∎

C3.2.1. Corollary. *Suppose that every nonzero object of C_X has a non-empty support in $\mathbf{Spec}(X)$. Then every closed point of $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$ is a closed point of $\mathbf{Spec}^1(X)$.*

Proof. Let $\mathcal{P}$ be a closed point of $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$; and let $\mathcal{P}_1$ be an element of $\mathbf{Spec}^1(X)$ such that $\mathcal{P}_1 \subseteq \mathcal{P}$. By C3.2, $\mathcal{P}/\mathcal{P}_1$ is a closed point of $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X/\mathcal{P}_1)$. But, $X/\mathcal{P}_1$ is a local 'space', hence it has a unique closed point – 0. This shows that $\mathcal{P}/\mathcal{P}_1 = 0$, i.e. $\mathcal{P} = \mathcal{P}_1$. ∎

C3.3. Proposition. *Suppose that C_X is an abelian category with the property (sup). Let $\{\mathcal{T}_i \mid i \in J\}$ be a finite set of Serre subcategories of C_X such that $\bigcap_{i\in J} \mathcal{T}_i = 0$. Then*

(a) A point $\mathcal{P}$ of $\mathbf{Spec}^-(X)$ is closed iff $\mathcal{P}/\mathcal{T}_i$ is a closed point of $\mathbf{Spec}^-(X/\mathcal{T}_i)$ for every $i \in J$ such that $\mathcal{T}_i \subseteq \mathcal{P}$.

(b) Suppose that every nonzero object of C_X has a nonzero support in $\mathbf{Spec}(X)$. Then a point $\mathcal{P}$ of $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$ is closed iff $\mathcal{P}/\mathcal{T}_i$ is a closed point of $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X/\mathcal{T}_i)$ for every $i \in J$ such that $\mathcal{T}_i \subseteq \mathcal{P}$.

Proof. (a) If $\mathcal{P} \in \mathbf{Spec}^-(X)$, then $\mathcal{P}/\mathcal{T}_i \in \mathbf{Spec}^-(X/\mathcal{T}_i)$ for all i such that $\mathcal{T}_i \subseteq \mathcal{P}$. And if $\mathcal{P}$ is a closed point, then $\mathcal{P}/\mathcal{T}_i$ is a closed point.

In fact, if $\mathcal{T}_i \subseteq \mathcal{P}$, then $\mathcal{P}/\mathcal{T}_i$ is an element of $\mathbf{Spec}^1(X/\mathcal{T}_i)$; and $\mathcal{P}/\mathcal{T}_i$ is a Serre subcategory of $C_X/\mathcal{T}_i$ (due to the reflectivity of the Serre subcategory $\mathcal{T}_i$ which is a consequence of the property (sup)). Therefore, it belongs to $\mathbf{Spec}^-(X/\mathcal{T}_i)$. If $\mathcal{P}' \in \mathbf{Spec}^-(X/\mathcal{T}_i)$ and $\mathcal{P}' \subseteq \mathcal{P}$, then the preimage $\mathcal{P}''$ of $\mathcal{P}'$ in C_X is a Serre subcategory which belongs to $\mathbf{Spec}^-(X)$ and is contained in $\mathcal{P}$. Thus, if $\mathcal{P}$ is a closed point of $\mathbf{Spec}^-(X/\mathcal{T}_i)$, then $\mathcal{P}'' = \mathcal{P}$, hence $\mathcal{P}' = \mathcal{P}/\mathcal{T}_i$.

(a1) Conversely, let $\mathcal{P}/\mathcal{T}_i$ be closed for all $i \in J$ such that $\mathcal{T}_i \subseteq \mathcal{P}$. Then we claim that $\mathcal{P}$ is closed. If the number $Card(J) = 1$, then the statement is true by a trivial reason. In the general case, let $\mathcal{P}'$ be an element of $\mathbf{Spec}^-(X)$ such that $\mathcal{P}' \subseteq \mathcal{P}$. And let $J^{\mathcal{P}'}$ denote the set $\{i \in J \mid \mathcal{T}_i \nsubseteq \mathcal{P}'\}$. Since J is finite, by 9.3, there exists $i \in J$ such that $\mathcal{T}_i \subseteq \mathcal{P}'$. Therefore $Card(J^{\mathcal{P}'}) < Card(J)$. By (the end of the argument of) C1.4.1(c) (or [R4, 4.2]),

$$\mathcal{P}' = (\bigcap_{i\in J} \mathcal{T}_i) \vee \mathcal{P}' = \bigcap_{i\in J} (\mathcal{T}_i \vee \mathcal{P}') = \bigcap_{i\in J^{\mathcal{P}'}} (\mathcal{T}_i \vee \mathcal{P}').$$

So that $\{\mathcal{T}'_i = (\mathcal{T}_i \vee \mathcal{P}')/\mathcal{P}',\ i \in J^{\mathcal{P}'}\}$ is a set of Serre subcategories of $C_X/\mathcal{P}'$ whose intersection is zero. The point $\widetilde{\mathcal{P}} = \mathcal{P}/\mathcal{P}'$ of $\mathbf{Spec}^-(X/\mathcal{P}')$ is such that that $\widetilde{\mathcal{P}}/\mathcal{T}'_i$ is a closed point of $\mathbf{Spec}^-(X/(\mathcal{T}_i \vee \mathcal{P}')$ for all $i \in J^{\mathcal{P}'}$ such that $\mathcal{T}'_i \subseteq \widetilde{\mathcal{P}}$. Since $Card(J^{\mathcal{P}'}) < Card(J)$, by induction hypothesis, $\widetilde{\mathcal{P}}$ is a closed point of $X/\mathcal{P}'$. The latter 'space' being local, this means that $\widetilde{\mathcal{P}} = 0$, or, equivalently, $\mathcal{P} = \mathcal{P}'$.

(b) If $\mathcal{P}$ is a closed point of $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$, then, by C3.2, $\mathcal{P}/\mathcal{T}_i$ is a closed point of $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X/\mathcal{T}_i)$ for every $i \in J$ such that $\mathcal{T}_i \subseteq \mathcal{P}$.

Conversely, suppose that $\mathcal{P} \in \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$ is such that $\mathcal{P}/\mathcal{T}_i$ is a closed point of the spectrum $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X/\mathcal{T}_i)$ if $\mathcal{T}_i \subseteq \mathcal{P}$. Let $\mathcal{P}'$ be an element of $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$ such that $\mathcal{P}' \subseteq \mathcal{P}$. By 9.3, there exists $i \in J$ such that $\mathcal{T}_i \subseteq \mathcal{P}'$; in particular, $\mathcal{T}_i \subseteq \mathcal{P}$. Since $\mathcal{P}'/\mathcal{T}_i$ is a point of $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X/\mathcal{T}_i)$ and $\mathcal{P}/\mathcal{T}_i$ is a closed point, the inclusion $\mathcal{P}'/\mathcal{T}_i \subseteq \mathcal{P}/\mathcal{T}_i$ implies that $\mathcal{P}'/\mathcal{T}_i$ and $\mathcal{P}/\mathcal{T}_i$ coincide. Therefore, $\mathcal{P}' = \mathcal{P}$. ∎

C3.3.1. Corollary. *Suppose that C_X is an abelian category with the property (sup). Let $\{\mathcal{T}_i \mid i \in J\}$ be a finite set of Serre subcategories such that $\bigcap_{i\in J} \mathcal{T}_i = 0$ and for every $i \in J$, any element of $\mathbf{Spec}^-(X/\mathcal{T}_i)$ contains a closed point of $\mathbf{Spec}^-(X/\mathcal{T}_i)$. Then every element of $\mathbf{Spec}^-(X)$ contains a closed point of $\mathbf{Spec}^-(X)$.*

Proof. Let $\mathcal{P} \in \mathbf{Spec}^-(X)$. Since $\bigcap_{i\in J} \mathcal{T}_i = 0$, there exists $J_{\mathcal{P}} = \{i \in J \mid \mathcal{T}_i \subseteq \mathcal{P}\}$ is non-empty. Fix an $i \in J_{\mathcal{P}}$. By hypothesis, $\mathcal{P}_i \subseteq \mathcal{P}$, where $\mathcal{P}_i/\mathcal{T}_i$ is a closed point of $\mathbf{Spec}^-(X/\mathcal{T}_i)$. If $J_{\mathcal{P}_i} = \{i\}$, then, by C3.3(a), $\mathcal{P}_i$ is a closed point of $\mathbf{Spec}^-(X)$. If $J_{\mathcal{P}_i}$ contains more than one element, we take $j \in J_{\mathcal{P}_i} - \{i\}$ and repeat the argument replacing $\mathcal{P}$ by $\mathcal{P}_i$; and so on. Since J is finite, the process stabilizes. As a result, we find an element $\mathcal{P}'$ of $\mathbf{Spec}^-(X)$ such that $\mathcal{P}' \subseteq \mathcal{P}$ and $\mathcal{P}'/\mathcal{T}_j$ is a closed point of $\mathbf{Spec}^-(X/\mathcal{T}_j$ for every $j \in J_{\mathcal{P}'}$. By C3.3(a), the latter means that $\mathcal{P}'$ is a closed point of $\mathbf{Spec}^-(X)$. ∎

C3.3.2. Corollary. *Suppose that C_X is an abelian category with the property (sup). Let $\{\mathcal{T}_i \mid i \in J\}$ be a finite set of Serre subcategories such that $\bigcap_{i\in J} \mathcal{T}_i = 0$ and for every $i \in J$, the set $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X/\mathcal{T}_i)_1$ of the closed points of $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X/\mathcal{T}_i)$ contains the set $\mathbf{Spec}^-(X/\mathcal{T}_i)_1$ of the closed points of $\mathbf{Spec}^-(X/\mathcal{T}_i)$. Then $\mathbf{Spec}^-(X)_1 \subseteq \mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)_1$.*

Suppose that, in addition, one of the following conditions holds:

(a) For all $i \in J$, every element of $\mathbf{Spec}^-(X/\mathcal{T}_i)$ contains a closed point.

(b) Every nonzero object of C_X has a non-empty support in $\mathbf{Spec}(X)$.

Then $\mathbf{Spec}^-(X)_1$ and $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)_1$ coincide.

Proof. Let $\mathcal{P}$ be a closed point of $\mathbf{Spec}^-(X)$. By C3.3, $\mathcal{P}/\mathcal{T}_i$ is a closed point of $\mathbf{Spec}^-(X/\mathcal{T}_i)$ for all $i \in J_{\mathcal{P}} = \{j \in J \mid \mathcal{T}_j \subseteq \mathcal{P}\}$. By hypothesis, $\mathcal{P}/\mathcal{T}_i$ is a closed point of $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X/\mathcal{T}_i)$ for all $i \in J_{\mathcal{P}}$. By 7.1, $\mathcal{P} \in \mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$. Since $\mathcal{P}$ is a closed point of the space $\mathbf{Spec}^-(X)$, it is, definitely, a closed point of its subspace $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$. This shows the inclusion $\mathbf{Spec}^-(X)_1 \subseteq \mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)_1$.

(a) Let $\mathcal{P} \in \mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$. Since $\mathcal{P}$ is an element of $\mathbf{Spec}^-(X)$, by C3.3.1, $\mathcal{P} \supseteq \mathcal{P}'$, where $\mathcal{P}'$ is a closed point of $\mathbf{Spec}^-(X)$. By C3.3(a), for every $i \in J_{\mathcal{P}'} = \{j \in J \mid \mathcal{T}_j \subseteq \mathcal{P}'\}$, the quotient subcategory $\mathcal{P}'/\mathcal{T}_i$ is a closed point of $\mathbf{Spec}^-(X/\mathcal{T}_i)$, hence, by hypothesis, it belongs to $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X/\mathcal{T}_i)$. By 7.1, the latter implies that $\mathcal{P}'$ belongs to $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$. Since it $\mathcal{P}'$ is a closed point of $\mathbf{Spec}^-(X)$, it is a closed point of $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$.

(b) If every nonzero object of C_X has a non-empty support, then, by C3.2.1, we have the inverse inclusion: $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)_1 \subseteq \mathbf{Spec}^-(X)_1$. ∎

C3.4. Proposition. *Let C_X be an abelian category and $\{\mathcal{T}_i \mid i \in J\}$ a finite set of thick subcategories such that $\bigcap_{i\in J} \mathcal{T}_i = 0$. Suppose that each category $C_{X/\mathcal{T}_i}$ has enough ob-*

jects of finite type. Then closed points of $\mathbf{Spec}(X)$ *are in a natural bijective correspondence with the isomorphism classes of simple objects of* C_X.

Proof. Let u_i^* denote the localization functor $C_X \longrightarrow C_X/\mathcal{T}_i$. Let M be an object of $Spec(X)$ such that $[M]$ be a closed point of $\mathbf{Spec}(X)$. Since $\bigcap_{i \in J} \mathcal{T}_i = 0$, there is an $i \in J$ such that $M \notin \mathcal{T}_i$. Therefore, $[u_i^*(M)]$ is a closed point of $\mathbf{Spec}(X/\mathcal{T}_i)$. Since the category $C_{X/\mathcal{T}_i} = C_X/\mathcal{T}_i$ has enough objects of finite type, all closed points of $\mathbf{Spec}(X/\mathcal{T}_i)$ correspond to simple objects. In particular, $u_i^*(M)$ is the direct sum of a finite number of copies of a simple object $u_i^*(L)$, and there is a monomorphism $u_i^*(L) \longrightarrow u_i^*(M)$. This monomorphism is described by a diagram $L \xleftarrow{s} L' \xrightarrow{\mathfrak{j}'} M$ such that $u_i^*(s)$ is an isomorphism and $Ker(\mathfrak{j}')$ belongs to $\mathcal{T}_i$. Since the object M is $\mathcal{T}_i$-torsion free, the object $L_i = L'/Ker(\mathfrak{j}')$ is $\mathcal{T}_i$-torsion free too. It follows that $u_i^*(L_i)$ is isomorphic to $u_i^*(L)$. In particular, L_i is a nonzero subobject of M. Since $M \in Spec(X)$, the object L_i also belongs to $Spec(X)$ and $[L_i] = [M]$. So, we replace M by L_i. Repeating this procedure consecutively for all $j \in J$ such that M does not belong to $\mathcal{T}_j$, we replace M by its subobject, N such that for any $j \in J$, the object $u_j^*(N)$ is either zero, or simple. Since N belongs to $Spec(X)$, it follows that for every nonzero monomorphism $N' \xrightarrow{h} N$, its image $u_i^*(h)$ is an isomorphism for every $i \in J$. The condition $\bigcap_{i \in J} \mathcal{T}_i = 0$ means that the family of localization functors $\{C_X \xrightarrow{u_i^*} C_X/\mathcal{T}_i \mid i \in J\}$ is conservative; hence h is an isomorphism. This shows that N is a simple object. Therefore, M is isomorphic to the coproduct of a finite number of copies of N. ■

The following proposition is a refinement of 1.6.2.

C3.5. Proposition. *Suppose that* C_X *is an abelian category with the property (sup). Let* $\{\mathcal{T}_i \mid i \in J\}$ *be a finite set of Serre subcategories of* C_X *such that* $\bigcap_{i \in J} \mathcal{T}_i = 0$, *and for every* $i \in J$, *the category* $C_{X/\mathcal{T}_i}$ *has enough objects of finite type. Then*

(a) The intersection $\mathbf{Spec}(X) \bigcap \mathbf{Spec}(X^o)$ *coincides with the set* $\mathbf{Spec}(X)_1$ *of closed points of* $\mathbf{Spec}(X)$, *and closed points of* $\mathbf{Spec}(X)$ *are of the form* $[M]$, *where* M *runs through simple objects of* C_X.

(b) Closed points of $\mathbf{Spec}^-(X)$*) are in bijective correspondence with the isomorphism classes of simple objects of* C_X.

Proof. (a) By C3.4, closed points of $\mathbf{Spec}(X)$ are of the form $[M]$, where M runs through simple objects of C_X. Since simple objects of C_X and $C_X^{op} = C_{X^o}$ are the same, the set $\mathbf{Spec}(X)_1$ of closed points of $\mathbf{Spec}(X)$ is contained in $\mathbf{Spec}(X) \cap \mathbf{Spec}(X^o)$.

(a1) Let $\mathfrak{U}$ denote the finite cover $\{U_i = X/\mathcal{T}_i \xrightarrow{u_i} X \mid i \in J\}$ associated with $\{\mathcal{T}_i \mid i \in J\}$. And let $\mathbf{Spec}^{1,1}_{\wp}(\mathfrak{U}) = \{\mathcal{P} \in \mathfrak{Th}(X) \mid \mathcal{P}/\mathcal{T}_i \in \mathbf{Spec}^{1,1}_{\mathfrak{t}}(U_i) \text{ if } \mathcal{T}_i \subseteq \mathcal{P}\}$. By 7.1, the natural map $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X) \longrightarrow \mathbf{Spec}^{1,1}_{\wp}(\mathfrak{U})$ is an isomorphism. This isomorphism and the embedding $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X^o) \longrightarrow \mathbf{Spec}^{1,1}_{\wp}(\mathfrak{U}^o)$ induce an injective map

$$\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)\bigcap \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X^o) \longrightarrow \mathbf{Spec}^{1,1}_{\wp}(\mathfrak{U})\bigcap \mathbf{Spec}^{1,1}_{\wp}(\mathfrak{U}^o), \tag{2}$$

where

$$\mathbf{Spec}^{1,1}_{\wp}(\mathfrak{U})\bigcap \mathbf{Spec}^{1,1}_{\wp}(\mathfrak{U}^o) =$$
$$\{\mathcal{P} \in \mathbf{Spec}^-(X) \mid \mathcal{P}/\mathcal{T}_i \in \mathbf{Spec}^{1,1}_{\mathfrak{t}}(U_i)\bigcap \mathbf{Spec}^{1,1}_{\mathfrak{t}}(U_i^o) \text{ if } \mathcal{T}_i \subseteq \mathcal{P}_i\}.$$

Since each category $C_{U_i} = C_X/\mathcal{T}_i$, $i \in J$, has enough objects of finite type, it follows from 1.6.2 and the isomorphism $\mathbf{Spec}(U_i) \xrightarrow{\sim} \mathbf{Spec}^{1,1}_{\mathfrak{t}}(U_i)$ (see 3.2(ii)) that the intersection $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(U_i)\bigcap \mathbf{Spec}^{1,1}_{\mathfrak{t}}(U_i^o)$ coincides with the set of closed points of $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(U_i)$ and these closed points are in bijective correspondence with isomorphism classes of simple objects of the category C_{U_i}. It follows now from (the argument of) C3.4 and the isomorphism $\mathbf{Spec}(X) \xrightarrow{\sim} \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$ (see 3.2(ii)) that the map (2) above is bijective.

(b) Notice that the conditions of this proposition imply the conditions (a) and (b) of C3.3.2. In particular, by C3.3.2, the spectra $\mathbf{Spec}^-(X)$ and $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$ have the same closed points. The assertion follows this fact and from (a) above. ∎

C3.6. Semilocal 'spaces'.

C3.6.1. Proposition. *Suppose that there is a finite subset $\{\mathcal{P}_i \mid i \in J\}$ of $\mathbf{Spec}^-(X)$ such that $\bigcap_{i\in J} \mathcal{P}_i = 0$. Then $\mathcal{P} \in \mathbf{Spec}^-(X)$ is a closed point iff it is a closed point of $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$, i.e. it is of the form $\mathcal{P} = \langle L\rangle$ for an object L of Spec(X).*

The set of closed points of $\mathbf{Spec}^-(X)$ coincides with the set of minimal elements of $\{\mathcal{P}_i \mid i \in J\}$.

Proof. Let $\mathcal{P}$ be a closed point of $\mathbf{Spec}^-(X)$). By 9.3, the set $J_{\mathcal{P}} = \{i \in J \mid \mathcal{P}_i \subseteq \mathcal{P}\}$ is not empty. Since $\mathcal{P}$ is a minimal element of $\mathbf{Spec}^-(X)$, the set $J_{\mathcal{P}}$ consists of all $i \in J$ such that $\mathcal{P}_i = \mathcal{P}$. Thus, $\mathcal{P}/\mathcal{P}_i$ is the zero subcategory of $C_X/\mathcal{P}_i$ which is the only closed point of the local space $X/\mathcal{P}_i = X/\mathcal{P}$. By 7.1, $\mathcal{P}$ is an element of $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$. Since $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$ is a subset of the spectrum $\mathbf{Spec}^-(X)$ and $\mathcal{P}$ is a closed point of the latter, it is a closed point of $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$.

This argument shows that the set of closed points of $\mathbf{Spec}^-(X)$ is a subset of the set of minimal elements of $\{\mathcal{P}_i \mid i \in J\}$, and that it is a subset of closed point of $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$.

Notice that every minimal element of $\{\mathcal{P}_i \mid i \in J\}$ is a closed point of $\mathbf{Spec}^-(X)$.

In fact, let $\mathcal{P}_j$ be a minimal element of the set $\{\mathcal{P}_i \mid i \in J\}$, and let $\mathcal{P}' \in \mathbf{Spec}^-(X)$ be a subcategory of $\mathcal{P}_j$. The set $J_{\mathcal{P}'} = \{i \in J \mid \mathcal{P}_i \subseteq \mathcal{P}'\}$ is non-empty, and $\mathcal{P}_m \subseteq \mathcal{P}' \subseteq \mathcal{P}_j$ for every $m \in J_{\mathcal{P}'}$. Since $\mathcal{P}_j$ is a minimal element, this implies that $\mathcal{P}_m = \mathcal{P}' = \mathcal{P}_j$.

Let $\mathcal{P} \in \mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$. The set $J_{\mathcal{P}} = \{i \in J \mid \mathcal{P}_i \subseteq \mathcal{P}\}$ is not empty. Let $\mathcal{P}_i$ be a minimal element of $\{\mathcal{P}_j \mid j \in J_{\mathcal{P}}\}$. Then, by the argument above, $\mathcal{P}_i$ is a closed point of $\mathbf{Spec}^-(X)$, hence it is a closed point of $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$ which is contained in $\mathcal{P}$. Thus, if $\mathcal{P}$ is a closed point of $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$, then $\mathcal{P}_i = \mathcal{P}$. ∎

C3.6.2. Corollary. *Let C_X be an abelian category. The following conditions are equivalent:*

(a) There is a finite subset $\{\mathcal{P}_i \mid i \in J\}$ of $\mathbf{Spec}^-(X)$ such that $\bigcap_{i \in J} \mathcal{P}_i = 0$.

(b) The set $\mathbf{Spec}^-(X)_1$ of closed points of $\mathbf{Spec}^-(X)$ is finite, and the intersection $\bigcap_{\mathcal{P} \in \mathbf{Spec}^-(X)_1} \mathcal{P}$ is zero.

(c) The set $\mathbf{Spec}^-(X)_1$ is finite, and the support in $\mathbf{Spec}^-(X)$ of any nonzero object of C_X contains a closed point.

(d) The set $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)_1$ of closed points of $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$ is finite, and the support in $\mathbf{Spec}(X)$ of every nonzero object of C_X contains a closed point.

Proof. Obviously, $(b) \Rightarrow (a)$. The implication $(a) \Rightarrow (b)$ follows from C3.6.1.

$(b) \Leftrightarrow (c)$. If $\bigcap_{\mathcal{P} \in \mathbf{Spec}^-(X)_1} \mathcal{P} = 0$, then for every nonzero object M of C_X, there exists a closed point $\mathcal{P}$ of $\mathbf{Spec}^-(X)$ such that $M \notin Ob\mathcal{P}$, which means precisely that $\mathcal{P} \in Supp^-(M)$. Conversely, if every nonzero object of C_X has an element of $\mathbf{Spec}^-(X)_1$ in its support, then $\bigcap_{\mathcal{P} \in \mathbf{Spec}^-(X)_1} \mathcal{P} = 0$.

$(d) \Rightarrow (a)$. The support in $\mathbf{Spec}(X)$ of a nonzero object M contains a point $\mathcal{Q}$, that is $\mathcal{Q} \subseteq [M]$, means precisely that $[M] \not\subseteq \widehat{\mathcal{Q}}$, or, equivalently, $M \notin Ob\widehat{\mathcal{Q}}$. By 3.2(ii), $\mathcal{Q}$ is a closed point of $\mathbf{Spec}(X)$ iff $\widehat{\mathcal{Q}} \in \mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)_1$. Therefore, the condition (d) implies that $\bigcap_{\mathcal{P} \in \mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)_1} \mathcal{P} = 0$.

The implication $(a) \Rightarrow (b)$ follows from C3.6.1. ∎

C3.6.3. Definition. Let C_X be an abelian category. We call the 'space' X *semi-local* if the equivalent conditions of C3.6.2 hold.

Chapter III
Spectra, Associated Points, and Representations.

Associated points. Each of the spectra gives rise to the corresponding notion of *associated points*. Let M be an object of the category C_X. An element $\mathcal{Q}$ of $\mathbf{Spec}(X)$ is called an *associated point* of M, if $\mathcal{Q} = [L]$ for a nonzero subobject L of M which is $\widehat{\mathcal{Q}}$-torsion free (that is L does not have nonzero subobjects which belong to $\widehat{\mathcal{Q}}$; equivalently, L is right orthogonal to $\widehat{\mathcal{Q}}$). We denote the set of associated points of M by $\mathfrak{Ass}(M)$.

The set $\mathfrak{Ass}^-(M)$ of *associated points in* $\mathbf{Spec}^-(X)$ of the object M consists of all $\mathcal{P} \in \mathbf{Spec}^-(X)$ such that the localization $M_{\mathcal{P}}$ of M at $\mathcal{P}$ has a closed associated point; that is $M_{\mathcal{P}}$ has a nonzero subobject which belongs to the smallest topologizing subcategory of $C_X/\mathcal{P}$. If $C_X/\mathcal{P}$ has simple objects (which is the case when C_X is locally noetherian, or, more generally, has a Gabriel-Krull dimension), then the condition means precisely that $M_{\mathcal{P}}$ has a nonzero socle. If C_X is the category of coherent sheaves on a noetherian scheme, then this notion coincides with the Grothendieck's notion of associated points (prime cycles) of a coherent sheaf.

Associated points in $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ are defined similarly to those in $\mathbf{Spec}(X)$, and the set of associated points of an object M is denoted by $\mathfrak{Ass}_{\mathfrak{c}}(M)$. The reader can now easily figure out what is the set $\mathfrak{Ass}^-_{\mathfrak{c}}(M)$ of associated points of M in $\mathbf{Spec}^-_{\mathfrak{c}}(X)$.

The natural embeddings

$$\begin{tikzcd} \mathbf{Spec}(X) \arrow[r] \arrow[d] & \mathbf{Spec}^0_{\mathfrak{c}}(X) \arrow[d] \\ \mathbf{Spec}^-(X) \arrow[r] & \mathbf{Spec}^-_{\mathfrak{c}}(X) \end{tikzcd} \qquad (1)$$

induce the corresponding embeddings of the associated points

$$\begin{tikzcd} \mathfrak{Ass}(M) \arrow[r] \arrow[d] & \mathfrak{Ass}_{\mathfrak{c}}(M) \arrow[d] \\ \mathfrak{Ass}^-(M) \arrow[r] & \mathfrak{Ass}^-_{\mathfrak{c}}(M) \end{tikzcd}$$

All four types of associated points have properties analogous to the known properties of associated points of modules over commutative rings (see C3).

Induction problem. Let $\mathfrak{X}$ and X be 'spaces' represented by abelian categories, resp. $C_{\mathfrak{X}}$ and C_X, $\mathfrak{X} \xrightarrow{f} X$ a continuous morphism of 'spaces', $\mathcal{P}$ a point of the spectrum of X. The *induction problem* is to find representatives M of the spectrum of $\mathfrak{X}$ such that

$\mathcal{P}$ is an associated point of $f_*(M)$. Here by the *spectrum* of a 'space' X we understand usually $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ or $\mathbf{Spec}(X)$ and sometimes one the remaining two spectra, $\mathbf{Spec}^-_{\mathfrak{c}}(X)$ or $\mathbf{Spec}^-(X)$, more precisely, their 'dual' versions, $\mathbf{Spec}^{\mathfrak{c}}_-(X)$ and $\mathbf{Spec}_-(X)$ – natural extensions of respectively $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ and $\mathbf{Spec}(X)$ introduced in 2.9.4.

This Chapter is concentrated around a (spectral version of the induction) construction which gives a solution of this problem in the case when f is a *locally affine* morphism and the pair $(f, \mathcal{P})$ satisfies certain additional conditions. We explain first its special case which can be formulated without preliminaries.

A special case of the construction. Let A and B be associative unital k-algebras, $C_{\mathfrak{X}} = B-mod$, $C_X = A-mod$, and the morphism $\mathfrak{X} \xrightarrow{f} X$ is induced by a k-algebra morphism $A \xrightarrow{\varphi} B$. Fix a simple A-module P. Let $\widetilde{B}_P$ denote the class of all A-subbimodules N of B which are flat as right A-modules and such that $N \otimes_A P$ is isomorphic to a direct sum of copies of P. We call the supremum, B_P, of the family $\widetilde{B}_P$ the *stabilizer of P in* B. Pick a simple B_P-module M whose restriction to A is isomorphic to the direct sum of copies of P. The B-module $B \otimes_{B_P} M$ has the largest B-submodule, $\mathfrak{t}_P(B \otimes_{B_P} M)$, whose restriction to A does not have any subquotients isomorphic to P. We denote by $\mathfrak{L}_P(M)$ the quotient module $B \otimes_{B_P} M/\mathfrak{t}_P(B \otimes_{B_P} M)$. Under certain additional conditions, the (multi-valued in general) map $P \longmapsto \mathfrak{L}_P(M)$ produces simple B-modules.

An effect of noncommutativity. Notice that the above construction is useless if the algebras are commutative, because in this case, the stabilizer B_P coincides with the whole algebra B. In general, the size of B_P over A can be regarded as a measure of the noncommutativity of the data $(A \to B, P)$. In the best, noncommutative, case, the stabilizer B_P coincides with the image of A which makes the construction look particularly familiar: $\mathfrak{L}_P(M) \simeq B \otimes_A M/\mathfrak{t}_P(B \otimes_A M)$.

The insufficiency of the special case. With rare exceptions, most of isomorphism classes of simple B-modules cannot be reached this way. But, under certain finiteness conditions, all isomorphism classes of simple B-modules, more generally, all points of the spectrum of $\mathfrak{X}$ (where $C_{\mathfrak{X}} = B - mod$), can be realized if we allow P run through representatives of all, not necessarily closed points of the spectrum of the 'space' X, where $C_X = A - mod$. The construction in this case becomes more subtle.

Besides, it is important to consider a non-affine version of this construction in order to include into the picture D-modules on (quantized and classical) flag varieties and other (commutative and noncommutative) schemes. Therefore, algebras are replaced by 'spaces' represented by abelian categories and morphisms of algebras by *locally affine* morphisms of 'spaces'. The meaning of the last words is explained above.

Reduction to the affine case and gluing. It follows from the results of [R7] that if a locally affine morphism $\mathfrak{X} \xrightarrow{f} X$ admits a finite affine cover $\{U_i \xrightarrow{u_i} \mathfrak{X} \mid i \in J\}$, then the problem can be split into solving it for each affine morphism $U_i \xrightarrow{f u_i} X$ and checking certain *glueing conditions* (explained in Section 7). If the *spectrum* is $\mathbf{Spec}^0_{\mathfrak{c}}(X)$, then the same holds for arbitrary infinite covers as well.

The construction. A natural setting consists of an abelian category C_X endowed with an action of a svelte monoidal category $\widetilde{\mathcal{E}}$ on C_X given by a monoidal functor $\widetilde{\Phi}$ with values in exact continuous (i.e. having a right adjoint) endofunctors of C_X. If C_X has small limits and colimits (say, it is a Grothendieck category), then the forgetful functor φ_* from the category $C_{\mathfrak{A}}$ of $\widetilde{\Phi}$-modules to the category C_X is a direct image functor of an affine morphism, $\mathfrak{A} \xrightarrow{\varphi} X$, hence (by Beck's theorem) the category $C_{\mathfrak{A}}$ can be replaced by the category of modules over the monad $\mathcal{F}_\varphi$ associated with (the inverse and direct image functors of) φ and functor φ_* by the forgetful functor $\mathcal{F}_\varphi - mod \longrightarrow C_X$.

To each point $\mathcal{P}$ of the spectrum of X, there corresponds its *stabilizer* which is the full monoidal subcategory $\widetilde{\mathcal{E}}_{(\mathcal{P})}$ of $\widetilde{\mathcal{E}}$ (defined in 2.1.1). The category $C_{\mathfrak{A}_\mathcal{P}}$ of modules over the (induced by $\widetilde{\Phi}$) action $\widetilde{\Phi}_{(\mathcal{P})}$ of $\widetilde{\mathcal{E}}_{(\mathcal{P})}$ is equivalent to the category of modules over a monad $\mathcal{F}_{\varphi_\mathcal{P}}$, which is also called the *stabilizer* of $\mathcal{P}$. Thus, we have a commutative diagram

$$\begin{tikzcd} \mathfrak{A} \arrow[rr, "\mathfrak{f}_\mathcal{P}"] \arrow[dr, "\varphi"'] & & \mathfrak{A}_\mathcal{P} \arrow[dl, "\varphi_\mathcal{P}"] \\ & X & \end{tikzcd}$$

of affine morphisms. Let $\mathfrak{L}_\mathcal{P}$ denote the composition of the functor $\mathfrak{f}^*_\mathcal{P}$ and the functor which assigns to every object of the category $C_{\mathfrak{A}}$ the quotient of this object by its $\varphi_*^{-1}(\widehat{\mathcal{P}})$-torsion, where $\widehat{\mathcal{P}}$ is the Serre subcategory of C_X corresponding to $\mathcal{P}$.

Let $Spec^0_{\mathfrak{c}}(\mathfrak{A})$ denote the family of all objects M such that $[M]_{\mathfrak{c}} = \mathcal{Q}$ is an element of the spectrum $\mathbf{Spec}^0_{\mathfrak{c}}(\mathfrak{A})$ and M is ${}^{\mathfrak{c}}\widehat{\mathcal{Q}}$-torsion free. In other words, objects of $Spec^0_{\mathfrak{c}}(\mathfrak{A})$ are *representatives* of elements of the spectrum. Let $Spec^{\mathcal{P}}_{\mathfrak{c}}(\mathfrak{A}_\mathcal{P})$ denote the family of all objects of $Spec^0_{\mathfrak{c}}(\mathfrak{A}_\mathcal{P})$ such that $\mathcal{P}$ is an associated point of their image in C_X. If the functor $\mathfrak{f}^*_\mathcal{P}$ is exact and faithful and the action $\widetilde{\Phi}$ satisfies certain 'ampleness' conditions, then the functor $\mathfrak{L}_\mathcal{P}$ transforms every object of $Spec^{\mathcal{P}}_{\mathfrak{c}}(\mathfrak{A}_\mathcal{P})$ into an object of the spectrum of the 'space' $\mathfrak{A}$. Moreover, every object of the spectrum of $\mathfrak{A}$ whose image in $C_{\mathfrak{A}_\mathcal{P}}$ has an associated point which belongs to $Spec^{\mathcal{P}}_{\mathfrak{c}}(\mathfrak{A}_\mathcal{P})$ is equivalent to the image of this associated point by the functor $\mathfrak{L}_\mathcal{P}$. The functor $\mathfrak{L}_\mathcal{P}$ maps simple objects from $Spec^{\mathcal{P}}_{\mathfrak{c}}(\mathfrak{A}_\mathcal{P})$ to simple objects of $C_{\mathfrak{A}}$ (see Theorem 2.2 for details).

Finiteness conditions. In the construction above, given a representative M of a point $\mathfrak{P}$ of the spectrum of $\mathfrak{A}$ such that $\varphi_*(M)$ has an associated point $\mathcal{P}$, one needs certain *finiteness conditions* which guarantee that $\mathfrak{P}$ can be obtained via the construction; i.e. that it coincides with $[\mathfrak{L}_\mathcal{P}(V)]$ for some object V of $Spec^{\mathcal{P}}_{\mathfrak{c}}\mathfrak{A}_\mathcal{P}$. The most straightforward finiteness conditions say that $\mathcal{P}$ is an associated point of $\varphi_*(M)$ *of finite multiplicity*. The latter means that the local category $C_X/\mathcal{P}$ has simple objects and the localization of $\varphi_*(M)$ at $\mathcal{P}$ has a finite socle. The length of this socle is called the *multiplicity* of the associated point $\mathcal{P}$ in $u_*(M)$. This finiteness condition works for the spectra $\mathbf{Spec}^-(-)$ and $\mathbf{Spec}^-_{\mathfrak{c}}(-)$ and, in certain cases, for $\mathbf{Spec}(-)$ and $\mathbf{Spec}^0_{\mathfrak{c}}(-)$.

Holonomic objects. Given a continuous morphism $\mathfrak{A} \xrightarrow{\varphi} X$, we call an object M of the category $C_{\mathfrak{A}}$ *holonomic over* X if each nonzero subquotient of $\varphi_*(M)$ has associated points in $\mathbf{Spec}^-_{\mathfrak{c}}(X)$ and all these associated points are of finite multiplicity.

If C_X is the category of quasi-coherent sheaves on a smooth scheme $\mathcal{X}$ and $C_{\mathfrak{A}}$ is the category of D-modules on $\mathcal{X}$, then holonomic objects are precisely holonomic D-modules.

If C_X is the category of quasi-coherent sheaves on the quantum flag variety of a semisimple Lie algebra $\mathfrak{g}$ and $C_{\mathfrak{A}}$ is the category of quasi-coherent $U_q(\mathfrak{g})$-modules on X (cf. [LR2]), then holonomic objects are called *holonomic* quantum D-modules.

All simple holonomic objects can be obtained via the described above construction (i.e. by applying the functors $\mathfrak{L}_{\mathcal{P}}$). Thanks to their functorial properties, the description of holonomic objects is directly reduced to their description on elements of any affine cover.

The first two sections contain preliminaries. Section 1 provides a short dictionary for 'spaces' and morphisms of 'spaces'. We remind the notions of *continuous, affine*, and *flat* morphisms of 'spaces' and basic facts about them needed in the main body of the text. Section 2 gives a short sketch of spectral theory of 'spaces' represented by abelian categories and related notions and facts.

Sections 3 and 4 are dedicated to the mentioned above construction of points of the spectrum $\mathbf{Spec}^0_{\mathfrak{c}}(\mathfrak{A})$. We conclude Section 4 with the reduction to the case when C_X is an element of $\mathbf{Spec}^0_{\mathfrak{c}}(X)$; i.e. C_X is the generic point of X. This reduction is useful for analyzing special cases. Two of them are considered in Section 3. The first one is when the functor $F_\varphi = \varphi_*\varphi^*$ is isomorphic to a direct sum of auto-equivalences. The second case is when the functor F_φ *differential* and exact. The functor F_φ being differential implies that F_φ (as well as every its subquotient) preserves each Serre subcategory of C_X. In combination with the exactness, this implies that F_φ is compatible with localization at any Serre subcategory. In each of these two cases, we are able to obtain a much more detailed picture and in the first case a convenient variant of Theorem 2.2.

Curiously, both cases (which are, in a sense, perpendicular to each other) appear in the example of the Weyl algebra A_n. Recall that A_n is the k-algebra generated by $x_i,\ y_i$ subject to the relations $[x_i, y_j] = \delta_{ij},\ [x_i, x_j] = 0 = [y_i, y_j]$ for all $1 \le i, j \le n$.

Taking as C_X the category of modules over the polynomial algebra $k[\mathbf{y}] = k[y_1, \ldots, y_n]$, and $C_{\mathfrak{A}} = A_n - mod$, we obtain a differential monad on X with $F_\varphi = A_n \otimes_{k[\mathbf{y}]} -$.

Taking as C_X the category of modules over the polynomial algebra $k[\xi] = k[\xi_1, \ldots, \xi_n]$, where $\xi_i = x_i y_i,\ 1 \le i \le n$, we obtain the functor $F_\varphi = A_n \otimes_{k[\xi]} -$ which is a direct sum of auto-equivalences of the category $k[\xi] - mod$.

This is discussed in more detail in Section C1 of "Complementary facts".

One of the main tools of studying spectra is the localization at appropriate Serre subcategories. The localization simplifies considerably the picture, so that in many cases it is not difficult to compute the spectrum of the quotient 'space'. But, unlike the commutative case, in general, not all points of the spectrum of the quotient 'space' corresponding to an 'open' subspace are localizations of points of the 'space' we started with. All we can say is that these points come from the counterpart $\mathbf{Spec}^{\mathfrak{c}}_{-}(X)$ of the spectrum $\mathbf{Spec}^{-}_{\mathfrak{c}}(X)$ (introduced in 2.9). Notice that $\mathbf{Spec}^{-}_{\mathfrak{c}}(-)$ and, therefore, $\mathbf{Spec}^{\mathfrak{c}}_{-}(-)$, are functorial with respect to localizations at Serre subcategories. These are some of the reasons why we need an analog of Theorem 2.2 for $\mathbf{Spec}^{\mathfrak{c}}_{-}(X)$ which is given in Section 4.

In Section 5, we remind local properties of the spectra which allow to construct elements of the spectra in the case of locally affine morphisms and simplify their construction in affine cases. We illustrate the general constructions of this work by a rough sketch of their applications to D-modules on classical and quantum flag varieties. In the classical case, the local properties of the spectra allow to reduce the study D-modules on the flag variety to the study of modules over the Weyl algebra A_n, where n is the dimension of the flag variety. Following the philosophy of this work, we study the spectrum of the affine scheme $\mathbf{Sp}(A_n)$ via hyperbolic coordinates, $k[\xi] \longrightarrow A_n$ mentioned above. Some details of this study are provided in "Complementary facts". It is worth to mention that Weyl algebras play also a crucial role in the representation theory of nilpotent Lie algebras: if $\mathfrak{g}$ is a finite-dimensional nilpotent Lie algebra over an algebraically closed field of zero characteristic, then the set of primitive ideals of its universal enveloping algebra $U(\mathfrak{g})$ is parameterized by the orbits of adjoint action on the dual space $\mathfrak{g}^*$; and for any primitive ideal $\mathfrak{J}$, the quotient algebra $U(\mathfrak{g})/\mathfrak{J}$ is isomorphic to the Weyl algebra A_n.

In "Complementary facts", besides of a fragment of the spectral theory of Weyl algebras obtained via their hyperbolic structure (sketched in Section C1), there are some remarks, in Section C2, about application of our induction machinery to natural subalgebras of the enveloping algebras and their quantum analogs. Thus, we observe that highest weight modules are recovered by applying our induction functor together with Harish-Chandra homomorphism to Cartan subalgebras. Similarly in the case of quantized enveloping algebras. More curious possibilities appear if we use upper triangular part instead. Section C3 is dedicated to associated points and produces a noncommutative version of the classical facts of commutative algebra. Section C4 contains facts on affine morphisms and differential monads, both play a big role in the main body of the text.

1. Actions of monoidal categories, stabilizers of points, induction functors.

1.1. Actions and continuous actions of monoidal categories. Let $\widetilde{\mathcal{E}} =$

$(\mathcal{E}, \odot, \mathbb{I}, a; \ell, \mathfrak{r})$ be a svelte monoidal category with the product $\odot$, the unit object $\mathbb{I}$, the associativity constraint a, and natural isomorphisms $\mathbb{I} \odot Id_{\mathcal{E}} \xleftarrow{\ \ell\ } Id_{\mathcal{E}} \xrightarrow{\ \mathfrak{r}\ } Id_{\mathcal{E}} \odot \mathbb{I}$.

An *action* of the monoidal category $\widetilde{\mathcal{E}}$ on a svelte category C_X is a monoidal functor $\widetilde{\Phi} = (\Phi, \phi, \phi_0)$ from $\widetilde{\mathcal{E}}$ to the monoidal category $\widetilde{End}(C_X) = (End(C_X), \circ, Id_{C_X})$ of endofunctors of the category C_X. Recall that here Φ is a functor $\mathcal{E} \longrightarrow End(C_X)$, ϕ a functorial morphism $\Phi(V) \circ \Phi(W) \longrightarrow \Phi(V \odot W)$, and ϕ_0 a morphism from Id_{C_X} (– the unit object of $\widetilde{End}(C_X)$) to $\Phi(\mathbb{I})$ – the image of the unit object of $\widetilde{\mathcal{E}}$. These morphisms are related via the commutative diagrams

$$\begin{array}{ccccc}
\Phi(\mathcal{V}) \circ \Phi(\mathcal{W}) \circ \Phi(\mathcal{Z}) & \xrightarrow{\ \phi_{\mathcal{V},\mathcal{W}}\Phi(\mathcal{Z})\ } & \Phi(\mathcal{V} \odot \mathcal{W}) \circ \Phi(\mathcal{Z}) & \xrightarrow{\ \phi_{\mathcal{V}\odot\mathcal{W},\mathcal{Z}}\ } & \Phi((\mathcal{V} \odot \mathcal{W}) \odot \mathcal{Z}) \\
{\scriptstyle id}\downarrow & & & & \wr\downarrow {\scriptstyle \Phi(a_{\mathcal{V},\mathcal{W},\mathcal{Z}})} \\
\Phi(\mathcal{V}) \circ \Phi(\mathcal{W}) \circ \Phi(\mathcal{Z}) & \xrightarrow{\ \Phi(\mathcal{V})\phi_{\mathcal{W},\mathcal{Z}}\ } & \Phi(\mathcal{V}) \circ \Phi(\mathcal{W} \odot \mathcal{Z}) & \xrightarrow{\ \phi_{\mathcal{V},\mathcal{W}\odot\mathcal{Z}}\ } & \Phi((\mathcal{V} \odot \mathcal{W}) \odot \mathcal{Z})
\end{array} \tag{1}$$

$$\begin{array}{ccccc}
\Phi(\mathcal{V}) \circ \Phi(\mathbb{I}) & \xleftarrow{\ \Phi(\mathcal{V})\phi_0\ } & \Phi(\mathcal{V}) & \xrightarrow{\ \phi_0\Phi(\mathcal{V})\ } & \Phi(\mathbb{I}) \circ \Phi(\mathcal{V}) \\
{\scriptstyle \phi_{\mathcal{V},\mathbb{I}}}\downarrow & & {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle \phi_{\mathbb{I},\mathcal{V}}} \\
\Phi(\mathcal{V} \odot \mathbb{I}) & \xleftarrow{\ \Phi(\ell_{\mathcal{V}})\ } & \Phi(\mathcal{V}) & \xrightarrow{\ \Phi(\mathfrak{r}_{\mathcal{V}})\ } & \Phi(\mathbb{I} \odot \mathcal{V})
\end{array} \tag{2}$$

for all $\mathcal{V}, \mathcal{W}, \mathcal{Z} \in Ob\mathcal{E}$.

An action $\widetilde{\mathcal{E}} \xrightarrow{\ \widetilde{\Phi}\ } \widetilde{End}(C_X)$ will be called *continuous* if $\widetilde{\Phi}$ takes values in the full monoidal subcategory $\widetilde{End}_{\mathfrak{c}}(C_X) = (End_{\mathfrak{c}}(C_X), \circ, Id_{C_X})$ of $\widetilde{End}(C_X)$ generated by all continuous endofunctors of the category C_X.

1.1.1. Example: actions of the trivial monoidal category and monads. Let $\widetilde{\mathcal{E}}_{\bullet}$ be the trivial monoidal category; i.e. the category consisting of one object and one (hence identical) morphism. The category of actions of $\widetilde{\mathcal{E}}_{\bullet}$ on the category C_X is isomorphic to the category $\mathfrak{Mon}(C_X)$ of monads on the category C_X.

In fact, each action $\widetilde{\Phi} = (\Phi, \phi, \phi_0)$ is determined by the image, $F = \Phi(\mathbb{I})$, of the unique (unit) object of the category $\mathcal{E}_{\bullet}$ and the morphism $F \circ F \xrightarrow{\ \phi\ } F$. The fact that $\widetilde{\Phi}$ is a monoidal functor, means precisely that ϕ is associative, i.e. $\phi \circ F\phi = \phi \circ \phi F$, and $Id_{C_X} \xrightarrow{\ \phi_0\ } F$ is the unit of $\mathcal{F} = \Phi(\mathbb{I})$: $\phi \circ F\phi_0 = id_F = \phi \circ \phi_0 F$.

The map $\widetilde{\Phi} \longmapsto \mathcal{F}_{\widetilde{\Phi}} = (\Phi(\mathbb{I}), \phi)$ extends naturally to an isomorphism from the category of actions of $\widetilde{\mathcal{E}}_{\bullet}$ on C_X and the category of monads on C_X. This isomorphism induces an isomorphism between the category of continuous actions of $\widetilde{\mathcal{E}}_{\bullet}$ on C_X and the category of continuous monads on C_X.

1.2. Modules over an action and the associated monad. Fix a continuous action $\widetilde{\Phi} = (\Phi, \phi, \phi_0)$ of a svelte monoidal category $\widetilde{\mathcal{E}} = (\mathcal{E}, \odot, \mathbb{I}, a)$ on the category C_X. The forgetful functor

$$(\widetilde{\Phi}/X) - mod \xrightarrow{\varphi_*} C_X$$

preserves small limits. Suppose that the category C_X is *small-complete* (i.e. it has small limits). Since the categories C_X and $\mathcal{E}$ are svelte, this implies, by Freyd adjoint functor theorem, the existence of a left adjoint, φ^*, to φ_*. The functor φ_* is exact and conservative. Therefore, by Beck's theorem, the category $(\widetilde{\Phi}/X) - mod$ is equivalent to the category of modules over a monad, $\mathcal{F}_\varphi = (F_\varphi, \mu_\varphi)$, where $F_\varphi = \varphi_*\varphi^*$. More precisely, the forgetful functor φ_* is equivalent to the forgetful functor $\mathcal{F}_\varphi - mod \longrightarrow C_X$.

Notice that the latter implies that the category $(\widetilde{\Phi}/X) - mod$ is small-complete too.

Assume in addition that the category C_X has small colimits. It follows from the fact that the functor Φ takes values in the category of *continuous* endofunctors of C_X that the functor φ_* preserves small colimits, hence it has a right adjoint, $\varphi^!$. The latter is equivalent to the fact that the monad $\mathcal{F}_\varphi$ is continuous.

1.3. Colimits of actions. Identifying the category $(\widetilde{\Phi}/X) - mod$ of $\widetilde{\Phi}$-modules with the category $(\mathcal{F}_\varphi/X) - mod$, we can take as φ^* the functor which assigns to every object V of the category C_X the $\mathcal{F}_\varphi$-module $\mathcal{F}_\varphi(V) = (F_\varphi(V), \mu_\varphi(V))$. On the other hand, $\varphi^*(V)$ is an $(\widetilde{\Phi}/X)$-module; that is we have an action $\Phi(-) \circ F_\varphi(V) \xrightarrow{\xi_\varphi(V)} F_\varphi(V)$ of $\widetilde{\mathcal{E}}$ on $\mathcal{F}_\varphi(V)$ which is functorial in V. Taking the composition of this action with the morphism

$$\Phi(-) = \Phi(-) \circ Id_{C_X} \xrightarrow{\Phi(-)\eta_\varphi} \Phi(-) \circ F_\varphi(V)$$

(where η_φ is an adjunction arrow), we obtain a cone $\Phi(-) \xrightarrow{\gamma_\varphi} F_\varphi$. Note that monads on C_X can be identified with constant monoidal functors from $\widetilde{\mathcal{E}}$ to $\widetilde{End}(C_X)$. One can see that the cone $\Phi(-) \xrightarrow{\gamma_\varphi} F_\varphi$ is a morphism of monoidal functors $\widetilde{\Phi} \longrightarrow \mathcal{F}_\varphi$.

Let $\mathfrak{MF}(\widetilde{\mathcal{E}}, \widetilde{\mathcal{E}}')$ denote the category of monoidal functors from $\widetilde{\mathcal{E}}$ to a monoidal category $\widetilde{\mathcal{E}}'$ and $\mathfrak{Mon}(C_X)$ the category of monads on C_X. Let $\mathfrak{J}_X^*$ denote the embedding

$$\mathfrak{Mon}(C_X) \longrightarrow \mathfrak{MF}(\widetilde{\mathcal{E}}, \widetilde{End}(C_X))$$

which assigns to every monoid on C_X the corresponding constant monoidal functor; and let $\mathfrak{J}_{X*}$ be functor which assigns to each monoidal functor $\widetilde{\Phi}$ from $\widetilde{\mathcal{E}}$ to $\widetilde{End}(C_X)$ the monad $\mathcal{F}_\varphi$. The map which assigns to every monoidal functor $\widetilde{\Phi}$ from $\widetilde{\mathcal{E}}$ to $\widetilde{End}(C_X)$ the morphism $\widetilde{\Phi} \xrightarrow{\gamma_\varphi} \mathcal{F}_\varphi$ is an adjunction arrow $Id \xrightarrow{\gamma} \mathfrak{J}_{X*}\mathfrak{J}_X^*$. The other adjunction arrow is the identical morphism. This means that the monad $\mathcal{F}_\varphi$ corresponding to a monoidal functor $\widetilde{\Phi} = (\Phi, \phi)$ is the *colimit of this monoidal functor.*

1.4. Colimits of continuous actions. Suppose now that the category C_X has small limits and colimits. Let $\mathfrak{MF}_\mathfrak{c}(\widetilde{\mathcal{E}}, \widetilde{End}(C_X))$ denote the full subcategory of $\mathfrak{MF}(\widetilde{\mathcal{E}}, \widetilde{End}(C_X))$ whose objects are *continuous* actions of $\widetilde{\mathcal{E}}$ on the category C_X. And let $\mathfrak{Mon}_\mathfrak{c}(C_X)$ denote the category of continuous monads on C_X. The embedding

$$\mathfrak{Mon}(C_X) \xrightarrow{\mathfrak{J}_X^*} \mathfrak{MF}(\widetilde{\mathcal{E}}, \widetilde{End}(C_X))$$

induces an embedding

$$\mathfrak{Mon}_\mathfrak{c}(C_X) \xrightarrow{{}^\mathfrak{c}\mathfrak{J}_X^*} \mathfrak{MF}_\mathfrak{c}(\widetilde{\mathcal{E}}, \widetilde{End}(C_X)).$$

Since the monad $\mathcal{F}_\varphi$ corresponding to the continuous action $\widetilde{\Phi}$ is continuous, the right adjoint $\mathfrak{J}_{X*}$ to $\mathfrak{J}_X^*$ induces a right adjoint

$$\mathfrak{MF}(\widetilde{\mathcal{E}}, \widetilde{End}(C_X)) \xrightarrow{{}^\mathfrak{c}\mathfrak{J}_{X*}} \mathfrak{Mon}_\mathfrak{c}(C_X)$$

to the functor ${}^\mathfrak{c}\mathfrak{J}_X^*$ which assigns to every continous action $\widetilde{\Phi} = (\Phi, \phi)$ of the monoidal category $\widetilde{\mathcal{E}}$ on the category C_X its colimit – a continuous monad $\mathcal{F}_\varphi = (F_\varphi, \mu_\varphi)$.

It follows from the fact that functor Φ takes values in the category of continuous endofunctors, that the functor $F_\varphi = \varphi_*\varphi^*$ is the colimit of Φ.

1.5. Functorialities. These correspondences are functorial in the following sense: if $\widetilde{\mathcal{E}}'$ is another svelte monoidal category and

$$\begin{tikzcd} \widetilde{\mathcal{E}}' \arrow[rr, "\widetilde{\Psi}"] \arrow[dr, "\widetilde{\Phi}'"'] & & \widetilde{\mathcal{E}} \arrow[dl, "\widetilde{\Phi}"] \\ & \widetilde{End}_\mathfrak{c}(C_X) & \end{tikzcd}$$

is a quasi-commutative diagram of monoidal functors, then the monoidal functor $\widetilde{\Psi}$ induces a pull-back functor $(\widetilde{\Phi}/X) - mod \xrightarrow{\mathfrak{f}_*} (\widetilde{\Phi}'/X) - mod$ such that the diagram

$$\begin{tikzcd} (\widetilde{\Phi}/X) - mod \arrow[rr, "\mathfrak{f}_*"] \arrow[dr, "\varphi_*"'] & & (\widetilde{\Phi}'/X) - mod \arrow[dl, "\varphi'_*"] \\ & C_X & \end{tikzcd} \qquad (1)$$

commutes. If the category C_X is small-complete, then, by the argument above, the functors $(\widetilde{\Phi}/X) - mod \xrightarrow{\varphi_*} C_X$ and $(\widetilde{\Phi}'/X) - mod \xrightarrow{\varphi'_*} C_X$ are equivalent to the forgetful functors, respectively $(\mathcal{F}_\varphi/X) - mod \longrightarrow C_X$ and $(\mathcal{F}_{\varphi'}/X) - mod \longrightarrow C_X$.

The functor $\mathfrak{f}_*$ corresponds to the restriction functor $\mathcal{F}_\varphi - mod \xrightarrow{\psi_*} \mathcal{F}_{\varphi'} - mod$ along a monad morphism $\mathcal{F}_{\varphi'} \xrightarrow{\psi} \mathcal{F}_\varphi$. In particular, the functor $\mathfrak{f}_*$ has a left adjoint, $\mathfrak{f}^*$. Thus, the diagram (1) is equivalent to the diagram of canonical direct image functors of the commutative diagram

$$\begin{array}{ccc} \mathbf{Sp}(\mathcal{F}_\varphi/X) & \xrightarrow{\mathbf{Sp}(\psi)} & \mathbf{Sp}(\mathcal{F}_{\varphi'}/X) \\ \varphi \searrow & & \swarrow \varphi' \\ & X & \end{array} \qquad (2)$$

corresponding to a monad morphism $\mathcal{F}_{\varphi'} \xrightarrow{\psi} \mathcal{F}_\varphi$. Notice that the monads $\mathcal{F}_\varphi$ and $\mathcal{F}_{\varphi'}$, being colimits of monoidal functors, are defined uniquely up to isomorphism. By the universal property of colimits, the monad morphism ψ is determined uniquely, once the monads $\mathcal{F}_\varphi$ and $\mathcal{F}_{\varphi'}$ are fixed. Therefore, the map which assigns to the diagram (1) the monad morphism $\mathcal{F}_{\varphi'} \xrightarrow{\psi} \mathcal{F}_\varphi$ is a functor, Γ_{X*}, from the category $\mathfrak{Act}_\mathfrak{c}(C_X)$ of continuous actions of (svelte) monoidal categories on the category C_X to the category $\mathfrak{Mon}(C_X)$ of monads on C_X. The functor Γ_{X*} has a right adjoint, $\Gamma^!_X$, which assigns to each monad $\mathcal{F} = (F, \mu)$ on C_X the forgetful strict monoidal functor

$$\widetilde{End}_\mathfrak{c}(C_X)/\mathcal{F} \xrightarrow{\Gamma^!_X(\mathcal{F})} \widetilde{End}_\mathfrak{c}(C_X).$$

Suppose that, in addition, the category C_X is small-cocomplete. Then the monads $\mathcal{F}_\varphi$ and $\mathcal{F}_{\varphi'}$ are continuous, or, equivalently, all morphisms of the diagram (2) are affine. The category $End_\mathfrak{c}(C_X)/F$ has a canonical final object – the pair (F, id_F), which implies that the adjunction arrow $\Gamma_{X*} \circ \Gamma^!_X \longrightarrow Id$ is an isomorphism, or, what is the same, the functor $\Gamma^!_X$ is fully faithful; i.e. Γ_{X*} is equivalent to a localization functor.

The functor Γ_{X*} has a left adjoint (forcibly fully faithful), Γ^*_X, which assigns to every monad $\mathcal{F}$ on C_X the monoidal functor from the trivial monoidal category to $\widetilde{End}_\mathfrak{c}(C_X)$ sending the unique object to F (cf. 1.1.1).

1.5.1. Example: the stabilizer of a set of subcategories. Let $\mathcal{B}$ be a set of full subcategories of the category C_X; and let $\mathcal{E}_\mathcal{B}$ be the full subcategory of the category $\mathcal{E}$ generated by all objects L such that $\Phi(L)(\mathcal{A}) \subseteq \mathcal{A}$ for each $\mathcal{A} \in \mathcal{B}$. It follows that $\mathcal{E}_\mathcal{B}$ is a monoidal subcategory of $\widetilde{\mathcal{E}}$ and the restriction $\widetilde{\Phi}_\mathcal{B}$ of the monoidal functor $\widetilde{\Phi}$ to the subcategory $\mathcal{E}_{\mathcal{B}}$ is a continuous action of $\widetilde{\mathcal{E}}_{\mathcal{B}}$ on C_X. Thus, we have the category of $\widetilde{\Phi}_{\mathcal{B}}$-modules and the restriction functor $(\widetilde{\Phi}/X)-mod \xrightarrow{\mathfrak{f}_{\mathcal{B}*}} (\widetilde{\Phi}_{\mathcal{B}}/X)-mod$ corresponding to the embedding $\widetilde{\mathcal{E}}_{\mathcal{B}} \longrightarrow \widetilde{\mathcal{E}}$.

If the category C_X is small-complete, then the functor $(\widetilde{\Phi}_{\mathcal{B}}/X)-mod \xrightarrow{\varphi_{\mathcal{B}*}} C_X$ is equivalent to the forgetful functor $\mathcal{F}_{\varphi_{\mathcal{B}}}-mod \longrightarrow C_X$ for a monad $\mathcal{F}_{\varphi_{\mathcal{B}}}$ on C_X and we obtain the commutative diagram

$$\begin{array}{ccc}
\mathbf{Sp}(\mathcal{F}_{\varphi}/X) & \xrightarrow{\psi_{\mathcal{B}}} & \mathbf{Sp}(\mathcal{F}_{\varphi_{\mathcal{B}}}/X) \\
\varphi \searrow & & \swarrow \varphi_{\mathcal{B}} \\
& X &
\end{array} \tag{3}$$

corresponding to a monad morphism $\mathcal{F}_{\varphi_{\mathcal{B}}} \xrightarrow{\psi_{\mathcal{B}}} \mathcal{F}_{\varphi}$.

If, in addition, the category C_X is small-cocomplete, then the monads $\mathcal{F}_{\varphi}$ and $\mathcal{F}_{\varphi_{\mathcal{B}}}$ are continuous and, therefore, all morphisms in the commutative diagram (3) are affine.

1.6. Stabilizers of points and related functors. We fix a svelte abelian category C_X together with a continuous action of a svelte monoidal category $\widetilde{\mathcal{E}} = (\mathcal{E}, \odot, \mathbb{I}, a)$ on C_X given by a monoidal functor $\widetilde{\Phi} = (\Phi, \phi, \phi_0)$ from $\widetilde{\mathcal{E}}$ to the monoidal category $\widetilde{\mathfrak{E}\mathfrak{x}_{\mathfrak{c}}}(C_X)$ of continuous *exact* additive endofunctors of C_X. We shall assume that the category C_X has small limits and colimits.

1.6.1. The stabilizer of a point of the spectrum. Fix a point $\mathcal{P}$ of $\mathbf{Spec}^0_{\mathfrak{c}}(X)$. We shall write $(\mathcal{P})$ for pair $\{\mathcal{P}, \widehat{\mathcal{P}}\}$, where $\widehat{\mathcal{P}}$ is the corresponding to $\mathcal{P}$ Serre subcategory. We define the *stabilizer of the point* $\mathcal{P}$ as the stabilizer $\mathcal{E}_{(\mathcal{P})}$ of the pair $(\mathcal{P}) = \{\mathcal{P}, \widehat{\mathcal{P}}\}$. We have a commutative diagram of affine morphisms

$$\begin{array}{ccc}
\mathfrak{A} = \mathbf{Sp}(\mathcal{F}_{\varphi}/X) & \xrightarrow{\mathfrak{f}_{\mathcal{P}}} & \mathbf{Sp}(\mathcal{F}_{\varphi_{\mathcal{P}}}/X) = \mathfrak{A}_{\mathcal{P}} \\
\varphi \searrow & & \swarrow \varphi_{\mathcal{P}} \\
& X &
\end{array} \tag{1}$$

where $\mathfrak{f}_{\mathcal{P}} = \mathbf{Sp}(\psi_{\mathcal{P}})$ for a monad morphism $\mathcal{F}_{\varphi_{\mathcal{P}}} \xrightarrow{\psi_{\mathcal{P}}} \mathcal{F}_{\varphi}$.

1.6.2. The functor $\mathfrak{L}_{\mathcal{P}}$. Fix an element $\mathcal{P}$ of $\mathbf{Spec}^0_{\mathfrak{c}}(X)$. We denote by $\mathfrak{L}_{\mathcal{P}}$ the composition of the functors $C_{\mathfrak{A}_{\mathcal{P}}} \xrightarrow{\mathfrak{f}^*_{\mathcal{P}}} C_{\mathfrak{A}}$ and

$$C_{\mathfrak{A}} \xrightarrow{\Psi_{\mathcal{P}}} C_{\mathfrak{A}}, \quad M \longmapsto M/tors_{\varphi_*^{-1}(\widehat{\mathcal{P}})}(M).$$

Notice that since $\widehat{\mathcal{P}}$ is a Serre subcategory of C_X and φ_* has a right adjoint, the preimage $\varphi_*^{-1}(\widehat{\mathcal{P}})$ of $\widehat{\mathcal{P}}$ is a Serre subcategory of $C_{\mathfrak{A}}$. Thanks to the property (sup), every Serre subcategory, $\mathbb{S}$, of $C_{\mathfrak{A}}$ is coreflective, i.e. the inclusion functor $\mathbb{S} \hookrightarrow C_{\mathfrak{A}}$ has a right adjoint, $tors_{\mathbb{S}} : C_{\mathfrak{A}} \longrightarrow \mathbb{S}$ which assigns to every object M its $\mathbb{S}$-torsion. In particular, $tors_{\varphi_*^{-1}(\widehat{\mathcal{P}})}(M)$ is well defined for all $M \in ObC_{\mathfrak{A}}$.

1.6.2.1. Proposition. *Let $\mathcal{P} \in \mathbf{Spec}^0_{\mathfrak{c}}(X)$ be such that an inverse image functor $\mathfrak{f}^*_{\mathcal{P}}$ of the morphism $\mathfrak{A} \xrightarrow{\mathfrak{f}_{\mathcal{P}}} \mathfrak{A}_{\mathcal{P}}$ is exact. Then the functor $\mathfrak{L}_{\mathcal{P}}$ is exact.*

Proof. The functor $\mathfrak{L}_{\mathcal{P}}$ is the composition of two right exact functors, $\mathfrak{f}^*_{\mathcal{P}}$ and $\Psi_{\mathcal{P}}$, hence it is right exact. It remains to verify that $\mathfrak{L}_{\mathcal{P}}$ maps monomorphisms to monomorphisms. Let $K \xrightarrow{\mathfrak{j}} M$ be a monomorphism in $C_{\mathfrak{A}_{\mathcal{P}}}$. Consider the commutative diagram

$$\begin{array}{ccc}
\mathfrak{f}^*_{\mathcal{P}}(K) & \xrightarrow{\mathfrak{f}^*_{\mathcal{P}}(\mathfrak{j})} & \mathfrak{f}^*_{\mathcal{P}}(M) \\
{\scriptstyle \mathfrak{e}_K}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{e}_M} \\
\mathfrak{L}_{\mathcal{P}}(K) & \xrightarrow{\mathfrak{L}_{\mathcal{P}}(\mathfrak{j})} & \mathfrak{L}_{\mathcal{P}}(M)
\end{array} \qquad (1)$$

and its image by the localization $C_{\mathfrak{A}} \xrightarrow{q^*} C_{\mathfrak{A}}/\varphi_*^{-1}(\widehat{\mathcal{P}})$. Since, by hypothesis, the functor $\mathfrak{f}^*_{\mathcal{P}}$ is exact and the localization functor q^* is exact, $q^*\mathfrak{f}^*_{\mathcal{P}}(K) \xrightarrow{q^*\mathfrak{f}^*_{\mathcal{P}}(\mathfrak{j})} q^*\mathfrak{f}^*_{\mathcal{P}}(M)$ is a monomorphism. The arrows $q^*\mathfrak{f}^*_{\mathcal{P}}(K) \xrightarrow{q^*(\mathfrak{e}_K)} q^*\mathfrak{L}_{\mathcal{P}}(K)$ and $q^*\mathfrak{f}^*_{\mathcal{P}}(M) \xrightarrow{q^*(\mathfrak{e}_M)} q^*\mathfrak{L}_{\mathcal{P}}(M)$ are isomorphisms. Therefore $q^*\mathfrak{L}_{\mathcal{P}}(K) \xrightarrow{q^*\mathfrak{L}_{\mathcal{P}}(\mathfrak{j})} q^*\mathfrak{L}_{\mathcal{P}}(M)$ is a monomorphism. Since the object $\mathfrak{L}_{\mathcal{P}}(K)$ is $Ker(q^*)$-torsion free, $\mathfrak{L}_{\mathcal{P}}(K) \xrightarrow{\mathfrak{L}_{\mathcal{P}}(\mathfrak{j})} \mathfrak{L}_{\mathcal{P}}(M)$ is a monomorphism. ∎

1.6.3. Remark. The notion of the stabilizer of a point, the definition of the functor $\mathfrak{L}_{\mathcal{P}}$, and Proposition 1.6.2.1 make sense if $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ is replaced by any of the remained spectra considered here: $\mathbf{Spec}(X)$, $\mathbf{Spec}_-(X)$, $\mathbf{Spec}^{\mathfrak{c}}_-(X)$, or $\mathbf{Spec}^0_{\mathfrak{s}}(X)$.

We need the following assertion which is of independent interest.

1.7. Proposition. *Let C_Y be an abelian category and $C_Y \xrightarrow{g^*} C_Z$ a functor having a right adjoint, g_*; and let $Id_{C_Y} \xrightarrow{\eta} g_*g^*$ be an adjunction arrow.*

(a) If the functor g^* is exact, then the adjunction morphism $M \xrightarrow{\eta(M)} g_*g^*(M)$ is a monomorphism for every $M \in Spec(Y)$ such that $g^*(M) \neq 0$.

(b) Suppose that the category C_Y satisfies (AB4), i.e. it has small coproducts and the coproduct of a set of monomorphisms is a monomorphism. If the functor g^* is exact and g_*

has a right adjoint, then $M \xrightarrow{\eta(M)} g_*g^*(M)$ *is a monomorphism for every* $M \in Spec^0_{\mathfrak{c}}(Y)$ *such that* $g^*(M) \neq 0$.

Proof. (a1) Let M be an arbitrary object of C_Y, and let $K \xrightarrow{\mathfrak{j}} M$ be the kernel of the adjunction morphism $M \xrightarrow{\eta(M)} g_*g^*(M)$. Consider the commutative diagram

$$\begin{array}{ccc} K & \xrightarrow{\eta(K)} & g_*g^*(K) \\ \mathfrak{j} \downarrow & & \downarrow g_*g^*(\mathfrak{j}) \\ M & \xrightarrow{\eta(M)} & g_*g^*(M) \end{array}$$

Since g^* is exact, the functor g_*g^* is left exact, in particular $g_*g^*(\mathfrak{j})$ is a monomorphism. Therefore, the equality $g_*g^*(\mathfrak{j}) \circ \eta(K) = \eta(M) \circ \mathfrak{j} = 0$ implies that $\eta(K) = 0$.

(a2) Suppose now that M belongs to $Spec(Y)$. If $K \neq 0$, then $K \succ M$, i.e. there exists a diagram $K^{\oplus n} \xleftarrow{\gamma} L \xrightarrow{\mathfrak{e}} M$ in which the left arrow is an epimorphism and the right arrow is a monomorphism. Consider the associated commutative diagram

$$\begin{array}{ccccc} K^{\oplus n} & \xleftarrow{\gamma} & L & \xrightarrow{\mathfrak{e}} & M \\ \eta(K^{\oplus n}) \downarrow & & \eta(L) \downarrow & & \downarrow \eta(M) \\ g_*g^*(K^{\oplus n}) & \xleftarrow{g_*g^*(\gamma)} & g_*g^*(L) & \xrightarrow{g_*g^*(\mathfrak{e})} & g_*g^*(M) \end{array} \qquad (2)$$

By (a1), the left vertical arrow in (2) is zero. Since $L \xrightarrow{\gamma} K^{\oplus n}$ is a monomorphism and the functor g_*g^* is, thanks to the exactness of g_*, left exact, $g_*g^*(\gamma)$ is a monomorphism. Therefore, the equality $g_*g^*(\gamma) \circ \eta(L) (= \eta(K^{\oplus n}) \circ \gamma) = 0$ implies that $\eta(L) = 0$. Then the commutativity of the right square of (2) yields the equality $\eta(M) \circ \mathfrak{e} = 0$. Since $\mathfrak{e}$ is an epimorphism, it follows that $\eta(M) = 0$. But, the equality $\eta(M) = 0$ means precisely that the object M belongs to the kernel of the functor g^*, i.e. $g^*(M) = 0$.

(b) Suppose that C_Y satisfies (AB4) and the functor g_* has a right adjoint.

By definition, an object M belongs to $Spec^0_{\mathfrak{c}}(Y)$ iff M is contained in the subcategory $[N]_{\mathfrak{c}}$ for any its nonzero subobject N. Since C_Y satisfies (AB4), each object of $[N]_{\mathfrak{c}}$ is a subquotients of the coproduct of a set of copies of the object N. In particular, if $K = Ker(\eta(M))$ is nonzero, there is a diagram $K^{\oplus J} \xleftarrow{\gamma} L \xrightarrow{\mathfrak{e}} M$, for some, infinite in general, set J, whose left (resp. right) arrow is a monomorphism (resp. an epimorphism). Thus, if $K \neq 0$, we have a commutative diagram

$$\begin{array}{ccccccc} K^{\oplus J} & \xrightarrow{id} & K^{\oplus J} & \xleftarrow{\gamma} & L & \xrightarrow{\mathfrak{e}} & M \\ \eta(K)^{\oplus J} \downarrow & & \eta(K^{\oplus J}) \downarrow & & \eta(L) \downarrow & & \downarrow \eta(M) \\ g_*g^*(K)^{\oplus J} & \xrightarrow{\sim} & g_*g^*(K^{\oplus J}) & \xleftarrow{g_*g^*(\gamma)} & g_*g^*(L) & \xrightarrow{g_*g^*(\mathfrak{e})} & g_*g^*(M) \end{array} \qquad (3)$$

in which the lower left horizontal arrow is an isomorphism and the lower right horizontal arrow is an epimorphisms. Both observations follow from the fact that, since g_* has a right adjoint, the composition g_*g^* has a right adjoint, hence it preserves arbitrary colimits.

It follows from the commutativity of the diagram (3) and the equality $\eta(K) = 0$ established in (a1) above, that $\eta(K^{\oplus J}) = 0$. Repeating the argument (a2), we obtain the equality $g^*(M) = 0$. ■

1.7.1. Corollary. *Let C_Y be an abelian category, $C_Y \xrightarrow{g^*} C_Z$ a functor having a right adjoint, g_*, and $Id_{C_Y} \xrightarrow{\eta} g_*g^*$ an adjunction arrow.*

(a) If the functor g^* is exact and faithful, then $M \xrightarrow{\eta(M)} g_*g^*(M)$ is a monomorphism for every $M \in Spec(Y)$.

(b) If the category C_Y satisfies (AB4), the functor g_* has a right adjoint, and the functor g^* is exact and faithful, then the adjunction morphism $M \xrightarrow{\eta(M)} g_*g^*(M)$ is a monomorphism for every $M \in Spec^0_{\mathfrak{c}}(Y)$.

Proof. Since the functor g^* is faithful, $g^*(M) \neq 0$ for any nonzero object M, in particular, for any $M \in Spec^0_{\mathfrak{c}}(Y)$. The assertion follows from 1.7. ■

1.8. Proposition. *Let $\mathcal{P}$ be an element of* $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ *such that the inverse image functor $\mathfrak{f}^*_{\mathcal{P}}$ of the morphism $\mathfrak{A} \xrightarrow{\mathfrak{f}_{\mathcal{P}}} \mathfrak{A}_{\mathcal{P}}$ is exact and faithful. Let $Id_{\mathfrak{A}} \xrightarrow{\mathfrak{r}_{\mathcal{P}}} \mathfrak{f}_{\mathcal{P}*}\mathfrak{L}_{\mathcal{P}}$ be the composition of the adjunction arrow $Id_{\mathfrak{A}} \longrightarrow \mathfrak{f}_{\mathcal{P}*}\mathfrak{f}^*_{\mathcal{P}}$ and the epimorphism $\mathfrak{f}_{\mathcal{P}*}\mathfrak{f}^*_{\mathcal{P}} \longrightarrow \mathfrak{f}_{\mathcal{P}*}\mathfrak{L}_{\mathcal{P}}$. The morphism*

$$M \xrightarrow{\mathfrak{r}_{\mathcal{P}}(M)} \mathfrak{f}_{\mathcal{P}*}\mathfrak{L}_{\mathcal{P}}(M) \tag{3}$$

is a monomorphism for every $M \in Spec(\mathfrak{A}_{\mathcal{P}})$ such that $\mathcal{P} \in Supp(\varphi^*_{\mathcal{P}}(M))$.

Proof. Let $K_M \xrightarrow{\mathfrak{j}_M} M$ denote the kernel of the morphism (3). The functor $\mathfrak{f}_{\mathcal{P}*}$ preserves colimits. By 1.7.1, the adjunction morphism $M \xrightarrow{\mathfrak{r}(M)} \mathfrak{f}_{\mathcal{P}*}\mathfrak{f}^*_{\mathcal{P}}(M)$ is a monomorphism for every $M \in Spec(\mathfrak{A}_{\mathcal{P}})$. Therefore $\varphi_{\mathcal{P}*}(K_M)$ is an object of $\widehat{\mathcal{P}}$. Since M belongs to the spectrum, if $K_M \neq 0$, then $M \in [K_M]_{\mathfrak{c}}$. The functor $\varphi_{\mathcal{P}*}$ is exact and preserves small colimits. Since for any object L, the subcategory $[L]_{\mathfrak{c}}$ is obtained from L by taking arbitrary small colimits and subobjects, $\varphi_{\mathcal{P}*}([L]_{\mathfrak{c}}) \subseteq [\varphi_{\mathcal{P}*}(L)]_{\mathfrak{c}}$. In particular, $\varphi_{\mathcal{P}*}(M)$ is an object of $[\varphi_{\mathcal{P}*}(K_M)]_{\mathfrak{c}}$. The latter implies that $\varphi_{\mathcal{P}*}(M)$ is also an object of the subcategory $\widehat{\mathcal{P}}$; that is $\mathcal{P} \notin Supp(\varphi_{\mathcal{P}*}(M))$. ■

2. Realization of points.

2.1. Assumptions and notations. We fix a Grothendieck category C_X together with a continuous action of a svelte monoidal category $\widetilde{\mathcal{E}} = (\mathcal{E}, \odot, \mathbb{I}, a)$ on C_X given by a monoidal functor $\widetilde{\Phi} = (\Phi, \phi, \phi_0)$ from $\widetilde{\mathcal{E}}$ to the full monoidal subcategory $\widetilde{\mathfrak{Ex}_{\mathfrak{c}}}(C_X)$ of $\widetilde{End}(C_X)$ generated by continuous *exact* endofunctors of C_X. Being a Grothendieck category, C_X has small limits and colimits, which guarantees that continuous actions of svelte monoidal categories on C_X have colimits, and these colimits are continuous monads.

In particular, there is a (determined uniquely up to isomorphism) continuous monad $\mathcal{F}_\varphi = (F_\varphi, \mu_\varphi)$ and a universal morphism (or universal cone) $\widetilde{\Phi} \xrightarrow{\gamma_\varphi} \mathcal{F}_\varphi$ whose pull-back functor $(\mathcal{F}_\varphi/X) - mod \xrightarrow{\gamma_{\varphi *}} (\widetilde{\Phi}/X) - mod$ is an equivalence between the category of $\widetilde{\Phi}$-modules and the category of $\mathcal{F}_\varphi$-modules (see 1.3). The morphism γ_φ gives rise to a monoidal functor $\widetilde{\mathcal{E}} \xrightarrow{\widetilde{\Psi}_\varphi} \widetilde{\mathfrak{Ex}_{\mathfrak{c}}}(C_X)/\mathcal{F}_\varphi$ so that $\widetilde{\Phi}$ is the composition of $\widetilde{\Psi}_\varphi$ and the forgetful (strict) monoidal functor $\widetilde{\mathfrak{Ex}_{\mathfrak{c}}}(C_X)/\mathcal{F}_\varphi \xrightarrow{\widetilde{\mathfrak{F}}_X} \widetilde{\mathfrak{Ex}_{\mathfrak{c}}}(C_X)$.

In what follows, the monoidal category $\widetilde{\mathcal{E}}$ can be identified with its image in the strict monoidal category $\widetilde{\mathfrak{Ex}_{\mathfrak{c}}}(C_X)/\mathcal{F}_\varphi$. So, we assume, for convenience, that $\widetilde{\mathcal{E}}$ is a monoidal subcategory of $\widetilde{\mathfrak{Ex}_{\mathfrak{c}}}(C_X)/\mathcal{F}_\varphi$ and $\widetilde{\Phi}$ is the restriction to $\widetilde{\mathcal{E}}$ of the forgetful functor $\widetilde{\mathfrak{F}}_X$.

2.1.1. $\mathbf{Spec}^{\mathcal{P}}_{\mathfrak{c}}(\mathfrak{A}_{\mathcal{P}})$ and $\mathbf{Spec}^{\mathcal{P}}_{\mathfrak{c}}(\mathfrak{A})$. Fix a point $\mathcal{P}$ of $\mathbf{Spec}^0_{\mathfrak{c}}(X)$. Let $\widetilde{\mathcal{E}}_{(\mathcal{P})} = \widetilde{\mathcal{E}}_{\{\mathcal{P}, \widehat{\mathcal{P}}\}}$ be the stabilizer of the point $\mathcal{P}$, i.e. the full subcategory of $\widetilde{\mathcal{E}}$ generated by all $(U, U \xrightarrow{\mathfrak{v}} F_\varphi)$ such that $U(\mathcal{P}) \subseteq \mathcal{P}$ and $U(\widehat{\mathcal{P}}) \subseteq \widehat{\mathcal{P}}$. Let $\widetilde{\Phi}_{(\mathcal{P})}$ be the restriction of $\widetilde{\Phi}$ to $\widetilde{\mathcal{E}}_{(\mathcal{P})}$ and $\mathcal{F}_{\varphi_{\mathcal{P}}}$ the corresponding monad – the colimit of $\widetilde{\Phi}_{(\mathcal{P})}$ (cf. 1.5.1). By 1.6.1, we have a commutative diagram of affine morphisms

$$\begin{array}{ccc} \mathfrak{A} = \mathbf{Sp}(\mathcal{F}_\varphi/X) & \xrightarrow{\mathfrak{f}_{\mathcal{P}}} & \mathbf{Sp}(\mathcal{F}_{\varphi_{\mathcal{P}}}/X) = \mathfrak{A}_{\mathcal{P}} \\ \varphi \searrow & & \swarrow \varphi_{\mathcal{P}} \\ & X & \end{array} \qquad (1)$$

corresponding to a monad morphism $\mathcal{F}_{\varphi_{\mathcal{P}}} \xrightarrow{\psi_{\mathcal{P}}} \mathcal{F}_\varphi$, where the 'space' $\mathfrak{A}_{\mathcal{P}}$ and the monad $\mathcal{F}_{\varphi_{\mathcal{P}}}$ (or, more precisely, the monad morphism $\psi_{\mathcal{P}}$) are called *stabilizers* of the point $\mathcal{P}$.

We denote by $Spec^{\mathcal{P}}_{\mathfrak{c}}(\mathfrak{A}_{\mathcal{P}})$ all objects $\widetilde{P}$ of $Spec^0_{\mathfrak{c}}(\mathfrak{A}_{\mathcal{P}})$ such that $\mathcal{P} \in Ass(\varphi^*_{\mathcal{P}}(\widetilde{P}))$, and we set $\mathbf{Spec}^{\mathcal{P}}_{\mathfrak{c}}(\mathfrak{A}_{\mathcal{P}}) = \{[\widetilde{P}]_{\mathfrak{c}} \mid \widetilde{P} \in Spec^{\mathcal{P}}_{\mathfrak{c}}(\mathfrak{A}_{\mathcal{P}})\}$.

Objects of $Spec^{\mathcal{P}}_{\mathfrak{c}}(\mathfrak{A})$ are all $M \in Spec_{\mathfrak{c}}(\mathfrak{A})$ such that the object $\mathfrak{f}^*_{\mathcal{P}}(M)$ has an associated point from $\mathbf{Spec}^{\mathcal{P}}_{\mathfrak{c}}(\mathfrak{A}_{\mathcal{P}})$. We set $\mathbf{Spec}^{\mathcal{P}}_{\mathfrak{c}}(\mathfrak{A}) = \{[M]_{\mathfrak{c}} \mid M \in Spec^{\mathcal{P}}_{\mathfrak{c}}(\mathfrak{A})\}$.

2.2. Theorem. *Let $\mathcal{P} \in \mathbf{Spec}^0_{\mathfrak{c}}(X)$ be such that the inverse image functor $\mathfrak{f}^*_{\mathcal{P}}$ of the morphism $\mathfrak{A} \xrightarrow{\mathfrak{f}_{\mathcal{P}}} \mathfrak{A}_{\mathcal{P}}$ is exact and faithful, and the following condition holds:*

(*) Let $P \in Spec^0_{\mathfrak{c}}(X)$ be representative of $\mathcal{P}$ and M a subobject of $\varphi^*(P)$ such that $\mathcal{P} \in Supp(\varphi_*(M))$. There exists $(U', \mathfrak{v}) \in Ob\mathcal{E}_{(\mathcal{P})}$ and a subobject P' of P such that the image of $U'(P')$ in $F_\varphi(P) = \varphi_*\varphi^*(P)$ is a subobject of $\varphi_*(M)$ whose support contains $\mathcal{P}$.

Then the functor $C_{\mathfrak{A}_\mathcal{P}} \xrightarrow{\mathfrak{L}_\mathcal{P}} C_\mathfrak{A}$ *induces a morphism*

$$\mathbf{Spec}_\mathfrak{c}^\mathcal{P}(\mathfrak{A}_\mathcal{P}) \xrightarrow{\mathfrak{L}_\mathcal{P}} \mathbf{Spec}_\mathfrak{c}^\mathcal{P}(\mathfrak{A}). \tag{1}$$

with the following properties:

(α) Every $[M] \in \mathbf{Spec}_\mathfrak{c}^0(\mathfrak{A})$ *such that the image* $\mathfrak{f}_\mathcal{P}^*(M)$ *of* M *in* $C_{\mathfrak{A}_\mathcal{P}}$ *has an associated point from* $\mathbf{Spec}_\mathfrak{c}^\mathcal{P}(\mathfrak{A}_\mathcal{P})$ *belongs to the image of the map (1).*

(β) The functor $\mathfrak{L}_\mathcal{P}$ *maps simple objects from* $Spec_\mathfrak{c}^\mathcal{P}(\mathfrak{A}_\mathcal{P})$ *to simple objects of* $C_\mathfrak{A}$*.*

Proof. (a) Let $\widetilde{P}$ be an object of $Spec_\mathfrak{c}^\mathcal{P}(\mathfrak{A}_\mathcal{P})$; i.e. $\widetilde{P}$ is an object of $Spec_\mathfrak{c}^0(\mathfrak{A}_\mathcal{P})$ and there exists a monomorphism $P \xrightarrow{\iota} \varphi_\mathcal{P}^*(\widetilde{P})$, where P is an object of $Spec_\mathfrak{c}^0(X)$ such that $\mathcal{P} = [P]_\mathfrak{c}$. The claim is that $\mathfrak{L}_\mathcal{P}(\widetilde{P})$ is an object of $Spec_\mathfrak{c}^\mathcal{P}(\mathfrak{A})$; i.e. $\mathfrak{L}_\mathcal{P}(\widetilde{P}) \in [M']_\mathfrak{c}$ for any nonzero subobject M' of $\mathfrak{L}_\mathcal{P}(\widetilde{P})$.

(i) Consider the composition $P \xrightarrow{\widehat{\mathfrak{v}}} \varphi_*\mathfrak{f}_\mathcal{P}^*(\widetilde{P})$ of the monomorphism $P \longrightarrow \varphi_\mathcal{P}^*(\widetilde{P})$ and the morphism $\varphi_\mathcal{P}^*(\widetilde{P}) \xrightarrow{\varphi_\mathcal{P}^*\eta(\widetilde{P})} \varphi_\mathcal{P}^*\mathfrak{f}_{\mathcal{P}*}\mathfrak{f}_\mathcal{P}^*(\widetilde{P}) = \varphi_*\mathfrak{f}_\mathcal{P}^*(\widetilde{P})$. By 1.7.1, the adjunction morphism $\widetilde{P} \xrightarrow{\eta(\widetilde{P})} \mathfrak{f}_{\mathcal{P}*}\mathfrak{f}_\mathcal{P}^*(\widetilde{P})$ is a monomorphism. Therefore its image by the exact functor $\varphi_\mathcal{P}^*$ is a monomorphism, which implies that $P \xrightarrow{\widehat{\mathfrak{v}}} \varphi_*\mathfrak{f}_\mathcal{P}^*(\widetilde{P})$ is a monomorphism.

In particular, the corresponding morphism $\varphi^*(P) \xrightarrow{\mathfrak{v}} \mathfrak{f}_\mathcal{P}^*(\widetilde{P})$ is nonzero.

(ii) Consider the cartesian square

$$\begin{array}{ccc} P_1 & \xrightarrow{h} & \widetilde{P} \\ \Big\downarrow & & \Big\downarrow \eta(\widetilde{P}) \\ \mathfrak{f}_{\mathcal{P}*}\varphi^*(P) & \xrightarrow{\mathfrak{f}_{\mathcal{P}*}(\mathfrak{v})} & \mathfrak{f}_{\mathcal{P}*}\mathfrak{f}_\mathcal{P}^*(\widetilde{P}) \end{array} \tag{2}$$

The functor $\varphi_\mathcal{P}^*$, being (left) exact, maps (2) to a cartesian square

$$\begin{array}{ccc} \varphi_\mathcal{P}^*(P_1) & \xrightarrow{h'} & \varphi_\mathcal{P}^*(\widetilde{P}) \\ \Big\downarrow & & \Big\downarrow \varphi_\mathcal{P}^*\eta(\widetilde{P}) \\ \varphi_*\varphi^*(P) & \xrightarrow{\varphi_*(\mathfrak{v})} & \varphi_*\mathfrak{f}_\mathcal{P}^*(\widetilde{P}) \end{array} \tag{3}$$

It follows from the commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{\iota} & \varphi_\mathcal{P}^*(\widetilde{P}) \\ \eta_u(P) \Big\downarrow & & \Big\downarrow \varphi_\mathcal{P}^*\eta(\widetilde{P}) \\ \varphi_*\varphi^*(P) & \xrightarrow{\varphi_*(\mathfrak{v})} & \varphi_*\mathfrak{f}_\mathcal{P}^*(\widetilde{P}) \end{array} \tag{4}$$

and the universal property of cartesian squares (applied to the square (3)) that there exists a unique morphism $P \xrightarrow{\rho} \varphi^*_{\mathcal{P}}(P_1)$ such that the diagram

$$\begin{array}{ccccc} P & \xrightarrow{\rho} & \varphi^*_{\mathcal{P}}(P_1) & \xrightarrow{\varphi^*_{\mathcal{P}}(h)} & \varphi^*_{\mathcal{P}}(\widetilde{P}) \\ {\scriptstyle \eta_u(P)}\downarrow & & \downarrow & & \downarrow{\scriptstyle \varphi^*_{\mathcal{P}}\eta(\widetilde{P})} \\ \varphi_*\varphi^*(P) & \xrightarrow{id} & \varphi_*\varphi^*(P) & \xrightarrow{\varphi_*(\mathfrak{v})} & \varphi_*\mathfrak{f}^*_{\mathcal{P}}(\widetilde{P}) \end{array} \tag{5}$$

commutes and the composition of $P \xrightarrow{\rho} \varphi^*_{\mathcal{P}}(P_1) \xrightarrow{\varphi^*_{\mathcal{P}}(h)} \varphi^*_{\mathcal{P}}(\widetilde{P})$ coincides with the monomorphism $P \xrightarrow{\iota} \varphi^*_{\mathcal{P}}(\widetilde{P})$ we started with. This shows, among other things, that the canonical morphism $P \xrightarrow{\rho} \varphi^*_{\mathcal{P}}(P_1)$ is a monomorphism and the morphism $P_1 \xrightarrow{h} \widetilde{P}$ is nonzero. Since $\widetilde{P}$ belongs to $Spec^0_{\mathfrak{c}}(\mathfrak{A}_{\mathcal{P}})$, the image, $\widetilde{P}_1$, of the morphism $P_1 \xrightarrow{h} \widetilde{P}$ is equivalent to $\widetilde{P}$, i.e. $[\widetilde{P}_1]_{\mathfrak{c}} = [\widetilde{P}]_{\mathfrak{c}}$. By 1.6.2.1, this implies that $[\mathfrak{L}_{\mathcal{P}}(\widetilde{P}_1)]_{\mathfrak{c}} = [\mathfrak{L}_{\mathcal{P}}(\widetilde{P})]_{\mathfrak{c}}$.

The decomposition of the morphism $P_1 \xrightarrow{h} \widetilde{P}$ into an epimorphism $P_1 \xrightarrow{h_1} \widetilde{P}_1$ and a monomorphism $\widetilde{P}_1 \longrightarrow \widetilde{P}$ induces the corresponding decomposition of (the right square of) the diagram (5):

$$\begin{array}{ccccccc} P & \xrightarrow{\rho} & \varphi^*_{\mathcal{P}}(P_1) & \xrightarrow{\varphi^*_{\mathcal{P}}(h_1)} & \varphi^*_{\mathcal{P}}(\widetilde{P}_1) & \longrightarrow & \varphi^*_{\mathcal{P}}(\widetilde{P}) \\ {\scriptstyle \eta_u(P)}\downarrow & & \downarrow & & \downarrow{\scriptstyle \varphi^*_{\mathcal{P}}\eta(\widetilde{P}_1)} & & \downarrow{\scriptstyle \varphi^*_{\mathcal{P}}\eta(\widetilde{P})} \\ \varphi_*\varphi^*(P) & \xrightarrow{id} & \varphi_*\varphi^*(P) & \xrightarrow{\varphi_*(\mathfrak{v}_1)} & \varphi_*\mathfrak{f}^*_{\mathcal{P}}(\widetilde{P}_1) & \longrightarrow & \varphi_*\mathfrak{f}^*_{\mathcal{P}}(\widetilde{P}) \end{array} \tag{5'}$$

Therefore, one can, replacing the object $\widetilde{P}$ by $\widetilde{P}_1$, assume that the morphism $P_1 \xrightarrow{h} \widetilde{P}$ is an epimorphism. We keep this assumption for the rest of the proof.

(iii) The fact that $P_1 \xrightarrow{h} \widetilde{P}$ is an epimorphism implies that the morphism $\varphi^*(P) \xrightarrow{\mathfrak{v}} \mathfrak{f}^*_{\mathcal{P}}(\widetilde{P})$ (defined in (i)) is an epimorphism.

Indeed, the diagram (2) is equivalent (via adjunction) to the commutative diagram

$$\begin{array}{ccc} \mathfrak{f}^*_{\mathcal{P}}(P_1) & \xrightarrow{\mathfrak{f}^*_{\mathcal{P}}(h)} & \mathfrak{f}^*_{\mathcal{P}}(\widetilde{P}) \\ \downarrow & & \downarrow{\scriptstyle id} \\ \varphi^*(P) & \xrightarrow{\mathfrak{v}} & \mathfrak{f}^*_{\mathcal{P}}(\widetilde{P}) \end{array}$$

The upper horizontal arrow is an epimorphism, because the functor $\mathfrak{f}^*_{\mathcal{P}}$ is right exact (as any functor having a right adjoint). Therefore $\varphi^*(P) \xrightarrow{\mathfrak{v}} \mathfrak{f}^*_{\mathcal{P}}(\widetilde{P})$ is an epimorphism.

(iii^{bis}) One can arrive to the above conclusions via a shorter argument taking into consideration that the morphism functor $\mathfrak{A}_{\mathcal{P}} \xrightarrow{\varphi_{\mathcal{P}}} X$ is continuous.

Indeed, let $\varphi^*_{\mathcal{P}}(P) \xrightarrow{\mathfrak{v}'} \widetilde{P}$ be the morphism of $C_{\mathfrak{A}_{\mathcal{P}}}$ corresponding to the monomorphism $P \longrightarrow \varphi_{\mathcal{P}*}(\widetilde{P})$. Since the morphism $\mathfrak{v}'$ is nonzero and $\widetilde{P}$ belongs to $Spec^0_{\mathfrak{c}}(\mathfrak{A}_{\mathcal{P}})$, the image of $\mathfrak{v}'$ is an object of $Spec(\mathfrak{A}_{\mathcal{P}})$ and is equivalent to $\widetilde{P}$. Thanks to 1.6.2.1, we can (and will) assume (replacing $\widetilde{P}$ by the image of $\mathfrak{v}'$) that $\mathfrak{v}'$ is an epimorphism. Since the functor $\mathfrak{f}^*_{\mathcal{P}}$ is right exact, $\mathfrak{f}^*_{\mathcal{P}}\varphi^*_{\mathcal{P}}(P) \xrightarrow{\mathfrak{f}^*_{\mathcal{P}}(\mathfrak{v}')} \mathfrak{f}^*_{\mathcal{P}}(\widetilde{P})$ is an epimorphism. Notice that $\varphi^* \simeq \mathfrak{f}^*_{\mathcal{P}}\varphi^*_{\mathcal{P}}$. Thus, we have an epimorphism $\varphi^*(P) \xrightarrow{\mathfrak{v}} \mathfrak{f}^*_{\mathcal{P}}(\widetilde{P})$.

(iv) We denote by $\varphi^*(P) \xrightarrow{\mathfrak{e}} \mathfrak{L}_{\mathcal{P}}(\widetilde{P})$ the composition of $\varphi^*(P) \xrightarrow{\mathfrak{v}} \mathfrak{f}^*_{\mathcal{P}}(\widetilde{P})$ and the epimorphism $\mathfrak{f}^*_{\mathcal{P}}(\widetilde{P}) \longrightarrow \mathfrak{L}_{\mathcal{P}}(\widetilde{P})$. Let $M' \xrightarrow{\mathfrak{j}} \mathfrak{L}_{\mathcal{P}}(\widetilde{P})$ be a nonzero monomorphism. Consider the cartesian square

$$\begin{array}{ccc}
\varphi^*(P) & \xrightarrow{\ \mathfrak{e}\ } & \mathfrak{L}_{\mathcal{P}}(\widetilde{P}) \\
{\scriptstyle \widetilde{\mathfrak{j}}}\uparrow & & \uparrow{\scriptstyle \mathfrak{j}} \\
M & \xrightarrow{\ \widetilde{\mathfrak{e}}\ } & M'
\end{array} \tag{6}$$

and define the morphisms $\widetilde{P}_M \longrightarrow \mathfrak{f}^*_{\mathcal{P}}(M)$ and $\widetilde{P}_{M'} \longrightarrow \mathfrak{f}^*_{\mathcal{P}}(M')$ via the cartesian squares

$$\begin{array}{ccccc}
\mathfrak{f}^*_{\mathcal{P}}(M) & \xrightarrow{\mathfrak{f}^*_{\mathcal{P}}(\widetilde{\mathfrak{e}})} & \mathfrak{f}^*_{\mathcal{P}}(M') & \xrightarrow{\mathfrak{f}^*_{\mathcal{P}}(\mathfrak{j})} & \mathfrak{f}^*_{\mathcal{P}}(\mathfrak{L}_{\mathcal{P}}(\widetilde{P})) \\
\uparrow & & \uparrow & & \uparrow \\
\widetilde{P}_M & \xrightarrow{\ \mathfrak{e}'\ } & \widetilde{P}_{M'} & \xrightarrow{\ \mathfrak{j}'\ } & \widetilde{P}
\end{array} \tag{7}$$

It follows from 1.8 that the right vertical arrow in the diagram (7) is a monomorphism. Therefore, by a well-known property of cartesian squares, the remaining vertical arrows are monomorphisms too.

Since $\mathfrak{f}^*_{\mathcal{P}}$ is an exact functor, $\mathfrak{f}^*_{\mathcal{P}}(\mathfrak{j})$ is a monomorphism and $\mathfrak{f}^*_{\mathcal{P}}(\mathfrak{e})$ is an epimorphism. Therefore, $\widetilde{P}_M \xrightarrow{\mathfrak{e}'} \widetilde{P}_{M'}$ is an epimorphism and $\widetilde{P}_{M'} \xrightarrow{\mathfrak{j}'} \widetilde{P}$ is a monomorphism.

(v) We claim that $\widetilde{P}_{M'} \neq 0$.

Notice that $\mathcal{P} \in Supp(\varphi_*(M'))$, because M' is a nonzero subobject of $\mathfrak{L}_{\mathcal{P}}(\widetilde{P})$, in particular it does not belong to the Serre subcategory $\varphi_*^{-1}(\widehat{\mathcal{P}})$. Since there is an epimorphism $M \longrightarrow M'$ (see (6) above) and φ_* is an exact functor, $\mathcal{P} \in Supp(\varphi_*(M))$.

By the condition (*), there exists $(U', \mathfrak{v}) \in \mathcal{E}_{(\mathcal{P})}$ and a subobject P' of P such that $U'(P') \longrightarrow F_{\varphi}(P)$ factors through $\varphi_*(M)$ and the support of its image contains $\mathcal{P}$. It follows from the construction that $U'(P')$ is a subobject of $\varphi^*_{\mathcal{P}}(\widetilde{P})$ and $\varphi_*(M)$. Therefore, $\mathcal{P} \in Supp(\varphi^*_{\mathcal{P}}(\widetilde{P}_M))$ which, in turn, implies that $\widetilde{P}_{M'} \neq 0$.

(vi) Consider the following commutative diagram

$$\begin{array}{ccc} \mathfrak{f}^*_{\mathcal{P}}(\widetilde{P}_{M'}) & \longrightarrow & M' \\ {\scriptstyle \mathfrak{f}^*_{\mathcal{P}}(\mathfrak{j}')}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{j}} \\ \mathfrak{f}^*_{\mathcal{P}}(\widetilde{P}) & \longrightarrow & \mathfrak{L}_{\mathcal{P}}(\widetilde{P}) \end{array} \qquad (4)$$

corresponding to the right square of (7). Since $M' \xrightarrow{\mathfrak{j}} \mathfrak{L}_{\mathcal{P}}(\widetilde{P})$ is a monomorphism, the morphism $\mathfrak{f}^*_{\mathcal{P}}(\widetilde{P}_{M'}) \longrightarrow M'$ in (4) factors through $\mathfrak{L}_{\mathcal{P}}(\widetilde{P}_{M'}) \longrightarrow M'$. Thus, (4) induces a commutative diagram

$$\begin{array}{ccc} \mathfrak{L}_{\mathcal{P}}(\widetilde{P}_{M'}) & \xrightarrow{\ \iota\ } & M' \\ {\scriptstyle \mathfrak{L}_{\mathcal{P}}(\mathfrak{j}')}\searrow & & \swarrow{\scriptstyle \mathfrak{j}} \\ & \mathfrak{L}_{\mathcal{P}}(\widetilde{P}) & \end{array}$$

By 1.6.2.1, the arrow $\mathfrak{L}_{\mathcal{P}}(\widetilde{P}_{M'}) \xrightarrow{\mathfrak{L}_{\mathcal{P}}(\mathfrak{j}')} \mathfrak{L}_{\mathcal{P}}(\widetilde{P})$ is a monomorphism, and $\mathfrak{L}_{\mathcal{P}}(\mathfrak{j}') = \mathfrak{j} \circ \iota$. Therefore $\mathfrak{L}_{\mathcal{P}}(\widetilde{P}_{M'}) \xrightarrow{\iota} M'$ is a monomorphism. In particular, $\mathfrak{L}_{\mathcal{P}}(\widetilde{P}_{M'}) \in [M']_{\mathfrak{c}}$. Since $\widetilde{P}_{M'}$ is a nonzero subobject of $\widetilde{P}$ and $\widetilde{P}$ belongs to $Spec^0_{\mathfrak{c}}(\mathfrak{A}_{\mathcal{P}})$, these objects are equivalent, that is $[\widetilde{P}_{M'}]_{\mathfrak{c}} = [\widetilde{P}]_{\mathfrak{c}}$. By 1.6.2.1, the functor $\mathfrak{L}_{\mathcal{P}}$ is exact, which implies that $\mathfrak{L}_{\mathcal{P}}(\widetilde{P}) \in \mathfrak{L}_{\mathcal{P}}(\widetilde{P}_{M'})$ (see the argument of 1.8). Therefore $\mathfrak{L}_{\mathcal{P}}(\widetilde{P}) \in [M']_{\mathfrak{c}}$.

(b) By Proposition 1.6.2.1, the functor $\mathfrak{L}_{\mathcal{P}}$ is exact. Therefore, by the argument of 1.8, the subcategory $[\mathfrak{L}_{\mathcal{P}}(M)]_{\mathfrak{c}}$ depends only on the subcategory $[M]_{\mathfrak{c}}$.

(c) *The inverse map.* Let $M \in Spec^{\mathcal{P}}_{\mathfrak{c}}(\mathfrak{A})$; i.e. M is an object of $Spec^0_{\mathfrak{c}}(\mathfrak{A})$, and there exists a monomorphism $\widetilde{P} \longrightarrow \mathfrak{f}^*_{\mathcal{P}}(M)$ such that $\mathcal{P} \in Ass(\varphi^*_{\mathcal{P}}(\widetilde{P}))$ and $\widetilde{P}$ belongs to $Spec^0_{\mathfrak{c}}(\mathfrak{A}_{\mathcal{P}})$. Note that the object M is $\varphi_*^{-1}(\mathcal{P})$-torsion free.

In fact, suppose that M has a nonzero subobject N which belongs to $\varphi_*^{-1}(\widehat{\mathcal{P}})$. Since $M \in Spec^0_{\mathfrak{c}}(\mathfrak{A})$, $M \in [N]_{\mathfrak{c}}$ which implies that $\varphi_*(M) \in Ob\widehat{\mathcal{P}}$. The latter contradicts to the fact that a representative of the subcategory $\mathcal{P}$ is a subobject of $\varphi_*(M)$.

Since the object M is $\varphi_*^{-1}(\widehat{\mathcal{P}})$-torsion free, the canonical morphism $\mathfrak{f}^*_{\mathcal{P}}(\widetilde{P}) \longrightarrow M$ factors through a morphism $\mathfrak{L}_{\mathcal{P}}(\widetilde{P}) \longrightarrow M$. Due to the fact that M belongs to the spectrum and the natural morphism $\mathfrak{L}_{\mathcal{P}}(\widetilde{P}) \longrightarrow M$ is nonzero, $M \in [\mathfrak{L}_{\mathcal{P}}(\widetilde{P})]_{\mathfrak{c}}$. To prove that $\mathfrak{L}_{\mathcal{P}}(\widetilde{P}) \in [M]_{\mathfrak{c}}$, it suffices to show that the morphism $\mathfrak{L}_{\mathcal{P}}(\widetilde{P}) \longrightarrow M$ is a monomorphism.

Consider the exact sequence $0 \longrightarrow K \xrightarrow{\kappa} \mathfrak{L}_{\mathcal{P}}(\widetilde{P}) \longrightarrow M$. It follows that the intersection of $\mathfrak{f}^*_{\mathcal{P}}(K) \xrightarrow{\mathfrak{f}^*_{\mathcal{P}}(\kappa)} \mathfrak{f}^*_{\mathcal{P}}(\mathfrak{L}_{\mathcal{P}}(\widetilde{P}))$ with the subobject $\widetilde{P} \longrightarrow \mathfrak{f}^*_{\mathcal{P}}(\mathfrak{L}_{\mathcal{P}}(\widetilde{P}))$ is zero. By the argument (v) above, this implies that $K = 0$. Therefore, the natural morphism $\mathfrak{L}_{\mathcal{P}}(\widetilde{P}) \longrightarrow M$ is a monomorphism.

(d) It remains to prove the last assertion of the theorem: if $\widetilde{P}$ is a simple object of the category $C_{\mathfrak{A}_{\mathcal{P}}}$ such that $\mathcal{P} \in Ass(\varphi^*_{\mathcal{P}}(\widetilde{P})$, then $\mathfrak{L}_{\mathcal{P}}(\widetilde{P})$ is a simple object of the category $\mathfrak{A}$.

In fact, let $K \xrightarrow{\mathfrak{j}} \mathfrak{L}_{\mathcal{P}}(\widetilde{P})$ be a nonzero monomorphism. By (v) above, the pull-back of monomorphisms $\mathfrak{f}^*_{\mathcal{P}}(K) \longrightarrow \mathfrak{f}^*_{\mathcal{P}}(\mathfrak{L}_{\mathcal{P}}(\widetilde{P})) \longleftarrow \widetilde{P}$ is nonzero. Since $\widetilde{P}$ is simple, it follows that the morphism $\widetilde{P} \longrightarrow \mathfrak{f}^*_{\mathcal{P}}(\mathfrak{L}_{\mathcal{P}}(\widetilde{P}))$ factors through $\mathfrak{f}^*_{\mathcal{P}}(K) \longrightarrow \mathfrak{f}^*_{\mathcal{P}}(\mathfrak{L}_{\mathcal{P}}(\widetilde{P}))$. Therefore the identical morphism $\mathfrak{L}_{\mathcal{P}}(\widetilde{P}) \longrightarrow \mathfrak{L}_{\mathcal{P}}(\widetilde{P})$ factors through $K \xrightarrow{\mathfrak{j}} \mathfrak{L}_{\mathcal{P}}(\widetilde{P})$ which shows that $K \xrightarrow{\mathfrak{j}} \mathfrak{L}_{\mathcal{P}}(\widetilde{P})$ is an isomorphism. ∎

2.2.1. The case of the trivial stabilizer. Suppose that the point $\mathcal{P}$ of the spectrum $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ has the trivial stabilizer. That is if $(U, U \to F_\varphi)$ is an object of the stabilizer of $\mathcal{P}$, then U is a subfunctor of the identical functor. In this case, the condition (*) in 2.2 is equivalent to the condition

(†) If a subobject M of $\varphi^*(P)$ is such that $\mathcal{P} = [P]_{\mathfrak{c}} \in Supp(\varphi_*(M))$, then $\mathcal{P}$ is an associated point of $\varphi_*(M)$.

Evidently, the condition (†) holds if $Supp(\varphi_*\varphi^*(P)) = Ass(\varphi_*\varphi^*(P))$.

The latter equality holds if the functor $F_\varphi = \varphi_*\varphi^*$ is *differential* (cf. C4.2, C4.3) and $\mathcal{P} = [P]_{\mathfrak{c}}$ is a closed point. In this case, $Supp(\varphi_*\varphi^*(P)) = \{\mathcal{P}\} = Ass(\varphi_*\varphi^*(P))$.

The condition (†) also holds if $\mathcal{P}$ is a closed point and the functor F_φ is a coproduct of auto-equivalences.

2.3. A reduction. Once a point $\mathcal{P}$ of $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ is fixed, one can avoid dealing with the irrelevant parts of the categories C_X and $C_{\mathfrak{U}}$ proceeding as follows. We define the 'space' $X_{\mathcal{P}}$ by $C_{X_{\mathcal{P}}} = \mathcal{P}$. If C_X is the category of quasi-coherent sheaves on a scheme, then $\mathcal{P}$ corresponds to a point of the underlying space of this scheme and the category $C_{X_{\mathcal{P}}} = \mathcal{P}$ is naturally equivalent to the category of quasi-coherent sheaves on the closure of the point $\mathcal{P}$. Thus, the 'space' $X_{\mathcal{P}}$ can be regarded as the closure of the point $\mathcal{P}$ in X.

The inclusion functor $C_{X_{\mathcal{P}}} \xrightarrow{\mathfrak{j}^*_{\mathcal{P}}} C_X$ has a right adjoint $C_X \xrightarrow{\mathfrak{j}_{\mathcal{P}*}} C_{X_{\mathcal{P}}}$ which assigns to every object of C_X its $\mathcal{P}$-torsion. Let $\mathfrak{A} \xrightarrow{u_{\mathcal{P}}} X_{\mathcal{P}}$ denote the composition of the morphisms $\mathfrak{A} \xrightarrow{\varphi} X$ and $X \xrightarrow{\mathfrak{j}_{\mathcal{P}}} X_{\mathcal{P}}$. The morphism $u_{\mathcal{P}}$, being a composition of two continuous morphisms, is continuous. Its direct image functor is not, in general, right exact, because the functor $C_X \xrightarrow{\mathfrak{j}_{\mathcal{P}*}} C_{X_{\mathcal{P}}}$ is not necessarily right exact. Notice that the functor $\mathfrak{j}_{\mathcal{P}*}$ preserves supremums of objects; in particular, it preserves infinite coproducts. Since $u_{\mathcal{P}*} \simeq \mathfrak{j}_{\mathcal{P}*} \circ \varphi_*$ and φ_* preserves arbitrary colimits, the functor $u_{\mathcal{P}*}$ also preserves infinite coproducts.

We replace the category $C_{\mathfrak{A}_{\mathcal{P}}}$ by its full subcategory $C_{\mathfrak{A}'_{\mathcal{P}}}$ generated by all $\mathcal{F}_{\varphi_{\mathcal{P}}}$-modules (M, ξ) such that M is an object of $C_{X_{\mathcal{P}}} = \mathcal{P}$. The inclusion functor $C_{\mathfrak{A}'_{\mathcal{P}}} \xrightarrow{\widetilde{\mathfrak{j}}^*_{\mathcal{P}}} C_{\mathfrak{A}_{\mathcal{P}}}$ has a right adjoint, $\widetilde{\mathfrak{j}}_{\mathcal{P}*}$, induced by the functor $C_X \xrightarrow{\mathfrak{j}_{\mathcal{P}*}} C_{X_{\mathcal{P}}}$. We define the functor $C_{\mathfrak{A}} \xrightarrow{\widetilde{\mathfrak{f}}_{\mathcal{P}*}} C_{\mathfrak{A}'_{\mathcal{P}}}$ as the composition of the pull-back functor $C_{\mathfrak{A}} \xrightarrow{\mathfrak{f}_{\mathcal{P}*}} C_{\mathfrak{A}_{\mathcal{P}}}$ and the functor

$C_{\mathfrak{A}_{\mathcal{P}}} \xrightarrow{\widetilde{\mathfrak{j}}_{\mathcal{P}*}} C_{\mathfrak{A}'_{\mathcal{P}}}$. Thus, we obtain a quasi-commutative diagram

$$\begin{tikzcd} C_{\mathfrak{A}} \arrow[rr, "u_{\mathcal{P}*}"] \arrow[dr, "\widetilde{\mathfrak{f}}_{\mathcal{P}*}"'] & & C_{X_{\mathcal{P}}} \\ & C_{\mathfrak{A}'_{\mathcal{P}}} \arrow[ur, "\widetilde{\varphi}_{\mathcal{P}*}"'] & \end{tikzcd} \tag{1}$$

interpreted as the diagram of direct image functors of the morphisms of the commutative diagram

$$\begin{tikzcd} \mathfrak{A} \arrow[rr, "u_{\mathcal{P}}"] \arrow[dr, "\widetilde{\mathfrak{f}}_{\mathcal{P}}"'] & & X_{\mathcal{P}} \\ & \mathfrak{A}'_{\mathcal{P}} \arrow[ur, "\widetilde{\varphi}_{\mathcal{P}}"'] & \end{tikzcd}$$

Let $\widetilde{\mathfrak{L}}_{\mathcal{P}}$ denote the restriction of the functor $\mathfrak{L}_{\mathcal{P}}$ to the subcategory $\mathfrak{A}'_{\mathcal{P}}$. The exactness of the functor $C_{\mathfrak{A}'_{\mathcal{P}}} \xrightarrow{\mathfrak{L}_{\mathcal{P}}} C_{\mathfrak{A}}$, depends now on the exactness of a left adjoint, $\widetilde{\mathfrak{f}}^*_{\mathcal{P}}$, to the functor $C_{\mathfrak{A}} \xrightarrow{\widetilde{\mathfrak{f}}_{\mathcal{P}*}} C_{\mathfrak{A}'_{\mathcal{P}}}$. The exactness of $\widetilde{\mathfrak{f}}^*_{\mathcal{P}}$ is a much weaker requirement than the exactness of a right adjoint $\mathfrak{f}^*_{\mathcal{P}}$ to the pull-back functor $C_{\mathfrak{A}} \xrightarrow{\mathfrak{f}_{\mathcal{P}*}} C_{\mathfrak{A}_{\mathcal{P}}}$ imposed in 2.2.

These considerations will be used in the following Sections.

3. Important special cases. Finiteness conditions.

3.1. In most of applications we have in mind (in particular, those mentioned in this work), the monad $\mathcal{F}_\varphi = (F_\varphi, \mu_\varphi)$ belongs to one of the following two classes:

(a) The functor F_φ is a direct sum of a family of auto-equivalences of the category C_X.

(b) The monad $\mathcal{F}_\varphi$ (i.e. the functor F_φ) is differential.

Below we consider each of these cases and give the corresponding specializations of Theorem 2.2.

3.2. The case of a direct sum of auto-equivalences. Let C_X be an abelian category, and let $\mathcal{F}_\varphi = (F_\varphi, \mu_\varphi)$ a monad on C_X such that $F_\varphi = \bigoplus_{\alpha \in \mathfrak{J}} \theta_\alpha$, where θ_α are auto-equivalences of the category C_X. We denote by $\mathfrak{A}$ the 'space' $\mathbf{Sp}(\mathcal{F}_\varphi/X)$ and by φ the canonical morphism $\mathfrak{A} \longrightarrow X$. We take as $\widetilde{\mathcal{E}}$ the full monoidal subcategory of monoidal category $\widetilde{\mathfrak{Ex}_{\mathfrak{c}}}(C_X)/\mathcal{F}_\varphi$ generated by the coprojections $\theta_\alpha \xrightarrow{\pi_\alpha} F_\varphi$, $\alpha \in \mathfrak{J}$.

We are going to use the reduction described in 2.3; hence we assume for the rest of this section that the category C_X has the property (sup).

Fix an element $\mathcal{P}$ of $\mathbf{Spec}^0_{\mathfrak{c}}(X)$. Following the pattern of 2.3, we obtain a quasi-commutative diagram of functors

$$\begin{tikzcd} C_{\mathfrak{A}} \arrow[rr, "u_{\mathcal{P}*}"] \arrow[dr, "\widetilde{\mathfrak{f}}_{\mathcal{P}*}"'] & & C_{X_{\mathcal{P}}} \\ & C_{\mathfrak{A}'_{\mathcal{P}}} \arrow[ur, "\widetilde{\varphi}_{\mathcal{P}*}"'] & \end{tikzcd} \qquad (1)$$

Here $C_{X_{\mathcal{P}}} = \mathcal{P}$, and $C_{\mathfrak{A}'_{\mathcal{P}}} = \mathbf{Sp}(\mathcal{F}_{\mathcal{P}}/X_{\mathcal{P}})$, where $\mathcal{F}_{\mathcal{P}}$ is a monad on $C_{X_{\mathcal{P}}}$ induced by the monad $\mathcal{F}_{\varphi_{\mathcal{P}}}$. In other words, $C_{\mathfrak{A}'_{\mathcal{P}}}$ is the full subcategory of the category $C_{\mathfrak{A}_{\mathcal{P}}}$ of $\mathcal{F}_{\varphi_{\mathcal{P}}}$-modules whose objects are modules (M,ξ) such that $M \in Ob\mathcal{P}$.

3.2.0. The Krull filtration of $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ and the associated filtration of X. Fix an abelian category C_X. For every cardinal α, we define a subset $\mathfrak{S}_\alpha(X)$ of $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ as follows.

$\mathfrak{S}_0(X) = \emptyset$;

if α is not a limit cardinal, then $\mathfrak{S}_\alpha(X)$ consists of all $\mathcal{P} \in \mathbf{Spec}^0_{\mathfrak{c}}(X)$ such that any $\mathcal{P}' \in \mathbf{Spec}^0_{\mathfrak{c}}(X)$ properly contained in $\mathcal{P}$ belongs to $\mathfrak{S}_{\alpha-1}(X)$;

if α is a limit cardinal, then $\mathfrak{S}_\alpha(X) = \bigcup\limits_{\beta<\alpha} \mathfrak{S}_\beta(X)$.

It follows from this definition (borrowed from [R, VI.6.3]) that $\mathfrak{S}_1(X)$ consists of all closed points of $\mathbf{Spec}^0_{\mathfrak{c}}(X)$.

We denote by $\mathfrak{S}_\omega(X)$ the union of all $\mathfrak{S}_\alpha(X)$. The filtration $\{\mathfrak{S}_\alpha(X)\}$ determines a filtration

$$C_{X_0} \hookrightarrow C_{X_1} \hookrightarrow \ldots C_{X_\alpha} \hookrightarrow \ldots \qquad (5)$$

of the category C_X (or the 'space' X) by taking as C_{X_α} the full subcategory of C_X generated by objects M such that $Supp^0_{\mathfrak{c}}(M) \subseteq \mathfrak{S}_\alpha(X)$. Recall that $Supp^0_{\mathfrak{c}}(M) = \{\mathcal{P} \in \mathbf{Spec}^0_{\mathfrak{c}}(X) \mid M \notin Ob\mathcal{P}\}$. In particular, C_{X_ω} is the full subcategory of C_X generated by all $M \in ObC_X$ such that $Supp^0_{\mathfrak{c}}(M) \subseteq \mathfrak{S}_\omega(X)$.

It follows from the general properties of supports that C_{X_α} is a Serre subcategory of C_X and $\mathbf{Spec}^0_{\mathfrak{c}}(X_\alpha)$ is naturally identified with $\mathfrak{S}_\alpha(X)$; in particular, $\mathbf{Spec}^0_{\mathfrak{c}}(X_\omega)$ is identified with $\mathfrak{S}_\omega(X)$.

3.2.0.1. Proposition. *For each cardinal α, the subset $\mathfrak{S}_\alpha(X)$ of the spectrum is stable under all auto-equivalences of the category C_X. Let $\mathcal{P} \in \mathfrak{S}_\omega(X)$. If θ is an auto-equivalence of the category C_X, such that $\theta(\mathcal{P}) \subseteq \mathcal{P}$, then $\theta(\mathcal{P}) = \mathcal{P}$.*

Proof. The assertion is true for $\mathcal{P} \in \mathfrak{S}_0(X)$, because any auto-equivalence maps spectral objects to spectral objects. So, if $\mathcal{P}$ is a closed point and $\theta(\mathcal{P}) \subseteq \mathcal{P}$, then $\mathcal{P} = \theta(\mathcal{P})$.

Suppose now that the fact is true if $\mathcal{P} \in \mathfrak{S}_\nu$ for any $\nu < \alpha$. The claim is that it holds for any $\mathcal{P} \in \mathfrak{S}_\alpha$. In fact, it holds by a trivial reason if α is a limit cardinal. Let α be a not a limit cardinal, $\mathcal{P} \in \mathfrak{S}_\alpha(X)$, and $\theta(\mathcal{P}) \subseteq \mathcal{P}$. If $\theta(\mathcal{P}) \neq \mathcal{P}$, then, by definition of $\mathfrak{S}_\alpha(X)$, the element $\theta(\mathcal{P})$ belongs to $\mathfrak{S}_{\alpha-1}(X)$. But then, by induction hypothesis, $\mathcal{P} \in \mathfrak{S}_{\alpha-1}(X)$, hence $\theta(\mathcal{P}) = \mathcal{P}$. ■

3.2.1. Proposition. *(a) Under the conditions above, the functor* $C_{\mathfrak{A}} \xrightarrow{\widetilde{\mathfrak{f}}_{\mathcal{P}*}} C_{\mathfrak{A}'_{\mathcal{P}}}$ *has a left adjoint; and the functor* $C_{\mathfrak{A}'_{\mathcal{P}}} \xrightarrow{\widetilde{\varphi}_{\mathcal{P}*}} C_{X_{\mathcal{P}}}$ *has a left adjoint which is faithfully flat.*

(b) Suppose that $\mathcal{P}$ *belongs to* $\mathfrak{S}_\omega(X)$. *Then the functor* $\mathfrak{f}^*_{\mathcal{P}}$ *is faithful.*

Proof. (a) Set $J_{\mathcal{P}} = \{\alpha \in J \mid \theta_\alpha \in \mathfrak{F}_{\mathcal{P}}\} = \{\alpha \in J \mid \theta_\alpha(\mathcal{P}) \subseteq \mathcal{P}\}$ and denote by $F_{\mathcal{P}}$ the endofunctor on $C_{X_{\mathcal{P}}} = \mathcal{P}$ (cf. 2.3) induced by $\bigoplus_{\alpha \in J_{\mathcal{P}}} \theta_\alpha$. The multiplication $F^2_\varphi \xrightarrow{\mu_\varphi} F_\varphi$ induces a multiplication $F^2_{\mathcal{P}} \xrightarrow{\mu_{\mathcal{P}}} F_{\mathcal{P}}$ on $F_{\mathcal{P}}$.

In fact, the monad structure on F_φ is determined by the compositions

$$\theta_\alpha \circ \theta_\beta \xrightarrow{\mu^\sigma_{\alpha,\beta}} \theta_\sigma, \quad \alpha, \beta, \sigma \in J, \tag{2}$$

of the embedding $\theta_\alpha \circ \theta_\beta \longrightarrow F_\varphi \circ F_\varphi$, the multiplication $F_\varphi \circ F_\varphi \xrightarrow{\mu_\varphi} F_\varphi$, and the projection $F_\varphi \longrightarrow \theta_\sigma$. Let $\alpha, \beta \in J_{\mathcal{P}} \not\ni \sigma$. Then the morphism $\theta_\alpha\theta_\beta(M) \xrightarrow{\mu^\sigma_{\alpha,\beta}(M)} \theta_\sigma(M)$ is zero for every object M of the subcategory $\mathcal{P}$.

(i) Suppose first that M is a representative of $\mathcal{P}$. Assume that $\mu^\sigma_{\alpha,\beta}(M) \neq 0$. Since θ_σ is an auto-equivalence and $M \in Spec^0_{\mathfrak{c}}(X)$, the object $\theta_\sigma(M)$ belongs to $Spec^0_{\mathfrak{c}}(X)$ too. Therefore, the existence of a nonzero morphism $\theta_\alpha\theta_\beta(M) \longrightarrow \theta_\sigma(M)$ implies that the subcategory $[\theta_\alpha\theta_\beta(M)]_{\mathfrak{c}}$ contains $\theta_\sigma(M)$. Since $\theta_\alpha\theta_\beta$ stabilizes $\mathcal{P} = [M]_{\mathfrak{c}}$, it follows that $\theta_\sigma(M)$ belongs to $[M]_{\mathfrak{c}}$, which means precisely that θ_σ stabilizes $\mathcal{P}$. This, in turn, implies that θ_σ stabilizes $\widehat{\mathcal{P}}$. In fact, θ_σ not stabilizing $\widehat{\mathcal{P}}$ means that there exists $N \in ObC_X$ such that $M \notin [N]_{\mathfrak{c}}$, but, $M \in [\theta_\sigma(N)]_{\mathfrak{c}}$. Since θ_σ is an auto-equivalence, it preserves the relation $M \notin [N]_{\mathfrak{c}}$, that is $\theta_\sigma(M) \notin [\theta_\sigma(N)]_{\mathfrak{c}}$. But, this contradicts to the fact that $\theta_\sigma(M) \in [M]_{\mathfrak{c}}$ and $[M]_{\mathfrak{c}} \subseteq [\theta_\sigma(N)]_{\mathfrak{c}}$.

(ii) Suppose now that M is an arbitrary object of $\mathcal{P}$. Let L be a representative of $\mathcal{P}$. Then there exists a diagram $L^{\oplus J} \longleftarrow K \longrightarrow M$ whose the left arrow is a monomorphism and the right arrow is an epimorphism. Thus, we have a commutative diagram

$$\begin{tikzcd}
\theta_\alpha\theta_\beta(L^{\oplus J}) \arrow[d] & \theta_\alpha\theta_\beta(K) \arrow[l] \arrow[r] \arrow[d] & \theta_\alpha\theta_\beta(M) \arrow[d] \\
\theta_\sigma(L^{\oplus J}) & \theta_\sigma(K) \arrow[l] \arrow[r] & \theta_\sigma(M)
\end{tikzcd}$$

whose left (resp. right) horizontal arrows are monomorphisms (resp. epimorphisms) and vertical arrows are values of the functor morphism $\mu^\sigma_{\alpha,\beta}$ on the objects respectively $L^{\oplus J}$, K

and M. Suppose that $\alpha, \beta \in J_{\mathcal{P}}$, but $\sigma \notin J_{\mathcal{P}}$. Since $[L^{\oplus J}]_{\mathfrak{c}} = [L]_{\mathfrak{c}}$ and, by hypothesis, $[L]_{\mathfrak{c}} = \mathcal{P}$, it follows from (i) that the left vertical arrow in the diagram above is the zero morphism; in particular, the composition of the central vertical arrow,

$$\theta_\alpha\theta_\beta(K) \xrightarrow{\mu^\sigma_{\alpha,\beta}(K)} \theta_\sigma(K),$$

and the monomorphism $\theta_\sigma(K) \longrightarrow \theta_\sigma(L^{\oplus J})$ is zero, hence $\mu^\sigma_{\alpha,\beta}(K) = 0$. This, in turn, implies that the composition of the epimorphism $\theta_\alpha\theta_\beta(K) \longrightarrow \theta_\alpha\theta_\beta(M)$ and the left vertical arrow, $\theta_\alpha\theta_\beta(M) \xrightarrow{\mu^\sigma_{\alpha,\beta}(M)} \theta_\sigma(M)$, is zero which means that $\mu^\sigma_{\alpha,\beta}(M) = 0$.

(iii) Set $J^\vee_{\mathcal{P}} = J - J_{\mathcal{P}}$ and $F^\vee_{\mathcal{P}} = \bigoplus_{\beta \in J^\vee_{\mathcal{P}}} \theta_\beta$. Then $F_\varphi = F_{\mathcal{P}} \oplus F^\vee_{\mathcal{P}}$. It follows from the above argument that the compostition of $F^2_{\mathcal{P}} \longrightarrow F^2_\varphi \xrightarrow{\mu_\varphi} F_\varphi$ with the projection $F_\varphi \xrightarrow{\pi} F^\vee_{\mathcal{P}}$ is zero. Therefore the composition of $F^2_{\mathcal{P}} \longrightarrow F^2_\varphi \xrightarrow{\mu_\varphi} F_\varphi$ factors through the embedding $F_{\mathcal{P}} \hookrightarrow F_\varphi$, i.e. there exists a unique morphism $F^2_{\mathcal{P}} \xrightarrow{\mu_{\mathcal{P}}} F_{\mathcal{P}}$ such that the diagram

$$\begin{array}{ccc}
F^2_{\mathcal{P}} & \xrightarrow{\mu_{\mathcal{P}}} & F_{\mathcal{P}} \\
\big\downarrow & & \big\downarrow \\
F^2_\varphi & \xrightarrow{\mu_\varphi} & F_\varphi
\end{array}$$

commutes. Thus, the morphisms $\{\theta_\alpha\theta_\beta \xrightarrow{\mu^\sigma_{\alpha,\beta}} \theta_\sigma \mid \alpha, \beta, \sigma \in J_{\mathcal{P}}\}$ determine an associative multiplication $F^2_{\mathcal{P}} \xrightarrow{\mu_{\mathcal{P}}} F_{\mathcal{P}}$ on $F_{\mathcal{P}}$.

(a1) The forgetful functor $\mathfrak{F}_{\mathcal{P}}-mod \xrightarrow{\widetilde{\varphi}_{\mathcal{P}*}} C_{X_{\mathcal{P}}} = \mathcal{P}$ has a left adjoint, $\widetilde{\varphi}^*_{\mathcal{P}}$, which assigns to every object M of the category $C_{X_{\mathcal{P}}}$ the pair $(F_{\mathcal{P}}(M), \widetilde{\mu})$, where $\widetilde{\mu}$ is the obvious action of $\widetilde{\mathcal{E}}_{(\mathcal{P})}$ on $F_{\mathcal{P}}(M)$. It follows that $\mathcal{F}_{\mathcal{P}} = (F_{\mathcal{P}}, \mu_{\mathcal{P}})$ is the monad associated with the pair $\widetilde{\varphi}^*_{\mathcal{P}}$, $\widetilde{\varphi}^*_{\mathcal{P}}$ of adjoint functors. Since the functor $\widetilde{\varphi}^*_{\mathcal{P}}$ is exact and conservative, the category $(\widetilde{\Phi}_{\mathcal{P}}/X_{\mathcal{P}}) - mod$ is naturally equivalent to the category $\mathcal{F}_{\mathcal{P}} - mod$ of $\mathcal{F}_{\mathcal{P}}$-modules.

(a2) The latter implies the existence of a left adjoint, $C_{\mathfrak{A}'_{\mathcal{P}}} \xrightarrow{\widetilde{\mathfrak{f}}^*_{\mathcal{P}}} C_{\mathfrak{A}}$, to the functor $C_{\mathfrak{A}} \xrightarrow{\widetilde{\mathfrak{f}}_{\mathcal{P}*}} C_{\mathfrak{A}'_{\mathcal{P}}}$ (defined in 2.3).

In fact, identifying the category $C_{\mathfrak{A}'_{\mathcal{P}}}$ with $\mathcal{F}_{\mathcal{P}} - mod$, we take as $\widetilde{\mathfrak{f}}^*_{\mathcal{P}}$ the functor

$$\mathcal{F}_\varphi \otimes_{\mathcal{F}_{\mathcal{P}}} : (\mathcal{F}_{\mathcal{P}}/X_{\mathcal{P}}) - mod \longrightarrow (\mathcal{F}_\varphi/X) - mod. \tag{3}$$

(b) If $\alpha, \beta \in J_{\mathcal{P}}$ and $\sigma \in J^\vee_{\mathcal{P}}$, then

$$\theta_\sigma\theta_\alpha(M) \xrightarrow{\mu^\beta_{\sigma,\alpha}(M)} \theta_\beta(M) \tag{4}$$

is zero for every $M \in Ob[\mathcal{P}]_{\mathfrak{c}}$.

Suppose first that M is a representative of $\mathcal{P}$. Since $\mathcal{P} \in \mathfrak{S}_\omega(X)$, the inclusion $[\theta_\beta(M)]_{\mathfrak{c}} \subseteq [M]_{\mathfrak{c}}$ implies, by 3.2.0.1, the equality $[\theta_\beta(M)]_{\mathfrak{c}} = [M]_{\mathfrak{c}}$. If the morphism (4) is nonzero, then $[\theta_\sigma(M)]_{\mathfrak{c}} \supseteq [\theta_\sigma\theta_\alpha(M)]_{\mathfrak{c}} \supseteq [\theta_\beta(M)]_{\mathfrak{c}} \supseteq [M]_{\mathfrak{c}}$ But, $[\theta_\sigma(M)]_{\mathfrak{c}} \supseteq [M]_{\mathfrak{c}} \Leftrightarrow [M]_{\mathfrak{c}} = [\theta_\sigma(M)]_{\mathfrak{c}}$, which means that $\sigma \in J_{\mathcal{P}}$.

If M is an arbitrary object of $\mathcal{P}$, the argument is the same as the argument (ii) above.

(b1) The argument similar to that of (iii) shows that the multiplication $F_\varphi^2 \xrightarrow{\mu_\varphi} F_\varphi$ induces a morphism $F_{\mathcal{P}}^\vee F_{\mathcal{P}} \xrightarrow{\gamma_{\mathcal{P}}} F_{\mathcal{P}}^\vee$ which is a structure of a right $(F_{\mathcal{P}}, \mu_{\mathcal{P}})$-module on $F_{\mathcal{P}}^\vee$.

(b2) If follows that, as $(F_{\mathcal{P}}, \mu_{\mathcal{P}})$-module, F_φ is the direct sum of $F_{\mathcal{P}}^\vee$ and $F_{\mathcal{P}}$. Therefore, for every $\mathcal{F}_{\mathcal{P}}$-module (M, ξ),

$$\mathfrak{f}_{\mathcal{P}*}\mathfrak{f}_{\mathcal{P}}^*(M,\xi) = F_\varphi \otimes_{\mathcal{F}_{\mathcal{P}}} (M,\xi) \simeq \big(F_{\mathcal{P}}^\vee \otimes_{\mathcal{F}_{\mathcal{P}}} (M,\xi)\big) \oplus (M,\xi),$$

which immediately implies that $\mathfrak{f}_{\mathcal{P}}^*$ is a faithful functor. ∎

The corresponding version of Theorem 2.2 is as follows.

3.2.2. Theorem. *Suppose that the category C_X has the property (sup). Let $F_\varphi = \bigoplus_{\alpha\in\mathfrak{J}} \theta_\alpha$, where θ_α are auto-equivalences of the category C_X, and let $\mathfrak{F} = \{\theta_\alpha \mid \alpha \in \mathfrak{J}\}$.*

Suppose that an element $\mathcal{P}$ of $\mathfrak{S}_\omega(X)$ is such that the functor $C_{\mathfrak{A}'_{\mathcal{P}}} \xrightarrow{\widetilde{\mathfrak{f}}_{\mathcal{P}}^*} C_{\mathfrak{A}}$ is exact and the following condition holds:

(*) If P is a representative of $\mathcal{P}$ and M is a subobject of $\varphi^*(P)$ such that $\mathcal{P} \in Supp(\varphi_*(M))$, then there exists a subobject P' of P and $\alpha \in \mathfrak{J}$ such that $\theta_\alpha(P')$ is a subobject of $\varphi_*(M)$ and $[P'] \subseteq [\theta_\alpha(P')]_{\mathfrak{c}}$.

Then

(a) The composition $C_{\mathfrak{A}'_{\mathcal{P}}} \xrightarrow{\mathcal{L}_{\mathcal{P}}} C_{\mathfrak{A}}$, of the functors $C_{\mathfrak{A}'_{\mathcal{P}}} \xrightarrow{\widetilde{\mathfrak{f}}_{\mathcal{P}}^*} C_{\mathfrak{A}}$, and

$$C_{\mathfrak{A}} \xrightarrow{\Psi_{\mathcal{P}}} C_{\mathfrak{A}}, \quad M \longmapsto M/tors_{\varphi_*^{-1}(\widehat{\mathcal{P}})}(M),$$

induces a morphism

$$\mathbf{Spec}_{\mathfrak{c}}^{\mathcal{P}}(\mathfrak{A}'_{\mathcal{P}}) \xrightarrow{\mathfrak{L}_{\mathcal{P}}} \mathbf{Spec}_{\mathfrak{c}}^{\mathcal{P}}(\mathfrak{A}). \tag{5}$$

with the following property:

Every $[M]_{\mathfrak{c}} \in \mathbf{Spec}_{\mathfrak{c}}^0(\mathfrak{A})$ such that the image $\mathfrak{f}_{\mathcal{P}}^*(M)$ of M in $C_{\mathfrak{A}_{\mathcal{P}}}$ has an associated point from $\mathbf{Spec}_{\mathfrak{c}}^{\mathcal{P}}(\mathfrak{A}_{\mathcal{P}})$ belongs to the image of the map (1).

(b) The functor $\mathcal{L}_{\mathcal{P}}$ maps simple objects to simple objects.

Proof. The condition (*) is the specialization of the condition (*) in 2.2. Thus, the assertion is a consequence of 3.2.1 and Theorem 2.2. ∎

3.2.3. Proposition. *Suppose that the category C_X has the property (sup). Each of the following conditions on a point $\mathcal{P}$ of $\mathfrak{S}_\omega(X)$ implies the condition (*) in 3.2.2:*

(a) the stabilizer of $\mathcal{P}$ is trivial;

(b) the local category $C_X/\widehat{\mathcal{P}}$ has simple objects.

Proof. (i) Set $J_{\mathcal{P}} = \{\alpha \mid [\theta_\alpha(P)] = \mathcal{P}\}$ and $J^{\mathcal{P}} = J - J_{\mathcal{P}}$. Let $M \hookrightarrow \varphi^*(P)$ be such that $\mathcal{P} \in Supp(\varphi_*(M))$. We denote by M' the kernel of the composition of the monomorphism $\varphi_*(M) \hookrightarrow F_\varphi(P)$ and the projection $F_\varphi(P) = \bigoplus_{\alpha \in J} \theta_\alpha(P)$ onto $\bigoplus_{\alpha \in J^{\mathcal{P}}} \theta_\alpha(P)$. It follows that M' is a subobject of $\bigoplus_{\alpha \in J_{\mathcal{P}}} \theta_\alpha(P)$. Since $Supp(\bigoplus_{\alpha \in J^{\mathcal{P}}} \theta_\alpha(P)) = \bigcup_{\alpha \in J^{\mathcal{P}}} Supp(\theta_\alpha(P))$ does not contain the point $\mathcal{P}$, and $Supp(\varphi_*(M)) = Supp(M') \bigcup Supp(M'')$, where M'' denotes the image of $\varphi_*(M)$ in $\bigoplus_{\alpha \in J^{\mathcal{P}}} \theta_\alpha(P)$, the condition $\mathcal{P} \in Supp(\varphi_*(M))$ is equivalent to that $\mathcal{P} \in Supp(M')$. In particular, $M' \neq 0$.

(ii) Suppose that $C_X/\widehat{\mathcal{P}}$ has simple objects. Replacing P with an appropriate subobject, we can and will assume that the image $q_{\widehat{\mathcal{P}}}(P)$ of P in $C_X/\widehat{\mathcal{P}}$ is a simple object. This implies that the image of $F_\varphi(P)$ (which coincides with the image of $\bigoplus_{\alpha \in J_{\mathcal{P}}} \theta_\alpha(P)$) in $C_X/\widehat{\mathcal{P}}$ is semisimple. Therefore, the image of M' in $C_X/\widehat{\mathcal{P}}$ is isomorphic to the image of $\bigoplus_{\alpha \in \mathcal{I}} \theta_\alpha(P)$ for some subset $\mathcal{I}$ of $J_{\mathcal{P}}$. This means that there exists a diagram

$$M' \xleftarrow{\ s\ } N \xrightarrow{\ t\ } \bigoplus_{\alpha \in \mathcal{I}} \theta_\alpha(P) \tag{6}$$

in C_X such that $q_{\widehat{\mathcal{P}}}(s)$ and $q_{\widehat{\mathcal{P}}}(t)$ are monomorphisms. Since M' and $\bigoplus_{\alpha \in \mathcal{I}} \theta_\alpha(P)$ are $\widehat{\mathcal{P}}$-torsion free objects, the object N in the diagram (6) can and will be chosen $\widehat{\mathcal{P}}$-torsion free. The latter means that the morphisms s and t are monomorphisms. Since $q_{\widehat{\mathcal{P}}}(t)$ is an isomorphism and the localization functor is exact, the intersection $P'_\alpha = N \bigcap \theta_\alpha(P)$ (i.e. the pull-back of the monomorphism $N \xrightarrow{t} \bigoplus_{\alpha \in \mathcal{I}} \theta_\alpha(P)$ and the coprojection $\theta_\alpha(P) \longrightarrow \bigoplus_{\alpha \in \mathcal{I}} \theta_\alpha(P)$) is nonzero for every $\alpha \in \mathcal{I}$. Setting $P' = \theta_\alpha^{-1}(P'_\alpha)$, we obtain a subobject of the object P satisfying the condition (*) of 3.2.2. ∎

3.2.4. Corollary. *Suppose that the category C_X has the property (sup). Let $F_\varphi = \bigoplus_{\alpha \in \mathfrak{J}} \theta_\alpha$, where θ_α are auto-equivalences of the category C_X, and let $\mathfrak{F} = \{\theta_\alpha \mid \alpha \in \mathfrak{J}\}$.*

Suppose that an element $\mathcal{P}$ of $\mathfrak{S}_\omega(X)$ is such that the functor $C_{\mathfrak{A}'_{\mathcal{P}}} \xrightarrow{\widetilde{\mathfrak{f}}^*_{\mathcal{P}}} C_{\mathfrak{A}}$ is exact and the quotient category $C_X/\widehat{\mathcal{P}}$ has simple objects. Then

(a) The composition $C_{\mathfrak{A}'_{\mathcal{P}}} \xrightarrow{\mathcal{L}_{\mathcal{P}}} C_{\mathfrak{A}}$, of the functors $C_{\mathfrak{A}'_{\mathcal{P}}} \xrightarrow{\widetilde{\mathfrak{f}}^*_{\mathcal{P}}} C_{\mathfrak{A}}$, and

$$C_{\mathfrak{A}} \xrightarrow{\Psi_{\mathcal{P}}} C_{\mathfrak{A}}, \quad M \longmapsto M/tors_{\varphi_*^{-1}(\widehat{\mathcal{P}})}(M),$$

induces a morphism

$$\mathbf{Spec}_{\mathfrak{c}}^{\mathcal{P}}(\mathfrak{A}'_{\mathcal{P}}) \xrightarrow{\mathfrak{L}_{\mathcal{P}}} \mathbf{Spec}_{\mathfrak{c}}^{\mathcal{P}}(\mathfrak{A}). \qquad (5)$$

with the following property:

Every $[M] \in \mathbf{Spec}(\mathfrak{A})$ *such that the image* $\mathfrak{f}_{\mathcal{P}}^*(M)$ *of* M *in* $C_{\mathfrak{A}_{\mathcal{P}}}$ *has an associated point from* $\mathbf{Spec}_{\mathfrak{c}}^{\mathcal{P}}(\mathfrak{A}_{\mathcal{P}})$ *belongs to the image of the map (1).*

(b) The functor $\mathcal{L}_{\mathcal{P}}$ *maps simple objects to simple objects.*

Proof. The assertion follows from 3.2.2 and 3.2.3. ∎

3.2.5. Proposition. *Suppose that the category* C_X *has the property (sup). Let* $F_\varphi = \bigoplus_{\alpha\in\mathfrak{J}} \theta_\alpha$, *where* θ_α *are auto-equivalences of the category* C_X.

Suppose that an element $\mathcal{P}$ *of* $\mathfrak{S}_\omega(X)$ *has a trivial stabilizer; i.e.* $[\theta_\alpha(\mathcal{P})] = \mathcal{P}$ *iff* $\alpha = 0$ *(here* $\theta_0 = Id_{C_X}$*). Then for every representative* P *of* $\mathcal{P}$, *the object* $\mathcal{L}_{\mathcal{P}}(P) = \varphi^*(P)/tors_{\varphi_*^{-1}(\widehat{\mathcal{P}})}(P)$ *belongs to* $Spec_{\mathfrak{c}}^0(\mathfrak{A})$. *If* P *is simple, then* $\mathcal{L}_{\mathcal{P}}(P)$ *is a simple object.*

Proof. We adopt the notations of the part (i) of the argument of 3.2.3. Thanks to the property (sup), there exists a finite subset I of $J_{\mathcal{P}}$ such that the intersection $\widetilde{M} = M' \bigcap \big(\bigoplus_{\alpha\in I}\theta_\alpha(P)\big)$ is nonzero. Since $[\theta_\alpha(P)]_{\mathfrak{c}} = \mathcal{P}$ for every $\alpha \in I$, the object $\widetilde{M}$ belongs to $Spec_{\mathfrak{c}}^0(X)$ and $[\widetilde{M}]_{\mathfrak{c}} = \mathcal{P}$.

The assertion follows now from the observation 2.2.1 and Theorem 2.2. ∎

3.3. Differential actions. For an abelian svelte category C_X, we denote by $\mathfrak{Der}_{\mathfrak{c}}(C_X)$ the full subcategory of the category $End(C_X)$ generated by all continuous exact differential endofunctors. Since the composition of differential endofunctors is a differential endofunctor, $\mathfrak{Der}_{\mathfrak{c}}(C_X)$ is a full monoidal subcategory of the monoidal category $\widetilde{End}(C_X)$.

We call an action $\widetilde{\Phi} = (\Phi, \phi, \phi_0)$ of a svelte monoidal category $\widetilde{\mathcal{E}} = (\mathcal{E}, \odot, \mathbb{I}, a; \ell, \mathfrak{r})$ on C_X *differential* if the functor Φ takes values in the subcategory $\mathfrak{Der}_{\mathfrak{c}}(C_X)$.

We assume until the end of the section that C_X is a Grothendieck category. This implies that C_X has small limits and colimits. Therefore, every continuous action $\widetilde{\Phi}$ of a svelte monoidal category $\widetilde{\mathcal{E}}$ has a colimit, $\mathcal{F}_\varphi = (F_\varphi, \mu_\varphi)$, which is a continuous monad. As in 2.1, we replace the monoidal category $\widetilde{\mathcal{E}}$ by its image in $\widetilde{\mathfrak{Ex}}(C_X)/\mathcal{F}_\varphi$ (determined by the universal cone $\widetilde{\Phi} \xrightarrow{\gamma_\varphi} \mathcal{F}_\varphi$) and identify the monoidal functor $\widetilde{\Phi}$ with the composition of the inclusion functor $\widetilde{\mathcal{E}} \longrightarrow \widetilde{\mathfrak{Ex}}(C_X)/\mathcal{F}_\varphi$ and the forgetful functor $\widetilde{\mathfrak{Ex}}(C_X)/\mathcal{F}_\varphi \longrightarrow \widetilde{\mathfrak{Ex}}(C_X)$.

If the action $\widetilde{\Phi}$ is differential, then $\widetilde{\mathcal{E}}$ is identified with a monoidal subcategory of $\widetilde{\mathfrak{Der}}(C_X)/\mathcal{F}_\varphi$ and the action $\widetilde{\Phi}$ with the restriction to $\widetilde{\mathcal{E}}$ of the forgetful monoidal functor $\widetilde{\mathfrak{Der}}(C_X)/\mathcal{F}_\varphi \longrightarrow \widetilde{\mathfrak{Der}}(C_X)$. In this case, the monad $\mathcal{F}_\varphi = (F_\varphi, \mu_\varphi)$ (that is the functor $F_\varphi = \varphi_*\varphi^*$) is differential (see C4.2, C4.3). Or, in other words, the affine morphism $\mathfrak{A} = \mathbf{Sp}(\mathcal{F}_\varphi/X) \xrightarrow{\varphi} X$ is differential.

For every $\mathcal{P} \in \mathbf{Spec}^0_{\mathfrak{c}}(X)$, we have a commutative diagram of affine morphisms

$$\begin{tikzcd} \mathfrak{A} = \mathbf{Sp}(\mathcal{F}_\varphi/X) \arrow[rr, "\mathfrak{f}_\mathcal{P}"] \arrow[dr, "\varphi"'] & & \mathbf{Sp}(\mathcal{F}_{\varphi_\mathcal{P}}/X) = \mathfrak{A}_\mathcal{P} \arrow[dl, "\varphi_\mathcal{P}"] \\ & X & \end{tikzcd}$$

corresponding to a monad morphism $\mathcal{F}_{\varphi_\mathcal{P}} \xrightarrow{\psi_\mathcal{P}} \mathcal{F}_\varphi$, where the 'space' $\mathfrak{A}_\mathcal{P}$ and the monad $\mathcal{F}_{\varphi_\mathcal{P}}$ (more precisely, the monad morphism $\psi_\mathcal{P}$) are *stabilizers* of the point $\mathcal{P}$ (see 2.1.1).

Therefore we have a well defined functor $C_{\mathfrak{A}_\mathcal{P}} \xrightarrow{\mathfrak{L}_\mathcal{P}} C_\mathfrak{A}$, which is the composition of $C_{\mathfrak{A}_\mathcal{P}} \xrightarrow{\mathfrak{f}^*_\mathcal{P}} C_\mathfrak{A}$, and the functor

$$C_\mathfrak{A} \xrightarrow{\Psi_\mathcal{P}} C_\mathfrak{A}, \quad M \longmapsto M/tors_{\varphi_*^{-1}(\widehat{\mathcal{P}})}(M).$$

Following the pattern of 2.3, consider the commutative diagram

$$\begin{tikzcd} \mathfrak{A} \arrow[rr, "u_\mathcal{P}"] \arrow[dr, "\widetilde{\mathfrak{f}}_\mathcal{P}"'] & & X_\mathcal{P} \\ & \mathfrak{A}'_\mathcal{P} \arrow[ur, "\widetilde{\varphi}_\mathcal{P}"'] & \end{tikzcd} \qquad (1)$$

associated with

$$\begin{tikzcd} \mathfrak{A} \arrow[rr, "\mathfrak{f}_\mathcal{P}"] \arrow[dr, "\varphi"'] & & \mathfrak{A}_\mathcal{P} \arrow[dl, "\varphi_\mathcal{P}"] \\ & X & \end{tikzcd}$$

Notice that the composition $\widetilde{\mathfrak{f}}^*_\mathcal{P}$ of the inclusion functor $C_{\mathfrak{A}'_\mathcal{P}} \longrightarrow C_{\mathfrak{A}_\mathcal{P}}$ and the functor $C_{\mathfrak{A}_\mathcal{P}} \xrightarrow{\mathfrak{f}^*_\mathcal{P}} C_\mathfrak{A}$ is a left adjoint to the functor $C_\mathfrak{A} \xrightarrow{\widetilde{\mathfrak{f}}_{\mathcal{P}*}} C_{\mathfrak{A}'_\mathcal{P}}$.

3.3.1. Lemma. *The functors $u^*_\mathcal{P}$ and $\widetilde{\mathfrak{f}}^*_\mathcal{P}$ take values in the full subcategory $C_{\mathfrak{A}[\mathcal{P}^-]}$ of the category $C_\mathfrak{A}$ formed by all $\mathcal{F}_\varphi$-modules (M,ξ) such that $M \in Ob\mathcal{P}^-$.*

Proof. Recall that $\mathcal{P}^-$ is the smallest Serre subcategory containing $\mathcal{P}$.

The assertion is due to the fact that every differential endofunctor of the category C_X preserves every Serre subcategory of C_X ([LR1]). A more detailed argument is as follows.

(a) The subcategory $C_{\mathfrak{A}[\mathcal{P}^-]}$ coincides with the preimage, $\varphi_*^{-1}(\mathcal{P}^-)$ of a Serre subcategory. Therefore it is a Serre subcategory, because the functor φ_* preserves small colimits.

(b) The functor $u^*_\mathcal{P}$ is a restriction of the functor $C_X \xrightarrow{\varphi^*} C_\mathfrak{A}$, $L \longmapsto (F_\varphi(L), \mu_\varphi(L))$, to the subcategory $C_{X_\mathcal{P}} = \mathcal{P}$. By hypothesis, the monad $\mathcal{F}_\varphi = (F_\varphi, \mu_\varphi)$ (i.e. the functor

$F_\varphi = \varphi_*\varphi^*$) is differential, hence $F_\varphi(L)$ is an object of $\mathcal{P}^-$ for every $L \in Ob\mathcal{P}^-$, in particular, for every $L \in Ob\mathcal{P}$.

(c) It follows from the construction of the functor $C_{\mathfrak{A}_\mathcal{P}} \xrightarrow{\mathfrak{f}^*_\mathcal{P}} C_\mathfrak{A}$ that for every $\widetilde{\mathfrak{F}}$-module $\mathcal{M} = (M, \widetilde{\xi})$ (– an object of the category $C_{\mathfrak{A}_\mathcal{P}}$), there is an $\mathcal{F}_\varphi$-module epimorphism $\varphi^*(M) = (F_\varphi(M), \mu_\varphi(M)) \longrightarrow \mathfrak{f}^*_\mathcal{P}(\mathcal{M})$. Since, by (c), $\varphi^*(M)$ is an object of the Serre subcategory $C_{\mathfrak{A}[\mathcal{P}^-]}$, its quotient object $\mathfrak{f}^*_\mathcal{P}(\mathcal{M})$ belongs to the subcategory $C_{\mathfrak{A}[\mathcal{P}^-]}$ too. ∎

The diagram (1) can be decomposed into a commutative diagram

$$\begin{array}{ccc} \mathfrak{A}[\mathcal{P}^-] & \xrightarrow{u_\mathcal{P}} & X_\mathcal{P} \\ \mathfrak{j}_\mathcal{P} \Big\uparrow & & \Big\uparrow \widetilde{\varphi}_\mathcal{P} \\ \mathfrak{A} & \xrightarrow{\widetilde{\mathfrak{f}}_\mathcal{P}} & \mathfrak{A}'_\mathcal{P} \end{array} \qquad (2)$$

Consider now the category $C_{\mathfrak{A}_\mathcal{P}}$. Its objects are pairs (M, ξ), where M is an object of the category $\mathcal{P}$ and ξ is an action of the differential monad $\mathcal{F}_{\varphi_\mathcal{P}}$ – the stabilizer of $\mathcal{P}$.

3.3.2. Proposition. *Let $C_{\mathcal{Y}_\mathcal{P}}$ be the full subcategory of the category $C_{\mathfrak{A}'_\mathcal{P}}$ formed by all (M, ξ) such that $M \in Ob\widehat{\mathcal{P}}$. Then $C_{\mathcal{Y}_\mathcal{P}}$ is a Serre subcategory of $C_{\mathfrak{A}'_\mathcal{P}}$ and $Spec(\mathfrak{A}'_\mathcal{P}) = Spec(\mathcal{Y}_\mathcal{P}) \coprod Spec^\mathcal{P}_\mathfrak{c}(\mathfrak{A}'_\mathcal{P})$. In particular,* $\mathbf{Spec}(\mathfrak{A}'_\mathcal{P}) = \mathbf{Spec}(\mathcal{Y}_\mathcal{P}) \coprod \mathbf{Spec}^\mathcal{P}_\mathfrak{c}(\mathfrak{A}'_\mathcal{P})$.

Proof (a) Let (M, ξ) be an object of $Spec(\mathfrak{A}'_\mathcal{P})$. Then either the object M is $\widehat{\mathcal{P}}$-torsion free, or $M \in Ob\widehat{\mathcal{P}}$, or, equivalently, $(M, \xi) \in ObC_{\mathcal{Y}_\mathcal{P}}$.

In fact, let $M_{\widehat{\mathcal{P}}}$ denote the $\widehat{\mathcal{P}}$-torsion of M. Any differential endofunctor of a category preserves all Serre subcategories of this category (see C4.3.3). Since objects of the monoidal category $\widetilde{\mathcal{E}}$, in particular objects of its subcategory $\widetilde{\mathcal{E}}_{(\mathcal{P})}$, are pairs $(U, U \to F_\varphi)$, where U is a differential endofunctor, the $\widehat{\mathcal{P}}$-torsion $M_{\widehat{\mathcal{P}}}$ of M is a submodule of the $\mathcal{F}_{\varphi_\mathcal{P}}$-module (M, ξ). Since (M, ξ) belongs to the spectrum, either $M_{\widehat{\mathcal{P}}} = 0$, or $[(M_{\widehat{\mathcal{P}}}, \xi')]_\mathfrak{c} \supseteq [(M, \xi)]_\mathfrak{c}$. Here ξ' denotes the induced $\mathcal{F}_{\varphi_\mathcal{P}}$-module structure. Thanks to the exactness of the forgetful functor $\varphi_{\mathcal{P}*}$, the latter implies that $[M_{\widehat{\mathcal{P}}}]_\mathfrak{c} \supseteq [M]_\mathfrak{c}$, hence $M \in Ob\widehat{\mathcal{P}}$, i.e. $M = M_{\widehat{\mathcal{P}}}$.

(b) Let (M, ξ) belong to $Spec(\mathfrak{A}'_\mathcal{P}) - Spec(\mathcal{Y}_\mathcal{P})$. By (a), this implies that M is an object of the subcategory $\mathcal{P} \bigcap \widehat{\mathcal{P}}^\perp$ formed by $\widehat{\mathcal{P}}$-torsion free objects of the $\mathcal{P}$. It follows that M has a nonzero subobject, $L \hookrightarrow M$, with $L \in Ob\mathcal{P} \bigcap \widehat{\mathcal{P}}^\perp$. Pick a representative, P', of $\mathcal{P}$. The inclusion $L \in Ob\mathcal{P}$ means that $[P']_\mathfrak{c} \supseteq [L]_\mathfrak{c}$. The fact that $L \notin Ob\widehat{\mathcal{P}}$ means precisely that $[L]_\mathfrak{c} \supseteq [P']_\mathfrak{c}$. Every nonzero subobject L' of L has the same properties: $[P']_\mathfrak{c} \supseteq [L']_\mathfrak{c} \supseteq [P']_\mathfrak{c}$. Therefore $[L']_\mathfrak{c} \supseteq [L]_\mathfrak{c}$. This shows that L belongs to the spectrum $Spec^0_\mathfrak{c}(X)$ and $[L]_\mathfrak{c} = [P']_\mathfrak{c} = \mathcal{P}$.

The argument above shows that every nonzero object M of $\mathcal{P}\bigcap\widehat{\mathcal{P}}^{\perp}$ is a representative of the point $\mathcal{P}$. In particular, $Ass(M)=\{\mathcal{P}\}$. ■

Now we shall make some observations related to the diagram (2) and the construction of the functor $\mathcal{L}_{\mathcal{P}}$.

Recall that an object M of the category C_X is called $\mathcal{P}$*-primary* if $Ass(M)=\{\mathcal{P}\}$.

3.3.3. Proposition. *Let $\mathbb{T}_{\mathcal{P}}$ denote the preimage $\varphi_*^{-1}(\widehat{\mathcal{P}})$ of the Serre subcategory $\widehat{\mathcal{P}}$ of C_X in $C_{\mathfrak{A}}$; and let $\mathcal{T}_{\mathcal{P}}$ denote the preimage in $C_{\mathfrak{A}[\mathcal{P}^-]}$ of the subcategory $\widehat{\mathcal{P}}\cap C_{X_{\mathcal{P}}}$ (cf. the diagram (2)).*

(a) An object $\mathcal{M}=(M,\xi)$ of $C_{\mathfrak{A}}=(\mathcal{F}_{\varphi}/X)-mod$ is $\mathbb{T}_{\mathcal{P}}$-torsion free iff the object $\varphi_*(\mathcal{M})=M$ is $\widehat{\mathcal{P}}$-torsion free.

(b) The image in $C_{X_{\mathcal{P}}}$ of every $\mathcal{T}_{\mathcal{P}}$-torsion free object of $C_{\mathfrak{A}[\mathcal{P}^-]}$ is $\mathcal{P}$-primary.

Proof. (a) Let $\mathcal{M}=(M,\xi)$ be an $(\mathcal{F}_{\varphi}/X)$-module, and let $M_{\widehat{\mathcal{P}}}$ denote the $\widehat{\mathcal{P}}$-torsion of the object M. Since $\mathcal{F}_{\varphi}=(F_{\varphi},\mu_{\varphi})$, where F_{φ} is a differential functor, and all differential functors preserve Serre subcategories, the action $F_{\varphi}(M)\xrightarrow{\xi}M$ induces an action, ξ', of F_{φ} on the subobject $M_{\widehat{\mathcal{P}}}$. Clearly, $(M_{\widehat{\mathcal{P}}},\xi')$ belongs to the Serre subcategory $\mathbb{T}_{\mathcal{P}}$.

(b) By definition, $C_{\mathfrak{A}[\mathcal{P}^-]}$ is a full subcategory of $C_{\mathfrak{A}}$ generated by $\mathcal{F}_{\varphi}$-modules (M,ξ) such that $M\in Ob\mathcal{P}^-$. Therefore, by (a), an object (M,ξ) of $C_{\mathfrak{A}[\mathcal{P}^-]}$ is $\mathcal{T}_{\mathcal{P}}$-torsion free iff M is an object of $\mathcal{P}^-\bigcap\widehat{\mathcal{P}}^{\perp}$. If M is nonzero, it contains (by the definition of $\mathcal{P}^-$) a nonzero subobject L which belongs to $\mathcal{P}\bigcap\widehat{\mathcal{P}}^{\perp}$. But, nonzero objects of $\mathcal{P}\bigcap\widehat{\mathcal{P}}^{\perp}$ are precisely all the representatives of $\mathcal{P}$ (see the part (b) of the argument of 3.3.2). This shows that $\mathcal{P}\in Ass(M)$.

Suppose $N\hookrightarrow M$ is a subobject of M such that $N\in Spec(X)$. Then N has a nonzero subobject L which belongs to $\mathcal{P}\bigcap\widehat{\mathcal{P}}^{\perp}$. Therefore $[N]_{\mathfrak{c}}=[L]_{\mathfrak{c}}=\mathcal{P}$; i.e. $\mathcal{P}$ is the only element of $Ass(M)$. ■

3.3.4. Proposition. *For every object $\mathcal{M}$ of $Spec^0_{\mathfrak{c}}(\mathfrak{A}[\mathcal{P}^-])$, its image in $C_{X_{\mathcal{P}}}$ either belongs to $\widehat{\mathcal{P}}$, or is $\mathcal{P}$-primary.*

Proof. Let $\mathcal{M}=(M,\xi)$ belong to $Spec(\mathfrak{A})$. By the argument of 3.3.3, the $\widehat{\mathcal{P}}$-torsion, $M_{\widehat{\mathcal{P}}}$, of the object M has a structure, ξ' of a submodule of $\mathcal{M}$. Therefore, if $M_{\widehat{\mathcal{P}}}\neq 0$, then $[(M_{\widehat{\mathcal{P}}},\xi')]_{\mathfrak{c}}\supseteq[(M,\xi)]_{\mathfrak{c}}$ which implies that $M=M_{\widehat{\mathcal{P}}}$ (see the part (a) of the argument of 3.3.2). If $M_{\widehat{\mathcal{P}}}=0$, then, by 3.3.3(b), the object M is $\mathcal{P}$-primary. ■

3.3.5. Proposition. *The functor $C_{\mathfrak{A}'_{\mathcal{P}}}\xrightarrow{\mathcal{L}_{\mathcal{P}}}C_{\mathfrak{A}}$ takes values in the full subcategory of $C_{\mathfrak{A}}$ generated by $\mathcal{F}_{\varphi}$-modules (M,ξ) such that M is an object of the category $\mathcal{P}^-\bigcap\widehat{\mathcal{P}}^{\perp}$. In particular, M is either zero, or $\mathcal{P}$-primary.*

Proof. Recall that the functor $C_{\mathfrak{A}_{\mathcal{P}}} \xrightarrow{\mathfrak{L}_{\mathcal{P}}} C_{\mathfrak{A}}$ is the composition of a left adjoint, $C_{\mathfrak{A}_{\mathcal{P}}} \xrightarrow{\mathfrak{f}^*_{\mathcal{P}}} C_{\mathfrak{A}}$, the forgetful functor $C_{\mathfrak{A}} \xrightarrow{\mathfrak{f}_{\mathcal{P}*}} C_{\mathfrak{A}_{\mathcal{P}}}$ and the functor

$$C_{\mathfrak{A}} \xrightarrow{\ \Psi_{\mathcal{P}}\ } C_{\mathfrak{A}}, \quad M \longmapsto M/tors_{\varphi_*^{-1}(\widehat{\mathcal{P}})}(M). \tag{3}$$

By 3.3.3(a), the functor (3) takes values in the full subcategory of $C_{\mathfrak{A}}$ generated by all $\mathcal{F}_{\varphi}$-modules (M,ξ) such that M is a $\widehat{\mathcal{P}}$-torsion free object of C_X.

The functor $C_{\mathfrak{A}'_{\mathcal{P}}} \xrightarrow{\mathcal{L}_{\mathcal{P}}} C_{\mathfrak{A}}$ is the composition of the functor $C_{\mathfrak{A}_{\mathcal{P}}} \xrightarrow{\mathfrak{L}_{\mathcal{P}}} C_{\mathfrak{A}}$ and the inclusion functor $C_{\mathfrak{A}'_{\mathcal{P}}} \longrightarrow C_{\mathfrak{A}_{\mathcal{P}}}$; that is $\mathcal{L}_{\mathcal{P}}$ is the composition of the three functors

$$C_{\mathfrak{A}'_{\mathcal{P}}} \longrightarrow C_{\mathfrak{A}_{\mathcal{P}}} \xrightarrow{\ \mathfrak{f}^*_{\mathcal{P}}\ } C_{\mathfrak{A}} \xrightarrow{\ \Psi_{\mathcal{P}}\ } C_{\mathfrak{A}}.$$

The composition of the first two functors takes values (thanks to the fact that F_{φ} is differential) in the subcategory $C_{\mathfrak{A}[\mathcal{P}^-]} = \varphi_*^{-1}(\mathcal{P}^-)$. Therefore the functor $\mathcal{L}_{\mathcal{P}}$ takes values in the preimage in $C_{\mathfrak{A}}$ of the subcategory $\mathcal{P}^- \bigcap \widehat{\mathcal{P}} \subseteq C_X$, which is the full subcategory of $C_{\mathfrak{A}}$ formed by all $\mathcal{F}_{\varphi}$-modules (M,ξ) such that M is an object of $\mathcal{P}^- \bigcap \widehat{\mathcal{P}}$. In particular, M is either zero, or $\mathcal{P}$-primary. ■

3.3.6. Localization. All exact differential endofunctors are compatible with localizations at Serre subcategories and induce exact differential endofunctors on the corresponding quotient categories (cf. C4.3.3). These endofunctors on quotient categories inherit *exactness* properties (like compatibility with limits or colimits of a certain class of diagrams, or having a right adjoint) of the initial endofunctors (see [KR2]). Thus, localization at any Serre subcategory $\mathbb{S}$ of the category C_X will transform our data (differential continuous monad $(F_{\varphi},\mu_{\varphi})$ and the family of exact continuous differential subfunctors of F_{φ}) to the same sort of data on $C_{X/\mathbb{S}}$. Taking an element $\mathcal{P}$ of the spectrum of $X/\mathbb{S}$, we obtain a relative version of the commutative diagram (1):

$$\begin{tikzcd} C_{\mathfrak{A}/\mathbb{S}''} \arrow[rr, "u_{\mathcal{P}*}"] \arrow[dr, "\widetilde{\mathfrak{f}}^*_{\mathcal{P}}"'] & & C_{(X/\mathbb{S})_{\mathcal{P}}} \\ & C_{(\mathfrak{A}/\mathbb{S})'_{\mathcal{P}}} \arrow[ur, "\widetilde{\varphi}^*_{\mathcal{P}}"'] & \end{tikzcd} \tag{4}$$

where $\mathbb{S}'' = \varphi_*^{-1}(\mathbb{S})$ and $\mathbb{S}' = {\varphi^*_{\mathcal{P}}}^{-1}(\mathbb{S})$. The category $C_{\mathfrak{A}/\mathbb{S}''}$ here is naturally identified with the category $\mathcal{F}_{\mathbb{S}}$-modules, where $\mathcal{F}_{\mathbb{S}}$ is the monad on $C_{X/\mathbb{S}}$ uniquely determined by the monad $\mathcal{F}_{\varphi} = (F_{\varphi},\mu_{\varphi})$.

Applying this observation to the Serre subcategory $\widehat{\mathcal{P}}$ and the unique closed point of the quotient local category $C_{X/\widehat{\mathcal{P}}}$, we replace X by the local 'space' $X/\widehat{\mathcal{P}}$ and obtain (using the decomposition (2) in 3.3.1) the diagram

$$\begin{tikzcd} C_{\mathfrak{A}_r[\mathcal{P}-]} \arrow[rr, "u_{\mathcal{P}*}"] \arrow[dr, "\widetilde{\mathfrak{f}}^*_{\mathcal{P}}"'] & & C_{X^r_{\mathcal{P}}} \\ & C_{\mathfrak{A}^r_{\mathcal{P}}} \arrow[ur, "\widetilde{\varphi}^*_{\mathcal{P}}"'] & \end{tikzcd} \qquad (5)$$

in which $X^r_{\mathcal{P}}$ is the residue 'space' of X at the point $\mathcal{P}$, $C_{\mathfrak{A}^r_{\mathcal{P}}}$ is the category of $\widetilde{\mathfrak{F}}$-modules $(L, \widetilde{\xi})$, where L is an object of the residue category $C_{X^r_{\mathcal{P}}}$, $C_{\mathfrak{A}_r[\mathcal{P}-]}$ is the category of $\mathcal{F}_{\widehat{\mathcal{P}}}$-modules (M, ξ), where M is an object of the smallest nonzero Serre subcategory of $C_{X/\widehat{\mathcal{P}}}$.

If the local category $C_{X/\widehat{\mathcal{P}}}$ has simple objects (which is always the case if X has a Gabriel-Krull dimension) and C_X has infinite coproducts, then the residue category is equivalent to the category of vector spaces over the residue field $k_{\mathcal{P}}$ of the point $\mathcal{P}$.

4. Computing $\mathbf{Spec}_-(X)$.

4.1. The construction. We assume the setting of 2.1. That is we fix a Grothendieck category C_X endowed with an action $\widetilde{\Phi}$ of a svelte monoidal category $\widetilde{\mathcal{E}}$ taking values in the monoidal category $\widetilde{\mathfrak{Ex}_{\mathfrak{c}}}(C_X)$ of exact continuous endofunctors of C_X. As in 2.1, we identify the monoidal category $\widetilde{\mathcal{E}}$ with its image in $\widetilde{\mathfrak{Ex}_{\mathfrak{c}}}(C_X)/\mathcal{F}_\varphi$, where the continuous monad $\mathcal{F}_\varphi = (F_\varphi, \mu_\varphi)$ is the colimit of the monoidal functor $\widetilde{\mathcal{E}} \xrightarrow{\widetilde{\Phi}} \widetilde{\mathfrak{Ex}_{\mathfrak{c}}}(C_X)$. With this identification, $\widetilde{\Phi}$ becomes the restriction to $\widetilde{\mathcal{E}}$ of the forgetful monoidal functor $\widetilde{\mathfrak{Ex}_{\mathfrak{c}}}(C_X)/\mathcal{F}_\varphi \longrightarrow \widetilde{\mathfrak{Ex}_{\mathfrak{c}}}(C_X)$.

Fix an element $\mathcal{P}$ of $\mathbf{Spec}^{\mathfrak{c}}_-(X)$. Applying the pattern of 2.1.1 to $\mathcal{P}$, we obtain the *stabilizer* of $\mathcal{P}$ which is, by definition, the stabilizer $\widetilde{\mathcal{E}}_{(\mathcal{P})}$ of the pair $(\mathcal{P}) = \{\mathcal{P}, \widehat{\mathcal{P}}\}$, and the commutative diagram of affine morphisms

$$\begin{tikzcd} \mathbf{Sp}(\mathcal{F}_\varphi/X) = \mathfrak{A} \arrow[rr, "\mathfrak{f}_{\mathcal{P}}"] \arrow[dr, "\varphi"'] & & \mathfrak{A}_{\mathcal{P}} = \mathbf{Sp}(\mathcal{F}_{\varphi_{\mathcal{P}}}/X) \arrow[dl, "\varphi_{\mathcal{P}}"] \\ & X & \end{tikzcd} \qquad (1)$$

corresponding to a monad morphism $\mathcal{F}_{\varphi_{\mathcal{P}}} \xrightarrow{\psi_{\mathcal{P}}} \mathcal{F}_\varphi$, where the 'space' $\mathfrak{A}_{\mathcal{P}}$ and the monad $\mathcal{F}_{\varphi_{\mathcal{P}}}$ (or, more precisely, the monad morphism $\psi_{\mathcal{P}}$) are called *stabilizers* of the point $\mathcal{P}$.

4.2. $\mathbf{Spec}^{\mathfrak{c}}_-(\mathfrak{A}_{\mathcal{P}})_{\mathcal{P}}$ and $\mathbf{Spec}^{\mathfrak{c}}_-(\mathfrak{A})_{\mathcal{P}}$. For an element $\mathcal{P}$ of $\mathbf{Spec}^{\mathfrak{c}}_-(X)$, we denote by $Spec^{\mathfrak{c}}_-(\mathfrak{A}_{\mathcal{P}})_{\mathcal{P}}$ the family of all objects $\widetilde{P}$ of $Spec^{\mathfrak{c}}_-(\mathfrak{A}_{\mathcal{P}})$ such that $\mathcal{P} \in Ass(\varphi^*_{\mathcal{P}}(\widetilde{P}))$. We denote by $\mathbf{Spec}^{\mathfrak{c}}_-(\mathfrak{A}_{\mathcal{P}})_{\mathcal{P}}$ the correponding subset of $\mathbf{Spec}^{\mathfrak{c}}_-(\mathfrak{A}_{\mathcal{P}})$.

Similarly, $Spec^{\mathfrak{c}}_{-}(\mathfrak{A})_{\mathcal{P}}$ will denote the family of all objects M of $Spec^{\mathfrak{c}}_{-}(\mathfrak{A})$ such that $\mathcal{P}$ is an associated point of $\varphi_*(M)$, and denote by $\mathbf{Spec}^{\mathfrak{c}}_{-}(\mathfrak{A})_{\mathcal{P}}$ the corresponding subset of the spectrum $\mathbf{Spec}^{\mathfrak{c}}_{-}(\mathfrak{A})$.

4.3. Theorem. *Let $\mathcal{P} \in \mathbf{Spec}_{-}(X)$ be such that the inverse image functor $\mathfrak{f}^*_{\mathcal{P}}$ of the morphism $\mathfrak{A} \xrightarrow{\mathfrak{f}_{\mathcal{P}}} \mathfrak{A}_{\mathcal{P}}$ is exact and faithful, and the following conditions hold:*

(*) If P is a representative of $\mathcal{P}$ and M is a subobject of $\varphi^*(P)$ such that $\mathcal{P} \in Supp(\varphi_*(M))$, then there exists $(U', \mathfrak{v}) \in \mathfrak{F}_{\mathcal{P}}$ and a subobject P' of P such that the image of $U'(P')$ in $F_\varphi(P) = \varphi_*\varphi^*(P)$ is a subobject of $\varphi_*(M)$ whose support contains $\mathcal{P}$.

Then the functor $C_{\mathfrak{A}_{\mathcal{P}}} \xrightarrow{\mathfrak{L}_{\mathcal{P}}} C_{\mathfrak{A}}$ induces a surjective morphism

$$\mathbf{Spec}^{\mathfrak{c}}_{-}(\mathfrak{A}_{\mathcal{P}})_{\mathcal{P}} \xrightarrow{\mathfrak{L}_{\mathcal{P}}} \mathbf{Spec}^{\mathfrak{c}}_{-}(\mathfrak{A})_{\mathcal{P}}. \tag{1}$$

The functor $\mathfrak{L}_{\mathcal{P}}$ maps simple objects to simple objects.

Proof. The argument is similar to the proof of 2.2. Details are left to the reader. ∎

4.4. Finiteness conditions.

4.4.1. Associated points of finite multiplicity. Let M be an object of C_X, and let $\mathcal{P} \in \mathbf{Spec}^{\mathfrak{c}}_{-}(X)$ be an associated point of M; i.e. M has a nonzero subobject which belongs to $\widehat{\mathcal{P}}^{\mathfrak{c}}_{\circledast} = \mathcal{P} \cap \widehat{\mathcal{P}}^{\perp}$. We say that the associated point $\mathcal{P}$ has a *finite multiplicity* if the $\widehat{\mathcal{P}}^{\circledast}_{\mathfrak{c}}/\widehat{\mathcal{P}}$-torsion of M belongs to $Spec(X/\widehat{\mathcal{P}})$.

If the quotient category $C_X/\widehat{\mathcal{P}}$ has simple objects, then the $\widehat{\mathcal{P}}^{\circledast}_{\mathfrak{c}}/\widehat{\mathcal{P}}$-torsion of the image of M in $C_X/\widehat{\mathcal{P}}$ coincides with its socle. The point $\mathcal{P}$ is of finite multiplicity in M iff this socle is of finite length. The latter is called *the multiplicity of $\mathcal{P}$ in M*.

4.4.2. Points of the spectrum finite over a point. Let $\mathfrak{A} \xrightarrow{\varphi} X$ be an affine morphism and $\mathcal{P}$ a point of $\mathbf{Spec}^{-}_{\mathfrak{c}}(X)$. It is not guaranteed, in general, that $Spec^{-}_{\mathcal{P}}(\mathfrak{A}_{\mathcal{P}})$ is nonempty. We denote by $Spec^{0}_{\mathcal{P},\mathfrak{f}}(\mathfrak{A})$ the preorder of all $M \in Spec^{\mathfrak{c}}_{-}(\mathfrak{A})$ such that $\mathcal{P}$ is an associated point of $\varphi_*(M)$ of finite multiplicity.

4.4.2.1. Proposition. *Suppose that $\mathcal{P} \in \mathbf{Spec}^{-}_{\mathfrak{c}}(X)$ is such that the category $C_X/\mathcal{P}$ has simple objects. Then for every $M \in Spec^{0}_{\mathcal{P},\mathfrak{f}}(\mathfrak{A})$, the object $f_{\mathcal{P}*}(M)$ has a subobject $\widetilde{P}$ which belongs to $Spec^{\mathcal{P}}_{\mathfrak{c}}(\mathfrak{A}_{\mathcal{P}})$. For any such object $\widetilde{P}$, the corresponding element of $\mathbf{Spec}^{-}_{\mathfrak{c}}(\mathfrak{A}_{\mathcal{P}})$ is an associated point of $f_{\mathcal{P}*}(M)$ of finite multiplicity, and $\mathcal{P}$ is an associated point of $\widetilde{P}$ of finite multiplicity.*

Proof. Consider the set $\Omega_{\mathcal{P}}$ of all subobjects L of $f_{\mathcal{P}*}(M)$ such that $\varphi_{\mathcal{P}*}(L)$ is a representative of $\mathcal{P}$. Those of them with the smallest rank of $\varphi_{\mathcal{P}*}(L)$ belong to $Spec^{\mathcal{P}}_{\mathfrak{c}}(\mathfrak{A}_{\mathcal{P}})$. Details and the remaining observations are left to the reader. ∎

4.5. Holonomic objects.

4.5.1. Definition. Let $\mathfrak{A} \xrightarrow{\varphi} X$ be a continuous morphism. We call an object M of the category $C_{\mathfrak{A}}$ *holonomic over* X (or, more precisely, φ*-holonomic*), if each nonzero subquotient of $\varphi_*(M)$ has associated points in $\mathbf{Spec}^{\mathfrak{c}}_{-}(X)$ and all these associated points are of finite multiplicity.

If C_X is the category of quasi-coherent sheaves on a smooth scheme $\mathcal{X}$ and $C_{\mathfrak{A}}$ is the category of D-modules on $\mathcal{X}$, then holonomic objects are precisely holonomic D-modules.

In the case C_X is the category of quasi-coherent sheaves on the quantum flag variety of a semisimple Lie algebra $\mathfrak{g}$ and $C_{\mathfrak{A}}$ is the category of quasi-coherent $U_q(\mathfrak{g})$-modules on X (cf. [LR2]), then holonomic objects are called *holonomic* quantum D-modules.

It follows from 4.4.2.1 that all holonomic objects over X which belong to $Spec^{\mathfrak{c}}_{-}(\mathfrak{A})$ are obtained via the construction of this work. Thanks to their functorial properties, the description of holonomic objects is directly reduced to their description on elements of an affine cover.

5. Local properties of spectra. Applications to D-modules on classical and quantum flag varieties.

5.1. Proposition. *Let $\{\mathcal{T}_i \mid i \in J\}$ be a set of coreflective thick subcategories of an abelian category C_X such that $\bigcap_{i\in J} \mathcal{T}_i = 0$; and let u_i^* denote the localization functor $C_X \longrightarrow C_X/\mathcal{T}_i$. The following conditions on a nonzero coreflective topologizing subcategory $\mathcal{Q}$ of C_X are equivalent:*

(a) $\mathcal{Q} \in \mathbf{Spec}^0_{\mathfrak{c}}(X)$,

(b) $[u_i^*(\mathcal{Q})]_{\mathfrak{c}} \in \mathbf{Spec}^0_{\mathfrak{c}}(X/\mathcal{T}_i)$ for every $i \in J$ such that $\mathcal{Q} \not\subseteq \mathcal{T}_i$.

Proof. See [R7, 10.4.3]. ∎

5.1.1. Note. The condition (b) of 5.1 can be reformulated as follows:
(b') For any $i \in J$, either $u_i^*(\mathcal{Q}) = 0$, or $[u_i^*(\mathcal{Q})]_{\mathfrak{c}} \in \mathbf{Spec}^0_{\mathfrak{c}}(X/\mathcal{T}_i)$.

5.2. Proposition. *Let C_X be an abelian category and $\mathfrak{U} = \{U_i \xrightarrow{u_i} X \mid i \in J\}$ a set of continuous morphisms such that $\{C_X \xrightarrow{u_i^*} C_{U_i} \mid i \in J\}$ is a conservative family of exact localizations.*

(a) The morphisms $U_{ij} = U_i \cap U_j \xrightarrow{u_{ij}} U_i$ are continuous for all $i, j \in J$.

(b) Let L_i be an object of $Spec^0_{\mathfrak{c}}(U_i)$; i.e. $[L_i]_{\mathfrak{c}} \in \mathbf{Spec}^0_{\mathfrak{c}}(U_i)$ and L_i is $\langle L_i\rangle$-torsion free. The following conditions are equivalent:

(i) $L_i \simeq u_i^*(L)$ for some $L \in Spec^0_{\mathfrak{c}}(X)$;

(ii) for any $j \in J$ such that $u_{ij}^*(L_i) \neq 0$, the object $u_{ji*}u_{ij}^*(L_i)$ of C_{U_j} has an associated point; i.e. it has a subobject L_{ij} which belongs to $Spec^0_{\mathfrak{c}}(U_j)$.

Proof. The assertion follows from 5.1 (the argument is similar to that of [R7, 9.7.1]. ∎

5.2.1. Note. If the cover $\mathfrak{U} = \{U_i \xrightarrow{u_i} X \mid i \in J\}$ in 5.2 is finite, then $Spec_c^0(-)$ and $\mathbf{Spec}_c^0(-)$ can be replaced by resp. $Spec(-)$ and $\mathbf{Spec}(-)$.

5.2.2. Examples. (a) If C_X is the category of quasi-coherent sheaves on a quasi-separated scheme $\mathcal{X}$ and each U_i is the category of quasi-coherent sheaves on an open subscheme of $\mathcal{X}$, then the gluing conditions of 5.2 hold for any $L_i \in Spec_{\mathfrak{c}}^0(U_i)$; i.e. the spectrum $\mathbf{Spec}_{\mathfrak{c}}^0(X)$ is naturally identified with $\bigcup\limits_{i\in J} \mathbf{Spec}_{\mathfrak{c}}^0(U_i)$.

(b) Similarly, if C_X is the category of holonomic modules over a sheaf of twisted differential operators on a smooth scheme $\mathcal{X}$, and $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ is a cover of X corresponding to an open Zariski cover of $\mathcal{X}$, then $\bigcup\limits_{i\in J} \mathbf{Spec}_{\mathfrak{c}}^0(U_i)$.

This is due to functoriality of sheaves of holonomic modules with respect to direct and inverse image functors of open immersions and the fact that holonomic modules are of finite length (hence they have associated closed points).

5.3. Proposition. *Let C_X be an abelian category and $\mathfrak{U} = \{U_i \xrightarrow{u_i} X \mid i \in J\}$ a finite set of morphisms of 'spaces' whose inverse image functors, $\{C_X \xrightarrow{u_i^*} C_{U_i} \mid i \in J\}$, form a conservative family of exact localizations, and $Ker(u_i^*)$ is a coreflective subcategory for every $i \in J$. Then $\mathbf{Spec}^-(X) = \bigcup\limits_{i\in J} \mathbf{Spec}^-(U_i)$ and $\mathbf{Spec}_{\mathfrak{c}}^-(X) = \bigcup\limits_{i\in J} \mathbf{Spec}_{\mathfrak{c}}^-(U_i)$.*

Proof. The first equality is proven in [R7, 9.5]. The argument for the second equality is similar to the proof of [R7, 9.5]. ∎

5.3.1. Proposition. *Let C_X be an abelian category and $\mathfrak{U} = \{U_i \xrightarrow{u_i} X \mid i \in J\}$ a set of continuous morphisms whose inverse image functors, $\{C_X \xrightarrow{u_i^*} C_{U_i} \mid i \in J\}$, form a conservative family of exact localizations. Suppose that $\mathbf{Spec}_{\mathfrak{c}}^-(X) = \bigcup\limits_{i\in J} \mathbf{Spec}_{\mathfrak{c}}^-(U_i)$ (e.g. J is finite) and C_{U_i} is a Grothendieck category with a Gabriel-Krull dimension (for instance, U_i is locally noetherian; say $U_i \simeq \mathbf{Sp}(A_i)$ for a left noetherian ring) for each $i \in J$. Then $\mathbf{Spec}_{\mathfrak{c}}^-(X)$ is isomorphic to the set of isomorphism classes of indecomposable injective objects of the category C_X.*

Proof. Each (isomorphism class of) indecomposable injective E of C_X corresponds to the element ${}^\perp E$ of $\mathbf{Spec}_{\mathfrak{c}}^-(X)$. Since direct image functors $C_{U_i} \xrightarrow{u_{i*}} C_X$ of morphisms u_i are right adjoints to exact functors, they map (indecomposable) injective objects to (resp. indecomposable) injective objects. For every 'space' Y such that C_Y is a Grothendieck category with a Gabriel-Krull dimension (in particular, for each U_i), the isomorphism classes of indecomposable injective objects are in bijective correspondence with elements of $\mathbf{Spec}^-_{\mathfrak{c}}(Y)$. ■

5.4. Towards some applications. The assertions above allow to apply the results of the previous sections to *locally affine* morphisms; i.e. morphisms of 'spaces' $\mathfrak{A} \xrightarrow{\mathfrak{f}} X$ endowed with a set $\mathfrak{U} = \{U_i \xrightarrow{u_i} X \mid i \in J\}$ of morphisms such that $\{C_X \xrightarrow{u_i^*} C_{U_i} \mid i \in J\}$ is a conservative family of exact localizations whose kernels are coreflective subcategories of $C_{\mathfrak{A}}$, and for every $i \in J$, the compositions $\mathfrak{f} \circ u_i$ is an affine morphism.

A slightly more general setting we are interested in consists of a family of commutative diagrams

$$\begin{array}{ccc} \mathfrak{U}_i & \xrightarrow{\widetilde{\mathfrak{u}}_i} & \mathfrak{X} \\ {\scriptstyle \mathfrak{f}_i}\downarrow & & \downarrow{\scriptstyle \mathfrak{f}} \\ U_i & \xrightarrow{u_i} & X \end{array} \qquad i \in J, \tag{1}$$

where $\{C_X \xrightarrow{u_i^*} C_{U_i} \mid i \in J\}$ and $\{C_{\mathfrak{X}} \xrightarrow{\widetilde{\mathfrak{u}}_i^*} C_{\mathfrak{U}_i} \mid i \in J\}$ are conservative families of exact localizations with coreflective kernels and morphisms $\mathfrak{U}_i \xrightarrow{\mathfrak{f}_i} U_i$ are locally affine for all $i \in J$. Even when the morphism $\mathfrak{A} \xrightarrow{\mathfrak{f}} X$ is affine, the propositions 5.1 – 5.3.1 help to simplify the problem by using appropriate covers. In the examples below, the morphisms $\mathfrak{f}$ and $\mathfrak{f}_i$ are affine. We start with differential morphisms.

5.4.1. Affine differential morphisms. Let $\mathfrak{X} \xrightarrow{\mathfrak{f}} X$ be a differential affine morphism whose inverse image functor is exact. This means that the 'space' $\mathfrak{X}$ is naturally isomorphic to $\mathbf{Sp}(\mathcal{F}_{\mathfrak{f}}/X)$, where $\mathcal{F}_{\mathfrak{f}} = (F_{\mathfrak{f}}, \mu_{\mathfrak{f}})$ is the monad associated with $\mathfrak{f}$, and the functor $F_{\mathfrak{f}} = \mathfrak{f}_*\mathfrak{f}^*$ is exact, differential, and has a right adjoint.

Let $U \xrightarrow{u} X$ be a flat (i.e. continuous and exact) localization, and let $C_X \xrightarrow{F} C_X$ be an exact differential functor. Then there exists a unique exact differential functor $C_U \xrightarrow{F_U} C_U$ such that $u^* \circ F = F_U \circ u^*$. The functor F_U is naturally isomorphic to the composition $u^* F u_*$. If the functor F is continuous, i.e. it has a right adjoint, $F^!$, then the functor F_U is continuous too: the composition $F_U^! = u^* F^! u_*$ is a right adjoint to F_U.

Let $\mathfrak{U} = \{U_i \xrightarrow{u_i} X \mid i \in J\}$ be a set of continuous morphisms whose inverse image functors $\{C_X \xrightarrow{u_i^*} C_{U_i} \mid i \in J\}$ form a conservative family of exact localizations. Then it follows from the discussion above and C4.1 that the differential affine morphism $\mathfrak{X} \xrightarrow{\mathfrak{f}} X$ gives rise to a uniquely determined commutative diagram (1) in which all morphisms $\mathfrak{f}_i$ are affine and differential.

5.4.2. Quasi-coherent sheaves of rings. Let $\mathcal{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a commutative scheme such that the embedding of each point of $\mathcal{X}$ into $\mathcal{X}$ has a direct image functor (e.g. $\mathcal{X}$ is quasi-separated). This condition implies that the scheme $\mathcal{X}$ can be canonically reconstructed (is naturally isomorphic to the *geometric center* of) the category $C_X = Qcoh_{\mathcal{X}}$ of quasi-coherent sheaves on $\mathcal{X}$. Let $\mathcal{A}_{\mathcal{X}}$ be a quasi-coherent sheaf of associative unital rings on $\mathcal{X}$ and $C_{\mathfrak{X}}$ the category of quasi-coherent sheaves of $\mathcal{A}_X$-modules. Let $\mathcal{O}_X \xrightarrow{\psi} \mathcal{A}_X$ be a morphism of sheaves of rings. The morphism ψ gives rise to an affine morphism $\mathfrak{X} \xrightarrow{\mathfrak{f}} X$ of 'spaces'. Fix an affine cover $\{\mathcal{U}_i \xrightarrow{\mathfrak{u}_i} \mathcal{X} \mid i \in J\}$ of $\mathcal{X}$. Then we have a commutative diagram

$$\begin{array}{ccc} \mathfrak{U}_i & \xrightarrow{\widetilde{\mathfrak{u}}_i} & \mathfrak{X} \\ {\scriptstyle \mathfrak{f}_i}\downarrow & & \downarrow{\scriptstyle \mathfrak{f}} \\ U_i & \xrightarrow{u_i} & X \end{array} \qquad i \in J, \tag{1}$$

where $U_i = \mathbf{Sp}(\mathcal{O}_{\mathcal{X}}(\mathcal{U}_i))$, $\mathfrak{U}_i = \mathbf{Sp}(\mathcal{A}_{\mathcal{X}}(\mathcal{U}_i))$, $\mathfrak{f}_i$ is the affine morphism corresponding to the ring morphism $\mathcal{O}_{\mathcal{X}}(\mathcal{U}_i) \xrightarrow{\psi(\mathcal{U}_i)} \mathcal{A}_{\mathcal{X}}(\mathcal{U}_i)$, and the morphisms u_i and $\widetilde{\mathfrak{u}}_i$ have restriction functors to the open subset $\mathcal{U}_i$ as inverse image functors. Since u_i^* and $\widetilde{\mathfrak{u}}_i^*$ are localization functors, the commutative diagram (1) shows that $\mathfrak{X}$ is a (noncommutative in general) scheme, $\{\mathfrak{U}_i \xrightarrow{\widetilde{\mathfrak{u}}_i} \mathfrak{X} \mid i \in J\}$ its affine cover, and $\mathfrak{X} \xrightarrow{\mathfrak{f}} X$ is a scheme morphism.

Fix $i \in J$ and pick a point x of the open set $\mathcal{U}_i$. To the point x, there corresponds an element $\mathcal{P}_x^i$ of $\mathbf{Spec}(U_i) = \mathbf{Spec}_{\mathfrak{c}}^0(U_i)$. Since $\mathcal{U}_i$ is a Zariski open subset of the commutative scheme $\mathcal{X}$, the point $\mathcal{P}_x^i$ is the image of a uniquely determined point $\mathcal{P}_x$ of X.

We assume that the ring morphism $\mathcal{O}_X(\mathcal{U}_i) \xrightarrow{\psi(\mathcal{U}_i)} \mathcal{A}_{\mathcal{X}}(\mathcal{U}_i)$ is flat; i.e. the functor $\mathfrak{f}_i^* = \mathcal{A}_{\mathcal{X}}(\mathcal{U}_i) \otimes_{\mathcal{O}_{\mathcal{X}}(\mathcal{U}_i)} -$ from C_{U_i} to $C_{\mathfrak{U}_i}$ is exact. The stabilizer of the point $\mathcal{P}_x^i$ can be identified with the subring $\mathcal{A}_{\mathcal{P}_x^i}$ of the ring $\mathcal{A}_{\mathcal{X}}(\mathcal{U}_i)$ which contains the image of $\mathcal{O}_{\mathcal{X}}(\mathcal{U}_i)$ and such that the induced morphism $\mathcal{O}_{\mathcal{X}}(\mathcal{U}_i) \longrightarrow \mathcal{A}_{\mathcal{P}_x^i}$ (– the corestriction of $\psi(U_i)$) is flat.

5.4.2.1. Finiteness conditions. Let $C_{\mathfrak{X}_{\mathfrak{f}}^x}$ denote the full subcategory of the category $C_{\mathfrak{X}}$ generated by all objects M of $C_{\mathfrak{X}}$ such that x is an associated point of $\mathfrak{f}_*(M)$ of finite multiplicity (or, what is the same, $\mathcal{P}_x$ is an associated point of $\mathfrak{f}_*(M)$ of finite multiplicity). It follows from generalities on associated points (see C3.2) that the subcategory $C_{\mathfrak{X}_{\mathfrak{f}}^x}$ is closed under extensions. It follows from 4.4.2.1 and 4.3 that every object M of the subcategory $C_{\mathfrak{X}_{\mathfrak{f}}^x}$ has an associated point of the form $\mathfrak{L}_{\mathcal{P}_x}(V)$, where V is an element of the spectrum of the stabilizer $\mathbf{Sp}(\mathcal{A}_{\mathcal{P}_x^i})$ of the point $\mathcal{P}_x^i$ whose image in C_X is an element of $Spec(X)$ representing the point $\mathcal{P}_x^i$. Therefore, if M is the point of the $Spec_-(\mathfrak{U}_i)$, then M is equivalent to $\mathfrak{L}_{\mathcal{P}_x^i}(V)$.

5.4.2.2. Example. Let now $\mathcal{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a smooth scheme over $Spec(k)$; and let $\mathcal{A}_{\mathcal{X}}$ be the sheaf of algebras of twisted differential operators on $\mathcal{X}$. Then $Spec_{\mathfrak{c}}^-(\mathfrak{X}) \bigcap C_{\mathfrak{X}_{\mathfrak{f}}^x}$, consists of all semisimple holonomic $\mathcal{A}_X$-modules whose simple components are isomorphic to each other.

5.4.3. Remark. Given a cover $\mathfrak{U} = \{U_i \xrightarrow{u_i} \mathfrak{X} \mid i \in J\}$, Proposition 5.2 suggests a way of constructing points of $\mathbf{Spec}^0_{\mathfrak{c}}(\mathfrak{X})$ starting from a point $\mathcal{P}$ of $\mathbf{Spec}^0_{\mathfrak{c}}(X)$, taking its image in $\mathbf{Spec}^0_{\mathfrak{c}}(U_i)$ for some U_i containing $\mathcal{P}$ (i.e. $u_i^*(\mathcal{P}) \neq 0$) and an object M_i of $Spec^0_{\mathfrak{c}}(\mathfrak{U}_i)$ such that its image in C_{U_i} has $u_i^*(\mathcal{P})$ as an associated point. Notice that the object M_i can be obtained via our induction procedure applied to some other affine morphism, $\mathfrak{V}_i \xrightarrow{\varphi_i} \mathfrak{U}_i$, and a point $\mathcal{Q}_i$ of $\mathbf{Spec}^0_{\mathfrak{c}}(\mathfrak{V}_i)$. All we need to know is that the image of M_i in C_{U_i} has an associated point of the form $u_i^*(\mathcal{P})$ for some $\mathcal{P} \in \mathbf{Spec}^0_{\mathfrak{c}}(X)$. Thus, the gluing data related to this approach is described by the diagram

$$\begin{array}{ccccc} \mathfrak{V}_i & \xleftarrow{\varphi_i} & \mathfrak{U}_i & \xrightarrow{\widetilde{\mathfrak{u}}_i} & \mathfrak{X} \\ & & {\scriptstyle \mathfrak{f}_i}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{f}} \\ & & U_i & \xrightarrow{u_i} & X \qquad i \in J, \end{array} \tag{3}$$

where $\{C_X \xrightarrow{u_i^*} C_{U_i} \mid i \in J\}$ and $\{C_{\mathfrak{X}} \xrightarrow{\widetilde{\mathfrak{u}}_i^*} C_{\mathfrak{A}_i} \mid i \in J\}$ are conservative families of continuous exact localizations and the morphisms $\mathfrak{V}_i \xleftarrow{\varphi_i} \mathfrak{U}_i \xrightarrow{\mathfrak{f}_i} U_i$ are affine for all $i \in J$.

5.4.4. Example: D-modules on flag varieties. Let $\mathfrak{g}$ be a semisimple Lie algebra over an algebraically closed field of zero characteristic, G a connected simply connected algebraic group whose Lie algebra is isomorphic to $\mathfrak{g}$. Let $\mathcal{B}$ be a Borel subgroup of G, and $\mathcal{W}$ its Weyl group. The sheaf $\mathcal{D}_{G/\mathcal{B}}$ of algebras of differential operators on $G/\mathcal{B}$ defines a noncommutative scheme $\mathfrak{X}_{G/\mathcal{B}}$ represented by the category of D-modules on $G/\mathcal{B}$, together with the affine morphism $\mathfrak{X}_{G/\mathcal{B}} \xrightarrow{\mathfrak{f}} X_{G/\mathcal{B}}$ corresponding to the morphism $\mathcal{O}_{G/\mathcal{B}} \longrightarrow \mathcal{D}_{G/\mathcal{B}}$ of sheaves of rings. Here $X_{G/\mathcal{B}}$ denotes the 'space' corresponding to the scheme $G/\mathcal{B}$, i.e. $C_{X_{G/\mathcal{B}}}$ is the category of quasi-coherent sheaves on $G/\mathcal{B}$. By Beilinson-Bernstein theorem, the category $C_{\mathfrak{X}_{B/\mathcal{B}}} = \mathcal{D}_{G/\mathcal{B}} - mod$ of D-modules on the flag variety $G/\mathcal{B}$ is equivalent to the category $U_\rho(\mathfrak{g}) - mod$ of $U(\mathfrak{g})$-modules with the trivial central character.

Consider the canonical affine cover $\{\mathcal{U}_w \xrightarrow{u_i} G/\mathcal{B} \mid w \in \mathcal{W}\}$ of the flag variety by the translations of the big cell. Each open subscheme $\mathcal{U}_w$ is isomorphic to the affine space $\mathbb{A}^n$. Therefore, for all $w \in \mathcal{W}$, the algebra $\mathcal{D}_{G/\mathcal{B}}(\mathcal{U}_w)$ is isomorphic to the Weyl algebra A_n. Thus we have commutative diagrams of 'spaces'

$$\begin{array}{ccccc} \mathbf{Sp}(A_n) & \xrightarrow{\widetilde{\psi}_w} & \mathfrak{U}_w & \xrightarrow{\widetilde{\mathfrak{u}}_w} & \mathfrak{X}_{G/\mathcal{B}} \\ {\scriptstyle \varphi_n}\big\downarrow & & {\scriptstyle \mathfrak{f}_w}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{f}} \\ \mathbf{Sp}(\Gamma(\mathbb{A}_n)) & \xrightarrow{\psi_w} & U_w & \xrightarrow{u_w} & X_{G/\mathcal{B}} \qquad w \in \mathcal{W}, \end{array} \tag{1}$$

where left horizontal arrows are isomorphisms, φ_n is a morphism corresponding to the embedding of the algebra $k[\mathbf{y}] = \Gamma(\mathbb{A}_n)$ of polynomials in n variables to the Weyl algebra.

By 5.2, the construction of points of $\mathbf{Spec}^0_{\mathfrak{c}}(\mathfrak{X}_{G/\mathcal{B}})$ is reduced to

(i) the construction of points of $\mathbf{Spec}^0_{\mathfrak{c}}(\mathfrak{U}_w) = \mathbf{Spec}(\mathfrak{U}_w) \simeq \mathbf{Spec}(\mathbf{Sp}(A_n))$,

(ii) verifying the gluing conditions of 5.2(b).

As it is observed in 5.2.2(b), the gluing conditions hold automatically if we study holonomic D-modules. We look at the first, most important, problem.

5.4.4.1. The standard approach. The diagram (1) invites to apply the developed here induction machinery to the morphism φ_n corresponding to the standard embedding $k[\mathbf{y}] \hookrightarrow A_n$. It follows from 3.3.1 that for every closed irreducible subvariety $\mathcal{V}$ of $\mathbb{A}^n$ (– a point of the spectrum of $k[\mathbf{y}]$), the functor $\mathfrak{L}_{\mathcal{V}}$ produces A_n-modules supported in $\mathcal{V}$. If the subvariety $\mathcal{V}$ is smooth, then the stabilizer of V in A_n coincides with the ring of differential operators on $\mathcal{V}$. In this case, it follows from the Kashiwara's theorem, that the induction functor establishes an equivalence between the category $\mathcal{D}_{\mathcal{V}} - mod$ of D-modules on $\mathcal{V}$ and the full subcategory $A_n - mod_{\mathcal{V}}$ of the category $A_n - mod$ whose objects are A_n-modules supported on $\mathcal{V}$.

5.4.4.2. Hyperbolic coordinates. They are given by the k-algebra embedding $k[\bar{\xi}] = k[\xi_1, \dots, \xi_n] \xrightarrow{\psi} A_n$ which maps each indeterminate ξ_i to the product $x_i y_i$. The main advantage of this choice is that only a countable number of points of $Spec(k[\bar{\xi}])$ have a nontrivial stabilizer, and their stabilizer can be easily described and taken into account. Thus, we extend the diagram (1) to the diagram

$$
\begin{array}{ccccccc}
\mathbf{Sp}(k[\xi]) & \xleftarrow{\widetilde{\psi}} & \mathbf{Sp}(A_n) & \xrightarrow{\widetilde{\psi}_w} & \mathfrak{U}_w & \xrightarrow{\widetilde{\mathfrak{u}}_w} & \mathfrak{X}_{G/\mathcal{B}} \\
 & & {\scriptstyle\varphi_n}\downarrow & & {\scriptstyle\mathfrak{f}_w}\downarrow & & \downarrow{\scriptstyle\mathfrak{f}} \\
 & & \mathbf{Sp}(\Gamma(\mathbb{A}_n)) & \xrightarrow{\psi_w} & U_w & \xrightarrow{u_w} & X_{G/\mathcal{B}} \qquad w \in \mathcal{W},
\end{array}
\tag{2}
$$

and use the morphism $\widetilde{\psi} = \mathbf{Sp}(\psi)$ for constructing elements of the spectrum of $\mathbf{Sp}(A_n)$.

5.5. Quantized D-modules on quantum flag varieties.

5.5.1. The cone of a non-unital ring. Let R_0 be a unital associative ring and R_+ an associative (non-unital in general) ring in the category of R_0-bimodules; i.e. R_+ is endowed with an R_0-bimodule morphism $R_+ \otimes_{R_0} R_+ \xrightarrow{m} R_+$ satisfying the associativity condition. We denote by R the augmented unital ring $R_0 \oplus R_+$ and by $\mathcal{T}_{R_+}$ the full subcategory of $R - mod$ whose objects are all R-modules annihilated by R_+.

We define the 'space' *cone of* R_+ by taking as $C_{\mathbf{Cone}(R_+)}$ the quotient category $R - mod/\mathcal{T}^-_{R_+}$ of $R - mod$ by the Serre subcategory spanned by $\mathcal{T}_{R_+}$. The localization functor $R-mod \xrightarrow{u^*} R-mod/\mathcal{T}^-_{R_+}$ is an inverse image functor of a morphism of 'spaces' $\mathbf{Cone}(R_+) \xrightarrow{u} \mathbf{Sp}(R)$. The functor u^* has a (necessarily fully faithful) right adjoint, i.e. the morphism u is continuous. If R_+ is a unital ring, then u is an isomorphism (see [KR2, C3.2.1]). The composition of the morphism u with the canonical affine morphism $\mathbf{Sp}(R) \longrightarrow \mathbf{Sp}(R_0)$ is a continuous morphism $\mathbf{Cone}(R_+) \longrightarrow \mathbf{Sp}(R_0)$. Its direct image functor is (regarded as) the *global sections functor*.

5.5.2. The graded version: $\mathbf{Proj}_{\mathcal{G}}$. Let $\mathcal{G}$ be a monoid and $R = R_0 \oplus R_+$ a $\mathcal{G}$-graded ring with zero component R_0. Then we have the category $gr_{\mathcal{G}}R-mod$ of $\mathcal{G}$-graded R-modules and its full subcategory $gr_{\mathcal{G}}\mathcal{T}_{R_+} = \mathcal{T}_{R_+} \cap gr_{\mathcal{G}}R-mod$ whose objects are graded modules annihilated by the ideal R_+. We define the 'space' $\mathbf{Proj}_{\mathcal{G}}(R)$ by setting

$$C_{\mathbf{Proj}_{\mathcal{G}}(R)} = gr_{\mathcal{G}}R-mod/gr_{\mathcal{G}}\mathcal{T}^-_{R_+}.$$

Here $gr_{\mathcal{G}}\mathcal{T}^-_{R_+}$ is the Serre subcategory of the category $gr_{\mathcal{G}}R-mod$ spanned by $gr_{\mathcal{G}}\mathcal{T}_{R_+}$. One can show that $gr_{\mathcal{G}}\mathcal{T}^-_{R_+} = gr_{\mathcal{G}}R-mod \cap \mathcal{T}^-_{R_+}$. Therefore, we have a canonical projection

$$\mathbf{Cone}(R_+) \xrightarrow{\mathfrak{p}} \mathbf{Proj}_{\mathcal{G}}(R).$$

The localization functor $gr_{\mathcal{G}}R-mod \longrightarrow C_{\mathbf{Proj}_{\mathcal{G}}(R_+)}$ is an inverse image functor of a continuous morphism $\mathbf{Proj}_{\mathcal{G}}(R) \xrightarrow{\mathfrak{v}} \mathbf{Sp}_{\mathcal{G}}(R)$. The composition $\mathbf{Proj}_{\mathcal{G}}(R) \xrightarrow{\mathfrak{v}} \mathbf{Sp}(R_0)$ of the morphism $\mathfrak{v}$ with the canonical morphism $\mathbf{Sp}_{\mathcal{G}}(R) \xrightarrow{\phi} \mathbf{Sp}(R_0)$ defines $\mathbf{Proj}_{\mathcal{G}}(R)$ as a 'space' over $\mathbf{Sp}(R_0)$. Its direct image functor is called the *global sections functor*.

5.5.2.1. Standard example: cone and Proj of a $\mathbb{Z}_+$-graded ring. Let $R = \bigoplus_{n\geq 0} R_n$ be a $\mathbb{Z}_+$-graded ring, R_+ its 'irrelevant' ideal. Thus, we have $\mathbf{Cone}(R_+)$, $\mathbf{Proj}(R) = \mathbf{Proj}_{\mathbb{Z}}(R)$, and the canonical morphism $\mathbf{Cone}(R_+) \longrightarrow \mathbf{Proj}(R)$.

5.5.3. The category of D-modules on the flag variety of a reductive Lie algebra. Let $\mathfrak{g}$ be a reductive Lie algebra over $\mathbb{C}$ and $U(\mathfrak{g})$ the enveloping algebra of $\mathfrak{g}$. Let $\mathcal{G}$ be the group of integral weights of $\mathfrak{g}$ and $\mathcal{G}_+$ the semigroup of nonnegative integral weights. Let $R = \oplus_{\lambda\in\mathcal{G}_+} R_\lambda$, where R_λ is the vector space of the (canonical) irreducible finite dimensional representation with the highest weight λ. The module R is a $\mathcal{G}$-graded algebra with the multiplication determined by the projections $R_\lambda \otimes R_\nu \longrightarrow R_{\lambda+\nu}$, for all $\lambda, \nu \in \mathcal{G}_+$. It is well known that the algebra R is isomorphic to the algebra of regular functions on the *base affine space* of $\mathfrak{g}$. Recall that G/U, where G is a connected simply connected algebraic group with the Lie algebra $\mathfrak{g}$, and U is its maximal unipotent subgroup.

5.5.3.1. Base affine space and flag variety. The category $C_{\mathbf{Cone}(R)}$ is equivalent to the category of quasi-coherent sheaves on the base affine space Y of the Lie algebra $\mathfrak{g}$.

The category $Proj_{\mathcal{G}}(R)$ is equivalent to the category of quasi-coherent sheaves on the flag variety of $\mathfrak{g}$.

5.5.3.2. D-modules on the flag variety. Consider the cross-product $U(\mathfrak{g})\#R$ associated with the Hopf action of $U(\mathfrak{g})$ on R. This is a $\mathcal{G}$-graded algebra (with the grading induced by the grading of the algebra R). One can show that the category $C_{\mathbf{Proj}_{\mathcal{G}}(U(\mathfrak{g})\#R)}$ is equivalent to the category $\mathcal{D} - mod_{G/\mathcal{B}}$ of D-modules on the flag variety of the Lie algebra $\mathfrak{g}$. In other words, the 'space' represented by the category of D-modules on the flag variety is isomorphic to $\mathbf{Proj}(U(\mathfrak{g})\#R)$.

5.5.4. The quantum base affine 'space' and quantum flag variety of a semisimple Lie algebra. Let now $\mathfrak{g}$ be a semisimple Lie algebra over a field k of zero characteristic, and let $U_q(\mathfrak{g})$ be the quantized enveloping algebra of $\mathfrak{g}$. Define the $\mathcal{G}$-graded algebra $\mathfrak{R} = \bigoplus_{\lambda\in\mathcal{G}_+} \mathfrak{R}_\lambda$ the same way as above, i.e. $\mathfrak{R}_\lambda$ is a simple finite-dimensional module with the highest weight λ. This time, however, the algebra $\mathfrak{R}$ is not commutative. If $\mathfrak{g} = sl_2$, then $\mathfrak{R}$ is isomorphic to the algebra $k_{\mathfrak{v}}[x,y] = k\langle x,y\rangle/(xy - \mathfrak{v}yx)$ for an appropriate $\mathfrak{v}$. Following the classical example (and representing 'spaces' by the categories of quasi-coherent sheaves on them), we call $\mathbf{Cone}(\mathfrak{R})$ *the quantum base affine 'space'* and $\mathbf{Proj}_{\mathcal{G}}(\mathfrak{R})$ the *quantum flag variety* of the Lie algebra $\mathfrak{g}$. We call $\mathfrak{R}$ the *algebra of functions* on the quantum base affine 'space'.

5.5.4.1. Canonical affine covers of the quantum base affine 'space' and the quantum flag variety. Let W be the Weyl group of the Lie algebra $\mathfrak{g}$. Fix a $w \in W$. For any $\lambda \in \mathcal{G}_+$, choose a nonzero w-extremal vector $e^\lambda_{w\lambda}$ generating the one dimensional vector subspace of $\mathfrak{R}_\lambda$ formed by the vectors of the weight $w\lambda$. Set $S_w = \{k^* e^\lambda_{w\lambda} | \lambda \in \mathcal{G}_+\}$. It follows from the Weyl character formula that $e^\lambda_{w\lambda} e^\mu_{w\mu} \in k^* e^{\lambda+\mu}_{w(\lambda+\mu)}$. Hence S_w is a multiplicative set. It was proved by Joseph [Jo] that S_w is a left and right Ore subset in $\mathfrak{R}$. The Ore sets $\{S_w | w \in W\}$ determine a conservative family of affine localizations

$$\mathbf{Sp}(S_w^{-1}\mathfrak{R}) \longrightarrow \mathbf{Cone}(\mathfrak{R}), \quad w \in W, \tag{4}$$

of the quantum base affine 'space' and a conservative family of affine localizations

$$\mathbf{Sp}_{\mathcal{G}}(S_w^{-1}\mathfrak{R}) \longrightarrow \mathbf{Proj}_{\mathcal{G}}(\mathfrak{R}), \quad w \in W, \tag{5}$$

of the quantum flag variety. Here $\mathbf{Sp}_{\mathcal{G}}(S_w^{-1}\mathfrak{R})$ is the 'space' represented by the category $gr_{\mathcal{G}} S_w^{-1}\mathfrak{R} - mod$ of $\mathcal{G}$-graded $gr_{\mathcal{G}} S_w^{-1}\mathfrak{R}$-modules.

We claim that the category $gr_{\mathcal{G}} S_w^{-1}\mathfrak{R} - mod$ is naturally equivalent to $(S_w^{-1}\mathfrak{R})_0 - mod$. By 1.5, it suffices to verify that the canonical functor $gr_{\mathcal{G}} S_w^{-1}\mathfrak{R} - mod \longrightarrow S_w^{-1}\mathfrak{R})_0 - mod$ which assigns to every graded $S_w^{-1}\mathfrak{R}$-module its zero component is faithful; i.e. the zero component of every nonzero $\mathcal{G}$-graded $S_w^{-1}\mathfrak{R}$-module is nonzero. This is, really, the case, because if z is a nonzero element of λ-component of a $\mathcal{G}$-graded $S_w^{-1}\mathfrak{R}$-module, then $(e_{w\lambda}^{\lambda})^{-1}z$ is a nonzero element of the zero component of this module.

This shows that for every $w \in W$, the morphism $\mathbf{Sp}_{\mathcal{G}}(S_w^{-1}\mathfrak{R}) \longrightarrow \mathbf{Proj}_{\mathcal{G}}(\mathfrak{R})$ is isomorphic to the morphism $\mathbf{Sp}((S_w^{-1}\mathfrak{R})_0) \xrightarrow{\mathfrak{u}_w} \mathbf{Proj}_{\mathcal{G}}(\mathfrak{R})$. The morphism $\mathfrak{u}_w$ form an affine cover

$$\mathbf{Sp}((S_w^{-1}\mathfrak{R})_0) \xrightarrow{\mathfrak{u}_w} \mathbf{Proj}_{\mathcal{G}}(\mathfrak{R}), \quad w \in W \tag{6}$$

of the quantum flag variety $\mathbf{Proj}_{\mathcal{G}}(\mathfrak{R})$ turning it into a noncommutative scheme.

5.5.5. The quantum flag D-variety. Similar to 5.5.3.2, we consider the cross-product $U_q(\mathfrak{g})\#\mathfrak{R}$, where $\mathfrak{R}$ is the algebra of functions on the quantum base affine 'space' defined in 5.5.4, with $\mathcal{G}$-grading induced by the $\mathcal{G}$-grading of $\mathfrak{R}$. We call the 'space' $\mathbf{Proj}(U_q(\mathfrak{g})\#\mathfrak{R})$ the quantum flag D-variety. The objects of the category representing $\mathbf{Proj}_{\mathcal{G}}(U_q(\mathfrak{g})\#\mathfrak{R})$ are called *quantum D-modules* on the quantum flag variety $\mathbf{Proj}_{\mathcal{G}}(\mathfrak{R})$.

The natural algebra morphism $\mathfrak{R} \longrightarrow U_q(\mathfrak{g})\#\mathfrak{R}$ induces an affine morphism

$$\mathbf{Proj}(U_q(\mathfrak{g})\#\mathfrak{R}) \xrightarrow{\mathfrak{f}} \mathbf{Proj}(\mathfrak{R}).$$

As every affine morphism, the morphism $\mathfrak{f}$ is isomorphic to the natural morphism

$$\mathbf{Sp}(\mathcal{F}_{\mathfrak{f}}/\mathbf{Proj}_{\mathcal{G}}(\mathfrak{R})) \xrightarrow{\widetilde{\mathfrak{f}}} \mathbf{Proj}_{\mathcal{G}}(\mathfrak{R})$$

for a monad $\mathcal{F}_{\mathfrak{f}}$. The monad $\mathcal{F}_{\mathfrak{f}}$ can be chosen canonically: it is uniquely determined by the action of $U_q(\mathfrak{g})$ on the category $gr_{\mathcal{G}}\mathfrak{R} - mod$ of $\mathcal{G}$-graded $\mathfrak{R}$-modules, because this action is compatible with the localization $gr_{\mathcal{G}}\mathfrak{R} - mod \longrightarrow \mathbf{Proj}_{\mathcal{G}}(\mathfrak{R})$.

Moreover, the action of $U_q(\mathfrak{g})$ on $gr_{\mathcal{G}}\mathfrak{R} - mod$ becomes *differential* in an appropriate sense (explained in [LR1] and [LR2]). This implies, among other things, that the action of $U_q(\mathfrak{g})$ on $gr_{\mathcal{G}}\mathfrak{R} - mod$ is compatible with localizations at the Ore sets S_w for each $w \in W$. So that the cover of $Proj_{\mathcal{G}}(\mathfrak{R})$ described in 5.5.4.1(6) induces a cover

$$\mathbf{Sp}((S_w^{-1}(U_q(\mathfrak{g})\#\mathfrak{R})_0) \xrightarrow{\mathfrak{u}_w^{\rho}} \mathbf{Proj}_{\mathcal{G}}(U_q(\mathfrak{g})\#\mathfrak{R}), \quad w \in W \tag{7}$$

of the 'space' $\mathbf{Proj}(U_q(\mathfrak{g})\#\mathfrak{R})$ such that the diagram

$$\begin{array}{ccc} \mathbf{Sp}((S_w^{-1}(U_q(\mathfrak{g})\#\mathfrak{R}))_0) & \xrightarrow{\mathfrak{u}_w^{\rho}} & \mathbf{Proj}_{\mathcal{G}}(U_q(\mathfrak{g})\#\mathfrak{R}) \\ \downarrow & & \downarrow \\ \mathbf{Sp}((S_w^{-1}\mathfrak{R})_0) & \xrightarrow{\mathfrak{u}_w} & \mathbf{Proj}_{\mathcal{G}}(\mathfrak{R}) \end{array} \qquad w \in W \tag{8}$$

whose all four arrows are affine morphisms, commutes. In particular, the cover (7) turns the 'space' $\mathbf{Proj}_{\mathcal{G}}(U_q(\mathfrak{g})\#\mathfrak{R})$ into a noncommutative separated scheme.

5.5.6. The global sections functor. For any $\mathcal{G}$-graded k-algebra $\mathcal{R}$, there is a canonical continuous morphism $\mathbf{Proj}_{\mathcal{G}}(\mathcal{R}) \xrightarrow{\gamma} \mathbf{Sp}(\mathcal{R}_0)$ whose direct image functor is the composition of the right adjoint $C_{\mathbf{Proj}(\mathcal{R})} \xrightarrow{\mathfrak{q}_*} gr_{\mathcal{G}}\mathcal{R} - mod$ to the localization functor $gr_{\mathcal{G}}\mathcal{R} - mod \xrightarrow{\mathfrak{q}^*} C_{\mathbf{Proj}(\mathcal{R})}$ and the functor

$$gr_{\mathcal{G}}\mathcal{R} - mod \xrightarrow{\mathfrak{p}_*} R_0 - mod$$

which assigns to every $\mathcal{G}$-graded $\mathcal{R}$-module M its zero component endowed with the action of the zero component $\mathcal{R}_0$ of the algebra $\mathcal{R}$. We call the direct image functor $\gamma_* = \mathfrak{p}_*\mathfrak{q}_*$ of the morphism γ the *global sections* functor.

Thus, if $\mathcal{R}$ is the algebra of functions on the quantum (or classical) flag variety of the Lie algebra $\mathfrak{g}$, then $\mathcal{R}_0 = k$. If $\mathcal{R} = U(\mathfrak{g})\#\mathfrak{R}$, then $\mathcal{R}_0 = U(\mathfrak{g})$; and the diagram

$$\begin{array}{ccc} \mathbf{Proj}_{\mathcal{G}}(U_q(\mathfrak{g})\#\mathfrak{R}) & \xrightarrow{\widetilde{\gamma}} & \mathbf{Sp}(U(\mathfrak{g})) \\ {\scriptstyle \mathfrak{f}}\big\downarrow & & \big\downarrow \\ \mathbf{Proj}_{\mathcal{G}}(\mathfrak{R}) & \xrightarrow{\gamma} & \mathbf{Sp}(k) \end{array} \qquad (9)$$

(where the right vertical arrow corresponds to the k-algebra structure on $U(\mathfrak{g})$) commutes.

By [LR2] (see also [T]), the morphism $\mathbf{Proj}_{\mathcal{G}}(U_q(\mathfrak{g})\#\mathfrak{R}) \xrightarrow{\widetilde{\gamma}} \mathbf{Sp}(U_q(\mathfrak{g}))$ is affine and its direct image function establishes an equivalence between the category $C_{\mathbf{Proj}_{\mathcal{G}}(U_q(\mathfrak{g})\#\mathfrak{R})}$ of quantum D-modules on the flag variety and the full subcategory $U_q(\mathfrak{g})_\rho - mod$ of $U_q(\mathfrak{g})$-modules with the trivial central character. Thus, we can replace the diagram (9) with the commutative diagram

$$\begin{array}{ccc} \mathbf{Proj}_{\mathcal{G}}(U_q(\mathfrak{g})\#\mathfrak{R}) & \xrightarrow{\gamma_\rho} & \mathbf{Sp}(U_q(\mathfrak{g})_\rho) \\ {\scriptstyle \mathfrak{f}}\big\downarrow & & \big\downarrow \\ \mathbf{Proj}_{\mathcal{G}}(\mathfrak{R}) & \xrightarrow{\gamma} & \mathbf{Sp}(k) \end{array} \qquad (10)$$

whose upper horizontal arrow is an isomorphism.Therefore, it induces isomorphisms between the corresponding spectra of these 'spaces'. In particular, the direct image functor $\gamma_{\rho *}$ of the morphism γ_ρ maps $Spec^0_{\mathfrak{c}}(\mathbf{Proj}_{\mathcal{G}}(U_q(\mathfrak{g})\#\mathfrak{R}))$ to $Spec^0_{\mathfrak{c}}(\mathbf{Sp}(U_q(\mathfrak{g})_\rho))$ and this map induces an isomorphism from $\mathbf{Spec}^0_{\mathfrak{c}}(\mathbf{Proj}_{\mathcal{G}}(U_q(\mathfrak{g})\#\mathfrak{R}))$ onto $\mathbf{Spec}^0_{\mathfrak{c}}(\mathbf{Sp}(U_q(\mathfrak{g})_\rho))$.

5.6. The twisted version. Fix a central character χ of the quantized enveloping algebra $U_q(\mathfrak{g})$ and consider the twisted cross-product $U_q(\mathfrak{g})\#_\chi\mathfrak{R}$. We call $\mathbf{Proj}_{\mathcal{G}}(U_q(\mathfrak{g})\#_\chi\mathfrak{R})$ the quantum D_χ-variety, or the quantum twisted D-variety. The constructions of 5.5 can be repeated literally for the twisted D-varieties and summarized in the commutative diagrams

$$\begin{array}{ccccc} \mathbf{Sp}((S_w^{-1}(U_q(\mathfrak{g})\#\mathfrak{R}))_0) & \xrightarrow{\mathfrak{u}_w^\chi} & \mathbf{Proj}_{\mathcal{G}}(U_q(\mathfrak{g})\#_\chi\mathfrak{R}) & \xrightarrow{\gamma_\chi} & \mathbf{Sp}(U_q(\mathfrak{g})_\chi) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{Sp}((S_w^{-1}\mathfrak{R})_0) & \xrightarrow{\mathfrak{u}_w} & \mathbf{Proj}_{\mathcal{G}}(\mathfrak{R}) & \xrightarrow{\gamma} & \mathbf{Sp}(k) \end{array} \qquad w \in W \tag{1}$$

It follows from [LR2] (and [T]) that if χ is regular, anti-dominant, then γ_χ is an isomorphism. In this case, computing the spectra of the twisted flag D-variety is the same as the computing the corresponding spectra of the affine scheme $\mathbf{Sp}(U_q(\mathfrak{g})_\chi)$.

As to the studying the spectra of the flag D_χ-variety, it is reduced to the study of the spectra of elements of the cover, $\mathbf{Sp}((S_w^{-1}(U_q(\mathfrak{g})\#\mathfrak{R}))_0)$, $w \in W$. The spectra of $\mathbf{Sp}((S_w^{-1}(U_q(\mathfrak{g})\#\mathfrak{R}))_0)$ can be studied via the affine morphism

$$\mathbf{Sp}((S_w^{-1}(U_q(\mathfrak{g})\#\mathfrak{R}))_0) \longrightarrow \mathbf{Sp}((S_w^{-1}\mathfrak{R})_0), \tag{2}$$

or, possibly, using a different affine morphism

$$\mathbf{Sp}((S_w^{-1}(U_q(\mathfrak{g})\#\mathfrak{R}))_0) \xrightarrow{\widetilde{\psi}_w} \mathbf{Sp}(\mathcal{A}_w). \tag{3}$$

5.7. Remarks.

5.7.1. These constructions for the usual enveloping algebras. If the quantized enveloping algebra $U_q(\mathfrak{g})$ is replaced by the enveloping algebra $U(\mathfrak{g})$ and the algebra $\mathfrak{R}$ of functions on the quantum base affine 'space' by the algebra R of functions on the base affine space, then the constructions of 5.5 and 5.6 become another, purely algebraic, description of D-modules on a flag variety, the related canonical covers of the flag variety, and the corresponding (twisted) D-scheme. In particular, the algebra $(S_w^{-1}R)_0$ is isomorphic to the polynomial algebra $k[\bar{y}] = k[y_1, \dots, y_n]$ – the coordinate algebra of the affine space $\mathbb{A}^n$, and $(S_w^{-1}(U(\mathfrak{g})\#R))_0$ is, therefore, isomorphic to the Weyl algebra A_n for all $w \in W$.

A sensible choice of the algebra $\mathcal{A}_w$ in (3) is the polynomial algebra $k[\bar{\xi}] = k[\xi_1, \dots, \xi_n]$ and the morphism (3) is induced by the algebra morphism $k[\bar{\xi}] \xrightarrow{\psi} A_n$ which maps each indeterminate ξ_i to the product $x_i y_i$ – *hyperbolic coordinates* (see 5.4.4.2). Why this choice is sensible is shown in Section C1 (see also [R, Chapters II and IV]).

5.7.2. Quantum hyperbolic coordinates. In the quantum case, the algebras $(S_w^{-1}\mathfrak{R})_0$ of functions on the *quantum* translations of the big cell are rather complicated noncommutative algebras, if $\mathfrak{g}$ is a simple Lie algebra of the rank higher than one. Finding their own spectra is already a problem, so that the *standard* method, i.e. using the morphism (2) for the construction (induction) of the points of the spectra of $(S_w^{-1}(U_q(\mathfrak{g})\#\mathfrak{R}))_0$ becomes unpractical. Amazingly, the second method, the induction along hyperbolic coordinates, survives. That is one can take as the algebra $\mathcal{A}_w$ in (3) the algebra of polynomials $k[\xi] = k[\xi_1, \dots, \xi_n]$ and a morphism

$$k[\xi] \xrightarrow{\psi_w} (S_w^{-1}(U_q(\mathfrak{g})\#\mathfrak{R}))_0 \tag{4}$$

which is a part of the hyperbolic structure. In the *classical limit* (i.e. after factorization by the ideal generated by $(q-1)$), the algebra $(S_w^{-1}(U_q(\mathfrak{g})\#\mathfrak{R}))_0$ becomes the Weyl algebra A_n and the morphism (4) turns into the canonical morphism $k[\bar{\xi}] \xrightarrow{\psi} A_n$ (see 5.4.4.2).

In the case when the Cartan matrix of the Lie algebra $\mathfrak{g}$ is of the type (A) or (C) and w is the longest element of the Weyl group, the construction of the hyperbolic structure on the algebra $(S_w^{-1}(U_q(\mathfrak{g})\#\mathfrak{R}))_0$, in particular the morphism (4), can be deduced from [Ha]. The construction is written explicitly (for a more general case) in [R, IV.C2.7].

The existence of the deformations (4) of the canonical map $k[\bar{\xi}] \xrightarrow{\psi} A_n$ (more precisely, of its composition with the isomorphism $A_n \xrightarrow{\sim} (S_w^{-1}(U(\mathfrak{g})\#\mathfrak{R}))_0$) implies that not only the highest weight simple $U_q(\mathfrak{g})$-modules are deformations of the highest weight simple $U(\mathfrak{g})$-modules (which is a well known result of G. Lusztig [L]), but also that 'almost all' representations of the quantized enveloping algebra $U_q(\mathfrak{g})$ parameterized by the points $\mathcal{P}$ of $Spec(k[\bar{\xi}])$ via the maps (4) and related functors $\mathfrak{L}_{\mathcal{P}}$ (hence these representations belong to the spectrum of the noncommutative 'space' $\mathbf{Sp}(U_q(\mathfrak{g}))$) are deformations of the representations of the enveloping algebra $U(\mathfrak{g})$ parameterized by the same points of $Spec(k[\bar{\xi}])$ via the maps $k[\bar{\xi}] \xrightarrow{\psi} A_n \xrightarrow{\sim} (S_w^{-1}(U(\mathfrak{g})\#\mathfrak{R}))_0$ and the functors $\mathfrak{L}_{\mathcal{P}}$ determined by the ring morphism ψ.

Note that the hyperbolic algebra structure works more or less the same way in all cases, so that the piece of spectral theory of $(S_w^{-1}(U_q(\mathfrak{g})\#\mathfrak{R}))_0$ (hence of $U_q(\mathfrak{g})$) related to the morphism (4) is produced approximately the same way as the piece of spectral theory of the Weyl algebra A_n related to hyperbolic coordinates $k[\bar{\xi}] \xrightarrow{\psi} A_n$. For the material supporting the latter assertion, we refer to the section C1 of this paper (see below) and Chapters II and IV of the monograph [R].

5.7.3. Hyperbolic coordinates and holonomic objects. One can show that all simple A_n-modules obtained via the functor $\mathfrak{L}_{\mathcal{P}}$ corresponding to the algebra morphism $k[\bar{\xi}] \xrightarrow{\psi} A_n$, where $\mathcal{P}$ runs through the closed points of $Spec(k[\bar{\xi}])$, are holonomic. This follows from the Roos criterium of the holonomicity, the formulas for the functors $\mathfrak{L}_{\mathcal{P}}$ in hyperbolic case, and the fact that the closed points of $Spec(k[\bar{\xi}])$ have the trivial stabilizer (see C1 below). Each simple holonomic module on an element of the cover (– a translation of the big cell) determines a simple holonomic D-module on the flag variety.

Similar facts hold in the quantum case for the algebra morphisms (4).

Complementary facts.

C1. Weyl and Heisenberg algebras.

The studying the spectra of universal enveloping algebra $U(\mathfrak{g})$ of a reductive Lie algebras over algebraically closed fields of zero characteristic is reduced (via the passage to the categories of quasi-coherent modules over sheaves of twisted differential operators on flag variety and using the standard cover of the latter by translations of the big cell) to studying modules over Weyl algebras (see 5.4.4).

Weyl algebras play also a crucial role in representation theory of nilpotent Lie algebras: if $\mathfrak{g}$ is a finite-dimensional nilpotent Lie algebra over an algebraically closed field of zero characteristic, then the set of primitive ideals of its universal enveloping algebra $U(\mathfrak{g})$ is parameterized by the orbits of adjoint action on the dual space $\mathfrak{g}^*$; and for any primitive ideal $\mathfrak{J}$, quotient algebra $U(\mathfrak{g})/\mathfrak{J}$ is isomorphic to the Weyl algebra A_n.

Recall that the Weyl algebra A_n is a k-algebra generated by x_i, y_i subject to the relations

$$[x_i, y_j] = \delta_{ij},\ [x_i, x_j] = 0 = [y_i, y_j] \quad \text{for all } 1 \le i, j \le n. \tag{3}$$

We assume that k is a field of zero characteristic.

C1.1. The standard realization. Let now C_X be the category of modules over the polynomial algebra $k[\mathbf{y}] = k[y_1, \dots, y_n]$, and $C_{\mathfrak{A}} = A_n - mod \xrightarrow{\varphi_*} C_X$ the pull-back functor corresponding to the embedding $k[\mathbf{y}] \hookrightarrow A_n$. Then $C_{\mathfrak{A}} = \mathcal{F}_\varphi - mod$, where $\mathcal{F}_\varphi = (F_\varphi, \mu_\varphi)$ is a differential monad on X; i.e. $F_\varphi = A_n \otimes_{k[\mathbf{y}]} -$ is a differential functor.

Fix a point $\mathcal{P}$ of $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ and consider the related commutative diagram (see (2) in 3.3)

$$\begin{tikzcd} C_{\mathfrak{A}[\mathcal{P}-]} \arrow[rr, "u_{\mathcal{P}*}"] \arrow[dr, "\widetilde{\mathfrak{f}}^*_{\mathcal{P}}"'] & & C_{X_{\mathcal{P}}} \\ & C_{\mathfrak{A}'_{\mathcal{P}}} \arrow[ur, "\widetilde{\varphi}^*_{\mathcal{P}}"'] & \end{tikzcd} \tag{4}$$

Let $V_{\mathcal{P}}$ denote the Zariski closed irreducible subspace of $Spec(k[\mathbf{y}])$ corresponding to $\mathcal{P}$. The category $C_{\mathfrak{A}'_{\mathcal{P}}}$ is equivalent to the category $\mathcal{D}(V_{\mathcal{P}}) - mod$ of modules over the ring $\mathcal{D}(V_{\mathcal{P}})$ of differential operators on the subvariety (corresponding to) $V_{\mathcal{P}}$. The category $C_{\mathfrak{A}[\mathcal{P}-]}$ is the category of A_n-modules whose support is contained in $V_{\mathcal{P}}$. If the subvariety $V_{\mathcal{P}}$ is smooth, then, by a Kashiwara's theorem, the functor $C_{\mathfrak{A}[\mathcal{P}-]} \xrightarrow{\widetilde{\mathfrak{f}}^*_{\mathcal{P}}} C_{\mathfrak{A}'_{\mathcal{P}}}$ in (4) is an equivalence of categories.

Thus, the problem of finding the part of the spectrum of $\mathfrak{A}$ corresponding to the point $\mathcal{P}$ such that $V_{\mathcal{P}}$ is a smooth subvariety, is reduced to the problem of classifying points of the spectrum of D-modules on the subvariety $V_{\mathcal{P}}$. If $\mathcal{P}$ is not a generic point, we reduce the dimension. The price to pay is studying D-modules on a possibly much more complicated variety.

Since we study only D-modules related to the point $\mathcal{P}$, we can localize at $\mathcal{P}$ and consider, together with the diagram (4), the diagram

$$\begin{tikzcd} C_{\mathfrak{A}_r[\mathcal{P}-]} \arrow[rr, "u_{\mathcal{P}*}"] \arrow[dr, "\widetilde{\mathfrak{f}}^*_{\mathcal{P}}"'] & & C_{X^r_{\mathcal{P}}} \\ & C_{\mathfrak{A}^r_{\mathcal{P}}} \arrow[ur, "\widetilde{\varphi}^*_{\mathcal{P}}"'] & \end{tikzcd} \tag{5}$$

Here $X^r_{\mathcal{P}}$ is the residue 'space' of X at the point $\mathcal{P}$; $C_{\mathfrak{A}^r_{\mathcal{P}}}$ is the category of $\widetilde{\mathfrak{F}}$-modules $(L, \widetilde{\xi})$, where L is an object of the residue category $C_{X^r_{\mathcal{P}}}$, and $C_{\mathfrak{A}_r[\mathcal{P}-]}$ is the category of $\mathcal{F}_{\widehat{\mathcal{P}}}$-modules (M, ξ), where M is an object of the *residue Serre subcategory* (which is by definition the smallest nonzero Serre subcategory) of $C_{X/\widehat{\mathcal{P}}}$ (cf. 3.3.6).

In the case of studying $\mathbf{Spec}_-(X)$, the diagram (4) can be replaced by (5).

The residue category $C_{X^r_{\mathcal{P}}}$ in (5) is equivalent to the category of vector spaces over the residue field $k_{\mathcal{P}}$ of the point $\mathcal{P}$. The category $C_{\mathfrak{A}^r_{\mathcal{P}}}$ is equivalent to the category of modules over the ring of differential operators on the subvariety $V_{\mathcal{P}}$ with rational coefficients. The category $C_{\mathfrak{A}_r[\mathcal{P}-]}$ is equivalent to the category of modules with support in the subvariety $V_{\mathcal{P}}$ over the algebra of differential operators with coefficients in the residue field $k_{\mathcal{P}}$.

If $\mathcal{P}$ is a generic point, then $V_{\mathcal{P}} = Spec(k[\mathbf{y}])$, $C_{\mathfrak{A}[\mathcal{P}-]} = C_{\mathfrak{A}'_{\mathcal{P}}} = C_{\mathfrak{A}}$, the residue field $k_{\mathcal{P}}$ is the field $k(\mathbf{y})$ of rational functions in $\mathbf{y} = (y_1, \dots, y_n)$.

Depending on the point $\mathcal{P}$ the algebras of differential operators with coefficients from the residue field $k_{\mathcal{P}}$, hence the categories of modules over them, might be quite complicated.

C1.2. The hyperbolic structure. Let C_X be the category of modules over the polynomial algebra $R = k[\xi_1, \dots, \xi_n]$, where $\xi_i = x_i y_i$, $1 \le i \le n$,. We take $C_{\mathfrak{A}} = A_n{-}mod$ and consider the morphism $\mathfrak{A} \xrightarrow{u} X$ corresponding to the embedding $k[\xi] \hookrightarrow A_n$. So the category $C_{\mathfrak{A}} = A_n - mod$ is realized as the category of modules over the monad $\mathcal{F}_\varphi = (F_\varphi, \mu_\varphi)$ on C_X, where $F_\varphi = A_n \otimes_R -$.

The algebra A_n is a free right R-module of rank $\mathbb{Z}^n$. Explicitly,

$$A_n = \bigoplus_{\mathbf{s},\mathbf{t} \in \mathbb{Z}^n_+,\ \mathbf{s}\cdot\mathbf{t}=0} \mathbf{x}^{\mathbf{s}} \mathbf{y}^{\mathbf{t}} R \tag{6}$$

Here $\mathbf{x}^{\mathbf{n}} = x_1^{s_1} \dots x_n^{s_n}$ and $\mathbf{s} \cdot \mathbf{t} = \sum_{1 \le i \le n} s_i t_i$.

The left R-module structure and the rest of multiplication are given by

$$\begin{aligned} & r\mathbf{x}^{\mathbf{s}}\mathbf{y}^{\mathbf{t}} = \mathbf{x}^{\mathbf{s}}\mathbf{y}^{\mathbf{t}}\vartheta^{\mathbf{t}-\mathbf{s}}(r) \quad \text{for all } r \in R; \\ & x_i y_i = \xi_i, \quad y_i x_i = \vartheta_i^{-1}(\xi_i) = \xi_i - 1, \\ & [x_i, y_j] = [x_i, x_j] = [y_i, y_j] = 0 \quad \text{for all} \quad 1 \le i, j \le n,\ i \ne j. \end{aligned} \tag{7}$$

Here $\vartheta^{\mathbf{s}} = \vartheta_1^{s_1} \circ \ldots \circ \vartheta_n^{s_n}$ and ϑ_i is an automorphism of the algebra R determined by $\vartheta_i(\xi_j) = \xi_j + \delta_{ij}$, where δ_{ij} is the Kronecker symbol.

It follows from this description that the functor $F_\varphi = A_n \otimes_R -$ is a direct sum of automorphisms of the category $C_X = R-mod$; namely, $F_\varphi = \bigoplus_{\mathbf{s}\in\mathbb{Z}^{\mathbf{n}}} \vartheta^{\mathbf{s}}$. The multiplication is defined by

$$\vartheta_i \circ \vartheta_j \xrightarrow{id} \vartheta_i\vartheta_j \quad \text{if } i \neq j, \quad \text{and} \quad \vartheta_i^n \circ \vartheta_i^m \xrightarrow{\xi_{n,m}^{(i)}} \vartheta_i^{n+m} \quad \text{for all } i,$$

where $\xi_{n,m}^{(i)} = id$ if n and m are both non-positive or both nonnegative. For $n \geq m \geq 1$, the morphisms $\xi_{n,-m}^{(i)}$ and $\xi_{-n,m}^{(i)}$ are defined by

$$\xi_{n,-m}^{(i)} = \xi_{n-1,-m+1}^{(i)} \circ \vartheta_i^{n-1}\xi_i\vartheta_i^{-m+1} \quad \text{and} \quad \xi_{-n,m}^{(i)} = \xi_{-n+1,m-1}^{(i)} \circ \vartheta_i^{-n+1}\xi_i\vartheta_i^{m-1}. \tag{8}$$

Here ξ_i is the endomorphism of the identical functor which assigns to every object N of C_X (– an R-module) the action of the element ξ_i on N.

C1.3. The non-degenerate part of the spectrum. Points $\mathcal{P}$ of the spectrum of C_X are in bijective correspondence with irreducible Zariski closed subspaces $V_{\mathcal{P}}$ of $Spec(R)$. The point $\mathcal{P}$ has a non-trivial stabilizer iff the subvariety $V_{\mathcal{P}}$ is stable by the transformation $\theta_1^{m_1} \ldots \theta_n^{m_n}$, where at least one of the integers m_i is nonzero. This shows that, generally, a point of $\mathbf{Spec}_{\mathfrak{c}}^0(X)$ has a trivial stabilizer.

C1.3.1. The description. If a point $\mathcal{P}$ of $\mathbf{Spec}_{\mathfrak{c}}^0(X)$ has a trivial stabilizer, then the functor $\mathfrak{f}_{\mathcal{P}}^*$ coincides with $\varphi^* : N \longmapsto (F_\varphi(N), \mu_\varphi(N))$. Let $M = R/p$, $p \in Spec(R)$, be a representative of $\mathcal{P}$. Then $M \xrightarrow{\lambda(M)} M$ is either zero or a monomorphism for any endomorphism λ of Id_{C_X}. In particular, either $\xi_i\vartheta_i^n(M)$ is a monomorphism for all n, or $\xi_i\vartheta_i^n(M) = 0$ for some unique n (see 8.1.3). The latter means that $\xi_i - n$ annihilates the R-module M; i.e. $\xi_i - n$ is an element of the prime ideal p.

If $\xi_i\vartheta_i^n(M)$ is a monomorphism for all n and all i, then one can show that the $\varphi_*(\langle M\rangle)$-torsion the $\mathcal{F}_\varphi$-module $\varphi^*(M) = (F_\varphi(M), \mu_\varphi(M))$ is zero. Therefore, by 2.2, $\varphi^*(M)$ is an object of $Spec(\mathfrak{A})$. The general case is as follows. We set

$$V_{i,n_i}(M) = \bigoplus_{m<n_i} \vartheta_i^m(M) \quad \text{if } n_i \geq 0, \quad \text{and} \quad V_{i,n_i}(M) = \bigoplus_{m\geq n_i} \vartheta_i^m(M) \quad \text{if } n_i < 0.$$

Let Ξ_M denote the set of all pairs (i, n_i) such that $\xi_i\vartheta_i^{n_i}(M) = 0$, or, equivalently, $\xi_i - n_i$ belongs to the prime ideal p. We set

$$V(M) = 0 \quad \text{if } \Xi_M = \emptyset, \quad \text{and} \quad V(M) = \bigoplus_{(i,n_i)\in\Xi_M} V_{i,n_i}(M) \quad \text{if } \Xi_M \neq \emptyset.$$

The $\mathcal{F}_\varphi$-submodule $\widetilde{V}(M)$ of $\varphi^*(M) = (F_\varphi(M), \mu_\varphi(M))$ generated by $V(M)$ coincides with the $\varphi_*^{-1}(\langle M\rangle)$-torsion of $\varphi^*(M)$. So, the quotient $\mathcal{F}_\varphi$-module $\varphi^*(M)/\widetilde{V}(M)$ is isomorphic to $\mathfrak{L}_{\mathcal{P}}(M)$. By 3.2.2, $\mathfrak{L}_{\mathcal{P}}(R/p)$ belongs to $Spec(\mathfrak{A})$.

C1.3.2. Note. We denote by $\mathbf{Spec}_{\varphi,0}(X)$ the subset of all points with trivial stabilizer and by $Spec_{\varphi,0}(R)$ the corresponding subset of $Spec(R)$. Let $\mathcal{P}_1,\ \mathcal{P}_2$ be points of $\mathbf{Spec}_{\varphi,0}(X)$, and let $p_1,\ p_2$ be the corresponding prime ideals – the elements of $Spec_{\varphi,0}(R)$. Set $M_i = R/p_i,\ i = 1,2$. It follows from the construction in C1.3.1 that if $\mathcal{P}_1 \subseteq \mathcal{P}_2$, or, equivalently, $p_2 \subseteq p_1$, then there is an epimorphism $L_{\mathcal{P}_1}(M_1) \longrightarrow L_{\mathcal{P}_2}(M_2)$. In particular, the point $[L_{\mathcal{P}_2}(M_2)]$ is a specialization of $[L_{\mathcal{P}_1}(M_1)]$.

C1.4. The degenerate part of the spectrum. For an element $\mathcal{P}$ of $\mathbf{Spec}^0_{\mathfrak{c}}(X)$, we set $\mathcal{G}_{\mathcal{P}} = \{\mathbf{t} \in \mathbb{Z}^n \mid \vartheta^{\mathbf{t}}(\mathcal{P}) = \mathcal{P}\}$. This is a subgroup of $\mathbb{Z}^n$ which we assume here to be nonzero, hence it is isomorphic to $\mathbb{Z}^m$ for some positive integer m. Let $\{\mathbf{t}_i \mid 1 \le i \le m\}$ be free generators of $\mathcal{G}_{\mathcal{P}}$. The category $C_{\mathfrak{A}_{\mathcal{P}}}$ is isomorphic to the category $R_{\mathcal{P}} - mod$ of left modules over the hyperbolic algebra $R_{\mathcal{P}}$ corresponding to the data $\{\widetilde{\vartheta}_i = \vartheta^{\mathbf{t}_i},\ \widetilde{\xi}_i = \xi(\mathbf{t}_i) \mid 1 \le i \le m\}$. Here $\xi(\mathbf{t}_i) = \prod\limits_{1\le j\le n} \xi_j(t_{ij})$, where t_{ij} is the j-th component of $\mathbf{t}_i$, and

$$\begin{aligned}
&\xi_j(\nu) = 1 \quad \text{if } \nu = 0,\\
&\xi_j(\nu) = \prod_{0\le s<\nu} \vartheta_j^s(\xi_j) = \prod_{0\le s<\nu} (\xi_j + s) \quad \text{if } \nu > 0, \quad \text{and}\\
&\xi_j(\nu) = \vartheta_j^{\nu}(\xi_j(-\nu) \prod_{1\le s\le -\nu} (\xi_j - s) \quad \text{if } \nu < 0.
\end{aligned} \tag{9}$$

That is $R_{\mathcal{P}}$ is generated by the algebra R and by the indeterminates $\widetilde{x}_i,\ \widetilde{y}_i$ subject to the relations

$$\begin{aligned}
&\widetilde{x}_i r = \widetilde{\vartheta}_i(r)\widetilde{x}_i, \quad r\widetilde{y}_i = \widetilde{y}_i\widetilde{\vartheta}_i(r),\\
&\widetilde{x}_i\widetilde{y}_i = \widetilde{\xi}_i, \quad \widetilde{y}_i\widetilde{x}_i = \widetilde{\vartheta}_i^{-1}(\widetilde{\xi}_i);\\
&[\widetilde{x}_i, \widetilde{y}_j] = [\widetilde{x}_i, \widetilde{x}_j] = [\widetilde{y}_i, \widetilde{y}_j] = 0 \quad \text{for all} \quad r \in R, \text{ and } 1 \le i,j \le m \text{ such that } i \ne j.
\end{aligned} \tag{10}$$

The functor $C_{\mathfrak{A}} \xrightarrow{\mathfrak{f}_{\mathcal{P}*}} C_{\mathfrak{A}_{\mathcal{P}}}$ corresponds to the algebra morphism $R_{\mathcal{P}} \longrightarrow A_n$ which is identical on R and maps $\widetilde{x}_i$ to $x^{\mathbf{t}_i^+} y^{\mathbf{t}_i^-}$ and $\widetilde{y}_i$ to $x^{\mathbf{t}_i^-} y^{\mathbf{t}_i^+}$, $1 \le i \le m$. Here $\mathbf{t}_i^+$ and $\mathbf{t}_i^-$ are elements of $\mathbb{Z}_+^n$ uniquely defined by the conditions: $\mathbf{t}_i = \mathbf{t}_i^+ - \mathbf{t}_i^-,\ \mathbf{t}_i^+ \cdot \mathbf{t}_i^- = 0$.

The category $C_{\mathfrak{A}'_{\mathcal{P}}}$ is naturally equivalent to the category $R_{\mathcal{P}}/(p) - mod$. Here p is the prime ideal in R corresponding to the point $\mathcal{P}$ and (p) denote the two-sided ideal in $R_{\mathcal{P}}$ generated by p.

The points of $\mathbf{Spec}_{\mathfrak{c}}^{\mathcal{P}}(\mathfrak{A}_{\mathcal{P}})$ are identified with those points of $\mathbf{Spec}(\mathfrak{A}'_{\mathcal{P}})$ which survive the localization at $\mathcal{P}$. The latter is given by the localization of the algebra R at p. Thus, $\mathbf{Spec}_{\mathfrak{c}}^{\mathcal{P}}(\mathfrak{A}_{\mathcal{P}})$ is identified with a subset of the spectrum of $\mathfrak{A}_{\mathcal{P}}^{\mathfrak{r}}$ (cf. 3.3.6). The category $C_{\mathfrak{A}_{\mathcal{P}}^{\mathfrak{r}}}$ is naturally equivalent to the category modules over the algebra $k_{\mathcal{P}}[(\widetilde{x}_i, \widetilde{x}_i^{-1}; \widetilde{\vartheta}_i)]$ of skew Laurent polynomials in $(\widetilde{x}_i \mid 1 \leq i \leq m)$ with coefficients in the residue field $k_{\mathcal{P}}$ of the point $\mathcal{P}$ which can be identified with the residue field $K(R/p)$ of the prime ideal p.

Here we used the fact that the elements $\widetilde{\xi}_i$, $1 \leq i \leq m$, do not belong to the ideal p.

Indeed, it follows from the formulas (9) that if $\widetilde{\xi}_i \in p$, then there is s such that $\xi_s + t \in p$ and $t_{is} \neq 0$. Since $\vartheta^{\mathbf{t}_i}(p) = p$, the element $\vartheta^{\ell \mathbf{t}_i}(\xi_s) = \xi_s + \ell t_{is}$ belongs to the ideal p for any $\ell \in \mathbb{Z}$. But, since $char(k) = 0$, this is impossible.

C1.4.1. The points of the spectrum over the generic point. Since $char(k) = 0$, the only $\mathbb{Z}^n$-invariant point of $\mathbf{Spec}_{\mathfrak{c}}^0(X)$ is the generic point $\mathcal{P}_0$ corresponding to the zero ideal of the k-algebra $R = k[\xi_1, \ldots, \xi_n]$.

The categories $C_{\mathfrak{A}}$, $C_{\mathfrak{A}_{\mathcal{P}_0}}$, and $C_{\mathfrak{A}'_{\mathcal{P}_0}}$ coincide, and the localization at $\mathcal{P}_0$ provides an embedding $\mathbf{Spec}_{\mathcal{P}_0}(\mathfrak{A}) \longrightarrow \mathbf{Spec}(\mathfrak{A}^{\mathfrak{r}}) = \mathbf{Spec}(\mathfrak{A}_{\mathcal{P}_0}^{\mathfrak{r}})$. The category $C_{\mathfrak{A}^{\mathfrak{r}}}$ here is equivalent to the category of modules over the algebra $k(\xi_1, \ldots, \xi_n)[x_1^{\pm 1}, \ldots, x_n^{\pm 1}; \theta_1, \ldots, \theta_n]$ of skew Laurent polynomials in $x_1, \ldots, x_n$ with coefficients in the field $k(\xi_1, \ldots, \xi_n)$ of rational functions in $\xi_1, \ldots, \xi_n$.

C1.5. Heisenberg algebras. Recall that the Heisenberg algebra $\mathcal{H}_n$ (– the enveloping algebra of the Heisenberg Lie algebra) is an associative k-algebra generated by x_i, y_i, and z subject to the relations

$$[x_i, y_j] = \delta_{ij} z, \quad [x_i, z] = [x_i, x_j] = 0 = [y_i, y_j] = [y_i, z] \quad \text{for all } 1 \leq i, j \leq n. \tag{1}$$

Let $R = k[z, \xi_1, \ldots, \xi_n]$. The Heisenberg algebra $\mathcal{H}_n$ is a free right R-module with the basis formed by $\mathbf{x}^{\mathbf{s}}\mathbf{y}^{\mathbf{t}}$, where $\mathbf{s} \in \mathbb{Z}_+^n \ni \mathbf{t}$ are such that $\mathbf{s} \cdot \mathbf{t} = \sum_{1 \leq i \leq n} s_i t_i = 0$, $\mathbf{x}^{\mathbf{s}} = x_1^{s_1} \ldots x_n^{s_n}$ (see C1.2):

$$\mathcal{H}_n = \bigoplus_{\mathbf{s}, \mathbf{t} \in \mathbb{Z}_+^n,\ \mathbf{s} \cdot \mathbf{t} = 0} \mathbf{x}^{\mathbf{s}}\mathbf{y}^{\mathbf{t}} R \tag{2}$$

The multiplication is given by

$$\begin{aligned} & r\mathbf{x}^{\mathbf{s}}\mathbf{y}^{\mathbf{t}} = \mathbf{x}^{\mathbf{s}}\mathbf{y}^{\mathbf{t}}\vartheta^{\mathbf{t}-\mathbf{s}}(r) \quad \text{for all } r \in R; \\ & x_i y_i = \xi_i, \quad y_i x_i = \vartheta_i^{-1}(\xi_i) = \xi_i - z, \\ & [x_i, y_j] = [x_i, x_j] = [y_i, y_j] = 0 \quad \text{for all} \quad 1 \leq i, j \leq n,\ i \neq j. \end{aligned} \tag{3}$$

Here $\vartheta^{\mathbf{s}} = \vartheta_1^{s_1} \circ \ldots \circ \vartheta_n^{s_n}$ and ϑ_i, $1 \leq i \leq n$, are automorphisms of the algebra R defined by $\vartheta_i(\xi_j) = \xi_j + \delta_{ij} z$, $\vartheta_i(z) = z$.

The spectral picture corresponding to the embedding $R \hookrightarrow \mathcal{H}_n$ is recovered the same way (and in the same degree) as the spectrum of the Weyl algebra A_n regarded as a hyperbolic algebra over the ring of polynomials. We leave details to the reader.

C2. Remarks on enveloping algebras.

C2.1. The Harish-Chandra homomorphism and the highest weight simple modules. Let $\mathfrak{g}$ be a semisimple Lie algebra over a field k of zero characteristic. Fix its Cartan subalgebra $\mathfrak{h}$. We take $C_X = U(\mathfrak{h}) - mod$, $C_{\mathfrak{A}} = U(\mathfrak{g}) - mod$, and the functor $C_{\mathfrak{A}} \xrightarrow{\varphi_*} C_X$ corresponding to the embedding $U(\mathfrak{h}) \longrightarrow U(\mathfrak{g})$.

We consider the canonical grading $U(\mathfrak{g}) = \bigoplus_{\lambda \in \mathcal{Q}} U(\mathfrak{g})_\lambda$ defined by the adjoint action of $\mathfrak{g}$ on $U(\mathfrak{g})$ (cf. [D, 7.4]). The subalgebra $U(\mathfrak{g})_0$ is the centralizer of $U(\mathfrak{h})$ in $U(\mathfrak{g})$.

Let $\mathcal{P}$ be a point of $\mathbf{Spec}^0_{\mathfrak{c}}(X) = \mathbf{Spec}(X)$, and p the corresponding prime ideal of $U(\mathfrak{h})$. The category $C_{\mathfrak{A}_\mathcal{P}}$ is equivalent to the category of modules over the $\mathcal{Q}_\mathcal{P}$-graded subalgebra $U(\mathfrak{g})_\mathcal{P} = \bigoplus_{\lambda \in \mathcal{Q}_\mathcal{P}} U(\mathfrak{g})_\lambda$, where $\mathcal{Q}_\mathcal{P}$ is the subgroup of $\mathcal{Q}$ stabilizing $\mathcal{P}$ (i.e. the ideal p). In particular, the centralizer $U(\mathfrak{g})_0$ of $U(\mathfrak{h})$ stabilizes the subcategory $\mathcal{P} = U(\mathfrak{h})/p - mod$ for every point $\mathcal{P}$. For most of points $\mathcal{P}$, the subgroup $\mathcal{Q}_\mathcal{P}$ is trivial, hence the category $C_{\mathfrak{A}_\mathcal{P}}$ is naturally equivalent to the category $U(\mathfrak{g})_0 - mod$. In particular, $C_{\mathfrak{A}_\mathcal{P}}$ is equivalent to $U(\mathfrak{g})_0 - mod$ for all closed points $\mathcal{P}$ of $\mathbf{Spec}^0_{\mathfrak{c}}(X) = \mathbf{Spec}(X)$.

Set $C_{\mathfrak{A}_0} = U(\mathfrak{g})_0 - mod$. The Harish-Chandra homomorphism $U(\mathfrak{g})_0 \xrightarrow{\varphi_\mathcal{H}} U(\mathfrak{h})$ induces a full embedding $C_X \xrightarrow{\varphi_{\mathcal{H}*}} C_{\mathfrak{A}_0}$ which identifies the category C_X with a coreflective topologizing subcategory of $C_{\mathfrak{A}_0}$. Therefore, the embedding $\varphi_{\mathcal{H}*}$ determines an embedding $\mathbf{Spec}(X) \longrightarrow \mathbf{Spec}(\mathfrak{A}_0)$. So that every element $\mathcal{P}$ of $\mathbf{Spec}(X)$ is identified with the corresponding element of $\mathbf{Spec}^\mathcal{P}_{\mathfrak{c}}(\mathfrak{A}_\mathcal{P})$.

Let $M = U(\mathfrak{h})/p$. Then the composition of the embedding

$$C_X = U(\mathfrak{h}) - mod \longrightarrow U(\mathfrak{g})_0 - mod = C_{\mathfrak{A}_\mathcal{P}}$$

with the functor $\mathfrak{L}_\mathcal{P}$ assigns to M the highest weight module corresponding to the ideal p.

C2.1.1. Example. If $\mathfrak{g} = sl_2$, then $U(\mathfrak{g})$ is generated by indeterminates x, y, z subject to the relations

$$[x, y] = z, \quad [x, z] = \alpha x, \quad [y, z] = -\alpha y, \tag{1}$$

where α is a nonzero element of the base field k. Thus, $U(\mathfrak{h}) = k[z]$, $U(\mathfrak{g})_0 = k[z, \xi]$, and the Harish-Chandra homomorphism $k[z, \xi] \longrightarrow k[z]$ assigns to every polynomial $f(z, \xi)$ the element $f(z, 0)$ of $k[z]$. The corresponding map $Spec(U(\mathfrak{h})) \longrightarrow Spec(U(\mathfrak{g})_0)$ assigns to any prime ideal p in $k[z]$ the prime ideal (p, ξ). If $\mathcal{P}$ is a closed point (i.e. p is a maximal ideal), then $U(\mathfrak{g})_0$ is the stabilizer of $\mathcal{P}$ in the sense that $C_{\mathfrak{A}_{\mathcal{P}}}$ is equivalent to the category $U(\mathfrak{g})_0 - mod = k[z, \xi] - mod$. The functor $\mathfrak{f}_{\mathcal{P}}^*$ is isomorphic to $U(\mathfrak{g}) \otimes_{U(\mathfrak{g})_0} -$. The functor $\mathfrak{L}_{\mathcal{P}}$ assigns to the simple $U(\mathfrak{g})_0$-module $M = U(\mathfrak{g})_0/(p, \xi) \simeq U(\mathfrak{h})/p$ the corresponding Verma module $U(\mathfrak{g})/(p, y) = \bigoplus_{m \geq 0} x^m M$, if $p \neq (z - n/2)$ for any nonnegative integer n.

If $p = (z - n/2)$ for some nonnegative integer n (there is only one such integer n), then the module $M = k[z]/(z - n/2)$ is one-dimensional and $\mathfrak{L}_{\mathcal{P}}(M) = \bigoplus_{0 \leq m \leq n} x^m M$ has dimension $n + 1$ over the field k. In particular, if $n = 0$, then $\mathfrak{L}_{\mathcal{P}}(M)$ is the unique one-dimensional representation of $U(sl_2)$.

Set $R = k[z, \xi] = U(\mathfrak{g})_0$. The relations (1) are equivalent to the relations

$$xy = \xi, \quad yx = \theta^{-1}(\xi); \quad xr = \theta(r)x, \quad ry = y\theta(r) \tag{2}$$

for all $r \in R$. Here θ is the automorphism of the algebra R is defined by $\theta(f)(z, \xi) = f(z + \alpha, \xi + z + \alpha)$. In terminology of [R, Ch.II], (2) is the representation of $U(sl_2)$ as a *hyperbolic* ring over R. We take $C_X = R - mod$, $C_{\mathfrak{A}} = U(sl_2) - mod$ and the functor $C_{\mathfrak{A}} \xrightarrow{\varphi_*} C_X$ corresponding to the embedding $R \longrightarrow U(sl_2)$. Application the functors $\mathfrak{L}_{\mathcal{P}}$ gives a fairly complete description of the rest of the picture. Closed points of $\mathbf{Spec}(X) \simeq Spec(R)$ have trivial stabilizer, and the functor $\mathfrak{L}_{\mathcal{P}}$ for such point $\mathcal{P}$ coincides with the induction functor. By 2.2, $\mathfrak{L}_{\mathcal{P}}$ maps a simple module R/p representing $\mathcal{P}$ to a simple $U(sl_2)$-module. If $\mathcal{P}$ is a curve, then $[\mathfrak{L}_{\mathcal{P}}(R/p)]$ is a noncommutative curve in $\mathbf{Spec}(\mathfrak{A})$. If $\mathcal{P}$ is the generic point, then we localize at the multiplicative set of nonzero elements of R and reduce the problem to the description of simple modules over a skew polynomial ring $k(z, \xi)[x, \theta]$ which is a Eucledian domain. Therefore, its simple modules correspond to irreducible (skew) polynomials. See [R, II.4.3] for details.

If $\mathfrak{g}$ has a higher rank (starting from $\mathfrak{g} = sl_3$), then $U(\mathfrak{g})_0$ is a rather complicated noncommutative subalgebra of $U(\mathfrak{g})$. In particular, it is not clear how to approach to the description of $\mathbf{Spec}_{\mathfrak{c}}^{\mathcal{P}}(\mathfrak{A}_{\mathcal{P}})$.

C2.1.2. Remark. Similar facts on the connection of the Harish-Chandra homomorphism and highest weight simple modules hold for quantized enveloping algebras $U_q(\mathfrak{g})$ in the case when q is not a root of one [XT]. Also, $U_q(sl_2)$ has a hyperbolic structure over the ring $R = k[z, z^{-1}, \xi]$ which allows to get a description to the spectrum of $\mathfrak{A} = \mathbf{Sp}(U_q(sl_2)$ (see [R,II.4.2]).

C3. Associated points and primary decomposition.

C3.1. Associated points. Fix an abelian category C_X. For every $M \in ObC_X$, the set $\mathfrak{Ass}(M)$ of associated points of M can be described as the set of all $\mathcal{Q} \in \mathbf{Spec}^0_{\mathfrak{c}}(X)$ such that there exists a nonzero monomorphism $L \hookrightarrow M$ with L from $\mathcal{Q} \cap^{\mathfrak{c}} \widehat{\mathcal{Q}}^{\perp}$.

We define $\mathfrak{Ass}^{1,1}_{\mathfrak{t}}(M)$ as the set of all $\mathcal{P} \in \mathfrak{Th}(X)$ such that there exists a nonzero monomorphism $L \hookrightarrow M$ with L from $\mathcal{P}^{\mathfrak{t}} \cap \mathcal{P}^{\perp}$. It follows that $\mathfrak{Ass}^{1,1}_{\mathfrak{t}}(M) \subseteq \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$.

We define $\mathfrak{Ass}^{-}(M)$ as the set of all $\mathcal{P} \in \mathfrak{Th}(X)$ such that there exists a nonzero monomorphism $L \hookrightarrow M$ with L from $\mathcal{P}_{\circledast} = \mathcal{P}^{\circledast} \cap \mathcal{P}^{\perp}$.

It follows that $\mathfrak{Ass}^{-}(M) \subseteq \mathbf{Spec}^{-}(X)$.

We denote by $\mathfrak{Ass}^{0,1}(M)$ the set of elements $\mathcal{P}_{\circledast}$ of $\mathbf{Spec}_{-}(X)$ such that there is a nonzero subobject $L \hookrightarrow M$ with $L \in Ob\mathcal{P}_{\circledast}$.

Finally, $\mathfrak{Ass}_{\mathfrak{L}}(M)$ is the set of all $\mathcal{P} \in \mathfrak{Th}(X)$ such that there exists a nonzero monomorphism $L \hookrightarrow M$ with L from $\mathcal{P}_{\star} = \mathcal{P}^{\star} \cap \mathcal{P}^{\perp}$. In particular, $\mathfrak{Ass}_{\mathfrak{L}}(M) \subseteq \mathbf{Spec}^{1,1}_{\mathfrak{Th}}(X)$.

We denote by $\mathfrak{Ass}^{0,1}_{\mathfrak{Th}}(M)$ the set whose elements are $\mathcal{P}_{\star} = \mathcal{P}^{\star} \cap \mathcal{P}^{\perp}$ of $\mathbf{Spec}^{0,1}_{\mathfrak{Th}}(X)$ such that M has a nonzero subobject which belongs to $\mathcal{P}_{\star}$.

It follows from these definitions that the commutative diagram

$$\begin{array}{ccccc}
\mathbf{Spec}^0_{\mathfrak{c}}(X) & \xrightarrow{\ \alpha\ } & \mathbf{Spec}_{-}(X) & \xrightarrow{\ \beta\ } & \mathbf{Spec}^{0,1}_{\mathfrak{Th}}(X) \\
\wr\downarrow & & \wr\downarrow & & \downarrow\wr \\
\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X) & \longrightarrow & \mathbf{Spec}^{-}(X) & \longrightarrow & \mathbf{Spec}^{1,1}_{\mathfrak{Th}}(X)
\end{array} \qquad (1)$$

(see C2.6(5)) induces for any object M of the category C_X a commutative diagram

$$\begin{array}{ccccc}
\mathfrak{Ass}(M) & \longrightarrow & \mathfrak{Ass}^{0,1}(M) & \longrightarrow & \mathfrak{Ass}^{0,1}_{\mathfrak{Th}}(M) \\
\wr\downarrow & & \wr\downarrow & & \downarrow\wr \\
\mathfrak{Ass}^{1,1}_{\mathfrak{t}}(M) & \longrightarrow & \mathfrak{Ass}^{-}(M) & \longrightarrow & \mathfrak{Ass}_{\mathfrak{L}}(M)
\end{array} \qquad (2)$$

whose horizontal arrows are embeddings and the vertical arrows are isomorphisms.

It follows that

$$\begin{aligned}
&\mathfrak{Ass}^{1,1}_{\mathfrak{t}}(M) = \mathfrak{Ass}^{-}(M) \bigcap \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X) = \mathfrak{Ass}_{\mathfrak{L}}(M) \bigcap \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X) \quad \text{and} \\
&\mathfrak{Ass}^{-}(M) = \mathfrak{Ass}^{1,1}_{\mathfrak{Th}}(M) \bigcap \mathbf{Spec}^{-}(X).
\end{aligned} \qquad (3)$$

C3.2. Remarks. (a) If X has a Gabriel-Krull dimension, then, by [R6, 8.7.1], the inclusion map $\mathbf{Spec}^{-}(X) \longrightarrow \mathbf{Spec}^{1,1}_{\mathfrak{Th}}$ in the diagram (1) is an isomorphism, hence the right horizontal arrows in the diagrams (1) and (2) are isomorphisms.

(b) The correspondence $M \longmapsto \mathfrak{Ass}_{\mathfrak{L}}(M)$ is studied in [R6, 10.8–10.10], where it is shown that $\mathfrak{Ass}_{\mathfrak{L}}(M)$ enjoys all general properties of associated points in the context of commutative algebra. Similar facts hold for the map $M \longmapsto \mathfrak{Ass}^{0,1}(M)$.

Here we sketch the facts about $M \longmapsto \mathfrak{Ass}(M)$ imitating [R6, 10.8–10.10] whenever it is possible to do.

C3.3. Proposition. *(a) For any exact sequence*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0,$$

$$\mathfrak{Ass}(M') \subseteq \mathfrak{Ass}(M) \subseteq \mathfrak{Ass}(M') \bigcup \mathfrak{Ass}(M'').$$

(b) Suppose X has the property (sup). Let an object M of C_X be a supremum of an ascending family, Ξ, of its subobjects. Then

$$\mathfrak{Ass}(M) = \bigcup_{M' \in \Xi} \mathfrak{Ass}(M').$$

(c) For every object M of C_X, any exact localization, $Y \xrightarrow{u} X$, induces an injective map $\mathfrak{Ass}(M) \cap \mathcal{U}_{\mathfrak{T}}(Ker(u^*)) \longrightarrow \mathfrak{Ass}(u^*(M))$. Here $\mathcal{U}_{\mathfrak{T}}(\mathbb{S}) = \{\mathbb{T} \in \mathfrak{T}(X) \mid \mathbb{T} \not\subseteq \mathbb{S}\}$.

(d) If M belongs to $Spec^0_{\mathfrak{c}}(X)$, then $\mathfrak{Ass}(M) = \{[M]\}$.

Proof. (a) The inclusion $\mathfrak{Ass}(M') \subseteq \mathfrak{Ass}(M)$ follows from definitions.

Let $\mathcal{P} \in \mathfrak{Ass}(M)$, i.e. there exists a nonzero subobject, L, of M such that $[L] = \mathcal{P}$. Suppose $L' = L \cap M' \neq 0$. Then L' is a nonzero subobject of M' and L. The latter implies that $[L'] = [L] = \mathcal{P}$, hence $\mathcal{P} \in \mathfrak{Ass}(M')$. If $L' = 0$, then the composition of $L \hookrightarrow M$ and the canonical epimorphism $M \longrightarrow M''$ is a monomorphism, hence $\mathcal{P} \in \mathfrak{Ass}(M'')$. This proves the inclusion $\mathfrak{Ass}(M) \subseteq \mathfrak{Ass}(M') \bigcup \mathfrak{Ass}(M'')$.

(b) It follows from (a) that the inclusion $\mathfrak{Ass}(M) \supseteq \bigcup_{M' \in \Xi} \mathfrak{Ass}(M')$ holds without any additional conditions on X.

Let $\mathcal{P} \in \mathfrak{Ass}(M)$, i.e. M has a nonzero subobject L such that $[L] = \mathcal{P}$. Since X has the property (sup), $L \cap M' \neq 0$ for some $M' \in \Xi$. Therefore $\mathcal{P} \in \mathfrak{Ass}(M')$ (see the argument in (a) above). This verifies the inverse inclusion, $\mathfrak{Ass}(M) \subseteq \bigcup_{M' \in \Xi} \mathfrak{Ass}(M')$.

(c) Let u^* be an inverse image functor of $Y \xrightarrow{u} X$. Set $Ker(u^*) = \mathbb{S}$. The claim is that the injective map $\mathcal{U}_{\mathfrak{T}}(X) \longrightarrow \mathfrak{Th}(Y)$, $\mathcal{P} \longrightarrow \mathcal{P}/\mathbb{S}$, induces a (forcibly injective) map $\mathfrak{Ass}(M) \cap \mathcal{U}_{\mathfrak{Th}}(\mathbb{S}) \longrightarrow \mathfrak{Ass}(u^*(M))$.

Let $\mathcal{P} \in \mathfrak{Ass}(M) \cap \mathcal{U}_{\mathfrak{T}}(\mathbb{S})$, that is $\mathcal{P} \not\subseteq \mathbb{S}$, and there exists a nonzero subobject L of M such that $[L] = \mathcal{P}$. Since $\mathcal{P} \not\subseteq \mathbb{S}$, the object L is $\mathbb{S}$-torsion free. Therefore, $u^*(L)$ is a nonzero subobject of $u^*(M)$ which belongs to $Spec^0_{\mathfrak{c}}(X)$.

(d) The assertion follows from the definition of $Spec^0_{\mathfrak{c}}(X)$. ∎

C3.4. Corollary. *(i) For any finite set, $\{M_i \mid i \in J\}$, of objects of C_X,*

$$\mathfrak{Ass}_{\mathfrak{L}}(\bigoplus_{i\in J} M_i) = \bigcup_{i\in J} \mathfrak{Ass}_{\mathfrak{L}}(M_i).$$

If X has the property (sup), then the finiteness condition can be dropped.

(ii) Let $\{L_i \mid i \in J\}$ be a finite set of subobjects of an object M such that $\bigcap_{i\in J} L_i = 0$. Then

$$\mathfrak{Ass}_{\mathfrak{L}}(M/(\bigcap_{i\in J} L_i)) \subseteq \bigcup_{i\in J} \mathfrak{Ass}_{\mathfrak{L}}(M/L_i).$$

Proof. (i) For a finite set $\{M_i \mid i \in J\}$, the assertion follows from C3.3(a). The infinite case is a consequence of C3.3(b).

(ii) The assertion follows from (i) and C3.3(a) applied to the canonical monomorphism $M/(\bigcap_{i\in J} L_i) \longrightarrow \bigoplus_{i\in J} M/L_i$. ∎

C3.5. Corollary. *The full subcategory, $C_{X^{\emptyset}_{\mathfrak{Ass}}}$, of the category C_X whose objects, M, have no associated points, $\mathfrak{Ass}_{\mathfrak{L}}(M) = \emptyset$, is closed under extensions, taking subobjects, and colimits of filtered diagrams of monoarrows.*

Proof. The assertion is a consequence of C3.3(a) and (b). ∎

C3.6. Proposition. *Let $Y \xrightarrow{u} X$ be an exact localization such that $\mathbb{S} = Ker(u^*)$ is a coreflective subcategory of the category C_X. Let $\mathcal{P} \in \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$ and $\mathbb{S} \subseteq \mathcal{P}$. Let M be an object of C_X such that $\mathfrak{Ass}(L) \neq \emptyset$ for any nonzero subobject, L, of M. Then the following conditions are equivalent:*

(a) $\mathfrak{Ass}^{1,1}_{\mathfrak{t}}(M) = \{\mathcal{P}\}$;

(b) $\mathfrak{Ass}^{1,1}_{\mathfrak{t}}(u^*(M)) = \{\mathcal{P}/\mathbb{S}\}$ and M is $\mathbb{S}$-torsion free.

Proof. (a)⇒(b). Let $\mathfrak{t}_{\mathbb{S}}M$ denote the $\mathbb{S}$-torsion of M. If $\mathfrak{t}_{\mathbb{S}}M \neq 0$, then, by hypothesis, $\mathfrak{Ass}(\mathfrak{t}_{\mathbb{S}}M) \neq \emptyset$, i.e. $\mathfrak{Ass}(\mathfrak{t}_{\mathbb{S}}M) = \{\mathcal{P}\}$. The latter means that $\mathfrak{t}_{\mathbb{S}}M$ has a nonzero subobject L such that to $\langle L\rangle = \mathcal{P}$; in particular, L is $\mathcal{P}$-, hence $\mathbb{S}$-torsion free, which contradicts to that L is a nonzero object of the subcategory $\mathbb{S}$.

Since M is $\mathbb{S}$-torsion free, it follows from C3.3(c) that $\mathfrak{Ass}^{1,1}_{\mathfrak{t}}(u^*(M)) = \{\mathcal{P}/\mathbb{S}\}$.

(b)⇒(a). There is a subobject N of M such that $\langle u^*(N)\rangle = \mathcal{P}/\mathbb{S}$. By hypothesis, since $N \neq 0$, $\mathfrak{Ass}(N) \neq \emptyset$; i.e. there exists a subobject $L \hookrightarrow N$ such that $[L] \in \mathbf{Spec}^0_{\mathfrak{c}}(X)$. Since L is $\mathcal{P}$-torsion free, it follows that $\mathcal{P} = \langle L\rangle$. ∎

C3.7. Proposition. *Suppose X has the property (sup). Let $M \in ObC_X$, and let Φ be a subset of $\mathfrak{Ass}(M)$. Then there exists a subobject $L \longrightarrow M$ such that*

$$\mathfrak{Ass}(M/L) = \mathfrak{Ass}(M) - \Phi \quad \textit{and} \quad \mathfrak{Ass}(L) = \Phi. \tag{4}$$

Proof. (a) Let $\mathfrak{D}_\Phi$ be the set of subobjects, M', of M such that $\mathfrak{Ass}(M') \subseteq \Phi$. The set $\mathfrak{D}_\Phi$ is not empty, because it contains the zero subobject. It follows from C3.3(b) that $sup\Xi \in \mathfrak{D}_\Phi$ for every filtered subset Ξ of $\mathfrak{D}_\Phi$. Therefore, by Zorn's lemma, there exists a maximal element (subobject), L, in $\mathfrak{D}_\Phi$. We claim that the subobject L satisfies the conditions (4). Thanks to C3.3(a), it suffices to show that $\mathfrak{Ass}(M/L) \subseteq \mathfrak{Ass}(M) - \Phi$.

(b) Let $\mathcal{P} \in \mathfrak{Ass}(M/L)$, i.e. M/L has a subobject, $N \longrightarrow M/L$ such that $\mathcal{P} = [N]$. Consider the short exact sequence

$$0 \longrightarrow L \longrightarrow \widetilde{N} = M \times_{M/L} N \longrightarrow N \longrightarrow 0. \tag{5}$$

associated with $N \longrightarrow M/L$. By C3.3(a), $\mathfrak{Ass}(\widetilde{N}) \subseteq \mathfrak{Ass}(L) \bigcup \mathfrak{Ass}(N)$. By C3.3(d), $\mathfrak{Ass}(N) = \{\mathcal{P}\}$. Since L is a maximal element of $\mathfrak{D}_\Phi$ and a proper subobject of $\widetilde{N}$, the latter does not belong to $\mathfrak{D}_\Phi$. Therefore $\mathcal{P} \in \mathfrak{Ass}(\widetilde{N}) - \Phi$. ∎

C3.8. Primary decomposition.

C3.8.1. Definition. Let M be an object of an abelian category C_X. We call a subobject N of M *primary*, or *$\mathcal{P}$-primary*, if $\mathfrak{Ass}(M/N)$ consists of one element, $\mathcal{P}$.

C3.8.2. Proposition. *Let $\{N_i \mid i \in J\}$ be a finite set of $\mathcal{P}$-primary subobjects of an object M of an abelian category C_X. Then $\bigcap\limits_{i \in J} N_i$ is a $\mathcal{P}$-primary subobject of M.*

Proof. The fact follows from C3.4(ii). ∎

C3.8.3. Definition. Let N be a subobject of an object M of the category C_X. A *primary decomposition of $N \hookrightarrow M$* is a finite set, $\{N_i \mid i \in J\}$, of primary subobjects of M such that N is a subobject of $\bigcap\limits_{i \in J} N_i$ and $\mathfrak{Ass}(\bigcap\limits_{i \in J} N_i/N) = \emptyset$.

C3.8.3.1. Note. It follows from this definition and C3.4(ii) that if a subobject N of M has a primary decomposition, then $\mathfrak{Ass}(M/N)$ is a subset of $\{\mathcal{P}_i \mid i \in J\}$, in particular, $\mathfrak{Ass}(M/N$ is finite. Here $\mathfrak{Ass}(M/N_i) = \{\mathcal{P}_i\}$.

C3.8.4. Proposition. *Let N be a subobject of an object M such that $\mathfrak{Ass}(M/N)$ is finite. Then there exists a primary decomposition, $\{N_\mathcal{P} \mid \mathcal{P} \in \mathfrak{Ass}(M/N)\}$, such that $N_\mathcal{P}$ is $\mathcal{P}$-primary for every $\mathcal{P} \in \mathfrak{Ass}(M/N)$.*

Proof. Replacing M by M/N, we can and will assume that $N = 0$. By C3.7, for every $\mathcal{P} \in \mathfrak{Ass}(M)$, there exists a subobject $N_{\mathcal{P}}$ of M such that $\mathfrak{Ass}(M/N_{\mathcal{P}}) = \{\mathcal{P}\}$ and $\mathfrak{Ass}(N_{\mathcal{P}}) = \mathfrak{Ass}(M) - \{\mathcal{P}\}$. Set $M_0 = \bigcap_{\mathcal{P}\in\mathfrak{Ass}(M)} N_{\mathcal{P}}$. For each $\mathcal{P} \in \mathfrak{Ass}(M)$, we have the inclusion $\mathfrak{Ass}(M_0) \subseteq \mathfrak{Ass}(N_{\mathcal{P}})$, hence $\mathfrak{Ass}(M_0) = \emptyset$. ■

C3.8.5. Definition. Let N be a subobject of an object M such that $\mathfrak{Ass}(M/N)$ is finite. Let $\{N_i \mid i \in J\}$ be a primary decomposition of N in M with $\mathfrak{Ass}(M/N_i = \{\mathcal{P}_i\}$. The primary decompsition $\{N_i \mid i \in J\}$ is called *reduced* if

(a) for any $i \in J$, $\mathfrak{Ass}(\bigcap_{J\ni j\neq i} N_j/\bigcap_{j\in J} N_j) \neq \emptyset$; in particular, the intersection $\bigcap_{J\ni j\neq i} N_j$ is not a subobject of N_i;

(b) if $i \neq j$, then $\mathcal{P}_i \neq \mathcal{P}_j$.

C3.8.5.1. Note. Starting with an arbitrary primary decomposition, one can obtain a reduced primary decomposition as follows. Let $\{N_i \mid i \in J\}$ be any primary decomposition of $N \hookrightarrow M$ with $\mathfrak{Ass}(M/N_i) = \{\mathcal{P}_i\}$, $i \in J$. Set $\Phi = \{\mathcal{P}_i \mid i \in J\}$. Let J_0 is a minimal element of the set of subsets, I, of J such that $\{N_i \mid i \in I\}$ is a primary decomposition. Clearly, $\{N_i \mid i \in J_0\}$ satisfies the condition (a). For each $\mathcal{P} \in \Phi$, let $N_{\mathcal{P}} = \bigcap_{\mathcal{P}_i=\mathcal{P}} N_i$. By C3.8.2, $N_{\mathcal{P}} \hookrightarrow M$ is $\mathcal{P}$-primary. Since $\bigcap_{\mathcal{P}\in\Phi} N_{\mathcal{P}} = \bigcap_{i\in J} N_i$, the set of subobjects $\{N_{\mathcal{P}} \mid \mathcal{P} \in \Phi\}$ is a reduced primary decomposition of $N \hookrightarrow M$.

C3.8.6. Proposition. *Let N be a subobject of an object M such that $\mathfrak{Ass}(M/N)$ is finite. Let $\{N_i \mid i \in J\}$ be a primary decomposition of N in M with $\mathfrak{Ass}(M/N_i) = \{\mathcal{P}_i\}$.*

(i) The following conditions are equivalent:

(a) The decomposition $\{N_i \mid i \in J\}$ is reduced.

(b) All $\mathcal{P}_i$ belong to $\mathfrak{Ass}(M/N)$ and $\mathcal{P}_i \neq \mathcal{P}_j$ if $i \neq j$.

(ii) If the equivalent conditions (a), (b) are fulfilled, then

$$\begin{aligned}&\mathfrak{Ass}(M/N) = \{\mathcal{P}_i \mid i \in J\} \quad \textit{and}\\ &\mathfrak{Ass}(N_i/N) = \{\mathcal{P}_j \mid j \in J,\ j \neq i\} \quad \textit{for all} \ \ i \in J.\end{aligned}$$

Proof. (a)⇒(b). Let $\{N_i \mid i \in J\}$ be a reduced primary decomposition. By C3.8.3.1, $\mathfrak{Ass}(M/N)$ is a subset of $\{\mathcal{P}_i \mid i \in J\}$. Set $N_i^\vee = \bigcap_{J\ni j\neq i} N_j$. We can and will assume that $N = \bigcap_{j\in J} N_j = N_i^\vee \cap N_i$. Since the decomposition $\{N_i \mid i \in J\}$ is reduced, $\mathfrak{Ass}(N_i^\vee/N) \neq \emptyset$.

Because $N_i^{\vee}/N$ is isomorphic to the subobject $sup(N_i^{\vee}, N_i)/N_i$ of M/N_i, this implies that $\mathfrak{Ass}(N_i^{\vee}/N) = \{\mathcal{P}_i\}$, whence the inverse inclusion: $\{\mathcal{P}_i \mid i \in J\} \subseteq \mathfrak{Ass}(M/N)$.

(b)$\Rightarrow$(a). If the condition (b) holds, $\{N_j \mid j \in J - \{i\}\}$ cannot be a primary decomposition, because this would imply that $\mathcal{P}_i \notin \mathfrak{Ass}(M/N)$. Therefore the primary decomposition $\{N_i \mid i \in J\}$ of $N \hookrightarrow M$ is reduced.

The equality $\mathfrak{Ass}(M/N) = \{\mathcal{P}_i \mid i \in J\}$ is already established. It remains to show that for any $i \in J$, $\mathfrak{Ass}(N_i/N) = \{\mathcal{P}_j \mid j \in J,\ j \neq i\}$. Applying C3.3(a) to the exact sequence

$$0 \longrightarrow N_i/N \longrightarrow M/N \longrightarrow M/N_i \longrightarrow 0,$$

we obtain inclusions

$$\mathfrak{Ass}(N_i/N) \subseteq \mathfrak{Ass}(M/N) \subseteq \mathfrak{Ass}(N_i/N) \bigcup \mathfrak{Ass}(M/N_i) = \mathfrak{Ass}(N_i/N) \bigcup \{\mathcal{P}_i\}.$$

This and the equality $\mathfrak{Ass}(M/N) = \{\mathcal{P}_j \mid j \in J\}$ imply that

$$\{\mathcal{P}_j \mid j \in J - \{i\}\} \subseteq \mathfrak{Ass}(N_i/N) \subseteq \{\mathcal{P}_j \mid j \in J\}.$$

On the other hand, since $N = \bigcap_{j \in J - \{i\}} (N_i \cap N_j)$, we have an inclusion

$$\mathfrak{Ass}(N_i/N) \subseteq \bigcup_{j \in J - \{i\}} \mathfrak{Ass}(N_i/(N_i \cap N_j)).$$

But, $N_i/(N_i \cap N_j)$ is isomorphic to the subobject $sup(N_i, N_j)/N_j$ of the object M/N_j, hence $\mathfrak{Ass}(N_i/(N_i \cap N_j)) \subseteq \mathfrak{Ass}(M/N_j) = \{\mathcal{P}_j\}$. This gives the inverse inclusion: $\mathfrak{Ass}(N_i/N) \subseteq \{\mathcal{P}_j \mid j \in J - \{i\}\}$. ■

C3.8.7. Corollary. *Let $\{N_i \mid i \in J\}$ be a primary decomposition of a subobject N of an object M. Then $Card(\mathfrak{Ass}(M/N)) \leq Card(J)$. The decomposition $\{N_i \mid i \in J\}$ is reduced iff $Card(\mathfrak{Ass}(M/N)) = Card(J)$.*

Proof. Following the procedure described in C3.8.5.1, one can obtain, starting from $\{N_i \mid i \in J\}$, a reduced primary decomposition, $\{\widetilde{N}_j \mid j \in I\}$ such that $Card(I) \leq Card(J)$. The rest follows from C3.8.6. ■

For any object M of the category C_X, let $\mathfrak{D}_{\wp}(M)$ denote the set of reduced primary decompositions of $0 \hookrightarrow M$. By C3.8.6, each element of $\mathfrak{D}_{\wp}(M)$ is a set, $\{N_{\mathcal{P}} \mid \mathcal{P} \in \mathfrak{Ass}(M)\}$ of subobjects of M such that $\mathfrak{Ass}(M/N_{\mathcal{P}}) = \{\mathcal{P}\}$ and $\mathfrak{Ass}\big(\bigcap_{\mathcal{P} \in \mathfrak{Ass}(M)} N_{\mathcal{P}}\big) = \emptyset$.

C3.8.8. Proposition. *Let* $\{N_{\mathcal{P}} \mid \mathcal{P} \in \mathfrak{Ass}(M)\}$ *and* $\{\widetilde{N}_{\mathcal{P}} \mid \mathcal{P} \in \mathfrak{Ass}(M)\}$ *be two elements of* $\mathfrak{D}_{\wp}(M)$*, and let* Φ *be a subset of* $\mathfrak{Ass}(M)$*. Then* $\{N_{\mathcal{P}} \mid \mathcal{P} \in \Phi\} \bigcup \{\widetilde{N}_{\mathcal{P}} \mid \mathcal{P} \in \mathfrak{Ass}(M) - \Phi\}$ *is an element of* $\mathfrak{D}_{\wp}(M)$*.*

Proof. Set $N_\Phi = \bigcap\limits_{\mathcal{P}\in\Phi} N_{\mathcal{P}}$ and $\widetilde{N}_\Phi^\vee = \bigcap\limits_{\mathcal{P}\in\mathfrak{Ass}(M)-\Phi} \widetilde{N}_{\mathcal{P}}$. Since $\mathfrak{Ass}(M/N_{\mathcal{P}}) = \{\mathcal{P}\}$ and $\mathfrak{Ass}(M/\widetilde{N}_{\mathcal{P}}) = \{\mathcal{P}\}$ for all $\mathcal{P} \in \mathfrak{Ass}(M)$, it suffices to verify (thanks to C3.8.6) that $\mathfrak{Ass}(N_\Phi \bigcap \widetilde{N}_\Phi^\vee) = \emptyset$.

By C3.8.6(ii), $\mathfrak{Ass}(N_{\mathcal{P}}) = \mathfrak{Ass}(M) - \{\mathcal{P}\}$, in particular, $\mathcal{P} \notin \mathfrak{Ass}(N_{\mathcal{P}})$. Therefore, every element of Φ does not belong to $\mathfrak{Ass}(N_\Phi)$, i.e. $\Phi \bigcap \mathfrak{Ass}(N_\Phi) = \emptyset$. Similarly, $(\mathfrak{Ass}(M) - \Phi) \bigcap \mathfrak{Ass}(\widetilde{N}_\Phi^\vee) = \emptyset$. Thus, $\mathfrak{Ass}(N_\Phi \bigcap \widetilde{N}_\Phi^\vee) \subseteq \Phi \bigcap (\mathfrak{Ass}(M) - \Phi) = \emptyset$. ■

C4. Monads and localizations. Differential monads.

C4.1. Localizations compatible with monadic morphisms. Fix a monadic morphism $X \xrightarrow{f} Z$ and a Serre localization $U \xrightarrow{u} Z$ (i.e. $C_Z \xrightarrow{u^*} C_U$ is the localization at a Serre subcategory) compatible with f. Here *compatible* means that the functor $F_f = f_* f^*$ maps $\Sigma_{u^*} \stackrel{\text{def}}{=} \{s \in HomC_Z \mid u^*(s) \in Iso(C_U)\}$ to Σ_{u^*}; or, equivalently, there exists a functor $C_U \xrightarrow{F_U} C_U$ such that $u^* \circ F_f = F_U \circ u^*$. Thanks to the universal property of localizations, the functor F_U is determined uniquely by the latter equality. The monad structure $F_f^2 \xrightarrow{\mu_f} F_f$ induces a monad structure $F_U \xrightarrow{\mu} F_U$, hence we obtain a monad $\mathcal{F}_U = (F_U, \mu)$. The localization functor u^* induces a functor $(\mathcal{F}_f/Z)-mod \xrightarrow{\widetilde{u}^*} (\mathcal{F}_U/U)-mod$ which maps an $\mathcal{F}_f$-module $(M, F_f(M) \xrightarrow{\xi} M)$ to the $\mathcal{F}_U$-module $(u^*(M), F_U u^*(M) \xrightarrow{u^*(\xi)} u^*(M))$.

It is easy to see that $\widetilde{u}^*$ is (isomorphic to) an exact localization and $Ker(\widetilde{u}^*)$ is generated by all $\mathcal{F}_f$-modules (M, ξ) with $M \in ObKer(u^*)$.

Suppose now that the localization φ is continuous, and let u_* is its direct image functor. The equality $F_U \circ u^* = u^* \circ F_f$ implies an isomorphism $u^* F_f u_* = F_U u^* u_* \xrightarrow{F_U \epsilon_u} F_U$, where ϵ_u is an adjunction isomorphism $u^* u_* \longrightarrow Id_{C_U}$. The compatibility of F_f with the localization functor u^* means precisely that the morphism $u^* F_f \xrightarrow{u^* F_f \eta_u} u^* F_f u_* u^*$, where η_u is an adjunction arrow $Id_{C_Z} \longrightarrow u_* u^*$, is an isomorphism. This isomorphism allows to write the multiplication $\widetilde{\mu}$ on $u^* F_f u_*$ as the composition of the isomorphism

$$(u^* F_f u_*)^2 = (u^* F_f u_* u^*) F_f u_* \xrightarrow{\sim} u^* F_f^2 u_* \quad \text{and} \quad u^* F_f^2 u_* \xrightarrow{u^* \mu_f u_*} u^* F_f u_*.$$

One can show that $\widetilde{\mu}$ is a monad structure on $u^* F_f u_*$ and the canonical isomorphism $u^* F_f u_* \xrightarrow{\sim} F_U$ described above is a monad isomorphism $(u^* F_f u_*, \widetilde{\mu}) \xrightarrow{\sim} (F_U, \mu)$.

One of the consequences of this isomorphism is a description of a canonical right adjoint $\widetilde{u}_*$ to the localization functor $\mathcal{F}_f - mod \xrightarrow{\widetilde{u}^*} \mathcal{F}_U - mod$.

In fact, let $\mathfrak{F}_U$ denote the monad $(u^*F_f u_*, \widetilde{\mu})$. Every morphism $u^*F_f u_*(M) \xrightarrow{\xi} M$ determines via adjunction (and is determined by) a morphism $F_f(u_*(M)) \xrightarrow{\widehat{\xi}} u_*(M)$. If ξ is an $(u^*F_f u_*, \widetilde{\mu})$-module structure, then $\widehat{\xi}$ is an $\mathcal{F}_f$-module structure. This defines a functor $\mathfrak{F}_U - mod \xrightarrow{\widetilde{u}_*} \mathcal{F}_f - mod$. The functor $\widetilde{u}_*$ is a right adjoint to the functor $\mathcal{F}_f - mod \xrightarrow{\widetilde{u}_*} \mathfrak{F}_U - mod$ which maps an $\mathcal{F}_f$-module (M, ζ) to the $\mathfrak{F}_U$-module $(u^*(M), \zeta_u)$, where ζ_u is the composition of the isomorphism $u^*F_f u_* u^*(M) \xrightarrow{\sim} u^*F_f(M)$ and $u^*F_f(M) \xrightarrow{u^*(\zeta)} u^*(M)$. One can verify that the adjunction morphisms $u^*u_* \xrightarrow{\epsilon_u} Id_{C_U}$ and $Id_{C_Z} \xrightarrow{\eta_u} u_*u^*$ give rise to the adjunction morphisms $\widetilde{u}^*\widetilde{u}_* \longrightarrow Id_{\mathfrak{F}_U - mod}$ and $Id_{\mathcal{F}_f - mod} \longrightarrow \widetilde{u}_*\widetilde{u}^*$. In particular, $\widetilde{u}^*\widetilde{u}_* \longrightarrow Id_{\mathfrak{F}_U - mod}$ is an isomorphism, which shows that $\widetilde{u}^*$ is a localization. It follows from this description that the diagram

$$\begin{array}{ccc}
\mathfrak{F}_U - mod & \xrightarrow{\widetilde{u}_*} & \mathcal{F}_f - mod \\
\mathfrak{f}_{u*} \downarrow & & \downarrow \mathfrak{f}_* \\
C_U & \xrightarrow{u_*} & C_Z
\end{array}$$

quasi-commutes. Here the vertical arrows are forgetful functors.

C4.1.1. Lemma. *Let $U \xrightarrow{u} X$ be a continuous morphism such that u^* is a localization, and $C_X \xrightarrow{F} C_X$ is a functor compatible with the localization u^*. If the functor F is continuous, then the induced endofunctor $C_U \xrightarrow{F_U} C_U$ is continuous.*

Proof. Let $F^!$ be a right adjoint to the functor F and $Id_{C_X} \xrightarrow{\eta} F^!F$, $FF^! \xrightarrow{\epsilon} Id_{C_X}$ adjunction arrows. By the argument above, the functor F_U uniquely determined by the equality $F_U \circ u^* = u^* \circ F$, is naturally isomorphic to u^*Fu_*, and the compatibility of F with the localization u^* (i.e. the existence of F_U is equivalent to that the natural morphism $u^*F \xrightarrow{u^*F\eta_u} u^*Fu_*u^*$ is an isomorphism. Here η_u is the adjunction arrow $Id_{C_X} \longrightarrow u_*u^*$. The claim is that the functor $u^*F^!u_*$ is a right adjoint to u^*Fu_* (hence to F_U).

In fact, there are natural morphisms

$$(u^*Fu_*)(u^*F^!u_*) = (u^*Fu_*u^*)(F^!u_*) \xrightarrow{\sim} u^*FF^!u_* \xrightarrow{u^*\epsilon u_*} u^*u_* \xrightarrow{\epsilon_u} Id_{C_U}$$

and

$$Id_{C_U} \xrightarrow{\epsilon_u^{-1}} u^*u_* \xrightarrow{u^*\eta u_*} u^*F^!Fu_* \xrightarrow{u^*F^!\eta_u Fu_*} u^*F^!u_*u^*F^u_*.$$

One can check that their respective compositions produce a pair of adjunction morphisms. Details are left to the reader. ■

C4.2. Infinitesimal neighborhoods of the diagonal. Differential calculus. Fix a monoidal category $\mathcal{A}^\sim = (\mathcal{A}, \otimes, \mathbf{1}, a)$. Here $\mathbf{1}$ denotes the unit object and a the associativity constraint. In order to simplify the exposition, we assume that the category $\mathcal{A}$ is quasi-abelian (i.e. it is additive and every morphism has a kernel and cokernel) and that the functor $M \otimes - : L \longmapsto M \otimes L$ preserves small colimits.

Fix a full monoidal subcategory T of $\mathcal{A}$ closed with respect to colimits taken in $\mathcal{A}$. The pair $(\mathcal{A}^\sim, T)$ is the initial data for differential calculus.

Objects of the n^{th} neighborhood $T^{(n+1)}$ of the subcategory T are called *T-differential objects of order* $\leq n$. In particular, zero objects are the only T-differential objects of the order -1, and T consists of T-differential objects of order ≤ 0. Sometimes we shall loosely call the subcategory T the 'diagonal'.

C4.2.1. Proposition. *The category* $T^{(\infty)} \stackrel{\mathrm{def}}{=} \bigcap\limits_{n \geq 1} T^{(n)}$ *of T-differential objects is a monoidal subcategory of* $\mathcal{A}^\sim$.

Proof. See [RL1]. ■

C4.2.2. Corollary. *The category T^∞ whose objects are colimits of objects of the category $T^{(\infty)}$ is a monoidal subcategory of $\mathcal{A}^\sim$.*

Proof. The assertion follows from C4.2.1 and the assumption that the functors $M \otimes :$ $L \longmapsto M \otimes L$, $M \in Ob\mathcal{A}$, preserve colimits. ■

C4.2.3. The smallest diagonal. We denote the *smallest 'diagonal'* (i.e. the monoidal subcategory of $\mathcal{A}^\sim$ closed with respect to colimits (taken in $\mathcal{A}$) and generated by the identity object $\mathbf{1}$) by $\Delta_{\mathcal{A}^\sim}$.

C4.3. Differential functors and differential monads.

C4.3.1. Differential functors and (co)monads. Let $\mathcal{A}^\sim$ be the monoidal category $\mathfrak{End}_\mathfrak{r}(C_X)$ of right exact endofunctors of an abelian category C_X and $\mathbb{T} = \Delta_{\mathcal{A}^\sim}$ the smallest diagonal of $\mathcal{A}^\sim$. Objects of the subcategory $\mathbb{T}^{(\infty)} = \Delta_{\mathcal{A}^\sim}^{(\infty)}$ are called *differential* functors.

A monad (F, μ) (resp. a comonad (G, δ) is called *differential* if the endofunctor F (resp. G) is differential.

C4.3.2. Differential bimodules. Let R be an associative unital ring and $\mathcal{A}^\sim$ the monoidal category of R-bimodules: $\mathcal{A}^\sim = R - bimod^\sim = (R - bimod, \otimes_R, R)$. In this case the smallest diagonal is the full subcategory of $R - bimod$ whose objects are all *central bimodules*, i.e. bimodules M generated by their center $C(M) \stackrel{\mathrm{def}}{=} \{z \in M |\ rz = zr \text{ for all } r \in R\}$. The corresponding differential objects are called differential bimodules.

Note that the monoidal category of differential bimodules is equivalent to the monoidal category of differential endofunctors $C_X \longrightarrow C_X$, where $C_X = R - mod$.

C4.3.3. Proposition. *(a) Let C_X be an abelian category and $C_X \xrightarrow{F} C_X$ a differential endofunctor. Then every thick subcategory $\mathbb{T}$ of the category C_X is F-stable, i.e. $F(\mathbb{T}) \subseteq \mathbb{T}$.*

(b) If, in addition, the functor F is exact, then there exists a unique endofunctor $F_{\mathbb{T}}$ of the quotient category $C_{X/\mathbb{T}}$ such that $F_{\mathbb{T}} \circ q^*_{\mathbb{T}} = q^*_{\mathbb{T}} \circ F$. Here $q_{\mathbb{T}}$ is the localization functor $C_X \longrightarrow C_{X/\mathbb{T}}$. The functor $F_{\mathbb{T}}$ is exact and differential.

(c) If the differential functor F is exact and continuous (i.e. it has a right adjoint), then for every continuous exact localization $C_X \xrightarrow{q_{\mathbb{T}}} C_{X/\mathbb{T}}$, the induced endofunctor $F_{\mathbb{T}}$ of $C_{X/\mathbb{T}}$ is continuous.

Proof. (a) If F belongs to the diagonal, then $F(\mathbb{S}) \subseteq \mathbb{S}$ for every full subcategory of C_X closed under coproducts and quotients (taken in C_X). In particular, every topologizing (hence every thick) subcategory of C_X is F-stable.

In general, an endofunctor F is differential iff it has an increasing filtration, $F_{-1} = 0 \hookrightarrow F_0 \hookrightarrow \ldots \hookrightarrow F_n = F$ such that all quotients F_i/F_{i-1}, $0 \leq i \leq n$, belong to the diagonal. In particular, for every object M of a thick subcategory $\mathbb{T}$, there is a filtration $0 \hookrightarrow F_0(M) \hookrightarrow \ldots \hookrightarrow F_n(M) = F(M)$ such that all quotients $F_i(M)/F_{i-1}(M)$, $0 \leq i \leq n$, belong to $\mathbb{T}$. Therefore, $F(M)$ is an object of $\mathbb{T}$.

(b) If a functor F stabilizes a thick subcategory $\mathbb{T}$ and is exact, then it determines a unique endofunctor $F_{\mathbb{T}}$ of the quotient category $C_{X/\mathbb{T}}$ such that $q^*_{\mathbb{T}} \circ F = F_{\mathbb{T}} \circ q^*_{\mathbb{T}}$. Since the functor $q^*_{\mathbb{T}} \circ F$ is exact, it follows from [GZ, 1.1.4] that the functor $F_{\mathbb{T}}$ is exact.

(c) If F_* is a right adjoint to the endofunctor F and $q_{\mathbb{T}*}$ is a right adjoint to the localization functor $C_X \xrightarrow{q_{\mathbb{T}}} C_{X/\mathbb{T}}$. The checking (or reading [KR2, C2.1]) is left to the reader. ■

Chapter IV
Geometry of 'Spaces' Represented by Triangulated Categories.

This Chapter contains a sketch of the beginning of one of the simplest forms of derived noncommutative geometry. Here 'spaces' are represented by svelte triangulated categories (we call them *t-'spaces'*) and morphisms by isomorphism classes of *triangle* functors. We start with pseudo-geometry following pattern of Chapter 1, that is we consider *continuous* morphisms and look for a triangulated version of Beck's theorem (which plays a central role for studying 'spaces' represented by ordinary categories, incorporating both affine schemes and, in the dual context, descent theory). The triangulated picture, turns to be much easier: the triangulated version of Beck's theorem on descent side states that every continuous morphism is the composition of a comonadic morphism and a continuous localization. In particular, any *faithfully flat* (in triangle sense) morphism is comonadic.

The geometric picture looks even better. There are two spectra, $\mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X})$ and $\mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X})$ which are triangulated analogs of the spectra respectively $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(\mathfrak{X})$ and $\mathbf{Spec}(\mathfrak{X})$. There is a natural bijective map $\mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X}) \xrightarrow{\sim} \mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X})$. But, unlike the bijection $\mathbf{Spec}(\mathfrak{X}) \xrightarrow{\sim} \mathbf{Spec}^{1,1}_{\mathfrak{t}}(\mathfrak{X})$ of II.3.3.2, this map does not preserve the specialization preorder $\supseteq$. The specialization preorder on $\mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X})$ is what we expect from specialization. So that the preorder $(\mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X}), \supseteq)$ is regarded as the "principal" spectrum of the t-'space' $\mathfrak{X}$. On the other hand, the points of the spectrum $\mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X})$ are closed with respect to the topology determined by the specialization preorder, or a natural version of Zariski topology on $\mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X})$. This gives certain technical advantages (which are not used here) and curious interpretations.

1. Preliminaries on triangulated categories.

1.0. $\mathbb{Z}$-Categories. Recall that a *$\mathbb{Z}$-category* is a category endowed with an action of $\mathbb{Z}$, where $\mathbb{Z}$ is regarded as a monoidal category: objects are elements and the tensor product is given by addition. In other words, a $\mathbb{Z}$-category is a category $C_{\mathfrak{X}}$ with an auto-equivalence $\theta_{\mathfrak{X}}$ and an associativity isomorphism $\theta_{\mathfrak{X}} \circ (\theta_{\mathfrak{X}} \circ \theta_{\mathfrak{X}}) \xrightarrow{\sim} (\theta_{\mathfrak{X}} \circ \theta_{\mathfrak{X}}) \circ \theta_{\mathfrak{X}}$ satisfying the usual cocycle conditions.

1.1. The category of triangulated categories. Triangulated k-linear categories are triples $(C_{\mathfrak{X}}, \theta_{\mathfrak{X}}; \mathfrak{Tr}_{\mathfrak{X}})$, where $(C_{\mathfrak{X}}, \theta_{\mathfrak{X}})$ is an additive k-linear $\mathbb{Z}$-category, and $\mathfrak{Tr}_{\mathfrak{X}}$ a full subcategory of the category of diagrams of the form

$$\mathcal{L} \longrightarrow \mathcal{M} \longrightarrow \mathcal{N} \longrightarrow \theta_{\mathfrak{X}}(\mathcal{L}).$$

The objects of the subcategory $\mathfrak{Tr}_{\mathfrak{X}}$ are called *triangles*. They satisfy to well known axioms due to Verdier [Ve1]. We denote a triangulated category $(C_{\mathfrak{X}}, \theta_{\mathfrak{X}}; \mathfrak{Tr}_{\mathfrak{X}})$ by $\mathcal{CT}_{\mathfrak{X}}$.

A *triangle* k-linear functor from a triangulated k-linear category $\mathcal{CT}_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}; \mathfrak{Tr}_{\mathfrak{X}})$ to a triangulated k-linear category $\mathcal{CT}_{\mathfrak{Y}} = (C_{\mathfrak{Y}}, \theta_{\mathfrak{Y}}; \mathfrak{Tr}_{\mathfrak{Y}})$ is a pair (F, ϕ), where F is a k-linear functor $C_{\mathfrak{X}} \longrightarrow C_{\mathfrak{Y}}$ and ϕ a functor isomorphism $\theta_{\mathfrak{Y}} \circ F \xrightarrow{\sim} F \circ \theta_{\mathfrak{X}}$ such that for any triangle $\mathcal{L} \longrightarrow \mathcal{M} \longrightarrow \mathcal{N} \longrightarrow \theta_{\mathfrak{X}}(\mathcal{L})$ of $\mathcal{C}\mathfrak{T}_{\mathfrak{X}}$, the diagram

$$F(\mathcal{L}) \longrightarrow F(\mathcal{M}) \longrightarrow F(\mathcal{N}) \longrightarrow \theta_{\mathfrak{Y}}(F(\mathcal{L})),$$

where $F(\mathcal{N}) \longrightarrow \theta_{\mathfrak{Y}}(F(\mathcal{L}))$ is the composition of $F(\mathcal{N} \longrightarrow \theta_{\mathfrak{X}}(\mathcal{L}))$ and the isomorphism $F\theta_{\mathfrak{X}}(\mathcal{L}) \xrightarrow{\phi(\mathcal{L})} \theta_{\mathfrak{Y}}(F(\mathcal{L}))$, is a triangle of the triangulated category $\mathcal{C}\mathfrak{T}_{\mathfrak{Y}}$.

We denote by $\mathfrak{TrCat}_k$ the category whose objects are svelte triangulated categories and morphisms are triangle functors between them.

1.3. Multiplicative systems in triangulated categories. Fix a triangulated category $\mathcal{CT}_{\mathfrak{X}} = (C_{\mathfrak{X}}, \gamma; \mathfrak{Tr}_{\mathfrak{X}})$. A multiplicative system Σ of (X, γ) is said to be *compatible with triangulation* if for any pair of triangles (L, M, N, u, v, w) and (L', M', N', u', v', w') and any commutative diagram

$$\begin{array}{ccc} L & \xrightarrow{u} & M \\ s \big\downarrow & & \big\downarrow s' \\ L' & \xrightarrow{u'} & M' \end{array}$$

where s and s' are elements of Σ, there exists a morphism $N \xrightarrow{t} N'$ which belongs to Σ and such that (s, s', t) is a morphism of triangles.

We shall use same notations: $\mathcal{SM}(\mathfrak{X})$ (resp. $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(\mathfrak{X})$) for the preorder of multiplicative (resp. saturated multiplicative) systems of the tr-'space' $\mathfrak{X}$. The dualization functor $\mathfrak{X} \longmapsto \mathfrak{X}^o$ induces an isomorphism of preorders

$$\mathcal{SM}(\mathfrak{X}) \xrightarrow{\sim} \mathcal{SM}(\mathfrak{X}^o) \quad \text{and} \quad \mathcal{S}^{\mathfrak{s}}\mathcal{M}(\mathfrak{X}) \xrightarrow{\sim} \mathcal{S}^{\mathfrak{s}}\mathcal{M}(\mathfrak{X}^o).$$

1.4. Triangulated subcategories. Recall that a full subcategory, $\mathbb{T}$, of the category $C_{\mathfrak{X}}$ is called a *triangulated subcategory* if it is stable by translations, and has a triangulated structure such that the inclusion functor $\mathbb{T} \hookrightarrow C_{\mathfrak{X}}$ is exact.

Let $\mathbb{T}$ be a full subcategory of $C_{\mathfrak{X}}$ stable by translations. The subcategory $\mathbb{T}$ admits a triangulated structure which makes it a triangulated subcategory of $\mathcal{CT}_{\mathfrak{X}}$ iff for any morphism $L \xrightarrow{f} M$ of $\mathbb{T}$, there exists a triangle (L, M, N, f, g, h) such that $N \in Ob\mathbb{T}$.

1.4.1. Definitions. (1) A full triangulated subcategory, $\mathbb{T}$, of $\mathcal{CT}_{\mathfrak{X}}$ is called *saturated* (in [Ve2]), if every direct summand (in $C_{\mathfrak{X}}$) of an object of $\mathbb{T}$ belongs to $\mathbb{T}$.

(2) A full triangulated subcategory, $\mathbb{T}$, of $\mathcal{CT}_{\mathfrak{X}}$ is called *thick* (in [Ve1] and everywhere else), if every triangle (L, M, N, u, v, w) such that $N \in Ob\mathbb{T}$ and $L \xrightarrow{u} M$ factors through an object of $\mathbb{T}$, belongs to $\mathbb{T}$ (that is L and M are objects of $\mathbb{T}$).

These two notions are equivalent: *A full triangulated subcategory of a triangulated category is thick iff it is saturated.*

1.5. Triangulated subcategories and multiplicative systems. For any full triangulated subcategory, $\mathbb{T}$, of the triangulated category $\mathcal{CT}_{\mathfrak{X}}$, let $\Sigma_{\mathbb{T}}$ denote the family of all morphisms $L \xrightarrow{u} M$ of $\mathcal{CT}_{\mathfrak{X}}$ such that there exists a triangle (L, M, N, u, v, w), where N is an object of $\mathbb{T}$.

1.5.1. Proposition [Ve2, 2.1.8]. *For any full triangulated subcategory of a triangulated category $\mathcal{CT}_{\mathfrak{X}}$, the family $\Sigma_{\mathbb{T}}$ is a multiplicative system. The system $\Sigma_{\mathbb{T}}$ is saturated iff the subcategory $\mathbb{T}$ is thick.*

For any multiplicative system Σ in the triangulated category $\mathcal{CT}_{\mathfrak{X}}$, let $\mathbb{T}_{\Sigma}$ denote the full subcategory of $\mathcal{CT}_{\mathfrak{X}}$ generated by objects N contained in a triangle (L, M, N, u, v, w) such that $L \xrightarrow{u} M$ belongs to Σ.

1.6. Proposition [Ve1, 2.1]. *The map $\Sigma \longmapsto \mathbb{T}_{\Sigma}$ is an isomorphism of the preorder $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(\mathfrak{X})$ of saturated multiplicative systems of a triangulated category $\mathcal{CT}_{\mathfrak{X}}$ onto the preorder $\mathfrak{Tht}(\mathfrak{X})$ of thick triangulated subcategories of $\mathcal{CT}_{\mathfrak{X}}$. The inverse isomorphism is given by the map $\mathbb{T} \longmapsto \Sigma_{\mathbb{T}}$.*

1.6.1. Corollary. *The intersection of any set of saturated multiplicative systems of a triangulated category is a saturated multiplicative system.*

Proof. The assertion follows from an easily checked fact that the intersection of any set of thick triangulated subcategories of a triangulated category is a thick triangulated subcategory. ∎

The following proposition (which is a part of [Ve2, 2.3.1]) is a convenient reference for the rest of this section.

1.7. Proposition. *Let $\mathcal{B}$ and $\mathcal{A}$ be full triangulated subcategories of a triangulated category $\mathcal{CT}_{\mathfrak{X}}$ such that $\mathcal{B} \subseteq \mathcal{A}$.*

(a) The canonical functor $\mathcal{A}/\mathcal{B} \longrightarrow \mathcal{CT}_{\mathfrak{X}}/\mathcal{B}$ is fully faithful and injective on objects. The image of this functor is $q^*_{\mathcal{B}}(\mathcal{A})$, where $q^*_{\mathcal{B}}$ is the canonical functor $\mathcal{CT}_{\mathfrak{X}} \longrightarrow \mathcal{CT}_{\mathfrak{X}}/\mathcal{B}$.

The subcategory $\mathcal{A}$ is thick iff the subcategory $q^*_{\mathcal{B}}(\mathcal{A})$ is thick.

(b) The map $\mathcal{A} \longmapsto q^*_{\mathcal{B}}(\mathcal{A})$ is an isomorphism of the preorder of strictly full triangulated subcategories of $\mathcal{CT}_{\mathfrak{X}}$ containing the kernel, $\mathcal{B}^{\mathfrak{t}}$ of the functor $q^*_{\mathcal{B}}$ onto the preorder of strictly full triangular subcategories of $\mathcal{CT}_{\mathfrak{X}}/\mathcal{B}$.

(c) The canonical functor $\mathcal{CT}_{\mathfrak{X}}/\mathcal{A} \longrightarrow (\mathcal{CT}_{\mathfrak{X}}/\mathcal{B})/(\mathcal{A}/\mathcal{B})$ is an isomorphism of triangulated categories.

1.7.1. Corollary. *Let $\mathcal{B}$ and $\mathcal{A}$ be full triangulated subcategories of a triangulated category $\mathcal{CT}_{\mathfrak{X}}$ such that $\mathcal{B} \subseteq \mathcal{A}$. Let $\mathcal{B}^{\mathrm{t}}$ be the thick envelope of $\mathcal{B}$ in $\mathcal{CT}_{\mathfrak{X}}$. Then $\mathcal{B}^{\mathrm{t}} \cap \mathcal{A}$ is the thick envelope of $\mathcal{B}$ in $\mathcal{A}$.*

Proof. Consider the commutative diagram

$$\begin{array}{ccccc} \mathcal{B}^{\mathrm{t}} & \longrightarrow & \mathcal{CT}_{\mathfrak{X}} & \longrightarrow & \mathcal{CT}_{\mathfrak{X}}/\mathcal{B} \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{B}^{\mathrm{t}}_{\mathcal{A}} & \longrightarrow & \mathcal{A} & \longrightarrow & \mathcal{A}/\mathcal{B} \end{array}$$

with exact rows. By 1.7(a), the functor $\mathcal{A}/\mathcal{B} \longrightarrow \mathcal{CT}_{\mathfrak{X}}/\mathcal{B}$ is faithful. Therefore, the kernel, $\mathcal{B}^{\mathrm{t}}_{\mathcal{A}}$, of the localization functor $\mathcal{A} \longrightarrow \mathcal{A}/\mathcal{B}$, which is the thick envelope of $\mathcal{B}$ in $\mathcal{A}$, coincides with $\mathcal{B}^{\mathrm{t}} \cap \mathcal{A}$. ∎

1.8. Preliminaries on orthogonality. For any subcategory $\mathcal{B}$ of $\mathcal{CT}_{\mathfrak{X}}$, the *left orthogonal*, ${}^{\perp}\mathcal{B}$, of $\mathcal{B}$ is the full subcategory of $\mathcal{CT}_{\mathfrak{X}}$ generated by all objects N such that $\mathcal{CT}_{\mathfrak{X}}(N, M) = 0$ for all $M \in Ob\mathcal{B}$. The right orthogonal of $\mathcal{B}$ is defined dually and is denoted by $\mathcal{B}^{\perp}$. Its objects are $\mathcal{B}$-torsion free objects of $\mathcal{CT}_{\mathfrak{X}}$. If $\mathcal{B}$ is a triangulated subcategory, then $\mathcal{B}^{\perp}$ and ${}^{\perp}\mathcal{B}$ are thick triangulated subcategories.

1.8.1. Proposition [Ve2, 2.3.3]. *Let $\mathcal{B}$ be a full triangulated subcategory of a triangulated category $\mathcal{CT}_{\mathfrak{X}}$ and $\mathcal{CT}_{\mathfrak{X}} \xrightarrow{q^*_{\mathcal{B}}} \mathcal{CT}_{\mathfrak{X}}/\mathcal{B}$ the canonical localization functor.*

(a) For every object M of $\mathcal{CT}_{\mathfrak{X}}$, the following conditions are equivalent:

(i) The object M is $q^*_{\mathcal{B}}$-free.

(ii) The object M is left closed for $\Sigma_{\mathcal{B}}$, i.e. $\mathcal{CT}_{\mathfrak{X}}(s, M)$ is an isomorphism for every $s \in \Sigma_{\mathcal{B}}$. Here $\Sigma_{\mathcal{B}}$ is the multiplicative system corresponding to $\mathcal{B}$ (cf. 1.5).

(iii) Every morphism $M \xrightarrow{s} N$ with $s \in \Sigma_{\mathcal{B}}$ admits a retraction.

(iv) The object M is $\mathcal{B}$-torsion free, that is for every $L \in Ob\mathcal{B}$, $\mathcal{CT}_{\mathfrak{X}}(L, M) = 0$.

(v) For every $N \in Ob\mathcal{CT}_{\mathfrak{X}}$, the map

$$\mathcal{CT}_{\mathfrak{X}}(N, M) \longrightarrow \mathcal{CT}_{\mathfrak{X}}/\mathcal{B}(q^*_{\mathcal{B}}(N), q^*_{\mathcal{B}}(M))$$

is an isomorphism.

(b) The full subcategory $\mathcal{L}(q_{\mathcal{B}})$ of $\mathcal{CT}_{\mathfrak{X}}$ generated by $q^*_{\mathcal{B}}$-free objects is a thick triangulated subcategory.

(c) The composition of the inclusion functor $\mathcal{CT}_{\mathcal{L}(q_{\mathcal{B}})} \hookrightarrow \mathcal{CT}_{\mathfrak{X}}$ and the localization functor $\mathcal{CT}_{\mathfrak{X}} \xrightarrow{q^*_{\mathcal{B}}} \mathcal{CT}_{\mathfrak{X}}/\mathcal{B}$ is a fully faithful functor injective on objects.

(d) Let $\mathcal{CT}_{\mathfrak{Tr}_{\mathfrak{X}}(q_{\mathcal{B}*})}$ be the full subcategory of the quotient triangulated category $\mathcal{CT}_{\mathfrak{X}}/\mathcal{B}$ generated by all objects M such that the functor $\mathcal{CT}_{\mathfrak{X}}/\mathcal{B}(q^*_{\mathcal{B}}(-), M)$ is representable. The subcategory $\mathcal{CT}_{\mathfrak{Tr}_{\mathfrak{X}}(q_{\mathcal{B}*})}$ is triangulated and strictly full. If infinite coproducts or products exist in $\mathcal{CT}_{\mathfrak{X}}$, then $\mathcal{CT}_{\mathfrak{Tr}_{\mathfrak{X}}(q_{\mathcal{B}*})}$ is thick.

(e) The localization functor $q^*_{\mathcal{B}}$ induces an equivalence of categories

$$\mathcal{CT}_{\mathcal{L}(q_{\mathcal{B}})} \longrightarrow \mathcal{CT}_{\mathfrak{Tr}_{\mathfrak{X}}(q_{\mathcal{B}*})}.$$

(f) An object N of $\mathcal{CT}_{\mathfrak{X}}$ belongs to the preimage, $\mathcal{CT}_{\mathfrak{R}(q_{\mathcal{B}})} = {q^*_{\mathcal{B}}}^{-1}(\mathcal{CT}_{\mathfrak{Tr}_{\mathfrak{X}}(q_{\mathcal{B}*})})$, of the subcategory $\mathcal{CT}_{\mathfrak{Tr}_{\mathfrak{X}}(q_{\mathcal{B}*})}$ iff there exists a morphism $N \xrightarrow{s} M$ such that M is $q^*_{\mathcal{B}}$-free and $q^*_{\mathcal{B}}(s)$ is invertible.

(g) The inclusion functor $\mathcal{CT}_{\mathcal{L}(q_{\mathcal{B}})} \longrightarrow \mathcal{CT}_{\mathfrak{R}(q_{\mathcal{B}})}$ has a left adjoint.

Proof. See [Ve2, 2.3.3]. ■

1.8.2. Corollary [Ve1, 6–3]. *Let $\mathbb{T}$ be a thick triangulated subcategory of the triangulated category $\mathcal{CT}_{\mathfrak{X}}$. The full subcategory of $\mathcal{CT}_{\mathfrak{X}}$ generated by objects which are left closed for $\Sigma_{\mathbb{T}}$, is the right orthogonal, $\mathbb{T}^{\perp}$, of the subcategory $\mathbb{T}$.*

Proof. The fact follows from the equivalence of (ii) and (iv) in 1.8.1. ■

1.8.3. Proposition [Ve1, 6–5]. *Let $\mathbb{T}$ be a thick triangulated subcategory of $\mathcal{CT}_{\mathfrak{X}}$, and let*

$$\mathbb{T} \xrightarrow{\iota^*_{\mathbb{T}}} \mathcal{CT}_{\mathfrak{X}} \xrightarrow{q^*_{\mathbb{T}}} \mathcal{CT}_{\mathfrak{X}}/\mathbb{T}$$

be the inclusion and localization functors. The following properties are equivalent:

(a) The functor $\iota^*_{\mathbb{T}}$ has a right adjoint.

(b) The functor $q^*_{\mathbb{T}}$ has a right adjoint.

1.9. The category of t-'spaces'. If $\mathcal{CT}_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}; \mathfrak{Tr}_{\mathfrak{X}})$ is a svelte Karoubian (that is the category $C_{\mathfrak{X}}$ is Karoubian) k-linear triangulated category, we say that it represents a *t-'space'* $\mathfrak{X}$. A morphism $\mathfrak{X} \xrightarrow{\mathfrak{f}} \mathfrak{Y}$ from a t-'space' $\mathfrak{X}$ to a t-'space' $\mathfrak{Y}$ is an isomorphism class of triangle functors from $\mathcal{CT}_{\mathfrak{Y}}$ to $\mathcal{CT}_{\mathfrak{X}}$. A representative of a morphism $\mathfrak{f}$ will be called an *inverse image* functor of $\mathfrak{f}$ and denoted, usually, by $\mathfrak{f}^*$. The composition $f \circ g$ is, by definition, the isomorphism class of the composition $g^* \circ f^*$ of inverse image functors of respectively g and f. This defines the category $\mathfrak{Esp}_{\mathfrak{Tr}}$ of t-'spaces'.

2. Triangulated categories and Frobenius $\mathbb{Z}$-categories.

We need some facts about *abelianization* of triangulated categories.

For any k-linear category $C_{\mathfrak{X}}$, we denote by $\mathcal{M}_k(\mathfrak{X})$ the abelian category of presheaves of k-modules on $C_{\mathfrak{X}}$ and by $C_{\mathfrak{X}_{\mathfrak{a}}}$ the full subcategory of $\mathcal{M}_k(\mathfrak{X})$ generated by all presheaves of k-modules which have a left resolution formed by representable presheaves. Since $C_{\mathfrak{X}_\mathfrak{a}}$ contains all representable presheaves, the Yoneda functor $C_{\mathfrak{X}} \xrightarrow{\mathfrak{h}_{\mathfrak{X}}} \mathcal{M}_k(\mathfrak{X})$ factors through the embedding $C_{\mathfrak{X}_\mathfrak{a}} \longrightarrow \mathcal{M}_k(\mathfrak{X})$. We denote by $\mathfrak{H}_{\mathfrak{X}}$ the corestriction $C_{\mathfrak{X}} \longrightarrow C_{\mathfrak{X}_\mathfrak{a}}$ of the Yoneda functor. Every k-linear functor $C_X \xrightarrow{F} C_Y$ induces a right exact functor $C_{X_\mathfrak{a}} \xrightarrow{F_\mathfrak{a}} C_{Y_\mathfrak{a}}$ such that the diagram

$$\begin{array}{ccc} C_X & \xrightarrow{F} & C_Y \\ {\scriptstyle h_X}\downarrow & & \downarrow{\scriptstyle h_Y} \\ C_{X_\mathfrak{a}} & \xrightarrow{F_\mathfrak{a}} & C_{Y_\mathfrak{a}} \end{array} \qquad (1)$$

commutes. The functor $F_\mathfrak{a}$ is determined uniquely up to isomorphism.

If $C_{\mathfrak{X}}$ is a $\mathbb{Z}$-category, then the categories $\mathcal{M}_k(\mathfrak{X})$ and $C_{\mathfrak{X}_\mathfrak{a}}$ inherit a $\mathbb{Z}$-action such that the functors $\mathfrak{h}_{\mathfrak{X}}$ and $\mathfrak{H}_{\mathfrak{X}}$ become $\mathbb{Z}$-functors. It follows that for every $\mathbb{Z}$-functor $C_X \xrightarrow{F} C_Y$, the functor $C_{X_\mathfrak{a}} \xrightarrow{F_\mathfrak{a}} C_{Y_\mathfrak{a}}$ is a $\mathbb{Z}$-functor.

2.1. Frobenius abelian $\mathbb{Z}$-categories. An exact k-linear $\mathbb{Z}$-category is called a *Frobenius* category if it has enough projective and injective obects and its projective and injective objects coincide. In this chapter, we are interested only in abelian Frobenius categories. We denote by $\mathfrak{F}_{\mathbb{Z}}\mathfrak{Cat}_k$ the category whose objects are svelte Frobenius k-linear abelian $\mathbb{Z}$-categories and morphisms are exact k-linear functors which map projective objects to projective objects.

2.2. Theorem. *(a) For any triangulated k-linear category $\mathcal{C}\mathfrak{T}_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}; \mathfrak{Tr}_{\mathfrak{X}})$, the category $C_{\mathfrak{X}_\mathfrak{a}}$ is a Frobenius abelian k-linear $\mathbb{Z}$-category. If the category $C_{\mathfrak{X}}$ is Karoubian, then the canonical functor $C_{\mathfrak{X}} \xrightarrow{\mathfrak{H}_{\mathfrak{X}}} C_{\mathfrak{X}_\mathfrak{a}}$ induces an equivalence between the category $C_{\mathfrak{X}}$ and the full subcategory of $C_{\mathfrak{X}_\mathfrak{a}}$ generated by its projective objects.*

(b) The correspondence $\mathcal{C}\mathfrak{T}_{\mathfrak{X}} \longmapsto C_{\mathfrak{X}_\mathfrak{a}}$ extends to a fully faithful functor from the category $\mathfrak{TrCat}_k$ to the category $\mathfrak{F}_{\mathbb{Z}}\mathfrak{Cat}_k$.

Proof. The assertion is equivalent to a part of Theorem 3.2.1 in [Ve2]. ∎

3. Localizations, continuous morphisms, and (co)monadic morphisms.

3.1. Localizations and conservative morphisms. Let $\mathfrak{X} \xrightarrow{\mathfrak{f}} \mathfrak{Y}$ be a morphism of t-'spaces'. Its inverse image functor $C_{\mathfrak{Y}} \xrightarrow{\mathfrak{f}^*} C_{\mathfrak{X}}$ is a composition of the localization at the thick subcategory $Ker(\mathfrak{f}^*)$ and a faithful triangle functor. In other words, we have a canonical decomposition $\mathfrak{f} = \mathfrak{p}_\mathfrak{f} \circ \mathfrak{f}_\mathfrak{c}$, where $\mathfrak{p}_\mathfrak{f}^*$ is the localization functor $C_{\mathfrak{Y}} \longrightarrow C_{\mathfrak{Y}}/Ker(\mathfrak{f}^*)$ and $\mathfrak{f}_\mathfrak{c}^*$ is a faithful triangle functor determined (uniquely once $\mathfrak{f}^*$ is fixed, hence) uniquely up to isomorphism.

We call a morphism of t-'spaces' $\mathfrak{X} \xrightarrow{\mathfrak{f}} \mathfrak{Y}$ a *localization* if $\mathfrak{f}_{\mathfrak{c}}$ is an isomorphism, or, equivalently, if its inverse image functor is a category equivalence.

3.2. Continuous morphisms. We call a morphism $\mathfrak{X} \xrightarrow{\mathfrak{f}} \mathfrak{Y}$ of t-'spaces' *continuous* if its inverse image functor, $\mathfrak{f}^*$ has a right adjoint, $\mathfrak{f}_*$, and this right adjoint is a triangle functor.

3.2.1. Proposition. *The following conditions are equivalent:*

(a) A morphism $\mathfrak{X} \xrightarrow{\mathfrak{f}} \mathfrak{Y}$ of t-'spaces' is continuous.

(b) The functor $C_{\mathfrak{X}_{\mathfrak{a}}} \xrightarrow{\mathfrak{f}_{\mathfrak{a}}^*} C_{\mathfrak{Y}_{\mathfrak{a}}}$ is an exact functor and has a right adjoint.

(c) The functor $C_{\mathfrak{Y}_{\mathfrak{a}}} \xrightarrow{\mathfrak{f}_{\mathfrak{a}*}} C_{\mathfrak{X}_{\mathfrak{a}}}$ is an exact functor and has a left adjoint.

Proof. $(a) \Rightarrow (b)$. Let $\mathfrak{X} \xrightarrow{\mathfrak{f}} \mathfrak{Y}$ be a continuous morphism of t-'spaces'; that is its inverse image functor $\mathfrak{f}^*$ has a right adjoint, $\mathfrak{f}_*$, which is a triangle functor. Then $\mathfrak{f}_{\mathfrak{a}*}$ is a right adjoint to the functor $\mathfrak{f}_{\mathfrak{a}}^*$ and it maps injective objects to injective objects. Since the category $C_{\mathfrak{Y}_{\mathfrak{a}}}$ has enough injective objects, the latter implies that the functor $\mathfrak{f}_{\mathfrak{a}}^*$ is exact.

$(b) \Rightarrow (a)$. Conversely, if the functor $C_{\mathfrak{X}_{\mathfrak{a}}} \xrightarrow{\mathfrak{f}_{\mathfrak{a}}^*} C_{\mathfrak{Y}_{\mathfrak{a}}}$ is exact and has a right adjoint, $\mathfrak{f}_{\mathfrak{a}*}$, then the latter maps injective objects to injective objects. Since injective objects in $C_{\mathfrak{X}_{\mathfrak{a}}}$ and $C_{\mathfrak{Y}_{\mathfrak{a}}}$ coincide with projective objects and the categories $C_{\mathfrak{X}_{\mathfrak{a}}}$ and $C_{\mathfrak{Y}_{\mathfrak{a}}}$ are Karoubian, the embeddings $C_{\mathfrak{X}} \longrightarrow C_{\mathfrak{X}_{\mathfrak{a}}}$ and $C_{\mathfrak{Y}} \longrightarrow C_{\mathfrak{Y}_{\mathfrak{a}}}$ induce equivalences between the category $C_{\mathfrak{X}}$ (resp. $C_{\mathfrak{Y}}$) and the full subcategory of the category $C_{\mathfrak{X}_{\mathfrak{a}}}$ (resp. $C_{\mathfrak{Y}_{\mathfrak{a}}}$) generated by projective objects. Therefore, the functor $\mathfrak{f}_{\mathfrak{a}*}$ induces a functor $C_{\mathfrak{Y}} \xrightarrow{\mathfrak{f}_*} C_{\mathfrak{X}}$ which is a right adjoint to $\mathfrak{f}^*$.

The implications $(a) \Leftrightarrow (c)$ follow by duality. ∎

3.3. Monads and comonads in triangulated categories. Let $\mathfrak{T}\mathcal{C}_{\mathfrak{X}} = (\mathcal{C}_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \mathfrak{Tr}_{\mathfrak{X}})$ be a triangulated category. A monad on $\mathfrak{T}\mathcal{C}_{\mathfrak{X}}$ (or a monad on the corresponding t-'space' $\mathfrak{X}$) is a monad $\mathcal{F} = (F, \mu)$ on the category $\mathcal{C}_{\mathfrak{X}}$ such that F is a triangle functor and $F^2 \xrightarrow{\mu} F$ is a morphism of triangle functors. Dually, a comonad on $\mathfrak{T}\mathcal{C}_{\mathfrak{X}}$ (or $\mathfrak{X}$) is a comonad $\mathcal{G} = (G, \delta)$ such that G is a triangle functor on $\mathfrak{T}\mathcal{C}_{\mathfrak{X}}$ and $G \xrightarrow{\delta} G^2$ is a morphism of triangle functors.

The category $(\mathcal{F}/\mathfrak{X}) - mod$ of $\mathcal{F}$-modules has a structure of triangulated category induced by the forgetful functor $(\mathcal{F}/\mathfrak{X}) - mod \xrightarrow{\mathfrak{f}_*} \mathcal{C}_{\mathfrak{X}}$.

The following assertion is the triangulated version of Beck's theorem.

3.4. Proposition. *Let $\mathfrak{T}\mathcal{C}_{\mathfrak{X}}$ and $\mathfrak{T}\mathcal{C}_{\mathfrak{Y}}$ be Karoubian triangulated categories and*

$$\mathfrak{T}\mathcal{C}_{\mathfrak{Y}} \xrightarrow{f^*} \mathfrak{T}\mathcal{C}_{\mathfrak{X}} \xrightarrow{f_*} \mathfrak{T}\mathcal{C}_{Y}$$

a pair of adjoint triangle functors with adjunction morphisms

$$f^*f_* \xrightarrow{\epsilon_f} Id_{\mathfrak{T}\mathcal{C}_{\mathfrak{X}}} \quad and \quad Id_{\mathfrak{T}\mathcal{C}_{\mathfrak{Y}}} \xrightarrow{\eta_f} f_*f^*.$$

(a) The canonical functor

$$\mathfrak{T}\mathcal{C}_{\mathfrak{Y}} \xrightarrow{\widetilde{\mathfrak{f}}^*} \mathcal{G}_f - Comod = (\mathfrak{X}\backslash\mathcal{G}_f) - Comod, \quad M \longmapsto (f^*(M), f^*\eta_f(M)), \qquad (1)$$

is a localization functor. It is a category equivalence iff the functor f^* is faithful.

(b) Dually, the canonical functor

$$\mathfrak{T}\mathcal{C}_{\mathfrak{X}} \xrightarrow{\widetilde{\mathfrak{f}}_*} \mathcal{F}_f - mod = (\mathcal{F}_f/\mathfrak{Y}) - mod, \quad L \longmapsto (f_*(L), f_*\epsilon_f(L)), \qquad (2)$$

is a localization. It is a category equivalence iff the functor f_* is faithful.

Here $\mathcal{G}_f = (G_f, \delta_f) = (f^*f_*, f^*\eta_f f_*)$ and $\mathcal{F}_f = (F_f, \mu_f) = (f_*f^*, f_*\epsilon_f f^*)$ are respectively the comonad and the monad associated with the pair of adjoint functors f^*, f_*.

Proof. It suffices to prove (a), because the two assertions are dual to each other.

Let $\mathcal{C}_{\mathfrak{X}_\mathfrak{a}}$ denote the *abelianization* of the triangulated category $\mathfrak{T}\mathcal{C}_{\mathfrak{X}}$. Any triangle functor $\mathfrak{T}\mathcal{C}_{\mathfrak{X}} \xrightarrow{F} \mathfrak{T}\mathcal{C}_{\mathfrak{Y}}$ gives rise to an exact $\mathbb{Z}$-functor $\mathcal{C}_{\mathfrak{X}_\mathfrak{a}} \xrightarrow{F_\mathfrak{a}} \mathcal{C}_{\mathfrak{Y}_\mathfrak{a}}$ between the corresponding abelian $\mathbb{Z}$-categories which maps injective objects to injective objects. In particular, we have a pair of adjoint exact $\mathbb{Z}$-functors

$$\mathcal{C}_{\mathfrak{Y}_\mathfrak{a}} \xrightarrow{f_\mathfrak{a}^*} \mathcal{C}_{\mathfrak{X}_\mathfrak{a}} \xrightarrow{f_{\mathfrak{a}*}} \mathcal{C}_{\mathfrak{Y}_\mathfrak{a}}$$

which map injective objects to injective objects. Thus, we have the canonical functor

$$\mathcal{C}_{\mathfrak{Y}_\mathfrak{a}} \xrightarrow{\widetilde{\mathfrak{f}}_\mathfrak{a}^*} (\mathfrak{X}_\mathfrak{a}\backslash\mathcal{G}_{f_\mathfrak{a}}) - Comod.$$

Since both adjoint functors, $f_\mathfrak{a}^*$ and $f_{\mathfrak{a}*}$ are exact functors between abelian categories, it follows from Beck's theorem that $\widetilde{\mathfrak{f}}_\mathfrak{a}^*$ is a localization functor. If the functor f^* is faithful, then the functor $f_\mathfrak{a}^*$ is faithful. This follows from the fact that every object M of the category $\mathcal{C}_{\mathfrak{X}_\mathfrak{a}}$ is a quotient object of an object N of $\mathcal{C}_{\mathfrak{X}}$ and a subobject of an object L of $\mathfrak{T}\mathcal{C}_{\mathfrak{X}}$. Therefore, the composition $N \xrightarrow{\beta} L$ of the epimorphism $N \longrightarrow M$ and a monomorphism $M \longrightarrow L$ is nonzero iff M is nonzero. Since the functor f^* is faithful, $f^*(\beta) \neq 0$ whenever $M \neq 0$, which, in turn, implies that $f_\mathfrak{a}^*(M) \neq 0$ if $M \neq 0$.

Since the functor $\mathcal{C}_{\mathfrak{Y}_{\mathfrak{a}}} \xrightarrow{f^*_{\mathfrak{a}}} \mathcal{C}_{\mathfrak{X}_{\mathfrak{a}}}$ is exact and the category $\mathcal{C}_{\mathfrak{Y}_{\mathfrak{a}}}$ is abelian, the faithfulness of $f^*_{\mathfrak{a}}$ is equivalent to its conservativeness. The fact that $f^*_{\mathfrak{a}}$ is conservative implies that $\widetilde{\mathfrak{f}}^*_{\mathfrak{a}}$ is conservative too. Therefore, being a localization functor, $\widetilde{\mathfrak{f}}^*_{\mathfrak{a}}$ is a category equivalence. ∎

3.5. Continuous and (co)monadic morphisms of t-spaces. Let $\mathfrak{X} \xrightarrow{\mathfrak{f}} \mathfrak{Y}$ be a continuous morphism of t-'spaces'. We call it *comonadic* if the canonical functor

$$\mathfrak{T}\mathcal{C}_{\mathfrak{Y}} \xrightarrow{\widetilde{\mathfrak{f}}^*} \mathcal{G}_f - Comod = (\mathfrak{X}\backslash\mathcal{G}_f) - Comod, \quad M \longmapsto (f^*(M), f^*\eta_f(M)), \tag{1}$$

is a category equivalence. Dually, $\mathfrak{f}$ is called a *monadic* morphism if

$$\mathfrak{T}\mathcal{C}_{\mathfrak{X}} \xrightarrow{\widetilde{\mathfrak{f}}_*} \mathcal{F}_f - mod = (\mathcal{F}_f/\mathfrak{Y}) - mod, \quad L \longmapsto (f_*(L), f_*\epsilon_f(L)), \tag{2}$$

is a category equivalence.

By 3.4, a continuous morphism $\mathfrak{X} \xrightarrow{\mathfrak{f}} \mathfrak{Y}$ is comonadic (resp. monadic) iff its inverse (resp. direct) image functor is faithful.

3.5.1. Decomposition. One can show that if $\mathfrak{X} \xrightarrow{\mathfrak{f}} \mathfrak{Y}$ is a continuous morphism, then both localization $\mathfrak{p}_{\mathfrak{f}}$ and the 'faithful' component $\mathfrak{f}_{\mathfrak{c}}$ in the decomposition $\mathfrak{f} = \mathfrak{p}_{\mathfrak{f}} \circ \mathfrak{f}_{\mathfrak{c}}$ (see 3.1) are continuous morphisms. It follows from 3.4 that every continuous morphism of t-'spaces' is a unique composition of a continuous localization and a comonadic morphism.

4. Presite of localizations.

4.1. Proposition. *Let $\mathcal{TC}_{\mathfrak{X}}$ be a svelte triangulated category and $\{T_i \mid i \in J\}$ a finite family of thick triangulated subcategories of $\mathcal{TC}_{\mathfrak{X}}$. Then*

$$\Big(\bigcap_{i\in J} T_i\Big) \sqcup S = \bigcap_{i\in J}(T_i \sqcup S)$$

for any thick triangulated subcategory S.

Proof. Let $\mathcal{C}_{\mathfrak{X}_{\mathfrak{a}}}$ denote the abelianization of the triangulated category $\mathcal{TC}_{\mathfrak{X}}$. For a triangulated subcategory $\mathcal{T}$ of $\mathcal{TC}_{\mathfrak{X}}$, let $\mathcal{T}^{\mathfrak{a}}$ denote the smallest thick $\mathbb{Z}$-subcategory of $\mathcal{C}_{\mathfrak{X}_{\mathfrak{a}}}$ generated by the image of $\mathcal{T}$ in $\mathcal{C}_{\mathfrak{X}_{\mathfrak{a}}}$.

(a) *If $\mathcal{T}$ is a thick triangulated subcategory of $\mathcal{TC}_{\mathfrak{X}}$, then $\mathcal{T} = \mathcal{T}^{\mathfrak{a}} \bigcap \mathcal{C}_{\mathfrak{X}}$.*

In fact, objects of the subcategory $\mathcal{T}^{\mathfrak{a}}$ are arbitrary subquotients of objects of $\mathcal{T}$. Let M be an object of $\mathcal{C}_{\mathfrak{X}}$ which is a subquotient of an object N of $\mathcal{C}_{\mathfrak{X}_{\mathfrak{a}}}$, i.e. there exists a diagram $N \xrightarrow{\mathfrak{j}} K \xrightarrow{\mathfrak{e}} M$ in which $\mathfrak{j}$ is a monomorphism and $\mathfrak{e}$ is an epimorphism. Since

M is a projective object, the epimorphism $\mathfrak{e}$ splits, i.e. there exists a morphism $M \xrightarrow{h} K$ such that $\mathfrak{e} \circ h = id_M$. Since M is an injective object of $\mathcal{C}_{\mathfrak{X}_{\mathfrak{a}}}$, the monomorphism $\mathfrak{j} \circ h$ splits. If the object N belongs to the subcategory $\mathcal{T}$, then M is also an object of $\mathcal{T}$, because thick subcategories contain all direct summands of all their objects.

(b) The equality $(\mathcal{S} \sqcup \mathcal{T})^{\mathfrak{a}} = \mathcal{S}^{\mathfrak{a}} \sqcup \mathcal{T}^{\mathfrak{a}}$ holds for any pair $\mathcal{S}$, $\mathcal{T}$ of thick triangulated subcategories of $\mathcal{T}\mathcal{C}_{\mathfrak{X}}$.

In fact, the squares

$$\begin{tikzcd} C_{\mathfrak{X}} \arrow[r] \arrow[d] & C_{\mathfrak{X}}/\mathcal{T} \arrow[d] \\ C_X/\mathcal{S} \arrow[r] & C_X/(\mathcal{S} \sqcup \mathcal{T}) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} C_{\mathfrak{X}_{\mathfrak{a}}} \arrow[r] \arrow[d] & C_{\mathfrak{X}_{\mathfrak{a}}}/\mathcal{T}^{\mathfrak{a}} \arrow[d] \\ C_{\mathfrak{X}_{\mathfrak{a}}}/\mathcal{S}^{\mathfrak{a}} \arrow[r] & C_{\mathfrak{X}_{\mathfrak{a}}}/(\mathcal{S}^{\mathfrak{a}} \sqcup \mathcal{T}^{\mathfrak{a}}) \end{tikzcd}$$

are cocartesian and the abelianization functor transforms cocartesian squares into cocartesian squares, which implies that the unique functor

$$C_{\mathfrak{X}_{\mathfrak{a}}}/(\mathcal{S}^{\mathfrak{a}} \sqcup \mathcal{T}^{\mathfrak{a}}) \longrightarrow C_{\mathfrak{X}_{\mathfrak{a}}}/(\mathcal{S} \sqcup \mathcal{T})^{\mathfrak{a}}$$

is a category equivalence.

(c) *The equality* $\bigcap_{i \in J} \mathbb{T}_i^{\mathfrak{a}} = \big(\bigcap_{i \in J} \mathbb{T}_i\big)^{\mathfrak{a}}$ *holds for any finite family* $\{\mathbb{T}_i \mid i \in J\}$ *of thick triangulated subcategories of* $\mathcal{T}\mathcal{C}_{\mathfrak{X}}$.

Replacing $\mathcal{T}\mathcal{C}_{\mathfrak{X}}$ by $\mathcal{T}\mathcal{C}_{\mathfrak{X}}/\mathcal{T}$ and $\mathcal{T}_i$ by $\mathcal{T}_i/\mathcal{T}$, where $\mathcal{T} = \bigcap_{i \in J} \mathbb{T}_i$, we reduce the assertion to the case $\bigcap_{i \in J} \mathbb{T}_i = 0$. In this case, the claim is $\bigcap_{i \in J} \mathbb{T}_i^{\mathfrak{a}} = 0$. The equality $\bigcap_{i \in J} \mathbb{T}_i = 0$ means precisely that the triangle functor

$$\mathcal{T}\mathcal{C}_{\mathfrak{X}} \longrightarrow \prod_{i \in J} \mathcal{T}\mathcal{C}_{\mathfrak{X}}/\mathcal{T}_i$$

induced by the localization functors $\{\mathcal{T}\mathcal{C}_{\mathfrak{X}} \longrightarrow \mathcal{T}\mathcal{C}_{\mathfrak{X}}/\mathcal{T}_i \mid i \in J\}$ is faithful. But, then its abelianization,

$$\mathcal{C}_{\mathfrak{X}_{\mathfrak{a}}} \longrightarrow \prod_{i \in J} \mathcal{C}_{\mathfrak{X}_{\mathfrak{a}}}/\mathcal{T}_i^{\mathfrak{a}},$$

is a faithful functor, i.e. its kernel, the intersection $\bigcap_{i \in J} \mathbb{T}_i^{\mathfrak{a}}$, equals to zero.

(d) It follows from (a), (b) and (c) that

$$(\bigcap_{i\in J}\mathcal{T}_i)\sqcup\mathcal{S} = ((\bigcap_{i\in J}\mathcal{T}_i^{\mathfrak{a}})\sqcup\mathcal{S}^{\mathfrak{a}})\bigcap\mathcal{C}_{\mathfrak{X}} = (\bigcap_{i\in J}(\mathcal{T}_i^{\mathfrak{a}}\sqcup\mathcal{S}^{\mathfrak{a}}))\bigcap\mathcal{C}_{\mathfrak{X}} =$$
$$\bigcap_{i\in J}((\mathcal{T}_i^{\mathfrak{a}}\sqcup\mathcal{S}^{\mathfrak{a}})\bigcap\mathcal{C}_{\mathfrak{X}}) = \bigcap_{i\in J}(\mathcal{T}_i\sqcup\mathcal{S}).$$

for any finite family $\{\mathcal{S},\ \mathcal{T}_i \mid i\in J\}$ of thick triangulated subcategories of $\mathcal{TC}_{\mathfrak{X}}$. ∎

4.2. Presite of exact localizations. Let $\mathfrak{Esp}^{\mathfrak{L}}_{\mathfrak{Tr}}$ denote the subcategory of the category $\mathfrak{Esp}_{\mathfrak{Tr}}$ of t-'spaces' whose objects are t-'spaces' and morphisms are localizations (i.e. their inverse image functors are compositions of localization functors and category equivalences). We call a set $\{\mathfrak{U}_i \stackrel{\mathfrak{u}_i}{\to} \mathfrak{X} \mid i\in J\}$ of morphisms of $\mathfrak{Esp}^{\mathfrak{L}}_{\mathfrak{Tr}}$ a *cover* of the t-'space' $\mathfrak{X}$ if there is a finite subset $\mathfrak{J}$ of J such that the family of inverse image functors $\{\mathfrak{TC}_{\mathfrak{X}} \xrightarrow{\mathfrak{u}_i^*} \mathfrak{TC}_{\mathfrak{U}_i} \mid i\in\mathfrak{J}\}$ is conservative. We denote the set of all such covers of $\mathfrak{X}$ by $\mathfrak{T}_{\mathfrak{f}}(\mathfrak{X})$.

4.2.1. Proposition. *The covers defined above form a pretopology, $\mathfrak{T}_{\mathfrak{f}}$, on $\mathfrak{Esp}^{\mathfrak{L}}_{\mathfrak{Tr}}$.*

Proof. The morphisms of the subcategory $\mathfrak{Esp}^{\mathfrak{L}}_{\mathfrak{Tr}}$ are determined, uniquely up to isomorphism, by the kernel of their inverse image functors. A family of inverse image functors $\{\mathfrak{TC}_{\mathfrak{X}} \xrightarrow{\mathfrak{u}_i^*} \mathfrak{TC}_{\mathfrak{U}_i} \mid i\in\mathfrak{J}\}$ is conservative iff the intersection of kernels of these inverse image functors is zero. The assertion follows now from 4.1. ∎

5. The spectra of t-'spaces'.

Fix a svelte triangulated category $\mathcal{CT}_{\mathfrak{X}} = (\mathcal{C}_{\mathfrak{X}},\theta_{\mathfrak{X}},\mathfrak{Tr}_{\mathfrak{X}})$. We denote by $\mathfrak{Tht}(\mathfrak{X})$ the preorder (with respect to the inclusion) of all thick triangulated subcategories of $\mathcal{CT}_{\mathfrak{X}}$. Recall that a full triangulated subcategory of $\mathcal{CT}_{\mathfrak{X}}$ is called *thick* if it contains all direct summands of its objects.

5.1. $\mathbf{Spec}^1_{\mathfrak{L}}(\mathfrak{X})$ and its decompositions. For any triangulated subcategory $\mathcal{T}$ of $\mathcal{CT}_{\mathfrak{X}}$, let $\mathcal{T}^\star$ denote the intersection of all thick triangulated subcategories of $\mathcal{CT}_{\mathfrak{X}}$ which contain $\mathcal{T}$ properly. And let $\mathcal{T}_\star$ be the intersection of $\mathcal{T}^\star$ and the subcategory $\mathcal{T}^\perp$ – the right orthogonal to $\mathcal{T}$. Recall that $\mathcal{T}^\perp$ is the full subcategory of $\mathcal{CT}_{\mathfrak{X}}$ generated by all objects N such that $\mathcal{CT}_{\mathfrak{X}}(N,M)=0$ for all $M\in Ob\mathcal{T}$. It follows that $\mathcal{T}^\perp$ is a triangulated subcategory of $\mathcal{CT}_{\mathfrak{X}}$ (for any subcategory $\mathcal{T}$ which is stable by the translation functor).

We denote by $\mathbf{Spec}^1_{\mathfrak{L}}(\mathfrak{X})$ the subpreorder of $\mathfrak{Tht}(\mathfrak{X})$ formed by all thick triangulated subcategories $\mathcal{P}$ for which $\mathcal{P}^\star\neq\mathcal{P}$. We have a decomposition

$$\mathbf{Spec}^1_{\mathfrak{L}}(\mathfrak{X}) = \mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X})\coprod\mathbf{Spec}^{1,0}_{\mathfrak{L}}(\mathfrak{X})$$

of $\mathbf{Spec}^1_{\mathfrak{L}}(\mathfrak{X})$ into a disjoint union of

$$\mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X}) = \{\mathcal{P} \in \mathfrak{Th}\mathfrak{t}(\mathfrak{X}) \mid \mathcal{P}_\star \neq 0\} \quad \text{and}$$
$$\mathbf{Spec}^{1,0}_{\mathfrak{L}}(\mathfrak{X}) = \{\mathcal{P} \in \mathbf{Spec}^1_{\mathfrak{L}}(\mathfrak{X}) \mid \mathcal{P}_\star = 0\}.$$

5.2. $\mathfrak{L}$-Local triangulated categories and $\mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X})$. We call a triangulated category $C\mathfrak{T}_{\mathfrak{Y}}$ $\mathfrak{L}$-*local* if it has the smallest nonzero thick triangulated subcategory.

5.2.1. Proposition. *Let $\mathcal{P} \in \mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X})$. Then*

(a) $\mathcal{P} = {}^\perp\mathcal{P}_*$.

(b) The triangulated category $\mathcal{P}^\perp$ is $\mathfrak{L}$-local and $\mathcal{P}_\star$ is its smallest nonzero thick triangulated subcategory.

Proof. (a) The condition $\mathcal{P}_\star \neq 0$ implies, obviously, that $\mathcal{P}^\star$ contains $\mathcal{P}$ properly, i.e. $\mathcal{P}$ is an object of $\mathbf{Spec}^1_{\mathfrak{L}}(\mathfrak{X})$

The inclusion $\mathcal{P}_\star \subseteq \mathcal{P}^\perp$ is equivalent to the inclusion $\mathcal{P} \subseteq^\perp \mathcal{P}_\star$. If the (thick triangulated) subcategory ${}^\perp\mathcal{P}_\star$ contains $\mathcal{P}$ properly, then ${}^\perp\mathcal{P}_\star/\mathcal{P}$ is a nonzero thick triangulated subcategory of $C\mathcal{T}_{\mathfrak{X}}/\mathcal{P}$, hence it contains the image, $\widetilde{\mathcal{P}}_\star$, of the subcategory $\mathcal{P}_\star$. This means that for every $L \in Ob\mathcal{P}_\star$, there exists an object, M, of ${}^\perp\mathcal{P}_\star$ and an isomorphism $q^*_{\mathcal{P}}(M) \xrightarrow{\sim} q^*_{\mathcal{P}}(L)$. The latter is determined by a diagram $M \xrightarrow{s'} K \xleftarrow{s} L$ whose both arrows belong to $\Sigma_{\mathcal{P}}$. Since L is an object of $\mathcal{P}^\perp$, it follows from the equivalence of (iii) and (iv) in 1.8.1 that the morphism $K \xleftarrow{s} L$ admits a retraction, $K \xrightarrow{s''} L$. Let $M \xrightarrow{t} L$ be the composition $s'' \circ s'$. Since the morphism t belongs to $\Sigma_{\mathcal{P}}$, there exists a triangle $M \xrightarrow{t} L \longrightarrow N$ such that $N \in Ob\mathcal{P}$. In particular, $N \in Ob^\perp\mathcal{P}_\star$. Thus, we have a triangle, $M \xrightarrow{t} L \longrightarrow N$, such that M and N are objects of the thick subcategory ${}^\perp\mathcal{P}_\star$. Therefore, L is an object of ${}^\perp\mathcal{P}_\star$, which cannot happen, unless $L = 0$. Thus, ${}^\perp\mathcal{P}_\star$ cannot contain $\mathcal{P}$ properly, i.e. $\mathcal{P} = {}^\perp\mathcal{P}_\star$.

(b) Let $\mathbb{T}$ be a nonzero thick triangulated subcategory of $\mathcal{P}^\perp$. Then the image, $q^*_{\mathcal{P}}(\mathbb{T})$, in the quotient category $C\mathcal{T}_{\mathfrak{X}}/\mathcal{P}$ is nonzero, hence its thick envelope contains the subcategory $\mathcal{P}^\star/\mathcal{P}$. In particular, it contains the image of the subcategory $\mathcal{P}_\star$. Since objects of the thick envelope of $q^*_{\mathcal{P}}(\mathbb{T})$ are direct summands of objects of $q^*_{\mathcal{P}}(\mathbb{T})$, this means that for every object L of $\mathcal{P}_\star$, there exists an object M of $\mathbb{T}$ such that $q^*_{\mathcal{P}}(L)$ is a direct summand of $q^*_{\mathcal{P}}(M)$. Since both objects, L and M, belong to the subcategory $\mathcal{P}^\perp$ and, by 1.8.1(c) (and 1.8.1(i) $\Leftrightarrow$ (iv)), the restriction of the localization functor $q^*_{\mathcal{P}}$ to the subcategory $\mathcal{P}^\perp$ is a fully faithful functor, it follows that L is a direct summand of M. Since $\mathbb{T}$ is a thick subcategory of $C\mathcal{T}_{\mathfrak{X}}$, it contains all direct summands of its objects. Thus $\mathcal{P}_\star \subseteq \mathbb{T}$. ∎

5.2.1.1. Corollary. *Let $\mathcal{P}$ be a thick triangulated subcategory of $C\mathcal{T}_{\mathfrak{X}}$ such that the intersection $\mathcal{P}_\star = \mathcal{P}^\perp \cap \mathcal{P}^\star$ is nonzero. Then $\mathcal{P}$ is closed under all colimits which exist in $C\mathcal{T}_{\mathfrak{X}}$, in particular, $\mathcal{P}$ is closed under all coproducts which exist in $C\mathcal{T}_{\mathfrak{X}}$.*

Proof. In fact, by 5.2.1, $\mathcal{P} = {}^{\perp}\mathcal{P}_{\star}$; and the left orthogonal to any subcategory is closed under arbitrary colimits which exist in $C\mathcal{T}_{\mathfrak{X}}$. ■

5.2.2. Proposition. *Suppose that infinite coproducts or products exist in $C\mathcal{T}_{\mathfrak{X}}$. Let $\mathcal{P}$ be a thick triangulated subcategory of $C\mathcal{T}_{\mathfrak{X}}$. Then the following properties of are equivalent:*

(i) $\mathcal{P}_{\star} = \mathcal{P}^{\perp} \cap \mathcal{P}^{\star}$ is nonzero, i.e. $\mathcal{P} \in \mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X})$;

(ii) $\mathcal{P}$ belongs to $\mathbf{Spec}^{1}_{\mathfrak{L}}(\mathfrak{X})$ and the composition of the inclusion $\mathcal{P}_{\star} \hookrightarrow C\mathcal{T}_{\mathfrak{X}}$ and the localization functor $C\mathcal{T}_{\mathfrak{X}} \xrightarrow{q^*_{\mathcal{P}}} C\mathcal{T}_{\mathfrak{X}}/\mathcal{P}$ induces an equivalence of triangulated categories $\mathcal{P}_{\star} \xrightarrow{\sim} \mathcal{P}^{\star}/\mathcal{P}$.

(iii) $\mathcal{P}$ belongs to $\mathbf{Spec}^{1}_{\mathfrak{L}}(\mathfrak{X})$ and the inclusion functor $\mathcal{P} \hookrightarrow \mathcal{P}^{\star}$ has a right adjoint.

(iv) $\mathcal{P}$ belongs to $\mathbf{Spec}^{1}_{\mathfrak{L}}(\mathfrak{X})$ and $\mathcal{P}^{\perp}$ is nonzero.

Proof. The implications $(iii) \Leftarrow (ii) \Rightarrow (i) \Rightarrow (iv) \Leftarrow (iii)$ hold by obvious reasons (see 3.1(a)). The implication $(iii) \Rightarrow (ii)$ follows from 1.8.3 (see also 1.8.2). Thus, $(iii) \Leftrightarrow (ii) \Rightarrow (i) \Rightarrow (iv)$ without any additional hypothesis on $C\mathcal{T}_{\mathfrak{X}}$. The existence of infinite coproducts or products is needed for the implication

$(iv) \Rightarrow (ii)$. Fix an object $\mathcal{P}$ of $\mathbf{Spec}^{1}_{\mathfrak{L}}(\mathfrak{X})$. By 1.8.1(e) (see also 1.8.1(i) $\Leftrightarrow$ (iv)), the composition of the localization functor $q^*_{\mathcal{P}}$ with the inclusion functor $\mathcal{P}^{\perp} \hookrightarrow C\mathcal{T}_{\mathfrak{X}}$ induces an equivalence of the triangulated categories $\mathcal{P}^{\perp} \longrightarrow C\mathcal{T}_{\mathfrak{Tr}_{\mathfrak{X}}(q_{\mathcal{P}*})}$. Here $C\mathcal{T}_{\mathfrak{Tr}_{\mathfrak{X}}(q_{\mathcal{P}*})}$ is the full subcategory of the quotient category $C\mathcal{T}_{\mathfrak{X}}/\mathcal{P}$ generated by all objects M such that the functor $C\mathcal{T}_{\mathfrak{X}}/\mathcal{P}(q^*_{\mathcal{P}}(-), M)$ is representable. By 1.8.1(d), if infinite coproducts, or infinite products exist in $C\mathcal{T}_{\mathfrak{X}}$, then $C\mathcal{T}_{\mathfrak{Tr}_{\mathfrak{X}}(q_{\mathcal{P}*})}$ is a thick triangulated subcategory of the $\mathfrak{L}$-local triangulated category $C\mathcal{T}_{\mathfrak{X}}/\mathcal{P}$. If $\mathcal{P}^{\perp} \neq 0$, then (and only then) the subcategory $C\mathcal{T}_{\mathfrak{Tr}_{\mathfrak{X}}(q_{\mathcal{P}*})}$ is nonzero, hence it contains the (smallest non-trivial thick) subcategory $\mathcal{P}^{\star}/\mathcal{P}$ which implies that $C\mathcal{T}_{\mathfrak{Tr}_{\mathfrak{X}}(q_{\mathcal{P}*})}$ is an $\mathfrak{L}$-local triangulated category having $\mathcal{P}^{\star}/\mathcal{P}$ as the smallest nonzero thick subcategory. This, in turn, implies that $\mathcal{P}_{\star} = \mathcal{P}^{\perp} \cap \mathcal{P}^{\star}$ is nonzero and, moreover, the localization $q^*_{\mathcal{P}}$ induces an equivalence between $\mathcal{P}_{\star}$ and $\mathcal{P}^{\star}/\mathcal{P}$. ■

5.2.3. Corollary. *Suppose that infinite coproducts or products exist in $C\mathcal{T}_{\mathfrak{X}}$. Then $\mathbf{Spec}^{1,0}_{\mathfrak{L}}(\mathfrak{X})$ consists of all $\mathcal{P} \in \mathbf{Spec}^{1}_{\mathfrak{L}}(\mathfrak{X})$ such that $\mathcal{P}^* \neq \mathcal{P}$ and $\mathcal{P}^{\perp} = 0$.*

5.2.4. Remark. Loosely, 5.2.3 says that the elements of $\mathbf{Spec}^{1,0}_{\mathfrak{L}}(\mathfrak{X})$ can be regarded as "fat" points – they generate (in a weak sense) the whole category $C\mathfrak{T}_{\mathfrak{X}}$.

The local properties of $\mathbf{Spec}^{1}_{\mathfrak{L}}(\mathfrak{X})$ and $\mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X})$ are described by the following proposition:

5.2.5. Proposition. *(a) Let $\{\mathcal{T}_i \mid i \in J\}$ be a finite set of thick subcategories of a triangulated category $\mathcal{T}C_{\mathfrak{X}}$ such that $\bigcap_{i \in J} \mathcal{T}_i = 0$. Then*

$$\mathbf{Spec}^{1}_{\mathfrak{L}}(\mathfrak{X}) = \bigcup_{i \in J} \mathbf{Spec}^{1}_{\mathfrak{L}}(\mathfrak{X}/\mathcal{T}_i) \tag{1}$$

(b) Suppose that ${}^\perp(\mathcal{T}_i^\perp) = \mathcal{T}_i$ *for all* $i \in J$*. Then*

$$\mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X}) = \bigcup_{i\in J} \mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X}/\mathcal{T}_i) \tag{2}$$

Proof. (a) The inclusion $\bigcup_{i\in J} \mathbf{Spec}^{1}_{\mathfrak{L}}(\mathfrak{X}/\mathcal{T}_i) \subseteq \mathbf{Spec}^{1}_{\mathfrak{L}}(\mathfrak{X})$ follows from the functoriality of $\mathbf{Spec}^{1}_{\mathfrak{L}}(-)$ with respect to localizations. Let $\mathcal{P} \in \mathbf{Spec}^{1}_{\mathfrak{L}}(\mathfrak{X})$. By 4.1,

$$\mathcal{P} = \big(\bigcap_{i\in J} T_i\big) \sqcup \mathcal{P} = \bigcap_{i\in J}(T_i \sqcup \mathcal{P}) \tag{3}$$

which implies that $\mathcal{T}_i \subseteq \mathcal{P}$ for some $i \in J$. In fact, if $\mathcal{T}_i \nsubseteq \mathcal{P}$ for all $i \in J$, then $T_i \sqcup \mathcal{P}$ contains properly $\mathcal{P}_i$ for all $i \in J$, hence the intersection $\bigcap_{i\in J}(T_i \sqcup \mathcal{P})$ contains properly $\mathcal{P}$, which contradicts to (3). This proves the inverse inclusion, that is the equality (1).

(b) The inclusion $\bigcup_{i\in J} \mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X}/\mathcal{T}_i) \subseteq \mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X})$ follows from the functoriality of $\mathbf{Spec}^{1,1}_{\mathfrak{L}}(-)$ with respect to localizations at thick subcategories $\mathcal{T}$ such that ${}^\perp(\mathcal{T}^\perp) = \mathcal{T}$. The inverse inclusion follows from (a). ∎

5.3. The spectrum $\mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X})$. Let $\mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X})$ denote the full subpreorder of $\mathfrak{Tht}(\mathfrak{X})$ whose objects are thick triangulated subcategories $\mathcal{Q}$ such that ${}^\perp\mathcal{Q}$ belongs to $\mathbf{Spec}^{1}_{\mathfrak{L}}(\mathfrak{X})$ and every thick triangulated subcategory of $\mathcal{CT}_{\mathfrak{X}}$ properly containing ${}^\perp\mathcal{Q}$ contains $\mathcal{Q}$; i.e. ${}^\perp\mathcal{Q} \vee \mathcal{Q}$ is the smallest thick triangulated subcategory of $\mathcal{CT}_{\mathfrak{X}}$ properly containing ${}^\perp\mathcal{Q}$.

5.3.1. Proposition. *(a) The map* $\mathcal{Q} \longmapsto {}^\perp\mathcal{Q}$ *induces a bijective map*

$$\mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X}) \xrightarrow{\sim} \mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X}). \tag{1}$$

(b) If $\mathcal{Q}$ *is an object of* $\mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X})$*, then* $\mathcal{Q}$ *is a minimal nonzero thick triangulated subcategory of* $\mathcal{CT}_{\mathfrak{X}}$*.*

(c) Suppose that $\mathcal{CT}_{\mathfrak{X}}$ *has infinite coproducts or products. Then the following properties of a thick triangulated subcategory* $\mathcal{Q}$ *are equivalent:*

(i) $\mathcal{Q}$ *belongs to* $\mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X})$*;*

(ii) $\mathcal{Q}$ *is a minimal nonzero thick triangulated subcategory of* $\mathcal{CT}_{\mathfrak{X}}$ *such that* ${}^\perp\mathcal{Q}$ *belongs to* $\mathbf{Spec}^{1}_{\mathfrak{L}}(\mathfrak{X})$*.*

Proof. (a) Let $\mathcal{Q}$ be an object of $\mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X})$. This means that ${}^{\perp}\mathcal{Q}$ belongs to $\mathbf{Spec}^{1}_{\mathfrak{L}}(\mathfrak{X})$ and $({}^{\perp}\mathcal{Q})^{\star} = {}^{\perp}\mathcal{Q} \vee \mathcal{Q}$. Since $\mathcal{Q}$ is contained in the intersection

$$\mathcal{Q}_1 = ({}^{\perp}\mathcal{Q})^{\perp} \cap ({}^{\perp}\mathcal{Q})^{\star} = ({}^{\perp}\mathcal{Q})^{\perp} \cap ({}^{\perp}\mathcal{Q} \vee \mathcal{Q})$$

and $\mathcal{Q} \neq 0$, the subcategory ${}^{\perp}\mathcal{Q}$ belongs to $\mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X})$.

By 3.1(b), the triangulated category $({}^{\perp}\mathcal{Q})^{\perp}$ is $\mathfrak{L}$-local and $\mathcal{Q}_1$ is its smallest nonzero thick triangulated subcategory. Therefore, $\mathcal{Q}_1 = \mathcal{Q}$. Thus, the composition of the map

$$\mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X}) \longrightarrow \mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X}), \qquad \mathcal{Q} \longmapsto {}^{\perp}\mathcal{Q}, \tag{1}$$

with the map $\mathcal{P} \longmapsto \mathcal{P}_{\star} = \mathcal{P}^{\perp} \cap \mathcal{P}^{\star}$ is identical. It follows from 3.1 that the correspondence $\mathcal{P} \longmapsto \mathcal{P}_{\star}$ defines a map

$$\mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X}) \longrightarrow \mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X}). \tag{2}$$

The argument above shows that the map (2) is inverse to the map (1).

(b) If $\mathcal{Q}$ is an object of $\mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X})$, then, by (a), $\mathcal{Q}$ is the smallest thick triangulated subcategory of the $\mathfrak{L}$-local category $({}^{\perp}\mathcal{Q})^{\perp}$; in particular, $\mathcal{Q}$ is a minimal nonzero thick triangulated subcategory of $\mathcal{CT}_{\mathfrak{X}}$.

(c) Suppose that $\mathcal{CT}_{\mathfrak{X}}$ has infinite coproducts or products. Let $\mathcal{Q}$ be a thick triangulated subcategory of $\mathcal{CT}_{\mathfrak{X}}$ such that ${}^{\perp}\mathcal{Q}$ belongs to $\mathbf{Spec}^{1}_{\mathfrak{L}}(\mathfrak{X})$. Then $({}^{\perp}\mathcal{Q})^{\perp}$ contains a nonzero subcategory $\mathcal{Q}$, hence it is nonzero. By 3.3(iv), this is equivalent to that $\mathcal{Q}_1 = ({}^{\perp}\mathcal{Q})^{\perp} \cap ({}^{\perp}\mathcal{Q})^{\star}$ is nonzero. By 3.1(b), $\mathcal{Q}_1$ is the smallest thick triangulated subcategory of the $\mathfrak{L}$-local triangulated category $({}^{\perp}\mathcal{Q})^{\perp}$. In particular, $\mathcal{Q}_1 \subseteq \mathcal{Q}$. If $\mathcal{Q}$ is a minimal nonzero thick triangulated subcategory of $\mathcal{CT}_{\mathfrak{X}}$, then the inclusion $\mathcal{Q}_1 \subseteq \mathcal{Q}$ implies that $\mathcal{Q}$ coincides with $\mathcal{Q}_1$. The assertion follows now from (a). ∎

5.3.2. Corollary. *(a) If $\mathcal{Q}$ is an object of $\mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X})$, then $\mathcal{Q} = [M]_{\mathfrak{t}}$ for any nonzero object M of $\mathcal{Q}$.*

(b) The following properties of an object M of the category $\mathcal{CT}_{\mathfrak{X}}$ are equivalent:

(i) The thick envelope, $[M]_{\mathfrak{t}}$, of M belongs to $\mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X})$.

(ii) ${}^{\perp}M$ belongs to $\mathbf{Spec}^{1}_{\mathfrak{L}}(\mathfrak{X})$, and if $\mathbb{T}$ is a thick triangulated subcategory of $\mathcal{CT}_{\mathfrak{X}}$ properly containing ${}^{\perp}M$, then $M \in Ob\mathbb{T}$.

(iii) $[M]_{\mathfrak{t}}$ is a minimal nonzero thick subcategory, and ${}^{\perp}M$ belongs to $\mathbf{Spec}^{1}_{\mathfrak{L}}(\mathfrak{X})$.

(c) The equivalent conditions (i), (ii), or (iii) imply the following property:

(iv) ${}^{\perp}M$ belongs to $\mathbf{Spec}^{1}_{\mathfrak{L}}(\mathfrak{X})$, and every nonzero thick triangulated subcategory of $({}^{\perp}M)^{\perp}$ contains M.

(d) If infinite coproducts or products exist in $\mathcal{CT}_{\mathfrak{X}}$, then (iv) is equivalent to the properties (i), (ii), and (iii).

Proof. (a) The assertion follows from the minimality of $\mathcal{Q}$ (see 5.3.1(b)).

(b) $(i) \Rightarrow (ii)$. Let $\mathcal{Q}$ be an object of $\mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X})$; and let M be a nonzero object of $\mathcal{Q}$. Since $\mathcal{Q} = [M]_{\mathfrak{t}}$ and ${}^{\perp}M = {}^{\perp}[M]_{\mathfrak{t}}$, the subcategory ${}^{\perp}M$ coincides with ${}^{\perp}\mathcal{Q}$. Since $\mathcal{Q}$ belongs to $\mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X})$, ${}^{\perp}\mathcal{Q} \vee \mathcal{Q}$ is the smallest thick subcategory of $\mathcal{CT}_{\mathfrak{X}}$ properly containing ${}^{\perp}\mathcal{Q}$. It contains the object M.

$(ii) \Rightarrow (i)$. The conditions (ii) mean that ${}^{\perp}M \vee [M]_{\mathfrak{t}}$ is the smallest thick triangulated subcategory of $\mathcal{CT}_{\mathfrak{X}}$ properly containing ${}^{\perp}M = {}^{\perp}[M]_{\mathfrak{t}}$.

The implications $(ii) \Leftrightarrow (iii)$ follow from 1.7.

(c) $(iii) \Rightarrow (iv)$. Let $\mathcal{Q} = [M]_{\mathfrak{t}}$. It follows from 1.7(ii) that $({}^{\perp}M)^{\perp} = ({}^{\perp}\mathcal{Q})^{\perp}$ is an $\mathfrak{L}$-local triangulated category and $\mathcal{Q} = [M]_{\mathfrak{t}}$ is its smallest nonzero thick subcategory. Clearly $M \in Ob\mathcal{Q}$.

(d) The implication $(iv) \Rightarrow (iii)$ follows from 1.7(c). ■

5.3.3. Corollary. *Let $\mathcal{TC}_{\mathfrak{X}}$ be a triangulated category with small coproducts or products. Then* $\mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X}) = \{\mathcal{P} \in \mathbf{Spec}^{1}_{\mathfrak{L}}(\mathfrak{X}) \mid {}^{\perp}(\mathcal{P}^{\perp}) = \mathcal{P}\}$.

Proof. The inclusion $\mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X}) \subseteq \{\mathcal{P} \in \mathbf{Spec}^{1}_{\mathfrak{L}}(\mathfrak{X}) \mid {}^{\perp}(\mathcal{P}^{\perp}) = \mathcal{P}\}$ holds without any conditions on the triangulated category $\mathcal{TC}_{\mathfrak{X}}$ and follows from 5.3.1, because if $\mathcal{P}$ is an element of $\mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X})$ and $\mathcal{Q} = \mathcal{P}^{\star} \cap \mathcal{P}^{\perp}$, then $\mathcal{P} \subseteq {}^{\perp}(\mathcal{P}^{\perp}) \subseteq {}^{\perp}\mathcal{Q} = \mathcal{P}$.

If the triangulated category $\mathcal{TC}_{\mathfrak{X}}$ has infinite products or coproducts, then, by 5.2.2, $\mathbf{Spec}^{1,0}_{\mathfrak{L}}(\mathfrak{X})$ consists of all $\mathcal{P} \in \mathbf{Spec}^{1}_{\mathfrak{L}}(\mathfrak{X})$ such that $\mathcal{P}^{\perp} = 0$, i.e. ${}^{\perp}(\mathcal{P}^{\perp}) = \mathcal{TC}_{\mathfrak{X}}$. In particular, ${}^{\perp}(\mathcal{P}^{\perp}) \neq \mathcal{P}$. ■

5.4. Flat spectra. Let $\mathfrak{Se}(\mathfrak{X})$ denote the family of all thick triangulated subcategories of the triangulated category $\mathcal{CT}_{\mathfrak{X}}$ which satisfy equivalent conditions of 1.8.3. We define the *complete flat spectrum* of $\mathfrak{X}$, $\mathbf{Spec}^{1}_{\mathfrak{fL}}(\mathfrak{X})$, by setting

$$\mathbf{Spec}^{1}_{\mathfrak{fL}}(\mathfrak{X}) = \mathbf{Spec}^{1}_{\mathfrak{L}}(\mathfrak{X}) \bigcap \mathfrak{Se}(\mathfrak{X}). \tag{1}$$

We define the *flat spectrum of* $\mathfrak{X}$ as a full subpreorder, $\mathbf{Spec}^{0}_{\mathfrak{fL}}(\mathfrak{X})$, of $\mathfrak{Tht}(\mathfrak{X})$ whose objects are all $\mathcal{P}$ such that $\widehat{\mathcal{P}} \in \mathbf{Spec}^{1}_{\mathfrak{fL}}(\mathfrak{X})\}$.

It follows from these definitions that the map $\mathcal{P} \longmapsto \widehat{\mathcal{P}}$ defines an injective morphism

$$\mathbf{Spec}^{0}_{\mathfrak{fL}}(\mathfrak{X}) \longrightarrow \mathbf{Spec}^{1}_{\mathfrak{fL}}(\mathfrak{X}). \tag{2}$$

Let $\mathbf{Spec}^{1/2}_{\mathfrak{fL}}(\mathfrak{X})$ denote the full subpreorder of $\mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X})$ whose objects are all $\mathcal{Q}$ such that ${}^{\perp}\mathcal{Q}$ belongs to $\mathfrak{Se}(\mathfrak{X})$.

5.4.2. Proposition. *(a) The map*

$$\mathfrak{Tht}(\mathfrak{X}) \longrightarrow \mathfrak{Tht}(\mathfrak{X}), \quad \mathcal{Q} \longmapsto {}^{\perp}\mathcal{Q},$$

induces an isomorphism

$$\mathbf{Spec}^{1/2}_{\mathfrak{f}\mathfrak{L}}(\mathfrak{X}) \xrightarrow{\sim} \mathbf{Spec}^{1}_{\mathfrak{f}\mathfrak{L}}(\mathfrak{X}). \tag{3}$$

(b) $\mathbf{Spec}^{0}_{\mathfrak{f}\mathfrak{L}}(\mathfrak{X}) = \mathbf{Spec}^{0}_{\mathfrak{L}}(\mathfrak{X}) \bigcap \mathbf{Spec}^{1/2}_{\mathfrak{f}\mathfrak{L}}(\mathfrak{X})$. *The canonical morphism (2) is the composition of the inclusion* $\mathbf{Spec}^{0}_{\mathfrak{f}\mathfrak{L}}(\mathfrak{X}) \hookrightarrow \mathbf{Spec}^{1/2}_{\mathfrak{f}\mathfrak{L}}(\mathfrak{X})$ *and the isomorphism (3).*

Proof. Notice that $\mathbf{Spec}^{1}_{\mathfrak{f}\mathfrak{L}}(\mathfrak{X}) \subseteq \mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X})$. This follows from 1.8.3 and the definitions of these spectra. Now the assertion becomes a consequence of 5,3.1. ∎

5.4.3. Proposition. *(a) If $\mathcal{Q}$ is an object of $\mathbf{Spec}^{1/2}_{\mathfrak{f}\mathfrak{L}}(\mathfrak{X})$, then $\mathcal{Q} = [M]_{\mathfrak{t}}$ (hence ${}^{\perp}\mathcal{Q} = {}^{\perp}M$) for any nonzero object M of $\mathcal{Q}$.*

(b) The following properties of an object M of the category $\mathcal{CT}_{\mathfrak{X}}$ are equivalent:

(i) The thick envelope, $[M]_{\mathfrak{t}}$, of M belongs to $\mathbf{Spec}^{1/2}_{\mathfrak{f}\mathfrak{L}}(\mathfrak{X})$.

(ii) ${}^{\perp}M$ belongs to $\mathbf{Spec}^{1}_{\mathfrak{f}\mathfrak{L}}(\mathfrak{X})$, and if $\mathbb{T}$ is a thick triangulated subcategory of $\mathcal{CT}_{\mathfrak{X}}$ properly containing ${}^{\perp}M$, then $M \in Ob\mathbb{T}$.

(iii) $[M]_{\mathfrak{t}}$ is a minimal nonzero thick subcategory, and ${}^{\perp}M$ belongs to $\mathbf{Spec}^{1}_{\mathfrak{f}\mathfrak{L}}(\mathfrak{X})$.

(iv) M is a nonzero object which belongs to every nonzero thick triangulated subcategory of $({}^{\perp}M)^{\perp}$ and such that the inclusion functor ${}^{\perp}M \longrightarrow \mathcal{CT}_{\mathfrak{X}}$ has a right adjoint.

Proof. (a) The assertion is a consequence of the minimality of $\mathcal{Q}$ (see 5.3.1(b)).

(b) The implications $(i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Rightarrow (iv)$ follow from the corresponding implications of 5.3.2.

$(iv) \Rightarrow (iii)$. By 1.6.3, the inclusion functor ${}^{\perp}M \longrightarrow \mathcal{CT}_{\mathfrak{X}}$ has a right adjoint iff the localization functor $\mathcal{CT}_{\mathfrak{X}} \longrightarrow \mathcal{CT}_{\mathfrak{X}}/{}^{\perp}M$ has a right adjoint. The latter implies that the quotient category $\mathcal{CT}_{\mathfrak{X}}/{}^{\perp}M$ is equivalent to the triangulated category $({}^{\perp}M)^{\perp}$. The condition that M is contained in every nonzero thick triangulated subcategory of $({}^{\perp}M)^{\perp}$ means that $({}^{\perp}M)^{\perp}$ is $\mathfrak{L}$-local and $[M]_{\mathfrak{t}}$ is its smallest thick triangulated subcategory. Therefore, ${}^{\perp}M$ belongs to $\mathbf{Spec}^{1}_{\mathfrak{f}\mathfrak{L}}(\mathfrak{X})$, and $[M]_{\mathfrak{t}}$ is a minimal nonzero thick triangulated subcategory of $\mathcal{CT}_{\mathfrak{X}}$. ∎

5.5. Supports and Zariski topology.

5.5.1. Supports. For any object M of the category C_X, the support of M in $\mathbf{Spec}^{1}_{\mathfrak{L}}(\mathfrak{X})$ is defined by $Supp^{1}_{\mathfrak{L}}(M) = \{\mathcal{P} \in \mathbf{Spec}^{1}_{\mathfrak{L}}(X) \mid M \notin Ob\mathcal{P}\}$. It follows that $Supp^{1}_{\mathfrak{L}}(\mathcal{L} \oplus \mathcal{M}) = Supp^{1}_{\mathfrak{L}}(\mathcal{L}) \bigcup Supp^{1}_{\mathfrak{L}}(\mathcal{M})$.

5.5.2. Topologies on $\mathbf{Spec}^{1}_{\mathfrak{L}}(X)$ and $\mathbf{Spec}^{1,1}_{\mathfrak{L}}(X)$. We follow the pattern of II.2.3 and II.2.4. Let Ξ be a class of objects of C_X closed under finite coproducts. For any set E of objects of Xi, let $\mathcal{V}^{1}_{\mathfrak{L}}(E)$ denote the intersection $\bigcap\limits_{M \in E} Supp^{1}_{\mathfrak{L}}(M)$. Then, for any family

$\{E_i \mid i \in \mathfrak{J}\}$ of such sets, we have, evidently,

$$\mathcal{V}_{\mathfrak{L}}(\bigcup_{i\in J} E_i) = \bigcap_{i\in J} \mathcal{V}_{\mathfrak{L}}(E_i).$$

It follows from the equality $Supp^1_{\mathfrak{L}}(M \oplus N) = Supp^1_{\mathfrak{L}}(M) \bigcup Supp^1_{\mathfrak{L}}(N)$ (see 2.2.1(a)) that $\mathcal{V}^1_{\mathfrak{L}}(E \oplus \widetilde{E}) = \mathcal{V}^1_{\mathfrak{L}}(E) \bigcup \mathcal{V}^1_{\mathfrak{L}}(\widetilde{E})$. Here $E \oplus \widetilde{E} \stackrel{\mathrm{def}}{=} \{M \oplus N \mid M \in E,\ N \in \widetilde{E}\}$.

This shows that the subsets $\mathcal{V}^1_{\mathfrak{L}}(E)$ of $\mathbf{Spec}^1_{\mathfrak{L}}(\mathfrak{X})$, where E runs through subsets of Ξ, are all closed sets of a topology, τ^1_{Ξ}, on the spectrum $\mathbf{Spec}^1_{\mathfrak{L}}(\mathfrak{X})$.

We denote by $\tau^{1,1}_{\Xi}$ the induced topology on $\mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X})$ and by $\tau^{\mathfrak{f}\mathfrak{L}}_{\Xi}$ the induced topology on $\mathbf{Spec}^1_{\mathfrak{f}\mathfrak{L}}$.

5.5.3. Compact topology. The class $\Xi_{\mathfrak{c}}(X)$ of compact objects of the category $\mathcal{C}_{\mathfrak{X}}$ is closed under finite coproducts, hence it defines a topology on $\mathbf{Spec}^1_{\mathfrak{L}}(\mathfrak{X})$, which we denote by $\tau_{\mathfrak{c}}$ and call the *compact* topology.

Restricting the compact topology to $\mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X})$ or to $\mathbf{Spec}^1_{\mathfrak{f}\mathfrak{L}}(\mathfrak{X})$, we obtain the compact topology on these spectra.

5.5.4. Zariski topology on $\mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X})$. We define the *Zariski* topology on the spectrum $\mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X})$ by taking as a base of closed sets the supports of compact objects and closures (i.e. the sets of all specializations) of points of $\mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X})$.

If the category $\mathcal{C}_{\mathfrak{X}}$ is generated by compact objects, then the Zariski topology coincides with the compact topology $\tau_{\mathfrak{c}}$.

5.5.5. Zariski topology on $\mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X})$. It is important to realize that the topologies we define are determined in the first place by the choice of a preorder on the set of thick subcategories (or topologizing subcategories in the case of abelian categories). And so far, the preorder was always the inverse inclusion.

Following these pattern, for any object M of a svelte triangulated category $\mathcal{C}\mathfrak{T}_{\mathfrak{X}}$, we define the *support* of M in $\mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X})$ as the set of all $\mathcal{Q} \in \mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X})$ such that the smallest thick triangulated subcategory $[M]_{\mathfrak{tr}}$ containing M contains also $\mathcal{Q}$.

We define the *Zariski* topology on $\mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X})$ by taking supports of compact objects and the finite subsets of $\mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X})$ as a base of its closed sets.

It follows from this definition of Zariski topology and 5.2.1(b) that all points of the spectrum $\mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X})$ are closed; that is Zariski topology on $\mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X})$ is a T_1-*topology*. The bijective map

$$\mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X}) \xrightarrow{\sim} \mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X}) \qquad (4)$$

is continuous, but, usually, not a homeomorphism.

5.5.6. Remark. Suppose that C_X is the heart of a t-structure on $\mathcal{C}\mathfrak{T}_{\mathfrak{X}}$. Then we have a commutative diagram

$$\begin{array}{ccc} \mathbf{Spec}(X) & \longrightarrow & \mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X}) \\ \wr\big\downarrow & & \big\downarrow\wr \\ \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X) & \longrightarrow & \mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X}) \end{array} \tag{5}$$

where horizontal arrows are embeddings and vertical arrows are canonical bijections. Thus, the Zariski topology on $\mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X})$ induces a T_1-topology on the spectrum $\mathbf{Spec}(X)$ of the 'space' represented by the abelian category C_X, which, obviously, differs from Zariski topology on $\mathbf{Spec}(X)$, unless $\mathbf{Spec}(X)$ is of zero Krull dimension.

5.6. A geometric realization of a triangulated category. We assign to a Karoubian triangulated category $\mathfrak{T}\mathcal{C}_{\mathfrak{X}}$ having a set of compact generators the contravariant pseudo-functor from the category of Zariski open subsets of the spectrum $\mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X})$ to the category of svelte triangulated categories. The associated stack is the stack of local triangulated categories.

5.7. The geometric center. We define the *center* of a svelte triangulated category $\mathfrak{T}\mathcal{C}_{\mathfrak{Y}} = (\mathcal{C}_{\mathfrak{Y}}, \theta_{\mathfrak{Y}}, \mathfrak{Tr}_{\mathfrak{Y}})$ as the subring $\mathcal{O}^{\mathfrak{T}}(\mathfrak{Y})$ of the center $\mathfrak{z}(\mathcal{C}_{\mathfrak{X}})$ of the category $\mathcal{C}_{\mathfrak{Y}}$ formed by $\theta_{\mathfrak{Y}}$-invariant endomorphisms of the identical functor of $\mathcal{C}_{\mathfrak{Y}}$. One can show that the ring $\mathcal{O}^{\mathfrak{T}}(\mathfrak{Y})$ is local if the triangulated category $\mathfrak{T}\mathcal{C}_{\mathfrak{Y}}$ is local.

Let $\mathfrak{T}\mathcal{C}_{\mathfrak{X}}$ be a Karoubian triangulated category with a set of compact generators and $\mathfrak{T}\mathfrak{F}^{\mathfrak{Z}}_{\mathfrak{X}}$ the corresponding stack of local triangulated categories (cf. 5.6). Assigning to each fiber of the stack $\mathfrak{T}\mathfrak{F}^{\mathfrak{Z}}_{\mathfrak{X}}$ its center, we obtain a presheaf of commutative rings on the spectrum $\mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X})$ endowed with the Zariski topology. The associated sheaf, $\mathcal{O}^{\mathfrak{T}}_{\mathfrak{X}}$, is a sheaf of local rings. We call the locally ringed topological space $(\mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X}), \mathcal{O}^{\mathfrak{T}}_{\mathfrak{X}})$ the *geometric centrum* of the triangulated category $\mathfrak{T}\mathcal{C}_{\mathfrak{X}}$.

5.7.1. Note. Similarly to the abelian case, one can define the *reduced geometric centrum* of $\mathfrak{T}\mathcal{C}_{\mathfrak{X}}$. Details of this construction are left to the reader.

5.8. On the spectra of a monoidal triangulated category.

5.8.1. A remark on spectral cuisine. There are certain rather simple general pattern of producing spectra starting from a preorder (they are outlined in Chapter VII). Here, these pattern are applied to the preorder $\mathfrak{Tht}(\mathfrak{X})$ of thick triangulated subcategories of the triangulated category $\mathcal{C}\mathfrak{T}_{\mathfrak{X}}$.

5.8.2. Application to monoidal triangulated categories. Suppose that a triangulated category $\mathfrak{T}\mathcal{C}_{\mathfrak{X}}$ has a structure of a monoidal category. Then, replacing the preorder of thick subcategories with the preorder of those thick subcategories which are ideals of $\mathfrak{T}\mathcal{C}_{\mathfrak{X}}$ and mimicking the definitions of $\mathbf{Spec}^1_{\mathfrak{L}}(\mathfrak{X})$ and $\mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X})$, we obtain the spectra respectively $\mathbf{Spec}^1_{\mathfrak{L},\otimes}(\mathfrak{X})$ and $\mathbf{Spec}^{1,1}_{\mathfrak{L},\otimes}(\mathfrak{X})$. If the monoidal category $\mathfrak{T}\mathcal{C}_{\mathfrak{X}}$ is symmetric, then $\mathbf{Spec}^1_{\mathfrak{L},\otimes}(\mathfrak{X})$ coincides with the spectrum introduced by P. Balmer in different terms, as a straightforward imitation of the notion of a prime ideal of a commutative ring.

However, triangulated categories associated with noncommutative 'spaces' of interest do not have any symmetric monoidal structure. A typical example is the monoidal category of continuous (that is having a right adjoint) endofunctors of a category C_X.

6. Functorialities.

6.1. Induction. Let $\mathfrak{X} \xrightarrow{\mathfrak{f}} \mathfrak{Y}$ be a continuous morphism of t-'spaces'. For every point $\mathcal{Q}$ of the spectrum $\mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{Y})$, we have a commutative diagram

$$\begin{array}{ccccc} C_{\mathfrak{X}} & \xrightarrow{\widetilde{\mathfrak{q}}^*} & C_{\mathfrak{X}/\mathfrak{f}_*^{-1}(\widehat{\mathcal{Q}})} & \xrightarrow{\widetilde{\mathfrak{f}}_{\mathcal{Q}*}} & C_{\mathfrak{X}_{\mathcal{Q},\mathfrak{f}_*}} \\ \mathfrak{f}_*\big\downarrow & & \mathfrak{f}_{\mathcal{Q}*}\big\downarrow & & \big\downarrow \widehat{\mathfrak{f}}_{\mathcal{Q}*} \\ C_{\mathfrak{Y}} & \xrightarrow{\mathfrak{q}^*_{\mathcal{Q}}} & C_{\mathfrak{Y}/\widehat{\mathcal{Q}}} & \xrightarrow{\mathfrak{j}_{\mathcal{Q}*}} & \widehat{\mathcal{Q}}^\star/\widehat{\mathcal{Q}} = \mathcal{Q} \end{array} \qquad (1)$$

in which the horizontal are locaization functors. The localization functors $\mathfrak{q}^*_{\mathcal{Q}}$ and $\widetilde{\mathfrak{q}}^*$ have right adjoint functors; the functor $\mathfrak{j}_{\mathcal{Q}*}$ is a right adjoint to the full embedding $\widehat{\mathcal{Q}}^\star/\widehat{\mathcal{Q}} = Q \xrightarrow{\mathfrak{j}^*_{\mathcal{Q}}} C_{\mathfrak{Y}/\widehat{\mathcal{Q}}}$ (hence it is a localization functor); the functor $\widetilde{\mathfrak{f}}_{\mathcal{Q}*}$ is the lozalization functor at the class of all arrows which the composition $\mathfrak{j}_{\mathcal{Q}*} \circ \mathfrak{f}_{\mathcal{Q}*}$ maps to isomorphisms.

All functors of the right square of (1) have a left adjoint and both vertical arrows are conservative functor. Therefore, by 3.4 (or 3.5), the diagram is isomorphic to the diagram

$$\begin{array}{ccccc} C_{\mathfrak{X}} & \xrightarrow{\widetilde{\mathfrak{q}}^*} & \mathcal{F}_{\mathfrak{f}_{\mathcal{Q}}} - mod & \xrightarrow{\widetilde{\mathfrak{f}}_{\mathcal{Q}*}} & \mathcal{F}_{\widehat{\mathfrak{f}}_{\mathcal{Q}}} - mod \\ \mathfrak{f}_*\big\downarrow & & \mathfrak{f}_{\mathcal{Q}*}\big\downarrow & & \big\downarrow \widehat{\mathfrak{f}}_{\mathcal{Q}*} \\ C_{\mathfrak{Y}} & \xrightarrow{\mathfrak{q}^*_{\mathcal{Q}}} & C_{\mathfrak{Y}/\widehat{\mathcal{Q}}} & \xrightarrow{\mathfrak{j}_{\mathcal{Q}*}} & \widehat{\mathcal{Q}}^\star/\widehat{\mathcal{Q}} = \mathcal{Q} \end{array} \qquad (2)$$

where $\mathcal{F}_{\mathfrak{f}_{\mathcal{Q}}}$ and $\mathcal{F}_{\widehat{\mathfrak{f}}_{\mathcal{Q}}}$ are monads on the triangulated categories respectively $C_{\mathfrak{Y}/\widehat{\mathcal{Q}}}$ and $\mathcal{Q}$ and $\mathfrak{f}_{\mathcal{Q}*}$, $\widehat{\mathfrak{f}}_{\mathcal{Q}*}$ the corresponding forgetful functors.

6.1.1. The stabilizer of a morphism at a point. The 'space' $\mathfrak{X}_{\mathcal{Q},\mathfrak{f}_*}$ over $\mathcal{Q}$ is called the *stabilizer* of the morphism $\mathfrak{f}$ at the point $\mathcal{Q}$ of the spectrum. We also call the monad $\mathcal{F}_{\widehat{\mathfrak{f}}_{\mathcal{Q}}}$ the *stabilizer* of the morphism $\mathfrak{f}$ at the point $\mathcal{Q}$.

6.1.2. The related maps of the spectra. The diagram (1) gives rise to the diagram

$$\begin{array}{ccc} \mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X}_{\mathcal{Q},\mathfrak{f}_*}) & \longrightarrow & \mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X}/\mathfrak{f}_*^{-1}(\widehat{\mathcal{Q}})) \\ \big\downarrow & & \big\downarrow \\ \{\mathcal{Q}\} & \longleftarrow & \mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X}) \end{array} \tag{3}$$

in which the upper horizontal and the right vertical arrows are embeddings.

Thus, to each point $\mathcal{Q}$ of the spectrum $\mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{Y})$, it is assigned a canonical embedding of the spectrum of the stabilizer $\mathfrak{X}_{\mathcal{Q},\mathfrak{f}_*}$ of the morphism $\mathfrak{f}$ at the point $\mathcal{Q}$ into the spectrum $\mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X})$ of the t-'space' $\mathfrak{X}$.

6.2. The covariant functoriality. Let $\mathfrak{X} \xrightarrow{\mathfrak{f}} \mathfrak{Y}$ be a continuous morphism of t-'spaces'. For every point $\mathcal{P}$ of the spectrum $\mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X})$, the set $\mathfrak{Ass}(\mathfrak{f}_*(L))$ does not depend on the choice of a nonzero object L of the category $\mathcal{P}$. Therefore, we denote this set by $\mathfrak{Ass}(\mathfrak{f}_*(\mathcal{P}))$. The correspondence

$$\mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X}) \longrightarrow 2^{\mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{Y})}, \quad \mathcal{P} \longmapsto \mathfrak{Ass}(\mathfrak{f}_*(\mathcal{P})) \tag{4}$$

expresses the covariant functoriality of the spectrum.

6.3. Dual notions. From the general nonsense point of view, the dual notions have the same rights. Thus, given a continuous morphism $\mathfrak{X} \xrightarrow{\mathfrak{f}} \mathfrak{Y}$ of t-'spaces', we have a (contravariant) correspondence

$$\mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{Y}) \longrightarrow 2^{\mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X})}, \quad \mathcal{Q} \longmapsto \mathfrak{Ass}(\mathfrak{f}^*(\mathcal{Q})). \tag{5}$$

Similarly, for any point $\mathcal{P}$ of $\mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X})$, we have the dual version of the diagram (1):

$$\begin{array}{ccccc} C_{\mathfrak{Y}} & \xrightarrow{\widetilde{\mathfrak{q}}^*} & C_{\mathfrak{Y}/\mathfrak{f}^{*-1}(\widehat{\mathcal{P}})} & \xrightarrow{\widetilde{\mathfrak{f}}_{\mathcal{P}*}} & C_{\mathfrak{Y}_{\mathcal{P},\mathfrak{f}^*}} \\ {\scriptstyle \mathfrak{f}^*}\big\downarrow & & {\scriptstyle \mathfrak{f}^*_{\mathcal{P}}}\big\downarrow & & \big\downarrow{\scriptstyle \widehat{\mathfrak{f}}^*_{\mathcal{P}}} \\ C_{\mathfrak{X}} & \xrightarrow{\mathfrak{q}^*_{\mathcal{Q}}} & C_{\mathfrak{X}/\widehat{\mathcal{P}}} & \xrightarrow{\mathfrak{j}_{\mathcal{P}*}} & \widehat{\mathcal{P}}^\star/\widehat{\mathcal{P}} = \mathcal{P} \end{array} \tag{6}$$

in which the right square is cartesian (in pseudo-functorial sense), its upper and lower horizontal arrows are localization functors having left adjoints, and its right vertical arrows are conservative functors having right adjoints. The latter implies, by Beck's theorem for triangulated categories (see 3.4), that the diagram (5) is isomorphic to the diagram

$$\begin{array}{ccccc} C_{\mathfrak{Y}} & \xrightarrow{\widetilde{\mathfrak{q}}^*} & \mathcal{G}_{\mathfrak{f}_{\mathcal{P}}} - comod & \xrightarrow{\widetilde{\mathfrak{f}}^*_{\mathcal{P}}} & \mathcal{G}_{\widehat{\mathfrak{f}}_{\mathcal{P}}} - comod \\ {\scriptstyle \mathfrak{f}^*}\big\downarrow & & {\scriptstyle \mathfrak{f}^*_{\mathcal{P}}}\big\downarrow & & \big\downarrow{\scriptstyle \widehat{\mathfrak{f}}^*_{\mathcal{P}}} \\ C_{\mathfrak{X}} & \xrightarrow{\mathfrak{q}^*_{\mathcal{Q}}} & C_{\mathfrak{X}/\widehat{\mathcal{P}}} & \xrightarrow{\mathfrak{j}_{\mathcal{P}*}} & \widehat{\mathcal{P}}^{\star}/\widehat{\mathcal{P}} = \mathcal{P} \end{array} \tag{7}$$

which is the dual version of the diagram (2) in 6.1.

The diagram (5) gives rise to the diagram

$$\begin{array}{ccc} \mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{Y}_{\mathcal{P},\mathfrak{f}^*}) & \longrightarrow & \mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{Y}/{\mathfrak{f}^*}^{-1}(\widehat{\mathcal{P}})) \\ \big\downarrow & & \big\downarrow \\ \{\mathcal{P}\} & \longleftarrow & \mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{Y}) \end{array} \tag{8}$$

in which the upper horizontal and the right vertical arrows are embeddings.

6.4. Multiplicities and finiteness conditions.

6.4.1. Multiplicities. Let $\mathfrak{Y}$ be a t-'space'. For any object L of the category $C_{\mathfrak{Y}}$ and any point $\mathcal{Q}$ of the spectrum $\mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{Y})$., we denote by $\mathfrak{m}_{\mathfrak{Y}}(L;\mathcal{Q})$ the image of the $\mathcal{Q}$-torsion of the object L in the K-group $K_0(\mathcal{Q})$. We call the element $\mathfrak{m}_{\mathfrak{Y}}(L;\mathcal{Q})$ the *multiplicity* of the object L at the point $\mathcal{Q}$. The map which assigns to any object L of $C_{\mathfrak{Y}}$ its multiplicity function, $\mathcal{Q} \longmapsto \mu_{\mathfrak{Y}}(L;\mathcal{Q})$, induces a group homomorphism

$$K_0(\mathfrak{T}\mathcal{C}_{\mathfrak{Y}}) \xrightarrow{\mathfrak{m}_{\mathfrak{Y}}} \prod_{\mathcal{Q}\in\mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{Y})} K_0(\mathcal{Q}). \tag{9}$$

6.4.2. Locally finite objects. It follows that, for every $L \in ObC_{\mathfrak{Y}}$, the support $Supp(\mathfrak{m}_{\mathfrak{Y}}(L,-))$ of the function $\mathcal{Q} \longmapsto \mathfrak{m}_{\mathfrak{Y}}(L;\mathcal{Q})$ (that is the set of $\mathcal{Q} \in \mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{Y})$ such that $\mathfrak{m}_{\mathfrak{Y}}(L;\mathcal{Q})$ is nonzero) is contained in $\mathfrak{Ass}(L)$. We call an object L of the category $C_{\mathfrak{Y}}$ *locally finite* if $Supp(\mathfrak{m}_{\mathfrak{Y}}(L,-))$ coincides with $\mathfrak{Ass}(L)$. In other words, every associated point of the object L appears with a *finite* non-trivial multiplicity.

6.4.3. Relatively locally finite objects. Let $\mathfrak{X} \xrightarrow{\mathfrak{f}} \mathfrak{Y}$ be a continuous morphism of t-'spaces'. To every object L of the category $C_{\mathfrak{X}}$ and every point $\mathcal{Q}$ of $\mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{Y})$, we assign the multiplicity $\mathfrak{m}_{\mathfrak{Y}}(\mathfrak{f}_*(L);\mathcal{Q})$ of the object $\mathfrak{f}_*(L)$ at the point $\mathcal{Q}$.

We call an object L of the category $C_{\mathfrak{X}}$ *locally finite* over $\mathfrak{Y}$ (or, more explicitly, $\mathfrak{f}$-*finite*), if its direct image, $\mathfrak{f}_*(L)$, is a locally finite object in $C_{\mathfrak{Y}}$.

Chapter V
Spectra Related with Localizations.

This Chapter can be regarded as an introduction to basic spectra associated with exact localizations of *general* 'spaces', i.e. 'spaces' represented by arbitrary categories regarded as categories of quasi-coherent sheaves. Section 1 contains preliminaries on localizations and multiplicative systems. In Section 2, we introduce the *spectrum of exact localizations*, or, shortly, the *$\mathfrak{L}$-spectrum*, of a 'space' and discuss its functorial properties. In Section 3, we define $\mathfrak{L}$-local 'spaces' and show that the localization at a 'point' of the $\mathfrak{L}$-spectrum is an $\mathfrak{L}$-local 'space'. In Section 4, we introduce the *complete $\mathfrak{L}$-spectrum* and show its functoriality with respect to exact localizations. In Section 5, we define the *closed spectrum* (resp. the *complete closed spectrum*) and the *flat spectrum* (resp. the *complete flat spectrum*) of a 'space'. In Section 6, we extend the notions of the spectra to the case of categories with an action of a monoidal category. This material, important by itself, is used further only in the simplest case of so called $\mathbb{Z}$-categories, in order to give a background to spectral theory of triangulated categories.

1. Preliminaries on localizations.

1.1. Multiplicative systems. A family of arrows Σ of a category C_X is called a *left multiplicative system* if it has the following properties:

(S1) Σ is closed under composition and contains all identical arrows of C_X.

(SL2) Every diagram $M' \xleftarrow{s} M \xrightarrow{f} L$, where $s \in \Sigma$, can be completed to a commutative square

$$\begin{array}{ccc} M & \xrightarrow{f} & L \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s'} \\ M' & \xrightarrow{f'} & L' \end{array}$$

where $s' \in \Sigma$.

(SL3) If $M \overset{f}{\underset{g}{\rightrightarrows}} N$ is a pair of arrows such that $f \circ s = g \circ s$ for some $s \in \Sigma$, then there exists a morphism $N \xrightarrow{t} N'$ of Σ such that $t \circ f = t \circ g$.

A family $\Sigma \subseteq HomC_X$ is a *right multiplicative system* if it has dual properties. Finally, Σ is called a *multiplicative system* if it is both right and left multiplicative.

We denote by $\mathcal{SM}_\ell(X)$ (resp. by $\mathcal{SM}_\mathfrak{r}(X)$) the family of all left (resp. right) multiplicative systems in C_X. We denote by $\mathcal{SM}(X)$ the family $\mathcal{SM}_\ell(X) \bigcap \mathcal{SM}_\mathfrak{r}(X)$ of all multiplicative systems in C_X.

We regard $\mathcal{SM}_\ell(X)$, $\mathcal{SM}_\mathfrak{r}(X)$, and $\mathcal{SM}(X)$ as preorders with respect to $\subseteq$.

1.1.1. Saturation. Let Σ be a family of morphisms of the category C_X. Let q_Σ be the localization morphism $\Sigma^{-1}X \longrightarrow X$ and $C_X \xrightarrow{q_\Sigma^*} C_{\Sigma^{-1}X} = \Sigma^{-1}C_X$ its canonical inverse image functor.

The family $\Sigma^\mathfrak{s} = \Sigma_{q_\Sigma}$ of all arrows of C_X which q_Σ^* transfers into isomorphisms (cf. 1.2) is called the *saturation* of Σ. A family of arrows Σ is called *saturated* if it coincides with its saturation.

1.1.2. Generalities on saturated families of arrows. It follows from the universal property of localizations, that for any morphism $Y \xrightarrow{f} X$, the family Σ_f of all arrows of C_X which f^* transforms to isomorphisms (see 1.2) is saturated. In particular, the saturation of any family of arrows is saturated.

Any set, $\{Y_i \xrightarrow{f_i} X \mid i \in J\}$, of morphisms of 'spaces' defines uniquely a morphism $\mathcal{Y} = \coprod_{i \in J} Y_i \xrightarrow{\mathbf{f}} X$ with an inverse image

$$C_X \xrightarrow{\ \mathbf{f}^*\ } C_\mathcal{Y} = \prod_{i \in J} C_{Y_i}$$

uniquely determined by a choice of inverse images, $C_X \xrightarrow{f_i^*} C_{Y_i}$, of morphisms f_i, $i \in J$. Evidently, $\Sigma_\mathbf{f} = \bigcap_{i \in J} \Sigma_{f_i}$. This shows that the intersection of any set of saturated families of morphisms is saturated.

1.1.3. Saturation of multiplicative systems. If Σ is a left multiplicative system, then its saturation, $\Sigma^\mathfrak{s}$, consists of all morphisms $L \xrightarrow{u} M$ which can be inserted in a commutative diagram of the form

$$\begin{tikzcd} L \arrow[r, "u"] \arrow[d, "s"'] & M \arrow[d, "t"] \arrow[dl, "v"] \\ V \arrow[r, "u_1"] & W \end{tikzcd}$$

where $s, t \in \Sigma$ (see [GZ, 1.1.3.5]).

It follows from this description that the saturation of a (left and right) multiplicative system Σ coincides with all arrows $s \in HomC_X$ such that there exist morphisms $\mathfrak{u}$ and $\mathfrak{v}$ such that $\mathfrak{u} \circ s \in \Sigma \ni s \circ \mathfrak{v}$.

1.1.3.1. Proposition. *The saturation of a (left and right) multiplicative system is a multiplicative system.*

Proof. Let Σ be a multiplicative system. It suffices to show that the saturation, $\Sigma^{\mathfrak{s}}$, of Σ has the properties (SL2) and (SL3).

Let $M \xrightarrow{s} M'$ be an element of $\Sigma^{\mathfrak{s}}$; i.e. there exist morphisms $M' \xrightarrow{\mathfrak{u}} M''$ and $N \xrightarrow{\mathfrak{v}} M$ such that $\mathfrak{u} \circ s \in \Sigma \ni s \circ \mathfrak{v}$. And let $M \xrightarrow{f} L$ be an arbitrary morphism.

By the property (SL3), the diagram $M'' \xleftarrow{\mathfrak{u} \circ s} M \xrightarrow{f} L$ can be inserted in a commutative diagram

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\mathfrak{u} \circ s"'] & L \arrow[d, "s'"] \\ M'' \arrow[r, "f'"] & L' \end{tikzcd} \qquad (1)$$

where $s' \in \Sigma$. The diagram (1) can be rewritten as

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "s"'] & L \arrow[d, "s'"] \\ M' \arrow[r, "f' \circ \mathfrak{u}"] & L' \end{tikzcd}$$

which proves (SL2).

Let $M' \underset{g}{\overset{f}{\rightrightarrows}} L$ is a pair of arrows such that $f \circ s = g \circ s$. In particular, $f \circ (s \circ \mathfrak{v}) = g \circ (s \circ \mathfrak{v})$. Since $s \circ \mathfrak{v} \in \Sigma$, there exists (by the property (SL3)) a morphism $L \xrightarrow{t} L'$ of Σ such that $t \circ f = t \circ g$. ∎

1.1.3.2. Note. The analogous assertion is not true, in general, for left (or right) multiplicative systems. It is true, however, if the category C_X has finite colimits (finite limits in the case of right multiplicative systems); see 1.2.1(b) and 1.2.2 below.

1.1.4. Notations. We denote by $\mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\ell}(X)$ (resp. by $\mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\mathfrak{r}}(X)$) the family of all saturated left (resp. right) multiplicative systems in C_X.

We denote by $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$ the family of all saturated (left and right) multiplicative systems in C_X; that is $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X) = \mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\ell}(X) \bigcap \mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\mathfrak{r}}(X)$.

We regard $\mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\ell}(X)$, $\mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\mathfrak{r}}(X)$, and $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$ as preorders with respect to $\subseteq$.

It follows from 1.1.3.1 that the saturation, $\Sigma \longmapsto \Sigma^{\mathfrak{s}}$ induces a functor

$$\mathcal{SM}(X) \longrightarrow \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$$

which is left adjoint to the inclusion functor $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X) \longrightarrow \mathcal{SM}(X)$.

1.2. Left exact, right exact, and exact morphisms. A morphism $X \xrightarrow{f} Y$ is called *right exact* (resp. *left exact*, resp *exact*), if its inverse image functor preserves colimits (resp. limits, resp. both limits and colimits) of arbitrary finite diagrams.

The following assertion is a reformulation of Propositions 1.3.1 and 1.3.4 in [GZ].

1.2.1. Proposition. *(a) Let Σ be a left multiplicative system in C_X. Then the canonical morphism $\Sigma^{-1}X \xrightarrow{q_\Sigma} X$ is right exact.*

(b) Let $f = p_f \circ f_{\mathfrak{c}}$ be the canonical decomposition of a morphism $X \xrightarrow{f} Y$ into a conservative morphism $X \xrightarrow{f_{\mathfrak{c}}} \Sigma_f^{-1}Y$ and a localization $\Sigma_f^{-1}Y \xrightarrow{p_f} Y$. Suppose C_Y has finite limits (resp. finite colimits). Then f is left exact (resp. right exact) iff the family of arrows Σ_f is a left (resp. right) multiplicative system. In this case both the localization p_f and the conservative morphism $f_{\mathfrak{c}}$ are left (resp. right) exact.

In particular, if the category C_Y has limits and colimits of finite diagrams, then f is exact iff both the localization p_f and the conservative component $f_{\mathfrak{c}}$ are exact. The exactness of p_f is equivalent to that $\Sigma_f \in \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$.

1.2.2. Corollary. *Suppose the category C_X has finite colimits. Then the saturation map, $\Sigma \longmapsto \Sigma^{\mathfrak{s}}$ induces a functor $\mathcal{SM}_\ell(X) \longrightarrow \mathcal{S}^{\mathfrak{s}}\mathcal{M}_\ell(X)$ which is left adjoint to the corresponding inclusion functor $\mathcal{S}^{\mathfrak{s}}\mathcal{M}_\ell(X) \longrightarrow \mathcal{SM}_\ell(X)$.*

1.2.3. Corollary. *Suppose C_X has finite colimits. Then the intersection of any set of saturated left multiplicative systems is a saturated left multiplicative system.*

1.3. Continuous morphisms and flat morphisms. A morphism f of $|Cat|^o$, or Cat^{op}, is called *continuous* if its inverse image functor has a right adjoint, f_*, which is called a *direct image functor* of f.

A morphism f is called *flat* if it is exact and continuous.

One can show that a morphism f is continuous iff both the localization p_f and the conservative component $f_{\mathfrak{c}}$ are continuous.

2. The $\mathfrak{L}$-spectrum.

Fix a 'space' X. Recall that $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$ denote the preorder (with resp. to $\subseteq$) of all saturated (left and right) multiplicative systems of the category C_X. The preorder $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$ has the initial object – the family $Iso(C_X)$ of all isomorphisms of C_X. Let $\mathcal{S}^{\mathfrak{s}}\mathcal{M}^\star(X)$ denote $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X) - \{Iso(C_X)\}$.

For any $\Sigma \subseteq HomC_X$, denote by $\widehat{\Sigma}$ the union of all saturated multiplicative systems of C_X which do not contain Σ. It follows that if $\Sigma_1 \subseteq \Sigma_2$, then $\widehat{\Sigma_2} \subseteq \widehat{\Sigma_1}$. Notice that if Σ_1 and Σ_2 are saturated multiplicative systems, then the inverse implication holds, i.e. $\Sigma_1 \subseteq \Sigma_2$ iff $\widehat{\Sigma_2} \subseteq \widehat{\Sigma_1}$.

2.1. Definition. The $\mathfrak{L}$-*spectrum*, $\mathbf{Spec}^0_{\mathfrak{L}}(X)$, of X consists of all saturated multiplicative systems Σ such that $\widehat{\Sigma}$ is a saturated multiplicative system.

In other words, elements of $\mathbf{Spec}^0_{\mathfrak{L}}(X)$, are saturated multiplicative systems Σ such that there exists the biggest saturated multiplicative system, $\widehat{\Sigma}$, which does not contain Σ. In particular, $\mathbf{Spec}^0_{\mathfrak{L}}(X) \subseteq \mathcal{S}^{\mathfrak{s}}\mathcal{M}^\star(X)$.

2.1.1. Note. If C_X is a groupoid, then $\mathcal{S}^{\mathfrak{s}}\mathcal{M}^\star(X)$ is empty, hence $\mathbf{Spec}^0_{\mathfrak{L}}(X) = \emptyset$.

2.1.2. Specialization preorder. We call the preorder, $\supseteq$, on $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$ the *specialization preorder*: Σ is a *specialization* of Σ' if $\Sigma \subseteq \Sigma'$.

It follows that if $\Sigma,\ \Sigma'$ are elements of $\mathbf{Spec}^0_{\mathfrak{L}}(X)$, then Σ is a specialication of Σ' iff the saturated multiplicative system $\widehat{\Sigma}$ is a specialization of $\widehat{\Sigma'}$.

2.2. Functorial properties of the $\mathfrak{L}$-spectrum. Let $\mathfrak{L}_{\mathfrak{e}}\mathfrak{Esp}$ denote the subcategory of $|Cat|^o$ formed by exact localizations (cf. 1.2). Since identical morphisms are exact localizations, $Ob\mathfrak{L}_{\mathfrak{e}}\mathfrak{Esp} = Ob|Cat|^o$. Let $\mathfrak{POrd}_\star$ denote the category of preorders with initial objects; its morphisms are morphisms of preorders mapping initial objects to initial objects.

2.2.1. Lemma. *The map $X \longmapsto \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$ gives a rise to a contravariant functor*

$$\mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\mathfrak{o}} : \mathfrak{L}_{\mathfrak{e}}\mathfrak{Esp}^{op} \longrightarrow \mathfrak{POrd}_\star$$

and to a covariant functor

$$\mathcal{S}^{\mathfrak{s}}\mathcal{M} : \mathfrak{L}_{\mathfrak{e}}\mathfrak{Esp} \longrightarrow \mathfrak{POrd}_\star.$$

Proof. Let $X \xrightarrow{u} Y$ be an exact localization and $C_Y \xrightarrow{u^*} C_X$ its inverse image functor. Set $\Sigma_u = \Sigma_{u^*} = \{s \in HomC_Y \mid u^*(s) \in IsoC_X\}$. The functor u^* induces a map

$$\mathcal{S}^{\mathfrak{s}}\mathcal{M}(Y) \xrightarrow{\mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\mathfrak{o}}(u)} \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$$

which assigns to a family $\Sigma \in \mathcal{S}^{\mathfrak{s}}\mathcal{M}(Y)$ the minimal saturated multiplicative system containing $u^*(\Sigma)$, and a map

$$\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X) \xrightarrow{\mathcal{S}^{\mathfrak{s}}\mathcal{M}(u)} \mathcal{S}^{\mathfrak{s}}\mathcal{M}(Y)$$

which sends any saturated multiplicative system Σ' to its preimage, $u^{*^{-1}}(\Sigma')$. Notice that $\mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\mathfrak{o}}(u) \circ \mathcal{S}^{\mathfrak{s}}\mathcal{M}(u)$ is the identical map. This shows that $\mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\mathfrak{o}}(u)$ and $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(u)$ induce an isomorphism between $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$ and the preorder $\mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\Sigma_u}(Y)$ of saturated multiplicative systems of C_Y containing Σ_u. Notice that the map $\mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\mathfrak{o}}(u)$ can be represented as the composition of the map

$$\mathcal{S}^{\mathfrak{s}}\mathcal{M}(Y) \longrightarrow \mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\Sigma_u}(Y), \quad \Sigma \longmapsto \Sigma \vee \Sigma_u,$$

and the restriction of $\mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\mathfrak{o}}(u)$ to $\mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\Sigma_u}(Y)$ (the inverse to $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(u)$). It is easy to see that both maps, $u \longmapsto \mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\mathfrak{o}}(u)$ and $u \longmapsto \mathcal{S}^{\mathfrak{s}}\mathcal{M}(u)$ are functorial. ∎

2.2.2. Extended $\mathfrak{L}$-spectrum. For any 'space' X, set $\mathbf{Spec}^0_{\mathfrak{L}\star}(X) = \mathbf{Spec}^0_{\mathfrak{L}}(X) \cup \{\star_X\}$, where $\star_X = Iso(C_X)$. We call $\mathbf{Spec}^0_{\mathfrak{L}\star}(X)$ the *extended spectrum* of X. Notice that $\widehat{Iso(C_X)} = \emptyset$. Thus, the added trivial multiplicative system $\star_X$ can be viewed as ∞ (with respect to the specialization preorder $\supseteq$).

2.2.3. Proposition. *Any exact localization $X \xrightarrow{u} Y$ induces a morphism of extended spectra $\mathbf{Spec}^0_{\mathfrak{L}\star}(Y) \longrightarrow \mathbf{Spec}^0_{\mathfrak{L}\star}(X)$. This correspondence defines a contravariant functor, $\mathbf{Spec}^0_{\mathfrak{L}\star}$, from the category $\mathfrak{L}_{\mathfrak{c}}\mathfrak{Esp}$ to the category $\mathfrak{POrd}_\star$ of preorders with initial objects.*

Proof. Fix an inverse image functor, $C_Y \xrightarrow{u^*} C_X$, of the morphism u. The map

$$\mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\mathfrak{o}}(u) : \mathcal{S}^{\mathfrak{s}}\mathcal{M}(Y) \longrightarrow \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$$

(cf. 2.2.1) induces a morphism of spectra $\mathbf{Spec}^0_{\mathfrak{L}\star}(Y) \longrightarrow \mathbf{Spec}^0_{\mathfrak{L}\star}(X)$.

In fact, let $\Sigma_P \in \mathbf{Spec}^0_{\mathfrak{L}}Y$ and $\Sigma_P \nsubseteq \Sigma_u$. Then $\Sigma_u \subseteq \widehat{\Sigma}_P$. By (the argument of) 2.2.1, the map $\mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\mathfrak{o}}(u)$ induces an isomorphism between $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$ and the preorder $\mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\Sigma_u}(Y)$ of saturated multiplicative systems of C_Y containing Σ_u. In particular, the image, $\widehat{\Sigma}'_P$, of $\widehat{\Sigma}_P$ is a saturated multiplicative system. It follows that $\widehat{\Sigma}'_P$ is the biggest saturated multiplicative system in C_X which does not contain the image, Σ'_P, of the saturated multiplicative system $\Sigma_u \vee \Sigma_P$. In fact, if $\Sigma'_P \nsubseteq \Sigma'$ for some saturated multiplicative system Σ', then Σ_P is not contained in the saturated multiplicative system $\Sigma = {u^*}^{-1}(\Sigma')$. Therefore $\Sigma \subseteq \widehat{\Sigma}_P$, hence the assertion. ∎

2.2.4. Remarks. (a) For any $S \in \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$, let $\mathcal{U}_{\mathfrak{L}}(S)$ denote $\{\Sigma \in \mathbf{Spec}^0_{\mathfrak{L}}(X) \mid \Sigma \nsubseteq S\}$ It follows that $\mathcal{U}_{\mathfrak{L}}(S) = \{\Sigma \in \mathbf{Spec}^0_{\mathfrak{L}}(X) \mid S \subseteq \widehat{\Sigma}\}$.

The argument of 2.2.3 proves that any exact localization, $X \xrightarrow{u} Y$, induces an injective map $\mathcal{U}_{\mathfrak{L}}(\Sigma_u) \longrightarrow \mathbf{Spec}^0_{\mathfrak{L}}Y$.

(b) In general, the map $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X) \xrightarrow{\mathcal{S}^{\mathfrak{s}}\mathcal{M}(u)} \mathcal{S}^{\mathfrak{s}}\mathcal{M}(Y)$ corresponding to an exact localization $X \xrightarrow{u} Y$ does not induce a map $\mathbf{Spec}^0_{\mathfrak{L}\star}(X) \longrightarrow \mathbf{Spec}^0_{\mathfrak{L}\star}(Y)$.

For any exact localization $U \xrightarrow{u} X$, set $\mathbf{Spec}^0_{\mathfrak{L}}(U; X) = \{\Sigma \in \mathbf{Spec}^0_{\mathfrak{L}}(X) \mid \Sigma_u \subseteq \widehat{\Sigma}\}$.

2.2.5. Proposition. *Let* $\{U_i \overset{u_i}{\to} X \mid i \in J\}$ *be a conservative set of exact localizations. Then* $\mathbf{Spec}^0_{\mathfrak{L}}(X) = \bigcup_{i \in J} \mathbf{Spec}^0_{\mathfrak{L}}(U_i; X)$.

Proof. By hypothesis, the family of localization functors, $\{C_X \xrightarrow{u_i^*} C_{U_i} \mid i \in J\}$, is conservative, i.e. $\bigcap_{i \in J} \Sigma_{u_i} = Iso(C_X)$. Therefore, for every $\Sigma \in \mathbf{Spec}^0_{\mathfrak{L}}(X)$, there exists $i \in J$ such that $\Sigma \nsubseteq \Sigma_{u_i}$ which means precisely that $\Sigma_{u_i} \subseteq \widehat{\Sigma}$ and the image of $\Sigma \vee \Sigma_{u_i}$ in $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(U_i)$ belongs to $\mathbf{Spec}^0_{\mathfrak{L}}(U_i)$ (see 2.2.3 and 2.2.4(a)), hence the assertion. ∎

3. $\mathfrak{L}$-Local 'spaces' and the spectrum $\mathbf{Spec}^0_{\mathfrak{L}}(X)$.

We call a 'space' X *$\mathfrak{L}$-local* (here $\mathfrak{L}$- stands for 'localization'), if $\mathcal{S}^{\mathfrak{s}}\mathcal{M}^\star(X)$ has the smallest element, or, equivalently, the intersection, Σ^X, of all non-trivial saturated multiplicative systems is a non-trivial saturated multiplicative system.

3.1. Proposition. *The following conditions on a 'space'* X *are equivalent:*
(a) The 'space' X *is* $\mathfrak{L}$*-local.*
(b) The family of arrows $\Sigma^X = \bigcap_{\Sigma \in \mathcal{S}^{\mathfrak{s}}\mathcal{M}^\star(X)} \Sigma$ *belongs to* $\mathbf{Spec}^0_{\mathfrak{L}}(X)$.
(c) The spectrum $\mathbf{Spec}^0_{\mathfrak{L}}(X)$ *has an element,* Σ'*, such that* $\widehat{\Sigma'} = Iso(C_X)$.

Proof. $(a) \Rightarrow (b)$ & $(a) \Rightarrow (c)$: If X is $\mathfrak{L}$-local and Σ is a saturated multiplicative system, then $\Sigma^X \nsubseteq \Sigma$ iff $\Sigma \notin \mathcal{S}^{\mathfrak{s}}\mathcal{M}^\star(X)$, that is if $\Sigma = Iso(C_X)$.

$(b) \Rightarrow (a)$ follows from definitions: if $\Sigma^X \in \mathbf{Spec}^0_{\mathfrak{L}}(X)$, then $\Sigma^X \in \mathcal{S}^{\mathfrak{s}}\mathcal{M}^\star(X)$, hence X is $\mathfrak{L}$-local.

$(c) \Rightarrow (a)$: Let $\Sigma' \in \mathbf{Spec}^0_{\mathfrak{L}}(X)$ be such that $\widehat{\Sigma'} = Iso(C_X)$. Then $\Sigma' \in \mathcal{S}^{\mathfrak{s}}\mathcal{M}^\star(X)$ and Σ' is contained in any non-trivial saturated multiplicative system, i.e. $\Sigma' = \Sigma^X$. ∎

3.2. Proposition. *For any* $\Sigma \in \mathbf{Spec}^0_{\mathfrak{L}}(X)$*, the 'space'* $\widehat{\Sigma}^{-1}X$ *is* $\mathfrak{L}$*-local.*

Proof. The localization functor $C_X \xrightarrow{q_P^*} \widehat{\Sigma}^{-1}C_X$ induces an isomorphism between the preorder of (non-trivial) saturated multiplicative systems of $C_{\widehat{\Sigma}^{-1}X} = \widehat{\Sigma}^{-1}C_X$ and saturated multiplicative systems of C_X which contain $\widehat{\Sigma}$ properly. Since every saturated multiplicative system which contains $\widehat{\Sigma}$ properly contains Σ as well, the preorder of saturated multiplicative systems properly containing $\widehat{\Sigma}$ coincides with the preorder of saturated multiplicative systems containing $\widehat{\Sigma} \vee \Sigma$. Therefore the image of $\widehat{\Sigma} \vee \Sigma$ in $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(\widehat{\Sigma}^{-1}X)$ is the smallest element of $\mathcal{S}^{\mathfrak{s}}\mathcal{M}^\star(\widehat{\Sigma}^{-1}X)$. ∎

4. The complete $\mathfrak{L}$-spectrum.

For any 'space' X, we define its *complete $\mathfrak{L}$-spectrum*, $\mathbf{Spec}^1_{\mathfrak{L}}(X)$, as follows. Elements of $\mathbf{Spec}^1_{\mathfrak{L}}(X)$ are saturated multiplicative systems, Σ_x, of C_X such that the 'space' of fractions $\Sigma_x^{-1}X$ is $\mathfrak{L}$-local. In other words, elements of $\mathbf{Spec}^1_{\mathfrak{L}}(X)$ are saturated multiplicative systems, Σ_x, such that the intersection of all saturated multiplicative systems properly containing Σ_x is a saturated multiplicative system which contains Σ_x properly too. We consider $\mathbf{Spec}^1_{\mathfrak{L}}(X)$ together with the preorder $\subseteq$. By 3.2, there is a morphism $\mathbf{Spec}^0_{\mathfrak{L}}(X) \longrightarrow \mathbf{Spec}^1_{\mathfrak{L}}(X)$ defined by $\Sigma \longmapsto \widehat{\Sigma}$.

4.1. Proposition. *The map* $X \longmapsto \mathbf{Spec}^1_{\mathfrak{L}}(X)$ *extends to a functor,* $\mathbf{Spec}^1_{\mathfrak{L}}$*, from the category* $\mathfrak{L}_{\mathfrak{e}}\mathfrak{Esp}$ *to the category* $\mathfrak{POrd}$ *of preorders.*

Proof. The map $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X) \xrightarrow{\mathcal{S}^{\mathfrak{s}}\mathcal{M}(u)} \mathcal{S}^{\mathfrak{s}}\mathcal{M}(Y)$, $\Sigma \longmapsto {u^*}^{-1}(\Sigma)$, corresponding to an exact localization $X \xrightarrow{u} Y$ induces a map $\mathbf{Spec}^1_{\mathfrak{L}}(X) \longrightarrow \mathbf{Spec}^1_{\mathfrak{L}}(Y)$.

Indeed, $\Sigma^{-1}X \simeq \mathcal{S}^{\mathfrak{s}}\mathcal{M}(u)(\Sigma)^{-1}Y$, so that $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(u)(\Sigma)^{-1}Y$ is $\mathfrak{L}$-local if $\Sigma^{-1}X$ is $\mathfrak{L}$-local, hence the assertion. ■

4.2. Note. For any 'space' X,

$$\mathbf{Spec}^1_{\mathfrak{L}}(X) = \bigcup_{\Sigma \in \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)} \mathbf{Spec}^0_{\mathfrak{L}}(\Sigma^{-1}X) = \bigcup_{\Sigma \in \mathbf{Spec}^1_{\mathfrak{L}}(X)} \mathbf{Spec}^0_{\mathfrak{L}}(\Sigma^{-1}X).$$

Here $\mathbf{Spec}^1_{\mathfrak{L}}(\Sigma^{-1}X)$ is identified with its image in $\mathbf{Spec}^1_{\mathfrak{L}}(X)$.

4.3. The extended complete $\mathfrak{L}$-spectrum. For any 'space' X, set $\mathbf{Spec}^1_{\mathfrak{L}\star}(X) = \mathbf{Spec}^1_{\mathfrak{L}}(X) \cup \{\star_X\}$, where $\star_X = Iso(C_X)$. We call $\mathbf{Spec}^1_{\mathfrak{L}\star}(X)$ the *extended complete $\mathfrak{L}$-spectrum* of X.

4.3.1. Proposition. *The map* $X \longmapsto \mathbf{Spec}^1_{\mathfrak{L}}(X)$ *gives rise to a contravariant functor*

$${}^a\mathcal{S}\mathfrak{p} : \mathfrak{L}_{\mathfrak{e}}\mathfrak{Esp}^{op} \longrightarrow \mathfrak{POrd}_\star$$

and to a covariant functor

$${}_a\mathcal{S}\mathfrak{p} : \mathfrak{L}_{\mathfrak{e}}\mathfrak{Esp} \longrightarrow \mathfrak{POrd}_\star$$

to the category $\mathfrak{POrd}_\star$ *of preorders with initial objects.*

Proof. The functor ${}_a\mathcal{S}\mathfrak{p} : \mathfrak{L}_{\mathfrak{e}}\mathfrak{Esp} \longrightarrow \mathfrak{POrd}_\star$ is the unique extension of the functor $\mathbf{Spec}^1_{\mathfrak{L}} : \mathfrak{L}_{\mathfrak{e}}\mathfrak{Esp} \longrightarrow \mathfrak{POrd}$ of 4.1.

Let $X \xrightarrow{u} Y$ be an exact localization. We define the map

$${}^a\mathcal{S}\mathfrak{p}(u) : \mathbf{Spec}^1_{\mathfrak{L}}(Y) \longrightarrow \mathbf{Spec}^1_{\mathfrak{L}}(X)$$

as follows. Let $\Sigma_x \in \mathbf{Spec}^1_{\mathfrak{L}}(Y)$. If $\Sigma_u \subseteq \Sigma_x$, then ${}^a\mathcal{S}\mathfrak{p}(u)(\Sigma_x)$ is the minimal saturated multiplicative system containing $u^*(\Sigma_x)$. By transitivity of localizations, ${}^a\mathcal{S}\mathfrak{p}(u)(\Sigma_x) \in$

$\mathbf{Spec}^1_{\mathfrak{L}}(X)$. If $\Sigma_u \nsubseteq \Sigma_x$, then ${}^a\mathcal{S}\mathfrak{p}(u)$ maps Σ_x to the trivial family, $\star_X = Iso(C_X)$. It is easy to check that the map $u \longmapsto^a \mathcal{S}\mathfrak{p}(u)$ is functorial. ∎

4.4. Remark. The dualization functor, $X \longmapsto X^o$, establishes an isomorphism between the preorder of left (saturated) multiplicative systems on X and right (saturated) multiplicative systems on X^o. This isomorphism induces an isomorphism of preorders of (saturated) multiplicative systems:

$$\mathcal{SM}(X) \xrightarrow{\sim} \mathcal{SM}(X^o) \quad \text{and} \quad \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X) \xrightarrow{\sim} \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X^o). \tag{1}$$

In particular, a 'space' X is $\mathfrak{L}$-local iff its dual, X^o is $\mathfrak{L}$-local.

Thus, the isomorphism $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X) \xrightarrow{\sim} \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X^o)$ induces isomorphisms of spectra

$$\mathbf{Spec}^1_{\mathfrak{L}}(X) \xrightarrow{\sim} \mathbf{Spec}^1_{\mathfrak{L}}(X^o) \quad \text{and} \quad \mathbf{Spec}^0_{\mathfrak{L}}(X) \xrightarrow{\sim} \mathbf{Spec}^0_{\mathfrak{L}}(X^o). \tag{2}$$

as well as the extended versions of these spectra.

The spectra $\mathbf{Spec}^1_{\mathfrak{L}}(X)$ and $\mathbf{Spec}^0_{\mathfrak{L}}(X)$ are too large, which is one of the reasons why the duality (2) takes place. In the next section, we single out smaller spectra inside of $\mathbf{Spec}^1_{\mathfrak{L}}(X)$ and $\mathbf{Spec}^0_{\mathfrak{L}}(X)$.

5. Closed spectra and flat spectra.

5.1. Σ-Torsion free objects. Let $\Sigma \subseteq HomC_X$. We say that an object, M, of the category C_X is Σ-*torsion free* if every morphism $M \xrightarrow{t} N$ which belongs to Σ is a monomorphism. We denote by C_{X_Σ} the full subcategory of the category C_X whose objects are Σ-torsion free.

5.1.1. Lemma. *Let $\Sigma \subseteq HomC_X$ be such that for every diagram $\widetilde{L} \xleftarrow{s} L \xrightarrow{g} M$, where $s \in \Sigma$ and g is a monomorphism, there exists a commutative diagram*

$$\begin{array}{ccc} L & \xrightarrow{g} & M \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle t} \\ \widetilde{L} & \xrightarrow{\widetilde{g}} & \widetilde{M} \end{array} \tag{1}$$

where $t \in \Sigma$ (e.g. Σ is a left multiplicative system).

Then any subobject of a Σ-free object is Σ-free.

Proof. Let $L \xrightarrow{g} M$ be a monomorphism, and $L \xrightarrow{s} \widetilde{L}$ a morphism of Σ. Then there exists a commutative diagram (1) in which $t \in \Sigma$. If M is Σ-torsion free, then $M \xrightarrow{t} \widetilde{M}$ is a monomorphism. Thus $t \circ g$ is a monomorphism. It follows from the equality $t \circ g = \widetilde{g} \circ s$ that s is a monomorphism, hence the assertion. ∎

5.2. Closed families of morphisms and closed spectra. Let $\Sigma \subseteq HomC_X$. We say that Σ is *closed*, or *right closed*, if for every $M \in ObC_X$, there exists a morphism $M \longrightarrow \widetilde{M}$ of Σ such that $\widetilde{M} \in ObC_{X_\Sigma}$.

5.2.1. Proposition. *Let $\Sigma \subseteq HomC_X$ be a left saturated multiplicative system such that the canonical morphism $\Sigma^{-1}X \xrightarrow{q_\Sigma} X$ is continuous; and let q_Σ^* and $q_{\Sigma *}$ be resp. its inverse and direct image functors. Then the following conditions on an object M of C_X are equivalent:*

(i) M is Σ-torsion free.

(ii) An adjunction morphism $M \xrightarrow{\eta_\Sigma(M)} q_{\Sigma *}q_\Sigma^*(M)$ is a monomorphism.

Proof. Let Σ be a saturated multiplicative system and $C_X \xrightarrow{q_\Sigma^*} C_{\Sigma^{-1}X} = \Sigma^{-1}C_X$ a localization functor at Σ. If the family Σ is flat, the functor q_Σ^* has a right adjoint, $q_{\Sigma *}$. For every $M \in ObC_X$, the adjunction arrow, $M \xrightarrow{\eta_\Sigma(M)} q_{\Sigma *}q_\Sigma^*(M)$, belongs to Σ. In particular, if M is a Σ-torsion free object, then the adjunction morphism $\eta_\Sigma(M)$ is a monomorphism.

Let $M \xrightarrow{s} N$ be a morphism from Σ. Then the upper horizontal arrow in the commutative diagram

$$\begin{array}{ccc} q_{\Sigma *}q_\Sigma^*(M) & \xrightarrow{q_{\Sigma *}q_\Sigma^*(s)} & q_{\Sigma *}q_\Sigma^*(N) \\ {\scriptstyle \eta_\Sigma(M)} \uparrow & & \uparrow {\scriptstyle \eta_\Sigma(N)} \\ M & \xrightarrow{s} & N \end{array}$$

is an isomorphism. If $\eta_\Sigma(M)$ is a monomorphism, then $\eta_\Sigma(N) \circ s = q_{\Sigma *}q_\Sigma^*(s) \circ \eta_\Sigma(M)$ is a monomorphism. Therefore s is a monomorphism. This shows that the object M is Σ-torsion free. ∎

5.2.2. Corollary. *Let Σ be a left saturated multiplicative system such that the canonical morphism $\Sigma^{-1}X \xrightarrow{q_\Sigma} X$ is continuous. Then Σ is closed.*

Proof. For every $M \in ObC_X$, the adjunction arrow, $M \xrightarrow{\eta_\Sigma(M)} q_{\Sigma *}q_\Sigma^*(M)$ belongs to Σ. If $\widetilde{M} = q_{\Sigma *}q_\Sigma^*(M)$, then the adjunction arrow $\eta_\Sigma(\widetilde{M})$ is an isomorphism, in particular it is a monomorphism. ∎

5.2.3. Closed spectra. We denote by $\mathfrak{CS}^{\mathfrak{s}}\mathcal{M}(X)$ the preorder of all closed saturated multiplicative systems on X. The *complete closed spectrum*, $\mathbf{Spec}^1_{\mathfrak{C}}(X)$, is defined by

$$\mathbf{Spec}^1_{\mathfrak{C}}(X) = \mathfrak{CS}^{\mathfrak{s}}\mathcal{M}(X) \bigcap \mathbf{Spec}^1_{\mathfrak{L}}(X);$$

that is elements of $\mathbf{Spec}^1_{\mathfrak{C}}(X)$ are closed saturated multiplicative systems, Σ, such that $\Sigma^{-1}X$ is $\mathfrak{L}$-local.

We call $\mathbf{Spec}^0_{\mathfrak{C}}(X) = \{\Sigma \in \mathbf{Spec}^0_{\mathfrak{L}}(X) \mid \widehat{\Sigma} \in \mathbf{Spec}^1_{\mathfrak{C}}(X)\}$ the *closed spectrum* of X.

5.3. Continuous localizations and flat spectra. Let $\Sigma \subseteq HomC_X$. Recall that an object M of C_X is called *left closed for* Σ if $C_X(s, M)$ is a bijection for each morphism s of Σ [GZ, I.4].

5.3.1. Lemma. *(a) Let $\Sigma \subseteq HomC_X$, and let M be an object of C_X such that $C_X(s, M)$ is a surjection for each morphism s of Σ. Then every morphism $M \xrightarrow{t} N$ which belongs to Σ is a retraction (i.e. $u \circ t = id_M$ for some morphism u). In particular, M is Σ-torsion free.*

(b) Suppose for any diagram $\widetilde{L} \xleftarrow{s} L \xrightarrow{g} M$ such that $s \in \Sigma$, there exists a commutative diagram

$$\begin{array}{ccc} L & \xrightarrow{f} & M \\ s \downarrow & & \downarrow t \\ \widetilde{L} & \xrightarrow{\widetilde{f}} & \widetilde{M} \end{array} \qquad (1)$$

where $t \in \Sigma$ (e.g. Σ is a left multiplicative system). Then $C_X(s, M)$ is a surjection for each morphism s of Σ iff every morphism $M \xrightarrow{t} N$ which belongs to Σ is a retraction.

Proof. (a) If $M \xrightarrow{t} N$ is a morphism of Σ, then the map

$$C_X(N, M) \longrightarrow C_X(M, M), \quad f \longmapsto f \circ t,$$

is surjective. In particular, there exists a morphism $N \xrightarrow{u} M$ such that $u \circ t = id_M$.

(b) Suppose that the object M is such that every morphism $M \longrightarrow N$ which belongs to Σ is a retraction. Then $C_X(s, M)$ is surjective for any morphism $L \xrightarrow{s} \widetilde{L}$ of Σ.

In fact, let $L \xrightarrow{f} M$ be an arbitrary morphism. By hypothesis, there is a commutative diagram (1), where $t \in \Sigma$. By condition, t is a retraction, i.e. there exists a morphism $\widetilde{M} \xrightarrow{u} M$ such that $u \circ t = id_M$. Then $(u \circ \widetilde{f}) \circ s = u \circ (t \circ f) = f$. This shows the surjectivity of $C_X(s, M)$. ∎

5.3.2. Proposition. *Suppose that $\Sigma \subseteq HomC_X$ is a left multiplicative system. Then the following conditions on an object M of C_X are equivalent:*

(a) M is left closed for Σ;

(b) $C_X(s, M)$ is surjective for any $s \in \Sigma$;

(c) any morphism $M \longrightarrow N$ which belongs to Σ is a retraction.

Proof. The implications $(b) \Leftrightarrow (c)$ follow from 5.3.1(b). The implication $(a) \Rightarrow (b)$ holds by definition. The implication $(b) \Rightarrow (a)$ is proved in [GZ, 1.4.1.1]. ■

5.3.3. Localizations and continuous localizations. Let $X \xrightarrow{f} Y$ be a morphism with an inverse image functor $C_Y \xrightarrow{f^*} C_X$. An object N of C_Y is called *f-free* over an object M of C_X, if there exists a morphism $f^*(N) \xrightarrow{u} M$ such that for any morphism $f^*(L) \xrightarrow{v} M$ there exists a unique morphism $L \xrightarrow{\widetilde{v}} N$ satisfying $v = u \circ f^*(\widetilde{v})$. In other words, $(N, f^*(N) \xrightarrow{u} M)$ is a final object of the category f^*/M, or, what is the same, the object N represents the functor $C_X(f^*(-), M) : C_Y^{op} \longrightarrow Sets$. We denote by $C_{\mathfrak{L}(f)}$ the full subcategory of the category C_Y generated by f-free objects.

Let $C_{\mathfrak{D}(f_*)}$ denote the full subcategory of C_X generated by all $M \in ObC_X$ such that the functor $C_X(f^*(-), M) : C_Y^{op} \longrightarrow Sets$ is representable. A choice for each $M \in ObC_{\mathfrak{D}(f_*)}$ of an object, $f_*(M)$, of the subcategory $C_{\mathfrak{L}(f)}$ representing the functor $C_X(f^*(-), M)$ extends uniquely to a functor $\mathfrak{D}(f_*) \longrightarrow C_Y$ taking values in $C_{\mathfrak{L}(f)}$. Let $C_{\mathfrak{R}(f_*)}$ denote ${f^*}^{-1}(\mathfrak{D}(f_*))$. The morphism f is continuous iff $\mathfrak{D}(f_*) = X$ and, therefore, $\mathfrak{R}(f_*) = Y$.

5.3.3.1. Proposition. *Suppose that $\Sigma \subseteq HomC_X$ is a left multiplicative system. And let $\Sigma^{-1}X \xrightarrow{q_\Sigma} X$ be the localization morphism. Then*

(a) $C_{\mathfrak{L}(q_\Sigma)}$ is the full subcategory of C_X generated by all objects which are left closed for Σ.

(b) The subcategory $C_{\mathfrak{R}(q_\Sigma)}$ is generated by all $M \in ObC_X$ such that there exists a morphism $M \xrightarrow{s} N$, where N is left closed for Σ and $q_\Sigma(s)$ is invertible.

(c) The composition of the inclusion $C_{\mathfrak{L}(q_\Sigma)} \longrightarrow C_X$ and the canonical localization functor $C_X \xrightarrow{q_\Sigma^*} \Sigma^{-1}C_X$ is a fully faithful functor injective on objects. This functor induces an isomorphism $\mathfrak{D}(q_\Sigma) \xrightarrow{\sim} \mathfrak{L}(q_\Sigma)$.

(d) The inclusion functor $C_{\mathfrak{L}(q_\Sigma)} \xrightarrow{\widetilde{q_{\Sigma *}}} C_{\mathfrak{R}(q_\Sigma)}$ has a left adjoint, $\widetilde{q}_\Sigma^*$.

Proof. (a) Let $Y \xrightarrow{f} X$ be a morphism, M an object of C_Y such that the functor $C(f^*(-), M)$ is representable. Then any object, N, representing $C(f^*(-), M)$ is, obviously, left closed for $\Sigma_f = \{s \in HomC_X \mid f^*(s) \in Iso(C_Y)\}$.

If $f = q_\Sigma$, then the converse is true: if $N \in ObC_X$ is left closed for Σ, then it follows from the universal property of the localization at Σ that the object N represents the functor $C_{\Sigma^{-1}X}(q_\Sigma^*(-), q_\Sigma^*(N))$.

(b) By definition, the subcategory $C_{\mathfrak{R}(q_\Sigma)}$ is generated by all $M \in ObC_X$ such that the functor $C_{\Sigma^{-1}X}(q_\Sigma^*(-), q_\Sigma^*(M))$ is representable by some object, N, of the category C_X. In particular, there exists a canonical morphism $M \xrightarrow{t} N$ corresponding to the identical arrow $q_\Sigma^*(M) \longrightarrow q_\Sigma^*(M)$. It follows that $q_\Sigma^*(t)$ is an isomorphism.

(c) The canonical localization functor $C_X \xrightarrow{q_\Sigma^*} C_{\Sigma^{-1}X}$ is identical on objects, hence the composition of q_Σ^* with the inclusion functor $C_{\mathcal{L}(q_\Sigma)} \longrightarrow C_X$ is injective on objects. For any $M \in ObC_{\mathcal{L}(q_\Sigma)}$ and any $L \in ObC_X$, we have a functorial isomorphism $C_{\Sigma^{-1}X}(q_\Sigma^*(L), q_\Sigma^*(M)) \simeq C_X(L, M)$. In particular, the composition of the embedding $C_{\mathcal{L}(q_\Sigma)} \longrightarrow C_X$ with q_Σ^* is a fully faithful functor.

(d) By (b), for any $M \in ObC_{\mathfrak{R}(q_\Sigma)}$, there exists a morphism $M \xrightarrow{t} N$, where N is left closed for Σ. It follows from 5.3.2 that the object N here is defined uniquely up to isomorphism. A choice of N for every $M \in ObC_{\mathfrak{R}(q_\Sigma)}$ defines a functor, $\widetilde{q}_\Sigma^*$, from $C_{\mathfrak{R}(q_\Sigma)}$ to $C_{\mathcal{L}(q_\Sigma)}$. This functor is a left adjoint to the inclusion functor $C_{\mathcal{L}(q_\Sigma)} \longrightarrow C_{\mathfrak{R}(q_\Sigma)}$. ∎

One of the corollaries of 5.3.3.1 is the following fact:

5.3.3.2. Proposition [GZ, 1.4.1]. *Suppose that $\Sigma \subseteq HomC_X$ is a left multiplicative system. Then the following conditions are equivalent:*

(a) The canonical morphism $\Sigma^{-1}X \xrightarrow{q_\Sigma} X$ is continuous.

(b) For every object M of the category C_X, there exists an object $\widetilde{M}$ left closed for Σ and a morphism $M \xrightarrow{s} \widetilde{M}$ such that $q_\Sigma(s)$ is invertible.

5.3.4. Continuous and flat multiplicative systems. We call $\Sigma \subseteq HomC_X$ *continuous* if it is a left multiplicative system and the canonical morphism $\Sigma^{-1}X \xrightarrow{q_\Sigma} X$ is continuous. It follows from [GZ, 1.1.3] that if Σ is continuous, then the saturation of Σ is a left multiplicative system, hence it is continuous. We denote by $\mathfrak{Lc}_\ell(X)$ the preorder of all continuous left saturated multiplicative systems and by $\mathfrak{Lc}(X)$ the preorder of all continuous saturated multiplicative systems, i.e. $\mathfrak{Lc}(X) = \mathfrak{Lc}_\ell(X) \cap \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$.

We will call continuous saturated multiplicative systems *flat*.

5.3.5. The flat spectra $\mathbf{Spec}^0_{\mathfrak{fL}}(X)$ and $\mathbf{Spec}^1_{\mathfrak{fL}}(X)$. The elements of the *flat complete $\mathfrak{L}$-spectrum* $\mathbf{Spec}^1_{\mathfrak{fL}}(X)$ are flat multiplicative systems Σ such that the 'space' of fractions $\Sigma^{-1}X$ is $\mathfrak{L}$-local.

We call $\mathbf{Spec}^1_{\mathfrak{fL}}(X)$ the *complete flat $\mathfrak{L}$-spectrum* of X.

The *flat $\mathfrak{L}$-spectrum*, $\mathbf{Spec}^0_{\mathfrak{fL}}(X)$, of the 'space' X is defined by setting

$$\mathbf{Spec}^0_{\mathfrak{fL}}(X) = \{\Sigma \in \mathbf{Spec}^0_{\mathfrak{L}}(X) \mid \widehat{\Sigma} \in \mathbf{Spec}^1_{\mathfrak{fL}}(X)\}.$$

It follows that

$$\mathbf{Spec}^0_{\mathfrak{fL}}(X) \subseteq \mathbf{Spec}^0_{\mathfrak{C}}(X) \quad \text{and} \quad \mathbf{Spec}^1_{\mathfrak{fL}}(X) \subseteq \mathbf{Spec}^1_{\mathfrak{C}}(X).$$

We leave to the reader the definition of the extended versions of these spectra.

5.3.6. Another description of flat localizations and $\mathbf{Spec}^1_{\mathfrak{fL}}(X)$. Fix a 'space' X. Consider the preorder $\mathfrak{fL}(X)$ of all strictly full subcategories C_Y of C_X such that the inclusion functor $C_Y \xrightarrow{\iota_{Y*}} C_X$ has an exact left adjoint $C_X \xrightarrow{\iota_Y^*} C_Y$. These functors are regarded as resp. direct and inverse image functors of a *strictly full embedding* $Y \overset{\iota_Y}{\hookrightarrow} X$. The map which assigns to every such subcategory the family of arrows $\Sigma_{\iota_Y^*} = \iota_Y^{*^{-1}}(Iso(C_Y))$ is an isomorphism of the preorder $(\mathfrak{fL}(X), \supseteq)$ onto the preorder $(\mathfrak{Lc}(X), \subseteq)$ of continuous saturated multiplicative systems.

Thus, the flat spectrum $\mathbf{Spec}^1_{\mathfrak{fL}}(X)$ can be identified with the preorder of all strictly full embeddings $Y \overset{\iota_Y}{\hookrightarrow} X$ such that Y is a local 'space' and ι_Y^* is an exact functor.

5.4. The spectrum $\mathbf{Spec}^1_{\mathfrak{Fl}}(X)$. Objects of $\mathbf{Spec}^1_{\mathfrak{Fl}}(X)$ are continuous saturated multiplicative systems Σ in C_X such that there exists the smallest continuous saturated multiplicative system properly containing Σ.

Let $\mathfrak{fL}^\star(X)$ denote the set $\mathfrak{fL}(X) - \{id_X\}$ of all proper strictly full embeddings. Thanks to the isomorphism $(\mathfrak{Lc}(X), \subseteq) \xrightarrow{\sim} (\mathfrak{fL}(X), \supseteq)$ (cf. 5.3.6), elements of $\mathbf{Spec}^1_{\mathfrak{Fl}}(X)$ can be identified with strictly full embeddings $Y \overset{\iota_Y}{\hookrightarrow} X$ such that ι_Y^* is an exact functor and the preorder $(\mathfrak{fL}^\star(Y), \subseteq)$ of proper strictly full embeddings into Y has the biggest element.

6. Actions of monoidal categories and their spectra.

We fix a monoidal category $\widetilde{C_T} = (C_T, \odot, \mathfrak{a}; 1, \phi_{\mathfrak{l}}, \phi_{\mathfrak{r}})$. Here $C_T \times C_T \xrightarrow{\odot} C_T$ is a monoidal structure ('tensor product'); $\mathfrak{a}$ is an associativity constraint, i.e. a functor isomorphism

$$\odot \circ (\odot \times Id_{C_T}) \longrightarrow \odot \circ (Id_{C_T} \times \odot)$$

satisfying certain natural compatibility conditions; 1 denotes the unit object,

$$1 \odot - \xrightarrow{\phi_{\mathfrak{l}}} Id_{C_T} \xleftarrow{\phi_{\mathfrak{r}}} - \odot 1$$

are canonical isomorphisms.

6.1. Actions. Let X be a 'space'. Fix an action, $C_T \times C_X \xrightarrow{\gamma^*} C_X$, of the monoidal category $\widetilde{C_T} = (C_T, \odot, 1)$ on the category C_X. The functor γ^* induces a functor

$$C_T \xrightarrow{\Gamma} \mathcal{E}nd(C_X), \quad a \longmapsto \Gamma_a,$$

where $\mathcal{E}nd(C_X)$ denote the category functors $C_X \longrightarrow C_X$, and $\Gamma_a(M) = \gamma^*(a, M)$. The functor γ^* being an 'action' means precisely that Γ is a monoidal functor, i.e. for any $a, b \in ObC_T$, there are natural morphisms

$$\Gamma_a \circ \Gamma_b \xrightarrow{\phi_{a,b}} \Gamma_{a \odot b} \quad \text{and} \quad \Gamma_1 \xrightarrow{\sim} Id_{C_X}$$

related in a natural way between themselves and with associativity constraint on $\widetilde{C_T}$.

A pair (C_X, γ^*), where γ^* is a $\widetilde{C_T}$-action, is called a *$\widetilde{C_T}$-category*. We call a pair (X, γ^*) a *$\widetilde{C_T}$-'space'*. A morphism (more precisely, a 1-morphism) between two $\widetilde{C_T}$-'spaces', $(X, \gamma^*) \longrightarrow (Y, \widetilde{\gamma}^*)$, is given by a pair (F, ϕ), where F is a functor $C_Y \xrightarrow{F} C_X$ such that the diagram

$$\begin{array}{ccc} C_T \times C_Y & \xrightarrow{Id\times F} & C_T \times C_X \\ \gamma^* \downarrow & & \downarrow \widetilde{\gamma}^* \\ C_Y & \xrightarrow{F} & C_X \end{array}$$

quasi-commutes, and ϕ is a functor isomorphism $\gamma^* \circ (Id \times F) \xrightarrow{\sim} F \circ \gamma^*$ satisfying a standard cocycle condition. The composition of morphisms is defined naturally.

6.1.1. Note. In the language of 'spaces', the monoidal structure, $C_T \times C_T \xrightarrow{\odot} C_T$, can be regarded as an inverse image of a morphism (a coaction) $T \longrightarrow T \coprod T$, and the action γ^* as an inverse image functor of a morphism $X \xrightarrow{\gamma} T \coprod X$.

6.1.2. Actions of a monoid. $\mathbb{Z}$-categories. Any monoid, $\mathcal{G}$, might be regarded as a discrete category with the monoidal structure given by multiplication. This defines an isomorphism between the category of monoids and the category of discrete 'small' monoidal categories which allows to define actions of monoids on categories. Thus, a *$\mathcal{G}$-category* is a pair (C_X, γ^*), where γ^* is a monoidal functor from $\mathcal{G}$ to the monoidal category $\mathfrak{End}(C_X)$ of functors $C_X \longrightarrow C_X$. If $\mathcal{G}$ is a group, the functor γ takes values in the monoidal subcategory $\mathbf{Pic}(X)$ of $\mathfrak{End}(C_X)$ formed by invertible functors and isomorphisms between them. The group $\mathbb{Z}$ is of particular interest because triangulated categories, categories of graded modules, and the category of quasi-coherent sheaves on (noncommutative) Proj are $\mathbb{Z}$-categories.

6.2. A graded category associated with an action. Suppose the category C_T is 'small'. For any pair of objects, L, M, of the category C_X and an object a of the category C_T, set $C_X^a(L, M) = C_X(L, \gamma^*(a, M)) = C_X(L, \Gamma_a(M))$. If $N \in ObC_X$ and $b \in ObC_T$, then we define a map

$$C_X^a(L, M) \times C_X^b(M, N) \longrightarrow C_X^{a\odot b}(L, N) \tag{1}$$

as the composition of the maps

$$\begin{array}{ccc} C_X(L, \Gamma_a(M)) \times C_X(M, \Gamma_b(N)) & \longrightarrow & C_X(L, \Gamma_a(M)) \times C_X(\Gamma_a(M), \Gamma_a \circ \Gamma_b(N)) \\ & & \downarrow \\ & & C_X(L, \Gamma_a(M)) \times C_X(\Gamma_a(M), \Gamma_{a\odot b}(N)), \end{array}$$

where the vertical arrow is induced by the functor morphism $\Gamma_a \circ \Gamma_b \xrightarrow{\phi_{a,b}} \Gamma_{a\odot b}$, and the composition map

$$C_X(L, \Gamma_a(M)) \times C_X(\Gamma_a(M), \Gamma_{a\odot b}(N)) \longrightarrow C_X(L, \Gamma_{a\odot b}(N)) = C_X^{a\odot b}(L, N)$$

This defines an *enriched* category, $C_{(X,\gamma^*)}$, with the same objects as C_X; morphisms between objects are $\widetilde{C_T}$-graded sets. To every morphism, $(X, \gamma^*) \longrightarrow (Y, \widetilde{\gamma}^*)$, of $\widetilde{C_T}$-actions, there corresponds a morphism, $C_{(X,\gamma^*)} \longrightarrow C_{(Y,\widetilde{\gamma}^*)}$, of enriched categories.

6.2.1. The case of 'spaces' over a monoid. Let $\mathcal{G}$ be a monoid and (X, γ^*) a $\mathcal{G}$-'space' such that C_X is a k-linear category for for some commutative ring k. Then $C_{(X,\gamma^*)}$ gives rise to an enriched category over the monoidal category of $\mathcal{G}$-graded k-modules. In particular, a $\mathbb{Z}$-'space' defines an enriched category of the monoidal category of $\mathbb{Z}$-graded k-modules.

6.3. Stable saturated multiplicative systems. Let $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X, \gamma^*)$ denote the family of saturated multiplicative systems in C_X which are invariant with respect to the action of $\widetilde{C_T}$. It follows from the universal property of localizations that for every $\Sigma \in \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X, \gamma^*)^\star$, the 'space' of fractions, $\Sigma^{-1}X$, inherits a $\widetilde{C_T}$-action uniquely defined by the condition that the canonical morphism $\Sigma^{-1}X \xrightarrow{q_\Sigma} X$ is a morphism of actions.

6.3.1. Proposition. *(a) If Σ is a $\widetilde{C_T}$-stable multiplicative system, then its saturation, $\Sigma^{\mathfrak{s}}$ is a $\widetilde{C_T}$-stable multiplicative system too.*

(b) If Σ_1 and Σ_2 are $\widetilde{C_T}$-stable saturated multiplicative systems, then the smallest saturated multiplicative system, $\Sigma_1 \vee \Sigma_2$, spanned by Σ_1 and Σ_2 is $\widetilde{C_T}$-stable.

(c) Suppose C_X has finite colimits. Then the intersection of any set of $\widetilde{C_T}$-stable saturated left multiplicative systems is a $\widetilde{C_T}$-stable saturated left multiplicative system.

Proof. (a) The saturation, $\Sigma^{\mathfrak{s}}$, of a multiplicative system Σ consists of all morphisms s of C_X such that $u \circ s \in \Sigma \ni s \circ v$ for some morphisms u and v (see 1.3.3). If Σ is $\widetilde{C_T}$-stable, then

$$\Gamma_a(u) \circ \Gamma_a(s) = \Gamma_a(u \circ s) \in \Sigma \ni \Gamma_a(s \circ v) = \Gamma_a(s) \circ \Gamma_a(v)$$

for all $a \in ObC_T$ (cf. 6.1 for notations); hence $\Gamma_a(s) \in \Sigma$ for all $a \in ObC_T$.

(b) We need several steps.

(i) Let Σ_1 and Σ_2 be two left multiplicative systems, and let $\Sigma_1 \sqcup \Sigma_2$ denote the smallest family of arrows closed under composition and containing Σ_1 and Σ_2. We claim that $\Sigma_1 \sqcup \Sigma_2$ is a left multiplicative system.

The family $\Sigma_1 \sqcup \Sigma_2$ consists of all possible compositions of arrows of $\Sigma_1 \bigcup \Sigma_2$. Since Σ_1 and Σ_2 are closed under composition and contain all identical morphisms, a generic element, s, of $\Sigma_1 \sqcup \Sigma_2$ can be represented as a composition

$$M_0 \xrightarrow{s_1} L_1 \xrightarrow{t_1} M_1 \xrightarrow{s_2} \ \ldots \ \xrightarrow{s_n} L_n \xrightarrow{t_n} M_n, \tag{2}$$

where $s_i \in \Sigma_1$ and $t_i \in \Sigma_2, \ i = 1, ..., n$.

Let s be an element of $\Sigma_1 \sqcup \Sigma_2$ given by the composition (2), and let $M_0 \xrightarrow{f} M_0'$ be an arbitrary morphism. By the property (SL2) (see 1.3), there exists a commutative square

$$\begin{array}{ccc} M_0 & \xrightarrow{s_1} & L_1 \\ f \downarrow & & \downarrow g_1 \\ M_0' & \xrightarrow{s_1'} & L_1' \end{array}$$

where $s_1' \in \Sigma_1$. Applying the property (SL2) to the pair of morphisms $L_1' \xleftarrow{g_1} L_1 \xrightarrow{t_1} M_1$, we complete it to a commutative diagram

$$\begin{array}{ccc} L_1 & \xrightarrow{t_1} & M_1 \\ g_1 \downarrow & & \downarrow f_1 \\ L_1' & \xrightarrow{t_1'} & M_1' \end{array}$$

where $t_1' \in \Sigma_2$. Continuing this process, we obtain a commutative diagram

$$\begin{array}{ccccccccccccc} M_0 & \xrightarrow{s_1} & L_1 & \xrightarrow{t_1} & M_1 & \xrightarrow{s_2} & \ldots & \xrightarrow{s_n} & L_n & \xrightarrow{t_n} & M_n \\ f \downarrow & & \downarrow g_1 & & \downarrow f_1 & & \ldots & & g_n \downarrow & & \downarrow f_n \\ M_0' & \xrightarrow{s_1'} & L_1' & \xrightarrow{t_1'} & M_1' & \xrightarrow{s_2'} & \ldots & \xrightarrow{s_n'} & L_n' & \xrightarrow{t_n'} & M_n' \end{array}$$

where $s_i' \in \Sigma_1$ and $t_i' \in \Sigma_2, \ i = 1, ..., n$. Thus, $\Sigma_1 \sqcup \Sigma_2$ has the property (SL2). It remains to verify the property (SL3) (see 1.3).

Let s be an element of $\Sigma_1 \sqcup \Sigma_2$ presented as the composition (2). And let $M_n \overset{f}{\underset{g}{\rightrightarrows}} N$ be a pair of arrows such that $f \circ s = g \circ s$. This equality can be presented as

$$(f \circ t_n \circ s_n \circ ... \circ t_1) \circ s_1 = (g \circ t_n \circ s_n \circ ... \circ t_1) \circ s_1.$$

By the property (SL3), there exists an element $s_1' \in \Sigma_1$ such that

$$s_1' \circ (f \circ t_n \circ s_n \circ ... \circ t_1) = s_1' \circ (g \circ t_n \circ s_n \circ ... \circ t_1). \tag{3}$$

Applying (SL3) to the equality (3) presented in the form

$$(s_1' \circ f \circ t_n \circ s_n \circ ... \circ s_2) \circ t_1 = (s_1' \circ g \circ t_n \circ s_n \circ ... \circ s_2) \circ t_1,$$

we find an element $t_1' \in \Sigma_2$ such that

$$t_1' \circ (s_1' \circ f \circ t_n \circ s_n \circ ... \circ s_2) = t_1' \circ (s_1' \circ g \circ t_n \circ s_n \circ ... \circ s_2).$$

By an induction argument, we obtain the equality

$$(t_n' \circ s_n' \circ ... \circ t_1' \circ s_1') \circ f = (t_n' \circ s_n' \circ ... \circ t_1' \circ s_1') \circ g,$$

where $s_i' \in \Sigma_1$ and $t_i' \in \Sigma_2,\ i = 1, ..., n$.

(ii) It follows from the description of $\Sigma_1 \sqcup \Sigma_2$ in (i) that if Σ_1 and Σ_2 are $\widetilde{C_T}$-stable, then $\Sigma_1 \sqcup \Sigma_2$ is $\widetilde{C_T}$-stable too.

(iii) The smallest saturated multiplicative system, $\Sigma_1 \vee \Sigma_2$, spanned by Σ_1 and Σ_2 is, evidently, the saturation of $\Sigma_1 \sqcup \Sigma_2$. By (ii), the left multiplicative system $\Sigma_1 \sqcup \Sigma_2$ is $\widetilde{C_T}$-stable. Therefore, by (a), its saturation, $\Sigma_1 \vee \Sigma_2$, is $\widetilde{C_T}$-stable.

(c) The assertion follows from 1.4.3. ■

6.4. Spectra. Clearly, the trivial multiplicative system, $Iso(C_X)$, is $\widetilde{C_T}$-invariant, i.e. $Iso(C_X) \in \mathcal{S}^\mathfrak{s}\mathcal{M}(X, \gamma^*)$. Let $\mathcal{S}^\mathfrak{s}\mathcal{M}(X, \gamma^*)^\star$ denote the family $\mathcal{S}^\mathfrak{s}\mathcal{M}(X, \gamma^*) - \{Iso(C_X)\}$ of non-trivial $\widetilde{C_T}$-invariant saturated multiplicative systems.

All notions and facts considered so far in this work are extended to 'spaces' with an action of a monoidal category $\widetilde{C_T} = (C_T, \odot, 1)$ by simply replacing $\mathcal{S}^\mathfrak{s}\mathcal{M}(X)^\star$ by $\mathcal{S}^\mathfrak{s}\mathcal{M}(X, \gamma^*)^\star$ and inserting "$\widetilde{C_T}$-invariant" whenever it is required.

Thus, $\mathfrak{L}$-*local* $\widetilde{C_T}$-'spaces' are those $\widetilde{C_T}$-'spaces' (X, γ^*) for which the intersection of all $\Sigma \in \mathcal{S}^\mathfrak{s}\mathcal{M}(X, \gamma^*)^\star$ belongs to $\mathcal{S}^\mathfrak{s}\mathcal{M}(X, \gamma^*)^\star$.

The *complete* $\mathfrak{L}$-*spectrum*, $\mathbf{Spec}^1_\mathfrak{L}(X, \gamma^*)$, of a $\widetilde{C_T}$-'space' (X, γ^*) consists of all $\Sigma \in \mathcal{S}^\mathfrak{s}\mathcal{M}(X, \gamma^*)^\star$ such that the $\widetilde{C_T}$-'space' of fractions $(\Sigma^{-1}X, \widetilde{\gamma}^*)$ is $\mathfrak{L}$-local. In other words, there exists the smallest $\widetilde{C_T}$-invariant saturated multiplicative system, $\Sigma^\star$, properly containing Σ.

The $\mathfrak{L}$-*spectrum*, $\mathbf{Spec}^0_\mathfrak{L}(X, \gamma^*)$, of a $\widetilde{C_T}$-'space' (X, γ^*) is formed by $\Sigma \in \mathcal{S}^\mathfrak{s}\mathcal{M}(X, \gamma^*)$ such that there exists the biggest $\widetilde{C_T}$-invariant saturated multiplicative system, $\widehat{\Sigma}$, which does not contain Σ.

6.4.1. Proposition. *The map* $\Sigma \longmapsto \widehat{\Sigma}$ *is an injective morphism from* $\mathbf{Spec}^0_\mathfrak{L}(X, \gamma^*)$ *to* $\mathbf{Spec}^1_\mathfrak{L}(X, \gamma^*)$.

Proof. Let $\Sigma \in \mathbf{Spec}^0_{\mathfrak{L}}(X,\gamma^*)$. By 6.3.1(b), $\Sigma \vee \widehat{\Sigma}$ is the smallest $\widetilde{C_T}$-invariant saturated multiplicative system which contains Σ and $\widehat{\Sigma}$. If Σ_1 is a $\widetilde{C_T}$-invariant saturated multiplicative system which properly contains $\widehat{\Sigma}$, then it contains Σ too, hence it contains $\Sigma\vee\widehat{\Sigma}$. Therefore, $\Sigma\vee\widehat{\Sigma}$ is the smallest $\widetilde{C_T}$-invariant saturated multiplicative system which properly contains $\widehat{\Sigma}$; in particular, $\widehat{\Sigma} \in \mathbf{Spec}^1_{\mathfrak{L}}(X,\gamma^*)$.

Injectivity of the map $\Sigma \longmapsto \widehat{\Sigma}$ follows from that $\Sigma_1 \subseteq \Sigma_2$ iff $\widehat{\Sigma}_1 \subseteq \widehat{\Sigma}_2$. ■

The definitions of the remaining spectra are even more straightforward.

6.4.2. Closed spectra and flat spectra. Elements of the *closed complete* $\mathfrak{L}$-*spectrum*, $\mathbf{Spec}^1_{\mathfrak{c}}(X,\gamma^*)$, are those $\Sigma \in \mathbf{Spec}^1_{\mathfrak{L}}(X,\gamma^*)$ which are closed in the sense of 5.2. The *closed* $\mathfrak{L}$-*spectrum*, $\mathbf{Spec}^0_{\mathfrak{c}}(X,\gamma^*)$, consists of all $\Sigma \in \mathbf{Spec}^0_{\mathfrak{L}}(X,\gamma^*)$ such that $\widehat{\Sigma}$ is closed.

The *flat complete* $\mathfrak{L}$-*spectrum*, $\mathbf{Spec}^1_{\mathfrak{f}\mathfrak{L}}(X,\gamma^*)$, is formed by $\Sigma \in \mathbf{Spec}^1_{\mathfrak{L}}(X,\gamma^*)$ such that the localization $\Sigma^{-1}X \longrightarrow X$ is continuous (i.e. it has a direct image functor).

The *flat* $\mathfrak{L}$-*spectrum*, $\mathbf{Spec}^0_{\mathfrak{f}\mathfrak{L}}(X,\gamma^*)$, is formed by $\Sigma \in \mathbf{Spec}^0_{\mathfrak{L}}(X,\gamma^*)$ such that $\widehat{\Sigma}$ belongs to $\mathbf{Spec}^1_{\mathfrak{f}\mathfrak{L}}(X,\gamma^*)$.

6.4.2.1. Note. For any $\Sigma \in \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X,\gamma^*)$, the full subcategory of the category C_X generated by objects which are left closed for Σ (cf. 5.3) is $\widetilde{C_T}$-stable. But, the full subcategory of C_X whose objects are Σ-torsion free objects of C_X is not, in general, $\widetilde{C_T}$-stable.

6.5. Locally trivial actions and spectra.

6.5.1. Proposition. *Let (X,γ^*) be a $\widetilde{C}_T$-'space' such that there exists a family $\{\Sigma_i \mid i \in J\}$ of saturated stable multiplicative systems with the following properties:*

(a) $\bigcap_{i\in J} \Sigma_i = Iso(C_X)$;

(b) every $\Sigma \in \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$ containing some of Σ_i is $\widetilde{C}_T$-stable.

Then the canonical map

$$\mathbf{Spec}^0_{\mathfrak{L}}(X) \longrightarrow \mathbf{Spec}^1_{\mathfrak{L}}(X), \quad \Sigma \mapsto \widehat{\Sigma},$$

takes values in $\mathbf{Spec}^1_{\mathfrak{L}}(X,\gamma^*)$.

Proof. Let U_i denote the 'space' $\Sigma_i^{-1}X$ and u_i the canonical morphism $U_i \longrightarrow X$, $i \in J$. Since each Σ_i is stable, the action γ^* induces a $\widetilde{C}_T$-action, γ_i^* on U_i. The condition (a) means that the family of localizations $\{U_i \xrightarrow{q_i} X \mid i \in J\}$ is conservative. By 2.2.5, $\mathbf{Spec}^0_{\mathfrak{L}}(X) = \bigcup_{i\in J} \mathbf{Spec}^0_{\mathfrak{L}}(U_i;X)$, where $\mathbf{Spec}^0_{\mathfrak{L}}(U_i;X) = \{\Sigma \in \mathbf{Spec}^0_{\mathfrak{L}}(X) \mid \Sigma_{u_i} \subseteq \widehat{\Sigma}\}$. The condition (b) means that for every $i \in J$, all $\Sigma \in \mathcal{S}^{\mathfrak{s}}\mathcal{M}(U_i)$ are $\widetilde{C}_T$-stable. In particular, $\mathbf{Spec}^0_{\mathfrak{L}}(U_i, \gamma_i^*) = \mathbf{Spec}^0_{\mathfrak{L}}(U_i)$ for every $i \in J$. This implies that for every $\Sigma \in \mathbf{Spec}^0_{\mathfrak{L}}(X)$, the multiplicative system $\widehat{\Sigma}$ is $\widetilde{C}_T$-stable. Thus, the map $\Sigma \longmapsto \widehat{\Sigma}$ is an embedding $\mathbf{Spec}^0_{\mathfrak{L}}(X) \longrightarrow \mathbf{Spec}^1_{\mathfrak{L}}(X, \gamma^*)$. ■

Chapter VI
Geometry of Right Exact 'Spaces'.

In this chapter, we extend the spectrum $\mathbf{Spec}(X)$ to 'spaces' represented by svelte *right exact* categories with *weak equivalences*. We call them *right exact 'spaces' with weak equivalences*, or, simply, *right exact 'spaces'*. By definition, a right exact category with weak equivalences is a triple $(C_X, \mathfrak{E}_X, \mathcal{W}_X)$, where C_X is a category, $\mathfrak{E}_X$ is a class of *strict epimorphisms* which forms a pretopology on C_X and $\mathcal{W}_X$ is a subpretopology of $\mathfrak{E}_X$. In other words, the class $\mathfrak{E}_X$ and its subclass $\mathcal{W}_X$ contain all isomorphisms of C_X and are closed under composition and pull-backs. Every exact category (in particular, any abelian category) is identified with a right exact category with trivial (that is consisting only of isomorphisms) class of weak equivalences, whose deflations are admissible epimorphisms. Right exact categories with the trivial class of weak equivalences came into life as a (half of the) base for a version of homological algebra developed in [R13] (and outlined in [R11]).

One of the motivations behind choosing right exact 'spaces' as a setting for spectral theory comes from the fact that they form a natural (although not the most general) domain for K-theory. Grothendieck introduced K-theory for studying cycles on commutative schemes. Having K-groups of right exact 'spaces' already defined [R13], [R11], the next question is what are cycles in this case. Other motivations are of a more pragmatical nature. Abelian categories are too restrictive even for commutative algebraic geometry. Already extending the spectral picture to 'spaces' represented by exact categories (which are a special case of right exact categories) considerably increases the area of potential applications, because, for instance, the category of Banach vector spaces has a natural exact structure. This exact structure coincides with the *finest right exact structure*, which exists on any category.

The spectral theory of ('spaces' represented by) abelian categories (sketched in Chapter II) is based on the notions of *topologizing, thick* and *Serre* subcategories and some of their basic properties. A starting observation behind the content of this chapter was that a subcategory of an abelian category can be replaced by the class of epimorphisms whose kernels are objects of this subcategory. So that the idea was to describe classes of epimorphisms corresponding to topologizing, thick and Serre subcategories, and then use this description for right exact categories. The realization of this program turned out to be way more subtle than it looked in the beginning.

We start, in Section 1, with preliminaries (borrowed from [R11]) on kernels of arrows in categories with initial objects and then continue with right exact categories with weak equivalences. An important new notion which appears here is that of *stable* class of deflations. In Section 2, we introduce topologizing, thick and Serre systems of deflations and establish their main properties, which in abelian case turn into the known properties of respectively topologizing, thick and Serre subcategories (discussed in Chapter II). In Section 3, we define the spectra of a right exact 'space' $(X, \bar{\mathfrak{E}}_X)$ with weak equivalences related with topologizing, thick and Serre systems of deflations. In particular, we define the spectrum $\mathbf{Spec}_{\mathfrak{t}}(X, \bar{\mathfrak{E}}_X)$ which, in the case of abelian category C_X with the standard exact structure, is naturally isomorphic to the spectrum $\mathbf{Spec}(X)$.

In Section 4, we sketch an alternative version of spectral theory based on the notions of *semitopologizing* and *strongly closed* (– a replacement of Serre) systems. This theory requires less conditions on the right exact categories and, therefore, is much more universal. In general, the spectra of 'spaces' differ from those defined in Section 3. Both spectral theories coincide in the abelian case. In Section 5, we introduce *strongly 'exact'* functor and, in particular, *strongly 'exact'* localizations, and establish their basic properties. In Section 6, we study functorial properties of the spectra with respect to strongly 'exact' localizations. We establish the so called *locality* theorems for the spectrum $\mathbf{Spec}_{\mathfrak{t}}(X, \bar{\mathfrak{E}}_X)$ and its semitopological analog $\mathbf{Spec}_{\mathfrak{st}}(X, \bar{\mathfrak{E}}_X)$, which is one of the most important properties of these spectra. In Section 7, the main notions and facts of the work are specialized for right exact categories with initial objects. In particular, we obtain a spectral theory of 'spaces' represented by exact categories and, in the abelian case, recover main constructions and facts of Chapter II. We conclude with a couple of examples of illustrative nature.

'Complementary facts' are dedicated to properties of kernels of morphisms.

1. Preliminaries on right exact categories.

1.1. Kernels of arrows.

Let C_X be a category with an initial object, x. For a morphism $M \xrightarrow{f} N$ we define the *kernel of f* as the upper horizontal arrow in a cartesian square

$$\begin{array}{ccc} Ker(f) & \xrightarrow{\mathfrak{k}(f)} & M \\ f' \downarrow & cart & \downarrow f \\ x & \longrightarrow & N \end{array}$$

when the latter exists.

Cokernels of morphisms are defined dually, via a cocartesian square

$$\begin{array}{ccc} N & \xrightarrow{\mathfrak{c}(f)} & Cok(f) \\ f \uparrow & cocart & \uparrow f' \\ M & \longrightarrow & y \end{array}$$

where y is a final object of C_X.

If C_X is a pointed category (i.e. its initial objects are final), then the notion of the kernel is equivalent to the usual one: the diagram $Ker(f) \xrightarrow{\mathfrak{k}(f)} M \underset{0}{\overset{f}{\rightrightarrows}} N$ is exact.

Dually, the cokernel of f makes the diagram $M \underset{0}{\overset{f}{\rightrightarrows}} N \xrightarrow{\mathfrak{c}(f)} Cok(f)$ exact.

1.1.1. Lemma. *Let C_X be a category with an initial object x.*

(a) Let a morphism $M \xrightarrow{f} N$ of C_X have a kernel. The canonical morphism $Ker(f) \xrightarrow{\mathfrak{k}(f)} M$ is a monomorphism, if the unique arrow $x \xrightarrow{i_N} N$ is a monomorphism.

(b) If $M \xrightarrow{f} N$ is a monomorphism, then $x \xrightarrow{i_M} M$ is the kernel of f.

Proof. The pull-backs of monomorphisms are monomorphisms. ■

1.2. Corollary. *Let C_X be a category with an initial object x. The following conditions are equivalent:*

(a) If $M \xrightarrow{f} N$ has a kernel, then the canonical arrow $Ker(f) \xrightarrow{\mathfrak{k}(f)} M$ is a monomorphism.

(b) The unique arrow $x \xrightarrow{i_M} M$ is a monomorphism for any $M \in ObC_X$.

Proof. $(a) \Rightarrow (b)$, because, by 1.1(b), the unique morphism $x \xrightarrow{i_M} M$ is the kernel of the identical morphism $M \longrightarrow M$. The implication $(b) \Rightarrow (a)$ follows from 1.1.1(a). ■

1.3. Note. The converse assertion is not true in general: a morphism might have a trivial kernel without being a monomorphism. It is easy to produce an example in the category of pointed sets.

1.4. Examples.

1.4.1. Kernels of morphisms of unital k-algebras. Let C_X be the category Alg_k of associative unital k-algebras. The category C_X has an initial object – the k-algebra k. For any k-algebra morphism $A \xrightarrow{\varphi} B$, we have a commutative square

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ {\scriptstyle \mathfrak{k}(\varphi)} \uparrow & & \uparrow \\ k \oplus K(\varphi) & \xrightarrow{\epsilon(\varphi)} & k \end{array}$$

where $K(\varphi)$ denote the kernel of the morphism φ in the category of non-unital k-algebras and the morphism $\mathfrak{k}(\varphi)$ is determined by the inclusion $K(\varphi) \longrightarrow A$ and the k-algebra structure $k \longrightarrow A$. This square is cartesian. In fact, if

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \gamma \uparrow & & \uparrow \\ C & \xrightarrow{\psi} & k \end{array}$$

is a commutative square of k-algebra morphisms, then C is an augmented algebra: $C = k \oplus K(\psi)$. Since the restriction of $\varphi \circ \gamma$ to $K(\psi)$ is zero, it factors uniquely through $K(\varphi)$. Therefore, there is a unique k-algebra morphism $C = k \oplus K(\psi) \xrightarrow{\beta} Ker(\varphi) = k \oplus K(\varphi)$ such that $\gamma = \mathfrak{k}(\varphi) \circ \beta$ and $\psi = \epsilon(\varphi) \circ \beta$.

This shows that each (unital) k-algebra morphism $A \xrightarrow{\varphi} B$ has a canonical kernel $Ker(\varphi)$ equal to the augmented k-algebra corresponding to the ideal $K(\varphi)$.

It follows from the description of the kernel $Ker(\varphi) \xrightarrow{\mathfrak{k}(\varphi)} A$ that it is a monomorphism iff the k-algebra structure $k \longrightarrow A$ is a monomorphism.

Notice that cokernels of morphisms are not defined in Alg_k, because this category does not have final objects.

1.4.2. Kernels and cokernels of maps of sets. Since the only initial object of the category *Sets* is the empty set $\emptyset$ and there are no morphisms from a non-empty set to $\emptyset$, the kernel of any map $X \longrightarrow Y$ is $\emptyset \longrightarrow X$. The cokernel of a map $X \xrightarrow{f} Y$ is the projection $Y \xrightarrow{\mathfrak{c}(f)} Y/f(X)$, where $Y/f(X)$ is the set obtained from Y by the contraction of $f(X)$ into a point. So that $\mathfrak{c}(f)$ is an isomorphism iff either $X = \emptyset$, or $f(X)$ is a one-point set.

1.4.3. Presheaves of sets. Let C_X be a svelte category and $C_X^{\wedge}$ the category of non-trivial presheaves of sets on C_X (that is we exclude the *trivial* presheaf which assigns to every object of C_X the empty set). The category $C_X^{\wedge}$ has a final object which is the constant presheaf with values in a one-element set. If C_X has a final object, y, then $\widehat{y} = C_X(-, y)$ is a final object of the category $C_X^{\wedge}$. Since $C_X^{\wedge}$ has small colimits, it has cokernels of arbitrary morphisms which are computed object-wise, that is using 1.4.2.

If the category C_X has an initial object, x, then the presheaf $\widehat{x} = C_X(-, x)$ is an initial object of the category $C_X^{\wedge}$. In this case, the category $C_X^{\wedge}$ has kernels of all its morphisms (because $C_X^{\wedge}$ has limits) and the Yoneda functor $C_X \xrightarrow{h} C_X^{\wedge}$ preserves kernels.

Notice that the initial object of $C_X^{\wedge}$ is not isomorphic to its final object unless the category C_X is pointed, i.e. initial objects of C_X are its final objects.

1.5. Some properties of kernels. See Appendix 1.

1.6. A construction. For a class of arrows $\mathcal{S}$ of a category C_X, we denote by $\mathcal{S}^{\bar{\wedge}}$ the class of all arrows $\mathfrak{s}$ of C_X such that some pull-back of $\mathfrak{s}$ belongs to $\mathcal{S}$.

1.6.1. Proposition. *Fix a category C_X.*

(a) $\bigcup_{i\in\mathcal{J}} \mathcal{T}_i^{\bar{\wedge}} = \big(\bigcup_{i\in\mathcal{J}} \mathcal{T}_i\big)^{\bar{\wedge}}$ *for any set $\{\mathcal{T}_i \mid i\in\mathcal{J}\}$ of classes of arrows of C_X.*

(b) $\mathcal{S}\subseteq\mathcal{S}^{\bar{\wedge}}$ *and* $\mathcal{S}^{\bar{\wedge}} = (\mathcal{S}^{\bar{\wedge}})^{\bar{\wedge}}$ *for any class of arrows $\mathcal{S}$ of the category C_X.*

(c) Suppose that the category C_X quasi-filtered in the sense that any diagram of the form $L\longrightarrow M\longleftarrow N$ in the category C_X can be completed to a commutative square

$$\begin{tikzcd} \widetilde{L} \arrow[r] \arrow[d] & L \arrow[d] \\ N \arrow[r] & M \end{tikzcd}$$

(for instance, C_X is a category with fibred products, or C_X has initial objects).

(i) Let $\mathcal{S}$ be a class of arrows of C_X stable under arbitrary pull-backs. Then the class $\mathcal{S}^{\bar{\wedge}}$ is stable under pull-backs.

(ii) Suppose that $\mathcal{S}$ is stable under pull-backs and satisfies the following condition:

(#) If in the commutative diagram

$$\begin{tikzcd} \widetilde{\mathcal{L}} \arrow[r, "\widetilde{\mathfrak{s}}"] \arrow[d, "\mathfrak{j}'"'] \arrow[dr, phantom, "cart"] & \mathcal{K} \arrow[d, "\mathfrak{j}"] & \\ \mathcal{L} \arrow[r, "\mathfrak{s}"] & \mathcal{M} \arrow[r, "\mathfrak{t}"] & \mathcal{K} \end{tikzcd}$$

with cartesian square $\mathfrak{t}\circ\mathfrak{j} = id_{\mathcal{N}}$ and morphisms $\mathfrak{t}$ and $\widetilde{\mathfrak{s}}$ belong to $\mathcal{S}$, then $\mathfrak{t}\circ\mathfrak{s}\in\mathcal{S}$.

Then the class $\mathcal{S}^{\bar{\wedge}}$ is multiplicative (that is closed under composition).

Proof. (a)&(b). The equality $\bigcup_{i\in\mathcal{J}} \mathcal{T}_i^{\bar{\wedge}} = \big(\bigcup_{i\in\mathcal{J}} \mathcal{T}_i\big)^{\bar{\wedge}}$, the inclusion $\mathcal{S}\subseteq\mathcal{S}^{\bar{\wedge}}$ and the equality $\mathcal{S}^{\bar{\wedge}} = (\mathcal{S}^{\bar{\wedge}})^{\bar{\wedge}}$ are obvious.

(i) Let $L\xrightarrow{\mathfrak{s}} M$ be a morphism of $\mathcal{S}^{\bar{\wedge}}$ and

$$\begin{tikzcd} \widetilde{L} \arrow[r, "\bar{\mathfrak{s}}"] \arrow[d, "\mathfrak{f}'"'] \arrow[dr, phantom, "cart"] & \widetilde{M} \arrow[d, "\mathfrak{f}"] \\ L \arrow[r, "\mathfrak{s}"] & M \end{tikzcd}$$

a cartesian square. Since $\mathfrak{s} \in \mathcal{S}^{\bar{\wedge}}$, there exists a cartesian square

$$\begin{array}{ccc} \mathfrak{L} & \xrightarrow{\mathfrak{t}} & \mathfrak{M} \\ {\scriptstyle \phi'}\downarrow & cart & \downarrow{\scriptstyle \phi} \\ L & \xrightarrow{\mathfrak{s}} & M \end{array}$$

with $\mathfrak{t} \in \mathcal{S}$. By condition, there exists a commutative square

$$\begin{array}{ccc} \widetilde{\mathfrak{M}} & \xrightarrow{\bar{\phi}} & \widetilde{M} \\ {\scriptstyle \mathfrak{f}''}\downarrow & & \downarrow{\scriptstyle \mathfrak{f}} \\ \mathfrak{M} & \xrightarrow{\phi} & M \end{array}$$

Set $\varphi = \mathfrak{f} \circ \widetilde{\phi} = \phi \circ \mathfrak{f}'$ and consider the cartesian square

$$\begin{array}{ccc} \widetilde{\mathfrak{L}} & \xrightarrow{\gamma} & \widetilde{\mathfrak{M}} \\ {\scriptstyle \varphi'}\downarrow & cart & \downarrow{\scriptstyle \varphi} \\ L & \xrightarrow{\mathfrak{s}} & M \end{array} \tag{1}$$

The equalities $\mathfrak{f} \circ \widetilde{\phi} = \varphi = \phi \circ \mathfrak{f}'$ imply two decompositions of the square (1),

$$\begin{array}{ccc} \widetilde{\mathfrak{L}} & \xrightarrow{\gamma} & \widetilde{\mathfrak{M}} \\ {\scriptstyle \bar{\phi}'}\downarrow & cart & \downarrow{\scriptstyle \bar{\phi}} \\ \widetilde{L} & \xrightarrow{\bar{\mathfrak{s}}} & \widetilde{M} \\ {\scriptstyle \mathfrak{f}'}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{f}} \\ L & \xrightarrow{\mathfrak{s}} & M \end{array} \quad \text{and} \quad \begin{array}{ccc} \widetilde{\mathfrak{L}} & \xrightarrow{\gamma} & \widetilde{\mathfrak{M}} \\ {\scriptstyle \widetilde{\mathfrak{f}}}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{f}''} \\ \mathfrak{L} & \xrightarrow{\mathfrak{t}} & \mathfrak{M} \\ {\scriptstyle \widetilde{\phi}}\downarrow & cart & \downarrow{\scriptstyle \phi} \\ L & \xrightarrow{\mathfrak{s}} & M \end{array} \tag{2}$$

Since the class $\mathcal{S}$ is stable under pull-backs and $\mathfrak{L} \xrightarrow{\mathfrak{t}} \mathfrak{M}$ belongs to $\mathcal{S}$, it follows from the upper cartesian square of the right diagram (2) that the morphism $\widetilde{\mathfrak{L}} \xrightarrow{\gamma} \widetilde{\mathfrak{M}}$ belongs to $\mathcal{S}$. The upper cartesian square of the left diagram (2) shows that the pull-back $\widetilde{L} \xrightarrow{\bar{\mathfrak{s}}} \widetilde{M}$ of the morphism $L \xrightarrow{\mathfrak{s}} M$ belongs to the class $\mathcal{S}^{\bar{\wedge}}$.

(ii) Let $L \xrightarrow{\mathfrak{s}} M$ and $M \xrightarrow{\mathfrak{t}} N$ be morphisms of $\mathcal{S}^{\bar{\wedge}}$; i.e. there exist cartesian squares

$$\begin{array}{ccc} \widetilde{L} & \xrightarrow{\bar{\mathfrak{s}}} & \widetilde{M} \\ {\scriptstyle \mathfrak{f}'}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{f}} \\ L & \xrightarrow{\mathfrak{s}} & M \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{M} & \xrightarrow{\bar{\mathfrak{t}}} & \mathcal{N} \\ {\scriptstyle g'}\downarrow & cart & \downarrow{\scriptstyle g} \\ M & \xrightarrow{\mathfrak{t}} & N \end{array}$$

whose upper horizontal arrows belong to $\mathcal{S}$. By hypothesis, there exists a commutative square

$$\begin{tikzcd} \mathfrak{M} \arrow[r, "\bar{\mathfrak{f}}"] \arrow[d, "g''"'] & \mathcal{M} \arrow[d, "g'"] \\ \widetilde{M} \arrow[r, "\mathfrak{f}"] & M \end{tikzcd}$$

which gives rise to a pair of diagrams

$$\begin{tikzcd} \widetilde{\mathfrak{M}} \arrow[r, "\widetilde{\mathfrak{s}}"] \arrow[d, "\bar{\phi}'"'] \arrow[dr, phantom, "cart"] & \mathfrak{M} \arrow[d, "g''"] \\ \widetilde{L} \arrow[r, "\bar{\mathfrak{s}}"] \arrow[d, "\mathfrak{f}'"'] \arrow[dr, phantom, "cart"] & \widetilde{M} \arrow[d, "\mathfrak{f}"] \\ L \arrow[r, "\mathfrak{s}"] & M \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathfrak{L} \arrow[r, "\gamma"] \arrow[d, "\varphi'"'] \arrow[dr, phantom, "cart"] & \mathfrak{M} \arrow[d, "\varphi"] \\ \mathcal{M} \arrow[r, "\bar{\mathfrak{t}}"] \arrow[d, "g'"'] \arrow[dr, phantom, "cart"] & \mathcal{N} \arrow[d, "g"] \\ M \arrow[r, "\mathfrak{t}"] & N \end{tikzcd} \tag{3}$$

with cartesian squares, where $\varphi = \bar{\mathfrak{t}} \circ \bar{\mathfrak{f}}$. The latter equality implies the existence of a unique arrow $\mathfrak{M} \xrightarrow{\mathfrak{j}} \widetilde{\mathfrak{M}}$ such that $\gamma \circ \mathfrak{j} = id_{\mathfrak{M}}$.

Notice that in the diagram

$$\begin{tikzcd} \widetilde{\mathcal{M}} \arrow[r, "\widetilde{\mathfrak{u}}"] \arrow[d, "\bar{\mathfrak{j}}"'] \arrow[dr, phantom, "cart"] & \mathfrak{M} \arrow[d, "\mathfrak{j}"] & \\ \widetilde{\mathfrak{L}} \arrow[r, "\mathfrak{u}"] \arrow[d, "\bar{\phi}"'] \arrow[dr, phantom, "cart"] & \mathfrak{L} \arrow[r, "\gamma"] \arrow[d, "\phi"] \arrow[dr, phantom, "cart"] & \mathfrak{M} \arrow[d, "\psi = g \circ \varphi"] \\ L \arrow[r, "\mathfrak{s}"] & M \arrow[r, "\mathfrak{t}"] & N \end{tikzcd} \tag{4}$$

built of cartesian squares, we can take $\widetilde{\mathcal{M}} = \widetilde{\mathfrak{M}}$ and $\widetilde{\mathfrak{u}} = \widetilde{\mathfrak{s}}$.

In fact, it follows from the equality $\phi \circ \mathfrak{j} = \mathfrak{f} \circ g''$, universal property of cartesian squares (and the fact that composition of cartesian squares is a cartesian square) that the cartesian square

$$\begin{tikzcd} \widetilde{\mathcal{M}} \arrow[r, "\widetilde{\mathfrak{u}}"] \arrow[d, "\bar{\phi} \circ \bar{\mathfrak{j}}"'] \arrow[dr, phantom, "cart"] & \mathfrak{M} \arrow[d, "\phi \circ \mathfrak{j}"] \\ L \arrow[r, "\mathfrak{s}"] & M \end{tikzcd}$$

is isomorphic to the cartesian square

$$\begin{array}{ccc} \widetilde{\mathfrak{M}} & \xrightarrow{\widetilde{\mathfrak{s}}} & \mathfrak{M} \\ {\scriptstyle \mathfrak{f}'\circ\bar{\phi}'}\big\downarrow & cart & \big\downarrow{\scriptstyle \mathfrak{f}\circ g''} \\ L & \xrightarrow{\mathfrak{s}} & M \end{array}$$

In particular, $\mathcal{M} \xrightarrow{\widetilde{\mathfrak{u}}} \mathfrak{M}$ is an arrow of $\mathcal{S}$. Applying the condition (#) to the subdiagram

$$\begin{array}{ccccc} \widetilde{\mathcal{M}} & \xrightarrow{\widetilde{\mathfrak{u}}} & \mathfrak{M} & & \\ {\scriptstyle \bar{\mathfrak{j}}}\big\downarrow & cart & \big\downarrow{\scriptstyle \mathfrak{j}} & & \\ \widetilde{\mathfrak{L}} & \xrightarrow{\mathfrak{u}} & \mathfrak{L} & \xrightarrow{\gamma} & \mathfrak{M} \end{array}$$

of the diagram (3), we obtain that the composition $\widetilde{\mathfrak{L}} \xrightarrow{\gamma\circ\mathfrak{u}} \mathfrak{M}$ belongs to $\mathcal{S}$. Since the square

$$\begin{array}{ccc} \widetilde{\mathfrak{L}} & \xrightarrow{\gamma\circ\mathfrak{u}} & \mathfrak{M} \\ {\scriptstyle \bar{\phi}}\big\downarrow & cart & \big\downarrow{\scriptstyle \psi} \\ L & \xrightarrow{\mathfrak{t}\circ\mathfrak{s}} & N \end{array}$$

(derived from the lower two squares of (3)) is cartesian, this means that $\mathfrak{t}\circ\mathfrak{s} \in \mathcal{S}^{\bar{\wedge}}$. ■

1.6.2. Remarks. (a) It follows that a class of arrows $\mathcal{S}$ satisfying the condition (#) of 1.6.1 is multiplicative, that is closed under composition.

(b) It follows from the argument of 1.6.1(ii) that it suffices to require in the condition (#) that the object $\mathcal{K}$ runs through a cofinal class of objects $\mathfrak{K}$. The word *cofinal* means that for any $M \in ObC_X$, there is an arrow $\mathcal{K} \longrightarrow M$ with $\mathcal{K} \in \mathfrak{K}$.

Thus, if C_X is a category with an initial object, $\mathfrak{x}$, then the condition (#) of 1.6.1 can be replaced by the following condition:

(#') If a morphism $\mathcal{L} \xrightarrow{\mathfrak{s}} \mathcal{M}$ is such that canonical arrow $Ker(\mathfrak{s}) \longrightarrow \mathfrak{x}$ belongs to $\mathcal{S}$ and $\mathcal{M} \xrightarrow{\mathfrak{t}} \mathfrak{x}$ is from $\mathcal{S}$, then the composition $\mathcal{M} \xrightarrow{\mathfrak{t}\circ\mathfrak{s}} \mathfrak{x}$ is a morphism of $\mathcal{S}$.

Notice that if $\mathcal{S}$ is a class of arrows of C_X stable under pull-backs along morphisms from initial objects (in particular, morphisms of $\mathcal{S}$ have kernels), then the class $\mathcal{S}^{\bar{\wedge}}$ consists of all arrows $\mathcal{M} \xrightarrow{\mathfrak{s}} \mathcal{L}$ such that the canonical morphism $Ker(\mathfrak{s}) \longrightarrow \mathfrak{x}$ (– the pull-back of $\mathfrak{s}$ along the unique arrow $\mathfrak{x} \longrightarrow \mathcal{L}$) belongs to $\mathcal{S}$.

(c) Let $\mathfrak{K}$ be a cofinal class of objects of the category C_X. Suppose that a functor $C_X \xrightarrow{F} C_Y$ is such that pull-backs of retracts $\mathcal{K} \longrightarrow \mathcal{M}$ from

$$\Sigma_F \stackrel{\mathrm{def}}{=} \{\mathfrak{s} \in HomC_X \mid F(\mathfrak{s}) \in Iso(C_Y)\}$$

with $\mathcal{K} \in \mathfrak{K}$ along arrows from $\Sigma_F^{\bar{\wedge}}$ belong to Σ_F. Then the system Σ_F satisfies the condition (#).

In fact, if the condition above holds and

$$\begin{tikzcd}
\widetilde{\mathcal{L}} \arrow[r, "\widetilde{\mathfrak{s}}"] \arrow[d, "\mathfrak{j}'"'] \arrow[dr, phantom, "cart"] & \mathcal{K} \arrow[d, "\mathfrak{j}"] & \\
\mathcal{L} \arrow[r, "\mathfrak{s}"] & \mathcal{M} \arrow[r, "\mathfrak{t}"] & \mathcal{K}
\end{tikzcd}$$

is a commutative diagram with cartesian square such that $\mathfrak{t} \circ \mathfrak{j} = id_{\mathcal{N}}$ and morphisms $\mathfrak{t}$ and $\widetilde{\mathfrak{s}}$ belong to Σ_F, then $\mathfrak{j}$ is a retract from Σ_F and, therefore, both vertical arrows and upper horizontal arrow of the diagram

$$\begin{tikzcd}
F(\widetilde{\mathcal{L}}) \arrow[r, "F(\widetilde{\mathfrak{s}})"] \arrow[d, "F(\mathfrak{j}')"'] & F(\mathcal{K}) \arrow[d, "F(\mathfrak{j})"] \\
\mathcal{L} \arrow[r, "F(\mathfrak{s})"] & F(\mathcal{M})
\end{tikzcd}$$

are invertible. Therefore $F(\mathfrak{s})$ is invertible, i.e. $\mathfrak{s} \in \Sigma_F$. In particular, $\mathfrak{t} \circ \mathfrak{s} \in \Sigma_F$.

(c') Suppose that the category C_X has an initial object $\mathfrak{x}$ and a functor $C_X \xrightarrow{F} C_Y$ preserves pull-backs along the morphisms from $\mathfrak{x}$ (for instance, C_Y has initial objects too and F preserves kernels of arrows). Then the system Σ_F satisfies the condition (#') above.

If the categories C_X, C_Y and the functor F are additive and all morphisms of C_X have kernels, then

$$[F \text{ preserves pull-backs of retracts}] \Leftrightarrow [F \text{ preserves kernels}] \Leftrightarrow [F \text{ is left exact}].$$

1.6.3. Morphisms with a trivial kernel. Let $Iso(C_X)$ denote the class of all isomorphisms of a category C_X. We call elements of $Iso(C_X)^{\bar{\wedge}}$ *morphisms with trivial kernel.* It follows from the observation 1.6.2(b) that if C_X is a category with initial objects, then $Iso(C_X)^{\bar{\wedge}} = \{\mathfrak{s} \in HomC_X \mid Ker(\mathfrak{s}) \text{ is an initial object}\}$. If the category C_X is additive, then the class $Iso(C_X)^{\bar{\wedge}}$ coincides with the class of all monomorphisms of the category

C_X. There are many non-additive categories having this property. One of them is the category Alg_k of unital associative k-algebras (see 1.4.1).

1.7. Proposition. *Suppose that C_X is a quasi-filtered category, i.e. any diagram of the form $L \longrightarrow M \longleftarrow N$ in the category C_X can be completed to a commutative square (say, C_X has initial objects). Then*

$$(\bigcap_{i\in J} \mathcal{S}_i)^{\bar{\wedge}} = \bigcap_{i\in J} \mathcal{S}_i^{\bar{\wedge}}$$

for any finite set $\{\mathcal{S}_i \mid i \in J\}$ of classes of arrows of C_X which are stable under pull-backs.

Proof. Evidently, $(\bigcap_{i\in J} \mathcal{S}_i)^{\bar{\wedge}} \subseteq \bigcap_{i\in J} \mathcal{S}_i^{\bar{\wedge}}$ for any set $\{\mathcal{S}_i \mid i \in J\}$ of classes of arrows of the category C_X. The claim is that the inverse inclusion holds when J is finite and each $\mathcal{S}_i$ is stable under pull-backs.

In fact, let $J = \{1, 2, \dots, n\}$, and let $\mathfrak{s} \in \bigcap_{i\in J} \mathcal{S}_i^{\bar{\wedge}}$. Then a pull-back, $\mathfrak{s}_1$, of $\mathfrak{s}$ belongs to $\mathcal{S}_1$. Since, by 1.6.1, each of the classes $\mathcal{S}_i^{\bar{\wedge}}$ is closed under pull-backs. So that $\mathfrak{s}_1$ is an element of $\mathcal{S}_1 \cap (\bigcap_{2\le i\le n} \mathcal{S}_i^{\bar{\wedge}})$. By a standard induction argument, $(\bigcap_{2\le i\le n} \mathcal{S}_i)^{\bar{\wedge}} = \bigcap_{i\in J} \mathcal{S}_i^{\bar{\wedge}}$. Therefore, there is a pull-back, $\widetilde{\mathfrak{s}}_1$ of $\mathfrak{s}_1$ which belongs to $\bigcap_{2\le i\le n} \mathcal{S}_i$. By hypothesis, $\mathcal{S}_1$ is stable under pull-backs, in particular, $\widetilde{\mathfrak{s}}_1 \in \bigcap_{1\le i\le n} \mathcal{S}_i$. Since $\widetilde{\mathfrak{s}}_1$ is a pull-back of $\mathfrak{s}$, this proves the desired inverse inclusion $(\bigcap_{i\in J} \mathcal{S}_i)^{\bar{\wedge}} \supseteq \bigcap_{i\in J} \mathcal{S}_i^{\bar{\wedge}}$. ∎

1.8. The dual construction. For a class $\mathcal{S}$ of arrows of a category C_X, we denote by $\mathcal{S}^{\underline{\vee}}$ the class of all arrows $\mathfrak{s}$ of C_X such that some push-forward of $\mathfrak{s}$ belongs to $\mathcal{S}$. The dual versions of the facts above are left to the reader.

We shall call the arrows of $Iso(C_X)^{\underline{\vee}}$ *morphisms with trivial cokernel.* It the category C_X is additive, $Iso(C_X)^{\underline{\vee}}$ coincides with the class of all epimorphisms of C_X (see 1.6.3). In this case, the intersection $Iso(C_X)^{\bar{\wedge}} \cap Iso(C_X)^{\underline{\vee}}$ consists of all bimorphisms of C_X.

1.9. Right exact 'spaces' with weak equivalences.

Right exact categories and 'spaces' (they represent) were introduced in [R13]. Here we need a slightly more flexible structure – right exact categories with *weak equivalences.*

1.9.1. Right exact categories and 'exact' functors. A *right exact category* is a pair $(C_X, \mathfrak{E}_X)$, where C_X is a category and $\mathfrak{E}_X$ is a Grothendieck pretopology on C_X whose covers are strict epimorphisms called (after P. Gabriel) *deflations.*

1.9.2. Right exact categories and 'spaces' with weak equivalences. We call this way triples $(C_X, \mathfrak{E}_X, \mathcal{W}_X)$, where $(C_X, \mathfrak{E}_X)$ is a right exact category and $(C_X, \mathcal{W}_X)$ is an exact subcategory of $(C_X, \mathfrak{E}_X)$ (i.e. $\mathcal{W}_X$ is a subpretopology of $\mathfrak{E}_X$). We call arrows of $\mathcal{W}_X$ *weak equivalences.* For convenience, we denote the pair $(\mathfrak{E}_X, \mathcal{W}_X)$ by $\bar{\mathfrak{E}}_X$ and write $(C_X, \bar{\mathfrak{E}}_X)$ instead of $(C_X, \mathfrak{E}_X, \mathcal{W}_X)$. An 'exact' functor from $(C_X, \bar{\mathfrak{E}}_X)$ to $(C_Y, \bar{\mathfrak{E}}_Y)$ is an 'exact' functor $(C_X, \mathfrak{E}_X) \xrightarrow{F} (C_Y, \mathfrak{E}_Y)$ such that $F(\mathcal{W}_X) \subseteq \mathcal{W}_Y$.

1.9.3. Examples. Fix a right exact category $(C_X, \mathfrak{E}_X)$.

(a) The smallest class of weak equivalences is $\mathcal{W}_X = Iso(C_X)$ – the class of all isomorphisms of the category C_X.

(b) An example essential for this work is

$$\mathcal{W}_X = \mathfrak{E}_X^{\circledast} \stackrel{\text{def}}{=} Iso(C_X)^{\bar{\wedge}} \cap \mathfrak{E}_X.$$

In other words, the class $\mathcal{W}_X$ consists of deflations with trivial kernels (see 1.6.3). If the category C_X is additive, then, as it is observed in 1.6.3, the class $Iso(C_X)^{\bar{\wedge}}$ consists of all monomorphisms of C_X. Since deflations are strict epimorphisms, it follows that $\mathfrak{E}_X^{\circledast} = Iso(C_X)$; i.e. weak equivalences are isomorphisms in this case. There are many natural examples of non-additive categories having this property.

(c) One of them is the category Alg_k of unital associative k-algebras with strict epimorphisms as deflations. In fact, a k-algebra morphism has a trivial kernel iff its kernel as a k-module morphism is trivial (see 1.4.1). So that algebra morphisms with trivial kernels are monomorphisms.

1.10. Stable classes of deflations. Fix a right exact category $(C_X, \mathfrak{E}_X)$. We call a class of deflations $\mathcal{S}$ of $(C_X, \mathfrak{E}_X)$ *stable* if it is closed under pull-backs and $\mathcal{S} = \mathfrak{E}_X \cap \mathcal{S}^{\bar{\wedge}}$.

1.10.1. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a right exact category such that the category C_X is quasi-filtered, i.e. any pair of arrows $L \longrightarrow M \longleftarrow N$ can be completed to a commutative square (for instance, C_X has initial objects, or it has fiber products). Then,*

(a) For any class $\mathcal{S}$ of arrows of the category C_X which is closed under pull-backs, the intersection $\mathfrak{E}_X \cap \mathcal{S}^{\bar{\wedge}}$ is a stable class.

(b) The union and the intersection of any set of stable classes are stable classes.

Proof. (a) It follows from 1.7 and the equality $\mathcal{S}^{\bar{\wedge}} = (\mathcal{S}^{\bar{\wedge}})^{\bar{\wedge}}$ (see 1.6.1) that

$$\mathfrak{E}_X \cap (\mathfrak{E}_X \cap \mathcal{S}^{\bar{\wedge}})^{\bar{\wedge}} = \mathfrak{E}_X \cap (\mathfrak{E}_X^{\bar{\wedge}} \cap (\mathcal{S}^{\bar{\wedge}})^{\bar{\wedge}}) = \mathfrak{E}_X \cap \mathcal{S}^{\bar{\wedge}}.$$

Since the category C_X is quasi-filtered, by 1.6.1, the class $\mathcal{S}^{\bar{\wedge}}$ inherits from $\mathcal{S}$ the stability under pull-backs. Therefore, the intersection $\mathfrak{E}_X \cap \mathcal{S}^{\bar{\wedge}}$ is stable under pull-backs too.

(b1) By 1.6.1(a), $\big(\bigcup_{i\in J}\mathcal{S}_i\big)^{\bar{\wedge}} = \bigcup_{i\in J}\mathcal{S}_i^{\bar{\wedge}}$ for any $\{\mathcal{S}_i \mid i\in J\}$ of classes of arrows of the category C_X. Therefore, if all classes $\mathcal{S}_i$ are stable, then

$$\bigcup_{i\in J}\mathcal{S}_i = \bigcup_{i\in J}(\mathfrak{E}_X\cap\mathcal{S}_i^{\bar{\wedge}}) = \mathfrak{E}_X\cap\big(\bigcup_{i\in J}\mathcal{S}_i\big)^{\bar{\wedge}}.$$

(b2) For any set $\{\mathcal{S}_i \mid i\in J\}$ of classes of arrows of the category C_X, there is an obvious inclusion $(\bigcap_{i\in J}\mathcal{S}_i)^{\bar{\wedge}} \subseteq \bigcap_{i\in J}\mathcal{S}_i^{\bar{\wedge}}$. If $\{\mathcal{S}_i \mid i\in J\}$ is a family of stable classes of deflations, then the inclusion above implies that

$$\bigcap_{i\in J}\mathcal{S}_i \subseteq \mathfrak{E}_X\cap\big(\bigcap_{i\in J}\mathcal{S}_i\big)^{\bar{\wedge}} \subseteq \mathfrak{E}_X\cap\big(\bigcap_{i\in J}\mathcal{S}_i^{\bar{\wedge}}\big) = \bigcap_{i\in J}(\mathfrak{E}_X\cap\mathcal{S}_i^{\bar{\wedge}}) = \bigcap_{i\in J}\mathcal{S}_i.$$

In particular, $\bigcap_{i\in J}\mathcal{S}_i = \mathfrak{E}_X\cap\big(\bigcap_{i\in J}\mathcal{S}_i\big)^{\bar{\wedge}}$, which means, by definition, that the intersection $\bigcap_{i\in J}\mathcal{S}_i$ is a stable class. ■

1.10.2. Proposition. *Let $(C_X,\bar{\mathfrak{E}}_X) = (C_X,\mathfrak{E}_X,\mathcal{W}_X)$ be a right exact category with a class of weak equivalences containing all deflations with trivial kernels and a left divisible class of deflations: $\mathfrak{s}\circ\mathfrak{t}\in\mathfrak{E}_X\ni\mathfrak{t}$ implies that $\mathfrak{s}\in\mathfrak{E}_X$. Then every class $\mathcal{S}$ of deflations of $(C_X,\bar{\mathfrak{E}}_X)$ which is closed under pull-backs and push-forwards (we assume that arbitrary push-forwards of arrows of $\mathcal{S}$ exist) and coincides with $\mathcal{W}_X\circ\mathcal{S}$ is stable.*

Proof. Let $\mathcal{L}\xrightarrow{\mathfrak{u}}\mathcal{M}$ be an arrow of $\mathcal{S}^{\bar{\wedge}}\cap\mathfrak{E}_X$; that is there exists a cartesian square

$$\begin{array}{ccc} \mathfrak{L} & \xrightarrow{\ \mathfrak{s}\ } & \mathfrak{M} \\ {\scriptstyle \mathfrak{f}'}\big\downarrow & cart & \big\downarrow{\scriptstyle \mathfrak{f}} \\ \mathcal{L} & \xrightarrow{\ \mathfrak{u}\ } & \mathcal{M} \end{array} \qquad (5)$$

whose upper horizontal arrow belongs to $\mathcal{S}$. Taking a push-forward of $\mathfrak{s}$ along the morphism $\mathfrak{f}'$, we obtain a decomposition of this diagram into a commutative diagram

$$\begin{array}{ccccc} \mathfrak{L} & \xrightarrow{\ \mathfrak{s}\ } & \mathfrak{M} & & \\ {\scriptstyle \mathfrak{f}'}\big\downarrow & (co)cart & \big\downarrow{\scriptstyle \tilde{\mathfrak{f}}_1} & & \\ \mathcal{L} & \xrightarrow{\ \mathfrak{u}_1\ } & \widetilde{\mathcal{M}} & \xrightarrow{\ \mathfrak{u}_2\ } & \mathcal{M} \end{array} \qquad (6)$$

with a cocartesian square and the morphism $\mathfrak{u}_2$ uniquely determined by the equalities $\mathfrak{u}_2 \circ \mathfrak{u}_1 = \mathfrak{u}$, $\mathfrak{u}_2 \circ \mathfrak{f}_1 = \mathfrak{f}$. Notice that the square in the diagram (6) is also cartesian, because the square (5) is cartesian. By hypothesis, the arrow $\mathcal{L} \xrightarrow{\mathfrak{u}_1} \widetilde{\mathcal{M}}$ belongs to $\mathcal{S}$ and the arrow $\mathfrak{u}_2$ is a deflation. Taking a pull-back of the deflation $\mathfrak{u}_2$ along the morphism $\mathfrak{M} \xrightarrow{\mathfrak{f}} \mathcal{M}$, we obtain a further decomposition of the diagram (5) into the diagram

$$\begin{array}{ccccc} \mathfrak{L} & \xrightarrow{\mathfrak{s}_1} & \widetilde{\mathfrak{M}} & \xrightarrow{\mathfrak{s}_2} & \mathfrak{M} \\ {\scriptstyle \mathfrak{f}'}\downarrow & (co)cart & \downarrow{\scriptstyle \widetilde{\mathfrak{f}}_1} & cart & \downarrow{\scriptstyle \mathfrak{f}} \\ \mathcal{L} & \xrightarrow{\mathfrak{u}_1} & \widetilde{\mathcal{M}} & \xrightarrow{\mathfrak{u}_2} & \mathcal{M} \end{array} \qquad (7)$$

whose both squares are cartesian and $\mathfrak{s} = \mathfrak{s}_2 \circ \mathfrak{s}_1$. A morphism $\mathcal{K} \xrightarrow{\mathfrak{v}} \widetilde{\mathfrak{M}}$ gives rise to the diagram

$$\begin{array}{ccccc} \mathcal{K}(\mathfrak{s}) & \xrightarrow{\lambda_{\mathfrak{s}}} & \mathcal{K} & & \\ {\scriptstyle id}\downarrow{\scriptstyle \wr} & cart & \downarrow{\scriptstyle \mathfrak{k}'} & & \\ \mathcal{K}(\mathfrak{s}) & \xrightarrow{\mathfrak{t}} & \mathcal{K}(\mathfrak{s}_2) & \xrightarrow{\lambda_{\mathfrak{s}_2}} & \mathcal{K} \\ {\scriptstyle \mathfrak{k}_1}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{k}} & cart & \downarrow{\scriptstyle \mathfrak{s}_2 \circ \mathfrak{v}} \\ \mathfrak{L} & \xrightarrow{\mathfrak{s}_1} & \widetilde{\mathfrak{M}} & \xrightarrow{\mathfrak{s}_2} & \mathfrak{M} \\ {\scriptstyle \mathfrak{f}'}\downarrow & (co)cart & \downarrow{\scriptstyle \widetilde{\mathfrak{f}}_1} & cart & \downarrow{\scriptstyle \mathfrak{f}} \\ \mathcal{L} & \xrightarrow{\mathfrak{u}_1} & \widetilde{\mathcal{M}} & \xrightarrow{\mathfrak{u}_2} & \mathcal{M} \end{array} \qquad (8)$$

built of cartesian squares. Here the arrow $\mathcal{K} \xrightarrow{\mathfrak{k}'} \mathcal{K}(\mathfrak{s}_2)$ is uniquely determined by the equalities $\lambda_{\mathfrak{s}_2} \circ \mathfrak{k}' = id_{\mathcal{K}}$, $\mathfrak{k} \circ \mathfrak{k}' = \mathfrak{v}$. The upper cartesian square (with the identical left vertical arrow) is due to the fact that the square in the diagram (6) is cartesian. All horizontal arrows of the diagram (8) are deflations, because $\mathfrak{u}_1$ and $\mathfrak{u}_2$ are deflations and each of the remaining arrows is a pull-back of either $\mathfrak{u}_1$, or $\mathfrak{u}_2$. In particular, $\mathcal{K}(\mathfrak{s}) \xrightarrow{\mathfrak{t}} \mathcal{K}(\mathfrak{s}_2)$ is a deflation. The fact that $\mathfrak{t} = \mathfrak{k}' \circ \lambda_{\mathfrak{s}}$ is a strict epimorphism implies that $\mathfrak{k}'$ is a strict epimorphism. On the other hand, $\mathfrak{k}'$ is a monomorphism, due to the equality $\lambda_{\mathfrak{s}_2} \circ \mathfrak{k}' = id_{\mathcal{K}}$; hence $\mathfrak{k}'$ is an isomorphism. Therefore, $\lambda_{\mathfrak{s}_2}$ is an isomorphism. The latter means that the arrow $\widetilde{\mathcal{M}} \xrightarrow{\mathfrak{u}_2} \mathcal{M}$ belongs to $\mathfrak{E}_X^{\circledast} = Iso(C_X)^{\bar{\wedge}} \cap \mathfrak{E}_X$, i.e. it is a deflation with a trivial kernel. By hypothesis all deflations with a trivial kernel are weak equivalences. Thus, our arbitrary element $\mathfrak{u}$ of $\mathcal{S}^{\bar{\wedge}} \cap \mathfrak{E}_X$ is the composition $\mathfrak{u}_2 \circ \mathfrak{u}_1$, where $\mathfrak{u}_1 \in \mathcal{S}$ and $\mathfrak{u}_2 \in \mathcal{W}_X$. Since $\mathcal{W}_X \circ \mathcal{S} = \mathcal{S}$, the arrow $\mathfrak{u}$ belongs to $\mathcal{S}$. ■

1.10.3. Right exact categories with stable classes of weak equivalences. We are particularly interested in the right exact categories $(C_X, \mathfrak{E}_X)$ with a *stable* class weak equivalences $\mathcal{W}_X$, that is $\mathcal{W}_X = \mathcal{W}_X^{\bar{\wedge}} \cap \mathfrak{E}_X$. Since any class of weak equivalences contains all isomorphisms of the category C_X, the smallest stable class coincides with the class $\mathfrak{E}_X^{\circledast} = Iso(C_X)^{\bar{\wedge}} \cap \mathfrak{E}_X$ of all deflations with trivial kernels.

1.11. Coimages of morphisms and deflations with trivial kernels.

1.11.1. Coimages of morphisms. Fix a category C_X with an initial object x. Let $M \xrightarrow{f} N$ be an arrow which has a kernel, i.e. we have a cartesian square

$$\begin{array}{ccc} Ker(f) & \xrightarrow{\mathfrak{k}(f)} & M \\ {\scriptstyle f'}\downarrow & cart & \downarrow{\scriptstyle f} \\ x & \xrightarrow{i_N} & N \end{array}$$

with which one can associate a pair of arrows $Ker(f) \overset{\mathfrak{k}(f)}{\underset{0_f}{\rightrightarrows}} M$, where 0_f is the composition of the projection f' and the unique morphism $x \xrightarrow{i_M} M$. Since $i_N = f \circ i_M$, the morphism f equalizes the pair $Ker(f) \overset{\mathfrak{k}(f)}{\underset{0_f}{\rightrightarrows}} M$. If the cokernel of this pair of arrows exists, it will be called the *coimage of* f and denoted by $Coim(f)$, or. loosely, $M/Ker(f)$.

Let $M \xrightarrow{f} N$ be a morphism such that $Ker(f)$ and $Coim(f)$ exist. Then f is the composition of the canonical strict epimorphism $M \xrightarrow{p_f} Coim(f)$ and a uniquely defined morphism $Coim(f) \xrightarrow{j_f} N$.

1.11.1.1. Lemma. *Let $M \xrightarrow{f} N$ be a morphism such that $Ker(f)$ and $Coim(f)$ exist. There is a natural isomorphism $Ker(f) \xrightarrow{\sim} Ker(p_f)$ and the kernel of the morphism $Coim(f) \xrightarrow{j_f} N$ is trivial.*

Proof. The outer square of the commutative diagram

$$\begin{array}{ccccc} Ker(f) & \xrightarrow{f'} & x & \longrightarrow & x \\ {\scriptstyle \mathfrak{k}(f)}\downarrow & cart & \downarrow & & \downarrow \\ M & \xrightarrow{p_f} & Coim(f) & \xrightarrow{j_f} & L \end{array} \qquad (1)$$

is cartesian. Therefore, its left square is cartesian which implies, by A.3, that $Ker(p_f)$ is isomorphic to $Ker(f')$. But, $Ker(f') \simeq Ker(f)$. ∎

1.11.2. Right exact categories with coimages of deflations.

1.11.2.1. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a right exact category and $\mathfrak{E}^{\mathfrak{c}}_X$ the class of deflations which are isomorphic to their coimage. The class $\mathfrak{E}^{\mathfrak{c}}_X$ is closed under composition and contains all isomorphisms of the category C_X.*

Proof. Consider the commutative diagram

$$\begin{array}{ccccccc}
Ker(\widetilde{f}) & \xrightarrow{\mathfrak{k}(\widetilde{f})} & Ker(gf) & \xrightarrow{\widetilde{f}} & Ker(g) & \xrightarrow{g'} & x \\
\wr\downarrow & & \mathfrak{k}(gf)\downarrow & cart & \downarrow\mathfrak{k}(g) & cart & \downarrow i_N \\
Ker(f) & \xrightarrow{\mathfrak{k}(f)} & L & \xrightarrow{f} & M & \xrightarrow{g} & N \\
& & \phi\downarrow & & & & \\
& & \mathcal{V} & & & &
\end{array} \qquad (2)$$

where f and g belong to $\mathfrak{E}^{\mathfrak{c}}_X$ and the morphism $L \xrightarrow{\phi} \mathcal{V}$ equalizes the pair of arrows $Ker(gf) \overset{\mathfrak{k}(gf)}{\underset{0_{gf}}{\rightrightarrows}} L$. It follows from the left square of the diagram (2) that ϕ equalizes the pair of arrows $Ker(f) \overset{\mathfrak{k}(f)}{\underset{0_f}{\rightrightarrows}} L$. Since, by hypothesis, $L \xrightarrow{f} M$ is an equalizer of this pair of arrows, there is a unique morphism $M \xrightarrow{\gamma_\phi} \mathcal{V}$ such that $\phi = \gamma_\phi \circ f$. Since ϕ equalizes the pair $Ker(gf) \overset{\mathfrak{k}(gf)}{\underset{0_{gf}}{\rightrightarrows}} L$, it follows from the commutativity of the central square of (2) that γ_ϕ equalizes the composition of the morphism $Ker(gf) \xrightarrow{\widetilde{f}} Ker(g)$ and the pair of arrows $Ker(g) \overset{\mathfrak{k}(g)}{\underset{0_g}{\rightrightarrows}} M$. Since $Ker(gf) \xrightarrow{\widetilde{f}} Ker(g)$ is a pull-back of a deflation, it is a deflation, in particular, it is a strict epimorphism. Therefore, the cokernel of this composition is the cokernel of the pair $Ker(g) \overset{\mathfrak{k}(g)}{\underset{0_g}{\rightrightarrows}} M$. Since $M \xrightarrow{g} N$ is an equalizer of the latter pair, there exists a unique morphism $N \xrightarrow{\lambda_\phi} \mathcal{V}$ such that $\gamma_\phi = \lambda_\phi \circ g$.

Thus, $\phi = (\lambda_\phi \circ \gamma_\phi) \circ (g \circ f)$. Since $g \circ f$ is an epimorphism, a morhism ξ such that $\phi = \xi \circ (g \circ f)$ is unique. Therefore, $g \circ f$ is a cokernel of the pair $Ker(gf) \overset{\mathfrak{k}(gf)}{\underset{0_{gf}}{\rightrightarrows}} L$. ■

1.11.2.2. Corollary. *Let $(C_X, \mathfrak{E}_X)$ be a right exact category such that every its deflation has a coimage which is also a deflation and the system of deflations $\mathfrak{E}_X$ is left divisible, i.e. if a composition $\mathfrak{t} \circ \gamma$ of two arrows is a deflation, then $\mathfrak{t}$ is a deflation. Then $\mathfrak{E}_X = \mathfrak{E}^{\circledast}_X \circ \mathfrak{E}^{\mathfrak{c}}_X$ and for every deflation, this decomposition is defined uniquely up to isomorphism.*

Proof. It follows from 1.11.1.1, 1.11.2.1 and the imposed conditions that every deflation $\mathfrak{e}$ is the composition $\widetilde{\mathfrak{e}} \circ \mathfrak{t}$, where $\mathfrak{t}$ coincides with its coimage and $\widetilde{\mathfrak{e}}$ is a morphism with a trivial kernel. Since, by hypothesis, $\mathfrak{E}_X$ is left divisible, $\widetilde{\mathfrak{e}}$ is a deflation. ■

1.11.3. Proposition. *Let $M \xrightarrow{\mathfrak{s}} N$ be a deflation from $\mathfrak{E}_X^{\mathfrak{c}}$. Any cartesian square*

$$\begin{array}{ccc} \widetilde{\mathcal{M}} & \xrightarrow{\mathfrak{t}} & \mathcal{N} \\ {\scriptstyle \mathfrak{f}'}\downarrow & & \downarrow{\scriptstyle \mathfrak{f}} \\ \mathcal{M} & \xrightarrow{\mathfrak{s}} & \mathcal{L} \end{array}$$

such that $\mathfrak{s}$ and $\mathfrak{t}$ are epimorphisms having kernels and $\mathfrak{s}$ is isomorphic to its coimage is a cocartesian square.

Proof. In fact, let

$$\begin{array}{ccc} \widetilde{\mathcal{M}} & \xrightarrow{\mathfrak{t}} & \mathcal{N} \\ {\scriptstyle \mathfrak{f}'}\downarrow & & \downarrow{\scriptstyle \xi_2} \\ \mathcal{M} & \xrightarrow{\xi_1} & \mathfrak{L} \end{array}$$

be a commutative square. It follows from the commutative diagram

$$\begin{array}{ccccc} Ker(\mathfrak{t}) & \xrightarrow{\mathfrak{k}(\mathfrak{t})} & \widetilde{\mathcal{M}} & \xrightarrow{\mathfrak{t}} & \mathcal{N} \\ \wr\downarrow & & {\scriptstyle \mathfrak{f}'}\downarrow & & \downarrow{\scriptstyle \xi_2} \\ Ker(\mathfrak{s}) & \xrightarrow{\mathfrak{k}(\mathfrak{s})} & \mathcal{M} & \xrightarrow{\xi_1} & \mathfrak{L} \end{array}$$

and the fact that the morphism $\mathfrak{t}$ (hence $\xi_2 \circ \mathfrak{t}$) equalizes the pair $Ker(\mathfrak{t}) \overset{\mathfrak{k}(\mathfrak{t})}{\underset{0_{\mathfrak{t}}}{\rightrightarrows}} \widetilde{\mathcal{M}}$, that $\mathcal{M} \xrightarrow{\xi_1} \mathfrak{L}$ equalizes the pair $Ker(\mathfrak{s}) \overset{\mathfrak{k}(\mathfrak{s})}{\underset{0_{\mathfrak{s}}}{\rightrightarrows}} \mathcal{M}$. Therefore, since, by hypothesis, $\mathfrak{s}$ is the cokernel of this pair of arrows $\xi_1 = \widetilde{\xi}_1 \circ \mathfrak{s}$ for a uniquely determined morphism $\mathcal{L} \xrightarrow{\widetilde{\xi}_1} \mathfrak{L}$. So that we have:

$$\xi_2 \circ \mathfrak{t} = \xi_1 \circ \mathfrak{f}' = \widetilde{\xi}_1 \circ \mathfrak{s} \circ \mathfrak{f}' = (\widetilde{\xi}_1 \circ \mathfrak{f}) \circ \mathfrak{t}.$$

Since $\mathfrak{t}$ is an epimorphism, the equality $\xi_2 \circ \mathfrak{t} = (\widetilde{\xi}_1 \circ \mathfrak{f}) \circ \mathfrak{t}$ implies that $\xi_2 = \widetilde{\xi}_1 \circ \mathfrak{f}$. ■

1.11.3.1. Corollary. *Let $M \xrightarrow{\mathfrak{s}} N$ be a deflation from $\mathfrak{E}_X^{\mathfrak{c}}$. Then any cartesian square*

$$\begin{array}{ccc} \widetilde{\mathcal{M}} & \xrightarrow{\mathfrak{t}} & \mathcal{N} \\ {\scriptstyle \mathfrak{f}'}\downarrow & & \downarrow{\scriptstyle \mathfrak{f}} \\ \mathcal{M} & \xrightarrow{\mathfrak{s}} & \mathcal{L} \end{array}$$

is a cocartesian square.

Proof. The fact follows from 1.11.3. ■

1.11.3.2. Note. Suppose that the conditions of 1.11.2.2 hold. Let

$$\begin{array}{ccc} \widetilde{\mathcal{M}} & \xrightarrow{\mathfrak{t}} & \mathcal{N} \\ \mathfrak{f}' \downarrow & cart & \downarrow \mathfrak{f} \\ \mathcal{M} & \xrightarrow{\mathfrak{s}} & \mathcal{L} \end{array} \tag{3}$$

be a cartesian square and $\mathcal{M} \xrightarrow{\mathfrak{s}} \mathcal{L}$ is a deflation. By 1.11.3, $\mathfrak{s} = \mathfrak{e}_{\mathfrak{s}} \circ \mathfrak{s}_{\mathfrak{c}}$, where $\mathfrak{s}_{\mathfrak{c}} \in \mathfrak{E}_X^{\mathfrak{c}}$ and $\mathfrak{e}_{\mathfrak{s}} \in \mathfrak{E}_X^{\circledast}$. To this decomposition, there corresponds a decomposition

$$\begin{array}{ccccc} \widetilde{\mathcal{M}} & \xrightarrow{\mathfrak{t}} & \mathcal{N}_{\mathfrak{c}} & \xrightarrow{\mathfrak{t}_{\mathfrak{c}}} & \mathcal{N} \\ \mathfrak{f}' \downarrow & (co)cart & \downarrow \mathfrak{f}_{\mathfrak{c}} & cart & \downarrow \mathfrak{f} \\ \mathcal{M} & \xrightarrow{\mathfrak{s}_{\mathfrak{c}}} & \mathfrak{L}_{\mathfrak{c}} & \xrightarrow{\mathfrak{e}_{\mathfrak{s}}} & \mathfrak{L} \end{array} \tag{4}$$

of the square (3) into two cartesian squares. Since the class $\mathfrak{E}_X^{\circledast}$ of deflations with trivial kernel is stable under pull-backs, the horizontal arrows of the right square belong to $\mathfrak{E}_X^{\circledast}$, in particular, they are weak equivalences. By 1.11.3, the left square of (4) is both cartesian and cocartesian.

2. Topologizing, thick and Serre systems.

2.0. Assumptions. Fix a right exact category with weak equivalences $(C_X, \bar{\mathfrak{E}}_X) = (C_X, \mathfrak{E}_X, \mathcal{W}_X)$. Below is the list of assumptions which appear (not necessarily simultaneously) in different assertions of this work.

(a) The category C_X is quasi-filtered, i.e. every pair of arrows $L \longrightarrow M \longleftarrow N$ can be completed to a commutative square.

(b) The class of weak equivalences is stable, i.e. $\mathcal{W}_X = \mathcal{W}_X^{\bar{\wedge}} \cap \mathfrak{E}_X$, and has two more properties:

(b1) If $\mathfrak{s} \circ \mathfrak{t} \in \mathcal{W}_X$ and both $\mathfrak{s}$ and $\mathfrak{t}$ are deflations, then $\mathfrak{s} \in \mathcal{W}_X \ni \mathfrak{t}$.

(b2) The class $\mathcal{W}_X$ is invariant under push-forwards along deflations; that is for any pair $L \xleftarrow{\mathfrak{e}} N \xrightarrow{\mathfrak{w}} M$ of deflations with $\mathfrak{w} \in \mathcal{W}_X$, there is a cocartesian square

$$\begin{array}{ccc} N & \xrightarrow{\mathfrak{w}} & M \\ \mathfrak{e} \downarrow & cocart & \downarrow \mathfrak{e}' \\ L & \xrightarrow{\mathfrak{v}} & \widetilde{\mathcal{N}} \end{array}$$

with both horizontal arrows from $\mathcal{W}_X$.

The class $\mathfrak{E}_X$ of deflations is *left divisible* in the sense that

(c) if $\mathfrak{t} \in \mathfrak{E}_X \ni \mathfrak{s} \circ \mathfrak{t}$, then $\mathfrak{s} \in \mathfrak{E}_X$,

and *weakly right divisible* in the following sense:

(d) if $\mathfrak{s} \in \mathfrak{E}_X \ni \mathfrak{s} \circ \mathfrak{t}$, then $\mathfrak{t} \in \mathcal{W}_X^{\bar{\wedge}} \circ \mathfrak{E}_X$.

(e) There is a multiplicative class $\mathfrak{D}_X$ of arrows of the category C_X which includes all deflations and $\mathcal{W}_X^{\bar{\wedge}}$ and a map which assigns to every $\mathfrak{s} \in \mathfrak{D}_X$ a decomposition $\mathfrak{s} = \gamma_{\mathfrak{s}} \circ \mathfrak{e}_{\mathfrak{s}}$, where $\mathfrak{e}_{\mathfrak{s}}$ is a strict epimorphism, such that $\gamma_{\mathfrak{s}} \in \mathcal{W}_X^{\bar{\wedge}}$ and $\mathfrak{e}_{\mathfrak{s}} \in \mathfrak{E}_X$, whenever $\mathfrak{s} \in \mathfrak{E}_X \circ \mathcal{W}_X^{\bar{\wedge}}$.

A more detailed version of this condition is obtained by adding to (e) the following:

(e') There is a multiplicative subclass $\mathcal{E}_X$ of the class of strict epimorphisms of the category C_X and a multiplicative subclass $\mathfrak{W}_X$ of $\mathcal{W}_X^{\bar{\wedge}}$ such that $\mathfrak{e}_{\mathfrak{s}} \in \mathcal{E}_X$ and $\gamma_{\mathfrak{s}} \in \mathfrak{W}_X$ for all $\mathfrak{s} \in \mathfrak{D}_X$; and the arrows $\lambda \in \mathfrak{W}_X$ and $\mathfrak{t} \in \mathcal{E}_X$ in the decomposition $\lambda \circ \mathfrak{t}$ are determined uniquely up to isomorphism. In particular, $\mathfrak{e}_{\mathfrak{s}}$ is isomorphic to $\mathfrak{s}$ for all $\mathfrak{s} \in \mathcal{E}_X$ and $\gamma_{\mathfrak{t}} \simeq \mathfrak{t}$ for all $\mathfrak{t} \in \mathfrak{W}_X$.

2.0.1. Comments. (a) The condition (a) holds automatically if the category C_X has initial objects, or if it has fiber products.

(b) Every stable class $\mathcal{W}_X$ of weak equivalences contains the class $\mathfrak{E}_X^{\circledast}$ of deflations with trivial kernel. The class $\mathfrak{E}_X^{\circledast}$ satisfies the condition (b1) and, if there exist push-forwards of deflations with trivial kernels along deflations with trivial kernels, it satisfies the condition (b2) as well.

(c) The largest class $\mathfrak{E}_X^{\mathfrak{st}}$ of deflations of the category C_X (which consists of all strict epimorphisms such that their arbitrary pull-backs exist and are strict epimorphisms) satisfies the condition (c). This follows from two observations:

(i) if $\mathfrak{s} \circ \mathfrak{t}$ is a strict epimorphism, then $\mathfrak{s}$ is a strict epimorphism;

(ii) if there exist arbitrary pull-backs of the composition $\mathfrak{s} \circ \mathfrak{t}$ and of the morphism $\mathfrak{t}$, then there are arbitrary pull-backs of the morphism $\mathfrak{s}$.

(d)&(e') Suppose that the class of all strict epimorphisms of the category C_X is stable under pull-backs and, for every morphism $\mathcal{M} \xrightarrow{\mathfrak{f}} \mathcal{N}$ of the category C_X, there exists a kernel pair $Ker_2(\mathfrak{f}) = \mathcal{M} \times_{\mathcal{N}} \mathcal{M} \overset{p_1}{\underset{p_2}{\rightrightarrows}} \mathcal{M}$ and the cokernel $\mathcal{M} \xrightarrow{\mathfrak{c}_2(\mathfrak{f})} Coim_2(\mathfrak{f})$ of the pair (p_1, p_2) which we call *2-coimage* of the morphism $\mathfrak{f}$.

Suppose that the class $\mathcal{W}_X^{\bar{\wedge}}$ contains all monomorphisms.

(d) Then the largest class of deflations $\mathfrak{E}_X^{\mathfrak{st}}$ (which coincides with the class of all strict epimorphisms) satisfies the conditions (d) by a trivial reason, because, under the conditions above, every morphism $\mathfrak{f}$ is the composition $\mathfrak{j}(\mathfrak{f}) \circ \mathfrak{c}_2(\mathfrak{f})$ of a strict epimorphism, $\mathfrak{c}_2(\mathfrak{f})$, and a monomorphism; and, by hypothesis, $\mathcal{W}_X^{\bar{\wedge}}$ contains all monomorphisms.

(e1) By the same reason, the class $\mathfrak{E}_X^{\mathfrak{st}}$ satisfies the condition (e') with $\mathcal{E}_X = \mathfrak{E}_X^{\mathfrak{st}}$ and with $\mathfrak{W}_X$ equal to the class of all monomorphisms of the category C_X.

Notice that $\mathcal{W}_X^{\bar{\wedge}}$ contains all monomorphisms automatically, if the category C_X has initial objects, because it contains all morphisms with a trivial kernel and monomorphisms have trivial kernels.

(d1) Actually, we need a weaker condition instead of 2.0(d) which is as follows. Let

$$\begin{tikzcd} L \arrow[r, "\mathfrak{t}"] & \mathcal{N} \arrow[r, "\mathfrak{s}_1"] \arrow[d, "\mathfrak{s}_2"'] \arrow[dr, phantom, "cart"] & \mathcal{M} \arrow[d, "\mathfrak{s}_2'"] \\ & \mathcal{L} \arrow[r, "\xi_1"] & \widetilde{\mathcal{N}} \end{tikzcd}$$

be a diagram whose square is cartesian and formed by deflations and the compositions $\mathfrak{s}_2 \circ \mathfrak{t}$ and $\mathfrak{s}_2 \circ \mathfrak{t}$ are deflations. Then $\mathfrak{t} \in \mathcal{W}_X^{\bar{\wedge}} \circ \mathfrak{E}_X$.

(e2) Suppose that the category C_X has initial objects and kernels and coimages of all morphisms. Then every morphism $\mathfrak{f}$ is the composition $\mathfrak{j}_\mathfrak{f} \circ \mathfrak{p}_\mathfrak{f}$ of its coimage, $\mathfrak{p}_\mathfrak{f}$ and a morphism $\mathfrak{j}_\mathfrak{f}$. The latter belongs to $Iso(C_X)^{\bar{\wedge}}$, hence it belongs to $\mathcal{W}_X^{\bar{\wedge}}$. Thus, if the coimage of any deflation is a deflation, then it follows from (the argument of) 1.11.2.1 and from 1.11.2.2 that the condition (e') holds if we take as $\mathcal{E}_X$ the class of all strict epimorphisms which coincide with their coimage and as $\mathfrak{W}_X$ the class of all morphisms with trivial kernel.

2.0.2. Examples. (i) Suppose that the category C_X is additive. Then the class $Iso(C_X)^{\bar{\wedge}}$ of all morphisms with a trivial kernel coincides with the class of all monomorphisms of the category C_X and a morphism has a kernel iff it has a 2-kernel. It follows also that the coimage of a morphism is the same as its image. Thus, if the class $\mathfrak{E}_X^{\mathfrak{s}}$ of all strict epimorphisms of the category C_X is closed under pull-backs, then $(C_X, \mathfrak{E}_X^{\mathfrak{s}}, Iso(C_X))$ satisfies all the conditions of 2.0.

(ii) Let C_X be the category Alg_k of associative unital k-algebras (see 1.4.1). Then, similarly to the additive case, the class of all strict epimorphism is stable under base change, the class of morphisms with trivial kernel coincides with the class of monomorphisms of the category C_X (see 1.9.3(c)) and $(C_X, \mathfrak{E}_X^{\mathfrak{s}}, Iso(C_X))$ satisfies all the conditions of 2.0.

2.1. Systems. We call a class of arrows $\mathcal{S}$ of C_X a *system* in $(C_X, \bar{\mathfrak{E}}_X)$ if

(a) $\mathcal{S}$ is closed under pull-backs,

(b) $\mathcal{W}_X \circ \mathcal{S} \circ \mathcal{W}_X = \mathcal{S} \supseteq \mathcal{W}_X$,

We denote the set of systems in $(C_X, \bar{\mathfrak{E}}_X)$ by $\mathfrak{S}(X, \bar{\mathfrak{E}}_X)$ and regard it as a preorder with respect to the inclusion. The smallest system, $\mathcal{W}_X$, will be referred as *trivial.*

We denote by $\mathfrak{S}^{\bar{\wedge}}(X, \bar{\mathfrak{E}}_X)$ the subpreorder of $\mathfrak{S}(X, \bar{\mathfrak{E}}_X)$ formed by *stable* systems of deflations, i.e. systems $\mathcal{S}$ such that $\mathcal{S} = \mathfrak{E}_X \cap \mathcal{S}^{\bar{\wedge}}$.

2.1.1. Proposition. *The set $\mathfrak{S}(X, \bar{\mathfrak{E}}_X)$ of systems and the set $\mathfrak{S}^{\bar{\wedge}}(X, \bar{\mathfrak{E}}_X)$ of stable systems in $(C_X, \bar{\mathfrak{E}}_X)$ are closed under compositions and arbitrary unions and intersections.*

Proof. The inclusion $\mathcal{T} \subseteq \mathcal{W}_X \circ \mathcal{T} \circ \mathcal{W}_X$ holds for any class of arrows $\mathcal{T}$ of the category C_X. If $\{\mathcal{S}_i \mid i \in J\}$ is a set of systems, then

$$\mathcal{W}_X \subseteq \bigcap_{i\in J} \mathcal{S}_i \subseteq \mathcal{W}_X \circ \big(\bigcap_{i\in J} \mathcal{S}_i\big) \circ \mathcal{W}_X \subseteq \bigcap_{i\in J} \mathcal{S}_i$$

and, evidently,

$$\mathcal{W}_X \subseteq \bigcup_{i\in J} \mathcal{S}_i \subseteq \mathcal{W}_X \circ \big(\bigcup_{i\in J} \mathcal{S}_i\big) \circ \mathcal{W}_X = \bigcup_{i\in J} (\mathcal{W}_X \circ \mathcal{S}_i \circ \mathcal{W}_X) = \bigcup_{i\in J} \mathcal{S}_i.$$

The fact that each $\mathcal{S}_i$ is invariant under base change implies that $\bigcup_{i\in J} \mathcal{S}_i$ and $\bigcap_{i\in J} \mathcal{S}_i$ have the same property. Therefore $\bigcup_{i\in J} \mathcal{S}_i$ and $\bigcap_{i\in J} \mathcal{S}_i$ are systems.

The similar assertion for stable systems follows from 1.10.1(b). ∎

2.2. Right and left divisible systems. We call a system $\mathcal{S}$ in $(C_X, \bar{\mathfrak{E}}_X)$ *right* (resp. *left*) *divisible* if $\mathfrak{s} \in \mathcal{S}$ (resp. $\mathfrak{t} \in \mathcal{S}$) whenever $\mathfrak{t} \circ \mathfrak{s} \in \mathcal{S}$.

We say that a system $\mathcal{S}$ is *right* (resp. *left*) *divisible in* $\mathfrak{E}_X$ if $\mathfrak{s} \in \mathcal{S}$ (resp. $\mathfrak{t} \in \mathcal{S}$) whenever $\mathfrak{t} \circ \mathfrak{s} \in \mathcal{S}$ and $\mathfrak{s} \in \mathfrak{E}_X$.

It follows that the class of right (resp. left) divisible systems is stable under arbitrary unions and intersections. Similarly for systems which are right (resp. left) divisible in $\mathfrak{E}_X$.

2.2.1. Proposition. *Suppose that the category C_X has pull-backs. Then, for any right (resp. left) divisible system $\mathcal{S}$, the system $\mathcal{S}^{\bar{\wedge}}$ is right (resp. left) divisible.*

If the system $\mathcal{S}$ is left divisible in $\mathfrak{E}_X$, then $\mathcal{S}^{\bar{\wedge}}$ is left divisible in $\mathfrak{E}_X$.

Proof. (a) In fact, let $\mathcal{M} \xrightarrow{\mathfrak{u}} \mathcal{N}$ be an arrow from $\mathcal{S}^{\bar{\wedge}}$, that is its pull-back along some arrow $\mathcal{L} \xrightarrow{f} \mathcal{N}$ belongs to $\mathcal{S}$. Let $\mathfrak{u} = \mathfrak{t} \circ \mathfrak{s}$. Then, since pull-backs exist in C_X, we have the decomposition

$$\begin{array}{ccccc} \widetilde{\mathcal{M}} & \xrightarrow{\widetilde{\mathfrak{s}}} & \widetilde{\mathcal{K}} & \xrightarrow{\widetilde{\mathfrak{t}}} & \mathcal{L} \\ {\scriptstyle f'}\downarrow & cart & \downarrow & cart & \downarrow{\scriptstyle f} \\ \mathcal{M} & \xrightarrow{\mathfrak{s}} & \mathcal{K} & \xrightarrow{\mathfrak{t}} & \mathcal{N} \end{array} \qquad (1)$$

of the pull-back of $\mathfrak{u}$ along f into cartesian squares. So, if $\mathcal{S}$ is right (resp. left) divisible, then $\widetilde{\mathfrak{s}} \in \mathcal{S}$ (resp. $\widetilde{\mathfrak{t}} \in \mathcal{S}$), hence $\mathfrak{s} \in \mathcal{S}^{\bar{\wedge}}$ (resp. $\mathfrak{t} \in \mathcal{S}^{\bar{\wedge}}$).

(b) Suppose now that $\mathcal{S}$ is a system which is left divisible in $\mathfrak{E}_X$, that is if $\mathfrak{u} = \mathfrak{t} \circ \mathfrak{s} \in \mathcal{S}$ and $\mathfrak{s}$ is a deflations, then $\mathfrak{t} \in \mathcal{S}$. Consider a pull-back of such $\mathfrak{u}$ which belongs to $\mathcal{S}$ and consider its decomposition described by the diagram (1) above. Since, by hypothesis, $\mathfrak{s}$ is a deflation, the arrow $\widetilde{\mathfrak{s}}$ in (1) is a deflation. Therefore, $\widetilde{\mathfrak{t}} \in \mathcal{S}$ which implies that $\mathfrak{t} \in \mathcal{S}^{\bar{\wedge}}$. ■

2.3. Orthogonal complements. For a class of arrows Σ containing $\mathcal{W}_X$, we define the *orthogonal complement* $\Sigma^{\perp}$ of Σ as the union of all right divisible systems $\mathcal{S}$ such that $\mathcal{S} \cap \Sigma = \mathcal{W}_X$. In other words, $\Sigma^{\perp}$ is the largest right divisible system having the trivial intersection with Σ.

2.3.1. Proposition. *Let $(C_X, \bar{\mathfrak{E}}_X)$ be a quasi-filtered right exact category with a stable class of weak equivalences; and let $\mathcal{S}$ be a class of deflations of $(C_X, \bar{\mathfrak{E}}_X)$ closed under pull-backs. If the category C_X has pull-backs, then $(\mathcal{S}^{\perp})^{\bar{\wedge}} = \mathcal{S}^{\perp}$.*

Proof. It follows from 1.6.1 and 1.7 that

$$\mathcal{S} \cap (\mathcal{S}^{\perp})^{\bar{\wedge}} \subseteq \mathfrak{E}_X \cap (\mathcal{S}^{\bar{\wedge}} \cap (\mathcal{S}^{\perp})^{\bar{\wedge}}) = \mathfrak{E}_X \cap (\mathcal{S} \cap \mathcal{S}^{\perp})^{\bar{\wedge}} = \mathfrak{E}_X \cap \mathcal{W}_X^{\bar{\wedge}}.$$

By hypothesis, the system of weak equivalences is stable, that is $\mathfrak{E}_X \cap \mathcal{W}_X^{\bar{\wedge}} = \mathcal{W}_X$.

Since $\mathcal{S}^{\perp}$ is right divisible and the category C_X has pull-backs, it follows from 2.2.1 that the system $(\mathcal{S}^{\perp})^{\bar{\wedge}}$ is right divisible. Therefore, $\mathcal{S}^{\perp} = (\mathcal{S}^{\perp})^{\bar{\wedge}}$. ■

2.3.2. Example. Let C_X be a category with an initial object $\mathfrak{x}$ and kernels of morphisms. For a strict subcategory $\mathcal{T}$ of the category C_X containing initial objects, we set

$$\Sigma_{\mathcal{T}} = \{\mathfrak{s} \in HomC_X \mid Ker(\mathfrak{s}) \in Ob\mathcal{T}\}.$$

It follows from the general properties of kernels that $\Sigma_{\mathcal{T}}$ is stable under base change.

Suppose that the kernel of any morphism $M \xrightarrow{\mathfrak{e}} N$ of C_X with $M \in Ob\mathcal{T}$ belongs to the subcategory $\mathcal{T}$. Then $\Sigma_{\mathcal{T}}$ is a right divisible system.

This observation follows from the commutative diagram

$$\begin{array}{ccccccc}
Ker(\mathfrak{s}) & \xrightarrow{\mathfrak{k}(\mathfrak{s})} & N & \xrightarrow{\mathfrak{s}} & M & \xrightarrow{\mathfrak{t}} & L \\
\wr\uparrow & & \mathfrak{k}(\mathfrak{s})\uparrow & cart & \uparrow & & \uparrow \\
Ker(\widetilde{\mathfrak{s}}) & \xrightarrow{\mathfrak{k}(\widetilde{\mathfrak{s}})} & Ker(\mathfrak{ts}) & \xrightarrow{\widetilde{\mathfrak{s}}} & Ker(\mathfrak{t}) & \longrightarrow & \mathfrak{x}
\end{array} \qquad (1)$$

with the cartesian central square.

Suppose that $\mathcal{W}_X = \mathfrak{E}_X^{\circledast}$ – the class of all deflations with trivial kernels (cf. 1.9.3(b)). Let $\mathcal{S} = \mathfrak{E}_{X,\mathcal{T}} \stackrel{\text{def}}{=} \Sigma_{\mathcal{T}} \cap \mathfrak{E}_X$. One can see that the *orthogonal complement* $\mathcal{S}^{\perp}$ to $\mathcal{S}$ coincides with $\Sigma_{\mathbb{T}}$, where $\mathbb{T}$ is the full subcategory of C_X determined by

$$Ob\mathbb{T} = \{\mathcal{M} \in ObC_X \mid Ker(\mathcal{M} \to \mathcal{N}) \notin Ob\mathcal{T} - \{\text{initial objects}\} \text{ for any arrow } \mathcal{M} \to \mathcal{N}\}.$$

In the case of an abelian category C_X, this description means that $Ob\mathbb{T}$ consists of all $\mathcal{T}$-torsion free objects of the category C_X.

2.4. Topologizing systems of deflations.

2.4.1. Conventions. We assume that $(C_X, \bar{\mathfrak{E}}_X) = (C_X, \mathfrak{E}_X, \mathcal{W}_X)$ is a svelte right exact category with a stable class of weak equivalences satisfying the condition 2.0(e); that is there exists a a multiplicative class $\mathfrak{D}_X$ of arrows of the category C_X which includes all deflations and $\mathcal{W}_X^{\bar\wedge}$ and a map which assigns to every $\mathfrak{s} \in \mathfrak{D}_X$ a decomposition $\mathfrak{s} = \gamma_{\mathfrak{s}} \circ \mathfrak{e}_{\mathfrak{s}}$, where $\mathfrak{e}_{\mathfrak{s}}$ is a strict epimorphism, such that $\gamma_{\mathfrak{s}} \in \mathcal{W}_X^{\bar\wedge}$ and $\mathfrak{e}_{\mathfrak{s}} \in \mathfrak{E}_X$, whenever $\mathfrak{s} \in \mathfrak{E}_X \circ \mathcal{W}_X^{\bar\wedge}$.

In some cases (like 2.4.4 below), we need a stronger assumption 2.0(e').

2.4.2. Left topologizing and right topologizing and topologizing systems of deflations. We call a system $\mathcal{S}$ of deflations of $(C_X, \bar{\mathfrak{E}}_X)$ *left topologizing* (resp. *right topologizing*, resp. *topologizing*) if it is left divisible (resp. right divisible, resp. divisible) in $\mathfrak{E}_X$ and the following conditions hold:

(a) If all arrows of a cartesian or a cocartesian square belong to $\mathcal{S}$, then the composition of the consequent arrows of this square belongs to $\mathcal{S}$.

(b) The system $\mathcal{S}$ is closed under push-forwards.

(c) For any $\mathfrak{s} \in \mathcal{S} \circ \mathcal{W}_X^{\bar\wedge}$, the deflation $\mathfrak{e}_{\mathfrak{s}}$ in the decomposition $\mathfrak{s} = \gamma_{\mathfrak{s}} \circ \mathfrak{e}_{\mathfrak{s}}$ belongs to the system $\mathcal{S}$. In particular, $\mathcal{S} \circ \mathcal{W}_X^{\bar\wedge} \subseteq \mathcal{W}_X^{\bar\wedge} \circ \mathcal{S}$.

2.4.3. Proposition. *Suppose that the class of deflations of $(C_X, \bar{\mathfrak{E}}_X)$ is left divisible and the class of weak equivalences stable. Then every left topologizing (resp. right topologizing, resp. topologizing) system is stable.*

Proof. Since the class of weak equivalences is stable, it contains the class $\mathfrak{E}_X^{\circledast}$ of deflations with a trivial kernel. By definition, every (left or/and right) topologizing system is closed under push-forwards and composition with weak equivalences. Therefore, the assertion follows from 1.10.2. ∎

We denote the preorder (with respect to the inclusion) of all left topologizing systems of $(C_X, \bar{\mathfrak{E}}_X)$ by $\mathfrak{T}_\ell(X, \bar{\mathfrak{E}}_X)$ and the preorder of topologizing systems by $\mathfrak{T}(X, \bar{\mathfrak{E}}_X)$.

It follows that the class $\mathcal{W}_X$ of weak equivalences is the smallest topologizing system.

One can see that $\mathfrak{T}_\ell(X, \bar{\mathfrak{E}}_X)$ and $\mathfrak{T}(X, \bar{\mathfrak{E}}_X)$ are closed under arbitrary intersections and filtered (with respect to the inclusion) unions.

2.4.4. Proposition. *Let $(C_X, \bar{\mathfrak{E}}_X) = (C_X, \mathfrak{E}_X, \mathcal{W}_X)$ be a right exact category with weak equivalences and left divisible system of deflations (see 2.0(c)). Suppose that the condition 2.0(e') holds. Then*

(a) The composition of left topologizing systems is a left topologizing system.

(b) Suppose that the class $\mathcal{W}_X$ *of weak equivalences is stable (the condition 2.0(b)) and the class* $\mathfrak{E}_X$ *of deflations is weakly right divisible (the condition 2.0(d)). Then the class* $\mathfrak{T}(X, \bar{\mathfrak{E}}_X)$ *of topologizing systems is closed under composition.*

Proof. (a) Let $\mathcal{S}$, $\mathcal{T}$ be left topologizing systems of deflations. By 2.1.1, the composition of two systems is a system. In particular, $\mathcal{T} \circ \mathcal{S}$ is a system. Since a push-forward of a composition of two arrows is the composition of the corresponding push-forwards, the system $\mathcal{T} \circ \mathcal{S}$ is closed under push-forwards.

The system $\mathcal{T} \circ \mathcal{S} \circ \mathcal{W}_X^{\bar{\wedge}}$ is preserved by the map $\mathfrak{u} \longmapsto \mathfrak{e}_{\mathfrak{u}}$.

In fact, let $\mathfrak{u} = \mathfrak{t} \circ \mathfrak{s} \circ \mathfrak{w}$, where $\mathfrak{t} \in \mathcal{T}$, $\mathfrak{s} \in \mathcal{S}$, and $\mathfrak{w} \in \mathcal{W}_X^{\bar{\wedge}}$. Then

$$\mathfrak{u} = \mathfrak{t} \circ (\gamma_{\mathfrak{s}\mathfrak{w}} \circ \mathfrak{e}_{\mathfrak{s}\mathfrak{w}}) = \gamma_{\mathfrak{t}\gamma_{\mathfrak{s}\mathfrak{w}}} \circ (\mathfrak{e}_{\mathfrak{t}\gamma_{\mathfrak{s}\mathfrak{w}}} \circ \mathfrak{e}_{\mathfrak{s}\mathfrak{w}}),$$

where $\mathfrak{e}_{\mathfrak{t}\gamma_{\mathfrak{s}\mathfrak{w}}} \in \mathcal{T}$ and $\mathfrak{e}_{\mathfrak{s}\mathfrak{w}} \in \mathcal{S}$, because the systems $\mathcal{T}$ and $\mathcal{S}$ are left topologizing, and $\gamma_{\mathfrak{t}\gamma_{\mathfrak{s}\mathfrak{w}}} \in \mathfrak{W}_X \subseteq \mathcal{W}_X^{\bar{\wedge}}$. By the condition 2.0(e'), the representation of a morphism as a product $\gamma \circ \mathfrak{e}$ of $\gamma \in \mathfrak{W}_X$ and $\mathfrak{e} \in \mathcal{E}_X$ is unique up to isomorphism. In particular, the arrows $\mathfrak{e}_{\mathfrak{t}\gamma_{\mathfrak{s}\mathfrak{w}}} \circ \mathfrak{e}_{\mathfrak{s}\mathfrak{w}} \in \mathcal{T} \circ \mathcal{S}$ and $\mathfrak{e}_{\mathfrak{u}}$ are isomorphic.

It remains to show that the system $\mathcal{T} \circ \mathcal{S}$ is left divisible in $\mathfrak{E}_X$. Let

$$\begin{array}{ccc} \mathcal{N} & \xrightarrow{\mathfrak{s}} & \mathcal{L} \\ {\scriptstyle \mathfrak{u}}\downarrow & & \downarrow{\scriptstyle \mathfrak{t}} \\ \mathcal{M} & \xrightarrow{\mathfrak{v}} & \mathfrak{N} \end{array} \qquad (4)$$

be a commutative square whose all arrows are deflations with $\mathfrak{s} \in \mathcal{S}$ and $\mathfrak{t} \in \mathcal{T}$. Since $\mathcal{S}$ is closed under push-forwards, the diagram (4) is decomposed into the diagram

$$\begin{array}{ccccc} \mathcal{N} & \xrightarrow{\mathfrak{s}} & \mathcal{L} & & \\ {\scriptstyle \mathfrak{t}}\downarrow & cocart & \downarrow{\scriptstyle \mathfrak{t}'} & & \\ \mathcal{M} & \xrightarrow{\widetilde{\mathfrak{s}}} & \mathfrak{M} & \xrightarrow{\lambda} & \mathfrak{N} \end{array} \qquad (5)$$

with a cocartesian square, where $\mathcal{M} \xrightarrow{\widetilde{\mathfrak{s}}} \mathfrak{M}$ is an element of $\mathcal{S}$ and the morphism $\mathfrak{M} \xrightarrow{\lambda} \mathfrak{N}$ is uniquely determined by the equalities $\lambda \circ \widetilde{\mathfrak{s}} = \mathfrak{v}$ and $\lambda \circ \mathfrak{t}' = \mathfrak{t}$. Since $\mathfrak{t}'$ and $\mathfrak{t}$ are deflations, $\mathfrak{t} \in \mathcal{T}$, and the system $\mathcal{T}$ is left divisible in $\mathfrak{E}_X$, it follows that $\lambda \in \mathcal{T}$. So, $\mathfrak{v} = \lambda \circ \widetilde{\mathfrak{s}} \in \mathcal{T} \circ \mathcal{S}$, which shows that the system $\mathcal{T} \circ \mathcal{S}$ is left divisible in $\mathfrak{E}_X$.

(b) Suppose now that the systems $\mathcal{T}$ and $\mathcal{S}$ are topologizing (that is left topologizing and divisible) and the conditions 2.0(b) and 2.0(d) hold. The claim is that these conditions imply that the system $\mathcal{T} \circ \mathcal{S}$ is right divisible (hence divisible) in $\mathfrak{E}_X$.

In fact, consider again the commutative square (4). This time, we decompose it by taking pullback of the arrow $\mathfrak{t} \in \mathcal{T}$ along $\mathcal{M} \xrightarrow{\mathfrak{v}} \mathfrak{N}$; that is we consider the diagram

$$\begin{array}{ccccc} \mathcal{N} & \xrightarrow{\ \gamma\ } & \widetilde{\mathfrak{M}} & \xrightarrow{\ \xi_{\mathfrak{s}}\ } & \mathcal{L} \\ & & {\scriptstyle \widehat{\mathfrak{t}}}\big\downarrow & cart & \big\downarrow{\scriptstyle \mathfrak{t}} \\ & & \mathcal{M} & \xrightarrow{\ \mathfrak{v}\ } & \mathfrak{N} \end{array} \tag{6}$$

with cartesian square and morphism $\mathcal{N} \xrightarrow{\gamma} \widetilde{\mathfrak{M}}$ uniquely determined by the equalities $\xi_{\mathfrak{s}} \circ \gamma = \mathfrak{s}$ and $\widehat{\mathfrak{t}} \circ \gamma = \mathfrak{u}$. Since $\xi_{\mathfrak{s}}$ and $\mathfrak{s}$ are deflations, it follows from the condition 2.0(b) that $\gamma = \mathfrak{w} \circ \mathfrak{e}_\gamma$, where $\mathfrak{w} \in \mathcal{W}_X^{\bar{\wedge}}$ and $\mathfrak{e}_\gamma$ is a deflation. Thus, $\mathfrak{u} = (\widehat{\mathfrak{t}} \circ \mathfrak{w}) \circ \mathfrak{e}_\gamma$. Since, by hypothesis, the system of deflations $\mathfrak{E}_X$ is left divisible and $\mathfrak{u} \in \mathfrak{E}_X$, it follows from this equality that $\widehat{\mathfrak{t}} \circ \mathfrak{w}$ is a deflation. On the other hand, it belongs to $\mathfrak{E}_X \cap (\mathcal{T} \circ \mathcal{W}_X^{\bar{\wedge}})$ and, since $\mathcal{T}$ is a topologizing system, $\mathcal{T} \circ \mathcal{W}_X^{\bar{\wedge}} \subseteq \mathcal{W}_X^{\bar{\wedge}} \circ \mathcal{T}$ (see 2.4.2(c)). Therefore,

$$\mathfrak{E}_X \cap (\mathcal{T} \circ \mathcal{W}_X^{\bar{\wedge}}) \subseteq \mathfrak{E}_X \cap (\mathcal{W}_X^{\bar{\wedge}} \circ \mathcal{T}) \subseteq (\mathfrak{E}_X \cap \mathcal{W}_X^{\bar{\wedge}}) \circ \mathcal{T} = \mathcal{W}_X \circ \mathcal{T} = \mathcal{T},$$

because the class of weak equivalences $\mathcal{W}_X$ is, by hypothesis, stable, i.e. $\mathcal{W}_X = \mathcal{W}_X^{\bar{\wedge}} \cap \mathfrak{E}_X$.

Altogether shows that $\widehat{\mathfrak{t}} \circ \mathfrak{w} \in \mathcal{T}$. The class of deflations $\mathfrak{E}_X$ being left divisible, the fact that $\mathfrak{s} = \xi_{\mathfrak{s}} \circ \gamma = (\xi_{\mathfrak{s}} \circ \mathfrak{w}) \circ \mathfrak{e}_\gamma$ implies that $\xi_{\mathfrak{s}} \circ \mathfrak{w}$ is a deflation. Since $\mathfrak{s} \in \mathcal{S}$ and the system $\mathcal{S}$ is right divisible, it follows from the equality $\mathfrak{s} = (\xi_{\mathfrak{s}} \circ \mathfrak{w}) \circ \mathfrak{e}_\gamma$ that $\mathfrak{e}_\gamma \in \mathcal{S}$. Therefore, $\mathfrak{u} = (\widehat{\mathfrak{t}} \circ \mathfrak{w}) \circ \mathfrak{e}_\gamma \in \mathcal{T} \circ \mathcal{S}$.

This shows that the system $\mathcal{T} \circ \mathcal{S}$ is right divisible. Since, by (a) above, $\mathcal{T} \circ \mathcal{S}$ is left topologizing, it is topologizing. ■

2.4.5. Proposition. *(a) Let $\{\mathcal{S}_i \mid i \in J\}$ be a finite family of systems of deflations which are right divisible in $\mathfrak{E}_X$ (that is if $\mathfrak{t} \circ \mathfrak{s} \in \mathcal{S}_i$ and $\mathfrak{s} \in \mathfrak{E}_X$, then $\mathfrak{s} \in \mathcal{S}$). Suppose that the class $\mathcal{W}_X$ of weak equivalences is stable and the condition 2.0(d1) holds. Then*

$$\bigcap_{i \in J} (\mathcal{T} \circ \mathcal{S}_i) = \mathcal{T} \circ \Big(\bigcap_{i \in J} \mathcal{S}_i \Big)$$

for any topologizing system $\mathcal{T}$.

(b) Let $\{\mathcal{S}_i \mid i \in J\}$ be a finite family of systems which are left divisible in $\mathfrak{E}_X$ (that is if $\mathfrak{s} \circ \mathfrak{e} \in \mathcal{S}_i$ and $\mathfrak{e} \in \mathfrak{E}_X$, then $\mathfrak{s} \in \mathcal{S}$). Suppose that $\mathcal{T}$ is a system of deflations such that for any pair $\mathcal{L} \xleftarrow{\mathfrak{s}} \mathcal{M} \xrightarrow{\mathfrak{t}} \mathcal{N}$ of arrows of $\mathcal{T}$, there exists a cocartesian square

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\ \mathfrak{s}\ } & \mathcal{L} \\ {\scriptstyle \mathfrak{t}}\big\downarrow & cocart & \big\downarrow{\scriptstyle \widetilde{\mathfrak{t}}} \\ \mathcal{N} & \xrightarrow{\ \mathfrak{s}'\ } & \mathfrak{N} \end{array}$$

with $\widetilde{\mathfrak{t}} \circ \mathfrak{s} \in \mathcal{T}$. *Then*

$$\bigcap_{i \in J} (\mathcal{S}_i \circ \mathcal{T}) = \big(\bigcap_{i \in J} \mathcal{S}_i \big) \circ \mathcal{T}.$$

Proof. The inclusions

$$\bigcap_{i \in J} (\mathcal{S}_i \circ \mathcal{T}) \supseteq \big(\bigcap_{i \in J} \mathcal{S}_i \big) \circ \mathcal{T} \quad \text{and} \quad \bigcap_{i \in J} (\mathcal{T} \circ \mathcal{S}_i) \supseteq \mathcal{T} \circ \big(\bigcap_{i \in J} \mathcal{S}_i \big)$$

hold for class of arrows $\mathcal{T}$ and any family of classes of arrows $\{\mathcal{S}_i \mid i \in J\}$. The claim is that the inverse inclusions hold under respective conditions (a) and (b).

(a) Let $\mathfrak{v}$ be an element of $\bigcap_{i \in J} (\mathcal{T} \circ \mathcal{S}_i)$, that is $\mathfrak{u} = \mathfrak{t}_i \circ \mathfrak{s}_i$, where $\mathfrak{s}_i \in \mathcal{S}_i$, $\mathfrak{t}_i \in \mathcal{T}$ and i runs through J. So that for any $i, j \in J$, we have a commutative square

$$\begin{array}{ccc}
\mathcal{M} & \xrightarrow{\mathfrak{s}_i} & \mathcal{M}_i \\
{\scriptstyle \mathfrak{s}_j} \downarrow & & \downarrow {\scriptstyle \mathfrak{t}_i} \\
\mathcal{M}_j & \xrightarrow{\mathfrak{t}_j} & \mathcal{N}
\end{array}$$

which is decomposed into the diagram

$$\begin{array}{ccccc}
\mathcal{M} & \xrightarrow{\gamma} & \mathfrak{M} & \xrightarrow{\widetilde{\mathfrak{t}}_i} & \mathcal{M}_i \\
& & {\scriptstyle \mathfrak{t}'_j} \downarrow & cart & \downarrow {\scriptstyle \mathfrak{t}_i} \\
& & \mathcal{M}_j & \xrightarrow{\mathfrak{t}_j} & \mathcal{N}
\end{array}$$

with a cartesian square, where the morphism $\mathcal{M} \xrightarrow{\gamma} \mathfrak{M}$ is uniquely determined by the equalities $\mathfrak{t}'_j \circ \gamma = \mathfrak{s}_j$ and $\widetilde{\mathfrak{t}}_i \circ \gamma = \mathfrak{s}_i$. Since $\mathfrak{s}_i$ and $\mathfrak{s}_j$ are deflations, it follows from the condition 2.0(d1) that $\gamma = \mathfrak{w} \circ \mathfrak{e}$, where $\mathfrak{e} \in \mathfrak{E}_X$ and $\mathfrak{w} \in \mathcal{W}_X^{\bar{\wedge}}$. Set $\mathfrak{u}_i = \widetilde{\mathfrak{t}}_i \circ \mathfrak{w}$ and $\mathfrak{u}_j = \mathfrak{t}'_j \circ \mathfrak{w}$. Since $\mathfrak{s}_i = \mathfrak{u}_i \circ \mathfrak{e}$, $\mathfrak{s}_j = \mathfrak{u}_j \circ \mathfrak{e}$ and the classes $\mathcal{S}_i$ and $\mathcal{S}_j$ are right divisible in $\mathfrak{E}_X$, the deflation $\mathfrak{e}$ belongs to $\mathcal{S}_i \cap \mathcal{S}_j$. The composition $\mathfrak{t} = \mathfrak{t}_i \circ \widetilde{\mathfrak{t}}_i$ belongs to $\mathcal{T}$ and $\mathfrak{t}_i \circ \mathfrak{u}_i = \mathfrak{t} \circ \mathfrak{w}$ is a deflation which belongs to $\mathcal{T} \circ \mathcal{W}_X^{\bar{\wedge}}$. But, by the argument 2.4.4(b), $\mathfrak{E}_X \cap (\mathcal{T} \circ \mathcal{W}_X^{\bar{\wedge}}) = \mathcal{T}$. Thus, the element $\mathfrak{u} = \mathfrak{t}_i \circ \mathfrak{s}_i$ equals to the composition $(\mathfrak{t} \circ \mathfrak{w}) \circ \mathfrak{e}$, where $\mathfrak{e} \in \mathcal{S}_i \cap \mathcal{S}_j$ and $\mathfrak{t} \circ \mathfrak{w} \in \mathcal{T}$. This proves the inclusion $\bigcap_{i \in J} (\mathcal{T} \circ \mathcal{S}_i) \subseteq \mathcal{T} \circ \big(\bigcap_{i \in J} \mathcal{S}_i \big)$ in the case when $|J| = 2$. By an induction argument, it follows for an arbitrary finite J.

(b) Suppose now that the conditions (b) hold. Let $\mathfrak{u}$ be an element of $\bigcap_{i\in J}(\mathcal{S}_i \circ \mathcal{T})$, that is $\mathfrak{u} = \mathfrak{s}_i \circ \mathfrak{t}_i$, where $\mathfrak{s}_i \in \mathcal{S}$, $\mathfrak{t}_i \in \mathcal{T}$ and i runs through J. Thus, for any $i, j \in J$, we have a commutative square

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\mathfrak{t}_i} & \mathcal{M}_i \\ {\scriptstyle \mathfrak{t}_j}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{s}_i} \\ \mathcal{M}_j & \xrightarrow{\mathfrak{s}_j} & \mathcal{N} \end{array}$$

which is decomposed into the diagram

$$\begin{array}{ccccc} \mathcal{M} & \xrightarrow{\mathfrak{t}_i} & \mathcal{M}_i & & \\ {\scriptstyle \mathfrak{t}_j}\big\downarrow & cocart & \big\downarrow{\scriptstyle \mathfrak{t}'_i} & & \\ \mathcal{M}_j & \xrightarrow{\widetilde{\mathfrak{t}}_j} & \mathcal{L} & \xrightarrow{\lambda} & \mathcal{N} \end{array}$$

with a cocartesian square, where the morphism $\mathcal{L} \xrightarrow{\lambda} \mathcal{N}$ is uniquely determined by the equalities $\lambda \circ \widetilde{\mathfrak{t}}_j = \mathfrak{s}_j$ and $\lambda \circ \mathfrak{t}'_i = \mathfrak{s}_i$. Since $\widetilde{\mathfrak{t}}_j$ and $\mathfrak{t}'_i$ are deflations and, by hypothesis, the classes $\mathcal{S}_i$ and $\mathcal{S}_j$ are left divisible in $\mathfrak{E}_X$, the morphism λ belongs to $\mathcal{S}_i \cap \mathcal{S}_j$. On the other hand, the composition $\widetilde{\mathfrak{t}}_j \circ \mathfrak{t}_j$ belongs to $\mathcal{T}$, because $\mathcal{T}$ is a topologizing system and both $\mathfrak{t}_i$ and $\mathfrak{t}_j$ are its elements. Thus, $\mathfrak{s}_i \circ \mathfrak{t}_i = \lambda \circ (\widetilde{\mathfrak{t}}_j \circ \mathfrak{t}_j) = \mathfrak{s}_j \circ \mathfrak{t}_j$. The rest of the proof is the standard induction argument. ∎

2.4.6. Proposition. *Suppose that $(C_X, \mathfrak{E}_X, \mathcal{W}_X)$ is such that $\mathcal{W}_X^{\bar{\wedge}} \circ \mathfrak{E}_X = HomC_X$, the class $\mathcal{W}_X$ of weak equivalences is stable and the condition 2.0(d1) holds. Then*

$$\bigcap_{i\in J}(\mathcal{T} \circ \mathcal{S}_i) = \mathcal{T} \circ \big(\bigcap_{i\in J} \mathcal{S}_i\big)$$

for any topologizing system $\mathcal{T}$ and any finite set $\{\mathcal{S}_i \mid i \in J\}$ of classes of morphisms which are right divisible in $\mathfrak{E}_X$.

Proof. The argument is similar to that of 2.4.5(a). Details are left to the reader. ∎

2.5. Thick systems of deflations. We call a system of deflations $\mathcal{S}$ of $(C_X, \bar{\mathfrak{E}}_X)$ *thick* if it is left and right divisible in $\mathfrak{E}_X$, closed under compositions and stable. We denote by $\mathfrak{M}(X, \bar{\mathfrak{E}}_X)$ the preorder (with respect to the inclusion) of thick systems of $(C_X, \bar{\mathfrak{E}}_X)$.

It follows that $\mathcal{W}_X$ is the smallest thick system of $(C_X, \bar{\mathfrak{E}}_X)$.

2.5.1. Example. Suppose that C_X has an initial object, $\mathfrak{x}$. Let $\mathcal{T}$ be a strictly full subcategory of C_X containing initial objects and

$$\mathcal{S} = \mathfrak{E}_{X,\mathcal{T}} \stackrel{\text{def}}{=} \Sigma_{\mathcal{T}} \cap \mathfrak{E}_X = \{\mathfrak{s} \in \mathfrak{E}_X \mid Ker(\mathfrak{s}) \in Ob\mathcal{T}\}$$

(see 2.3.2). Suppose that the kernel of any deflation $M \xrightarrow{\mathfrak{e}} N$ with $M \in Ob\mathcal{T}$ belongs to the subcategory $\mathcal{T}$. Then it follows from the diagram 2.3.2(1) that the system $\mathcal{S} = \mathfrak{E}_{X,\mathcal{T}}$ is right divisible in $\mathfrak{E}_X$. Notice that the cartesian square (2) gives rise to a cartesian square

$$\begin{array}{ccc} Ker(\mathfrak{t}\circ\mathfrak{s}') & \longrightarrow & Ker(\mathfrak{t}) \\ \Big\downarrow & cart & \Big\downarrow \\ Ker(\mathfrak{s}) & \longrightarrow & \mathfrak{x} \end{array} \qquad (3)$$

with all arrows in $\mathfrak{E}_{X,\mathcal{T}}$. The condition of 2.4.2 holds iff for each cartesian square (2) with arrows from $\mathfrak{E}_{X,\mathcal{T}}$, the composition $Ker(\mathfrak{t}\circ\mathfrak{s}') \longrightarrow \mathfrak{x}$ of consecutive arrows of (3) belongs to $\mathfrak{E}_{X,\mathcal{T}}$. Notice that the object $(Ker(\mathfrak{t}\circ\mathfrak{s}'), Ker(\mathfrak{t}\circ\mathfrak{s}') \to \mathfrak{x})$ of the category $C_X/\mathfrak{x}$ is the product of $(Ker(\mathfrak{s}), Ker(\mathfrak{s}) \to \mathfrak{x})$ and $(Ker(\mathfrak{t}), Ker(\mathfrak{t}) \to \mathfrak{x})$.

2.5.2. Proposition. *Suppose that each deflation has a coimage which is also a deflation, every morphism to an initial object is a deflations, and the class of deflations $\mathfrak{E}_X$ is left divisible.*

(a) The system $\mathfrak{E}_{X,\mathcal{T}}$ is topologising iff the subcategory $\mathcal{T}/\mathfrak{x}$ is closed under finite products (taken in $C_X/\mathfrak{x}$) and for any deflation $\mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{N}$ with $\mathcal{M} \in Ob\mathcal{T}/\mathfrak{x}$, both $Ker(\mathfrak{e})$ and $\mathcal{N}$ are objects of $\mathcal{T}/\mathfrak{x}$.

(b) The system $\mathfrak{E}_{X,\mathcal{T}}$ is thick iff for any deflation $M \xrightarrow{\mathfrak{e}} N$ such that N has arrows to initial objects, M is an object of the subcategory $\mathcal{T}$ then and only then the objects N and $Ker(\mathfrak{e})$ belong to $\mathcal{T}$.

Proof. The argument for (a) follows from the discussion above. The proof of (b) uses the commutative diagram 2.3.2(1). Details are left to the reader. ■

2.5.3. The case of a pointed category. If $\mathfrak{x}$ is also a final object of C_X, then the categories $C_X/\mathfrak{x}$ and C_X are naturally isomorphic and, therefore, $K(\mathfrak{t}\circ\mathfrak{s}')$ is isomorphic to the product of $Ker(\mathfrak{t})$ and $Ker(\mathfrak{s})$.

2.5.4. Topologizing and thick subcategories of exact and abelian categories. It follows that if $(C_X, \mathfrak{E}_X)$ is an exact category, then $\mathfrak{E}_{X,\mathcal{T}}$ is a topologizing system iff the subcategory $\mathcal{T}$ is closed under finite products and admissible subquotients. In particular, if $(C_X, \mathfrak{E}_X)$ is an abelian category, then $\mathfrak{E}_{X,\mathcal{T}}$ is a topologizing system iff $\mathcal{T}$ is a topologizing subcategory of C_X in the sense of Gabriel.

It follows from 2.5.2 that any thick subcategory of an exact category $(C_X, \mathfrak{E}_X)$ is topologizing. If $(C_X, \mathfrak{E}_X)$ is abelian, then thick categories are thick in the usual sense.

2.6. Serre systems.

Fix a svelte right exact category with weak equivalences $(C_X, \bar{\mathfrak{E}}_X) = (C_X, \mathfrak{E}_X, \mathcal{W}_X)$.

2.6.1. The closure. For a class $\mathcal{S}$ of deflations of $(C_X, \bar{\mathfrak{E}}_X)$, let $\mathfrak{R}_{\mathcal{S}}$ denote the set of all systems of deflations Σ divisible in $\mathfrak{E}_X$ such that any *non-trivial* right divisible subsystem Σ' of Σ has a non-trivial intersection with $\mathcal{S}$ (that is $\mathcal{S} \cap \Sigma' - \mathcal{W}_X$ is non-empty). We denote by $\mathcal{S}^-$ the union of all $\Sigma \in \mathfrak{R}_{\mathcal{S}}$ and call it the *closure* of S.

2.6.2. Proposition. *(a) $\mathcal{S}^-$ belongs to $\mathfrak{R}_{\mathcal{S}}$ (hence it is the largest element of $\mathfrak{R}_{\mathcal{S}}$).*

(b) $(\mathcal{S}^-)^- = \mathcal{S}^-$.

(c) The system $\mathcal{S}^-$ is closed under the composition.

(d) Suppose that the class $\mathcal{W}_X$ of weak equivalences is stable. Then the system $\mathcal{S}^-$ is stable, that is $\mathcal{S}^- = \mathfrak{E}_X \cap (\mathcal{S}^-)^{\bar{\wedge}}$.

Proof. (a) Since divisible systems are closed under arbitrary unions, $\mathcal{S}^-$ is a divisible system. Let Σ be a non-trivial right divisible subsystem of $\mathcal{S}^-$. Then there exists $\Sigma' \in \mathfrak{R}_{\mathcal{S}}$ such that $\Sigma' \cap \Sigma$ is a non-trivial right divisible system. Since it is a subsystem of Σ' and $\Sigma' \in \mathfrak{R}_{\mathcal{S}}$, the intersection $\Sigma' \cap \Sigma \cap \mathcal{S}$ is non-trivial. In particular, $\Sigma \cap \mathcal{S}$ is non-trivial.

(b) It follows from the argument (a) that $\mathfrak{R}_{\mathcal{S}} = \mathfrak{R}_{\mathcal{S}^-}$; hence $(\mathcal{S}^-)^- = \mathcal{S}^-$.

(c) Let Σ be a non-trivial right divisible in $\mathfrak{E}_X$ system contained in $\mathcal{S}^- \circ \mathcal{S}^-$. Let $\mathfrak{t}$, $\mathfrak{s}$ be elements of $\mathcal{S}^-$ such that $\mathfrak{t} \circ \mathfrak{s} \in \Sigma - \mathcal{W}_X$. Since Σ is right divisible, it contains $\mathfrak{s}$. Suppose that $\mathfrak{s}$ is non-trivial, that is $\mathfrak{s} \notin \mathcal{W}_X$. Take any right divisible subsystem of Σ containing element $\mathfrak{s}$ and denote by $\widetilde{\Sigma}$ its intersection with $\mathcal{S}^-$. Thus, $\widetilde{\Sigma}$ is a non-trivial right divisible subsystem of $\mathcal{S}^- \cap \Sigma$, hence it has a non-trivial intersection with $\mathcal{S}$. If $\mathfrak{s} \in \mathcal{W}_X$, then Σ contains $\mathfrak{t} \circ \mathfrak{s}$, and we apply the argument above to $\mathfrak{t} \circ \mathfrak{s}$ itself. This shows that $\mathcal{S}^- \circ \mathcal{S}^- \in \mathfrak{F}_{\mathcal{S}}$, or, equivalently, $\mathcal{S}^- \circ \mathcal{S}^- = \mathcal{S}^-$.

(d) It follows from the definition of $\mathcal{S}^-$ that it coincides with the union of all divisible systems of deflations $\mathcal{T}$ such that $\mathcal{T} \cap \mathcal{S}^{\perp} = \mathcal{W}_X$. One can consider only *stable* systems $\mathcal{T}$, because, by 2.3.1,

$$(\mathfrak{E}_X \cap \mathcal{T}^{\bar{\wedge}}) \cap \mathcal{S}^{\perp} = \mathfrak{E}_X \cap (\mathcal{T}^{\bar{\wedge}} \cap (\mathcal{S}^{\perp})^{\bar{\wedge}}) = \mathfrak{E}_X \cap (\mathcal{T} \cap \mathcal{S}^{\perp})^{\bar{\wedge}} = \mathfrak{E}_X \cap \mathcal{W}_X^{\bar{\wedge}} = \mathcal{W}_X.$$

It follows from 1.6.1 that the union of stable systems is a stable system, hence $\mathcal{S}^-$ is a stable system. ∎

2.6.2.1. Note. One can see that $\mathcal{W}_X^- = \mathcal{W}_X$. In fact, $\mathcal{W}_X^{\perp}$ coincides with $HomC_X$, whence the equality $\mathcal{T} \cap \mathcal{W}_X^{\perp} = \mathcal{W}_X$ for a system $\mathcal{T}$ means precisely that $\mathcal{T} = \mathcal{W}_X$.

2.6.3. Serre systems of deflations. We call a class $\mathcal{S}$ of deflations of $(C_X, \bar{\mathfrak{E}}_X)$ a *Serre system* of deflations if $\mathcal{S}^- = \mathcal{S}$. We denote by $\mathfrak{Se}(X, \bar{\mathfrak{E}}_X)$ the preorder (with respect to the inclusion) of all Serre systems of deflations of $(C_X, \bar{\mathfrak{E}}_X)$.

It follows from this definition and 2.6.2 that Serre systems of deflations are thick.

2.6.4. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a svelte right exact category.*

(a) The intersection of any family of Serre systems of deflations of $(C_X, \mathfrak{E}_X)$ is a Serre system.

(b) Let $\{\mathcal{S}_i \mid i \in J\}$ be a finite set of right divisible systems of deflations of $(C_X, \mathfrak{E}_X)$. Then $\bigcap\limits_{i\in J} \mathcal{S}_i^- = \big(\bigcap\limits_{i\in J} \mathcal{S}_i\big)^-$.

Proof. (a) Let $\{\Sigma_j \mid j \in \mathfrak{I}\}$ be a set of Serre systems of deflations of $(C_X, \mathfrak{E}_X)$. Let $\mathcal{S}$ be a divisible system of deflations such that every non-trivial right divisible subsystem $\mathfrak{S}$ of $\mathcal{S}$ has a non-trivial intersection with $\bigcap\limits_{j\in\mathfrak{I}} \Sigma_j$. In particular, $\mathfrak{S} \cap \Sigma_j$ is non-trivial for every $j \in \mathfrak{I}$. Since $\Sigma_j = \Sigma_j^-$ for all $j \in \mathfrak{I}$, it follows that $\mathcal{S} \subseteq \Sigma_j$ for all $j \in \mathfrak{I}$; that is $\mathcal{S} \subseteq \bigcap\limits_{j\in\mathfrak{I}} \Sigma_j$. This shows that $\bigcap\limits_{j\in\mathfrak{I}} \Sigma_j = \big(\bigcap\limits_{j\in\mathfrak{I}} \Sigma_j\big)^-$.

(b) Let $\{\mathcal{S}_i \mid i \in J\}$ be a set of right divisible systems of deflations of $(C_X, \mathfrak{E}_X)$. Then, evidently, $\bigcap\limits_{i\in J} \mathcal{S}_i^- \supseteq \big(\bigcap\limits_{i\in J} \mathcal{S}_i\big)^-$. If J is finite, then the inverse inclusion holds.

Since, by (a) above, $\bigcap\limits_{i\in J} \mathcal{S}_i^-$ is a Serre system of deflations, it suffices to show that any non-trivial right divisible subsystem of $\bigcap\limits_{i\in J} \mathcal{S}_i^-$ has a non-trivial intersection with $\bigcap\limits_{i\in J} \mathcal{S}_i$.

Let $J = \{1, 2, \ldots, n\}$, and let $\mathcal{T}$ be a non-trivial right divisible subsystem of $\bigcap\limits_{i\in J} \mathcal{S}_i^-$. In particular, $\mathcal{T}$ is a non-trivial right divisible subsystem of $\mathcal{S}_1^-$. Therefore, $\mathcal{T} \cap \mathcal{S}_1$ is a non-trivial right divisible subsystem of $\bigcap\limits_{2\le i\le n} \mathcal{S}_i^-$. By a standard induction argument, this implies that $(\mathcal{T} \cap \mathcal{S}_1) \cap \big(\bigcap\limits_{2\le i\le n} \mathcal{S}_i\big) = \mathcal{T} \cap \big(\bigcap\limits_{i\in J} \mathcal{S}_i\big)$ is a non-trivial right divisible system. Therefore, $\mathcal{T} \subseteq \big(\bigcap\limits_{i\in J} \mathcal{S}_i\big)^-$. In particular, $\bigcap\limits_{i\in J} \mathcal{S}_i^- \subseteq \big(\bigcap\limits_{i\in J} \mathcal{S}_i\big)^-$. ∎

2.6.5. The lattice of Serre systems. Fix a svelte right exact category $(C_X, \mathfrak{E}_X)$. For any pair Σ_1, Σ_2 of Serre systems of deflations, we denote by $\Sigma_1 \vee \Sigma_2$ the smallest Serre system containing Σ_1 and Σ_2.

2.6.5.1. Proposition. *Let $\{\mathcal{S}_i \mid i \in J\}$ be a finite set of Serre systems of deflations of $(C_X, \mathfrak{E}_X)$. Then $\Sigma \vee \big(\bigcap\limits_{i\in J} \mathcal{S}_i\big) = \bigcap\limits_{i\in J}(\Sigma \vee \mathcal{S}_i)$ for any Serre system of deflations Σ.*

Proof. There are the equalities

$$\bigcap_{i\in J}(\Sigma\vee\mathcal{S}_i)=\bigcap_{i\in J}(\Sigma\cup\mathcal{S}_i)^-=\Big(\bigcap_{i\in J}(\Sigma\cup\mathcal{S}_i)\Big)^-=\Big(\big(\bigcap_{i\in J}\mathcal{S}_i\big)\cup\Sigma\Big)^-=\Big(\bigcap_{i\in J}\mathcal{S}_i\Big)\vee\Sigma.$$

Here the second equality follows from 2.6.4. ∎

2.7. Serre subcategories of a right exact category with initial objects. Suppose that the category C_X has an initial object, $\mathfrak{x}$ and all morphisms to $\mathfrak{x}$ are deflations. Let $\mathcal{S}$ be a class of deflations of $(C_X,\bar{\mathfrak{E}}_X)$. We denote by $\widetilde{\mathcal{T}}_{\mathcal{S}}$ the full subcategory of the category C_X generated by all $M\in ObC_X$ having the following property: for any pair of deflations $M\xrightarrow{\mathfrak{e}}L\xrightarrow{\mathfrak{t}}\mathfrak{x}$ such that $\mathfrak{t}$ is non-trivial (i.e. $\mathfrak{t}\notin\mathcal{W}_X$), there exists a decomposition $\mathfrak{t}=\mathfrak{u}\circ\mathfrak{s}$, where $\mathfrak{u}$ and $\mathfrak{s}$ are deflations and $\mathfrak{s}$ is a non-trivial element of $\mathcal{S}$. We denote by $\mathcal{T}_{\mathcal{S}}$ the full subcategory of C_X generated by all $M\in ObC_X$ such that for any pair of deflations $M\xrightarrow{\mathfrak{u}}L\xrightarrow{\mathfrak{t}}\mathfrak{x}$ the object $Ker(\mathfrak{u})$ belongs to the subcategory $\widetilde{\mathcal{T}}_{\mathcal{S}}$.

It follows from the definition of $\mathcal{T}_{\mathcal{S}}$ that if M is an object of $\mathcal{T}_{\mathcal{S}}$ and $M\xrightarrow{\mathfrak{e}}L\xrightarrow{\mathfrak{t}}\mathfrak{x}$ are deflations, then $L\in Ob\mathcal{T}_{\mathcal{S}}$.

In fact, let $L\xrightarrow{\mathfrak{u}}N\xrightarrow{\mathfrak{t}}\mathfrak{x}$ be deflations. Then we have a commutative diagram

$$\begin{tikzcd}
Ker(\mathfrak{u}\circ\mathfrak{e}) \arrow[r, "\mathfrak{e}'"] \arrow[d] \arrow[dr, phantom, "cart"] & Ker(\mathfrak{u}) \arrow[r, "\lambda_{\mathfrak{u}}"] \arrow[d] \arrow[dr, phantom, "cart"] & \mathfrak{x} \arrow[d] & \\
M \arrow[r, "\mathfrak{e}"] & L \arrow[r, "\mathfrak{u}"] & N \arrow[r] & \mathfrak{x}
\end{tikzcd}$$

in which all horizontal arrows are deflations and $Ker(\mathfrak{u}\circ\mathfrak{e})$ is an object of $\widetilde{\mathcal{T}}_{\mathcal{S}}$. Therefore, $Ker(\mathfrak{u})\in Ob\widetilde{\mathcal{T}}_{\mathcal{S}}$, which implies that $L\in Ob\mathcal{T}_{\mathcal{S}}$.

By 2.3.2, the latter property implies that $\Sigma_{\mathcal{T}_{\mathcal{S}}}$ is a left divisible system.

2.7.1. Proposition. *Let $(C_X,\bar{\mathfrak{E}}_X)$ be a right exact category with an initial object $\mathfrak{x}$ and the class of weak equivalences coinciding with $\mathfrak{E}_X^{\circledast}=\{\mathfrak{e}\in\mathfrak{E}_X\mid Ker(\mathfrak{e})\simeq\mathfrak{x}\}$.*

If $\mathcal{S}$ is a class of deflations of $(C_X,\bar{\mathfrak{E}}_X)$ closed under pull-backs, then $\Sigma_{\mathcal{T}_{\mathcal{S}}}=\mathcal{S}^-$.

Proof. (a) The system $\Sigma_{\mathcal{T}_{\mathcal{S}}}$ belongs to $\mathfrak{R}_{\mathcal{S}}$; in particular, $\Sigma_{\mathcal{T}_{\mathcal{S}}}\subseteq\mathcal{S}^-$.

In fact, let $M\xrightarrow{\mathfrak{t}}N$ be an element of $\Sigma_{\mathcal{T}_{\mathcal{S}}}-\mathcal{W}_X$. By condition, this means that $Ker(\mathfrak{t})$ is non-trivial (i.e. it is not an initial) object of the subcategory $\mathcal{T}_{\mathcal{S}}$. Therefore, the canonical morphism $Ker(\mathfrak{t})\xrightarrow{\mathfrak{e}_{\mathfrak{t}}}\mathfrak{x}$ is the composition of a non-trivial arrow $Ker(\mathfrak{t})\xrightarrow{\mathfrak{s}}L$ of $\mathcal{S}$ and a deflation $L\longrightarrow\mathfrak{x}$. This shows that any right divisible system containing the arrow $M\xrightarrow{\mathfrak{t}}N$ has a non-trivial morphism from $\mathcal{S}$; hence $\Sigma_{\mathcal{T}_{\mathcal{S}}}\subseteq\mathcal{S}^-$.

(b) It remains to show that $\mathcal{S}^-\subseteq\Sigma_{\mathcal{T}_{\mathcal{S}}}$.

Suppose this is not true, and let $M \xrightarrow{\mathfrak{s}} L$ be an arrow from $\mathcal{S}^-$ which does not belong to $\Sigma_{\mathcal{T}_\mathcal{S}}$; that is $Ker(\mathfrak{s})$ is not an object of $\mathcal{T}_\mathcal{S}$, which means that the canonical deflation $Ker(\mathfrak{s}) \xrightarrow{\mathfrak{e}_\mathfrak{s}} \mathfrak{x}$ factors through a deflation $Ker(\mathfrak{s}) \xrightarrow{\mathfrak{v}} N$ such that $Ker(\mathfrak{v}) \xrightarrow{\lambda_\mathfrak{v}} \mathfrak{x}$ is a composition of two deflations, $Ker(\mathfrak{v}) \xrightarrow{\mathfrak{u}} L$ and $\mathfrak{L} \xrightarrow{\mathfrak{t}} \mathfrak{x}$, where $\mathfrak{t}$ is non-trivial and $\mathcal{S}$-*torsion free* in the sense that if $\mathfrak{t} = \mathfrak{t}' \circ \gamma$ and $\gamma \in \mathcal{S}$, then $\gamma \in \mathcal{W}_X$. Since $\mathcal{S}^-$ is a left divisible system of deflations, the deflation $\mathfrak{L} \xrightarrow{\mathfrak{t}} \mathfrak{x}$ belongs to $\mathcal{S}^-$.

Consider the smallest right divisible system generated by the morphism $L \xrightarrow{\mathfrak{t}} \mathfrak{x}$. It consists of all deflations $\mathcal{M} \xrightarrow{\mathfrak{e}} \widetilde{\mathcal{N}}$ such that there is a deflation $\widetilde{\mathcal{N}} \xrightarrow{\mathfrak{w}} \mathcal{N}$ and a cartesian square

$$\begin{tikzcd}
L \arrow[r, "\mathfrak{t}"] & \mathfrak{x} \\
\mathcal{M} \arrow[u] \arrow[r, "\mathfrak{w}\circ\mathfrak{e}"'] \arrow[ur, phantom, "cart"] & \mathcal{N} \arrow[u]
\end{tikzcd}$$

Since the composition of cartesian squares is a cartesian square, we have a decomposition

$$\begin{tikzcd}
L \arrow[r] \arrow[d, "\mathfrak{t}"'] \arrow[dr, phantom, "cart"] & \mathcal{M} \arrow[r] \arrow[d, "\mathfrak{w}\mathfrak{e}"'] \arrow[dr, phantom, "cart"] & L \arrow[d, "\mathfrak{t}"] \\
\mathfrak{x} \arrow[r] & \mathcal{N} \arrow[r] & \mathfrak{x}
\end{tikzcd}
\quad \text{of} \quad
\begin{tikzcd}
L \arrow[r, "id"] \arrow[d, "\mathfrak{t}"'] & L \arrow[d, "\mathfrak{t}"] \\
\mathfrak{x} \arrow[r, "id"] & \mathfrak{x}
\end{tikzcd}$$

and a decomposition

$$\begin{tikzcd}
L \arrow[r, "\widetilde{\mathfrak{e}}"] \arrow[d] \arrow[dr, phantom, "cart"] & Ker(\mathfrak{w}) \arrow[r, "\mathfrak{e}_\mathfrak{w}"] \arrow[d] \arrow[dr, phantom, "cart"] & \mathfrak{x} \arrow[d] \\
\mathcal{M} \arrow[r, "\mathfrak{e}"] & \widetilde{\mathcal{N}} \arrow[r, "\mathfrak{w}"] & \mathcal{N}
\end{tikzcd}
\quad \text{of} \quad
\begin{tikzcd}
L \arrow[r, "\mathfrak{t}"] \arrow[d] \arrow[dr, phantom, "cart"] & \mathfrak{x} \arrow[d] \\
\mathcal{M} \arrow[r, "\mathfrak{w}\circ\mathfrak{e}"] & \mathcal{N}
\end{tikzcd}$$

If $\mathcal{M} \xrightarrow{\mathfrak{e}} \widetilde{\mathcal{N}}$ is a non-trivial element of $\mathcal{S}$, then, since (by hypothesis, $Ker(\mathfrak{e})$ is non-trivial and) $Ker(\widetilde{\mathfrak{e}}) \simeq Ker(\mathfrak{e})$, the arrow $L \xrightarrow{\widetilde{\mathfrak{e}}} Ker(\mathfrak{w})$ is a non-trivial element of $\mathcal{S}$, which contradicts to the condition on $L \xrightarrow{\mathfrak{t}} \mathfrak{x}$. ∎

2.8. Coreflective systems and Serre systems. Let $(C_X, \bar{\mathfrak{E}}_X)$ be a right exact category with weak equivalences and $\mathcal{S}$ a class of its deflations containing $\mathcal{W}_X$. We call the class $\mathcal{S}$ *coreflective* if every deflation $M \xrightarrow{\mathfrak{e}} L$ is the composition of an arrow $M \xrightarrow{\mathfrak{s}_\mathfrak{e}} N$ of $\mathcal{S}$ and a deflation $N \xrightarrow{\gamma_\mathfrak{e}} L$ such that any other decomposition $M \xrightarrow{\mathfrak{t}} \mathfrak{N} \xrightarrow{\mathfrak{u}} L$ of $\mathfrak{e}$ with $\mathfrak{t} \in \mathcal{S}$ factors through $M \xrightarrow{\mathfrak{s}_\mathfrak{e}} N \xrightarrow{\gamma_\mathfrak{e}} L$. The latter means that there exists a deflation $\mathfrak{N} \xrightarrow{\mathfrak{v}} N$ such that $\mathfrak{s}_\mathfrak{e} = \mathfrak{v} \circ \mathfrak{t}$ and $\mathfrak{u} = \gamma_\mathfrak{e} \circ \mathfrak{v}$. Since $\mathfrak{t}$ is an epimorphism, the first equality implies that $\mathfrak{v}$ is uniquely defined.

2.8.1. Proposition. *Every coreflective system of deflations which is stable under base change and closed under compositions is a Serre system.*

Proof. In fact, each deflation $M \xrightarrow{\mathfrak{e}} L$ has the biggest decomposition $M \xrightarrow{\mathfrak{s}_\mathfrak{e}} N \xrightarrow{\gamma_\mathfrak{e}} L$, where $\mathfrak{s}_\mathfrak{e} \in \mathcal{S}$. Since $\mathcal{S}$ is closed under composition, $\gamma_\mathfrak{e}$ has only a trivial decomposition. Therefore, $\mathcal{S}^- = \mathcal{S}$. ∎

3. The spectra related with topologizing, thick and Serre systems.

Fix a svelte right exact category with a stable class of weak equivalences $(C_X, \bar{\mathfrak{E}}_X) = (C_X, \mathfrak{E}_X, \mathcal{W}_X)$. Recall that $\mathfrak{T}(X, \bar{\mathfrak{E}}_X)$ denotes the preorder of all topologizing systems of deflations of $(C_X, \bar{\mathfrak{E}}_X)$, $\mathfrak{Se}(X, \bar{\mathfrak{E}}_X)$ the preorder of all Serre systems and $\mathfrak{M}(X, \bar{\mathfrak{E}}_X)$ the preorder of all thick systems of $(C_X, \bar{\mathfrak{E}}_X)$. We denote by $\mathfrak{M}_\mathfrak{T}(X, \bar{\mathfrak{E}}_X)$ (resp. by $\mathfrak{Se}_\mathfrak{T}(X, \bar{\mathfrak{E}}_X)$) the subpreorder of all thick (resp. Serre) *topologizing* systems. That is

$$\mathfrak{M}_\mathfrak{T}(X, \bar{\mathfrak{E}}_X) = \mathfrak{M}(X, \bar{\mathfrak{E}}_X) \cap \mathfrak{T}(X, \bar{\mathfrak{E}}_X) \quad \text{and} \quad \mathfrak{Se}_\mathfrak{T}(X, \bar{\mathfrak{E}}_X) = \mathfrak{Se}(X, \bar{\mathfrak{E}}_X) \cap \mathfrak{T}(X, \bar{\mathfrak{E}}_X).$$

3.1. The support in topologizing systems. For any class $\mathcal{S}$ of the deflations of $(C_X, \bar{\mathfrak{E}}_X)$ containing the class $\mathcal{W}_X$ of weak equivalences, we denote by $Supp_\mathfrak{T}(\mathcal{S})$ the subpreorder of $\mathfrak{T}(X, \bar{\mathfrak{E}}_X)$ formed by all topologizing systems which do not contain $\mathcal{S}$, and call it *the support of $\mathcal{S}$ in topologizing systems.*

We denote by $\widehat{\mathcal{S}}$ the union of all systems of $Supp_\mathfrak{T}(\mathcal{S})$. It follows that the inclusion $\mathcal{S}_1 \subseteq \mathcal{S}_2$ implies that $\widehat{\mathcal{S}_1} \subseteq \widehat{\mathcal{S}}_2$. If $\mathcal{S}_2$ is topologizing, then the inverse implication holds: if $\widehat{\mathcal{S}_1} \subseteq \widehat{\mathcal{S}}_2$, then $\mathcal{S}_1 \subseteq \mathcal{S}_2$ (because, if $\mathcal{S}_1 \not\subseteq \mathcal{S}_2$, then $\mathcal{S}_2 \subseteq \widehat{\mathcal{S}}_1$, but, $\mathcal{S}_2 \not\subseteq \widehat{\mathcal{S}}_2$). Let $[\mathcal{S}]$ denote the smallest topologizing system containing $\mathcal{S}$. It is clear that $\widehat{\mathcal{S}} = \widehat{[\mathcal{S}]}$.

Finally, notice that if $\mathcal{S} \supseteq \mathcal{S}_1 \not\subseteq \widehat{\mathcal{S}}$, then $\widehat{\mathcal{S}} \subseteq \widehat{\mathcal{S}}_1 \subseteq \widehat{\mathcal{S}}$, that is $\widehat{\mathcal{S}}_1 = \widehat{\mathcal{S}}$.

The system $\widehat{\mathcal{S}}$ is the largest element of $Supp_\mathfrak{T}(\mathcal{S})$ whenever $\widehat{\mathcal{S}}$ is topologizing. The following assertion provides sufficient conditions for this occurrence.

3.1.1. Lemma. *Let $(C_X, \bar{\mathfrak{E}}_X) = (C_X, \mathfrak{E}_X, \mathcal{W}_X)$ be a svelte right exact category with a stable class of weak equivalences (condition 2.0(b)) and with a left divisible a weakly right divisible class $\mathfrak{E}_X$ of deflations (conditions 2.0(c) and 2.0(d)). Suppose also that the condition 2.0(e') holds. If $\mathcal{S}$ is a class of deflations such that the system $\widehat{\mathcal{S}}$ is multiplicative, then $\widehat{\mathcal{S}}$ is topologizing.*

Proof. Let $\mathcal{T}_1$, $\mathcal{T}_2$ be topologizing systems from the support of $\mathcal{S}$. If $\widehat{\mathcal{S}}$ is multiplicative, then $\mathcal{T}_1 \circ \mathcal{T}_2 \subseteq \widehat{\mathcal{S}}$. By 2.4.4, the system $\mathcal{T}_1 \circ \mathcal{T}_2$ is topologizing and it contains $\mathcal{T}_1$ and $\mathcal{T}_2$. This shows that the support of $\mathcal{S}$ is filtered, hence $\widehat{\mathcal{S}}$ is a topologizing system. ∎

3.2. The spectrum $\mathbf{Spec}_\mathfrak{t}(X, \bar{\mathfrak{E}}_X)$. The elements of the spectrum $\mathbf{Spec}_\mathfrak{t}(X, \bar{\mathfrak{E}}_X)$ are all topologizing systems $\mathcal{S}$ such that $\widehat{\mathcal{S}}$ is a Serre system, i.e. $\widehat{\mathcal{S}} = \widehat{\mathcal{S}}^-$. We endow $\mathbf{Spec}_\mathfrak{t}(X, \bar{\mathfrak{E}}_X)$ with the preorder $\supseteq$ called (with a good reason) the *specialization* preorder.

3.3. The spectra $\mathbf{Spec}^1_{\mathfrak{t}}(X,\bar{\mathfrak{E}}_X)$ and $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X,\bar{\mathfrak{E}}_X)$. For any system $\mathcal{S}$ of deflations of $(C_X,\bar{\mathfrak{E}}_X)$, let $\mathcal{S}^*$ denote the intersection of all topologizing systems properly containing $\mathcal{S}$. We denote by $\mathbf{Spec}^1_{\mathfrak{t}}(X,\bar{\mathfrak{E}}_X)$ the preorder (with respect to $\supseteq$) of all thick topologizing systems of deflations Σ such that $\Sigma^* \neq \Sigma$ and set

$$\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X,\bar{\mathfrak{E}}_X) = \mathbf{Spec}^1_{\mathfrak{t}}(X,\bar{\mathfrak{E}}_X)\bigcap \mathfrak{Se}(X,\bar{\mathfrak{E}}_X).$$

Thus, the spectrum $\mathbf{Spec}^1_{\mathfrak{t}}(X,\bar{\mathfrak{E}}_X)$ is the disjoint union of

$$\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X,\bar{\mathfrak{E}}_X) = \{\Sigma\in\mathfrak{T}(X,\bar{\mathfrak{E}}_X)\mid \Sigma=\Sigma^-\subsetneqq\Sigma^*\}\quad\text{and}$$
$$\mathbf{Spec}^{1,0}_{\mathfrak{t}}(X,\bar{\mathfrak{E}}_X) = \{\Sigma\in\mathfrak{M}_{\mathfrak{T}}(X,\bar{\mathfrak{E}}_X)\mid \Sigma\neq\Sigma^*\subseteq\Sigma^-\}.$$

3.4. Proposition. *Suppose that $(C_X,\bar{\mathfrak{E}}_X)=(C_X,\mathfrak{E}_X,\mathcal{W}_X)$ is a svelte right exact category with a stable class of weak equivalences (condition 2.0(b)) and with a left divisible a weakly right divisible class $\mathfrak{E}_X$ of deflations (conditions 2.0(c) and 2.0(d)). Then there is a natural isomorphism*

$$\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X,\bar{\mathfrak{E}}_X)\xrightarrow{\sim}\mathbf{Spec}_{\mathfrak{t}}(X,\bar{\mathfrak{E}}_X).$$

Proof. Consider the map which assigns to each $\Sigma\in\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X,\bar{\mathfrak{E}}_X)$ the union Σ_* of all right divisible in $\mathfrak{E}_X$ subsystems of Σ^* which have trivial intersection with Σ. Notice that, since Σ is a Serre system, the right divisible system Σ_* is a non-trivial. The claim is that the topologizing system $[\Sigma_*]$ spanned by Σ_* (which is a topologizing subsystem of the topologizing system Σ^*) is an element of the spectrum $\mathbf{Spec}_{\mathfrak{t}}(X,\bar{\mathfrak{E}}_X)$.

(i) Observe that $\Sigma\subseteq\widehat{\Sigma_*}$, because the system Σ is topologizing and the equality $\Sigma_*\cap\Sigma=\mathcal{W}_X$ combined with the non-triviality of the system Σ_* implies that $\Sigma_*\nsubseteq\Sigma$.

(ii) On the other hand, if $\mathcal{S}$ is a topologizing system of deflations which is not contained in the system Σ, then $\Sigma_*\subseteq\mathcal{S}$.

In fact, suppose that $\mathcal{S}\nsubseteq\Sigma$. Then, by 2.4.4(b), the composition $\mathcal{S}\circ\Sigma$ is a topologizing system properly containing Σ. Therefore,

$$\Sigma_*\subseteq(\mathcal{S}\circ\Sigma)\cap\Sigma^{\perp}\subseteq\mathcal{S}\circ(\Sigma\cap\Sigma^{\perp})=\mathcal{S}\circ\mathcal{W}_X=\mathcal{S}.$$

In other words, if $\mathcal{S}$ is a topologizing system which does not contain Σ_*, then $\mathcal{S}\subseteq\Sigma$. This proves the inverse inclusion, $\widehat{\Sigma_*}\subseteq\Sigma$., hence the equality $\Sigma^-=\Sigma=\widehat{\Sigma_*}$. As it is observed in 3.2, $\widehat{\Sigma_*}=\widehat{[\Sigma_*]}$; so that $[\Sigma_*]$ is an element of the spectrum $\mathbf{Spec}_{\mathfrak{t}}(X,\bar{\mathfrak{E}}_X)$. Thus, we obtained a map

$$\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X,\bar{\mathfrak{E}}_X)\longrightarrow\mathbf{Spec}_{\mathfrak{t}}(X,\bar{\mathfrak{E}}_X),\quad\Sigma\longmapsto[\Sigma_*].\tag{1}$$

Let now $\mathcal{S} \in \mathbf{Spec}_{\mathfrak{t}}(X, \bar{\mathfrak{E}}_X)$, that is $\mathcal{S}$ is a topologizing system such that $\widehat{\mathcal{S}}$ is a Serre system. Since, by 2.6.2(c), any Serre system is multiplicative, it follows from 3.1.1 that the system $\widehat{\mathcal{S}}$ is topologizing. If Σ is a topologizing system properly containing $\widehat{\mathcal{S}}$, then Σ contains $\mathcal{S}$. This shows that $\widehat{\mathcal{S}}^*$ coincides with the smallest topologizing system containing $\widehat{\mathcal{S}} \cup \mathcal{S}$; in particular, $\widehat{\mathcal{S}}^- = \widehat{\mathcal{S}} \neq \widehat{\mathcal{S}}^*$, i.e. $\widehat{\mathcal{S}}$ is an element of the spectrum $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X, \bar{\mathfrak{E}}_X)$. One can see that the map

$$\mathbf{Spec}_{\mathfrak{t}}(X, \bar{\mathfrak{E}}_X) \longrightarrow \mathbf{Spec}_{\mathfrak{t}}^{1,1}(X, \bar{\mathfrak{E}}_X), \quad \mathcal{S} \longmapsto \widehat{\mathcal{S}},$$

is inverse to the map (1). ■

3.5. Remark. It follows from the argument of 3.4 that the map

$$\Sigma \longmapsto \Sigma_*, \quad \Sigma \in \mathbf{Spec}_{\mathfrak{t}}^{1,1}(X, \bar{\mathfrak{E}}_X),$$

gives a canonical realization of $\mathbf{Spec}_{\mathfrak{t}}(X, \bar{\mathfrak{E}}_X)$ as the preorder of systems of deflations $\mathcal{S}$ which are characterized by the following properties:

(a) $\widehat{\mathcal{S}}$ is a Serre system and $\mathcal{S} \cap \widehat{\mathcal{S}} = \mathcal{W}_X$;

(b) if a system $\mathcal{T}$ of deflations is such that $\widehat{\mathcal{T}} = \widehat{\mathcal{S}}$ and $\mathcal{T} \cap \widehat{\mathcal{S}} = \mathcal{W}_X$, then $\mathcal{T} \subseteq \mathcal{S}$.

Notice that that for every such system $\mathcal{S}$, the corresponding Serre system $\widehat{\mathcal{S}}$ coincides with the union $\check{\mathcal{S}}$ of all topologizing systems of deflations Σ such that $\mathcal{S} \cap \Sigma = \mathcal{W}_X$.

3.6. Local right exact 'spaces' and categories with weak equivalences. Let $(C_X, \bar{\mathfrak{E}}_X) = (C_X, \mathfrak{E}_X, \mathcal{W}_X)$ be a svelte right exact category with weak equivalences. We call $(C_X, \bar{\mathfrak{E}}_X)$ (and the *right exact* 'space' $(X, \bar{\mathfrak{E}}_X)$ it represents) *local* if there is the smallest non-trivial topologizing system, or, equivalently, the intersection $\mathcal{W}_X^*$ of all non-trivial topologizing systems of $(C_X, \mathfrak{E}_X)$ is non-trivial.

It follows that a right exact 'space' $(X, \bar{\mathfrak{E}}_X)$ is *local* iff the spectrum $\mathbf{Spec}_{\mathfrak{t}}(X, \bar{\mathfrak{E}}_X)$ has a unique closed point, and this closed point belongs to the support of any non-trivial divisible system of $(C_X, \bar{\mathfrak{E}}_X)$.

3.7. The spectrum $\mathbf{Spec}_{\mathfrak{M}}^{1,1}(X, \bar{\mathfrak{E}}_X)$. The elements of this spectrum are all Serre systems Σ such that the intersection $\Sigma^\star$ of all thick systems of deflations of $(C_X, \bar{\mathfrak{E}}_X)$ properly containing Σ is not equal to Σ. Equivalently, $\mathbf{Spec}_{\mathfrak{M}}^{1,1}(X, \bar{\mathfrak{E}}_X)$ consists of all Serre systems Σ such that $\Sigma_\star \overset{\mathrm{def}}{=} \Sigma^\star \cap \Sigma^\perp$ is non-trivial. As all other spectra, the spectrum $\mathbf{Spec}_{\mathfrak{M}}^{1,1}(X, \bar{\mathfrak{E}}_X)$ is endowed with the *specialization* preorder $\supseteq$.

One of the most essential properties of the spectrum $\mathbf{Spec}_{\mathfrak{M}}^{1,1}(X, \bar{\mathfrak{E}}_X)$ is the following.

3.7.1. Proposition. *Let $(X, \bar{\mathfrak{E}}_X)$ be a right exact 'space' and $\Sigma \in \mathbf{Spec}_{\mathfrak{M}}^{1,1}(X, \bar{\mathfrak{E}}_X)$. For any finite family $\{\mathcal{S}_i \mid i \in \mathfrak{J}\}$ of right divisible in $\mathfrak{E}_X$ systems of deflations, $\mathcal{S}_i \nsubseteq \Sigma$ for all $i \in \mathfrak{J}$ iff $\bigcap_{i \in \mathfrak{J}} \mathcal{S}_i \nsubseteq \Sigma$.*

Proof. By 2.6.4, $\bigcap_{i\in J} \mathcal{S}_i^- = \big(\bigcap_{i\in J} \mathcal{S}_i\big)^-$, and by 2.6.5.1, $\Sigma \vee \big(\bigcap_{i\in J} \mathcal{S}_i^-\big) = \bigcap_{i\in J}(\Sigma \vee \mathcal{S}_i^-)$. Therefore,

$$\Sigma \vee \big(\bigcap_{i\in J} \mathcal{S}_i\big)^- = \bigcap_{i\in J}(\Sigma \vee \mathcal{S}_i^-). \tag{2}$$

If $\mathcal{S}_i \nsubseteq \Sigma$ for all $i \in J$, then each of the strongly closed systems $\mathcal{S}_i^- \vee \Sigma$ contains Σ properly. Since Σ is an element of the spectrum $\mathbf{Spec}^{1,1}_{\mathfrak{sc}}(X, \bar{\mathfrak{E}}_X)$, the intersection of $\mathcal{S}_i^- \vee \Sigma$, $i \in J$, contains in Σ properly. Then it follows from the equality (5) that the intersection $\bigcap_{i\in J} \mathcal{S}_i$ is not contained in Σ. ■

Since the spectrum $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X, \bar{\mathfrak{E}}_X)$ is contained in $\mathbf{Spec}^{1,1}_{\mathfrak{M}}(X, \bar{\mathfrak{E}}_X)$, the elements of $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X, \bar{\mathfrak{E}}_X)$ have the property described in 3.7.1.

4. Semitopologizing systems and the related spectral theory.

The topological systems defined in 2.4 might be inconvenient in some situations, because they require invariance of deflations under push-forwards, which is not necessarily available in right exact, or even exact categories. There is a different setting based on the notion of a *semitopological system*,which does not require push-forwards and still recovers the abelian theory. It is sketched below.

4.0. Conventions. We fix a svelte right exact category with a stable class of weak equivalences $(C_X, \bar{\mathfrak{E}}_X) = (C_X, \mathfrak{E}_X, \mathcal{W}_X)$ such that $\mathcal{W}_X \circ \mathcal{W}_X^{\bar{\wedge}} = \mathcal{W}_X^{\bar{\wedge}}$. We assume that the category C_X has fiber products.

4.1. Strongly stable, cartesian complete and semitopologizing systems.

(i) A class $\mathcal{S}$ of deflations of $(C_X, \bar{\mathfrak{E}}_X)$ will be called *strongly stable* if it is invariant under pull-backs, stable (that is $\mathcal{S} = \mathfrak{E}_X \cap \mathcal{S}^{\bar{\wedge}}$) and, in addition,

$$\mathcal{S} = \mathfrak{E}_X \cap (\mathcal{S} \circ \mathcal{W}_X^{\bar{\wedge}}). \tag{1}$$

(ii) We call a class of deflations $\mathcal{S}$ *cartesian complete* if, for any cartesian square with arrows in $\mathcal{S}$, the composition of two consecutive arrows belongs to $\mathcal{S}$.

(iii) We call a system of deflations *right semitopologizing* (resp. *left semitopologizing*) if it is cartesian complete, strongly stable, and right (resp. left) divisible in $\mathfrak{E}_X$.

We say that a system *semitopologizing* if it is both left and right semitopologizing.

4.1.1. Topologizing and semitopologizing systems. Suppose that the class $\mathfrak{E}_X$ of deflations is *left divisible* in the following sense: if $\mathfrak{t}\circ\mathfrak{s} \in \mathfrak{E}_X \ni \mathfrak{s}$, then $\mathfrak{t} \in \mathfrak{E}_X$. Then every left (resp. right) topologizing system of deflations is left (resp. right) semitopologizing.

In fact, any left (resp. right) topologizing system is, by definition, cartesian complete and, by 2.4.3 (or 1.10.2), stable. If $\mathcal{T}$ is a left (or/and right) topologizing system, then $\mathcal{T} \circ \mathcal{W}_X^{\bar{\wedge}} \subseteq \mathcal{W}_X^{\bar{\wedge}} \circ \mathcal{T}$ (see 2.4.2(c)), so that

$$\mathcal{T} \subseteq \mathfrak{E}_X \cap (\mathcal{T} \circ \mathcal{W}_X^{\bar{\wedge}}) \subseteq \mathfrak{E}_X \cap (\mathcal{W}_X^{\bar{\wedge}} \circ \mathcal{T}) = (\mathfrak{E}_X \cap \mathcal{W}_X^{\bar{\wedge}}) \circ \mathcal{T} = \mathcal{W}_X \circ \mathcal{T} = \mathcal{T},$$

whence $\mathcal{T} = \mathfrak{E}_X \cap (\mathcal{T} \circ \mathcal{W}_X^{\bar{\wedge}})$. Here the first equality is due to the left divisibility of $\mathfrak{E}_X$ and the second one to the stability of $\mathcal{W}_X$.

4.1.2. About cartesian completeness. Let C_X have an initial object, $\mathfrak{x}$, and let $\mathbb{T}$ be a full subcategory of the category $C_X/\mathfrak{x}$. Consider the system $\mathcal{S}_{\mathbb{T}}$ of all deflations $\mathfrak{s}$ of $(C_X, \bar{\mathfrak{E}}_X)$ such that $(Ker(\mathfrak{s}), Ker(\mathfrak{s}) \longrightarrow \mathfrak{x})$ is an object of $\mathbb{T}$. The class of arrows $\mathcal{S}_{\mathbb{T}}$ is cartesian complete iff $\mathbb{T}$ is a category with finite products.

This follows from the observation that to every cartesian square

$$\begin{array}{ccc} \mathfrak{M} & \xrightarrow{\ \mathfrak{s}'\ } & \mathcal{M} \\ {\scriptstyle \widetilde{\mathfrak{t}}}\big\downarrow & cart & \big\downarrow{\scriptstyle \mathfrak{t}} \\ \mathcal{L} & \xrightarrow{\ \mathfrak{s}\ } & \mathcal{N} \end{array} \qquad (2)$$

there corresponds a cartesian square

$$\begin{array}{ccc} Ker(\mathfrak{s} \circ \widetilde{\mathfrak{t}}) & \longrightarrow & Ker(\mathfrak{t}) \\ \big\downarrow & cart & \big\downarrow \\ Ker(\mathfrak{s}) & \longrightarrow & \mathfrak{x} \end{array}$$

obtained via pulling back the square (2) along the unique arrow $\mathfrak{x} \longrightarrow \mathcal{N}$.

4.1.3. Proposition. *(a) Let $\mathcal{S}$ be a system of deflations satisfying the equality $\mathcal{S} = \mathfrak{E}_X \cap (\mathcal{S} \circ \mathcal{W}_X^{\bar{\wedge}})$. Then the stable envelope $\mathfrak{E}_X \cap \mathcal{S}^{\bar{\wedge}}$ of the system $\mathcal{S}$ has this property; that is the class $\mathfrak{E}_X \cap \mathcal{S}^{\bar{\wedge}}$ is strongly stable.*

(b) The family of strongly stable classes of deflations is closed under arbitrary intersections and unions.

(c) The family of cartesian complete classes of deflations is closed under arbitrary intersections and filtered (with respect to the inclusion) unions. Similarly for left or/and right semitopologizing systems.

Proof. (a) Since, by hypothesis, the category C_X has fiber products, for any pair of classes of arrows $\mathcal{S}$, $\mathcal{T}$, there is an obvious inclusion $\mathcal{S}^{\bar{\wedge}} \circ \mathcal{T}^{\bar{\wedge}} \subseteq (\mathcal{S} \circ \mathcal{T}^{\bar{\wedge}})^{\bar{\wedge}}$. In particular,

$\mathcal{S}^{\bar\wedge} \circ \mathcal{W}_X^{\bar\wedge} \subseteq (\mathcal{S} \circ \mathcal{W}_X^{\bar\wedge})^{\bar\wedge}$. Therefore, for any system $\mathcal{S}$ such that $\mathcal{S} = \mathfrak{E}_X \cap (\mathcal{S} \circ \mathcal{W}_X^{\bar\wedge})$, we obtain the following:

$$\mathfrak{E}_X \cap \mathcal{S}^{\bar\wedge} \subseteq \mathfrak{E}_X \cap ((\mathfrak{E}_X \cap \mathcal{S}^{\bar\wedge}) \circ \mathcal{W}_X^{\bar\wedge}) \subseteq \mathfrak{E}_X \cap (\mathcal{S}^{\bar\wedge} \circ \mathcal{W}_X^{\bar\wedge}) \subseteq \mathfrak{E}_X \cap (\mathcal{S} \circ \mathcal{W}_X^{\bar\wedge})^{\bar\wedge} =$$
$$\mathfrak{E}_X \cap (\mathfrak{E}_X^{\bar\wedge} \cap (\mathcal{S} \circ \mathcal{W}_X^{\bar\wedge})^{\bar\wedge}) = \mathfrak{E}_X \cap (\mathfrak{E}_X \cap (\mathcal{S} \circ \mathcal{W}_X^{\bar\wedge}))^{\bar\wedge} = \mathfrak{E}_X \cap \mathcal{S}^{\bar\wedge}.$$

(b) Let $\{\mathcal{T}_i \mid i \in J\}$ be a set of classes of arrows such that $\mathcal{T}_i = \mathfrak{E}_X \cap (\mathcal{T}_i \circ \mathcal{W}_X^{\bar\wedge})$ for all $i \in J$. Then

$$\bigcap_{i \in J} \mathcal{T}_i \subseteq \mathfrak{E}_X \cap \big(\big(\bigcap_{i \in J} \mathcal{T}_i\big) \circ \mathcal{W}_X^{\bar\wedge}\big) \subseteq \mathfrak{E}_X \cap \big(\bigcap_{i \in J} \mathcal{T}_i \circ \mathcal{W}_X^{\bar\wedge}\big) = \bigcap_{i \in J} (\mathfrak{E}_X \cap (\mathcal{T}_i \circ \mathcal{W}_X^{\bar\wedge})) = \bigcap_{i \in J} \mathcal{T}_i.$$

Similarly,

$$\bigcup_{i \in J} \mathcal{T}_i \subseteq \mathfrak{E}_X \cap \big(\big(\bigcup_{i \in J} \mathcal{T}_i\big) \circ \mathcal{W}_X^{\bar\wedge}\big) = \mathfrak{E}_X \cap \big(\bigcup_{i \in J} \mathcal{T}_i \circ \mathcal{W}_X^{\bar\wedge}\big) = \bigcup_{i \in J} (\mathfrak{E}_X \cap (\mathcal{T}_i \circ \mathcal{W}_X^{\bar\wedge})) = \bigcup_{i \in J} \mathcal{T}_i.$$

By 2.1, the class of (right or/and left) stable systems is closed under arbitrary intersections and unions.

(c) The assertion follows from (b). ∎

4.1.4. Proposition. *(a) For any system of deflations $\mathcal{T}$, the intersection*

$$\mathfrak{E}_X \cap (\mathcal{T} \circ \mathcal{W}_X^{\bar\wedge})^{\bar\wedge}$$

is the smallest strongly stable system containing $\mathcal{T}$.

(b) If the system $\mathcal{T}$ is cartesian complete, then the smallest strongly stable system containing $\mathcal{T}$ is cartesian complete.

(c) Suppose that the condition 2.0(d) holds. Then, for any right divisible system of deflations $\mathcal{T}$, the smallest strongly stable system containing $\mathcal{T}$ is right divisible.

In particular, if $\mathcal{T}$ is a right divisible and cartesian complete system, then its strongly stable envelope $\mathfrak{E}_X \cap (\mathcal{T} \circ \mathcal{W}_X^{\bar\wedge})^{\bar\wedge}$ is a right semitopological system.

Proof. (a) By 1.6.1(ii), if a system $\mathcal{S}$ satisfies the equality $\mathcal{S} = \mathfrak{E}_X \cap (\mathcal{S} \circ \mathcal{S}^{\bar\wedge})$, then the class of arrows $\mathcal{S}^{\bar\wedge}$ is multiplicative. In particular, the class of arrows $\mathcal{W}_X^{\bar\wedge}$ is closed under composition. So that, for any system $\mathcal{S}$, we have

$$\mathfrak{E}_X \cap (\mathcal{S} \circ \mathcal{W}_X^{\bar\wedge}) \subseteq \mathfrak{E}_X \cap ((\mathfrak{E}_X \cap (\mathcal{S} \circ \mathcal{W}_X^{\bar\wedge})) \circ \mathcal{W}_X^{\bar\wedge}) \subseteq$$
$$\mathfrak{E}_X \cap ((\mathcal{S} \circ \mathcal{W}_X^{\bar\wedge}) \circ \mathcal{W}_X^{\bar\wedge}) = \mathfrak{E}_X \cap (\mathcal{S} \circ \mathcal{W}_X^{\bar\wedge}),$$

which shows that the system $\mathcal{T} = \mathfrak{E}_X \cap (\mathcal{S} \circ \mathcal{W}_X^{\bar{\wedge}})$ satisfies the equality $\mathcal{T} = \mathfrak{E}_X \cap (\mathcal{T} \circ \mathcal{W}_X^{\bar{\wedge}})$. Evidently, $\mathcal{T}$ is the smallest system containing $\mathcal{S}$ and satisfying this equality.

By 4.1.2(a), the system $\mathfrak{E}_X \cap \mathcal{T}^{\bar{\wedge}}$ is the smallest strongly stable system containing $\mathcal{T}$. Notice that

$$\mathfrak{E}_X \cap \mathcal{T}^{\bar{\wedge}} = \mathfrak{E}_X \cap (\mathfrak{E}_X \cap (\mathcal{S} \circ \mathcal{W}_X^{\bar{\wedge}})^{\bar{\wedge}} = \mathfrak{E}_X \cap (\mathfrak{E}_X^{\bar{\wedge}} \cap (\mathcal{S} \circ \mathcal{W}_X^{\bar{\wedge}})^{\bar{\wedge}}) = \mathfrak{E}_X \cap (\mathcal{S} \circ \mathcal{W}_X^{\bar{\wedge}})^{\bar{\wedge}}$$

hence the assertion.

(b1) If a class of deflations $\mathcal{T}$ is cartesian complete, then $\mathfrak{E}_X \cap (\mathcal{T} \circ \mathcal{W}_X^{\bar{\wedge}})$ has this property.

In fact, for any pair of arrows $\mathfrak{t}_i \circ \mathfrak{w}_i \in \mathfrak{E}_X$, $i = 1, 2$, such that $\mathfrak{t}_i \in \mathcal{T}$ and $\mathfrak{w}_i \in \mathcal{W}_X^{\bar{\wedge}}$, $i = 1, 2$, we have diagram

$$\begin{tikzcd}
\mathfrak{L} \arrow[r, "\mathfrak{w}_2''"] \arrow[d, "\widetilde{\mathfrak{w}}_1"'] \arrow[dr, phantom, "cart"] & \mathfrak{M} \arrow[r, "\mathfrak{t}_2''"] \arrow[d, "\mathfrak{w}_1'"] \arrow[dr, phantom, "cart"] & \mathcal{L}_1 \arrow[d, "\mathfrak{w}_1"] \\
\mathfrak{L}' \arrow[r, "\mathfrak{w}_2'"] \arrow[d, "\widetilde{\mathfrak{t}}_1"'] \arrow[dr, phantom, "cart"] & \mathfrak{M}' \arrow[r, "\mathfrak{t}_2'"] \arrow[d, "\mathfrak{t}_1'"] \arrow[dr, phantom, "cart"] & \mathcal{M}_1 \arrow[d, "\mathfrak{t}_1"] \\
\mathcal{L}_2 \arrow[r, "\mathfrak{w}_2"] & \mathcal{M}_2 \arrow[r, "\mathfrak{t}_2"] & \mathcal{N}
\end{tikzcd}$$

built out of cartesian squares. So that

$$(\mathfrak{t}_1 \circ \mathfrak{w}_1) \circ (\mathfrak{t}_2'' \circ \mathfrak{w}_2'') = (\mathfrak{t}_1 \circ \mathfrak{t}_2') \circ (\mathfrak{w}_1' \circ \mathfrak{w}_2'') \in \mathfrak{E}_X \cap (\mathcal{T} \circ \mathcal{W}_X^{\bar{\wedge}}),$$

because, by hypothesis, $\mathcal{T}$ is cartesian complete and $\mathcal{W}_X^{\bar{\wedge}} \circ \mathcal{W}_X^{\bar{\wedge}} = \mathcal{W}_X^{\bar{\wedge}}$.

(c) Consider a commutative square

$$\begin{tikzcd}
\mathcal{L} \arrow[r, "\mathfrak{w}"] \arrow[d, "\mathfrak{s}_1"'] & \mathcal{M} \arrow[d, "\mathfrak{t}"] \\
\mathfrak{L} \arrow[r, "\mathfrak{s}_2"] & \mathfrak{M}
\end{tikzcd}$$

where $\mathfrak{t} \in \mathcal{T}$, $\mathfrak{w} \in \mathcal{W}_X$, and $\mathfrak{t} \circ \mathfrak{w}$, $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are deflations. The square is decomposed into the diagram

$$\begin{tikzcd}
\mathcal{L} \arrow[r, "\mathfrak{f}"] & \widetilde{\mathfrak{L}} \arrow[r, "\mathfrak{w}_2"] \arrow[d, "\mathfrak{t}_1"'] \arrow[dr, phantom, "cart"] & \mathcal{M} \arrow[d, "\mathfrak{t}"] \\
& \mathfrak{L} \arrow[r, "\mathfrak{s}_2"] & \mathfrak{M}
\end{tikzcd}$$

with cartesian square such that $\widetilde{\mathfrak{s}}_2 \circ \mathfrak{f} = \mathfrak{w}$ and $\mathfrak{t}_1 \circ \mathfrak{f} = \mathfrak{s}_1$. By the condition 2.0(d), the latter equality implies that $\mathfrak{f} = \mathfrak{w}_1 \circ \mathfrak{e}$, where $\mathfrak{e}$ is a deflation and $\mathfrak{w}_1 \in \mathcal{W}_X^{\bar{\wedge}}$. It follows from the fact that $\mathfrak{w} = (\mathfrak{w}_2 \circ \mathfrak{w}_1) \circ \mathfrak{e} \in \mathcal{W}_X$ and $\mathfrak{e}$ is a deflation that $\mathfrak{w}_2 \circ \mathfrak{w}_1 \in \mathcal{W}_X$ and $\mathfrak{e} \in \mathcal{W}_X$. Therefore, $\mathfrak{s}_1 = \mathfrak{t}_1 \circ (\mathfrak{w}_1 \circ \mathfrak{e})$, where $\mathfrak{t}_1 \in \mathcal{T}$ and $\mathfrak{w}_1 \circ \mathfrak{e} \in \mathcal{W}_X^{\bar{\wedge}}$. ■

4.2. Strongly thick systems. We call a system of deflations $\mathcal{S}$ *strongly thick* if it is divisible in $\mathfrak{E}_X$, stable, and $\mathcal{S} \circ \mathcal{S}^{\bar{\wedge}} = \mathcal{S}^{\bar{\wedge}}$.

We denote by $\mathfrak{M}_{\mathfrak{s}}(X, \bar{\mathfrak{E}}_X)$ the preorder (with respect to the inclusion) of all strongly thick systems of $(C_X, \bar{\mathfrak{E}}_X)$. It follows from our assumptions (see 4.0) that the class $\mathcal{W}_X$ of weak equivalences is the smallest element of $\mathfrak{M}_{\mathfrak{s}}(X, \bar{\mathfrak{E}}_X)$.

4.2.1. Observations. (a) Thanks to the existence of fiber products in C_X, for any class of arrows $\mathcal{S}$ which is invariant under pull-backs, the inclusion $\mathcal{S} \circ \mathcal{S}^{\bar{\wedge}} \subseteq \mathcal{S}^{\bar{\wedge}}$ is equivalent to the multiplicativity of $\mathcal{S}^{\bar{\wedge}}$, that is the inclusion $\mathcal{S}^{\bar{\wedge}} \circ \mathcal{S}^{\bar{\wedge}} \subseteq \mathcal{S}^{\bar{\wedge}}$.

In particular, a system $\mathcal{S}$ is strongly thick iff it is stable and $\mathcal{S}^{\bar{\wedge}} \circ \mathcal{S}^{\bar{\wedge}} = \mathcal{S}^{\bar{\wedge}}$.

(b) Every strongly thick system of deflation $\mathcal{S}$ is multiplicative, because

$$\mathcal{S} \circ \mathcal{S} \subseteq \mathfrak{E}_X \cap (\mathcal{S} \circ \mathcal{S}^{\bar{\wedge}}) = \mathfrak{E}_X \cap \mathcal{S}^{\bar{\wedge}} = \mathcal{S}.$$

(c) Every strongly thick system $\mathcal{S}$ is semitopologizing, because it is multiplicative and

$$\mathfrak{E}_X \cap (\mathcal{S} \circ \mathcal{W}_X^{\bar{\wedge}}) \subseteq \mathfrak{E}_X \cap (\mathcal{S} \circ \mathcal{S}^{\bar{\wedge}}) = \mathfrak{E}_X \cap \mathcal{S}^{\bar{\wedge}} = \mathcal{S}.$$

(d) A stable, divisible in $\mathfrak{E}_X$ system of deflations $\mathcal{S}$ is strongly thick, when it satisfies the condition

(#) If in the commutative diagram

$$\begin{array}{ccccc}
\widetilde{\mathcal{L}} & \xrightarrow{\widetilde{\mathfrak{s}}} & \mathcal{K} & & \\
{\scriptstyle \mathfrak{j}'}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{j}} & & \\
\mathcal{L} & \xrightarrow{\mathfrak{s}} & \mathcal{M} & \xrightarrow{\mathfrak{t}} & \mathcal{K}
\end{array} \qquad (3)$$

with cartesian square $\mathfrak{t} \circ \mathfrak{j} = id_{\mathcal{N}}$ and morphisms $\mathfrak{t}$ and $\widetilde{\mathfrak{s}}$ belong to $\mathcal{S}$, then $\mathfrak{t} \circ \mathfrak{s} \in \mathcal{S}$.

Indeed, if the condition (#) holds, then, by 1.6.1(ii), the class $\mathcal{S}^{\bar{\wedge}}$ is multiplicative.

(e) Evidently, the class of deflations $\mathfrak{E}_X$ is strongly thick iff $\mathfrak{E}_X \circ \mathfrak{E}_X^{\bar{\wedge}} = \mathfrak{E}_X^{\bar{\wedge}}$.

(f) It follows from 4.1.2 that the preorder $\mathfrak{M}_{\mathfrak{s}}(X, \bar{\mathfrak{E}}_X)$ of strongly thick systems is closed under arbitrary intersections and filtered unions.

4.2.2. Note. One can see that the condition (#) from 4.2.1(d) holds for $\mathfrak{E}_X$, if the class $\mathfrak{E}_X$ is left divisible, because $(\mathfrak{t} \circ \mathfrak{s}) \circ \mathfrak{j}' = \widetilde{\mathfrak{s}} \in \mathfrak{E}_X$ (see the diagram (3) above).

In general, the condition (#) provides an effective tool for finding if a system is stable or not. The following assertion shows that in most of cases of interest the condition (#) is a criterium.

4.2.3. Proposition. *Suppose that the class $\mathfrak{E}_X$ of deflations satisfies the condition (#) (say, it is left divisible). Let $\mathcal{S}$ be a stable class of deflations invariant under pull-backs. Then the following conditions are equivalent:*
(a) $\mathcal{S}^{\overline{\wedge}} \circ \mathcal{S}^{\overline{\wedge}} \subseteq \mathcal{S}^{\overline{\wedge}}$,
(b) $\mathcal{S}$ satisfies the condition (#).

Proof. $(a) \Rightarrow (b)$. The morphism $\mathfrak{t} \circ \mathfrak{s}$ in the condition (#) belongs to the intersection of $\mathcal{S} \circ \mathcal{S}^{\overline{\wedge}}$ and $\mathfrak{E}_X$ due to the fact that the arrows $\widetilde{\mathfrak{s}}$ and $\mathfrak{t}$ in the condition (#) are deflations and $\mathfrak{E}_X$ satisfies (#). Therefore, if the condition (b) holds and $\mathcal{S}$ is stable, $\mathfrak{t} \circ \mathfrak{s}$ belongs to $\mathcal{S}^{\overline{\wedge}} \cap \mathfrak{E}_X = \mathcal{S}$.
The implication $(b) \Rightarrow (a)$ follows from 1.6.1(ii). ■

4.2.4. Proposition. *Suppose that $\mathfrak{E}_X^{\overline{\wedge}} \subseteq \mathcal{W}_X^{\overline{\wedge}} \circ \mathfrak{E}_X$. Then a strongly stable divisible in $\mathfrak{E}_X$ system of deflations $\mathcal{S}$ is strongly thick iff it is multiplicative.*
In other words, a thick system $\mathcal{S}$ is strongly thick iff $\mathcal{S} = \mathfrak{E}_X \cap (\mathcal{S} \circ \mathcal{W}_X^{\overline{\wedge}})$.

Proof. By 4.2.1(b), any strongly thick system is multiplicative.
The claim is that any multiplicative strongly stable right divisible in $\mathfrak{E}_X$ class of deflations satisfies the condition (#).
In fact, the inclusion $\mathfrak{E}_X^{\overline{\wedge}} \subseteq \mathcal{W}_X^{\overline{\wedge}} \circ \mathfrak{E}_X$ allows to replace the diagram in (#) by the diagram

$$\begin{tikzcd}
\widetilde{\mathcal{L}} \arrow[r, "\widetilde{\mathfrak{s}}_1"] \arrow[d, "\mathsf{j}'"'] \arrow[dr, phantom, "cart"] & \mathcal{K}_1 \arrow[r, "\widetilde{\mathfrak{w}}"] \arrow[d, "\mathsf{j}_1"] \arrow[dr, phantom, "cart"] & \mathcal{K} \arrow[d, "\mathsf{j}"] & \\
\mathcal{L} \arrow[r, "\mathfrak{s}_1"] & \mathcal{M}_1 \arrow[r, "\mathfrak{w}"] & \mathcal{M} \arrow[r, "\mathfrak{t}"] & \mathcal{K}
\end{tikzcd} \qquad (4)$$

with cartesian squares, where $\widetilde{\mathfrak{w}} \circ \widetilde{\mathfrak{s}}_1 = \widetilde{\mathfrak{s}}$, $\mathfrak{w} \circ \mathfrak{s}_1 = \mathfrak{s}$, $\mathfrak{s}_1 \in \mathfrak{E}_X$, and $\mathfrak{w} \in \mathcal{W}_X^{\overline{\wedge}}$.
Since the system of deflations $\mathfrak{E}_X$ is left divisible in $\mathfrak{E}_X$, the morphism $\widetilde{\mathfrak{w}}$ is a deflation (hence it belongs to $\mathcal{W}_X$ thanks to the stabiligy of $\mathcal{W}_X$). The composition $\mathfrak{t}\circ\mathfrak{w}$ is a deflation, because, by hypothesis, the system of deflations satisfies (#) and both $\widetilde{\mathfrak{w}}$ and $\mathfrak{t}$ are deflations (look at the diagram (3) ignoring its left square). The equality $\mathfrak{E}_X \cap (\mathcal{S} \circ \mathcal{W}_X^{\overline{\wedge}}) = \mathcal{S}$ implies that $\mathfrak{t} \circ \mathfrak{w} \in \mathcal{S}$. Since the system $\mathcal{S}$ is right divisible in $\mathfrak{E}_X$, the arrow $\widetilde{\mathfrak{s}}$ belongs to $\mathcal{S}$, whence $\mathfrak{s}_1 \in \mathfrak{E}_X \cap \mathcal{S}^{\overline{\wedge}} = \mathcal{S}$ (thanks to the stability of $\mathcal{S}$). Finally, the multiplicativity of $\mathcal{S}$ implies that $\mathfrak{t} \circ \mathfrak{s} = (\mathfrak{t} \circ \mathfrak{w}) \circ \mathfrak{s}_1 \in \mathcal{S}$. ■

4.2.5. Note. For any class of arrows $\mathcal{S}$ invariant under pull-backs and such that $\mathcal{W}_X \circ \mathcal{S} = \mathcal{S}$, there is the inclusion $\mathcal{W}_X^{\overline{\wedge}} \circ \mathcal{S} \subseteq \mathcal{S}^{\overline{\wedge}}$. In particular, $\mathcal{W}_X^{\overline{\wedge}} \circ \mathfrak{E}_X \subseteq \mathfrak{E}_X^{\overline{\wedge}}$. Therefore, the inclusion $\mathfrak{E}_X^{\overline{\wedge}} \subseteq \mathcal{W}_X^{\overline{\wedge}} \circ \mathfrak{E}_X$ used in 4.2.4 is equivalent to the equality $\mathfrak{E}_X^{\overline{\wedge}} = \mathcal{W}_X^{\overline{\wedge}} \circ \mathfrak{E}_X$.

The equality $\mathfrak{E}_X^{\bar{\wedge}} = \mathcal{W}_X^{\bar{\wedge}} \circ \mathfrak{E}_X$ holds when $\mathfrak{E}_X$ coincides with the class of all strict epimorphisms of the category C_X, in C_X, there exist 2-coimages of arbitrary arrows (see 2.0.1(d)&(e)), and $\mathcal{W}_X^{\bar{\wedge}}$ contains all monomorphisms of C_X (say, C_X has initial objects).

Indeed, in this case $\mathcal{W}_X^{\bar{\wedge}} \circ \mathfrak{E}_X$ (hence $\mathfrak{E}_X^{\bar{\wedge}}$) coincides with $HomC_X$.

4.3. Strongly closed systems.

4.3.1. The strong closure. For a class of deflations $\mathcal{S}$ of $(C_X, \bar{\mathfrak{E}}_X)$, let $\mathfrak{R}^{\mathfrak{s}}_{\mathcal{S}}$ denote the set of all systems $\mathcal{T}$ divisible in $\mathfrak{E}_X$ such that $\mathcal{T} = \mathfrak{E}_X \cap (\mathcal{T} \circ \mathcal{W}_X^{\bar{\wedge}})$ and $\mathcal{T} \cap \mathcal{S}^{\perp} = \mathcal{W}_X$. We denote by $\mathcal{S}^{\dagger}$ the union of all $\mathcal{T} \in \mathfrak{R}^{\mathfrak{s}}_{\mathcal{S}}$.

The construction $\mathcal{S} \longmapsto \mathcal{S}^{\dagger}$ has the properties similar to those of the closure $\mathcal{S} \longmapsto \mathcal{S}^{-}$.

4.3.2. Proposition. *(a) For any class of deflations $\mathcal{S}$, the system $\mathcal{S}^{\dagger}$ belongs to $\mathfrak{R}^{\mathfrak{s}}_{\mathcal{S}}$ (hence it is the largest element of $\mathfrak{R}^{\mathfrak{s}}_{\mathcal{S}}$).*

(b) $(\mathcal{S}^{\dagger})^{\dagger} = \mathcal{S}^{\dagger}$.

(c) The system $\mathcal{S}^{\dagger}$ is closed under the composition.

(d) The system $\mathcal{S}^{\dagger}$ is stable (hence strongly stable), that is $\mathcal{S}^{\dagger} = \mathfrak{E}_X \cap (\mathcal{S}^{\dagger})^{\bar{\wedge}}$.

Proof. (a) The assertion follows from 4.1.2(b).

(b) It follows that $\mathfrak{R}^{\mathfrak{s}}_{\mathcal{S}^{\dagger}} \subseteq \mathfrak{R}^{\mathfrak{s}}_{\mathcal{S}}$. On the other hand, $\mathcal{S}^{\dagger} \in \mathfrak{R}^{\mathfrak{s}}_{\mathcal{S}^{\dagger}}$. Hence the equality.

(c) The argument is similar to that of 2.6.2(c).

(d) If $\mathcal{T} \in \mathfrak{R}^{\mathfrak{s}}_{\mathcal{S}}$, then the associated stable system, $\mathfrak{E}_X \cap \mathcal{T}^{\bar{\wedge}}$, belongs to $\mathfrak{R}^{\mathfrak{s}}_{\mathcal{S}}$.

In fact, by 4.1.2(a), the system $\mathfrak{E}_X \cap \mathcal{T}^{\bar{\wedge}}$ is strongly stable. On the other hand, the equality $\mathcal{S}^{\perp} = (\mathcal{S}^{\perp})^{\bar{\wedge}}$ implies that

$$(\mathfrak{E}_X \cap \mathcal{T}^{\bar{\wedge}}) \cap \mathcal{S}^{\perp} = \mathfrak{E}_X \cap \mathcal{T}^{\bar{\wedge}} \cap (\mathcal{S}^{\perp})^{\bar{\wedge}} = \mathfrak{E}_X \cap (\mathcal{T} \cap \mathcal{S}^{\perp})^{\bar{\wedge}} = \mathfrak{E}_X \cap \mathcal{W}_X^{\bar{\wedge}} = \mathcal{W}_X.$$

This shows that $\mathfrak{E}_X \cap \mathcal{T}^{\bar{\wedge}}$ belongs to $\mathfrak{R}^{\mathfrak{s}}_{\mathcal{S}}$. In particular, $\mathfrak{E}_X \cap (\mathcal{S}^{\dagger})^{\bar{\wedge}}$ belongs to $\mathfrak{R}^{\mathfrak{s}}_{\mathcal{S}}$, which implies the stability of $\mathcal{S}^{\dagger}$. ■

4.3.3. Observations. (a) One can see that $\mathcal{S}^{\dagger}$ is the largest divisible in $\mathfrak{E}_X$ strongly stable subsystem of $\mathcal{S}^{-}$. So that $\mathcal{S}^{-} = \mathcal{S}^{\dagger}$ iff $\mathfrak{E}_X \cap (\mathcal{S}^{-} \circ \mathcal{W}_X^{\bar{\wedge}}) = \mathcal{S}^{-}$.

It follows from 4.1.1 that if the class of deflations $\mathfrak{E}_X$ is left divisible (in the sense of 4.1.1) and $\mathcal{S}^{-}$ is a topological system, then $\mathcal{S}^{-} = \mathcal{S}^{\dagger}$.

In particular, $\mathcal{S}^{-} = \mathcal{S}^{\dagger}$ in the case of an abelian category $(C_X, \bar{\mathfrak{E}}_X)$.

(b) Since, by hypothesis, the class $\mathcal{W}_X$ of weak equivalences is strongly stable and, by 2.6.2.1, $\mathcal{W}_X^{-} = \mathcal{W}_X$, it follows that $\mathcal{W}_X = \mathcal{W}_X^{\dagger}$.

4.3.4. Proposition. *Let $\{\mathcal{S}_i \mid i \in J\}$ be a finite set of right divisible in $\mathfrak{E}_X$ systems of deflations of $(C_X, \mathfrak{E}_X)$. Then $\bigcap_{i \in J} \mathcal{S}_i^{\dagger} = \big(\bigcap_{i \in J} \mathcal{S}_i\big)^{\dagger}$.*

Proof. The argument is similar to the proof of 2.6.4(b). ■

4.3.5. Strongly closed systems of deflations. We call a class $\mathcal{S}$ of deflations of $(C_X, \bar{\mathfrak{E}}_X)$ a *strongly closed* system if $\mathcal{S} = \mathcal{S}^{\dagger}$. Strongly closed systems of $(C_X, \bar{\mathfrak{E}}_X)$ form a preorder with respect to the inclusion, which we denote by $\mathfrak{S}\mathfrak{e}_{\mathfrak{s}}(X, \mathfrak{E}_X)$.

By 4.3.2, strongly closed systems are strongly stable and multiplicative; and, by definition, they are divisible in $\mathfrak{E}_X$. Therefore, every strongly closed system is semitopological.

4.3.6. Proposition. *(a) The intersection of any set of strongly closed systems of deflations of* $(C_X, \mathfrak{E}_X)$ *is a strongly closed system.*

(b) Suppose that $\mathfrak{E}_X^{\bar{\wedge}} = \mathcal{W}_X^{\bar{\wedge}} \circ \mathfrak{E}_X$. *Then every strongly closed system of deflations is strongly thick.*

Proof. (a) The argument is similar to that of 2.6.4(a).

(b) The assertion follows from the multiplicativity of strongly closed systems (see 4.3.2(c)) and 4.2.4. ■

4.3.7. The lattice of strongly closed systems. Fix a svelte right exact category $(C_X, \mathfrak{E}_X)$. For any pair Σ_1, Σ_2 of strongly closed systems of deflations, we denote by $\Sigma_1 \sqcup \Sigma_2$ the smallest strongly closed system containing Σ_1 and Σ_2.

4.3.7.1. Proposition. *Let* $\{\mathcal{S}_i \mid i \in J\}$ *be a finite set of strongly closed systems of deflations of* $(C_X, \mathfrak{E}_X)$. *Then* $\Sigma \sqcup \big(\bigcap_{i\in J} \mathcal{S}_i\big) = \bigcap_{i\in J}(\Sigma \sqcup \mathcal{S}_i)$ *for any strongly closed system of deflations* Σ.

Proof. There are the equalities

$$\bigcap_{i\in J}(\Sigma \sqcup \mathcal{S}_i) = \bigcap_{i\in J}(\Sigma \cup \mathcal{S}_i)^{\dagger} = \Big(\bigcap_{i\in J}(\Sigma \cup \mathcal{S}_i)\Big)^{\dagger} = \Big(\big(\bigcap_{i\in J} \mathcal{S}_i\big) \cup \Sigma\Big)^{\dagger} = \Big(\bigcap_{i\in J} \mathcal{S}_i\Big) \sqcup \Sigma.$$

Here the second equality follows from 2.6.4. ■

4.4. Spectra. For every class of deflations $\mathcal{S}$, we denote by $\mathcal{S}^{\mathfrak{st}}$ the intersection of all semitopologizing systems properly containing $\mathcal{S}$ and by $\mathcal{S}^{\mathfrak{sc}}$ the intersection of all *strongly stable* thick systems properly containing the class $\mathcal{S}$.

We denote by $\mathbf{Spec}^{1,1}_{\mathfrak{sc}}(X, \bar{\mathfrak{E}}_X)$ the preorder (with respect to $\supseteq$) formed by those strongly closed systems of deflations Σ for which $\Sigma^{\mathfrak{sc}} \neq \Sigma$, or, equivalently, the intersection $\Sigma_{\mathfrak{sc}} \stackrel{\mathrm{def}}{=} \Sigma^{\mathfrak{sc}} \cap \Sigma^{\perp}$ is a non-trivial system of deflations.

Similarly, we define the spectrum $\mathbf{Spec}^{1,1}_{\mathfrak{st}}(X, \bar{\mathfrak{E}}_X)$ as the preorder formed by all strongly closed systems Σ for which $\Sigma^{\mathfrak{st}} \neq \Sigma$, or, what is the same, the system of deflations

$$\Sigma_{\mathfrak{st}} \stackrel{\mathrm{def}}{=} \Sigma^{\mathfrak{st}} \cap \Sigma^{\perp}$$

is non-trivial. It follows from the definitions that the spectrum $\mathbf{Spec}^{1,1}_{\mathfrak{st}}(X,\bar{\mathfrak{E}}_X)$ is a subpreorder of the spectrum $\mathbf{Spec}^{1,1}_{\mathfrak{st}}(X,\bar{\mathfrak{E}}_X)$.

The following useful fact is a direct analog uf 3.7.1.

4.4.1. Proposition. *Let $\Sigma \in \mathbf{Spec}^{1,1}_{\mathfrak{sc}}(X,\bar{\mathfrak{E}}_X)$. For any finite family $\{\mathcal{S}_i \mid i \in \mathfrak{J}\}$ of right divisible in $\mathfrak{E}_X$ systems of deflations, $\mathcal{S}_i \nsubseteq \Sigma$ for all $i \in \mathfrak{J}$ iff $\bigcap\limits_{i\in\mathfrak{J}} \mathcal{S}_i \nsubseteq \Sigma$.*

Proof. The argument below is similar to the proof of 3.7.1.

By 4.3.4, $\bigcap\limits_{i\in J} \mathcal{S}_i^\dagger = \big(\bigcap\limits_{i\in J} \mathcal{S}_i\big)^\dagger$, and by 4.3.7.1, $\Sigma \sqcup \big(\bigcap\limits_{i\in J} \mathcal{S}_i^\dagger\big) = \bigcap\limits_{i\in J} (\Sigma \sqcup \mathcal{S}_i^\dagger)$. Therefore,

$$\Sigma \sqcup \big(\bigcap_{i\in J} \mathcal{S}_i\big)^\dagger = \bigcap_{i\in J} (\Sigma \sqcup \mathcal{S}_i^\dagger). \tag{5}$$

If $\mathcal{S}_i \nsubseteq \Sigma$ for all $i \in J$, then each of the strongly closed systems $\mathcal{S}_i^\dagger \sqcup \Sigma$ contains Σ properly. Since Σ is an element of the spectrum $\mathbf{Spec}^{1,1}_{\mathfrak{sc}}(X,\bar{\mathfrak{E}}_X)$, the intersection of $\mathcal{S}_i^\dagger \sqcup \Sigma$, $i \in J$, contains in Σ properly. Then it follows from the equality (5) that the intersection $\bigcap\limits_{i\in J} \mathcal{S}_i$ is not contained in Σ. ■

5. Strongly 'exact' functors and localizations.

5.0. Strongly 'exact' functors. Let $(C_X,\bar{\mathfrak{E}}_X) = (C_X,\mathfrak{E}_X,\mathcal{W}_X)$ and $(C_Y,\mathfrak{E}_Y,\mathcal{W}_Y)$ be right exact categories with weak equivalences. Recall that an 'exact' functor from $(C_Y,\bar{\mathfrak{E}}_Y)$ to $(C_X,\bar{\mathfrak{E}}_X)$ is given by a functor $C_Y \longrightarrow C_X$ which maps deflations to deflations, weak equivalences to weak equivalences and preserves pull-backs of deflations.

We say that an 'exact' functor $(C_Y,\bar{\mathfrak{E}}_Y) \xrightarrow{F} (C_X,\bar{\mathfrak{E}}_X)$ is *strongly 'exact'* if any cartesian square

$$\begin{array}{ccc}
\widetilde{\mathcal{M}} & \xrightarrow{\ \widetilde{\mathfrak{f}}\ } & \widetilde{\mathcal{N}} \\
{\scriptstyle \mathfrak{s}'}\big\downarrow & cart & \big\downarrow{\scriptstyle \mathfrak{s}} \\
\mathcal{M} & \xrightarrow{\ \mathfrak{f}\ } & \mathcal{N}
\end{array}$$

whose left vertical arrow is a deflation can be completed by a pull-back of this deflation

$$\begin{array}{ccccc}
\widetilde{\mathcal{L}} & \xrightarrow{\ \widetilde{\xi}\ } & \widetilde{\mathcal{M}} & \xrightarrow{\ \widetilde{\mathfrak{f}}\ } & \widetilde{\mathcal{N}} \\
{\scriptstyle \mathfrak{s}''}\big\downarrow & cart & {\scriptstyle \mathfrak{s}'}\big\downarrow & cart & \big\downarrow{\scriptstyle \mathfrak{s}} \\
\mathcal{L} & \xrightarrow{\ \xi\ } & \mathcal{M} & \xrightarrow{\ \mathfrak{f}\ } & \mathcal{N}
\end{array}$$

such that F maps the outer cartesian square

$$\begin{array}{ccc}
\widetilde{\mathcal{L}} & \xrightarrow{\widetilde{\mathfrak{f}}\circ\widetilde{\xi}} & \widetilde{\mathcal{N}} \\
{\scriptstyle \mathfrak{s}''}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{s}} \\
\mathcal{L} & \xrightarrow{\xi\circ\mathfrak{f}} & \mathcal{N}
\end{array}$$

to a cartesian square.

In particular, any functor $C_Y \xrightarrow{F} C_X$ which maps deflations to deflations and preserves cartesian squares having at least one deflation among its arrows is strongly 'exact'.

5.0.1. Strong 'exactness' and preserving kernels. This seemingly technical notion has a transparent meaning in the case the category C_X has initial objects and morphisms to initial objects are deflations. In this case,

an 'exact' functor $(C_Y, \bar{\mathfrak{E}}_Y) \xrightarrow{F} (C_X, \bar{\mathfrak{E}}_X)$ *is strongly exact iff the functor* $C_Y \xrightarrow{F} C_X$ *preserves kernels of arrows.*

5.1. Proposition. *Let* $(C_X, \bar{\mathfrak{E}}_X) \xrightarrow{F} (C_Y, \bar{\mathfrak{E}}_Y)$ *be a strongly 'exact' functor.*

(a) $F(\mathcal{S}^{\bar{\wedge}}) \subseteq F(\mathcal{S})^{\bar{\wedge}}$ *for any class of deflations* $\mathcal{S}$ *of the category* C_X.

In particular, $F(\mathcal{W}_X^{\bar{\wedge}}) \subseteq F(\mathcal{W}_X)^{\bar{\wedge}} \subseteq \mathcal{W}_Y^{\bar{\wedge}}$ *and* $F(\mathfrak{E}_X^{\bar{\wedge}}) \subseteq F(\mathfrak{E}_X)^{\bar{\wedge}} \subseteq \mathfrak{E}_Y^{\bar{\wedge}}$.

(b) The map $\mathcal{T} \longmapsto \mathfrak{E}_X \cap F^{-1}(\mathcal{T})$ *transfers stable, strongly stable, thick and semi-topologizing systems of deflations to systems of deflations of the same kind.*

(c) Suppose that one of the following conditions holds:

(i) If in the commutative diagram

$$\begin{array}{ccccc}
\widetilde{\mathcal{L}} & \xrightarrow{\widetilde{\mathfrak{s}}} & \mathcal{K} & & \\
{\scriptstyle \mathfrak{j}'}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{j}} & & \\
\mathcal{L} & \xrightarrow{\mathfrak{s}} & \mathcal{M} & \xrightarrow{\mathfrak{t}} & \mathcal{K}
\end{array}$$

in C_X *or* C_Y *the square is cartesian,* $\mathfrak{t}\circ\mathfrak{j} = id_{\mathcal{N}}$ *and morphisms* $\mathfrak{t}$ *and* $\widetilde{\mathfrak{s}}$ *are deflations, then the composition* $\mathfrak{t}\circ\mathfrak{s}$ *is a deflation.*

(ii) $\mathfrak{E}_X^{\bar{\wedge}} = \mathcal{W}_X^{\bar{\wedge}} \circ \mathfrak{E}_X$.

Then the map $\mathcal{T} \longmapsto \mathfrak{E}_X \cap F^{-1}(\mathcal{T})$ *preserves strongly thick systems.*

Proof. (a) The inclusions follow from definitions.

(b1) Suppose that $\mathcal{T}$ is a class of arrows of C_Y satisfying $\mathcal{T} = \mathfrak{E}_Y \cap (\mathcal{T} \circ \mathcal{W}_Y^{\bar{\wedge}})$. Then

$$F(\mathfrak{E}_X \cap ((\mathfrak{E}_X \cap F^{-1}(\mathcal{T})) \circ \mathcal{W}_X^{\bar{\wedge}})) \subseteq \mathfrak{E}_Y \cap F(F^{-1}(\mathcal{T})) \circ \mathcal{W}_X^{\bar{\wedge}}) \subseteq$$
$$\mathfrak{E}_Y \cap (\mathcal{T} \circ F(\mathcal{W}_X^{\bar{\wedge}})) \subseteq \mathfrak{E}_Y \cap (\mathcal{T} \circ \mathcal{W}_Y^{\bar{\wedge}}) = \mathcal{T},$$

which implies the inclusion

$$\mathfrak{E}_X \cap ((\mathfrak{E}_X \cap F^{-1}(\mathcal{T})) \circ \mathcal{W}_X^{\bar{\wedge}}) \subseteq \mathfrak{E}_X \cap F^{-1}(\mathcal{T}).$$

Since the inverse inclusion holds (for any class of arrows $\mathcal{T}$), we obtain the equality

$$\mathfrak{E}_X \cap ((\mathfrak{E}_X \cap F^{-1}(\mathcal{T})) \circ \mathcal{W}_X^{\bar{\wedge}}) = \mathfrak{E}_X \cap F^{-1}(\mathcal{T}).$$

(b2) Similarly with the stability: if $\mathcal{T} = \mathfrak{E}_Y \cap \mathcal{T}^{\bar{\wedge}}$, then

$$F(\mathfrak{E}_X \cap (\mathfrak{E}_X \cap F^{-1}(\mathcal{T}))^{\bar{\wedge}}) = F(\mathfrak{E}_X \cap (\mathfrak{E}_X^{\bar{\wedge}} \cap (F^{-1}(\mathcal{T}))^{\bar{\wedge}}) =$$
$$F(\mathfrak{E}_X \cap (F^{-1}(\mathcal{T}))^{\bar{\wedge}}) \subseteq \mathfrak{E}_Y \cap F((F^{-1}(\mathcal{T}))^{\bar{\wedge}}) \subseteq \mathfrak{E}_Y \cap \mathcal{T}^{\bar{\wedge}} = \mathcal{T},$$

which implies the inclusions

$$\mathfrak{E}_X \cap F^{-1}(\mathcal{T}) \subseteq \mathfrak{E}_X \cap (\mathfrak{E}_X \cap F^{-1}(\mathcal{T}))^{\bar{\wedge}} \subseteq \mathfrak{E}_X \cap F^{-1}(\mathcal{T})$$

equivalent to the stability of the class $\mathfrak{E}_X \cap F^{-1}(\mathcal{T})$.

(b3) The fact that the map $\mathcal{T} \longmapsto \mathfrak{E}_X \cap F^{-1}(\mathcal{T})$ respects strong stability implies, obviously, that it maps semitopologizing systems to semitopologizing systems.

(c1) Suppose that the class of deflations $\mathfrak{E}_X$ satisfies the condition (i). Let $\mathcal{S}$ be a stable class of deflations of $(C_Y, \bar{\mathfrak{E}}_Y)$ invariant under pull-backs and satisfying the condition

(#) If in the commutative diagram

$$\begin{tikzcd}
\widetilde{\mathcal{L}} \arrow[r, "\widetilde{\mathfrak{s}}"] \arrow[d, "\mathfrak{j}'"'] \arrow[dr, phantom, "cart"] & \mathcal{K} \arrow[d, "\mathfrak{j}"] & \\
\mathcal{L} \arrow[r, "\mathfrak{s}"] & \mathcal{M} \arrow[r, "\mathfrak{t}"] & \mathcal{K}
\end{tikzcd} \qquad (1)$$

with cartesian square $\mathfrak{t} \circ \mathfrak{j} = id_{\mathcal{N}}$ and morphisms $\mathfrak{t}$ and $\widetilde{\mathfrak{s}}$ belong to $\mathcal{S}$, then $\mathfrak{t} \circ \mathfrak{s} \in \mathcal{S}$.

Then the class $\mathfrak{E}_X \cap F^{-1}(\mathcal{S})$ satisfies this condition.

In fact, let (1) be a diagram whose square is cartesian and $\widetilde{\mathfrak{s}}$ and $\mathfrak{t}$ are arrows from $\mathfrak{E}_X \cap F^{-1}(\mathcal{S})$. Since the functor F is strongly 'exact' and the arrow $\widetilde{\mathfrak{s}}$ in the diagram (1) is a deflation, there exists a pull-back of $\widetilde{\mathfrak{s}}$ along some arrow $\mathfrak{K} \xrightarrow{\gamma} \mathcal{K}$ such that F maps the pull-back of $\mathcal{L} \xrightarrow{\mathfrak{s}} \mathcal{M}$ along $\mathfrak{K} \xrightarrow{\mathfrak{j} \circ \gamma} \mathcal{M}$ to a pull-back of $F(\mathfrak{s})$ along $F(\mathfrak{j} \circ \gamma)$.

By hypothesis, the class of deflations $\mathfrak{E}_X$ satisfies the condition (#). So that the composition $\mathfrak{t} \circ \mathfrak{s}$ is a deflation. Taking a pull-back of the deflation $\mathfrak{t} \circ \mathfrak{s}$ along the morphism

$\mathfrak{K} \xrightarrow{\gamma} \mathcal{K}$, we obtain the diagram

$$\begin{array}{ccccc}
\widetilde{\mathfrak{L}} & \xrightarrow{\widehat{\mathfrak{s}}} & \mathfrak{K} & & \\
{\scriptstyle \lambda'}\downarrow & cart & \downarrow{\scriptstyle \lambda} & & \\
\mathfrak{L} & \xrightarrow{\bar{\mathfrak{s}}} & \mathfrak{M} & \xrightarrow{\bar{\mathfrak{t}}} & \mathfrak{K} \\
{\scriptstyle \gamma''}\downarrow & cart & \downarrow{\scriptstyle \gamma'} & cart & \downarrow{\scriptstyle \gamma} \\
\mathcal{L} & \xrightarrow{\mathfrak{s}} & \mathcal{M} & \xrightarrow{\mathfrak{t}} & \mathcal{K}
\end{array} \qquad (2)$$

built of cartesian squares, where the morphism $\mathfrak{K} \xrightarrow{\lambda} \mathfrak{M}$ is uniquely determined by the equalities $\gamma' \circ \lambda = \mathfrak{j} \circ \gamma$ and $\bar{\mathfrak{t}} \circ \lambda = id_{\mathfrak{K}}$. Since the functor F is 'exact' and the arrows $\mathfrak{t}$ and $\mathfrak{t} \circ \mathfrak{s}$ are deflations, F preserves pull-backs of this arrows which implies that it maps the lower two cartesian squares of the diagram (2) to cartesian squares. Since, by construction, F preserves the pull-back of the arrow $\mathcal{L} \xrightarrow{\mathfrak{s}} \mathcal{M}$ along the morphism $\gamma' \circ \lambda = \mathfrak{j} \circ \gamma$, it follows that F maps the upper square of (2) to a cartesian square as well. By the condition (#), $F(\bar{\mathfrak{t}} \circ \bar{\mathfrak{s}}) = F(\bar{\mathfrak{t}}) \circ F(\bar{\mathfrak{s}}) \in \mathcal{S}$, that is $\bar{\mathfrak{t}} \circ \bar{\mathfrak{s}} \in F^{-1}(\mathcal{S}) \cap \mathfrak{E}_X$. Since, by hypothesis, the class $\mathcal{S}$ is stable and $\bar{\mathfrak{t}} \circ \bar{\mathfrak{s}}$ is a pull-back of the deflation $\mathfrak{t} \circ \mathfrak{s}$, it follows from the assertion (b) that $\mathfrak{t} \circ \mathfrak{s} \in F^{-1}(\mathcal{S}) \cap \mathfrak{E}_X$.

(c2) Suppose that the condition (i) holds, that is the both classes of deflations, $\mathfrak{E}_X$ and $\mathfrak{E}_Y$, satisfy the condition (#). The fact that $\mathfrak{E}_Y$ satisfies (#) implies, by 4.2.3, that any class $\mathcal{S}$ of deflations of $(C_Y, \bar{\mathfrak{E}}_Y)$ invariant under pull-backs and such that $\mathcal{S}^{\bar{\wedge}}$ is multiplicative satisfies the condition (#). If, in addition, the class $\mathcal{S}$ is stable, then, by (c1) above, the class $\mathfrak{E}_X \cap F^{-1}(\mathcal{S})$ satisfies the condition (#), which implies the multiplicativity of the class $(\mathfrak{E}_X \cap F^{-1}(\mathcal{S}))^{\bar{\wedge}}$. Since, by (b), the map $\mathcal{T} \longmapsto \mathfrak{E}_X \cap F^{-1}(\mathcal{T})$ preserves strongly stable systems, we obtain that it preserves strongly thick systems.

(c3) If $\mathfrak{E}_X^{\bar{\wedge}} = \mathcal{W}_X^{\bar{\wedge}} \circ \mathfrak{E}_X$, then, by 4.2.4, a strongly stable divisible in $\mathfrak{E}_X$ system is *strongly thick* iff it is multiplicative. Evidently, the $\mathcal{T} \longmapsto \mathfrak{E}_X \cap F^{-1}(\mathcal{T})$ maps multiplicative systems to multiplicative systems. Therefore, it maps strongly stable multiplicative systems (in particular, strongly thick systems) to strongly thick systems. ∎

5.2. Proposition. *Let* $(C_Y, \bar{\mathfrak{E}}_Y) \xrightarrow{F} (C_Z, \bar{\mathfrak{E}}_Z)$ *be a strongly 'exact' functor. Set* $\mathfrak{E}_{Y,F} = \Sigma_F \cap \mathfrak{E}_Y = \{\mathfrak{s} \in \mathfrak{E}_Y \mid F(\mathfrak{s}) \in Iso(C_Z)\}$.

(a) Suppose that the class of deflations $\mathfrak{E}_Y$ *satisfies the condition (i) of 5.1. Then the class* $(\mathfrak{E}_{Y,F})^{\bar{\wedge}}$ *is multiplicative.*

(b) If all deflations of $(C_Z, \bar{\mathfrak{E}}_Z)$ *having a trivial kernel are isomorphisms, then* $\mathfrak{E}_{Y,F}$ *is a stable class, that is* $\mathfrak{E}_{Y,F} = (\mathfrak{E}_{Y,F})^{\bar{\wedge}} \cap \mathfrak{E}_Y$.

Proof. (a) It suffices to show that if $\mathcal{L} \xrightarrow{\mathfrak{s}} \mathcal{M}$ and $\mathcal{M} \xrightarrow{\mathfrak{t}} \mathcal{N}$ are morphisms of C_X such that $\mathfrak{s} \in \mathfrak{E}_{Y,F}^{\bar{\wedge}}$ and $\mathfrak{t} \in \mathfrak{E}_{Y,F}$, then $\mathfrak{t} \circ \mathfrak{s} \in \mathfrak{E}_{Y,F}^{\bar{\wedge}}$. Since $\mathfrak{s} \in \mathfrak{E}_{Y,F}^{\bar{\wedge}}$ and the functor F is strongly 'exact', there is a cartesian square

$$\begin{array}{ccc} \widetilde{\mathcal{L}} & \xrightarrow{\ \widetilde{\mathfrak{s}}\ } & \mathcal{K} \\ {\scriptstyle \mathfrak{j}'}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{j}} \\ \mathcal{L} & \xrightarrow{\ \mathfrak{s}\ } & \mathcal{M} \end{array} \qquad (3)$$

such that $\widetilde{\mathfrak{s}} \in \mathfrak{E}_{Y,F}$ and the functor F maps it to a cartesian square. Taking pull-back of $\mathcal{L} \xrightarrow{\mathfrak{t}\circ\mathfrak{s}} \mathcal{N}$ along the morphism $\mathcal{K} \xrightarrow{\mathfrak{t}\circ\mathfrak{j}} \mathcal{N}$, we obtain a diagram

$$\begin{array}{ccccc} \widetilde{\mathcal{L}} & \xrightarrow{\ \widetilde{\mathfrak{s}}\ } & \mathcal{K} & & \\ {\scriptstyle \beta'}\downarrow & cart & \downarrow{\scriptstyle \beta} & & \\ \mathfrak{L} & \xrightarrow{\ \bar{\mathfrak{s}}\ } & \mathfrak{M} & \xrightarrow{\ \bar{\mathfrak{t}}\ } & \mathcal{K} \\ {\scriptstyle \gamma''}\downarrow & cart & \downarrow{\scriptstyle \gamma'} & cart & \downarrow{\scriptstyle \gamma = \mathfrak{t}\circ\mathfrak{j}} \\ \mathcal{L} & \xrightarrow{\ \mathfrak{s}\ } & \mathcal{M} & \xrightarrow{\ \mathfrak{t}\ } & \mathcal{K} \end{array} \qquad (4)$$

built of cartesian squares with the arrow $\mathcal{K} \xrightarrow{\beta} \mathfrak{M}$ uniquely determined by the equalities $\bar{\mathfrak{t}} \circ \beta = id_{\mathcal{K}}$, $\gamma' \circ \beta = \mathfrak{j}$ (and with $\gamma'' \circ \beta'$ equal to the left vertical arrow $\mathfrak{j}'$ in the cartesian square (3)). Since $\mathcal{M} \xrightarrow{\mathfrak{t}} \mathcal{K}$ is a deflation, the functor F, being 'exact', maps the right cartesian square of the diagram (4) to a cartesian square. In particular, $\bar{\mathfrak{t}} \in \mathfrak{E}_{Y,F}$. By construction, F transfers the square (3) to a cartesian square, which implies that it maps the upper square of the diagram (4) to a cartesian square. The equality $\bar{\mathfrak{t}} \circ \beta = id_{\mathcal{K}}$ together with the fact that $F(\bar{\mathfrak{t}})$ is an isomorphism, implies that $F(\beta)$ is an isomorphism. Therefore, $F(\beta')$ is an isomorphism. Since $\widetilde{\mathfrak{s}} \in \mathfrak{E}_{Y,F}$ by construction, we obtain that $F(\bar{\mathfrak{s}})$ is an isomorphism. So that $F(\bar{\mathfrak{t}} \circ \bar{\mathfrak{s}})$ is an isomorphism. By hypothesis, the class of deflations $\mathfrak{E}_Y$ satisfies the condition (i) of 5.1, which implies that $\bar{\mathfrak{t}} \circ \bar{\mathfrak{s}}$ is a deflation. Since $\bar{\mathfrak{t}} \circ \bar{\mathfrak{s}}$ is a pull-back of $\mathfrak{t} \circ \mathfrak{s}$, the latter belongs to $\mathfrak{E}_{Y,F}^{\bar{\wedge}}$.

(b) For any strongly 'exact' functor $(C_Y, \bar{\mathfrak{E}}_Y) \xrightarrow{F} (C_Z, \bar{\mathfrak{E}}_Z)$ and any class of deflations $\mathcal{S}$ of $(C_Y, \bar{\mathfrak{E}}_Y)$, we have, by 5.1(a), the inclusions

$$F(\mathcal{S}^{\bar{\wedge}}) \subseteq (F(\mathcal{S}))^{\bar{\wedge}} \quad \text{and} \quad F(\mathcal{S}^{\bar{\wedge}} \cap \mathfrak{E}_Y) \subseteq (F(\mathcal{S}))^{\bar{\wedge}} \cap \mathfrak{E}_Z.$$

So that if $\mathcal{S} \subseteq \mathfrak{E}_{Y,F} = \{\mathfrak{s} \in \mathfrak{E}_Y \mid F(\mathfrak{s}) \in Iso(C_Z)\}$, then $F(\mathcal{S}^{\bar{\wedge}} \cap \mathfrak{E}_Y)$ is contained in the class $\mathfrak{E}_Z^{\circledast} \stackrel{\text{def}}{=} Iso(C_Z)^{\bar{\wedge}} \cap \mathfrak{E}_Z$ of deflations with a trivial kernel. Therefore, if $\mathfrak{E}_Z^{\circledast} = Iso(C_Z)$, then $\mathcal{S} \subseteq \mathcal{S}^{\bar{\wedge}} \cap \mathfrak{E}_Y \subseteq \Sigma_F \cap \mathfrak{E}_Y = \mathfrak{E}_{Y,F}$, hence $\mathfrak{E}_{Y,F}$ is a stable class of deflations. ∎

5.3. 'Exact' and strongly 'exact' localizations. An 'exact' (resp. strongly 'exact') functor $(C_Y, \bar{\mathfrak{E}}_Y) \xrightarrow{\mathfrak{q}^*} (C_X, \bar{\mathfrak{E}}_X)$ will be called an *'exact'* (resp. *strongly 'exact'*) *localization*, if $C_Y \xrightarrow{\mathfrak{q}^*} C_X$ is a localization and the essential image of $\mathfrak{E}_Y$ (resp. the essential image of $\mathcal{W}_Y$) coincides with $\mathfrak{E}_X$ (resp. with $\mathcal{W}_X$).

5.3.1. Note. Since $C_Y \xrightarrow{\mathfrak{q}^*} C_X$ is a localization functor, it is determined by the class of arrows

$$\Sigma_{\mathfrak{q}^*} \stackrel{\text{def}}{=} \{\mathfrak{s} \in HomC_Y \mid \mathfrak{q}^*(\mathfrak{s}) \in Iso(C_X)\}.$$

The fact that $(C_Y, \bar{\mathfrak{E}}_Y) \xrightarrow{\mathfrak{q}^*} (C_X, \bar{\mathfrak{E}}_X)$ is a 'exact' localization means that the class of deflations $\mathfrak{E}_{Y,\mathfrak{q}^*} \stackrel{\text{def}}{=} \Sigma_{\mathfrak{q}^*} \cap \mathfrak{E}_Y$ is invariant under pull-backs.

5.4. Strongly 'exact' saturation. Every strongly 'exact' functor

$$(C_Y, \bar{\mathfrak{E}}_Y) \xrightarrow{F} (C_Z, \bar{\mathfrak{E}}_Z)$$

factors through a strongly 'exact' localization $(C_Y, \bar{\mathfrak{E}}_Y) \xrightarrow{\mathfrak{q}^*} (C_X, \bar{\mathfrak{E}}_X)$ uniquely determined by the equality $\Sigma_{\mathfrak{q}^*_F} = \Sigma_F$. This implies that for any class of arrows $\mathcal{S}$ of a right exact category $(C_Y, \bar{\mathfrak{E}}_Y)$, there exists the smallest strongly 'exact' localization $\mathfrak{q}^*_{\mathcal{S}}$ which maps all arrows of $\mathcal{S}$ to isomorphisms.

In fact, we consider the family $\Xi_{\mathcal{S}}$ of all strongly 'exact' functors from $(C_Y, \bar{\mathfrak{E}}_Y)$ which map all arrows of $\mathcal{S}$ to isomorphisms. Since the category C_Y is svelte, the family $\{\Sigma_F \mid F \in \Xi_{\mathcal{S}}\}$ is a set. Therefore, there is a subset $\widetilde{\Xi}_{\mathcal{S}}$ of $\Xi_{\mathcal{S}}$ such that $\{\Sigma_F \mid F \in \widetilde{\Xi}_{\mathcal{S}}\} = \{\Sigma_F \mid F \in \Xi_{\mathcal{S}}\}$. The set of 'exact' functors $\widetilde{\Xi}_{\mathcal{S}}$ defines an 'exact' functor $\Phi_{\mathcal{S}}$ to the product of the corresponding right exact categories. Evidently, $\Sigma_{\Phi_{\mathcal{S}}} = \bigcap\limits_{F \in \widetilde{\Xi}_{\mathcal{S}}} \Sigma_F$.

We denote $\Sigma_{\mathfrak{q}^*_{\mathcal{S}}} = \Sigma_{\Phi_{\mathcal{S}}}$ by $\bar{\mathcal{S}}$ and call it the *strongly 'exact' saturation* of $\mathcal{S}$.

5.5. Saturated multiplicative classes of deflations. We call a class of deflations $\mathcal{S}$ of a right exact category with weak equivalences $(C_X, \bar{\mathfrak{E}}_X)$ *saturated* if $\bar{\mathcal{S}} \cap \mathfrak{E}_X = \mathcal{S}$.

It follows that, for any class of deflations $\mathcal{S}$, the intersection $\bar{\mathcal{S}} \cap \mathfrak{E}_X$ is the smallest saturated class of deflations containing $\mathcal{S}$.

Since the localization at $\bar{\mathcal{S}}$ is an 'exact' functor, in particular it maps deflations to deflations, the class $\bar{\mathcal{S}}$ is left and right divisible in $\mathfrak{E}_X$ in the sense that if $\mathfrak{s} \circ \mathfrak{e} \in \bar{\mathcal{S}}$ and $\mathfrak{e}$ is a deflation, then both $\mathfrak{s}$ and $\mathfrak{e}$ are elements of $\mathcal{S}$. In particular, the system of deflations $\bar{\mathcal{S}} \cap \mathfrak{E}_X$ is divisible in $\mathfrak{E}_X$.

5.5.1. Proposition. *(a) Suppose that the class of deflations $\mathfrak{E}_X$ satisfies the condition (i) of 5.1. Then, for any saturated system $\mathcal{S}$ of deflations of $(C_X, \bar{\mathfrak{E}}_X)$, the class $\mathcal{S}^{\wedge}$ is multiplicative.*

(b) If the system $\mathcal{S}$ is stable (that is $\mathcal{S} = \mathcal{S}^{\bar{\wedge}} \cap \mathfrak{E}_X$), then deflations with trivial kernel of the quotient right exact category $(C_{\bar{\mathcal{S}}^{-1}X}, \bar{\mathfrak{E}}_{\bar{\mathcal{S}}^{-1}X})$ are isomorphisms.

Proof. (a) Let $(C_X, \bar{\mathfrak{E}}_X) \xrightarrow{\mathfrak{q}^*} (C_Z, \bar{\mathfrak{E}}_Z)$ be the localization at the saturation $\bar{\mathcal{S}}$ of $\mathcal{S}$. Since, by definition of $\bar{\mathcal{S}}$, the functor $\mathfrak{q}^*$ is strongly 'exact', it follows from 5.2 that the system $\mathcal{S}^{\bar{\wedge}}$ is multiplicative.

(b) Let $M \xrightarrow{\mathfrak{s}'} N$ be a deflation of $(C_Z, \bar{\mathfrak{E}}_Z)$ with a trivial kernel. The latter means that there exists a cartesian square

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\mathfrak{t}'} & \mathcal{N} \\ {\scriptstyle \mathfrak{f}'}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{f}} \\ M & \xrightarrow{\mathfrak{s}'} & N \end{array} \qquad (1)$$

whose upper horizontal arrow is an isomorphism. Since $\mathfrak{q}^*$ is an 'exact' *localization* functor, every arrow of $\mathfrak{E}_Z$ is isomorphic to the image of an arrow of $\mathfrak{E}_Y$, there is a deflation $\mathfrak{M} \xrightarrow{\mathfrak{s}} \mathfrak{N}$ and an arrow $\widetilde{\mathfrak{N}} \xrightarrow{\phi} \mathfrak{N}$ such that the pair of arrows $\mathfrak{q}^*(\mathfrak{M} \xrightarrow{\mathfrak{s}} \mathfrak{N} \xleftarrow{\phi} \widetilde{\mathfrak{N}})$ is isomorphic to the pair of arrows $M \xrightarrow{\mathfrak{s}'} N \xleftarrow{\mathfrak{f}} N$. Therefore, the functor $\mathfrak{q}^*$ maps the cartesian square

$$\begin{array}{ccc} \widetilde{\mathfrak{M}} & \xrightarrow{\widetilde{\mathfrak{s}}} & \widetilde{\mathfrak{N}} \\ {\scriptstyle \phi'}\downarrow & cart & \downarrow{\scriptstyle \phi} \\ \mathfrak{M} & \xrightarrow{\mathfrak{s}} & \mathfrak{N} \end{array} \qquad (2)$$

to a square isomorphic to the cartesian square (1). In particular, $\widetilde{\mathfrak{s}}$ is a deflation which $\mathfrak{q}^*$ maps to an isomorphism; that is $\widetilde{\mathfrak{s}} \in \mathcal{S}$. Since, by hypothesis, $\mathcal{S}$ is stable, the lower horizontal arrow of (2), $\mathfrak{M} \xrightarrow{\mathfrak{s}} \mathfrak{N}$, belongs to $\mathcal{S}$ too. Therefore, the arrow $M \xrightarrow{\mathfrak{s}'} N$ in the diagram (1) is an isomorphism. ■

5.6. Stable saturated classes. For a svelte right exact category with weak equivalences $(C_X, \bar{\mathfrak{E}}_X) = (C_X, \mathfrak{E}_X, \mathcal{W}_X)$, we denote by $\mathfrak{M}_{\mathfrak{s}}(X, \mathfrak{E}_X)$ the preorder (with respect to the inclusion) formed by stable saturated classes of deflations of $(C_X, \bar{\mathfrak{E}}_X)$ and by $\widetilde{\mathfrak{M}}_{\mathfrak{s}}(X, \mathfrak{E}_X)$ the (isomorphic to $\mathfrak{M}_{\mathfrak{s}}(X, \mathfrak{E}_X)$) preorder formed by the strongly 'exact' saturations $\{\bar{\mathcal{S}} \mid \mathcal{S} \in \mathfrak{M}_{\mathfrak{s}}(X, \mathfrak{E}_X)\}$ of these classes.

It follows that every stable saturated class of deflations containing the class $\mathcal{W}_X$ of weak equivalences is thick. Notice that, since any saturated class of deflations is stable and contains all isomorphisms, each element of $\mathfrak{M}_{\mathfrak{s}}(X, \bar{\mathfrak{E}}_X)$ automatically contains $\mathcal{W}_X$, if the latter consists of deflations with trivial kernels.

6. Functorial properties of spectra.

6.1. Proposition. *Let* $(C_X, \bar{\mathfrak{E}}_X) \xrightarrow{\mathfrak{u}^*} (C_U, \bar{\mathfrak{E}}_U)$ *be an 'exact' localization having the following properties:*

(1) $\Sigma_{\mathfrak{u}^*} \subseteq (\Sigma_{\mathfrak{u}^*} \cap \mathfrak{E}_X)^{\bar{\wedge}}$,

(2) $\Sigma_{\mathfrak{u}^*}$ *is closed under push-forwards of deflations along arrows of* $\Sigma_{\mathfrak{u}^*}$,

(3) If $\mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L} \xleftarrow{\mathfrak{t}} \mathcal{N}$ *are deflations such that the arrows* $\mathfrak{u}^*(\mathfrak{e}) = \mathfrak{u}^*(\mathfrak{t}) \circ \phi$ *for some isomorphism* ϕ, *then there exists a pull-back* $\widetilde{\mathcal{M}} \xrightarrow{\widetilde{\mathfrak{e}}} \widetilde{\mathcal{L}} \xleftarrow{\widetilde{\mathfrak{t}}} \widetilde{\mathcal{N}}$ *of these arrows along some morphism* $\widetilde{\mathcal{L}} \longrightarrow \mathcal{L}$ *and a commutative diagram*

$$\begin{array}{ccc}
\mathfrak{M} & \xrightarrow{\mathfrak{s}} & \widetilde{\mathcal{M}} \\
{\scriptstyle \mathfrak{s}'}\big\downarrow & & \big\downarrow{\scriptstyle \widetilde{\mathfrak{e}}} \\
\widetilde{\mathcal{N}} & \xrightarrow{\widetilde{\mathfrak{t}}} & \widetilde{\mathcal{L}}
\end{array}$$

where $\mathfrak{s}$, $\mathfrak{s}'$ *are arrows of* $\Sigma_{\mathfrak{u}^*}$ *and* $\mathfrak{s}$ *is a deflation.*

Let Σ *be a system of deflations containing* $\mathfrak{E}_X \cap \Sigma_{\mathfrak{u}^*}$. *Then for any strongly stable system* $\mathcal{T}$ *of deflations of* $(C_X, \bar{\mathfrak{E}}_X)$, *there is the equality*

$$\mathfrak{T} \cap \Sigma^{\perp} = \mathfrak{E}_X \cap \mathfrak{u}^{*^{-1}}([\mathfrak{u}^*(\mathfrak{T} \cap \Sigma^{\perp})]) \cap \Sigma^{\perp}. \tag{2}$$

Proof. The inclusion $\mathfrak{T}' \subseteq \mathfrak{u}^{*^{-1}}([\mathfrak{u}^*(\mathfrak{T}')])$ for any class of arrows $\mathfrak{T}'$ imply, in particular, that

$$\mathfrak{T} \cap \Sigma^{\perp} \subseteq \mathfrak{E}_X \cap \big(\mathfrak{u}^{*^{-1}}([\mathfrak{u}^*(\mathfrak{T} \cap \Sigma^{\perp})]) \cap \Sigma^{\perp}. \tag{3}$$

The claim is that the inverse inclusion holds.

In fact, let $\mathcal{L} \xrightarrow{\xi} \mathcal{M}$ be an element of $\mathfrak{E}_X \cap \mathfrak{u}^{*^{-1}}([\mathfrak{u}^*(\mathfrak{T} \cap \Sigma^{\perp})]) \cap \Sigma^{\perp}$. This means that $\xi \in \mathfrak{E}_X \cap \Sigma^{\perp}$ and there exists an isomorphism $\mathfrak{u}^*(\xi) \simeq \mathfrak{u}^*(\mathfrak{t})$ for some $\mathfrak{t} \in \mathfrak{T} \cap \Sigma^{\perp}$. This isomorphism is represented by a diagram

$$\begin{array}{ccc}
\mathcal{L} & \xrightarrow{\xi} & \mathcal{M} \\
{\scriptstyle \sigma}\big\uparrow & & \big\uparrow{\scriptstyle \gamma} \\
\mathcal{L}' & & \mathcal{M}' \\
{\scriptstyle \sigma'}\big\downarrow & & \big\downarrow{\scriptstyle \gamma'} \\
\mathcal{L} & \xrightarrow{\mathfrak{t}} & \mathcal{M}
\end{array}$$

whose vertical arrows belong to $\Sigma_{\mathfrak{u}^*}$ and, in addition, the upper vertical arrows, σ and γ are deflations. Using the fact that ξ and $\mathfrak{t}$ are deflations, we can form two cartesian squares

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{\xi} & \mathcal{M} \\ {\scriptstyle\widetilde{\gamma}}\uparrow & cart & \uparrow{\scriptstyle\gamma} \\ \mathfrak{L} & \xrightarrow{\xi'} & \mathcal{M}' \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathfrak{L}' & \xrightarrow{\mathfrak{t}'} & \mathcal{M}' \\ {\scriptstyle\widetilde{\gamma}'}\downarrow & cart & \downarrow{\scriptstyle\gamma'} \\ \mathcal{L} & \xrightarrow{\mathfrak{t}} & \mathcal{M} \end{array}$$

whose vertical arrows belong to $\Sigma_{\mathfrak{u}^*}$ and, besides, all arrows of the left square are deflations.

The arrow $\mathfrak{L}' \xrightarrow{\mathfrak{t}'} \mathcal{M}'$ belongs to $\mathfrak{T} \cap \Sigma^\perp$, because $\mathfrak{t} \in \mathfrak{T} \cap \Sigma^\perp$ and both $\mathfrak{T}$ and $\Sigma^\perp$ are base change invariant. One can see that the arrows $\mathfrak{u}^*(\xi')$ and $\mathfrak{u}^*(\mathfrak{t}')$ are isomorphic. By hypothesis, there exists a pull-back $\widetilde{\mathfrak{L}}' \xrightarrow{\widetilde{\mathfrak{t}}'} \widetilde{\mathcal{M}}' \xleftarrow{\widetilde{\xi}} \widetilde{\mathfrak{L}}$ of these two arrows along some morphism $\widetilde{\mathcal{M}}' \longrightarrow \mathcal{M}'$ and an isomorphism between them which can be represented by a commutative diagram

$$\begin{array}{ccc} \mathfrak{L}'' & \xrightarrow{\mathfrak{s}} & \widetilde{\mathfrak{L}} \\ {\scriptstyle\mathfrak{s}'}\downarrow & & \downarrow{\scriptstyle\widetilde{\xi}} \\ \widetilde{\mathfrak{L}}' & \xrightarrow{\widetilde{\mathfrak{t}}'} & \widetilde{\mathcal{M}}' \end{array} \tag{4}$$

whose upper horizontal and left vertical arrows belong to $\Sigma_{\mathfrak{u}^*}$ and both horizontal and the right vertical arrows are deflations. In particular, there exists a kernel pair

$$Ker_2(\mathfrak{s}) = \widetilde{\mathfrak{L}} \prod_{\mathfrak{s},\mathfrak{s}} \widetilde{\mathfrak{L}} \underset{p_2}{\overset{p_1}{\rightrightarrows}} \widetilde{\mathfrak{L}}.$$

of the morphism $\widehat{\mathfrak{L}} \xrightarrow{\mathfrak{s}} \mathfrak{L}$. Since $\mathfrak{s}$ is a deflation, there exists a cocartesian square

$$\begin{array}{ccc} \widetilde{\mathfrak{L}} & \xrightarrow{\mathfrak{s}} & \mathfrak{L} \\ {\scriptstyle\mathfrak{s}'}\downarrow & cocart & \downarrow{\scriptstyle\mathfrak{s}''} \\ \mathfrak{L}' & \xrightarrow{\mathfrak{e}} & \mathfrak{M}' \end{array}$$

whose arrows belong to $\Sigma_{\mathfrak{u}^*}$. It is easy to see that the arrow $\mathfrak{L}' \xrightarrow{\mathfrak{e}} \mathfrak{M}'$ is the cokernel of the pair

$$Ker_2(\mathfrak{s}) \underset{\mathfrak{s}'p_2}{\overset{\mathfrak{s}'p_1}{\rightrightarrows}} \mathfrak{L}'.$$

It follows from the commutativity of the diagram (4) that $\widetilde{\mathfrak{t}}' = \bar{\mathfrak{t}} \circ \mathfrak{e}$ and $\widetilde{\xi} = \bar{\mathfrak{t}} \circ \mathfrak{s}''$ for a uniquely determined morphism $\mathfrak{M}' \xrightarrow{\bar{\mathfrak{t}}} \mathcal{M}'$. Since $\mathfrak{t}' \in \mathfrak{T}$ and the system $\mathfrak{T}$ is left divisible in $\mathfrak{E}_X$, the morphism $\bar{\mathfrak{t}}$ belongs to $\mathfrak{T}$. This shows that an appropriate pull-back of the morphism ξ belongs to $\mathfrak{T} \circ \Sigma_{\mathfrak{u}^*}$, that is $\xi \in (\mathfrak{T} \circ \Sigma_{\mathfrak{u}^*})^{\bar{\wedge}}$. So that we obtained the inclusion ${\mathfrak{u}^*}^{-1}([\mathfrak{u}^*(\mathfrak{T})]) \subseteq (\mathfrak{T} \circ \Sigma_{\mathfrak{u}^*})^{\bar{\wedge}}$ which implies the inclusion

$$\mathfrak{E}_X \cap \big({\mathfrak{u}^*}^{-1}([\mathfrak{u}^*(\mathfrak{T})])\big) \cap \Sigma^{\perp} \subseteq \mathfrak{E}_X \cap \big((\mathfrak{T} \circ \Sigma_{\mathfrak{u}^*})^{\bar{\wedge}}\big) \cap \Sigma^{\perp}. \tag{5}$$

By 2.3.1, $\Sigma^{\perp} = (\Sigma^{\perp})^{\bar{\wedge}}$. Therefore, we have

$$\begin{aligned}&\mathfrak{E}_X \cap (\mathfrak{T} \circ \Sigma_{\mathfrak{u}^*})^{\bar{\wedge}}\big) \cap \Sigma^{\perp} = \mathfrak{E}_X \cap (\mathfrak{T} \circ \Sigma_{\mathfrak{u}^*})^{\bar{\wedge}} \cap (\Sigma^{\perp})^{\bar{\wedge}} = \\ &\mathfrak{E}_X \cap ((\mathfrak{T} \circ \Sigma_{\mathfrak{u}^*}) \cap \Sigma^{\perp})^{\bar{\wedge}} = \mathfrak{E}_X \cap ((\mathfrak{T} \circ \Sigma_{\mathfrak{u}^*}) \cap \Sigma^{\perp})^{\bar{\wedge}}.\end{aligned}$$

Since $\Sigma^{\perp}$ is right divisible,

$$(\mathfrak{T} \circ \Sigma_{\mathfrak{u}^*}) \cap \Sigma^{\perp} = \mathfrak{T} \circ (\Sigma_{\mathfrak{u}^*} \cap \Sigma^{\perp}) \cap \Sigma^{\perp}. \tag{6}$$

By hypothesis, $\Sigma_{\mathfrak{u}^*} \subseteq (\Sigma_{\mathfrak{u}^*} \cap \mathfrak{E}_X)^{\bar{\wedge}}$. Therefore,

$$\Sigma_{\mathfrak{u}^*} \cap \Sigma^{\perp} \subseteq (\Sigma_{\mathfrak{u}^*} \cap \mathfrak{E}_X)^{\bar{\wedge}} \cap \Sigma^{\perp} \subseteq \Sigma^{\bar{\wedge}} \cap \Sigma^{\perp} \subseteq (\Sigma \cap \Sigma^{\perp})^{\bar{\wedge}} = \mathcal{W}_X^{\bar{\wedge}}. \tag{7}$$

The last inclusion, $\Sigma^{\bar{\wedge}} \cap \Sigma^{\perp} \subseteq (\Sigma \cap \Sigma^{\perp})^{\bar{\wedge}}$, is due to the fact that if $\mathfrak{s}$ is an element of the intersection $\Sigma^{\bar{\wedge}} \cap \Sigma^{\perp}$, then some pull-back of $\mathfrak{s}$ is an element of $\Sigma \cap \Sigma^{\perp} = \mathcal{W}_X$.

Applying the inclusion $\Sigma_{\mathfrak{u}^*} \cap \Sigma^{\perp} \subseteq \mathcal{W}_X^{\bar{\wedge}}$ from (7) to (6), we obtain the inclusion

$$(\mathfrak{T} \circ \Sigma_{\mathfrak{u}^*}) \cap \Sigma^{\perp} \subseteq \big(\mathfrak{T} \circ \mathcal{W}_X^{\bar{\wedge}}\big) \cap \Sigma^{\perp}. \tag{8}$$

It follows from (8) that

$$(\mathfrak{T} \circ \Sigma_{\mathfrak{u}^*})\big) \cap \Sigma^{\perp} \subseteq (\mathfrak{T} \circ \mathcal{W}_X^{\bar{\wedge}}) \cap \Sigma^{\perp}. \tag{9}$$

Combining all above (starting with (5)) and using the stability of $\mathfrak{T}$ and the equality $\mathfrak{E}_X \cap (\mathcal{T} \circ \mathcal{W}_X^{\bar{\wedge}}) = \mathcal{T}$, we obtain

$$\begin{aligned}&\mathfrak{E}_X \cap \big({\mathfrak{u}^*}^{-1}([\mathfrak{u}^*(\mathfrak{T})])\big) \cap \Sigma^{\perp} \subseteq \mathfrak{E}_X \cap (\mathcal{T} \circ \mathcal{W}_X^{\bar{\wedge}})^{\bar{\wedge}} \cap \Sigma^{\perp} = \\ &\mathfrak{E}_X \cap \mathfrak{E}_X^{\bar{\wedge}} \cap (\mathcal{T} \circ \mathcal{W}_X^{\bar{\wedge}})^{\bar{\wedge}} \cap \Sigma^{\perp} = \mathfrak{E}_X \cap (\mathfrak{E}_X \cap (\mathcal{T} \circ \mathcal{W}_X^{\bar{\wedge}}))^{\bar{\wedge}} \cap \Sigma^{\perp} = \\ &(\mathfrak{E}_X \cap \mathcal{T}^{\bar{\wedge}}) \cap \Sigma^{\perp} = \mathcal{T} \cap \Sigma^{\perp}\end{aligned}$$

whence the equality (2). ■

6.2. Proposition. *Let* $(C_X, \bar{\mathfrak{E}}_X) \xrightarrow{\mathfrak{u}^*} (C_U, \bar{\mathfrak{E}}_U)$ *be a strongly 'exact' localization satisfying the conditions (1)–(3) of 6.1 and such that* $\mathfrak{E}_X \cap \Sigma_{\mathfrak{u}^*}$ *is a stable system.*

Then, for any $\mathcal{Q} \in \mathbf{Spec}^{1,1}_{\mathfrak{sc}}(X, \bar{\mathfrak{E}}_X)$ *such that* $\Sigma_{\mathfrak{u}^*} \cap \mathfrak{E}_X \subseteq \mathcal{Q}$*, the system* $[\mathfrak{u}^*(\mathcal{Q})]$ *belongs to the spectrum* $\mathbf{Spec}^{1,1}_{\mathfrak{sc}}(U, \bar{\mathfrak{E}}_U)$*. If* $\mathcal{Q} \in \mathbf{Spec}^{1,1}_{\mathfrak{st}}(X, \bar{\mathfrak{E}}_X)$*, then* $[\mathfrak{u}^*(\mathcal{Q})]$ *belongs to the spectrum* $\mathbf{Spec}^{1,1}_{\mathfrak{st}}(U, \bar{\mathfrak{E}}_U)$*.*

Proof. $\mathcal{Q} \in \mathbf{Spec}^{1,1}_{\mathfrak{sc}}(X, \bar{\mathfrak{E}}_X)$ and $\mathfrak{E}_X \cap \Sigma_{\mathfrak{u}^*} \subseteq \mathcal{Q}$. Let $\mathbb{T}$ be a strongly stable thick system of deflations of $(C_U, \bar{\mathfrak{E}}_U)$ properly containing $[\mathfrak{u}^*(\mathcal{Q})]$. Since $\mathfrak{E}_U = [\mathfrak{u}^*(\mathfrak{E}_X)]$, this means precisely that ${\mathfrak{u}^*}^{-1}(\mathbb{T})$ contains $\mathcal{Q}$ properly, hence it contains $\mathcal{Q}^{\mathfrak{sc}}$. So that $[\mathfrak{u}^*(\mathcal{Q}^{\mathfrak{sc}})] \subseteq \mathbb{T}$. On the other hand, $[\mathfrak{u}^*(\mathcal{Q})] \subsetneq [\mathfrak{u}^*(\mathcal{Q}^{\mathfrak{sc}})]$, because, by 6.1,

$$\mathfrak{E}_X \cap {\mathfrak{u}^*}^{-1}([\mathfrak{u}^*(\mathfrak{Q} \cap \mathcal{Q}^{\perp})]) \cap \mathcal{Q}^{\perp} = \mathcal{Q} \cap \mathcal{Q}^{\perp} = \mathcal{W}_X,$$

while

$$\mathfrak{E}_X \cap {\mathfrak{u}^*}^{-1}([\mathfrak{u}^*(\mathfrak{Q}^{\mathfrak{sc}} \cap \mathcal{Q}^{\perp})]) \cap \mathcal{Q}^{\perp} = \mathcal{Q}^{\mathfrak{sc}} \cap \mathcal{Q}^{\perp} \stackrel{\text{def}}{=} \mathcal{Q}_{\mathfrak{sc}}$$

and, since $\mathcal{Q} \in \mathbf{Spec}^{1,1}_{\mathfrak{sc}}(X, \bar{\mathfrak{E}}_X)$, the system $\mathcal{Q}_{\mathfrak{sc}}$ is non-trivial.

Same argument (with $\mathbb{T}$ a semitopological system) shows that $[\mathfrak{u}^*(\mathcal{Q})] \in \mathbf{Spec}^{1,1}_{\mathfrak{st}}(U, \bar{\mathfrak{E}}_U)$ for any $\mathcal{Q} \in \mathbf{Spec}^{1,1}_{\mathfrak{st}}(X, \bar{\mathfrak{E}}_X)$. ∎

6.3. Covers. We call a set $\{(U_i, \bar{\mathfrak{E}}_{U_i}) \xrightarrow{\mathfrak{u}_i} (U, \bar{\mathfrak{E}}_U) \mid i \in J\}$ of 'exact' localizations a *cover* of $(X, \bar{\mathfrak{E}}_X)$ if $\mathfrak{E}_X \cap \big(\bigcap_{i \in J} \Sigma_{\mathfrak{u}_i^*}\big) = \mathcal{W}_X$. Below we consider only covers which have finite subcovers whose elements satisfy the conditions of 6.1.

6.4. Proposition. *Let* $\mathfrak{U} = \{(U_i, \bar{\mathfrak{E}}_{U_i}) \xrightarrow{\mathfrak{u}_i} (X, \bar{\mathfrak{E}}_X) \mid i \in J\}$ *be a cover of the right exact 'space'* $(X, \bar{\mathfrak{E}}_X)$ *by strongly 'exact' localizations which has a finite subcover* $\mathfrak{U} = \{(U_i, \bar{\mathfrak{E}}_{U_i}) \xrightarrow{\mathfrak{u}_i} (X, \bar{\mathfrak{E}}_X) \mid i \in \mathfrak{I}\}$ *with the following properties:*

(1) $\Sigma_{\mathfrak{u}_i^*} \subseteq (\Sigma_{\mathfrak{u}_i^*} \cap \mathfrak{E}_X)^{\wedge}$,

(2) $\Sigma_{\mathfrak{u}_i^*}$ *is closed under push-forwards of deflations along arrows of* $\Sigma_{\mathfrak{u}_i^*}$,

(3) If $\mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L} \xleftarrow{\mathfrak{t}} \mathcal{N}$ *are deflations such that the arrows* $\mathfrak{u}_i^*(\mathfrak{e}) = \mathfrak{u}_i^*(\mathfrak{t}) \circ \phi$ *for some isomorphism* ϕ*, then there exists a pull-back* $\widetilde{\mathcal{M}} \xrightarrow{\widetilde{\mathfrak{e}}} \widetilde{\mathcal{L}} \xleftarrow{\widetilde{\mathfrak{t}}} \widetilde{\mathcal{N}}$ *of these arrows along some morphism* $\widetilde{\mathcal{L}} \longrightarrow \mathcal{L}$ *and a commutative diagram*

$$\begin{array}{ccc}
\mathfrak{M} & \xrightarrow{\mathfrak{s}} & \widetilde{\mathcal{M}} \\
{\scriptstyle \mathfrak{s}'}\big\downarrow & & \big\downarrow{\scriptstyle \widetilde{\mathfrak{e}}} \\
\widetilde{\mathcal{N}} & \xrightarrow{\widetilde{\mathfrak{t}}} & \widetilde{\mathcal{L}}
\end{array}$$

where $\mathfrak{s}$, $\mathfrak{s}'$ *are arrows of* $\Sigma_{\mathfrak{u}_i^*}$ *and* $\mathfrak{s}$ *is a deflation.*

Then the following conditions on a Serre system Σ *of deflations of* $(C_X, \bar{\mathfrak{E}}_X)$ *are equivalent:*

(a) $\Sigma \in \mathbf{Spec}^{1,1}_{\mathfrak{st}}(X, \bar{\mathfrak{E}}_X)$*,*

(b) $\Sigma \in \mathbf{Spec}^{1,1}_{\mathfrak{tc}}(X, \bar{\mathfrak{E}}_X)$ *and* $[\mathfrak{u}_i^*(\Sigma)] \in \mathbf{Spec}^{1,1}_{\mathfrak{st}}(U_i, \bar{\mathfrak{E}}_{U_i})$ *whenever* $\mathfrak{E}_X \cap \Sigma_{\mathfrak{u}_i^*} \subseteq \Sigma$*.*

Proof. The implication $(a) \Rightarrow (b)$ follows from 6.2.

$(b) \Rightarrow (a)$. Fix a finite subcover $\mathfrak{U}_{\mathfrak{J}} = \{(U_i, \bar{\mathfrak{E}}_{U_i}) \xrightarrow{\mathfrak{u}_i} (X, \bar{\mathfrak{E}}_X) \mid i \in \mathfrak{J}\}$ of the cover $\mathfrak{U}$. Set $\mathfrak{J}_\Sigma = \{j \in \mathfrak{J} \mid \mathfrak{E}_{X,\mathfrak{u}_j^*} \subseteq \Sigma\}$. Let Σ be an element of $\mathbf{Spec}^{1,1}_{\mathfrak{M}}(X, \bar{\mathfrak{E}}_X)$ such that $[\mathfrak{u}_i^*(\Sigma)] \in \mathbf{Spec}^{1,1}_{\mathfrak{t}}(U_i, \bar{\mathfrak{E}}_{U_i})$ for every $i \in \mathfrak{J}_\Sigma$. The claim is that $\Sigma \in \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X, \bar{\mathfrak{E}}_X)$.

For every $i \in \mathfrak{J}_\Sigma$, we denote by $\widetilde{\mathcal{S}}_i$ the intersection $\mathfrak{E}_X \cap {\mathfrak{u}_i^*}^{-1}([\mathfrak{u}_i^*(\Sigma)]^*) \cap \Sigma^\perp$.

Recall that $\Sigma^\perp$ is the largest right divisible system having the trivial intersection with Σ (cf. 2.2). Since Σ is a Serre system of deflations, the right divisible system of deflations $\widetilde{\mathcal{S}}_i$ is non-trivial, and $\widetilde{\mathcal{S}}_i \nsubseteq \Sigma$. By 4.4.1, this implies that $\widetilde{\mathcal{S}} = \bigcap\limits_{i \in \mathfrak{J}_\Sigma} \widetilde{\mathcal{S}}_i$ is not contained in Σ. Since $\widetilde{\mathcal{S}} \subseteq \Sigma^\perp$, this means precisely that $\widetilde{\mathcal{S}}$ is a non-trivial system.

We consider each of the two cases: $\mathfrak{J}_\Sigma = \mathfrak{J}$ and $\mathfrak{J}_\Sigma \neq \mathfrak{J}$.

(i) Suppose that $\mathfrak{J}_\Sigma = \mathfrak{J}$. Set $\widetilde{\mathcal{S}} = \bigcap\limits_{i \in \mathfrak{J}_\Sigma} \widetilde{\mathcal{S}}_i$. The claim is that $\langle \widetilde{\mathcal{S}} \rangle = \Sigma$ which implies that $\Sigma \in \mathbf{Spec}^{1,1}_{\mathfrak{st}}(X, \bar{\mathfrak{E}}_X)$. The equality $\langle \widetilde{\mathcal{S}} \rangle = \Sigma$ means precisely that if $\mathfrak{T}$ is a semitopologizing system of deflations of $(X, \bar{\mathfrak{E}}_X)$ such that $\widetilde{\mathcal{S}} \nsubseteq \mathfrak{T}$, then $\mathfrak{T} \subseteq \Sigma$.

Since $\widetilde{\mathcal{S}} \subseteq \Sigma^\perp$, the fact $\widetilde{\mathcal{S}} \nsubseteq \mathfrak{T}$ is equivalent to $\widetilde{\mathcal{S}} \nsubseteq \mathfrak{T} \cap \Sigma^\perp$.

It follows from 6.1 that

$$
\begin{aligned}
&\mathfrak{T} \cap \Sigma^\perp = \mathfrak{E}_X \cap {\mathfrak{u}_i^*}^{-1}([\mathfrak{u}_i^*(\mathfrak{T} \cap \Sigma^\perp)]) \cap \Sigma^\perp \quad \text{and} \\
&\widetilde{\mathcal{S}} = \widetilde{\mathcal{S}} \cap \Sigma^\perp = \mathfrak{E}_X \cap {\mathfrak{u}_i^*}^{-1}([\mathfrak{u}_i^*(\widetilde{\mathcal{S}})]) \cap \Sigma^\perp
\end{aligned} \tag{2}
$$

for every $i \in \mathfrak{J}$. The equality (2) implies that if $\widetilde{\mathfrak{S}} \nsubseteq \mathfrak{T} \cap \Sigma^\perp$, then $[\mathfrak{u}_i^*(\widetilde{\mathcal{S}})] \nsubseteq [\mathfrak{u}_i^*(\mathfrak{T})]$. But, then $[\mathfrak{u}_i^*(\mathfrak{T})] \subseteq [\mathfrak{u}_i^*(\Sigma)]$, whence $\mathfrak{T} \subseteq {\mathfrak{u}_i^*}^{-1}([\mathfrak{u}_i^*(\Sigma)]) \cap \mathfrak{E}_X = \Sigma$.

(ii) Suppose now that $\mathfrak{J}_\Sigma \neq \mathfrak{J}$. Set $\mathfrak{J}^\Sigma = \mathfrak{J} - \mathfrak{J}_\Sigma$ and $\mathfrak{E}_X^\Sigma = \bigcap\limits_{i \in \mathfrak{J}^\Sigma} \mathfrak{E}_{X,\mathfrak{u}_i^*}$. Since, by the definition of $\mathfrak{J}^\Sigma$, $\mathfrak{E}_{X,\mathfrak{u}_i^*} \nsubseteq \Sigma$ for all $i \in \mathfrak{J}^\Sigma$, it follows from 4.4.1 that $\mathfrak{E}_X^\Sigma \nsubseteq \Sigma$.

Set $\mathfrak{S} = \widetilde{\mathcal{S}} \cap \mathfrak{E}_X^\Sigma$. The claim is that $\langle \mathfrak{S} \rangle = \Sigma$.

Indeed, if $\mathfrak{T}$ is a semitopologizing system of deflations of $(X, \bar{\mathfrak{E}}_X)$ such that $\mathfrak{S} \nsubseteq \mathfrak{T}$, then it follows from the argument (i) above that $[\mathfrak{u}_i^*(\mathfrak{S})] \nsubseteq [\mathfrak{u}_i^*(\mathfrak{T})]$ for some $i \in \mathfrak{J}$. Notice that this i belongs to $\mathfrak{J}_\Sigma$, because $\mathfrak{S} \subseteq \mathfrak{E}_X^\Sigma$, hence $\mathfrak{u}_i^*(\mathfrak{S}) \subseteq Iso(C_{U_i}) \subseteq [\mathfrak{u}_i^*(\mathfrak{T})]$ for every $i \in \mathfrak{J}^\Sigma$. Therefore, the end of the argument of (i) applies. ■

Similar fact (but, with additional assumptions) holds for the spectrum $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X, \bar{\mathfrak{E}}_X)$.

6.5. Proposition. *Let $(C_X,\bar{\mathfrak{E}}_X) = (C_X,\mathfrak{E}_X,\mathcal{W}_X)$ be a svelte right exact category with a stable class of weak equivalences (that is $\mathcal{W}_X = \mathcal{W}_X^{\bar{\wedge}} \cap \mathfrak{E}_X$) and a left divisible and weakly right divisible class of deflations (conditions 2.0(c) and 2.0(d)). Let $\mathfrak{U} = \{(U_i,\bar{\mathfrak{E}}_{U_i}) \xrightarrow{\mathfrak{u}_i} (X,\bar{\mathfrak{E}}_X) \mid i \in J\}$ be a cover of the right exact 'space' $(X,\bar{\mathfrak{E}}_X)$ by strongly 'exact' localizations which has a finite subcover $\mathfrak{U} = \{(U_i,\bar{\mathfrak{E}}_{U_i}) \xrightarrow{\mathfrak{u}_i} (X,\bar{\mathfrak{E}}_X) \mid i \in \mathfrak{I}\}$ with the following properties:*

(1) $\Sigma_{\mathfrak{u}_i^*} \subseteq (\Sigma_{\mathfrak{u}_i^*} \cap \mathfrak{E}_X)^{\bar{\wedge}}$,

(2) $\Sigma_{\mathfrak{u}_i^*}$ is closed under push-forwards of deflations along arrows of $\Sigma_{\mathfrak{u}_i^*}$,

(3) If $\mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L} \xleftarrow{\mathfrak{t}} \mathcal{N}$ are deflations such that the arrows $\mathfrak{u}_i^*(\mathfrak{e}) = \mathfrak{u}_i^*(\mathfrak{t}) \circ \phi$ for some isomorphism ϕ, then there exists a pull-back $\widetilde{\mathcal{M}} \xrightarrow{\widetilde{\mathfrak{e}}} \widetilde{\mathcal{L}} \xleftarrow{\widetilde{\mathfrak{t}}} \widetilde{\mathcal{N}}$ of these arrows along some morphism $\widetilde{\mathcal{L}} \longrightarrow \mathcal{L}$ and a commutative diagram

$$\begin{array}{ccc}
\mathfrak{M} & \xrightarrow{\ \mathfrak{s}\ } & \widetilde{\mathcal{M}} \\
{\scriptstyle \mathfrak{s}'}\big\downarrow & & \big\downarrow{\scriptstyle \widetilde{\mathfrak{e}}} \\
\widetilde{\mathcal{N}} & \xrightarrow{\ \widetilde{\mathfrak{t}}\ } & \widetilde{\mathcal{L}}
\end{array}$$

where $\mathfrak{s}$, $\mathfrak{s}'$ are arrows of $\Sigma_{\mathfrak{u}_i^*}$ and $\mathfrak{s}$ is a deflation.

(4) The functors $\mathfrak{u}_i^*$ preserve push-forwards of deflations.

Then the following conditions on a Serre system Σ of deflations of $(C_X,\bar{\mathfrak{E}}_X)$ are equivalent:

(a) $\Sigma \in \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X,\bar{\mathfrak{E}}_X)$,

(b) $\Sigma \in \mathbf{Spec}^{1,1}_{\mathfrak{M}}(X,\bar{\mathfrak{E}}_X)$ and $[\mathfrak{u}_i^*(\Sigma)] \in \mathbf{Spec}^{1,1}_{\mathfrak{t}}(U_i,\bar{\mathfrak{E}}_{U_i})$ whenever $\mathfrak{E}_X \cap \Sigma_{\mathfrak{u}_i^*} \subseteq \Sigma$.

Proof. The argument is similar to that of 6.4. Details are left to the reader. ∎

6.6. Comments about the conditions on localizations. In the assertions of this section, we consider strongly 'exact' localizations $(C_X,\bar{\mathfrak{E}}_X) \xrightarrow{\mathfrak{u}^*} (C_U,\bar{\mathfrak{E}}_U)$ such that $\Sigma_{\mathfrak{u}^*} \cap \mathfrak{E}_X$ is stable and the following properties hold:

(1) $\Sigma_{\mathfrak{u}^*} \subseteq (\Sigma_{\mathfrak{u}^*} \cap \mathfrak{E}_X)^{\bar{\wedge}}$,

(2) $\Sigma_{\mathfrak{u}^*}$ is closed under push-forwards of deflations along arrows of $\Sigma_{\mathfrak{u}^*}$,

(3) If $\mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L} \xleftarrow{\mathfrak{t}} \mathcal{N}$ are deflations such that $\mathfrak{u}^*(\mathfrak{e}) = \mathfrak{u}^*(\mathfrak{t}) \circ \phi$ for some isomorphism ϕ, then there exists a pull-back $\widetilde{\mathcal{M}} \xrightarrow{\widetilde{\mathfrak{e}}} \widetilde{\mathcal{L}} \xleftarrow{\widetilde{\mathfrak{t}}} \widetilde{\mathcal{N}}$ of these arrows along some morphism $\widetilde{\mathcal{L}} \longrightarrow \mathcal{L}$ and a commutative diagram

$$\begin{array}{ccc}
\mathfrak{M} & \xrightarrow{\ \mathfrak{s}\ } & \widetilde{\mathcal{M}} \\
{\scriptstyle \mathfrak{s}'}\big\downarrow & & \big\downarrow{\scriptstyle \widetilde{\mathfrak{e}}} \\
\widetilde{\mathcal{N}} & \xrightarrow{\ \widetilde{\mathfrak{t}}\ } & \widetilde{\mathcal{L}}
\end{array}$$

where $\mathfrak{s}$, $\mathfrak{s}'$ are arrows of $\Sigma_{\mathfrak{u}^*}$ and $\mathfrak{s}$ is a deflation.

(1) The condition (1) in combination with the stability of $\mathfrak{E}_{X,\mathfrak{u}^*} = \Sigma_{\mathfrak{u}^*} \cap \mathfrak{E}_X$ implies that the system of deflations $\mathfrak{E}_{X,\mathfrak{u}^*}$ is *saturated* (cf. 5.5).

(2) The condition (2) holds if $\Sigma_{\mathfrak{u}^*}$ is closed under taking cokernels of pairs of arrows $\mathcal{M} \overset{\mathfrak{t}_1}{\underset{\mathfrak{t}_2}{\rightrightarrows}} \mathcal{N}$ such that $\mathfrak{u}^*(\mathfrak{t}_1) = \mathfrak{u}^*(\mathfrak{t}_2)$ (see the argument of 6.1).

The condition (2) holds, if the functor $\mathfrak{u}^*$ preserves push-forwards of deflations.

(3) It follows from the condition (1) that there exists a pull-back $\widetilde{\mathcal{M}} \xrightarrow{\widetilde{\mathfrak{e}}} \widetilde{\mathcal{L}} \xleftarrow{\widetilde{\mathfrak{t}}} \widetilde{\mathcal{N}}$ of the deflations $\mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L} \xleftarrow{\mathfrak{t}} \mathcal{N}$ along some morphism $\widetilde{\mathcal{L}} \longrightarrow \mathcal{L}$ such that the isomorphism ϕ is described by a diagram

$$\begin{array}{ccc} \mathfrak{M} & \xrightarrow{\ \mathfrak{s}\ } & \widetilde{\mathcal{M}} \\ {\scriptstyle \mathfrak{s}'}\downarrow & & \downarrow{\scriptstyle \widetilde{\mathfrak{e}}} \\ \widetilde{\mathcal{N}} & \xrightarrow{\ \widetilde{\mathfrak{t}}\ } & \widetilde{\mathcal{L}} \end{array}$$

whose upper horizontal and left vertical arrows belong to $\Sigma_{\mathfrak{u}^*}$, which the functor $\mathfrak{u}^*$ transforms to a commutative diagram. The condition (3) holds for sure if the category C_X is pointed (or, more generally, C_X has initial objects and, for any object of C_X, there is at most one morphism to an initial object): it suffices to take a pull-back of the square above along the unique arrow $\mathfrak{x} \longrightarrow \widetilde{\mathfrak{L}}$ from an initial object.

Notice that the conditions (1), (2), (3) stand finite intersections. So that one talk about covers and the corresponding pretopology. We shall not go into details of this here.

7. Spectra of right exact 'spaces' over a point.

We start with 'spaces' represented by right exact categories with stable class of weak equivalences and initial objects and gather together different facts and observations scattered in the previous sections.

7.0. Right exact 'spaces' over a point. A "point", $\mathbf{x}$, is represented by the *trivial* right exact category, that is the category $C_{\mathbf{x}}$ with only one (hence identical) arrow.

A right exact 'space' over a point $\mathbf{x}$ is a pair $((X, \bar{\mathfrak{E}}_X), \gamma)$, where γ is a *continuous* morphism $(X, \bar{\mathfrak{E}}_X) \longrightarrow \mathbf{x}$. Right exact 'spaces' over the point $\mathbf{x}$ form a category in a standard way: morphisms from $((X, \bar{\mathfrak{E}}_X), \gamma)$ to $(Y, \bar{\mathfrak{E}}_Y), \widetilde{\gamma})$ are given by morphisms of 'spaces' $X \xrightarrow{\mathfrak{f}} Y$ whose inverse image functor $\mathfrak{f}^*$ is an 'exact' functor from $(C_Y, \bar{\mathfrak{E}}_Y)$ to $(C_X, \bar{\mathfrak{E}}_X)$ and such that $\widetilde{\gamma} \circ \mathfrak{f} = \gamma$, which means that $\mathfrak{f}^* \circ \widetilde{\gamma}^* \simeq \gamma^*$.

Recall that *continuous* means that an inverse image functor $C_{\mathbf{x}} \xrightarrow{\gamma^*} C_X$ of the morphism γ has a right adjoint. One can see that this condition means precisely that γ^* maps the unique object of the category $C_{\mathbf{x}}$ to an initial object of the category C_X. It follows that morphisms of right exact 'spaces' over a point are precisely those morphisms of right exact 'spaces' whose inverse image functor preserves initial objects.

7.0.1. Conventions. We fix a right exact 'space' $((X, \bar{\mathfrak{E}}_X), \gamma)$ over a point $\mathbf{x}$ together with an inverse image functor the morphism γ. The latter means that we fix an initial object $\mathfrak{x}$ of the category C_X. We assume that $(C_X, \bar{\mathfrak{E}}_X)$ has a stable class of weak equivalences and that all split epimorphisms of the category C_X are deflations. In particular, every morphism to an initial object is a deflation.

Since in a general right exact category deflations are not invariant under push-forwards, we look at the version of the spectral theory based on the notion of a *semitopological* system (see Sections 4 and 5). Fix an initial object $\mathfrak{x}$ of the category C_X.

7.1. Stable systems of deflations and subcategories of $C_X/\mathfrak{x}$. Following general pattern, we consider the correspondence which assigns to any class of deflations $\mathcal{S}$ of $(C_X, \bar{\mathfrak{E}}_X)$ the full subcategory $\mathbb{T}_{\mathcal{S}}$ of the category $C_X/\mathfrak{x}$ whose objects are pairs $(M, M \xrightarrow{\mathfrak{s}} \mathfrak{x})$ with $\mathfrak{s} \in \mathcal{S}$. In other words, $\mathbb{T}_{\mathcal{S}}$ is generated by the kernels of arrows of $\mathcal{S}$. Here by a kernel of a morphism $\mathcal{M} \xrightarrow{\mathfrak{f}} \mathcal{N}$ we understand the pair $(Ker(\mathfrak{f}), Ker(\mathfrak{f}) \to \mathfrak{x})$ (– an object of $C_X/\mathfrak{x}$), where $Ker(\mathfrak{e}) \to \mathfrak{x}$ is the canonical morphism.

The stability of a class $\mathcal{S}$ of deflations of $(C_X, \bar{\mathfrak{E}}_X)$ means that

$$\mathcal{S} = \{\mathfrak{s} \in \mathfrak{E}_X \mid Ker(\mathfrak{s}) \in \mathbb{T}_{\mathcal{S}}\}.$$

The correspondence $\mathcal{S} \longmapsto \mathbb{T}_{\mathcal{S}}$ establishes an isomorphism between the preorder of stable systems invariant under pull-backs and the preorder formed by strictly full subcategories of the category $C_X/\mathfrak{x}$ containing kernels of weak equivalences; in particular, they contain initial objects. The inverse maps assigns to a strictly full subcategory $\mathcal{T}$ of the category $C_X/\mathfrak{x}$ the class $\mathfrak{E}_X^{\mathcal{T}}$ of all deflations $\mathfrak{s}$ such that $Ker(\mathfrak{s}) \in \mathcal{T}$.

Given a strictly full subcategory $\mathcal{T}$ of the category $C_X/\mathfrak{x}$, let $\mathfrak{K}_X^{\mathcal{T}}$ denote the class of all arrows of C_X which have a kernel from $\mathcal{T}$. By definition, $\mathfrak{E}_X^{\mathcal{T}} = \mathfrak{E}_X \cap \mathfrak{K}_X^{\mathcal{T}}$. It follows that $(\mathfrak{E}_X^{\mathcal{T}})^{\bar{\wedge}} = \mathfrak{K}_X^{\mathcal{T}}$. So that if $\mathcal{S}$ is a stable system of deflations, then $\mathcal{S}^{\bar{\wedge}} = \mathfrak{K}_X^{\mathbb{T}_{\mathcal{S}}}$, i.e. $\mathcal{S}^{\bar{\wedge}}$ consists of all arrows of C_X whose kernel exists and belongs to $\mathbb{T}_{\mathcal{S}}$.

7.2. Cartesian closedness and divisibility. A stable system $\mathcal{S}$ of deflations of $(C_X, \bar{\mathfrak{E}}_X)$ is cartesian closed iff the corresponding full subcategory $\mathbb{T}_{\mathcal{S}}$ of the category $C_X/\mathfrak{x}$ is closed under finite products (taken in $C_X/\mathfrak{x}$).

A system $\mathcal{S}$ is *left divisible* iff for any $M \in Ob\mathbb{T}_{\mathcal{S}}$ and any deflation $M \longrightarrow N$ (in $C_X/\mathfrak{x}$), the object N belongs to $\mathbb{T}_{\mathcal{S}}$. A system $\mathcal{S}$ is *right divisible* if for any object M of $\mathbb{T}_{\mathcal{S}}$, the kernel of any deflation $M \xrightarrow{\mathfrak{e}} N$ belongs to $\mathbb{T}_{\mathcal{S}}$.

7.3. Strong stability. The class $\mathcal{W}_X^{\bar{\wedge}}$ contains all morphisms with trivial kernel, in particular, all monomorphisms. Therefore, the condition $\mathcal{S} = \mathfrak{E}_X \cap (\mathcal{S} \circ \mathcal{W}_X^{\bar{\wedge}})$ (which makes a difference between the strong stability and stability) implies that the subcategory $\mathbb{T}_{\mathcal{S}}$ is closed under taking arbitrary (not only "admissible") subobjects. If morphisms with trivial kernel are isomorphisms and all weak equivalences are isomorphisms, then the system $\mathcal{S}$ is strongly stable iff the corresponding subcategory is closed under taking arbitrary subobjects.

7.4. Semitopologizing systems and strongly topologizing subcategories. Summarizing all above, one can see that the map $\mathcal{S} \longmapsto \mathbb{T}_{\mathcal{S}}$ induces an isomorphism between the preorder of semitopologizing systems of deflations of $(C_X, \bar{\mathfrak{E}}_X)$ and the preorder of full subcategories $\mathcal{T}$ of C_X which are closed under finite products and subobjects (taken in C_X) and such that for any deflation $\mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{N}$ with $\mathcal{M} \in Ob\mathcal{T}$, the object $\mathcal{N}$ belongs to $\mathcal{T}$. We call such subcategories *strongly topologizing*.

7.4.1. Note. We use here *strongly topologizing*, because the name "topologizing subcategories" was given (years ago) to the most straightforward generalization of this notion for exact categories [R, Ch.5]. We recall it for completeness: a subcategory $\mathcal{T}$ of an exact category is called *topologizing* if it is closed under finite products and for any deflation $\mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{N}$ with $\mathcal{M} \in Ob\mathcal{T}$, both N and $Ker(\mathfrak{e})$ are objects of the subcategory $\mathcal{T}$.

7.5. Thick systems and thick subcategories. A system $\mathcal{S}$ of deflations of $(C_X, \bar{\mathfrak{E}}_X)$ is thick iff the corresponding subcategory $\mathbb{T}_{\mathcal{S}}$ is *thick* in the most expected, ordinary sense: if $M \xrightarrow{\mathfrak{e}} N$ is a deflation in $C_X/\mathfrak{x}$, then M is an object of $\mathbb{T}_{\mathcal{S}}$ iff both N and $Ker(\mathfrak{e})$ are objects of $\mathbb{T}_{\mathcal{S}}$. In other words, the subcategory $\mathbb{T}_{\mathcal{S}}$ is topologizing and closed under extensions.

7.6. Strongly thick systems and strongly thick subcategories. A system of deflations $\mathcal{S}$ is strongly thick iff the corresponding subcategory $\mathbb{T}_{\mathcal{S}}$ is strongly topologizing and closed under extensions; or, what is the same, strongly topologizing and thick.

7.7. Orthogonal complements. We call objects of the subcategory $\mathbb{T}_{\mathcal{W}_X}$ *trivial*. If $\mathcal{W}_X$ consists of arrows with trivial kernel, then objects of $\mathbb{T}_{\mathcal{W}_X}$ are pairs $(V, V \to \mathfrak{x})$, where V runs through initial objects of C_X (i.e. $V \to \mathfrak{x}$ is an isomorphism).

Let $\mathcal{T}$ be a strictly full subcategory of the category $C_X/\mathfrak{x}$ containing $\mathbb{T}_{\mathcal{W}_X}$. We denote by $\mathcal{T}^{\perp}$ the full subcategory of the category C_X generated by all objects $\mathcal{M}$ of C_X such that the kernel of a deflation $\mathcal{M} \longrightarrow \mathcal{N}$ belongs to $\mathcal{T}$ iff it is trivial. It follows that, for any stable system of deflations $\mathcal{S}$, its orthogonal complement $\mathcal{S}^{\perp}$ contains $\mathfrak{K}_X^{\mathbb{T}_{\mathcal{S}}^{\perp}}$ and is contained in $\mathfrak{K}_X^{\mathbb{T}_{\mathcal{S}}^{\perp}} \bigcup \{$morphisms of C_X without kernel$\}$.

In particular, if the category C_X has kernels of all morphisms, then $\mathcal{S}^{\perp} = \mathfrak{K}_X^{\mathbb{T}_{\mathcal{S}}^{\perp}}$.

7.8. Serre systems of deflations and Serre subcategories of $C_X/\mathfrak{x}$. For any subcategory $\mathcal{T}$ of the category $C_X/\mathfrak{x}$, let $\mathcal{T}^-$ denote the full subcategory of $C_X/\mathfrak{x}$ generated by all objects $\mathcal{M}$ having the following property: if $\mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{N}$ is a non-trivial deflation (that is $Ker(\mathfrak{e})$ is non-trivial), then there exists a non-trivial deflation $Ker(\mathfrak{e}) \xrightarrow{\xi} \mathcal{L}$ (in $C_X/\mathfrak{x}$) with $Ker(\xi) \in Ob\mathcal{T}$. We call a subcategory $\mathcal{T}$ of $C_X/\mathfrak{x}$ a *Serre* subcategory if $\mathcal{T} = \mathcal{T}^-$.

There is the equality $(\mathfrak{E}_{X,\mathcal{T}})^- = \mathfrak{E}_{X,\mathcal{T}^-}$.

In particular, a system of deflations $\mathcal{S}$ of Σ of deflations of $(C_X, \bar{\mathfrak{E}}_X)$ is a Serre system iff it is stable and $\mathbb{T}_{\mathcal{S}} = \mathbb{T}_{\mathcal{S}}^-$. This establishes an isomorphism between the preorders of Serre systems of deflations of $(C_X, \bar{\mathfrak{E}}_X)$ and Serre subcategories of $C_X/\mathfrak{x}$.

7.9. Strongly closed systems of deflations and strongly closed subcategories of $C_X/\mathfrak{x}$. For any subcategory $\mathcal{T}$ of the category $C_X/\mathfrak{x}$, let $\mathcal{T}^\dagger$ denote the full subcategory of $C_X/\mathfrak{x}$ generated by objects $\mathcal{M} \in \mathcal{T}^-$ such that for any morphism $L \longrightarrow M$ from $\mathcal{W}_X^{\bar{\wedge}}$, the object L belongs to $\mathcal{T}^-$.

We call a subcategory $\mathcal{T}$ of $C_X/\mathfrak{x}$ *strongly closed* if $\mathcal{T} = \mathcal{T}^\dagger$.

It follows from 7.8 and the observation 4.3.3(a) that $(\mathfrak{E}_{X,\mathcal{T}})^\dagger = \mathfrak{E}_{X,\mathcal{T}^\dagger}$. In particular, a system of deflations $\mathcal{S}$ is strongly closed iff it is stable and $\mathbb{T}_{\mathcal{S}} = \mathbb{T}_{\mathcal{S}}^\dagger$.

7.10. Strongly 'exact' functors. An 'exact' functor $(C_X, \bar{\mathfrak{E}}_X) \xrightarrow{F} (C_Y, \bar{\mathfrak{E}}_Y)$ is *strongly 'exact'* iff it maps cartesian squares of the form

$$\begin{array}{ccc} Ker(\mathfrak{f}) & \longrightarrow & \mathfrak{x} \\ \downarrow & cart & \downarrow \\ \mathcal{M} & \xrightarrow{\mathfrak{f}} & \mathcal{N} \end{array} \qquad (1)$$

to cartesian squares. If F maps initial objects to initial objects, this condition means that F preserves kernels of arrows. Since localizations map initial objects to initial objecs, an 'exact' localization is strongly 'exact' iff it preserves kernels.

7.10.1. Remark. Since morphisms to initial objects in C_X are deflations, it follows from the diagram (1) that, for a strongly 'exact' functor $(C_X, \bar{\mathfrak{E}}_X) \xrightarrow{F} (C_Y, \bar{\mathfrak{E}}_Y)$, the class of arrows $\Sigma_F = \{\mathfrak{s} \in HomC_X \mid F(\mathfrak{s}) \text{ is invertible}\}$ is contained in $(\Sigma_F \cap \mathfrak{E}_X)^{\bar{\wedge}} = \mathfrak{E}_{X,F}^{\bar{\wedge}}$ iff all arrows of Σ_F have kernels. So that, in the case when all arrows of the category C_X have kernels, $\Sigma_F \subseteq \mathfrak{E}_{X,F}^{\bar{\wedge}}$ for any strongly 'exact' functor F.

7.10.2. Kernels of strongly 'exact' functors. Suppose that a strongly 'exact' functor $(C_X, \bar{\mathfrak{E}}_X) \xrightarrow{F} (C_Y, \bar{\mathfrak{E}}_Y)$ maps initial objects to initial objects. Then F induces a functor $(C_{X_\mathfrak{x}}, \bar{\mathfrak{E}}_{X_\mathfrak{x}}) \xrightarrow{F_\mathfrak{x}} (C_{Y_\mathfrak{y}}, \bar{\mathfrak{E}}_{Y_\mathfrak{y}})$, where $C_{X_\mathfrak{x}} = C_X/\mathfrak{x}$, $C_{Y_\mathfrak{y}} = C_Y/\mathfrak{y}$, $\mathfrak{y} = F(\mathfrak{x})$.

One can see that the subcategory $\mathbb{T}_{\mathfrak{E}_{X,F}}$ coincides with the kernel of the functor $C_X/\mathfrak{x} \xrightarrow{F_\mathfrak{x}} C_Y/\mathfrak{y}$, and the latter is naturally equivalent to the full subcategory of kernel of the functor F generated by all objects N of $Ker(F)$ having a morphism to $\mathfrak{x}$.

We denote the kernel of the functor $F_{\mathfrak{x}}$ by $\mathfrak{Ker}(F)$.

7.10.3. Covers by strongly 'exact' localizations. Elements of the covers we consider here are morphisms $(U, \bar{\mathfrak{E}}_U) \xrightarrow{\mathfrak{u}} (X, \bar{\mathfrak{E}}_X)$ whose inverse image functors are strongly 'exact' localizations such that all arrows of $\Sigma_{\mathfrak{u}^*}$ have kernels and the intersection $\mathfrak{E}_{X,\mathfrak{u}^*} = \mathfrak{E}_X \cap \Sigma_{\mathfrak{u}^*}$ is a stable system of deflations. We call a set

$$\{(U_i, \bar{\mathfrak{E}}_{U_i}) \xrightarrow{\mathfrak{u}_i} (X, \bar{\mathfrak{E}}_X) \mid i \in \mathfrak{I}\}$$

of such morphisms a *cover* of the right exact 'space' $(X, \bar{\mathfrak{E}}_X)$ if $\mathfrak{E}_X \cap \big(\bigcap_{i \in J} \Sigma_{\mathfrak{u}_i^*}\big) = \mathcal{W}_X$, or, equivalently, $\bigcap_{i \in J} \mathbb{T}_{\mathfrak{E}_{X,\mathfrak{u}_i^*}} = \mathbb{T}_{\mathcal{W}_X}$. Taking into consideration the discussion and notation of 7.10.2, we can rewrite the latter equality as $\bigcap_{i \in J} \mathfrak{Ker}(\mathfrak{u}_i^*) = \mathbb{T}_{\mathcal{W}_X}$. If the class $\mathcal{W}_X$ consists of deflations with a trivial kernel (which is a standard choice), then the trivial subcategory $\mathbb{T}_{\mathcal{W}_X}$ is trivial in the usual sense: all its objects are initial.

7.11. The spectra. For every subcategory $\mathcal{T}$ of the category $C_X/\mathfrak{x}$, we denote by $\mathcal{T}^\star$ the intersection of all strongly thick subcategories of $C_X/\mathfrak{x}$ which contain properly the subcategory $\mathcal{T}$. We denote by $\mathcal{T}_\star$ the intersection $\mathcal{T}^\star \cap \mathcal{T}^\perp$.

We denote by $\mathbf{Spec}^{1,1}_{\mathfrak{S}\mathfrak{C}}(X, \bar{\mathfrak{E}}_X)$ the preorder (with respect to the inverse inclusion) formed by all strongly closed subcategories $\mathcal{T}$ of the category $C_X/\mathfrak{x}$ for which $\mathcal{T}^\star \neq \mathcal{T}$, or, equivalently, the subcategory $\mathcal{T}_\star$ is non-trivial.

Similarly, we denote by $\mathcal{T}^*$ the intersection of all strongly topologizing subcategories of $C_X/\mathfrak{x}$ properly containing $\mathcal{T}$ and set $\mathcal{T}_* = \mathcal{T}^* \cap \mathcal{T}^\perp$. We denote by $\mathbf{Spec}^{1,1}_{\mathfrak{S}\mathfrak{T}}(X, \bar{\mathfrak{E}}_X)$ the subpreorder of $\mathbf{Spec}^{1,1}_{\mathfrak{S}\mathfrak{C}}(X, \bar{\mathfrak{E}}_X)$ formed by those strongly closed subcategories $\mathcal{T}$ of the category $C_X/\mathfrak{x}$ for which $\mathcal{T}^* \neq \mathcal{T}$, or, equivalently, $\mathcal{T}_*$ is a non-trivial subcategory of $C_X/\mathfrak{x}$.

7.11.1. Proposition. *The map $\mathcal{S} \longmapsto \mathbb{T}_{\mathcal{S}}$ induces isomorphisms*

$$\begin{aligned} \mathbf{Spec}^{1,1}_{\mathfrak{sc}}(X, \bar{\mathfrak{E}}_X) &\xrightarrow{\sim} \mathbf{Spec}^{1,1}_{\mathfrak{S}\mathfrak{C}}(X, \bar{\mathfrak{E}}_X) \\ \mathbf{Spec}^{1,1}_{\mathfrak{st}}(X, \bar{\mathfrak{E}}_X) &\xrightarrow{\sim} \mathbf{Spec}^{1,1}_{\mathfrak{S}\mathfrak{T}}(X, \bar{\mathfrak{E}}_X) \end{aligned} \tag{2}$$

between the spectra defined in terms of systems of deflations (4.4) and the spectra defined in terms of strongly closed subcategories.

Proof. The assertion follows from the sketched above dictionary between the stable systems of deflations of different kind and the subcategories of the category $C_X/\mathfrak{x}$. ∎

7.11.2. Proposition. *Let $\mathfrak{U} = \{(U_i, \bar{\mathfrak{E}}_{U_i}) \xrightarrow{\mathfrak{u}_i} (X, \bar{\mathfrak{E}}_X) \mid i \in J\}$ be a cover of the right exact 'space' $(X, \bar{\mathfrak{E}}_X)$ by strongly 'exact' localizations which has a finite subcover*

$\mathfrak{U} = \{(U_i, \bar{\mathfrak{E}}_{U_i}) \xrightarrow{\mathfrak{u}_i} (X, \bar{\mathfrak{E}}_X) \mid i \in \mathfrak{I}\}$ *such that, for every* $i \in \mathfrak{I}$*, the subcategory* $\mathfrak{Ker}(\mathfrak{u}_i^*)$ *of* $C_X/\mathfrak{x}$ *is invariant under push-forwards of deflations (which holds if* $\mathfrak{Ker}(\mathfrak{u}_i^*)$ *is invariant under cokernels of pairs of arrows).*

Then the following conditions on a strongly closed subcategory $\mathcal{P}$ *of the category* $C_X/\mathfrak{x}$ *are equivalent:*

(a) $\mathcal{P} \in \mathbf{Spec}^{1,1}_{\mathfrak{S}\mathfrak{T}}(X, \bar{\mathfrak{E}}_X)$,

(b) $\mathcal{P} \in \mathbf{Spec}^{1,1}_{\mathfrak{S}\mathfrak{C}}(X, \bar{\mathfrak{E}}_X)$ *and* $[\mathfrak{u}_i^*(\mathcal{P})] \in \mathbf{Spec}^{1,1}_{\mathfrak{S}\mathfrak{T}}(U_i, \bar{\mathfrak{E}}_{U_i})$ *whenever* $\mathfrak{Ker}(\mathfrak{u}_i^*) \subseteq \mathcal{P}$.

Proof. The claim is that the assertion follows from 6.4.

If fact, since, by the definition of covers, the arrows of $\Sigma_{\mathfrak{u}_i^*}$ have kernels for all $i \in J$, the condition (1) of 6.4 holds: $\Sigma_{\mathfrak{u}_i^*} \subseteq (\Sigma_{\mathfrak{u}_i^*} \cap \mathfrak{E}_X)^{\bar{\wedge}}$ for all $i \in J$ (see 7.10.1).

It follows from isomorphisms of 7.11.1 that one can, replacing the category C_X by $C_X/\mathfrak{x}$, assume that the category is pointed. Therefore, the condition (3) holds (see 6.6(3)).

Finally, the invariance of the subcategories $\mathfrak{Ker}(\mathfrak{u}_i^*)$ under push-forwards of deflations is what remains of the condition (2) of 6.4. ■

8. Special cases, some examples.

8.1. The abelian case. Let $(C_X, \bar{\mathfrak{E}}_X)$ be an abelian category; that is C_X is an abelian category, deflations are arbitrary epimorphisms and weak equivalences are isomorphisms. Then (as it was already mentioned in the text) semitopological classes of deflations become topological and, therefore, the spectral theories outlined in Sections 3 and 4 coincide. To every class $\mathcal{S}$ of epimorphisms of the category C_X, we assign a full subcategory $\mathbb{T}_{\mathcal{S}}$ of C_X whose objects are kernels of morphisms from $\mathcal{S}$. The correspondence $\mathcal{S} \longmapsto \mathbb{T}_{\mathcal{S}}$ induces isomorphisms between the preorder of topological systems of deflations and the preorder (with respect to the inclusion) of topological subcategories of the category C_X in the sense of Gabriel (– full subcategories of C_X closed under taking subquotients and finite products). Similarly, $\mathcal{S} \longmapsto \mathbb{T}_{\mathcal{S}}$ induces an isomorphism between the preorder of thick (resp. Serre) systems and the preorder of thick (resp. Serre) subcategories of C_X.

Strongly 'exact' functors between abelian categories are the same as 'exact' functors and the latter are just *exact* functors in the usual sense. For any exact functor $C_X \xrightarrow{F} C_Y$, the class of arrows $\Sigma_F = \{\mathfrak{s} \in HomC_X \mid F(\mathfrak{s}) \text{ is an isomorphism}\}$ satisfies all the conditions which appear in the main assertions of Section 6 (and are discussed in 6.6).

It follows from this isomorphisms and coincidences that the results of this Chapter (translated into the language of topological, thick and Serre subcategories) recover all essential facts of Chapter II.

8.2. Spectra of 'spaces' represented by exact categories. Suppose that $(C_X, \bar{\mathfrak{E}}_X)$ is an exact category. In this case, weak equivalences are isomorphisms and deflations are called sometimes *admissible epimorphisms*. Since, for a general exact category, deflations are not invariant under push-forwards, we look at the version of the spectral theory based on the notion of a *semitopological* system (see Sections 4 and 5).

Following general pattern, we consider the correspondence which assigns to any class of deflations $\mathcal{S}$ the full subcategory $\mathbb{T}_{\mathcal{S}}$ of the category C_X generated by all objects M such that morphism from M to the zero object belongs to $\mathcal{S}$.

Notice that, since the category C_X is additive and weak equivalences are isomorphisms, the class $\mathcal{W}_X^{\wedge}$ consists of all monomorphisms of the category C_X. Therefore, a system $\mathcal{S}$ of deflations of $(C_X, \bar{\mathfrak{E}}_X)$ is strongly stable iff the subcategory $\mathbb{T}_{\mathcal{S}}$ is closed under taking arbitrary subobjects (cf. 7.3).

The map $\mathcal{S} \longmapsto \mathbb{T}_{\mathcal{S}}$ induces an isomorphism between the preorder of semitopologizing systems of deflations of $(C_X, \bar{\mathfrak{E}}_X)$ and the preorder of *strongly topologizing* subcategories $\mathcal{T}$ of C_X which are full subcategories of C_X closed under finite products and subobjects (taken in C_X) and such that for any deflation $\mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{N}$ with $\mathcal{M} \in Ob\mathcal{T}$, the object $\mathcal{N}$ belongs to $\mathcal{T}$ (see 7.4).

8.3. A note about the spectral theory of the category of algebras. Let C_X be the category Alg_k of associative unital algebras over a commutative unital ring k, deflations are strict (that is surjective) epimorphisms of algebras and weak equivalences are isomorphisms. The k-algebra k is the canonical initial object of the category Alg_k and the category Alg_k/k is isomorphic to the category of augmented algebras. The category Alg_k/k of augmented k-algebras is naturally equivalent to the category Alg_k^1 of non-unital k-algebras. The equivalence is given by the functor $Alg_k^1 \xrightarrow{\iota_k^*} Alg_k/k$ which assigns to a non-unital k-algebra $\mathcal{R}$ the augmented algebra $(k\oplus\mathcal{R}, k\oplus\mathcal{R} \to k)$. Its quasi-inverse functor maps an augmented algebra (A, ξ_A) to its augmentation ideal $Ker(\xi_A)$.

We have a commutative diagram of functors

$$\begin{array}{ccccc} Alg_k^1 & \xrightarrow{\iota_k^*} & Alg_k/k & \xrightarrow{\mathfrak{j}_k^*} & Alg_k \\ \widetilde{\mathfrak{f}}^* \Big\uparrow \Big\downarrow \widetilde{\mathfrak{f}}_* & & \bar{\mathfrak{f}}^* \Big\uparrow \Big\downarrow \bar{\mathfrak{f}}_* & & \mathfrak{f}^* \Big\uparrow \Big\downarrow \mathfrak{f}_* \\ k-mod & \xrightarrow{Id} & k-mod & \xrightarrow{Id} & k-mod \end{array} \qquad (1)$$

where $\widetilde{\mathfrak{f}}_*$ is the forgetful functor, $\widetilde{\mathfrak{f}}^*$ its left adjoint which assigns to every k-module V the irrelevant ideal $T_k^{\geq 1}(V) = \bigoplus_{n\geq 1} V^{\otimes_k n}$ of the tensor algebra $T_k(V)$ of the module V; $\mathfrak{j}_k^*$ is the canonical forgetful functor and the functor $\bar{\mathfrak{f}}^*$ assigns to each k-module V the tensor algebra $T_k(V)$ with the canonical augmentation and $\mathfrak{f}_*$ is the forgetful functor.

By 7.11.1, the spectral theory outlined here requires only the category Alg_k/k of augmented k-algebras. There is another pair of adjoint functors

$$k-mod \underset{\varphi^*}{\overset{\varphi_*}{\rightleftarrows}} Alg_k/k, \qquad (2)$$

where the functor φ_* assigns to each k-module V the k-algebra $k \oplus V$ with $V \cdot V = 0$. Its left adjoint functor, φ^*, assigns to every augmented k-algebra $(A, A \overset{\xi_A}{\to} k)$ the k-module $Ker(\xi_A)/Ker(\xi_A)^2$. The composition $\varphi^* \circ \varphi_*$ is the identical functor, which means that φ_* is a fully faithful functor and, therefore, φ^* is a localization functor. The functor φ_* is 'exact' and induces a natural equivalence between the category $k-mod$ and the topologizing subcategory $\mathbb{T}_0$ of the category Alg_k/k whose objects are those augmented algebras $(A, A \overset{\xi_A}{\to} k)$ for which $Ker(\xi_A)^2 = 0$. Thus, the spectrum of the 'space' $\mathbf{Sp}(k)$ (which is isomorphic to $Spec(k)$ – the prime spectrum of k, is embedded into $\mathbf{Spec}^{1,1}_{\mathfrak{S}\mathfrak{T}}(X_{\mathfrak{r}})$.

The picture looks slightly simpler in terms of the category $\mathcal{C}_{\mathfrak{A}^1_X} = Alg_k^1$ of non-unital k-algebras. Namely, the pair of adjoint functors (2) corresponds to the functors

$$k-mod \underset{\gamma^*}{\overset{\gamma_*}{\rightleftarrows}} Alg_k^1, \qquad (2')$$

where γ_* assigns to each k-module V the same k-module with the zero multiplication and the functor γ^* maps each non-unital k-algebra A to the k-module A/A^2. The kernel of the localization functor γ^* is the full subcategory $\mathcal{C}_{\mathfrak{A}^{\mathfrak{f}}_X} = Alg^{\mathfrak{f}}_k$ of the category $Alg^1_k = \mathcal{C}_{\mathfrak{A}^1_X}$ whose objects are so-called *firm* algebras A defined by the equality $A^2 = A$ (evidently, every unital algebra is firm). One can show that the spectrum $\mathbf{Spec}^{1,1}_{\mathfrak{S}\mathfrak{T}}(\mathfrak{A}^1_X, \bar{\mathfrak{E}}_{\mathfrak{A}^1_X})$ is the disjoint union of the image of the prime spectrum of k and the spectrum $\mathbf{Spec}^{1,1}_{\mathfrak{S}\mathfrak{T}}(\mathfrak{A}^{\mathfrak{f}}_X, \bar{\mathfrak{E}}_{\mathfrak{A}^{\mathfrak{f}}_X})$ of the right exact 'space' represented by the subcategory of firm k-algebras.

8.3.1. Generalizations. Let $\widetilde{\mathcal{C}}_X = (\mathcal{C}_X, \odot, \mathbb{I})$ be a monoidal category with 'tensor' product $\odot$ and the unit object $\mathbb{I}$. We assume that $\mathcal{C}_X$ is a pointed category with countable colimits preserved by 'tensor product; and tensoring any object by a zero object produces a zero object. Besides, $\mathcal{C}_X$ is endowed with a right exact structure $\mathfrak{E}_X$ with weak equivalences $\mathcal{W}_X$ such that all split epimorphisms of C_X are deflations, both respected by the 'tensor' product $\odot$. The latter means that $\alpha \odot \beta$ is a deflation (resp. a weak equivalence), if α and β are deflations (resp. weak equivalences).

Let $C_{\mathfrak{A}_X}$ denote the category $Alg_{\widetilde{\mathcal{C}}_X}$ of algebras in the $\widetilde{\mathcal{C}}_X$ (in classical sources, like [ML], the objects of $Alg_{\widetilde{\mathcal{C}}_X}$ are called monoids). Thanks to the existence of countable coproducts and the compatibility of 'tensor' product with them, the forgetful functor $C_{\mathfrak{A}_X} \xrightarrow{\mathfrak{f}_*} \mathcal{C}_X$ has a left adjoint, $\mathfrak{f}^*$ which assigns to every object V of the monoidal category $\widetilde{\mathcal{C}}_X$ its *tensor algebra* $(T(V), \mu_V)$, where $T(V) = \bigoplus_{n\geq 0} V^{\odot n}$ and the multiplication μ_V is given by the canonical isomorphisms $V^{\odot n} \odot V^{\odot m} \xrightarrow{\sim} V^{\odot n+m}$. Here $V^{\odot 0} \stackrel{\mathrm{def}}{=} \mathbb{I}$.

The category of algebras has a natural initial object – the 'unit' algebra $\mathbb{I}$. We denote the category $Alg_{\widetilde{\mathcal{C}}_X}/\mathbb{I}$ of *augmented* algebras by $C_{\mathfrak{A}^1_X}$.

If the category $\mathcal{C}_X$ is additive, then the category of augmented algebras is naturally equivalent to the category $Alg^1_{\widetilde{\mathcal{C}}_X}$ of non-unital algebras.

Like in the case of k-algebras, we have a commutative diagram of functors

$$
\begin{array}{ccc}
C_{\mathfrak{A}^1_X} & \xrightarrow{\mathfrak{j}^*_X} & C_{\mathfrak{A}_X} \\
\bar{\mathfrak{f}}^* \uparrow \downarrow \bar{\mathfrak{f}}_* & & \mathfrak{f}^* \uparrow \downarrow \mathfrak{f}_* \\
\mathcal{C}_X & \xrightarrow{Id} & \mathcal{C}_X
\end{array}
\qquad (3)
$$

where $\mathfrak{j}^*_X$ is the functor which forgets augmentation.

We define deflations on the category $C_{\mathfrak{A}_X}$ of algebras by setting $\mathfrak{E}_{\mathfrak{A}_X} = \mathfrak{f}_*^{-1}(\mathfrak{E}_X)$ and weak equivalences by $\mathcal{W}_{\mathfrak{A}_X} = \mathfrak{f}_*^{-1}(\mathcal{W}_X)$. The right exact structure on the category of augmented algebras induced via the forgetful functor $\mathfrak{j}^*_X$, so that $\mathfrak{E}_{\mathfrak{A}^1_X} = \bar{\mathfrak{f}}_*^{-1}(\mathfrak{E}_X)$ and $\mathcal{W}_{\mathfrak{A}^1_X} = \bar{\mathfrak{f}}_*^{-1}(\mathcal{W}_X)$ (see the diagram (3)).

There is also the embedding $\mathcal{C}_X \xrightarrow{\phi_*} C_{\mathfrak{A}^1_X}$, which assigns to every object V of $\mathcal{C}_X$ the algebra $(\mathbb{I} \oplus V, \mu^0_V)$, where μ^0_V is the multiplication trivial on V. This embedding has a left adjoint, $C_{\mathfrak{A}_X} \xrightarrow{\phi^*} C_{\mathfrak{A}^1_X}$, which maps every augmented algebra $(A, A \overset{\xi_A}{\to} \mathbb{I})$ to the object $Ker(\xi_A)/(Ker(\xi_A)^2$. Notice that the object $Ker(\xi_A)$ exists because ξ_A is a split epimorphism, hence a deflation.

The embedding $\mathcal{C}_X \xrightarrow{\phi_*} C_{\mathfrak{A}^1_X}$ induces an embedding of the spectrum $\mathbf{Spec}^{1,1}_{\mathfrak{S}\mathfrak{T}}(X, \bar{\mathfrak{E}}_X)$ of the 'space' $(X, \bar{\mathfrak{E}}_X)$ into the spectrum $\mathbf{Spec}^{1,1}_{\mathfrak{S}\mathfrak{T}}(\mathfrak{A}^1_X, \bar{\mathfrak{E}}_{\mathfrak{A}^1_X})$ of the 'space' $(\mathfrak{A}^1_X, \bar{\mathfrak{E}}_{\mathfrak{A}^1_X})$ represented by the right exact category of augmented algebras.

8.3.2. Example. Let R be an associative unital k-algebra and $\mathcal{C}_X$ the monoidal category of R-bimodules endowed with the standard exact structure. Algebras in this monoidal category are associative unital k-algebras A endowed with an algebra morphism $R \xrightarrow{\psi_A} A$. In other words, the category of algebras is isomorphic to the category $R\backslash Alg_k$ of k-algebras over R. Augmented algebras are triples (ψ_A, A, ξ_A), where $A \xrightarrow{\xi_A} R$ is the left inverse to ψ_A, that is $\xi_A \circ \psi_A = id_R$. The right exact structure on the category of (augmented) algebras over R is standard: deflations are surjective morphisms and weak equivalences are isomorphisms. We have the natural embedding of the spectrum $\mathbf{Spec}^{1,1}_{\mathfrak{S}\mathfrak{T}}(X, \bar{\mathfrak{E}}_X)$ of the 'space' $(X, \bar{\mathfrak{E}}_X)$ represented by the category of R-bimodules into the spectrum $\mathbf{Spec}^{1,1}_{\mathfrak{S}\mathfrak{T}}(\mathfrak{A}^1_X, \bar{\mathfrak{E}}_{\mathfrak{A}^1_X})$ of the 'space' $(\mathfrak{A}^1_X, \bar{\mathfrak{E}}_{\mathfrak{A}^1_X})$ represented by the right exact category of augmented algebras. The complement to the image of $\mathbf{Spec}^{1,1}_{\mathfrak{S}\mathfrak{T}}(X, \bar{\mathfrak{E}}_X)$ is the spectrum of the 'subspace' $(\mathfrak{A}^{\mathfrak{f}}_X, \bar{\mathfrak{E}}_{\mathfrak{A}^{\mathfrak{f}}_X})$ of the right exact 'space' $(\mathfrak{A}^1_X, \bar{\mathfrak{E}}_{\mathfrak{A}^1_X})$ represented by the subcategory of all augmented rings (ψ_A, A, ξ_A) such that $Ker(\xi_A)^2 = Ker(\xi_A)$. These augmented algebras correspond to the firm non-unital k-algebras over the algebra $\mathcal{R}$.

Complements: some properties of kernels.

C.1. Proposition. *Let $M \xrightarrow{f} N$ be a morphism of C_X which has a kernel pair, $M \times_N M \underset{p_2}{\overset{p_1}{\rightrightarrows}} M$. Then the morphism f has a kernel iff p_1 has a kernel, and these two kernels are naturally isomorphic to each other.*

Proof. Suppose that f has a kernel, i.e. there is a cartesian square

$$\begin{array}{ccc} Ker(f) & \xrightarrow{\mathfrak{k}(f)} & M \\ {\scriptstyle f'}\downarrow & cart & \downarrow{\scriptstyle f} \\ x & \xrightarrow{i_N} & N \end{array} \qquad (1)$$

Then we have the commutative diagram

$$\begin{array}{ccccc} Ker(f) & \xrightarrow{\gamma} & M\times_N M & \xrightarrow{p_2} & M \\ {\scriptstyle f'}\downarrow & & {\scriptstyle p_1}\downarrow & cart & \downarrow{\scriptstyle f} \\ x & \xrightarrow{i_M} & M & \xrightarrow{f} & N \end{array} \tag{2}$$

which is due to the commutativity of (1) and the fact that the unique morphism $x \xrightarrow{i_N} N$ factors through the morphism $M \xrightarrow{f} N$. The morphism γ is uniquely determined by the equality $p_2 \circ \gamma = \mathfrak{k}(f)$. The fact that the square (1) is cartesian and the equalities $p_2 \circ \gamma = \mathfrak{k}(f)$ and $i_N = f \circ i_M$ imply that the left square of the diagram (2) is cartesian, i.e. $Ker(f) \xrightarrow{\gamma} M \times_N M$ is the kernel of the morphism p_1.

Conversely, if p_1 has a kernel, then we have a diagram

$$\begin{array}{ccccc} Ker(p_1) & \xrightarrow{\mathfrak{k}(p_1)} & M\times_N M & \xrightarrow{p_2} & M \\ {\scriptstyle p_1'}\downarrow & cart & {\scriptstyle p_1}\downarrow & cart & \downarrow{\scriptstyle f} \\ x & \xrightarrow{i_M} & M & \xrightarrow{f} & N \end{array}$$

which consists of two cartesian squares. Therefore the square

$$\begin{array}{ccc} Ker(p_1) & \xrightarrow{\mathfrak{k}(f)} & M \\ {\scriptstyle p_1'}\downarrow & cart & \downarrow{\scriptstyle f} \\ x & \xrightarrow{i_N} & N \end{array}$$

with $\mathfrak{k}(f) = p_2 \circ \mathfrak{k}(p_1)$ is cartesian. ∎

C.2. Remarks. (a) Needless to say that the picture obtained in (the argument of) C.1 is symmetric, i.e. there is an isomorphism $Ker(p_1) \xrightarrow{\tau_f'} Ker(p_2)$ which is an arrow in the commutative diagram

$$\begin{array}{ccccc} Ker(p_1) & \xrightarrow{\mathfrak{k}(p_1)} & M\times_N M & \xrightarrow{p_1} & M \\ {\scriptstyle \tau_f'}\downarrow\wr & & {\scriptstyle \tau_f}\downarrow\wr & & \downarrow{\scriptstyle id_M} \\ Ker(p_2) & \xrightarrow{\mathfrak{k}(p_2)} & M\times_N M & \xrightarrow{p_2} & M \end{array}$$

(b) Let a morphism $M \xrightarrow{f} N$ have a kernel pair, $M \times_N M \underset{p_2}{\overset{p_1}{\rightrightarrows}} M$, and a kernel. Then, by C.1, there exists a kernel of p_1, so that we have a morphism $Ker(p_1) \xrightarrow{\mathfrak{k}(p_1)} M \times_N M$ and the diagonal morphism $M \xrightarrow{\Delta_M} M \times_N M$. Since the left square of the commutative diagram

$$\begin{tikzcd} x \arrow[r] \arrow[d] \arrow[dr, phantom, "cart"] & Ker(p_1) \arrow[r, "p_1'"] \arrow[d, "\mathfrak{c}(p_1)"'] & x \arrow[d] \\ M \arrow[r, "\Delta_M"] & M \times_N M \arrow[r, "p_1"] & M \end{tikzcd}$$

is cartesian and compositions of the horizontal arrows are identical morphisms, it follows that its left square is cartesian too. Loosely, one can say that the intersection of $Ker(p_1)$ with the diagonal of $M \times_N M$ is zero. If there exists a coproduct $Ker(p_1) \coprod M$, then the pair of morphisms $Ker(p_1) \xrightarrow{\mathfrak{k}(p_1)} M \times_N M \xleftarrow{\Delta_M} M$ determine a morphism

$$Ker(p_1) \coprod M \longrightarrow M \times_N M.$$

If the category C_X is additive, then this morphism is an isomorphism, or, what is the same, $Ker(f) \coprod M \simeq M \times_N M$. In general, it is rarely the case, as the reader can find out looking at the examples of 1.4.

C.3. Proposition. *Let*

$$\begin{tikzcd} \widetilde{M} \arrow[r, "\widetilde{f}"] \arrow[d, "\widetilde{g}"'] \arrow[dr, phantom, "cart"] & \widetilde{N} \arrow[d, "g"] \\ M \arrow[r, "f"] & N \end{tikzcd} \qquad (3)$$

be a cartesian square. Then $Ker(f)$ exists iff $Ker(\widetilde{f})$ exists, and they are naturally isomorphic to each other.

C.4. The kernel of a composition and related facts. Fix a category C_X with an initial object x.

C.4.1. The kernel of a composition. Let $L \xrightarrow{f} M$ and $M \xrightarrow{g} N$ be morphisms such that there exist kernels of g and $g \circ f$. Then the argument similar to that of C.3 shows that we have a commutative diagram

$$\begin{tikzcd} Ker(gf) \arrow[r, "\widetilde{f}"] \arrow[d, "\mathfrak{k}(gf)"'] \arrow[dr, phantom, "cart"] & Ker(g) \arrow[r, "g'"] \arrow[d, "\mathfrak{k}(g)"] \arrow[dr, phantom, "cart"] & x \arrow[d, "i_N"] \\ L \arrow[r, "f"] & M \arrow[r, "g"] & N \end{tikzcd} \qquad (1)$$

whose both squares are cartesian and all morphisms are uniquely determined by f, g and the (unique up to isomorphism) choice of the objects $Ker(g)$ and $Ker(gf)$.

Conversely, if there is a commutative diagram

$$\begin{array}{ccccc}
K & \xrightarrow{u} & Ker(g) & \xrightarrow{g'} & x \\
t \big\downarrow & cart & \big\downarrow \mathfrak{k}(g) & & \big\downarrow i_N \\
L & \xrightarrow{f} & M & \xrightarrow{g} & N
\end{array}$$

whose left square is cartesian, then its left vertical arrow, $K \xrightarrow{t} L$, is the kernel of the composition $L \xrightarrow{g\circ f} N$.

C.4.2. Remarks. (a) It follows from C.3 that the kernel of $L \xrightarrow{f} M$ exists iff the kernel of $Ker(gf) \xrightarrow{\widetilde{f}} Ker(g)$ exists and they are isomorphic to each other. More precisely, we have a commutative diagram

$$\begin{array}{ccccccc}
Ker(\widetilde{f}) & \xrightarrow{\mathfrak{k}(\widetilde{f})} & Ker(gf) & \xrightarrow{\widetilde{f}} & Ker(g) & \xrightarrow{g'} & x \\
\wr\big\downarrow & & \mathfrak{k}(gf) \big\downarrow & cart & \big\downarrow \mathfrak{k}(g) & cart & \big\downarrow i_N \\
Ker(f) & \xrightarrow{\mathfrak{k}(f)} & L & \xrightarrow{f} & M & \xrightarrow{g} & N
\end{array}$$

whose left vertical arrow is an isomorphism.

(b) Suppose that $(C_X, \mathfrak{E}_X)$ is a right exact category (with an initial object x). If the morphism f above is a deflation, then it follows from this observation that the canonical morphism $Ker(gf) \xrightarrow{\widetilde{f}} Ker(g)$ is a deflation too. In this case, $Ker(f)$ exists, and we have a commutative diagram

$$\begin{array}{ccccc}
Ker(\widetilde{f}) & \xrightarrow{\mathfrak{k}(\widetilde{f})} & Ker(gf) & \xrightarrow{\widetilde{f}} & Ker(g) \\
\wr\big\downarrow & & \mathfrak{k}(gf) \big\downarrow & cart & \big\downarrow \mathfrak{k}(g) \\
Ker(f) & \xrightarrow{\mathfrak{k}(f)} & L & \xrightarrow{f} & M
\end{array}$$

whose rows are conflations.

The following observations is useful (and are used) for analysing diagrams.

C.4.3. Proposition. *(a) Let $M \xrightarrow{g} N$ be a morphism with a trivial kernel. Then a morphism $L \xrightarrow{f} M$ has a kernel iff the composition $g \circ f$ has a kernel, and these two kernels are naturally isomorphic one to another.*

(b) Let

$$\begin{array}{ccc} L & \xrightarrow{f} & M \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle g} \\ \widetilde{M} & \xrightarrow{\phi} & N \end{array}$$

be a commutative square such that the kernels of the arrows f and ϕ exist and the kernel of g is trivial. Then the kernel of the composition $\phi \circ \gamma$ is isomorphic to the kernel of the morphism f, and the left square of the commutative diagram

$$\begin{array}{ccccccc} Ker(f) & \xrightarrow{\sim} & Ker(\phi\gamma) & \xrightarrow{\mathfrak{k}(f)} & L & \xrightarrow{f} & M \\ & & {\scriptstyle\widetilde{\gamma}}\downarrow & cart & {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle g} \\ & & Ker(\phi) & \xrightarrow{\mathfrak{k}(\phi)} & \widetilde{M} & \xrightarrow{\phi} & N \end{array}$$

is cartesian.

Proof. (a) Since the kernel of g is trivial, the diagram C.4.1(1) specializes to the diagram

$$\begin{array}{ccccc} Ker(gf) & \xrightarrow{\widetilde{f}} & x & \xrightarrow{id_x} & x \\ {\scriptstyle\mathfrak{k}(gf)}\downarrow & cart & \downarrow{\scriptstyle\mathfrak{k}(g)} & & \downarrow{\scriptstyle i_N} \\ L & \xrightarrow{f} & M & \xrightarrow{g} & N \end{array}$$

with cartesian squares. The left cartesian square of this diagram is the definition of $Ker(f)$. The assertion follows from C.4.1.

(b) Since the kernel of g is trivial, it follows from (a) that $Ker(f)$ is naturally isomorphic to the kernel of $g \circ f = \phi \circ \gamma$. The assertion follows now from C.4.1. ■

C.4.4. Corollary. *Let C_X be a category with an initial object x. Let $L \xrightarrow{f} M$ be a strict epimorphism and $M \xrightarrow{g} N$ a morphism such that $Ker(g) \xrightarrow{\mathfrak{k}(g)} M$ exists and is a monomorphism. Then the composition $g \circ f$ is a trivial morphism iff g is trivial.*

C.4.4.1. Note. The following example shows that the requirement "$Ker(g) \longrightarrow M$ is a monomorphism" in C.4.4 cannot be omitted.

Let C_X be the category Alg_k of associative unital k-algebras, and let $\mathfrak{m}$ be an ideal of the ring k such that the epimorphism $k \longrightarrow k/\mathfrak{m}$ does not split. Then the identical morphism $k/\mathfrak{m} \longrightarrow k/\mathfrak{m}$ is non-trivial, while its composition with the projection $k \longrightarrow k/\mathfrak{m}$ is a trivial morphism.

Chapter VII
Spectral Cuisine for the Working Mathematicians.

The main construction is presented in Section 1. Roughly, it runs as follows. With any category, $\mathfrak{H}$, we associate two spectra, $\mathfrak{Spec}^0(\mathfrak{H})$ and $\mathfrak{Spec}^1(\mathfrak{H})$. These spectra are subcategories of $\mathfrak{H}$. And, by construction, there is a natural functor $\mathfrak{Spec}^0(\mathfrak{H}) \longrightarrow \mathfrak{H}$ (different from the inclusion functor). Given a functor $\mathfrak{G} \xrightarrow{F} \mathfrak{H}$, we define (in 1.6) two spectra, $\mathfrak{Spec}^0(\mathfrak{G}, F)$ and $\mathfrak{Spec}^1(\mathfrak{G}, F)$, of the pair $(\mathfrak{G}, F)$ as pullbacks of $\mathfrak{Spec}^0(\mathfrak{H})$ and $\mathfrak{Spec}^1(\mathfrak{H})$ along F. If $\mathfrak{H}$ is a preorder (which is the case of our main examples), then there exists a canonical morphism $\mathfrak{Spec}^0(\mathfrak{G}, F) \longrightarrow \mathfrak{Spec}^1(\mathfrak{G}, F)$. In particular, there is a canonical morphism $\mathfrak{Spec}^0(\mathfrak{H}) \longrightarrow \mathfrak{Spec}^1(\mathfrak{H})$.

Taking as F the inclusion map of the preorder of Serre subcategories to the preorder of topologizing subcategories of an abelian category, we recover the spectrum $\mathbf{Spec}(X)$ of Chapter II. If $\mathfrak{H}$ is the preorder of saturated multiplicative systems of a category (resp. triangulated category) and F is the identical functor, we recover the basic spectra of an arbitrary category (resp. triangulated category). These and some other applications of the general construction are sketched in Section 2.

Spectra considered in Section 2 are related with saturated (left and right) multiplicative systems, or what is the same, with exact (i.e. preserving finite limits and colimits) localizations. In the case of an abelian or triangulated category, they correspond to thick subcategories. There are categories with only trivial saturated multiplicative systems. A fundamental example is the category *Sets* of sets which belong to a given universe. It has no non-trivial right multiplicative systems, but has plenty of saturated left multiplicative systems. The latter are in bijective correspondence with right exact (– preserving colimits) localizations. In Section 3 we apply the pattern of Section 1 to the preorder of saturated left multiplicative systems of a category and obtain, as a result, left versions of the spectra discussed in Section 2 (and in Chapter II).

In Section 4, we look at injective objects and related localizations and spectra, in particular, the Gabriel's spectrum. Injective objects play important role not only in abelian (Grothendieck) categories, but also in a large class of non-additive categories which includes toposes. Therefore, the exposition here is not restricted to abelian or even additive categories. We define a *left exact* multiplicative system as a saturated left multiplicative systems such that the corresponding localization functor preserves strict monomorphisms. In the case of abelian categories, left exact multiplicative systems are precisely saturated (left and right) multiplicative systems. On the other hand, in the case of the category *Sets*, every saturated left system is left exact, but, as it is mentioned above, there are no non-trivial right saturated multiplicative systems.

To any injective object E of a category C, there corresponds a left exact multiplicative system Σ_E which consists of all arrows s such that $Hom_C(s, E)$ is an isomorphism. Injective spectra, in particular (the non-additive version of) the Gabriel's spectrum, are obtained by applying to this correspondence the general formalism of Section 1.

It is worth to mention that left exact multiplicative systems are usually more important, at least from spectral prospective, than injective objects. For instance, if C is the category $Sets^1$ of non-empty sets, then there are only trivial left multiplicative systems of the form Σ_E. In particular, injective spectra are trivial. On the other hand, the preorder of left exact multiplicative systems is isomorphic to the order of infinite cardinals. Both spectra, $\mathfrak{Spec}^0$ and $\mathfrak{Spec}^1$, of this preorder are naturally isomorphic to the order of non-limit infinite cardinals.

The purpose of this Chapter is to explain what stands behind the known constructions of spectra and give a couple of curious examples. There is no attempt to make the list of applications and examples complete (i.e. include all applications which seem to be important ones) and, with more reason, no attempt to impose choices. The reader might make a different choice of the functor $\mathfrak{G} \xrightarrow{F} \mathfrak{H}$ and use the 'spectral cuisine' of Section 1 to produce other spectra which could be appropriate for something.

1. General pattern.

Fix a category $\mathfrak{H}$. Let $\mathfrak{H}_0$ denote the full subcategory of $\mathfrak{H}$ whose objects are initial objects of $\mathfrak{H}$. Thus, $\mathfrak{H}_0$ is either empty, or a groupoid. Let $\mathfrak{H}^1$ denote the full subcategory of $\mathfrak{H}$ defined by $Ob\mathfrak{H}^1 = Ob\mathfrak{H} - Ob\mathfrak{H}_0$.

1.1. Definition. We call $\mathfrak{H}$ *local* if the category $\mathfrak{H}^1$ has an initial object.

1.1.1. Note. It follows that if $\mathfrak{H}$ is local, than $\mathfrak{H}$ has initial objects, i.e. $\mathfrak{H}_0 \neq \emptyset$.

1.1.2. Example. The preorder $\{x \to y\}$ is local, since $\mathfrak{H}^1$ has only one object, y, and one morphism, id_y.

1.1.3. Example. Let R be an associative commutative unital ring, and let IR denote the set of its ideals. The preorder $(IR, \supseteq)$ is local iff the ring R is local, i.e. there exists a maximal ideal in R which contains all other proper ideals.

1.2. The spectrum $\mathfrak{Spec}^1(\mathfrak{H})$. We denote by $\mathfrak{Spec}^1(\mathfrak{H})$ the full subcategory of the category $\mathfrak{H}$ generated by all $x \in Ob\mathfrak{H}$ such that the category $x\backslash\mathfrak{H}$ is local. We call $\mathfrak{Spec}^1(\mathfrak{H})$ the *local spectrum* of $\mathfrak{H}$.

In other words, an object x of $\mathfrak{H}$ belongs to $\mathfrak{Spec}^1(\mathfrak{H})$ iff there exists an object $x^\star$ of $\mathfrak{H}$ and an arrow $x \xrightarrow{\gamma_x} x^\star$ such that γ_x is not an isomorphism and if $x \xrightarrow{f} y$ is not an isomorphism, then there exists a unique arrow $x^\star \xrightarrow{\bar{f}} y$ such that $f = \bar{f}\circ\gamma_x$. The morphism

$x \xrightarrow{\gamma_x} x^\star$ (in particular, the object $x^\star$) is determined by these conditions uniquely up to isomorphism.

1.2.1. Note. It follows from this definition and 1.1.1 that $\mathfrak{H}$ is local iff it has initial objects and they belong to $\mathfrak{Spec}^1(\mathfrak{H})$.

1.2.1.1. Example. Let $\mathfrak{H} = Sets$. The category $Sets$ has one initial object – $\emptyset$. It is local: every one element set is an initial object of the category $Sets^1$. Notice that the spectrum $\mathfrak{Spec}^1(Sets^1)$ is empty. Therefore $\mathfrak{Spec}^1(Sets)$ consists of one point which is the initial object $\emptyset$.

1.2.2. Maximal proper objects and $\mathfrak{Spec}^1(\mathfrak{H})$. We call an object x of the category $\mathfrak{H}$ *proper* if there exists an arrow $x \longrightarrow y$ which is not an isomorphism. We call a proper object, x, *maximal* if any two proper morphisms, $y_1 \xleftarrow{s} x \xrightarrow{t} y_2$, are isomorphic; that is there exists an isomorphism $y_1 \xrightarrow{u} y_2$ such that $t = u \circ s$.

We denote by $\mathcal{M}ax(\mathfrak{H})$ the full subcategory of $\mathfrak{H}$ generated by maximal proper objects. It follows that $\mathcal{M}ax(\mathfrak{H})$ is a groupoid which is connected iff all maximal proper objects are isomorphic to each other.

If for every proper object, y, there is an arrow from y to a maximal proper object, then the groupoid $\mathcal{M}ax(\mathfrak{H})$ is connected iff $\mathfrak{H}^{op}$ is a local category.

1.2.2.1. Proposition. $\mathcal{M}ax(\mathfrak{H}) \subseteq \mathfrak{Spec}^1(\mathfrak{H})$.

Proof. In fact, if x is an object of $\mathcal{M}ax(\mathfrak{H})$, then the category $x\backslash\mathfrak{H}$ is equivalent to the preorder $\{x \to y\}$, hence it is local (cf. 1.1.2). ■

1.2.2.2. Example. If $\mathfrak{H}$ is the preorder $(I_\ell R, \subseteq)$ of left ideals of an associative unital ring R, then $\mathcal{M}ax(\mathfrak{H})$ coincides with the set $Max_\ell R$ of left maximal ideals of R regarded as a discrete category. The category $Max_\ell R$ is connected iff R has only one left maximal ideal, μ. Notice that in this case the left ideal μ is two-sided, because for every $r \in R - \mu$, the ideal $(\mu : r) = \{a \in R \mid ar \in \mu\}$ is a maximal left ideal, hence it coincides with μ.

1.2.3. Minimal proper objects. We call $x \in Ob\mathfrak{H}$ a *minimal proper* object of the category $\mathfrak{H}$ if x is a maximal proper object of $\mathfrak{H}^{op}$. We denote by $\mathcal{M}in(\mathfrak{H})$ the full subcategory of $\mathfrak{H}$ generated by minimal proper objects. By definition, $\mathcal{M}in(\mathfrak{H})$ is isomorphic to $\mathcal{M}ax(\mathfrak{H}^{op})$. In particular, $\mathcal{M}in(\mathfrak{H})$ is a groupoid which is connected iff $\mathfrak{H}$ is a local category. By 1.2.2.1, $\mathcal{M}in(\mathfrak{H}) \subseteq \mathfrak{Spec}^1(\mathfrak{H}^{op})$.

1.2.3.1. Example. Let C_X be a category with an initial object, and let $C_{\mathfrak{M}(X)}$ be the subcategory of C_X formed by all monoarrows of C_X. Then $\mathcal{M}in(C_{\mathfrak{M}(X)})$ is the groupoid of all simple objects of the category C_X. Isomorphism classes of simple objects can be regarded as a *naive spectrum* of C_X.

The groupoid $\mathcal{M}in(C_{\mathfrak{M}(X)})$ is connected (that is the category $C_{\mathfrak{M}(X)}$ is local) iff the category C_X has a unique, up to isomorphism, simple object.

1.2.3.2. Note. A useful version of 1.2.3.1 is obtained by taking instead of $C_{\mathfrak{M}(X)}$ the subcategory $C_{\mathfrak{M}_\mathfrak{s}(X)}$ generated by all strict monomorphisms of the category C_X.

Recall that a monomorphism $L \xrightarrow{h} M$ is called *strict* if every arrow $L' \longrightarrow M$ which equalizes all pairs of arrows $M \overset{g_1}{\underset{g_2}{\rightrightarrows}} N$ equalized by h is represented as the composition of h and an arrow $L' \longrightarrow L$ uniquely determined by this property. If an arrow $L \xrightarrow{h} M$ is such that there exists a fibred coproduct $M \coprod_L M$, then h is a strict monomorphism iff the canonical diagram $L \xrightarrow{h} M \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} M \coprod_L M$ is exact.

The groupoid $\mathcal{M}in(C_{\mathfrak{M}_\mathfrak{s}(X)})$ is generated by objects which are simple in a "strict" sense. For instance, if C_X is the category of continuous representations of a topological algebra in topological vector spaces, objects of $\mathcal{M}in(C_{\mathfrak{M}_\mathfrak{s}(X)})$ are topologically irreducible representations of this algebra.

1.2.3.3. Example: injectives and the Gabriel's spectrum. Let C_X be a category with finite limits. An object E of the category C_X is called *injective* if the functor $C_X(-,E) : C_X^{op} \longrightarrow Sets$ preserves strict epimorphisms (in other words, $C_X(\mathfrak{j},E)$ is a surjective map for any strict monomorphism $\mathfrak{j}$). We denote by $C_{\mathfrak{I}(X)}$ the subcategory of C_X formed by injective objects and strict monomorphisms (see 1.2.3.2) If follows that if E is an injective object, than any strict monomorphism $E \xrightarrow{g} M$ is a split monomorphism; i.e. $h \circ g = id_E$ for some $M \xrightarrow{h} E$.

We call an arrow in C_X a *zero morphism* if it factors through an initial object (if any).

We call an injective object E of the category C_X *indecomposable* if the only nonzero idempotent $E \longrightarrow E$ is the identical morphism. Equivalently, any strict monomorphism $E_1 \longrightarrow E$ with E_1 injective and non-initial, is an isomorphism.

Objects of the groupoid $\mathcal{M}in(C_{\mathfrak{I}(X)})$ are precisely indecomposable injective objects of the full subcategory C_X^1 of the category C_X formed by non-initial objects. Isomorphism classes of indecomposable injective objects are points of the Gabriel's spectrum.

1.2.4. Functorial properties. Let $\mathfrak{H} \xrightarrow{F} \widetilde{\mathfrak{H}}$ be a functor. For any $x \in Ob\mathfrak{H}$, the functor F induces a functor $x\backslash\mathfrak{H} \xrightarrow{F_x} F(x)\backslash\widetilde{\mathfrak{H}}$. Suppose that the functor F is such that F_x is an equivalence of categories. Then F induces a functor $\mathfrak{Spec}^1(\mathfrak{H}) \longrightarrow \mathfrak{Spec}^1(\widetilde{\mathfrak{H}})$.

A typical example is the functor

$$y\backslash\mathfrak{H} \xrightarrow{f_\star} z\backslash\mathfrak{H}, \quad (y, y \xrightarrow{g} v) \longmapsto (z, z \xrightarrow{gf} v),$$

corresponding to a morphism $z \xrightarrow{f} y$, or the canonical functor $y\backslash\mathfrak{H} \longrightarrow \mathfrak{H}$.

1.3. Supports. For any $x \in Ob\mathfrak{H}$, we denote by $\mathfrak{Supp}_{\mathfrak{H}}(x)$ the full subcategory of $\mathfrak{H}$ generated by all $y \in Ob\mathfrak{H}$ such that $\mathfrak{H}(x,y) = \emptyset$. We call $\mathfrak{Supp}_{\mathfrak{H}}(x)$ the *support of x in $\mathfrak{H}$*.

1.3.1. Proposition. *(a) For any two objects, x and y, of the category $\mathfrak{H}$, there exists an arrow $x \to y$ iff $\mathfrak{Supp}_{\mathfrak{H}}(x) \subseteq \mathfrak{Supp}_{\mathfrak{H}}(y)$.*

(b) Let $\{x_i \mid i \in J\}$ be a set of objects of $\mathfrak{H}$ such that there exists a coproduct, $\coprod_{i \in J} x_i$. Then

$$\mathfrak{Supp}_{\mathfrak{H}}\Big(\coprod_{i \in J} x_i\Big) = \bigcup_{i \in J} \mathfrak{Supp}_{\mathfrak{H}}(x_i). \tag{1}$$

Proof. (a) If there exists a morphism $x \longrightarrow y$ and $\mathfrak{H}(x,z) = \emptyset$, then, obviously, $\mathfrak{H}(y,z) = \emptyset$, hence $\mathfrak{Supp}_{\mathfrak{H}}(x) \subseteq \mathfrak{Supp}_{\mathfrak{H}}(y)$.

If $\mathfrak{H}(x,y) = \emptyset$, i.e. $y \in \mathfrak{Supp}_{\mathfrak{H}}(x)$, then, since $y \notin Ob\mathfrak{Supp}_{\mathfrak{H}}(y)$, the inclusion $\mathfrak{Supp}_{\mathfrak{H}}(x) \subseteq \mathfrak{Supp}_{\mathfrak{H}}(y)$ does not hold.

(b) Since $\mathfrak{H}(\coprod_{i \in J} x_i, z) \simeq \prod_{i \in J} \mathfrak{H}(x_i, z)$, it follows that $\mathfrak{H}(\coprod_{i \in J} x_i, z) = \emptyset$ iff $\mathfrak{H}(x_i, z) = \emptyset$ for some $i \in J$, whence the equality (1). ∎

1.3.2. Support in $\mathfrak{Spec}^1(\mathfrak{H})$. For any $x \in Ob\mathfrak{H}$, we denote the intersection $\mathfrak{Supp}_{\mathfrak{H}}(x) \bigcap \mathfrak{Spec}^1(\mathfrak{H})$ by $\mathfrak{Supp}^1_{\mathfrak{H}}(x)$ and call it the *support of x in* $\mathfrak{Spec}^1(\mathfrak{H})$. Evidently, 1.3.1(b) is still true if $\mathfrak{Supp}_{\mathfrak{H}}(x)$ is replaced by $\mathfrak{Supp}^1_{\mathfrak{H}}(x)$, as well as a half of 1.3.1(a): if $\mathfrak{H}(x,y)$ is not empty, then $\mathfrak{Supp}^1_{\mathfrak{H}}(x) \subseteq \mathfrak{Supp}^1_{\mathfrak{H}}(y)$.

1.4. The spectrum $\mathfrak{Spec}^0(\mathfrak{H})$.

We denote by $\mathfrak{Spec}^0(\mathfrak{H})$ the full subcategory of $\mathfrak{H}$ generated by $x \in Ob\mathfrak{H}$ such that $\mathfrak{Supp}_{\mathfrak{H}}(x)$ is not empty and has a final object, $\widehat{x}$.

1.4.1. Proposition. *Let $\mathfrak{H}$ be local. Then initial objects of $\mathfrak{H}^1$ belong to $\mathfrak{Spec}^0(\mathfrak{H})$.*

Proof. Let $\mathfrak{H}_0$ be the full subcategory (groupoid) of $\mathfrak{H}$ generated by all initial objects of $\mathfrak{H}$. If x is an initial object of the category $\mathfrak{H}^1$, then $\mathfrak{Supp}_{\mathfrak{H}}(x)$ coincides with $\mathfrak{H}_0$.

In fact, suppose that there is an arrow, $x \xrightarrow{f} y$, for some $y \in Ob\mathfrak{H}_0$. Since y is an initial object of the category $\mathfrak{H}$, there exists a unique morphism $y \xrightarrow{g} x$. By the universal property of y, the composition $y \xrightarrow{fg} y$ is the identical morphism. Since x is an initial object of the category $\mathfrak{H}^1$, the composition $x \xrightarrow{gf} x$ is the identical morphism too. This means that the morphism $x \xrightarrow{f} y$ is an isomorphism which contradicts to the fact that x is not an initial object of the category $\mathfrak{H}$.

Thus, $\mathfrak{H}_0$ is a subcategory of $\mathfrak{Supp}_{\mathfrak{H}}(x)$. Since for every $z \in Ob\mathfrak{H}^1 = Ob\mathfrak{H} - Ob\mathfrak{H}_0$ there is a (unique) morphism $x \longrightarrow z$, the subcategory $\mathfrak{Supp}_{\mathfrak{H}}(x)$ is contained in $\mathfrak{H}_0$; i.e. $\mathfrak{Supp}_{\mathfrak{H}}(x) = \mathfrak{H}_0$.

Since $\mathfrak{H}_0$ is a connected groupoid, every object of $\mathfrak{H}_0$ is final. ■

1.4.2.1. Example. The spectrum $\mathfrak{Spec}^0(Sets)$ coincides with the subcategory $Sets^1$ of all non-empty sets, because the support of any non-empty set consists of only $\emptyset$.

1.4.2. Proposition. *The full subcategory, $\mathfrak{Spec}^0(\mathfrak{H})_0$, generated by initial objects of $\mathfrak{Spec}^0(\mathfrak{H})$ coincides with be the full subcategory $\mathfrak{H}_1 = (\mathfrak{H}^1)_0$ of the category $\mathfrak{H}$ generated by initial objects of the subcategory $\mathfrak{H}^1$.*

Proof. By definition, the subcategory $\mathfrak{Supp}_{\mathfrak{H}}(x)$ (cf. 1.4) is not empty for every object x of $\mathfrak{Spec}^0(\mathfrak{H})$. Therefore initial objects of the category $\mathfrak{H}$ do not belong to $\mathfrak{Spec}^0(\mathfrak{H})$, i.e. $\mathfrak{Spec}^0(\mathfrak{H})$ is contained in the subcategory $\mathfrak{H}^1$. In particular, the subcategory $\mathfrak{Spec}^0(\mathfrak{H})_0$ of initial objects of $\mathfrak{Spec}^0(\mathfrak{H})$ is contained in $\mathfrak{H}_1 = (\mathfrak{H}^1)_0$. The converse inclusion is a consequence of 1.4.1. ■

1.4.3. Corollary. *Let $|\mathfrak{Spec}^1(\mathfrak{H})|$ denote the set of isomorphism classes of objects of $\mathfrak{Spec}^1(\mathfrak{H})$. Then*

$$Ob\mathfrak{Spec}^1(\mathfrak{H}) = \bigcup_{x\in|\mathfrak{Spec}^1(\mathfrak{H})|} \big\{y \mid (y, x \to y) = \widehat{z},\ z \in Ob\mathfrak{Spec}^0(x\backslash\mathfrak{H})_0\big\}. \tag{1}$$

In particular,

$$Ob\mathfrak{Spec}^1(\mathfrak{H}) \subseteq \bigcup_{x\in|\mathfrak{Spec}^1(\mathfrak{H})|} \big\{y \mid (y, x \to y) = \widehat{z},\ z \in Ob\mathfrak{Spec}^0(x\backslash\mathfrak{H})\big\}. \tag{2}$$

Here $\widehat{z}$ is a final object of the category $\mathfrak{Supp}_{x\backslash\mathfrak{H}}(z)$ (cf. 1.4).

Proof. The formula (1) follows from 1.4.2 applied to the category $x\backslash\mathfrak{H}$. ■

1.4.4. Lemma. *A choice for every $x \in Ob\mathfrak{Spec}^0(\mathfrak{H})$ of a final object, $\widehat{x}$, of the category $\mathfrak{Supp}_{\mathfrak{H}}(x)$ extends to a functor $\mathfrak{Spec}^0(\mathfrak{H}) \xrightarrow{\vartheta_{\mathfrak{H}}} \mathfrak{H}$.*

Proof. In fact, if $x, y \in Ob\mathfrak{Spec}^0(\mathfrak{H})$, and there is a morphism $x \longrightarrow y$, then $\mathfrak{Supp}_{\mathfrak{H}}(x) \subseteq \mathfrak{Supp}_{\mathfrak{H}}(y)$. Therefore there exists a unique morphism $\widehat{x} \longrightarrow \widehat{y}$. ■

1.4.5. Remark. Notice that the functor $\mathfrak{Spec}^0(\mathfrak{H}) \xrightarrow{\vartheta_{\mathfrak{H}}} \mathfrak{H}$ is faithful iff $\mathfrak{H}$ is a preorder, i.e. for any pair of objects, x, y, of $\mathfrak{H}$, there is at most one morphism $x \longrightarrow y$.

1.4.6. Proposition. *Suppose the category $\mathfrak{H}$ is a preorder with finite coproducts (i.e. supremums of pairs of objects). Then the functor $\mathfrak{Spec}^0(\mathfrak{H}) \xrightarrow{\vartheta_{\mathfrak{H}}} \mathfrak{H}$ takes values in $\mathfrak{Spec}^1(\mathfrak{H})$, i.e. it induces a functor $\mathfrak{Spec}^0(\mathfrak{H}) \xrightarrow{\theta_{\mathfrak{H}}} \mathfrak{Spec}^1(\mathfrak{H})$.*

Proof. For any $x \in Ob\mathfrak{Spec}^0(\mathfrak{H})$, the final object, $\widehat{x}$, of the category $\mathfrak{Supp}_{\mathfrak{H}}(x)$ belongs to $\mathfrak{Spec}^1(\mathfrak{H})$. More explicitly, we claim that the canonical coprojection, $\widehat{x} \longrightarrow x \sqcup \widehat{x}$, is an initial object of the category $(\widehat{x}\backslash\mathfrak{H})^1$.

In fact, let $\widehat{x} \xrightarrow{g} y$ be a morphism. Then one of two things happens: either $y \in Ob\mathfrak{Supp}_{\mathfrak{H}}(x)$, or not. If $y \in Ob\mathfrak{Supp}_{\mathfrak{H}}(x)$, then, since $\widehat{x}$ is a final object of the category $\mathfrak{Supp}_{\mathfrak{H}}(x)$, there is a unique morphism $y \xrightarrow{h} \widehat{x}$. It follows from the universal property of $\widehat{x}$ that $h \circ g = id_{\widehat{x}}$. By hypothesis, $\mathfrak{H}$ is a preorder, in particular, h is a monomorphism. Therefore, h is an isomorphism inverse to g.

If $y \notin Ob\mathfrak{Supp}_{\mathfrak{H}}(x)$, then there exists an arrow $x \longrightarrow y$ which, together with $\widehat{x} \xrightarrow{g} y$, determines (and is determined by) a morphism $(x \sqcup \widehat{x}, \widehat{x} \to x \sqcup \widehat{x}) \longrightarrow (y, \widehat{x} \xrightarrow{g} y)$. Since $\mathfrak{H}$ is a preorder, this is all we need. ■

1.4.6.1. Example. Let $\mathfrak{H}$ be a category with initial objects and such that $\mathfrak{H}(x,y) \neq \emptyset$ for every $x \in Ob\mathfrak{H}$ and $\mathfrak{H}(y,z) = \emptyset$ if y is not an initial object and z is an initial object. Then $\mathfrak{Supp}_{\mathfrak{H}}(x) = \mathfrak{H}_0$ for any $x \in Ob\mathfrak{H}^1$. Therefore, $\mathfrak{Spec}^0(\mathfrak{H})$ coincides with $\mathfrak{H}^1$. If $\mathfrak{H}$ is a preorder, then, under conditions, $\mathfrak{H}^1$ is a connected groupoid, hence $\mathfrak{Spec}^1(\mathfrak{H}) = \mathfrak{H}_0$ and the functor $\mathfrak{Spec}^0(\mathfrak{H}) \xrightarrow{\theta_{\mathfrak{H}}} \mathfrak{Spec}^1(\mathfrak{H})$ is a category equivalence.

What might happen if $\mathfrak{H}$ is not a preorder is illustrated by the following.

1.4.6.2. Example. If $\mathfrak{H} = Sets$, then, by 1.2.1.1, $\mathfrak{Spec}^1(\mathfrak{H}) = \{\emptyset, id_{\emptyset}\}$ and by 14.2.1, $\mathfrak{Spec}^0(\mathfrak{H}) = Sets^1$ – the category of all non-empty sets. There is only one functor $\mathfrak{Spec}^0(\mathfrak{H}) \longrightarrow \mathfrak{Spec}^1(\mathfrak{H})$.

1.4.7. Proposition. *Suppose that the category $\mathfrak{H}$ is a preorder with coproducts. Then*

$$Ob\mathfrak{Spec}^1(\mathfrak{H}) = \bigcup_{x \in |\mathfrak{Spec}^1(\mathfrak{H})|} \big\{y \mid (y, x \to y) = \widehat{z},\ z \in Ob\mathfrak{Spec}^0(x\backslash\mathfrak{H})\big\}. \tag{3}$$

Proof. The assertion follows from 1.4.6 and 1.4.3(2). ■

1.4.8. Support in $\mathfrak{Spec}^0(\mathfrak{H})$. For any object x of the category $\mathfrak{H}$, we denote by $\mathfrak{Supp}^0_{\mathfrak{H}}(x)$ the preimage of $\mathfrak{Supp}_{\mathfrak{H}}(x)$ by the functor $\mathfrak{Spec}^0(\mathfrak{H}) \longrightarrow \mathfrak{H}$, $z \longmapsto \widehat{z}$, (cf. 1.4.4) and call it the *support of x in* $\mathfrak{Spec}^0(\mathfrak{H})$. This means that $\mathfrak{Supp}^0_{\mathfrak{H}}(x)$ is a full subcategory of $\mathfrak{H}$ whose objects are all objects z of $\mathfrak{Spec}^0(\mathfrak{H})$ such that $\mathfrak{H}(x, \widehat{z}) = \emptyset$, or, equivalently, $\mathfrak{Supp}_{\mathfrak{H}}(z) \subseteq \mathfrak{Supp}_{\mathfrak{H}}(x)$. By 1.3.1, the latter means precisely that there exists a morphism

$z \to x$. Thus, $\mathfrak{Supp}^0_{\mathfrak{H}}(x)$ is a full subcategory of $\mathfrak{H}$ generated by all objects z of $\mathfrak{Spec}^0(\mathfrak{H})$ such that $z \to x$.

1.4.9. Proposition. *(a) The map $x \longmapsto \mathfrak{Supp}^0_{\mathfrak{H}}(x)$ is functorial: if there exists an arrow $x \to y$, then $\mathfrak{Supp}^0_{\mathfrak{H}}(x) \subseteq \mathfrak{Supp}^0_{\mathfrak{H}}(y)$.*

(b) Let $\{x_i \mid i \in J\}$ be a set of objects of $\mathfrak{H}$ such that there exists a coproduct, $\coprod_{i\in J} x_i$. Then

$$\mathfrak{Supp}^0_{\mathfrak{H}}\big(\coprod_{i\in J} x_i\big) = \bigcup_{i\in J} \mathfrak{Supp}^0_{\mathfrak{H}}(x_i). \tag{1}$$

Proof. (a) The assertion follows from the fact that

$$Ob\mathfrak{Supp}^0_{\mathfrak{H}}(x) = \{z \in Ob\mathfrak{Spec}^0(\mathfrak{H}) \mid \mathfrak{H}(z,x) \neq \emptyset\}$$

(see the discussion in 1.4.8).

(b) An object z of $\mathfrak{Spec}^0(\mathfrak{H})$ belongs to $\mathfrak{Supp}^0_{\mathfrak{H}}\big(\coprod_{i\in J} x_i\big)$ iff $\mathfrak{H}\big(\coprod_{i\in J} x_i, \widehat{z}\big) = \emptyset$. Since $\mathfrak{H}\big(\coprod_{i\in J} x_i, \widehat{z}\big) = \prod_{i\in J} \mathfrak{H}(x_i, \widehat{z})$, this occurs iff $\mathfrak{H}(x_i, \widehat{z}) = \emptyset$ for some $i \in J$. ∎

1.5. Topologies and spectra.

1.5.1. Generalities on topologies. Let τ be a topology on $\mathfrak{H}^{op}$, i.e. τ is a function which assigns to every object x of $\mathfrak{H}$ a set, $\tau(x)$, of subfunctors of the functor $\mathfrak{H}(x,-)$ (called the *refinements* of x) satisfying the following conditions:

(a) for every arrow $x \xrightarrow{f} y$ of $\mathfrak{H}$ and every $R \in \tau(x)$, the fibre product, R^f, of $R \longrightarrow \mathfrak{H}(x,-) \xleftarrow{\mathfrak{H}(f,-)} \mathfrak{H}(y,-)$ belongs to $\tau(y)$;

(b) If $R \in \tau(x)$ and E is a subfunctor of $\mathfrak{H}(x,-)$ such that $E^f \in \tau(y)$ for any $f \in R(y)$ and any y, then $E \in \tau(x)$.

1.5.1.1. Cocovers. A family of arrows $\widetilde{x} = \{x \xrightarrow{u_i} x_i \mid i \in J\}$ generates a subfunctor, $R_{\widetilde{x}}$, of $\mathfrak{H}(x,-)$ defined as follows: $R_{\widetilde{x}}(y)$ consists of all arrows $x \longrightarrow y$ which factor through $x \xrightarrow{u_i} x_i$ for some $i \in J$. The family $\widetilde{x} = \{x \xrightarrow{u_i} x_i \mid i \in J\}$ is called a *cocover* (or a cover in $\mathfrak{H}^{op}$) for the topology τ if $R_{\widetilde{x}} \in \tau(x)$.

1.5.1.2. Sheaves. Subcanonical topologies. A functor $\mathfrak{H} \xrightarrow{F} Sets$ (viewed as a presheaf of sets on $\mathfrak{H}^{op}$) is called a sheaf (on $(\mathfrak{H}^{op}, \tau)$) if for every $x \in Ob\mathfrak{H}$ and any refinement R of x, the map $F(x) \longrightarrow Hom(R,F)$ induced by the embedding $R \hookrightarrow \mathfrak{H}(x,-)$ and the Yoneda isomorphism $F(x) \simeq Hom(\mathfrak{H}(x,-),F)$, is a bijection.

The topology τ on $\mathfrak{H}^{op}$ is called *subcanonical* if every representable presheaf, i.e. a functor of the form $\mathfrak{H}(x,-)$, $x \in Ob\mathfrak{H}$, is a sheaf.

1.5.1.3. Cosieves. It is convenient sometimes to describe topologies on $\mathfrak{H}^{op}$ in terms of cosieves. Recall that a *cosieve* in a category $\mathcal{A}$ is a full subcategory, $\mathcal{B}$, of $\mathcal{A}$ such that for every $x \in Ob\mathcal{B}$, all arrows $x \longrightarrow y$ belong to $\mathcal{B}$.

Let x be an object of the category $\mathfrak{H}$. There is a one-to-one correspondence between cosieves of the category $x\backslash\mathfrak{H}$ and subfunctors of the functor $\mathfrak{H}(x,-)$. Namely, an object $(y, x \xrightarrow{\xi} y)$ of $x\backslash\mathfrak{H}$ belongs to the cosieve $\widetilde{R}$ corresponding to a subfunctor R of $\mathfrak{H}(x,-)$ iff the morphism $x \xrightarrow{\xi} y$ is an element of $R(y)$.

Thus, a topology, τ, on $\mathfrak{H}^{op}$ can be described as a function which assigns to each object of $\mathfrak{H}$ a non-empty family of cosieves of the category $x\backslash\mathfrak{H}$, which are also called *refinements of* x, satisfying the conditions reflecting properties (a) and (b) of 1.5.1.

One can see that a topology τ on $\mathfrak{H}^{op}$ is subcanonical iff for every $x \in Ob\mathfrak{H}$ each refinement of x is a terminal cone. In other words, for every refinement $\widetilde{R}$ of x, the limit of the canonical functor $R \longrightarrow \mathfrak{H}$, $(y, x \xrightarrow{\xi} y) \longmapsto y$, is isomorphic to x.

1.5.1.4. Pretopologies on $\mathfrak{H}^{op}$. A *pretopology* on $\mathfrak{H}^{op}$ is a function, τ, which assigns to each object x of $\mathfrak{H}$ a family, τ_x, of sets of arrows $\{x \to x_i \mid i \in J\}$ (in $\mathfrak{H}$) having the following properties:

(a) for every $x \in Ob\mathfrak{H}$, $\{id_x\} \in \tau_x$;

(b) if $\{x \to x_i \mid i \in J\} \in \tau_x$ and $\{x_i \to x_{ij} \mid j \in J_i\} \in \tau_{x_i}$ for every $i \in J$, then $\{x \to x_{ij} \mid i \in J,\ j \in J_i\} \in \tau_x$;

(c) for any $\widetilde{x} = \{x \to x_i \mid i \in J\} \in \tau_x$ and any morphism $x \xrightarrow{\phi} y$, there exists $\widetilde{y} = \{y \xrightarrow{v_j} y_j \mid j \in I\} \in \tau_y$ such that the morphism ϕ can be lifted to a morphism $\widetilde{x} \xrightarrow{\widetilde{\phi}} \widetilde{y}$. The latter means that for every $j \in I$, the composition of $x \xrightarrow{\phi} y$ and $y \longrightarrow y_j$ factors through $x \longrightarrow x_i$ for some $i \in J$.

Elements of τ_x are called *cocovers* of x. Arrows which belong to cocovers are interpreted as closed subsets. The corresponding arrows in $\mathfrak{H}^{op}$ are viewed as open subsets.

Every pretopology determines a topology obtained by taking cosieves (or subfunctors of representable functors) associated with cocovers.

1.5.2. Proposition. *Suppose that τ is a subcanonical topology on $\mathfrak{H}^{op}$; and let $\widetilde{x} = \{x \xrightarrow{u_i} x_i \mid i \in J\}$ be a cocover for τ. Then*

$$\mathfrak{Spec}^1(x\backslash\mathfrak{H}) = \bigcup_{i \in J} \mathfrak{Spec}^1(x_i\backslash\mathfrak{H}).$$

Here $\mathfrak{Spec}^1(x_i\backslash\mathfrak{H})$ is identified with its image in $x\backslash\mathfrak{H}$ via the morphism $x \xrightarrow{u_i} x_i$.

Proof. Let an object $\widetilde{z} = (z, x \xrightarrow{\mathfrak{f}} z)$ belong to $\mathfrak{Spec}^1(x\backslash\mathfrak{H})$. By definition, this means that the subcategory $(\widetilde{z}\backslash(x\backslash\mathfrak{H}))^1$ has an initial object. Notice that the category $\widetilde{z}\backslash(x\backslash\mathfrak{H})$ is isomorphic to the category $z\backslash\mathfrak{H}$. Thus, the category $(z\backslash\mathfrak{H})^1$ has an initial object, $(z^\star, z \xrightarrow{\xi} z^\star)$. By condition, ξ is not an isomorphism.

Let $R_{\widetilde{x}}$ be a refinement of x associated with the cocover $\widetilde{x} = \{x \xrightarrow{u_i} x_i \mid i \in J\}$; and let $R^{\xi}_{\widetilde{x}}$ be the corresponding refinement of the object z. Notice that there exists $y \in Ob\mathfrak{H}$ and an element $z \xrightarrow{g} y$ of $R^{\xi}_{\widetilde{x}}(y)$ such that $(y, z \xrightarrow{g} y) \notin Ob(z\backslash\mathfrak{H})^1$.

In fact, if such element would not exist, then, since $(z^\star, z \xrightarrow{\xi} z^\star)$ is an initial object of $(z\backslash\mathfrak{H})^1$, every element $z \xrightarrow{g} y$ of $R^{\xi}_{\widetilde{x}}(y)$ factors through $z \xrightarrow{\xi} z^\star$, and this factorization is unique. Since the topology τ is subcanonical, (z, id_z) is an initial object of the sieve $\widehat{R}^{\xi}_{\widetilde{x}}$ associated with $R^{\xi}_{\widetilde{x}}$. Therefore, there exists a unique morphism $(z^\star, \xi) \longrightarrow (z, id_z)$. But, this cannot happen (see the argument of 1.4.6).

Thus, there exists an element $z \xrightarrow{g} y$ of $R^{\xi}_{\widetilde{x}}(y)$ such that $(y, z \xrightarrow{g} y) \notin Ob(z\backslash\mathfrak{H})^1$, or, what is the same, the arrow $z \xrightarrow{g} y$ is an isomorphism. By the definition of $\widehat{R}^{\xi}_{\widetilde{x}}$, there exists a commutative diagram

$$\begin{array}{ccc} x & \xrightarrow{\ \mathfrak{f}\ } & z \\ {\scriptstyle u_i}\downarrow & & \downarrow{\scriptstyle g} \\ x_i & \xrightarrow{\ \mathfrak{f}_i\ } & y \end{array} \qquad (4)$$

for some $i \in J$. Since the arrow g in (4) is an isomorphism, it follows from (4) that $x \xrightarrow{\mathfrak{f}} z$ factors through the element $x \xrightarrow{u_i} x_i$ of the cocover $\widetilde{x}$. Therefore, the object $(z, \mathfrak{f})$ of $\mathfrak{Spec}^1(x\backslash\mathfrak{H})$ is the image of an object $(z, g\mathfrak{f}_i)$ of $\mathfrak{Spec}^1(x_i\backslash\mathfrak{H})$; hence the assertion. ■

For any $x \in Ob\mathfrak{H}$, let $\mathcal{U}_{\mathfrak{H}}(x)$ denote the full subcategory of $\mathfrak{H}$ generated by all $y \in Ob\mathfrak{H}$ such that $\mathfrak{H}(y, x) = \emptyset$. Thus, $\mathcal{U}_{\mathfrak{H}}(x)$ coincides with $\big(\mathfrak{Supp}_{\mathfrak{H}^{op}}(x)\big)^{op}$.

1.5.3. Proposition. *Let $x \in Ob\mathfrak{H}$ be such that for any $z \in Ob\mathfrak{H}$, there exists a coproduct, $x \sqcup z$. Then the map $z \longmapsto (x \sqcup z, x \to x \sqcup z)$ defines a functor*

$$\mathfrak{Spec}^0(\mathfrak{H}) \bigcap \mathcal{U}_{\mathfrak{H}}(x) \longrightarrow \mathfrak{Spec}^0(x\backslash\mathfrak{H}).$$

Proof. For any $y \in Ob\mathfrak{H}$, we set $f^*(y) = (y \sqcup x, x \to y \sqcup x)$. The map $y \mapsto f^*(y)$ extends to a functor, $\mathfrak{H} \xrightarrow{f^*} x\backslash\mathfrak{H}$, which is left adjoint to the functor $x\backslash\mathfrak{H} \xrightarrow{f_*} \mathfrak{H}$, $(v, x \to v) \mapsto v$.

Let z be an object of the subcategory $\mathfrak{Spec}^0(\mathfrak{H}) \bigcap \mathcal{U}_{\mathfrak{H}}(x)$, and let $\widehat{z}$ be a final object of the category $\mathfrak{Supp}_{\mathfrak{H}}(z)$. Since $z \in Ob\mathcal{U}_{\mathfrak{H}}(x)$, i.e. $\mathfrak{H}(z, x) = \emptyset$, there exists a unique morphism $x \to \widehat{z}$. We claim that the $(\widehat{z}, x \to \widehat{z})$ is a final object of the category $\mathfrak{Supp}_{x\backslash\mathfrak{H}}(f^*(z))$.

In fact, $x\backslash\mathfrak{H}(f^*(z),(y,x \to y)) \simeq \mathfrak{H}(z,y)$ which shows that $(y, x \to y)$ is an object of $\mathfrak{Supp}_{x\backslash\mathfrak{H}}(f^*(z))$ iff y is an object of $\mathfrak{Supp}_{\mathfrak{H}}(z)$. Therefore, $(\widehat{z}, x \to \widehat{z})$ belongs to $\mathfrak{Supp}_{x\backslash\mathfrak{H}}(f^*(z))$ and, moreover, is a final object of this category. ∎

1.5.4. Corollary. *Let $x \overset{u}{\to} y$ be a morphism of $\mathfrak{H}$ such that for any other morphism, $x \overset{v}{\to} z$, there exists a fibred coproduct, $y \sqcup_x z$. Then the functor*

$$y\backslash\mathfrak{H} \xrightarrow{u_*} x\backslash\mathfrak{H}, \quad (z, y \overset{g}{\to} z) \longmapsto (z, x \overset{gu}{\to} z),$$

has a left adjoint, u^*; and u^* induces a functor

$$\mathfrak{Spec}^0(x\backslash\mathfrak{H}) \bigcap \mathcal{U}_{x\backslash\mathfrak{H}}(y,u) \longrightarrow \mathfrak{Spec}^0(y\backslash\mathfrak{H}).$$

Proof. The fact is a consequence of 1.5.3 applied to the category $x\backslash\mathfrak{H}$. ∎

1.5.5. Proposition. *Let $\mathbf{x} = \{x \overset{u_i}{\to} x_i \mid i \in J\}$ be a set of arrows such that the cone $x \longrightarrow \widetilde{R}_{\mathbf{x}}$ is terminal (i.e. $x = \lim \widetilde{R}_{\mathbf{x}}$) and for any arrow $x \longrightarrow y$, there exist fibred coproducts $x_i \sqcup_x y$. If $\mathfrak{H}$ is a preorder, then the image of the canonical map $\mathfrak{Spec}^0(x\backslash\mathfrak{H}) \longrightarrow \mathfrak{Spec}^1(x\backslash\mathfrak{H})$ is contained in the union of images of $\mathfrak{Spec}^0(x_i\backslash\mathfrak{H})$, $i \in J$, in $\mathfrak{Spec}^1(x\backslash\mathfrak{H})$.*

Proof. By 1.5.4, there are natural functors

$$\mathfrak{Spec}^0(x\backslash\mathfrak{H}) \bigcap \mathcal{U}_{x\backslash\mathfrak{H}}(x_i,u_i) \longrightarrow \mathfrak{Spec}^0(x_i\backslash\mathfrak{H})$$

Therefore, it suffices to show that for any object (z,ξ) of $\mathfrak{Spec}^0(x\backslash\mathfrak{H})$, there exists $i \in J$ such that there are no morphisms from (z,ξ) to (x_i,u_i).

Suppose that for each $i \in J$, there is an arrow $(z,\xi) \longrightarrow (x_i,u_i)$. Since $\mathfrak{H}$ is a preorder, these arrows determine a cone $z \longrightarrow \widetilde{R}_{\mathbf{x}}$. By hypothesis, $x = \lim \widetilde{R}_{\mathbf{x}}$, hence there exists a morphism $(z,\xi) \longrightarrow (x, id_x)$. Since (x, id_x) is an initial object of the category $x\backslash\mathfrak{H}$, this means that for any object of $x\backslash\mathfrak{H}$ there is an arrow from (z,ξ) to this object, which cannot happen, because (z,ξ) belongs to $\mathfrak{Spec}^0(x\backslash\mathfrak{H})$. Thus, there exists $i \in J$ such that there are no morphisms from (z,ξ) to (x_i,u_i). ∎

1.5.6. The spectrum of a precosite. Let τ be a pretopology on $\mathfrak{H}^{op}$. We call the pair $(\mathfrak{H},\tau)$ a *precosite*. Let $\mathfrak{H}_\tau$ denote the subcategory of $\mathfrak{H}$ formed by arrows which belong to some cocovers. For every $x \in Ob\mathfrak{H}$, we denote by τ_x the induced pretopology on $\mathfrak{H}^{op}/x = (x\backslash\mathfrak{H})^{op}$. We denote by $\mathfrak{Spec}^0(x\backslash\mathfrak{H},\tau_x)$ the full subcategory of $x\backslash\mathfrak{H}$ generated by the images of $\mathfrak{Spec}^0(y\backslash\mathfrak{H})$ for all arrows $x \to y$ of the subcategory $\mathfrak{H}_\tau$. We call $\mathfrak{Spec}^0(x\backslash\mathfrak{H}, \tau_x)$ the *spectrum* of the precosite $(x\backslash\mathfrak{H}, \tau_x)$.

Thus, if the category $\mathfrak{H}$ has an initial object, we obtain the spectrum, $\mathfrak{Spec}^0(\mathfrak{H}, \tau)$, of the precosite $(\mathfrak{H}, \tau)$. If, in addition, all arrows of the subcategory $\mathfrak{H}_\tau$ are isomorphisms (for instance, the pretopology τ is discrete), then $\mathfrak{Spec}^0(\mathfrak{H}, \tau)$ coincides with $\mathfrak{Spec}^0(\mathfrak{H})$.

1.5.6.1. Proposition. *Suppose that $\mathfrak{H}_\tau = \mathfrak{H}$ and $\mathfrak{H}$ is a preorder with finite coproducts and an initial object. Then $\mathfrak{Spec}^0(\mathfrak{H}, \tau)$ is isomorphic to $\mathfrak{Spec}^1(\mathfrak{H})$.*

Proof. The assertion is a consequence of 1.4.6. ∎

1.5.6.2. Proposition. *To any morphism, $x \longrightarrow y$, of the subcategory $\mathfrak{H}_\tau$, there corresponds an inclusion $\mathfrak{Spec}^0(y\backslash\mathfrak{H}, \tau_y) \subseteq \mathfrak{Spec}^0(x\backslash\mathfrak{H}, \tau_x)$, i.e. the map $x \longmapsto \mathfrak{Spec}^0(x\backslash\mathfrak{H}, \tau_x)$ extends to a functor $\mathfrak{H}_\tau^{op} \longrightarrow Cat$.*

Proof. The fact follows from definitions. ∎

1.6. Relative spectra.

Let $\mathfrak{G} \xrightarrow{F} \mathfrak{H}$ be a functor. We define the relative spectra, $\mathfrak{Spec}^1(\mathfrak{G}, F)$ and $\mathfrak{Spec}^0(\mathfrak{G}, F)$, via cartesian squares

$$
\begin{array}{ccc}
\mathfrak{Spec}^1(\mathfrak{G}, F) & \xrightarrow{\theta_F^1} & \mathfrak{G} \\
{\scriptstyle \pi_1^F}\downarrow & & \downarrow{\scriptstyle F} \\
\mathfrak{Spec}^1(\mathfrak{H}) & \xrightarrow{\theta_{\mathfrak{H}}^1} & \mathfrak{H}
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\mathfrak{Spec}^0(\mathfrak{G}, F) & \xrightarrow{\vartheta_F} & \mathfrak{G} \\
{\scriptstyle \pi_0^F}\downarrow & & \downarrow{\scriptstyle F} \\
\mathfrak{Spec}^0(\mathfrak{H}) & \xrightarrow{\vartheta_{\mathfrak{H}}} & \mathfrak{H}
\end{array}
\qquad (1)
$$

(in the bicategorical sense, i.e. the squares quasi-commute), where $\mathfrak{Spec}^0(\mathfrak{H}) \xrightarrow{\vartheta_{\mathfrak{H}}} \mathfrak{H}$ is the canonical functor of 1.4.4.

Explicitly, objects of the category $\mathfrak{Spec}^1(\mathfrak{G}, F)$ are triples $(z, x; \phi)$, where z is an object of $\mathfrak{Spec}^1(\mathfrak{H})$, $x \in Ob\mathfrak{G}$, and ϕ is an isomorphism $z \xrightarrow{\sim} F(x)$. Morphisms from $(z, x; \phi)$ to $(z', x'; \phi')$ are given by pairs of arrows, $z \xrightarrow{g} z'$ and $x \xrightarrow{h} x'$ such that the diagram

$$
\begin{array}{ccc}
z & \xrightarrow{g} & z' \\
{\scriptstyle \phi}\downarrow\wr & & \wr\downarrow{\scriptstyle \phi'} \\
F(x) & \xrightarrow{F(h)} & F(x')
\end{array}
$$

commutes. The projections $\mathfrak{Spec}^1(\mathfrak{H}) \xleftarrow{\pi_1^F} \mathfrak{Spec}^1(\mathfrak{G}, F) \xrightarrow{\theta_F^1} \mathfrak{G}$ in the left diagram (1) are defined by $\pi_1^F(z, x; \phi) = z$ and $\theta_F^1(z, x; \phi) = x$.

Similarly, objects of the category $\mathfrak{Spec}^0(\mathfrak{G}, F)$ are triples $(z, x; \psi)$, where z is an object of $\mathfrak{Spec}^0(\mathfrak{H})$, $x \in Ob\mathfrak{G}$, and ψ is an isomorphism $\vartheta_{\mathfrak{H}}(z) \xrightarrow{\sim} F(x)$.

1.6.1. Proposition. *Let* i *be* 0 *or* 1. *The map* $(\mathfrak{G}, F) \longmapsto \mathfrak{Spec}^i(\mathfrak{G}, F)$ *extends to a pseudo-functor* $\mathfrak{Spec}^i : Cat/\mathfrak{H} \longrightarrow Cat$.

Proof. The assertion follows from the universal property of cartesian squares. ∎

1.6.2. Proposition. *Suppose* $\mathfrak{H}$ *is a preorder with finite coproducts. Then for every functor* $\mathfrak{G} \xrightarrow{F} \mathfrak{H}$, *there is a canonical functor*

$$\mathfrak{Spec}^0(\mathfrak{G}, F) \xrightarrow{\vartheta_{(\mathfrak{G},F)}} \mathfrak{Spec}^1(\mathfrak{G}, F). \tag{2}$$

The family $\{\vartheta_{(\mathfrak{G},F)} \mid (\mathfrak{G}, F) \in ObCat/\mathfrak{H}\}$ *is a morphism of pseudo-functors,*

$$\mathfrak{Spec}^0 \xrightarrow{\vartheta} \mathfrak{Spec}^1. \tag{3}$$

Proof. Since $\mathfrak{H}$ is a preorder with finite coproducts, the functor $\mathfrak{Spec}^0(\mathfrak{H}) \xrightarrow{\vartheta_{\mathfrak{H}}} \mathfrak{H}$ takes values in $\mathfrak{Spec}^1(\mathfrak{H})$, hence it factors through the embedding $\mathfrak{Spec}^1(\mathfrak{H}) \longrightarrow \mathfrak{H}$ (see 1.4.6). By the universal property of cartesian squares, there exists a unique functor (2) such that $\theta^1_F \circ \vartheta_{(\mathfrak{G},F)} = \vartheta_F$ and $\pi^F_1 \circ \vartheta_{(\mathfrak{G},F)} = \pi^F_0$ (see the diagram (2)).

It is useful to have an explicit description of the functor (2) in terms of the descriptions of $\mathfrak{Spec}^0(\mathfrak{G}, F)$ and $\mathfrak{Spec}^1(\mathfrak{G}, F)$ given above. The functor $\vartheta_{(\mathfrak{G},F)}$ maps an object $(z, x; \psi)$ of $\mathfrak{Spec}^0(\mathfrak{G}, F)$ to the object $(\vartheta_{\mathfrak{H}}(z), x; \psi)$ of $\mathfrak{Spec}^1(\mathfrak{G}, F)$.

It follows from this description that $\vartheta = \{\vartheta_{(\mathfrak{G},F)} \mid (\mathfrak{G}, F) \in ObCat/\mathfrak{H}\}$ is a morphism of pseudo-functors. ∎

1.7. The strict support and the spectrum $\mathfrak{Spec}(\mathfrak{H})$.

We fix a category $\mathfrak{H}$ with an initial object $\mathfrak{x}$. For any pair of objects y, z of $\mathfrak{H}$, we shall write $y \cap z = \mathfrak{x}$ if any diagram $y \leftarrow w \rightarrow z$ factors through $y \leftarrow \mathfrak{x} \rightarrow z$; that is there exists a morphism $w \rightarrow \mathfrak{x}$ such that the diagram

$$\begin{array}{ccccc} y & \longleftarrow & w & \longrightarrow & z \\ {\scriptstyle id}\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle id} \\ y & \longleftarrow & \mathfrak{x} & \longrightarrow & z \end{array}$$

commutes.

1.7.1. Observations. (a) It follows that $y \cap z = \mathfrak{x}$ if y is an initial object.

(b) Recall that a morphism $y \longrightarrow z$ is called *trivial* if it factors through an initial object. If $y \cap z = \mathfrak{x}$, then there are no non-trivial arrows between y and z.

(b') In particular, if all arrows of $\mathfrak{H}$ are monomorphisms (say, $\mathfrak{H}$ is a preorder), then $y \cap z = \mathfrak{x}$ implies that either y is an initial object, or $\mathfrak{H}(y, z) = \emptyset$.

(c) If all arrows of $\mathfrak{H}$ are monomorphisms, then $y \leftarrow w \rightarrow z$ factors through $y \leftarrow \mathfrak{x} \rightarrow z$ iff the object w is trivial.

1.7.2. The strict support. For any $x \in Ob\mathfrak{H}$, we denote by $\mathfrak{Supp}^{\mathfrak{s}}_{\mathfrak{H}}(x)$ the full subcategory of $\mathfrak{H}$ generated by all $y \in Ob\mathfrak{H}$ such that $y \cap x = \mathfrak{x}$. We call the subcategory $\mathfrak{Supp}^{\mathfrak{s}}_{\mathfrak{H}}(x)$ the *strict support* of x.

It follows that $\mathfrak{Supp}^{\mathfrak{s}}_{\mathfrak{H}}(x)$ is a *strictly* full subcategory of $\mathfrak{H}$ containing all initial objects. In particular, $\mathfrak{Supp}^{\mathfrak{s}}_{\mathfrak{H}}(x)$ is non-empty for all objects x (unlike the support $\mathfrak{Supp}_{\mathfrak{H}}(x)$ of 1.3 which might be empty for some objects x).

Notice that if x is an initial object, then $\mathfrak{Supp}^{\mathfrak{s}}_{\mathfrak{H}}(x) = \mathfrak{H}$.

We denote the final object of $\mathfrak{Supp}^{\mathfrak{s}}_{\mathfrak{H}}(x)$ (if any) by $\check{x}$.

1.7.2.1. Note. It follows from 1.7.1(b') that if all arrows of $\mathfrak{H}$ are monomorphisms, then $\mathfrak{Supp}^{\mathfrak{s}}_{\mathfrak{H}}(x) \subseteq \mathfrak{Supp}_{\mathfrak{H}}(x)$ for all $x \in Ob\mathfrak{H}_1$. If both a final object $\check{x}$ of $\mathfrak{Supp}^{\mathfrak{s}}_{\mathfrak{H}}(x)$ and a final object $\widehat{x}$ of $\mathfrak{Supp}_{\mathfrak{H}}(x)$ exist, then there is a unique arrow $\check{x} \longrightarrow \widehat{x}$. This arrow is an isomorphism if $x \cap \widehat{x} = \mathfrak{x}$.

1.7.2.2. Lemma. *The map $x \longmapsto \mathfrak{Supp}^{\mathfrak{s}}_{\mathfrak{H}}(x)$ is a contravariant functor: if $x \to y$, then $\mathfrak{Supp}^{\mathfrak{s}}_{\mathfrak{H}}(y) \subseteq \mathfrak{Supp}^{\mathfrak{s}}_{\mathfrak{H}}(x)$.*

1.7.3. The closure. For an object x of $\mathfrak{H}$, let $\{x\}^-$ denote the full subcategory of $\mathfrak{H}$ generated by all objects y such that if $z \to y$ is a non-trivial morphism, then $z \cap x \neq \mathfrak{x}$.

It follows from this definition that, for any object x of $\mathfrak{H}$, the subcategory $\{x\}^-$ contains the full subcategory $\mathfrak{H}_0$ generated by all initial objects of $\mathfrak{H}$, and if x is an initial object, then $\{x\}^-$ coincides with $\mathfrak{H}_0$. It follows from 1.7.1(b) that x is an object of $\{x\}^-$.

We denote the final object of $\{x\}^-$ (if any) by x^-.

1.7.4. Proposition. *Let $\mathfrak{H}$ be a preorder with initial objects.*

(a) If there exists a morphism $x \to y$, then $\{x\}^- \subseteq \{y\}^-$.

(b) Let $x \in Ob\mathfrak{H}$ be such that x^- exists. Then $(x^-)^- \simeq x^-$.

(c) Let $x \in Ob\mathfrak{H}$ be such that the final object $\check{x}$ of $\mathfrak{Supp}^{\mathfrak{s}}_{\mathfrak{H}}(x)$ exists. Then $\check{x}^- = \check{x}$.

Proof. (a) Suppose that an object z' belongs to $\{x\}^-$, that is for any non-trivial arrow $z \to z'$ there exists a diagram $x \leftarrow w \to z$, where w is not an initial object. But then we have the diagram $y \leftarrow w \to z$ (thanks to the arrow $x \to y$), hence $z' \in Ob\{y\}^-$.

(b) By definition, $Ob\{x^-\}^-$ consists of all objects y of $\mathfrak{H}$ such that if $z \to y$ is a non-trivial arrow, then $z \cap x^- \neq \mathfrak{x}$. The latter means that there exists a pair of arrows $z \leftarrow w \to x^-$ with non-trivial w. Since x^- is an object of $\{x\}^-$, the intersection $w \cap x$

is non-trivial; i.e. there exists a diagram $w \leftarrow v \rightarrow x$ with non-initial v. Composing it with $z \leftarrow w$, we obtain the diagram $z \leftarrow v \rightarrow x$. Therefore, $y \in Ob\{x\}^-$. This proves the inclusion $\{x^-\}^- \subseteq \{x\}^-$. The inverse inclusion follows from the existence of the morphism $x \rightarrow x^-$ (because $x \in Ob\{x\}^-$ and x^- is a final object of $\{x\}^-$) and (a) above.

(c) Let x be an object of $\mathfrak{H}$ such that there exists a final object, $\check{x}$, of $\mathfrak{Supp}^{\mathfrak{s}}_{\mathfrak{H}}(x)$. Let y be an object of $\{\check{x}\}^-$, that is for any non-trivial morphism $z \rightarrow y$, the intersection $z \cap \check{x}$ is non-trivial. The claim is that y belongs to $\mathfrak{Supp}^{\mathfrak{s}}_{\mathfrak{H}}(x)$, i.e. $y \cap x = \mathfrak{x}$.

Suppose that, on the contrary, $y \cap x$ is non-trivial; i.e. there is a diagram $y \leftarrow z \rightarrow x$ with non-initial z. Then $z \cap \check{x}$ is non-trivial, that is there exists a diagram $z \leftarrow w \rightarrow \check{x}$ with a non-initial w. Composing with $z \rightarrow x$, we obtain a diagram $x \leftarrow w \rightarrow \check{x}$, which contradicts to the fact that $x \cap \check{x} = \mathfrak{x}$. ∎

1.7.5. Note. For any subset $\mathcal{B}$ of objects of $\mathfrak{H}$, set $\mathfrak{Supp}^{\mathfrak{s}}_{\mathfrak{H}}(\mathcal{B}) \stackrel{\mathrm{def}}{=} \bigcap_{x \in \mathcal{B}} \mathfrak{Supp}^{\mathfrak{s}}_{\mathfrak{H}}(x)$. For a subcategory $\mathfrak{B}$ of $\mathfrak{H}$, we set $\mathfrak{Supp}^{\mathfrak{s}}_{\mathfrak{H}}(\mathfrak{B}) \stackrel{\mathrm{def}}{=} \mathfrak{Supp}^{\mathfrak{s}}_{\mathfrak{H}}(Ob\mathfrak{B})$.

One can see that $\{x\}^- = \mathfrak{Supp}^{\mathfrak{s}}_{\mathfrak{H}}(\mathfrak{Supp}^{\mathfrak{s}}_{\mathfrak{H}}(x))$ for any $x \in Ob\mathfrak{H}$.

1.8. The relative version. Fix a functor $\mathfrak{G} \xrightarrow{F} \mathfrak{H}$.

1.8.1. Relative closure. For any $x \in Ob\mathfrak{H}$, we define the *closure* $\{x\}^-_F$ of x in $\mathfrak{G}$ via the cartesian square

$$\begin{array}{ccc} \{x\}^-_F & \longrightarrow & \mathfrak{G} \\ \Big\downarrow & cart & \Big\downarrow F \\ \{x\}^- & \longrightarrow & \mathfrak{H} \end{array}$$

In other words, $\{x\}^-_F$ is the preimage of $\{x\}^-$ in $\mathfrak{G}$.

1.8.1.1. Example. Let C_X be a svelte abelian category and $\mathfrak{H}$ the preorder (with resp. to the inclusion) of its strictly full subcategories closed under taking subobjects, $\mathfrak{G}$ the preorder of strictly full subcategories of C_X closed under taking subquotients, F the inclusion functor $\mathfrak{G} \hookrightarrow \mathfrak{H}$. Then for each subcategory $\mathcal{S} \in \mathfrak{H}$, the subcategory $\mathcal{S}^-$ exists and coincides with the *Serre envelope* of the subcategory $\mathcal{S}$. Objects of $\mathcal{S}^-$ are all objects M of C_X whose nonzero subquotients have nonzero subobjects from $\mathcal{S}$. Thus, $\mathcal{S}^-$ is a *Serre* (in particular, thick) subcategory of C_X.

1.8.1.2. Example: the closure in Serre subcategories. Let $\mathfrak{H}$ be as above, $\mathfrak{G}$ the preorder $\mathfrak{Se}(X)$ of Serre subcategories of C_X, and F the inclusion functor. Although $\{\mathcal{S}\}^-_F$ is, usually, not the same as in the previous setting, the final object is the same – the Serre envelope $\mathcal{S}^-$ of the subcategory $\mathcal{S}$.

1.8.2. The relative strict support. Similarly, the relative *strict support* of $x \in Ob\mathfrak{H}$ is defined via the cartesian square

$$\begin{array}{ccc} \mathfrak{Sup}^F(x) & \longrightarrow & \mathfrak{G} \\ \Big\downarrow & cart & \Big\downarrow F \\ \mathfrak{Supp}^{\mathfrak{s}}_{\mathfrak{H}}(x) & \longrightarrow & \mathfrak{H} \end{array}$$

i.e. objects of $\mathfrak{Sup}^F(x)$ are all $y \in Ob\mathfrak{G}$ such that $F(y) \cap x = \mathfrak{x}$.

1.8.2.1. Example. Consider the setting of 1.8.1.2; that is $\mathfrak{H}$ the preorder of strictly full subcategories of a svelte abelian category C_X which are closed under taking subobjects, $\mathfrak{G}$ is the preorder $\mathfrak{Se}(X)$ of Serre subcategories of C_X, and F the inclusion functor. The subpreorder $\mathfrak{Sup}^F(x)$ has a final object iff the smallest topologizing subcategory $[x]$ spanned by x belongs to the spectrum $\mathbf{Spec}(X)$.

There is a relative version of 1.7.4 which looks as follows.

1.8.3. Proposition. *Let $\mathfrak{H}$ be a preorder with initial objects and $\mathfrak{G} \xrightarrow{F} \mathfrak{H}$ a functor.*
(a) If $x,\ y$ are objects of $\mathfrak{H}$ and there exists a morphism $x \to y$, then $\{x\}^-_F \subseteq \{y\}^-_F$.
(b) Suppose that the closure $\{x\}^-_F$ of an object x of $\mathfrak{H}$ in $\mathfrak{G}$ has a final object, x^-. Then the closure $\{F(x^-)\}^-$ of $F(x^-)$ in $\mathfrak{G}$ has a final object which is isomorphic to x^-.
(c) Let $x \in Ob\mathfrak{H}$ be such that the support $\mathfrak{Sup}^F(x)$ of x in $\mathfrak{G}$ has a final object, $\check{x}$. The object $\check{x}$ is a final object of the relative closure of $F(\check{x})$.

Proof. (a) The assertion follows from 1.7.4(a) and the definition of the relative support.
(b) and (c). The arguments are adaptations of the corresponding arguments of 1.7.4. Details are left to the reader. ■

1.9. The spectra.

Fix a preorder $\mathfrak{H}$ with an initial object $\mathfrak{x}$. Recall that $\mathfrak{H}_1$ denote the full subcategory of $\mathfrak{H}$ generated by non-initial objects.

1.9.1. The spectrum $\mathfrak{Spec}(\mathfrak{H})$. Recall that the spectrum $\mathfrak{Spec}^0(\mathfrak{H})$ is a full subcategory of $\mathfrak{H}$ whose objects are those $x \in Ob\mathfrak{H}_1$ for which $\mathfrak{Supp}_{\mathfrak{H}}(x)$ has a final object, $\widehat{x}$. We denote by $\mathfrak{Spec}^\vee(\mathfrak{H})$ the full subcategory of $\mathfrak{Spec}(\mathfrak{H})$ generated by all x such that $x \cap \widehat{x} = \mathfrak{x}$. Consider the functor

$$\mathfrak{Spec}^\vee(\mathfrak{H}) \xrightarrow{\phi_{\mathfrak{H}}} \mathfrak{H}, \quad x \longmapsto \widehat{x},$$

and the class $\Sigma_{\phi_{\mathfrak{H}}}$ of all arrows of $\mathfrak{Spec}^\vee(\mathfrak{H})$ which $\phi_{\mathfrak{H}}$ transforms into isomorphisms.
We define the spectrum $\mathfrak{Spec}(\mathfrak{H})$ as the localization $\Sigma^{-1}_{\phi_{\mathfrak{H}}}\mathfrak{Spec}^\vee(\mathfrak{H})$.

1.9.2. The 'strict' spectrum $\mathfrak{Spec}_{\mathfrak{s}}(\mathfrak{H})$. We denote by $\mathfrak{Spec}^{\vee}_{\mathfrak{s}}(\mathfrak{H})$ the full subcategory of $\mathfrak{H}$ generated by all $x \in Ob\mathfrak{H}_1$ such that $\mathfrak{Supp}^{\mathfrak{s}}_{\mathfrak{H}}(x)$ has a final object, $\check{x}$. Let $\mathfrak{Spec}^{\vee}_{\mathfrak{s}}(\mathfrak{H}) \xrightarrow{\psi_{\mathfrak{H}}} \mathfrak{H}$ be the functor which assigns to every object x of $\mathfrak{Spec}^{\vee}_{\mathfrak{s}}(\mathfrak{H})$ the final object $\check{x}$ of the strict support $\mathfrak{Supp}^{\mathfrak{s}}_{\mathfrak{H}}(x)$ of x. We denote by $\mathfrak{Spec}_{\mathfrak{s}}(\mathfrak{H})$ the quotient category $\Sigma^{-1}_{\psi_{\mathfrak{H}}}\mathfrak{Spec}^{\vee}_{\mathfrak{s}}(\mathfrak{H})$ and call it the *strict* spectrum of $\mathfrak{H}$.

It follows from 1.7.2.1 that the embedding $\mathfrak{Spec}^{\vee}(\mathfrak{H}) \subseteq \mathfrak{Spec}^{\vee}_{\mathfrak{s}}(\mathfrak{H})$ induces a functor $\mathfrak{Spec}(\mathfrak{H}) \longrightarrow \mathfrak{Spec}_{\mathfrak{s}}(\mathfrak{H})$.

1.9.3. The spectrum $\mathfrak{Spe}(\mathfrak{H})$. We denote by $\mathfrak{Sp}(\mathfrak{H})$ the full subcategory of $\mathfrak{H}_1$ generated by objects x such that if $y \to x$ is an arrow of $\mathfrak{H}_1$, then $\mathfrak{Supp}^{\mathfrak{s}}_{\mathfrak{H}}(y) = \mathfrak{Supp}^{\mathfrak{s}}_{\mathfrak{H}}(x)$.

The spectrum $\mathfrak{Spe}(\mathfrak{H})$ is defined as the localization of $\mathfrak{Sp}(\mathfrak{H})$ at the class $\Sigma_{\gamma_{\mathfrak{H}}}$ of all arrows which the functor

$$\mathfrak{Sp}(\mathfrak{H}) \xrightarrow{\gamma_{\mathfrak{H}}} 2^{\mathfrak{H}}, \quad x \longmapsto \mathfrak{Supp}^{\mathfrak{s}}_{\mathfrak{H}}(x),$$

maps to isomorphisms. Here $2^{\mathfrak{H}}$ denotes the preorder (with respect to the inclusion) of all strictly full subcategories of $\mathfrak{H}$.

1.9.4. The relative versions. Fix a functor $\mathfrak{G} \xrightarrow{F} \mathfrak{H}$. Then we have the relative spectrum $\mathfrak{Spec}(\mathfrak{G}, F)$ defined via the canonical cartesian square

$$\begin{array}{ccc}
\mathfrak{Spec}(\mathfrak{G}, F) & \longrightarrow & \mathfrak{G} \\
\big\downarrow & cart & \big\downarrow F \\
\mathfrak{Spec}(\mathfrak{H}) & \xrightarrow{\phi_{\mathfrak{H}}} & \mathfrak{H}
\end{array}$$

Similarly, the spectrum $\mathfrak{Spec}_{\mathfrak{s}}(\mathfrak{G}, F)$ is defined by the cartesian square

$$\begin{array}{ccc}
\mathfrak{Spec}_{\mathfrak{s}}(\mathfrak{G}, F) & \longrightarrow & \mathfrak{G} \\
\big\downarrow & cart & \big\downarrow F \\
\mathfrak{Spec}_{\mathfrak{s}}(\mathfrak{H}) & \xrightarrow{\psi_{\mathfrak{H}}} & \mathfrak{H}
\end{array}$$

Finally, the relative spectrum $\mathfrak{Spe}(\mathfrak{G}, F)$ is defined via the cartesian square

$$\begin{array}{ccc}
\mathfrak{Spe}(\mathfrak{G}, F) & \longrightarrow & 2^{\mathfrak{G}} \\
\big\downarrow & cart & \big\downarrow 2^F \\
\mathfrak{Spe}(\mathfrak{H}) & \xrightarrow{\gamma_{\mathfrak{H}}} & 2^{\mathfrak{H}}
\end{array}$$

where 2^F is the induced by the functor F morphism from the preorder of strictly full subcategories of $\mathfrak{G}$ to the preorder of strictly full subcategories of $\mathfrak{H}$.

1.9.5. Examples. (a) Let $\mathfrak{H}$ be the preorder of full subcategories of a svelte abelian category C_X closed under taking subobjects, $\mathfrak{G}$ is preorder $\mathfrak{Se}(X)$ of Serre subcategories of C_X, and F the inclusion functor. Then each of the three spectra, $\mathfrak{Spec}(\mathfrak{G},F)$, $\mathfrak{Spec}_{\mathfrak{s}}(\mathfrak{G},F)$, and $\mathfrak{Spe}(\mathfrak{G},F)$, is naturally isomorphic to the spectrum $\mathbf{Spec}(X)$ of the 'space' X.

The isomorphism $\mathfrak{Spec}(\mathfrak{G},F) \xrightarrow{\sim} \mathbf{Spec}(X)$ assigns to every element of $\mathfrak{Spec}(\mathfrak{G},F)$ the smallest topologizing subcategory $[x]$ containing a representative x of this element.

Notice that there is the biggest representative of the class equal (therefore) to the union of all its representatives. If x is the biggest representative, (or any other representative which is closed under coproducts), then the smallest topologizing subcategory $[x]$ containing x is generated by all possible quotients of objects of x.

Same map gives isomorphisms $\mathfrak{Spec}_{\mathfrak{s}}(\mathfrak{G},F) \xrightarrow{\sim} \mathbf{Spec}(X) \xleftarrow{\sim} \mathfrak{Spe}(\mathfrak{G},F)$.

(b) Let $\mathfrak{H}$ be the preorder of full coreflective subcategories of C_X closed under taking subobjects, $\mathfrak{G}$ is preorder $\mathfrak{Th}_{\mathfrak{c}}(X)$ of thick coreflective subcategories of C_X, and F the inclusion functor. Then each of the three spectra, $\mathfrak{Spec}(\mathfrak{G},F)$, $\mathfrak{Spec}_{\mathfrak{s}}(\mathfrak{G},F)$, and $\mathfrak{Spe}(\mathfrak{G},F)$, is naturally isomorphic to the spectrum $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ of the 'space' X.

In the case of $\mathfrak{Spec}(\mathfrak{G},F)$, the isomorphism in question assigns to every element of $\mathfrak{Spec}(\mathfrak{G},F)$ the smallest coreflective topologizing subcategory of C_X containing a representative of this element. Similarly for $\mathfrak{Spec}_{\mathfrak{s}}(\mathfrak{G},F)$ and $\mathfrak{Spe}(\mathfrak{G},F)$.

1.10. Complementary facts: relative support and associated points.

1.10.1. Relative support. Fix a functor $\mathfrak{G} \xrightarrow{F} \mathfrak{H}$. For an $x \in Ob\mathfrak{G}$, we call $\mathfrak{Supp}_{\mathfrak{H}}(F(x))$ the *support of x in $\mathfrak{H}$*, or the *relative support of x*.

1.10.2. Weakly associated points. For any $x \in Ob\mathfrak{G}$, we denote by $Ass^1_{(\mathfrak{G},F)}(x)$ the full subcategory of $\mathfrak{Spec}^1(\mathfrak{H})$ generated by all objects z for which there exists $\widetilde{x} \longrightarrow x$ such that $z \in Ob\mathfrak{Supp}_{\mathfrak{H}}(F(\widetilde{x}))$ and there is an arrow $F(\widetilde{x}) \longrightarrow z^\star$. As before, $(z, z \to z^\star)$ denotes an initial object of $(z\backslash\mathfrak{H})^1$.

It follows from this definition that $Ass^1_{(\mathfrak{G},F)}(x)$ is a subcategory of the relative support, $\mathfrak{Supp}_{\mathfrak{H}}(F(x))$, of the object x.

We call objects of $Ass^1_{(\mathfrak{G},F)}(x)$ *weakly associated points of x in* $(\mathfrak{G},F)$.

If F is the identical functor $\mathfrak{H} \longrightarrow \mathfrak{H}$, we shall write $Ass^1_{\mathfrak{H}}(x)$ instead of $Ass^1_{(\mathfrak{H},Id_{\mathfrak{H}})}(x)$ and call objects of this category *weakly associated points of x*. It follows that objects of $Ass^1_{\mathfrak{H}}(x)$ are $z \in Ob\mathfrak{Spec}^1(\mathfrak{H})$ such that there exist arrows $z^\star \longleftarrow \widetilde{x} \longrightarrow x$ and z belongs to $\mathfrak{Supp}_{\mathfrak{H}}(\widetilde{x})$.

1.10.3. Associated points. Fix a functor $\mathfrak{G} \xrightarrow{F} \mathfrak{H}$. For an object x of $\mathfrak{G}$, we denote by $\mathfrak{Ass}^1_{(\mathfrak{G},F)}(x)$ the full subcategory of $\mathfrak{Spec}^1(\mathfrak{H})$ generated by all objects z such that there exists $\widetilde{x} \in Ob\mathfrak{G}$ having the following properties:

(a) there exist arrows $\widetilde{x} \longrightarrow x$ and $F(\widetilde{x}) \longrightarrow z^\star$;

(b) if there exists an arrow $y \longrightarrow \widetilde{x}$, then $\mathfrak{H}(F(y), z) = \emptyset$.
We call objects of $\mathfrak{Ass}^1_{(\mathfrak{G},F)}(x)$ *associated points* of the object x in $(\mathfrak{G}, F)$.

It follows from the condition (b) that z belongs to $\mathfrak{Supp}_{\mathfrak{H}}(F(\widetilde{x}))$. Together with the condition (a), this means that $\mathfrak{Ass}^1_{(\mathfrak{G},F)}(x) \subseteq Ass^1_{(\mathfrak{G},F)}(x)$.

If F is the identical functor $\mathfrak{H} \longrightarrow \mathfrak{H}$, we shall write $\mathfrak{Ass}^1_{\mathfrak{H}}(x)$ instead of $\mathfrak{Ass}^1_{(\mathfrak{H},Id_{\mathfrak{H}})}(x)$. It follows that objects of $\mathfrak{Ass}^1_{\mathfrak{H}}(x)$ are $z \in Ob\mathfrak{Spec}^1(\mathfrak{H})$ for which there exists a pair of arrows $z^\star \longleftarrow \widetilde{x} \longrightarrow x$ with $\widetilde{x}$ such that there is no diagram of the form $z \longleftarrow y \longrightarrow \widetilde{x}$.

1.10.4. Associated points and weakly associated points in $\mathfrak{Spec}^0(\mathfrak{H})$. We define $\mathfrak{Ass}^0_{(\mathfrak{G},F)}(x)$, resp. $Ass^0_{(\mathfrak{G},F)}(x)$, as full subcategories of $\mathfrak{H}$ generated by all $z \in Ob\mathfrak{Spec}^0(\mathfrak{H})$ such that $\widehat{z}$ is an object of $\mathfrak{Ass}^1_{(\mathfrak{G},F)}(x)$, resp. an object of $Ass^1_{(\mathfrak{G},F)}(x)$.

Consider the following two properties:

(sup1) If $x \in Ob\mathfrak{H}$ is the supremum of a filtered system, $\{x_i \mid i \in J\}$, of its subobjects, then for any morphism $\widetilde{x} \longrightarrow x$, there exists a cofinal subset $I \subseteq J$ such that for every $i \in I$, there exists a fibre product, $\widetilde{x}_i = \widetilde{x} \times_x x_i$, and the canonical arrow $colim(\widetilde{x}_i|i \in I) \longrightarrow \widetilde{x}$ is an isomorphism.

(sup2) If $x \in Ob\mathfrak{H}$ is the supremum of a filtered system, $\{x_i \mid i \in J\}$, of its subobjects, then for any morphism $\widetilde{x} \longrightarrow x$, there exists a diagram $\widetilde{x} \longleftarrow y \longrightarrow x_i$ for some $i \in J$.

1.10.5. Proposition. *(a) If $x = colim(x_i \mid i \in J)$, then*

$$\bigcup_{i \in J} Ass^1_{(\mathfrak{G},F)}(x_i) \subseteq Ass^1_{(\mathfrak{G},F)}(x) \quad \textit{and} \quad \bigcup_{i \in J} \mathfrak{Ass}^1_{(\mathfrak{G},F)}(x_i) \subseteq \mathfrak{Ass}^1_{(\mathfrak{G},F)}(x).$$

(b) Let $x \in Ob\mathfrak{H}$ be the supremum of a filtered system, $\{x_i \mid i \in J\}$, of its subobjects.

(i) If $\mathfrak{H}$ is a preorder with the property (sup1) and F preserves colimits of filtered systems, then

$$Ass^1_{(\mathfrak{G},F)}(x) = \bigcup_{i \in J} Ass^1_{(\mathfrak{G},F)}(x_i).$$

(ii) Suppose $\mathfrak{G}$ possesses the property (sup2). Then

$$\mathfrak{Ass}^1_{(\mathfrak{G},F)}(x) = \bigcup_{i \in J} \mathfrak{Ass}^1_{(\mathfrak{G},F)}(x_i).$$

Proof. (a) Let $z \in ObAss^1_{(\mathfrak{G},F)}(x_i)$, that is z belongs to $\mathfrak{Spec}^1(\mathfrak{H})$ and there exists a pair of arrows, $z^\star \longleftarrow F(\widetilde{x}_i)$ and $\widetilde{x}_i \longrightarrow x_i$, such that $z \in Ob\mathfrak{Supp}_{\mathfrak{H}}(F(\widetilde{x}_i))$. Since there is an arrow $x_i \longrightarrow x$, same $\widetilde{x}_i$ serves for x. Therefore $Ass^1_{(\mathfrak{G},F)}(x_i) \subseteq Ass^1_{(\mathfrak{G},F)}(x)$ for all $i \in J$. Similar argument shows the inclusion $\bigcup_{i \in J} \mathfrak{Ass}^1_{(\mathfrak{G},F)}(x_i) \subseteq \mathfrak{Ass}^1_{(\mathfrak{G},F)}(x)$.

(b)(i) Suppose that $(\mathfrak{G}, F)$ possesses the property (sup1). Let z be an object of $Ass^1_{(\mathfrak{G},F)}(x)$; i.e. z belongs to $\mathfrak{Spec}^1(\mathfrak{H})$ and there exists a pair of arrows, $z^\star \longleftarrow F(\widetilde{x})$ and $\widetilde{x} \longrightarrow x$, such that $z \in Ob\mathfrak{Supp}_{\mathfrak{H}}(F(\widetilde{x}))$. By the property (sup1), a fibre product $\widetilde{x}_i = \widetilde{x} \times_x x_i$ exists for $i \in I$, where I is a cofinal subset of J, and the canonical arrow $colim(\widetilde{x}_i | i \in I) \longrightarrow \widetilde{x}$ is an isomorphism. If for every $i \in I$, there is an arrow $F(\widetilde{x}_i) \longrightarrow z$, then there is an arrow $F(\widetilde{x}) \longrightarrow z$, that is z does non belong to $\mathfrak{Supp}_{\mathfrak{H}}(F(\widetilde{x}))$, which contradicts to the hypothesis. Thus, $z \in Ob\mathfrak{Supp}_{\mathfrak{H}}(F(\widetilde{x}_i))$ for some $i \in I$, which means that z is an object of $Ass^1_{(\mathfrak{G},F)}(x_i)$.

(ii) Let now $(\mathfrak{G}, F)$ have the property (sup2). Let z be an object of $\mathfrak{Ass}^1_{(\mathfrak{G},F)}(x)$; i.e. z belongs to $\mathfrak{Spec}^1(\mathfrak{H})$ and there exists a pair of arrows, $z^\star \longleftarrow F(\widetilde{x})$ and $\widetilde{x} \longrightarrow x$, such that if $\mathfrak{H}(F(y), z) \neq \emptyset$, then $\mathfrak{G}(y, \widetilde{x}) = \emptyset$. By the property (sup2), for some $i \in J$, there exists a pair of arrows $\widetilde{x} \longleftarrow \widetilde{x}_i \longrightarrow x_i$. It follows that if $\mathfrak{H}(F(y), z) \neq \emptyset$, then $\mathfrak{G}(y, \widetilde{x}_i) = \emptyset$, hence z belongs to $\mathfrak{Ass}^1_{(\mathfrak{G},F)}(x_i)$. ■

1.10.6. Example: supports and associated points of a family of arrows. Fix a svelte category C_X. Let $\mathfrak{H}$ be the preorder $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$ of saturated multiplicative systems of C_X. Let $\mathfrak{G}$ be the preorder (with respect to the inclusion) of non-empty families of arrows of the category C_X, and let $\mathfrak{G} \xrightarrow{F} \mathfrak{H}$ be the functor which assigns to each family S the intersection, $[S]_\bullet$, of all saturated multiplicative systems containing S.

The support, $\mathfrak{Supp}(F(S))$, of a family of arrows S in $\mathfrak{H} = \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$ consists of all saturated multiplicative systems Σ which do not contain S.

Weakly associated points of S are saturated multiplicative systems Σ such that there exists $\widetilde{S} \subseteq S$ which is not contained in Σ, but is contained in $\Sigma^\star$. Here $\Sigma^\star$ is the intersection of all saturated multiplicative systems of C_X properly containing Σ.

Thus, the preorder $Ass_{(\mathfrak{G},F)}(S)$ of weakly associated points of a family of arrows S coincides with the preorder $Ass_{\mathfrak{L}}(S)$ of weakly associated points of S defined in [R6, 9.4.2].

Notice that in this case, *associated points* and weakly associated points of S coincide.

In fact, by definition 1.10.3, *associated points* of S are saturated multiplicative systems Σ having the following property: there exists a non-empty subfamily, T, of S such that $T \cap \Sigma = \emptyset$. Let Σ be a weakly associated point of S, i.e. there exists $\widetilde{S} \subseteq S$ such that $\Sigma \not\supseteq \widetilde{S} \subseteq \Sigma^\star$. The family $T = \widetilde{S} - \Sigma$ is non-empty, and $T \cap \Sigma = \emptyset$. This means, precisely, that Σ is an associated point of S.

1.10.7. Supports and associated points of objects. Let $\mathfrak{G}$, $\mathfrak{H}$ and F be same as in 1.10.6; i.e. $\mathfrak{G}$ is a preorder of families of arrows of a category C_X, $\mathfrak{H}$ is the preorder $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$ of saturated multiplicative systems of C_X, and F maps every family S to the smallest saturated multiplicative system containing S. Fix a functor $C_Y \xrightarrow{\phi^*} C_X$. The functor ϕ^* determines, for every $M \in ObC_Y$, a functor $C_Y/M \xrightarrow{\phi^*_M} C_X/\phi^*(M)$. The map

$$ObC_Y \longrightarrow \mathfrak{G}, \quad M \longmapsto \phi^*_M(Ob(C_Y/M)),$$

defines a functor $C_Y \xrightarrow{\Phi} \mathfrak{G}$. Let $C_Y \xrightarrow{\Upsilon_\phi} \mathfrak{H}$ denote the composition of the functor Φ with the functor F. The functor Υ_ϕ provides the notions of the support, weakly associated points and associated points in $\mathfrak{H} = \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$ of any object M of the category C_Y.

The support of an object M consists of all saturated multiplicative systems Σ such that $\phi^*(\xi) \notin \Sigma$ for some arrow $L \xrightarrow{\xi} M$.

A saturated multiplicative system Σ of C_X is a *weakly associated point* of an object M of the category C_Y iff there exists a morphism $L \longrightarrow M$ such that $\phi^*(C_Y(N, L)) \subseteq \Sigma^\star$ for all $N \in ObC_Y$ and $\phi^*(\xi) \notin \Sigma$ for some arrow $M' \xrightarrow{\xi} M$.

Finally, Σ is an *associated point* of M iff there exists a morphism $L \longrightarrow M$ such that $\Sigma \not\ni \phi^*(\xi) \in \Sigma^\star$ for every arrow $N \xrightarrow{\xi} L$.

1.10.8. A canonical setting. Let C_Y denote the subcategory of C_X formed by all non-initial objects of C_X and strict monomorphisms; and let ϕ^* be the inclusion functor $C_Y \hookrightarrow C_X$. Applying the construction of 1.10.7, we obtain non-trivial notions of the support and weakly associated and associated points of any non-initial object of the category C_X.

1.10.8.1. The case of an abelian category. If C_X is an abelian category, these notions are equivalent to those introduced in [R6, 9.4, 10.8]. The equivalence is given by the isomorphism between the preorder $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$ of saturated multiplicative systems and the preorder $\mathfrak{Th}(X)$ of thick subcategories of the category C_X. By definition, a saturated system $\Sigma_{\mathbb{T}}$ corresponding to a thick subcategory $\mathbb{T}$ belongs to the support of an object M of C_X iff there exists a monomorphism $N \xrightarrow{g} M$ which does not belong to $\Sigma_{\mathbb{T}}$. This means, precisely, that $Cok(g)$ does not belong to the subcategory $\mathbb{T}$. Thus, $\Sigma_{\mathbb{T}}$ belongs to the support of M iff M does not belong to $\mathbb{T}$.

A multiplicative system $\Sigma_{\mathbb{T}}$ is a *weakly associated point* of an object M if there exists a subobject N of M such that all monoarrows $L \longrightarrow N$ belong to $\Sigma^\star_{\mathbb{T}} = \Sigma_{\mathbb{T}^\star}$, but some of them do not belong $\Sigma_{\mathbb{T}}$. This means that N is an object of the subcategory $\mathbb{T}^\star$ which does not belong to $\mathbb{T}$. Here $\mathbb{T}^\star$ is the intersection of all thick subcategories of C_X properly containing $\mathbb{T}$.

A multiplicative system $\Sigma_{\mathbb{T}}$ is an *associated point* of an object M if there is nonzero subobject N of M such that every nonzero monoarrow $L \longrightarrow N$ belongs to $\Sigma_{\mathbb{T}^\star}$ and does not belong $\Sigma_{\mathbb{T}}$. This means that the subobject N belongs to $\mathbb{T}^\star$ and is $\mathbb{T}$-torsion free.

1.10.8.2. The direct description. Fix an abelian category C_X. Let $\mathfrak{G}$ be the subcategory $C_{\mathfrak{M}^\star(X)}$ of C_X formed by all nonzero monomorphisms and all nonzero objects of C_X. Let $\mathfrak{H}$ be the preorder $\mathfrak{Th}(X)$ of thick subcategories of the category C_X. The functor $\mathfrak{G} \xrightarrow{F} \mathfrak{H}$ assigns to every object M of the category $C_{\mathfrak{M}^\star(X)}$ the smallest thick subcategory, $[M]_\bullet$, containing M.

The support, $\mathfrak{Supp}(F(M))$, of the object M in $\mathfrak{H} = \mathfrak{Th}(X)$ consists of all thick subcategories $\mathbb{T}$ such that $F(M) = [M]_\bullet \not\subseteq \mathbb{T}$, or, equivalently, $M \notin Ob\mathbb{T}$.

A thick subcategory $\mathbb{T}$ is a *weakly associated point* of a nonzero object M iff there exists a subobject $\widetilde{M}$ of M which belongs to $Ob\mathbb{T}^\star - Ob\mathbb{T}$.

A thick subcategory $\mathbb{T}$ is an *associated point* of a nonzero object M iff there exists a nonzero subobject $\widetilde{M}$ of M which belongs to $\mathbb{T}^\star$ and is $\mathbb{T}$-torsion free.

Thus the preorder $Ass^1_{(\mathfrak{G},F)}(M)$ coincides with the preorder $Ass^1_{\mathfrak{Th}}(M)$ of weakly associated points of M in the sense of [R6, 10.1]. The preorder $\mathfrak{Ass}^1_{(\mathfrak{G},F)}(M)$ coincides with the preorder $\mathfrak{Ass}^1_{\mathfrak{L}}(M)$ introduced in [R6, 10.8].

2. Applications: spectra of 'spaces'.

2.1. The spectra of exact localizations.

Let C_X be a svelte category. We take as $\mathfrak{H}$ the preorder $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$ of saturated multiplicative systems of C_X and set

$$\mathbf{Spec}^1_{\mathfrak{L}}(X) = \mathfrak{Spec}^1(\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)) \quad \text{and} \quad \mathbf{Spec}^0_{\mathfrak{L}}(X) = \mathfrak{Spec}^0(\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)).$$

Since $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$ is a preorder, there exists a canonical injective morphism

$$\mathbf{Spec}^0_{\mathfrak{L}}(X) \longrightarrow \mathbf{Spec}^1_{\mathfrak{L}}(X), \quad \Sigma \longmapsto \widehat{\Sigma}$$

(cf. 1.4.6). Notice that the support, $\mathfrak{Supp}_{\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)}(\Sigma)$, of Σ consists of all saturated multiplicative systems of C_X which do not contain Σ, and its final object, $\widehat{\Sigma}$, is the union of all multiplicative systems which belong to the support of Σ.

2.2. Closed spectra and flat spectra.

The *closed* spectra of a 'space' X are relative spectra corresponding to the inclusion functor, $\mathfrak{C}\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X) \xrightarrow{\mathfrak{J}_{\mathfrak{C}}(X)} \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$, where $\mathfrak{C}\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$ is the preorder of all closed saturated multiplicative systems of C_X (cf. 2.0.5). Thus,

$$\mathbf{Spec}^i_{\mathfrak{C}}(X) = \mathfrak{Spec}^i(\mathfrak{C}\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X), \mathfrak{J}_{\mathfrak{C}}(X)), \quad i = 0, 1.$$

Similarly, the *flat* spectra of X are relative spectra of $(\mathfrak{Lc}(X), \mathfrak{J}_{\mathfrak{L}}(X))$, where $\mathfrak{Lc}(X)$ is the preorder of all flat saturated multiplicative systems of C_X (cf. 2.0.6) and $\mathfrak{J}_{\mathfrak{L}}(X)$ is the inclusion functor $\mathfrak{Lc}(X) \hookrightarrow \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$. Thus,

$$\mathbf{Spec}^i_{\mathfrak{fL}}(X) = \mathfrak{Spec}^i(\mathfrak{Lc}(X), \mathfrak{J}_{\mathfrak{L}}(X)), \quad i = 0, 1.$$

2.3. Spectra of 'spaces' represented by abelian categories.

Fix a 'space' X such that C_X is an abelian category.

2.3.1. Thick spectra. We take as $\mathfrak{H}$ the preorder $\mathfrak{Th}(X)$ of thick subcategories of C_X and set
$$\mathbf{Spec}^i_{\mathfrak{Th}}(X) = \mathfrak{Spec}^i(\mathfrak{Th}(X)), \quad i = 0, 1.$$

Since the preorder $\mathfrak{Th}(X)$ is naturally isomorphic to the preorder $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$ of saturated multiplicative systems, the isomorphism $\mathfrak{Th}(X) \xrightarrow{\sim} \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$ induces isomorphisms

$$\mathbf{Spec}^i_{\mathfrak{Th}}(X) \xrightarrow{\sim} \mathbf{Spec}^i_{\mathfrak{L}}(X), \quad i = 0, 1$$

such that the diagram

$$\begin{array}{ccc}
\mathbf{Spec}^0_{\mathfrak{Th}}(X) & \xrightarrow{\sim} & \mathbf{Spec}^0_{\mathfrak{L}}(X) \\
\downarrow & & \downarrow \\
\mathbf{Spec}^1_{\mathfrak{Th}}(X) & \xrightarrow{\sim} & \mathbf{Spec}^1_{\mathfrak{L}}(X)
\end{array}$$

commutes. Here the vertical arrows are canonical embeddings of 1.4.6.

2.3.2. Representatives of $\mathbf{Spec}^0_{\mathfrak{Th}}(X)$. Let $C_{\mathfrak{M}(X)}$ denote the subcategory of C_X formed by all monomorphisms of C_X. The map which assigns to each object, M, of the category C_X the smallest thick subcategory, $[M]_\bullet$, containing M defines a functor $C_{\mathfrak{M}(X)} \xrightarrow{\mathfrak{F}_X} \mathfrak{Th}(X)$. We denote by $Spec^0_{\mathfrak{Th}}(X)$ the preimage, $\mathfrak{F}_X^{-1}(\mathbf{Spec}^0_{\mathfrak{Th}}(X))$, of the spectrum $\mathbf{Spec}^0_{\mathfrak{Th}}(X)$. An object M of $Spec^0_{\mathfrak{Th}}(X)$ is regarded as a representative of the object $[M]_\bullet$ of $\mathbf{Spec}^0_{\mathfrak{Th}}(X)$.

It follows from [R6, 7.1.1] that the functor

$$Spec^0_{\mathfrak{Th}}(X) \longrightarrow \mathbf{Spec}^0_{\mathfrak{Th}}(X), \quad M \longmapsto [M]_\bullet, \tag{1}$$

is surjective. Namely, if $\mathcal{P}$ is an object of $\mathbf{Spec}^0_{\mathfrak{Th}}(X)$, then $\mathcal{P} = [M]_\bullet$ for any $M \in Ob\mathcal{P} - Ob\widehat{\mathcal{P}}$. Here $\widehat{\mathcal{P}}$ is the union of all thick subcategories of C_X which do not contain $\mathcal{P}$.

2.3.3. Closed and flat spectra. Let $\mathfrak{CTh}(X)$ be the preorder of coreflective thick subcategories of the category C_X, and $\mathfrak{J}_{\mathfrak{CT}}(X)$ the inclusion functor $\mathfrak{CTh}(X) \hookrightarrow \mathfrak{Th}(X)$. Recall that a full subcategory $\mathbb{T}$ of C_X is *coreflective* if the inclusion functor $\mathbb{T} \longrightarrow C_X$ has a right adjoint. In other words, every object of C_X has the biggest subobject which belongs to $\mathbb{T}$.

The coreflective spectra of X are defined by

$$\mathbf{Spec}^i_{\mathfrak{CTh}}(X) = \mathfrak{Spec}^i(\mathfrak{CTh}(X), \mathfrak{J}_{\mathfrak{CT}}(X)), \quad i = 0, 1.$$

By [R6, 7.2.1], the isomorphism $\mathfrak{Th}(X) \xrightarrow{\sim} \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$ induces an isomorphism of the preorder $\mathfrak{CTh}(X)$ and the preorder $\mathfrak{C}\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$ of closed saturated multiplicative systems. Therefore, $\mathbf{Spec}^i_{\mathfrak{CTh}}(X)$ is isomorphic to the closed spectrum, $\mathbf{Spec}^i_{\mathfrak{C}}(X)$ defined in 2.2.

Let $\mathfrak{Th}_{\mathfrak{c}}(X)$ denote the preorder of all thick subcategories $\mathbb{T}$ such that the localization functor $C_X \longrightarrow C_X/\mathbb{T}$ has a right adjoint. And let $\mathfrak{J}_{\mathfrak{c}}(X)$ denote the inclusion functor $\mathfrak{Th}_{\mathfrak{c}}(X) \hookrightarrow \mathfrak{Th}(X)$. Since the preorder $\mathfrak{Th}_{\mathfrak{c}}(X)$ is isomorphic to the preorder $\mathfrak{Lc}(X)$ of flat saturated systems of C_X (cf. 2.2), the spectrum

$$\mathbf{Spec}^i_{\mathfrak{Th}_{\mathfrak{c}}}(X) = \mathfrak{Spec}^i(\mathfrak{Th}_{\mathfrak{c}}(X), \mathfrak{J}_{\mathfrak{c}}(X))$$

is isomorphic to the corresponding flat spectrum $\mathbf{Spec}^i_{\mathfrak{fL}}(X)$ defined in 2.2. Here $i = 0, 1$.

2.3.4. Relative Serre spectrum. Fix an abelian category C_X. Let $\mathfrak{Se}(X)$ be the preorder of Serre subcategories of C_X and $\mathfrak{J}_{\mathfrak{Se}}(X)$ the inclusion functor $\mathfrak{Se}(X) \hookrightarrow \mathfrak{Th}(X)$. The *Serre spectra*, or, shortly, S-spectra of X are defined by

$$\mathbf{Spec}^i_{\mathfrak{Se}}(X) = \mathfrak{Spec}^i(\mathfrak{Se}(X), \mathfrak{J}_{\mathfrak{Se}}(X)), \quad i = 0, 1.$$

Since $\mathfrak{Th}_{\mathfrak{c}}(X) \subseteq \mathfrak{CTh}(X) \subseteq \mathfrak{Se}(X)$, there are inclusions of spectra,

$$\mathbf{Spec}^i_{\mathfrak{Th}_{\mathfrak{c}}}(X) \subseteq \mathbf{Spec}^i_{\mathfrak{CTh}}(X) \subseteq \mathbf{Spec}^i_{\mathfrak{Se}}(X), \quad i = 0, 1.$$

By 2.3.4.1, if C_X is a category with the property (sup), then $\mathfrak{CTh}(X) = \mathfrak{Se}(X)$, in particular the spectra $\mathbf{Spec}^i_{\mathfrak{Se}}(X)$ and $\mathbf{Spec}^i_{\mathfrak{CTh}}(X)$ coincide. If C_X is a Grothendieck category, then $\mathfrak{Th}_{\mathfrak{c}}(X) = \mathfrak{CTh}(X) = \mathfrak{Se}(X)$, hence in this case,

$$\mathbf{Spec}^i_{\mathfrak{Th}_{\mathfrak{c}}}(X) = \mathbf{Spec}^i_{\mathfrak{CTh}}(X) = \mathbf{Spec}^i_{\mathfrak{Se}}(X), \quad i = 0, 1.$$

2.3.4.3. The category $Spec^0_{\mathfrak{s}}(X)$. We denote by $Spec^0_{\mathfrak{s}}(X)$ the full subcategory of the category $C_{\mathfrak{M}(X)}$ generated by nonzero objects M such that $M \in Ob[N]_\bullet$ if there exists a nonzero arrow $N \longrightarrow M$ (see [R6, 7.3.4]). By [R6, 7.3.5], $Spec^0_{\mathfrak{s}}(X) \subseteq Spec^0_{\mathfrak{Se}}(X)$.

It is easy to see that a nonzero object M belongs to $Spec^0_{\mathfrak{s}}(X)$ iff $[L]_\bullet = [M]_\bullet$ for any nonzero subobject L of M.

2.3.4.4. Proposition. *The image,* $\mathbf{Spec}^0_{\mathfrak{s}}(X)$*, of the map*

$$Spec^0_{\mathfrak{s}}(X) \longrightarrow \mathbf{Spec}^0_{\mathfrak{Th}}(X), \quad M \longmapsto [M]_\bullet, \tag{2}$$

contains $\mathbf{Spec}^0_{\mathfrak{CTh}}(X)$*. If the category C_X has the property (sup), then the image of (2) coincides with* $\mathbf{Spec}^0_{\mathfrak{CTh}}(X)$*.*

Proof. Let $\mathcal{P}$ is an object of $\mathbf{Spec}^0_{\mathfrak{C}\mathfrak{T}\mathfrak{h}}(X)$; i.e. $\mathcal{P}$ is an object of $\mathbf{Spec}^0_{\mathfrak{T}\mathfrak{h}}(X)$ such that the thick subcategory $\widehat{\mathcal{P}}$ is coreflective. Let $M \in Ob\mathcal{P} - Ob\widehat{\mathcal{P}}$. Since $\widehat{\mathcal{P}}$ is coreflective, M has a $\widehat{\mathcal{P}}$-torsion, $\mathfrak{t}_{\widehat{\mathcal{P}}}M$. Replacing M by the quotient $M/\mathfrak{t}_{\widehat{\mathcal{P}}}M$, we can assume that M is $\widehat{\mathcal{P}}$-torsion free. Since $\widehat{\mathcal{P}} = \langle M \rangle_\bullet$, it follows from [R6, 7.3.6] that M is an object of $Spec^0_{\mathfrak{s}}(X)$ such that $[M]_\bullet = \mathcal{P}$.

By [R6, 7.3.5], the subcategory $\langle M \rangle_\bullet$ is a Serre subcategory for every object M of $Spec^0_{\mathfrak{s}}(X)$. If the category C_X has the property (sup), then, by [R6, 7.3.8], every Serre subcategory of C_X is coreflective. ∎

2.3.5. Spectra related to topologizing subcategories. Let $\mathfrak{T}(X)$ denote the preorder of all topologizing subcategories of the category C_X. Let $\mathfrak{J}^{\mathfrak{t}}_X$ denote the inclusion functor $\mathfrak{T}\mathfrak{h}(X) \longrightarrow \mathfrak{T}(X)$.

Thus, we have two spectra associated to this functor,

$$\mathbf{Spec}^i_{\mathfrak{t}}(X) = \mathfrak{Spec}^i(\mathfrak{T}\mathfrak{h}(X), \mathfrak{J}^{\mathfrak{t}}_X), \quad i = 0, 1.$$

and a canonical morphism from one to another, $\mathbf{Spec}^0_{\mathfrak{t}}(X) \longrightarrow \mathbf{Spec}^1_{\mathfrak{t}}(X)$.

2.3.5.1. Proposition. *(a) There is a natural map* $\mathbf{Spec}^1_{\mathfrak{t}}(X) \longrightarrow \mathbf{Spec}^1_{\mathfrak{T}\mathfrak{h}}(X)$.

(b) The functor $\mathfrak{T}(X) \longrightarrow \mathfrak{T}\mathfrak{h}(X)$ *which assigns to every topologizing subcategory* $\mathbb{T}$ *the smallest thick subcategory,* $\mathbb{T}_\bullet$*, containing* $\mathbb{T}$*, induces a functor* $\mathbf{Spec}^0_{\mathfrak{t}}(X) \longrightarrow \mathbf{Spec}^0_{\mathfrak{T}\mathfrak{h}}(X)$.

Proof. (a) The category $\mathbf{Spec}^1_{\mathfrak{t}}(X)$ is defined by the cartesian square

$$\begin{array}{ccc} \mathbf{Spec}^1_{\mathfrak{t}}(X) & \longrightarrow & \mathfrak{T}\mathfrak{h}(X) \\ \downarrow & & \downarrow \\ \mathfrak{Spec}^1(\mathfrak{T}(X)) & \longrightarrow & \mathfrak{T}(X) \end{array}$$

in which the right vertical arrow and the lower horizontal arrow are inclusions (see 1.6). It follows from this description (or from the explicit description of relative spectra in 1.6) that objects of $\mathbf{Spec}^1_{\mathfrak{t}}(X)$ are naturally identified with thick subcategories, $\mathcal{P}$, which are objects of $\mathfrak{Spec}^1(\mathfrak{T}(X))$. The latter means that that there exists the smallest topologizing subcategory, $\mathcal{P}^{\mathfrak{t}}$, properly containing $\mathcal{P}$. Therefore, the smallest thick subcategory, $[\mathcal{P}^{\mathfrak{t}}]_\bullet$, containing $\mathcal{P}^{\mathfrak{t}}$ is the smallest thick subcategory *properly* containing $\mathcal{P}$, hence $\mathcal{P}$ belongs to $\mathbf{Spec}^1_{\mathfrak{T}\mathfrak{h}}(X)$.

(b) The spectrum $\mathbf{Spec}^0_{\mathfrak{t}}(X)$ is defined by the cartesian square

$$\begin{array}{ccc} \mathbf{Spec}^0_{\mathfrak{t}}(X) & \longrightarrow & \mathfrak{T}\mathfrak{h}(X) \\ \downarrow & & \downarrow \\ \mathfrak{Spec}^0(\mathfrak{T}(X)) & \longrightarrow & \mathfrak{T}(X) \end{array} \qquad (1)$$

(see 1.6). By definition, objects of $\mathfrak{Spec}^0(\mathfrak{T}(X))$ are topologizing subcategories, $\mathcal{P}$, such that $\mathfrak{Supp}_{\mathfrak{T}(X)}(\mathcal{P})$ has a final object. This means, precisely, that the union, $\widehat{\mathcal{P}}^{\mathfrak{t}}$, of all topologizing subcategories which do not contain $\mathcal{P}$ is a topologizing subcategory. The lower horizontal arrow of the diagram (1) maps an element $\mathcal{P}$ to $\widehat{\mathcal{P}}^{\mathfrak{t}}$.

It follows that objects of $\mathbf{Spec}^0_{\mathfrak{t}}(X)$ can be identified with topologizing subcategories, $\mathcal{P}$, such that $\widehat{\mathcal{P}}^{\mathfrak{t}}$ is a thick subcategory. Therefore, $\widehat{\mathcal{P}}^{\mathfrak{t}}$ is a final object of the support $\mathfrak{Supp}_{\mathfrak{Th}(X)}(\mathcal{P}_\bullet)$, where $\mathcal{P}_\bullet$ is the smallest thick subcategory containing $\mathcal{P}$. This implies that the map

$$\mathbf{Spec}^0_{\mathfrak{t}}(X) \longrightarrow \mathfrak{Th}(X), \quad \mathcal{P} \longmapsto \mathcal{P}_\bullet,$$

takes values in $\mathbf{Spec}^0_{\mathfrak{Th}}(X)$, hence the assertion. ■

2.3.5.2. Representatives of $\mathbf{Spec}^0_{\mathfrak{t}}(X)$. Let $C_{\mathfrak{M}(X)}$ denote the subcategory of C_X formed by all monomorphisms of C_X. The map which assigns to each object, M, of the category C_X the smallest topologizing subcategory, $[M]$, containing M defines a functor $C_{\mathfrak{M}(X)} \xrightarrow{\mathfrak{F}_X} \mathfrak{T}(X)$. We denote by $Spec^0_{\mathfrak{t}}(X)$ the preimage, $\mathfrak{F}_X^{-1}(\mathbf{Spec}^0_{\mathfrak{t}}(X))$, of the spectrum $\mathbf{Spec}^0_{\mathfrak{t}}(X)$. An object M of $Spec^0_{\mathfrak{t}}(X)$ is regarded as a representative of the object $[M]$ of $\mathbf{Spec}^0_{\mathfrak{t}}(X)$.

2.3.5.3. Proposition. *The functor*

$$Spec^0_{\mathfrak{t}}(X) \longrightarrow \mathbf{Spec}^0_{\mathfrak{t}}(X), \quad M \longmapsto [M],$$

is surjective.

Proof. Let $\mathcal{P}$ be an object of $\mathbf{Spec}^0_{\mathfrak{t}}(X)$. For any $M \in Ob\mathcal{P} - Ob\widehat{\mathcal{P}}^{\mathfrak{t}}$, the union, $\langle M \rangle$, of all topologizing subcategories of C_X which do not contain the object M coincides with $\widehat{\mathcal{P}}^{\mathfrak{t}}$. In fact, $\langle M \rangle \subseteq \widehat{\mathcal{P}}^{\mathfrak{t}}$, because $M \in Ob\mathcal{P}$; and $\widehat{\mathcal{P}}^{\mathfrak{t}} \subseteq \langle M \rangle$, because $M \notin Ob\widehat{\mathcal{P}}^{\mathfrak{t}}$.

It remains to notice that $[M] = \mathcal{P}$. Clearly $[M] \subseteq \mathcal{P}$. The inverse inclusion, $\mathcal{P} \subseteq [M]$, holds because if $\mathcal{P} \nsubseteq [M]$, then $[M] \subseteq \langle M \rangle$ which is impossible by the definition of $\langle M \rangle$. ■

2.3.6. Closed spectra defined by topologizing subcategories. Let $\mathfrak{J}^{\mathfrak{s}}_X$ be the inclusion functor $\mathfrak{CTh}(X) \longrightarrow \mathfrak{T}(X)$. This functor creates two spectra,

$$\mathbf{Spec}^i_{\mathfrak{ct}}(X) = \mathfrak{Spec}^i(\mathfrak{CTh}(X), \mathfrak{J}^{\mathfrak{s}}_X), \quad i = 0, 1.$$

and a canonical morphism from one to another, $\mathbf{Spec}^0_{\mathfrak{ct}}(X) \longrightarrow \mathbf{Spec}^1_{\mathfrak{ct}}(X)$.

2.3.6.1. The spectrum $Spec(X)$. We denote by $Spec(X)$ the full subcategory of the category $C_{\mathfrak{M}(X)}$ generated by nonzero objects M such that $M \in Ob[N]$ if there exists a nonzero morphism $N \longrightarrow M$.

Since $[N] \subseteq [N]_\bullet$ for every object N, it follows that $Spec(X) \subseteq Spec^0_{\mathfrak{s}}(X)$. In particular, $Spec(X) \subseteq Spec^0_{\mathfrak{Se}}(X)$.

It is easy to see that a nonzero object M belongs to $Spec(X)$ iff $[L] = [M]$ for every nonzero subobject L of M. In other words, the functor

$$C_{\mathfrak{M}(X)} \longrightarrow \mathfrak{T}(X), \quad M \longmapsto [M],$$

maps every nonzero (mono)morphism $L \longrightarrow M$ to the (identical) isomorphism.

2.3.6.2. Proposition. *The image* $\mathbf{Spec}(X)$ *of the map*

$$Spec(X) \longrightarrow \mathbf{Spec}^0_{\mathfrak{t}}(X), \quad M \longmapsto [M], \tag{1}$$

contains $\mathbf{Spec}^0_{\mathfrak{Ct}}(X)$. *If the category* C_X *has the property (sup), then the image of (1) coincides with* $\mathbf{Spec}^0_{\mathfrak{Ct}}(X)$.

Proof. Let $\mathcal{P}$ is an object of $\mathbf{Spec}^0_{\mathfrak{Ct}}(X)$; i.e. $\mathcal{P}$ is an object of $\mathbf{Spec}^0_{\mathfrak{t}}(X)$ such that $\widehat{\mathcal{P}}^{\mathfrak{t}}$ is a coreflective thick subcategory. Let $M \in Ob\mathcal{P} - Ob\widehat{\mathcal{P}}^{\mathfrak{t}}$. Since $\widehat{\mathcal{P}}^{\mathfrak{t}}$ is coreflective, M has a $\widehat{\mathcal{P}}^{\mathfrak{t}}$-torsion, $\mathfrak{t}_{\widehat{\mathcal{P}}^{\mathfrak{t}}}M$. Replacing M by the quotient $M/\mathfrak{t}_{\widehat{\mathcal{P}}^{\mathfrak{t}}}M$, we assume that M is $\widehat{\mathcal{P}}^{\mathfrak{t}}$-torsion free. Since $\widehat{\mathcal{P}}^{\mathfrak{t}}$ is the union, $\langle M\rangle$, of all topologizing subcategories of C_X which do not contain M, it follows that M is an object of $Spec(X)$ such that $[M] = \mathcal{P}$.

Since $\langle M\rangle$ is thick, it coincides with $\langle M\rangle_\bullet$. In particular, it is (by [R6, 7.3.5]) a Serre subcategory. If the category C_X has the property (sup), then, by [R6, 7.3.8], every Serre subcategory of C_X is coreflective. ■

2.4. Spectra defined by Serre subcategories.

Let $\mathfrak{H}$ be the preorder $\mathfrak{Se}(X)$ of all Serre subcategories of the category C_X. Thus, we have two spectra and an embedding:

$$\mathfrak{Spec}^0(\mathfrak{Se}(X)) \longrightarrow \mathfrak{Spec}^1(\mathfrak{Se}(X)).$$

2.4.1. Proposition. *There are natural functors*

$$\mathbf{Spec}^i_{\mathfrak{Se}}(X) \longrightarrow \mathfrak{Spec}^i(\mathfrak{Se}(X)), \quad i = 0, 1,$$

such that the diagram

$$\begin{array}{ccc} \mathbf{Spec}^0_{\mathfrak{Se}}(X) & \longrightarrow & \mathfrak{Spec}^0(\mathfrak{Se}(X)) \\ \big\downarrow & & \big\downarrow \\ \mathbf{Spec}^1_{\mathfrak{Se}}(X) & \longrightarrow & \mathfrak{Spec}^1(\mathfrak{Se}(X)) \end{array}$$

commutes.

Proof. The functor $\mathbf{Spec}^1_{\mathfrak{S}\mathfrak{e}}(X) \longrightarrow \mathfrak{Spec}^1(\mathfrak{Se}(X))$ is the inclusion. The functor $\mathbf{Spec}^0_{\mathfrak{S}\mathfrak{e}}(X) \longrightarrow \mathfrak{Spec}^0(\mathfrak{Se}(X))$ assigns to each object $\mathcal{P}$ of $\mathbf{Spec}^0_{\mathfrak{S}\mathfrak{e}}(X)$ the Serre subcategory $\mathcal{P}^-$. ∎

2.5. Spectra defined by closed cosubspaces.

Let $\mathfrak{CT}(X)$ denote the preorder of all coreflective topologizing subcategories of the category C_X. Let $\mathfrak{J}^{\mathfrak{ct}}_X$ denote the inclusion functor $\mathfrak{CTh}(X) \longrightarrow \mathfrak{CT}(X)$.

Thus, we have two spectra associated to this functor,

$$\mathbf{Spec}^i_{\mathfrak{c}}(X) = \mathfrak{Spec}^i(\mathfrak{Th}(X), \mathfrak{J}^{\mathfrak{ct}}_X), \quad i = 0, 1.$$

and a canonical morphism from one to another, $\mathbf{Spec}^0_{\mathfrak{c}}(X) \longrightarrow \mathbf{Spec}^1_{\mathfrak{c}}(X)$.

If the category C_X has the property (sup), then the preorder $\mathfrak{CTh}(X)$ of coreflective thick subcategories coincides with the preorder $\mathfrak{Se}(X)$ of Serre subcategories.

2.5.1. Proposition. *Suppose that the category C_X has the property (sup).*
(a) There is a natural map $\mathbf{Spec}^1_{\mathfrak{c}}(X) \longrightarrow \mathfrak{Spec}^1(\mathfrak{Se}(X)$.
(b) The functor

$$\mathfrak{CT}(X) \longrightarrow \mathfrak{Se}(X), \ \mathbb{T} \longmapsto \mathbb{T}^-,$$

(see 2.4) induces a functor $\mathbf{Spec}^0_{\mathfrak{c}}(X) \longrightarrow \mathfrak{Spec}^0(\mathfrak{Se}(X)$.

Proof. The argument is similar to that of 2.3.5.1. Details are left to the reader. ∎

2.5.2. The spectrum $Spec^0_{\mathfrak{c}}(X)$**.** We denote by $Spec^0_{\mathfrak{c}}(X)$ the full subcategory of the category $C_{\mathfrak{M}(X)}$ generated by nonzero objects M such that $M \in Ob[N]_{\mathfrak{c}}$ if there exists a nonzero morphism $N \longrightarrow M$. Here $[N]_{\mathfrak{c}}$ denotes the smallest coreflective topologizing subcategory of C_X containing the object N.

Since $[N]_{\mathfrak{c}} \subseteq [N]^-$ for every object N, it follows that $Spec^0_{\mathfrak{c}}(X) \subseteq Spec^0_{\mathfrak{s}}(X)$. In particular, $Spec(X) \subseteq Spec^0_{\mathfrak{S}\mathfrak{e}}(X)$.

2.5.2.1. Remarks. (a) It is easy to show that a nonzero object M belongs to $Spec^0_{\mathfrak{c}}(X)$ iff $[L]_{\mathfrak{c}} = [M]_{\mathfrak{c}}$ for every nonzero subobject L of M. In other words, the functor

$$C_{\mathfrak{M}(X)} \longrightarrow \mathfrak{CT}(X), \quad M \longmapsto [M]_{\mathfrak{c}},$$

maps every nonzero (mono)morphism $L \longrightarrow M$ to the (identical) isomorphism.

(b) Suppose the category C_X has infinite coproducts. Then one can show that, for any object $M \in ObC_X$, objects of the subcategory $[M]_{\mathfrak{c}}$ are subquotients of a coproduct of a set of copies of M, while objects of the subcategory $[M]$ are subquotients of a coproduct of a *finite* set of copies of M.

Thus, a nonzero object M belongs to $Spec^0_{\mathfrak{c}}(X)$ iff M is a subquotient of a coproduct of a set of copies of any of its nonzero subobjects. And a nonzero object M' belongs to $Spec(X)$ iff M' is a subquotient of a coproduct of a finite set of copies of any of its nonzero subobjects.

2.5.3. Proposition. *Suppose that the category C_X has the property (sup). Then the map*

$$Spec^0_{\mathfrak{c}}(X) \longrightarrow \mathbf{Spec}^0_{\mathfrak{c}}(X), \quad M \longmapsto [M]_{\mathfrak{c}}, \tag{1}$$

is surjective.

Proof. Let $\mathcal{P}$ is an object of $\mathbf{Spec}^0_{\mathfrak{c}}(X)$; i.e. $\mathcal{P}$ is a coreflective topologizing subcategory such that the union, $\widehat{\mathcal{P}}^{\mathfrak{ct}}$, of all coreflective topologizing subcategories of C_X which do not contain $\mathcal{P}$ is a Serre subcategory. Let $M \in Ob\mathcal{P} - Ob\widehat{\mathcal{P}}^{\mathfrak{ct}}$. Since $\widehat{\mathcal{P}}^{\mathfrak{ct}}$ is coreflective, we can and will assume M that M is $\widehat{\mathcal{P}}^{\mathfrak{ct}}$-torsion free. Since $\widehat{\mathcal{P}}^{\mathfrak{ct}}$ is the union, $\langle M\rangle_{\mathfrak{c}}$, of all topologizing subcategories of C_X which do not contain M, it follows that M is an object of $Spec^0_{\mathfrak{c}}(X)$ such that $[M]_{\mathfrak{c}} = \mathcal{P}$. ■

Recall that an object M of a category C_X is *of finite type* if the functor $C_X(M,-)$ preserves colimits of filtered systems of monomorphisms. If the category C_X has the property (sup), then M is of finite type iff the following condition holds: if M is the supremum of a family, $\mathfrak{F}$, of its subobjects, then M is the supremum of a finite subfamily of $\mathfrak{F}$. If C_X is the category of modules over some associative ring, then its objects of finite type are finitely generated modules.

2.5.4. Proposition. *Suppose that the category C_X has the property (sup) and every nonzero object of C_X has a nonzero subobject of finite type. Then* $\mathbf{Spec}^0_{\mathfrak{c}}(X) = \mathbf{Spec}(X)$.

Proof. The inclusion $\mathbf{Spec}(X) \subseteq \mathbf{Spec}^0_{\mathfrak{c}}(X)$ holds without any additional hypothesis. The inverse inclusion is a consequence of the following observations.

(a) Thanks to the property (sup), the smallest coreflective subcategory spanned by a topologizing subcategory, $\mathbb{T}$, is generated by objects which are supremums of objects of $\mathbb{T}$. In particular, for any object N of the category C_X, objects of the subcategory $[N]_{\mathfrak{c}}$ are supremums (of a filtered family) of their subobjects which belong to $[N]$. This implies that every object of finite type of the category $[N]_{\mathfrak{c}}$ belongs to $[N]$.

(b) Let $\mathcal{P}$ be an object of $\mathbf{Spec}^0_{\mathfrak{c}}(X)$. By 2.5.3, $\mathcal{P} = [M]_{\mathfrak{c}}$ for some object M of $Spec^0_{\mathfrak{c}}(X)$. Suppose that M is of finite type. Then M belongs to $Spec(X)$.

In fact, M belongs to the subcategory $[N]_{\mathfrak{c}}$ for any nonzero subobject N of M. By (a), since M is of finite type, it belongs to $[N]$. This means that M is an object of $Spec(X)$.

(c) Since $[M]_{\mathfrak{c}} = [L]_{\mathfrak{c}}$ for any nonzero subobject, L, of M, and, by hypothesis, M has a nonzero subobject of finite type, we can choose M to be of finite type. ■

2.5.4.1. Corollary. *If C_X is the category of left (or right) modules over an associative unital ring, then* $\mathbf{Spec}^0_{\mathfrak{c}}(X) = \mathbf{Spec}(X)$.

2.6. Spectra of 'spaces' represented by triangulated categories.

2.6.1. The spectra of exact localizations. Let $\mathcal{CT}_{\mathfrak{X}} = (C_X, \gamma; \mathfrak{D})$ be a triangulated category. Let $\mathfrak{Tht}(\mathfrak{X})$ denote the preorder of all thick triangulated subcategories of $\mathcal{CT}_{\mathfrak{X}}$. Following the standard procedure, we associate with the preorder $\mathfrak{Tht}(\mathfrak{X})$ two spectra of the 'space' $\mathfrak{X}$ represented by the triangulated category $\mathcal{CT}_{\mathfrak{X}}$

$$\mathbf{Spec}^i_{\mathfrak{Tht}}(\mathfrak{X}) = \mathfrak{Spec}^i(\mathfrak{Tht}(\mathfrak{X})), \quad i = 0, 1.$$

and a canonical morphism from one to another,

$$\mathbf{Spec}^0_{\mathfrak{Tht}}(\mathfrak{X}) \longrightarrow \mathbf{Spec}^1_{\mathfrak{Tht}}(\mathfrak{X}) \tag{1}$$

which assigns to every object $\mathcal{P}$ of $\mathbf{Spec}^0_{\mathfrak{Tht}}(\mathfrak{X})$ the union, $\widehat{\mathcal{P}}^{\mathfrak{tr}}$, of all thick triangulated subcategories of $\mathcal{CT}_{\mathfrak{X}}$ which do not contain the subcategory $\mathcal{P}$.

2.6.2. Flat spectra. Let $\mathfrak{Se}(\mathfrak{X})$ denote the family of all thick triangulated categories $\mathbb{T}$ of C_X such that the localization functor $C_X \xrightarrow{q^*_{\mathbb{T}}} C_X/\mathbb{T}$ has a right adjoint, $q_{\mathbb{T}*}$.

The *flat* spectra of $\mathfrak{X}$ are relative spectra

$$\mathbf{Spec}^i_{\mathfrak{fL}}(\mathfrak{X}) = \mathfrak{Spec}^i(\mathfrak{Se}(\mathfrak{X}), \mathfrak{J}_{\mathfrak{S}}), \quad i = 0, 1,$$

corresponding to the inclusion functor $\mathfrak{Se}(\mathfrak{X}) \xrightarrow{\mathfrak{J}_{\mathfrak{S}}} \mathfrak{Tht}(\mathfrak{X})$. The morphism (1) induces a canonical morphism

$$\mathbf{Spec}^i_{\mathfrak{fL}}(\mathfrak{X}) \longrightarrow \mathbf{Spec}^i_{\mathfrak{fL}}(\mathfrak{X}). \tag{2}$$

Let $\mathbf{Spec}^{1/2}_{\mathfrak{fL}}(\mathfrak{X})$ denote the full subpreorder of $\mathfrak{Tht}(\mathfrak{X})$ whose objects are thick triangulated subcategories $\mathcal{Q}$ such that ${}^\perp\mathcal{Q}$ belongs to $\mathbf{Spec}^1_{\mathfrak{fL}}(\mathfrak{X})$ and every thick triangulated subcategory of $\mathcal{CT}_{\mathfrak{X}}$ properly containing ${}^\perp\mathcal{Q}$ contains $\mathcal{Q}$; i.e. ${}^\perp\mathcal{Q} \vee \mathcal{Q}$ is the smallest thick triangulated subcategory of $\mathcal{CT}_{\mathfrak{X}}$ properly containing ${}^\perp\mathcal{Q}$. Here ${}^\perp\mathcal{Q}$ is the *left orthogonal to* $\mathcal{Q}$, i.e. the full subcategory of $\mathcal{CT}_{\mathfrak{X}}$ generated by all objects L such that $\mathcal{CT}_{\mathfrak{X}}(L, M) = 0$ for every $M \in Ob\mathcal{Q}$.

3. The left spectra.

There are 'spaces' with only trivial saturated multiplicative systems. They might be called *simple* in the same sense as a ring with only trivial two-sided ideals is called simple. If X is such a 'space', then $\mathbf{Spec}^1_{\mathfrak{L}}(X) = \{Iso(C_X)\}$ and $\mathbf{Spec}^0_{\mathfrak{L}}(X) = \{HomC_X\}$. It follows that all other spectra introduced above (closed, flat, etc.) are one-element sets too. Some of simple 'spaces' have quite meaningful left spectra. The latter are associated with the preorder of saturated left multiplicative systems.

A fundamental example of a simple 'space' is the 'space' represented by the category $Sets = Sets_{\mathfrak{U}}$ of sets which belong to a given universe $\mathfrak{U}$.

3.1. Basic left spectra. Let X be a 'space'. We take as $\mathfrak{H}$ the preorder $\mathcal{S}^{\mathfrak{s}}\mathcal{M}_\ell(X)$ of saturated left multiplicative systems on C_X and set

$$\mathbf{Spec}^i_{\mathfrak{L},\ell}(X) = \mathfrak{Spec}^i(\mathcal{S}^{\mathfrak{s}}\mathcal{M}_\ell(X)), \quad i = 0, 1.$$

By 1.4.6, there exists a canonical injective map

$$\mathbf{Spec}^0_{\mathfrak{L},\ell}(X) \longrightarrow \mathbf{Spec}^1_{\mathfrak{L},\ell}(X), \quad \Sigma \longmapsto \widehat{\Sigma}.$$

3.1.1. The spectrum $\mathbf{Spec}^1_{\mathfrak{L},\ell}(X)$ and left local quotient 'spaces'. We call a 'space' Y *left local* if the preorder $(\mathcal{S}^{\mathfrak{s}}\mathcal{M}_\ell(Y), \subseteq)$ is local, i.e. there is the smallest non-trivial saturated left multiplicative system on C_Y. It follows that $\mathbf{Spec}^1_{\mathfrak{L},\ell}(X)$ is formed by all $\Sigma \in \mathcal{S}^{\mathfrak{s}}\mathcal{M}_\ell(X)$ such that the quotient 'space' $\Sigma^{-1}X$ is left local.

3.2. Closed left spectra. Let $\mathfrak{C}\mathcal{S}^{\mathfrak{s}}\mathcal{M}_\ell(X)$ denote the preorder of all closed saturated left multiplicative systems on C_X (cf. 2.0.5). They give rise to the spectra

$$\mathbf{Spec}^i_{\mathfrak{f},\ell}(X) = \mathfrak{Spec}^i(\mathfrak{C}\mathcal{S}^{\mathfrak{s}}\mathcal{M}_\ell(X)), \quad i = 0, 1.$$

and the relative spectra

$$\mathbf{Spec}^i_{\mathfrak{C},\ell}(X) = \mathfrak{Spec}^i(\mathfrak{C}\mathcal{S}^{\mathfrak{s}}\mathcal{M}_\ell(X), \mathfrak{J}_{\mathfrak{C},\ell}), \quad i = 0, 1,$$

where $\mathfrak{J}_{\mathfrak{C},\ell}$ denotes the embedding $\mathfrak{C}\mathcal{S}^{\mathfrak{s}}\mathcal{M}_\ell(X) \hookrightarrow \mathcal{S}^{\mathfrak{s}}\mathcal{M}_\ell(X)$.

They are related by canonical injective maps

$$\mathbf{Spec}^0_{\mathfrak{f},\ell}(X) \longrightarrow \mathbf{Spec}^1_{\mathfrak{f},\ell}(X),$$

$$\mathbf{Spec}^0_{\mathfrak{C},\ell}(X) \longrightarrow \mathbf{Spec}^1_{\mathfrak{C},\ell}(X)$$

(see 1.4.6 and 1.6.2).

3.3. Continuous left multiplicative systems and continuous left spectra. A left multiplicative system Σ in C_X is called *continuous* if the corresponding localization functor $C_X \longrightarrow \Sigma^{-1}C_X$ has a right adjoint.

Let $\mathfrak{Lc}_\ell(X)$ denote the preorder of continuous saturated left multiplicative systems. By [R6, 5.2.1, 5.2.2], every continuous saturated left multiplicative system is closed, i.e. $\mathfrak{Lc}_\ell(X) \subseteq \mathfrak{C}\mathcal{S}^{\mathfrak{s}}\mathcal{M}_\ell(X)$.

Let $\mathfrak{J}_{\mathfrak{c},\ell}$ denote the embedding $\mathfrak{Lc}_\ell(X) \hookrightarrow \mathcal{S}^{\mathfrak{s}}\mathcal{M}_\ell(X)$. This data provides us with the *continuous left spectra*

$$\mathbf{Spec}^i_{\mathfrak{Fl},\ell}(X) = \mathfrak{Spec}^i(\mathfrak{Lc}_\ell(X)), \quad i = 0, 1.$$

and the *relative continuous left spectra*

$$\mathbf{Spec}^i_{\mathfrak{fL},\ell}(X) = \mathfrak{Spec}^i(\mathfrak{Lc}_\ell(X), \mathfrak{J}_{\mathfrak{c},\ell}), \quad i = 0, 1,$$

together with the canonical injective maps

$$\mathbf{Spec}^0_{\mathfrak{Fl},\ell}(X) \longrightarrow \mathbf{Spec}^1_{\mathfrak{Fl},\ell}(X),$$

$$\mathbf{Spec}^0_{\mathfrak{fL},\ell}(X) \longrightarrow \mathbf{Spec}^1_{\mathfrak{fL},\ell}(X).$$

3.3.1. Another realization of continuous localizations and continuous left spectra. Fix a 'space' X. Consider the preorder $\mathfrak{fL}_\ell(X)$ of all strictly full subcategories C_Y of C_X such that the inclusion functor $C_Y \xrightarrow{\iota_{Y*}} C_X$ has a left adjoint $C_X \xrightarrow{\iota_Y^*} C_Y$. These functors are regarded as resp. direct and inverse image functors of a *continuous strictly full embedding* $Y \overset{\iota_Y}{\hookrightarrow} X$. The map which assigns to every such subcategory the family of arrows $\Sigma_{\iota_Y^*} = {\iota_Y^*}^{-1}(Iso(C_Y))$ is an isomorphism of the preorder $(\mathfrak{fL}_\ell(X), \supseteq)$ onto the preorder $(\mathfrak{Lc}_\ell(X), \subseteq)$ of continuous saturated left multiplicative systems.

Thus, the continuous left spectrum $\mathbf{Spec}^1_{\mathfrak{fL}}(X)$ can be identified with the preorder of all continuous strictly full embeddings $Y \overset{\iota_Y}{\hookrightarrow} X$ such that Y is a left local 'space'.

Let $\mathfrak{fL}^\star_\ell(X)$ denote the set $\mathfrak{fL}(X) - \{id_X\}$ of all proper continuous strictly full embeddings. Thanks to the isomorphism $(\mathfrak{Lc}(X), \subseteq) \xrightarrow{\sim} (\mathfrak{fL}(X), \supseteq)$ elements of $\mathbf{Spec}^1_{\mathfrak{Fl},\ell}(X)$ can be identified with continuous strictly full embeddings $Y \overset{\iota_Y}{\hookrightarrow} X$ such that the preorder $(\mathfrak{fL}^\star(Y), \subseteq)$ of proper strictly full continuous embeddings into Y has the biggest element.

3.4. The left spectra of Sets. Let $\mathcal{E}$ denote the 'space' represented by the category $Sets = Sets_{\mathfrak{U}}$ of sets which belong to a fixed universe $\mathfrak{U}$, i.e. $C_{\mathcal{E}} = Sets$.

The preorder $\mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\mathfrak{r}}(\mathcal{E})$ of right saturated multiplicative systems on $Sets$ is trivial: it consists only of $Iso(C_{\mathcal{E}})$ and $HomC_{\mathcal{E}}$. In particular, the preorder $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(\mathcal{E})$ of saturated multiplicative systems on $Sets$ is trivial.

For an infinite cardinal number α, let Σ_α denote the family of all maps $M \xrightarrow{f} N$ (– morphisms of $C_{\mathcal{E}}$) such that

(a) $M \neq \emptyset$ if $N \neq \emptyset$,

(b) $Card(N - f(M)) < \alpha$,

(c) There exists a subset M' of M such that $Card(M - M') < \alpha$ and the restriction of the map f to M' is injective.

Let $\Sigma_{\alpha*}$ denote the family of all maps $M \xrightarrow{f} N$ satisfying (b) and (c) only. So that $\Sigma_\alpha \subset \Sigma_{\alpha*}$. Explicitly, $\Sigma_{\alpha*} = \Sigma_\alpha \bigcup \{\emptyset \longrightarrow N \mid Card(N) < \alpha\}$.

Both Σ_α and $\Sigma_{\alpha*}$ are saturated left multiplicative systems on *Sets*. Moreover, every saturated left multiplicative system on *Sets* is either Σ_α or $\Sigma_{\alpha*}$ for a suitable infinite cardinal α (see [GZ], I.2.5f) and I.3.5).

Let $\mathfrak{Sp}(\mathfrak{U})$ denote the order of non-limit cardinals which belong to the universe $\mathfrak{U}$.

3.4.1. Proposition. *(a) Let α be an infinite cardinal number. Then the following conditions are equivalent:*

(i) α is a non-limit cardinal,

(ii) Σ_α belongs to $\mathbf{Spec}^0_{\mathfrak{L},\ell}(\mathcal{E})$.

(b) The map $\alpha \longmapsto \Sigma_\alpha$ defines an isomorphism of preorders $\mathfrak{Sp}(\mathfrak{U}) \xrightarrow{\varrho^0} \mathbf{Spec}^0_{\mathfrak{L},\ell}(\mathcal{E})$.

(c) If α is a non-limit infinite cardinal, then $\Sigma_{\alpha-1*}$ belongs to $\mathbf{Spec}^1_{\mathfrak{L},\ell}(\mathcal{E})$. The map $\alpha \longmapsto \Sigma_{\alpha-1*}$ defines an isomorphism of preorders $\mathfrak{Sp}(\mathfrak{U}) \xrightarrow{\varrho^1} \mathbf{Spec}^1_{\mathfrak{L},\ell}(\mathcal{E})$ such that the diagram

$$\begin{tikzcd} \mathbf{Spec}^0_{\mathfrak{L},\ell}(\mathcal{E}) \arrow[rr, "\theta_\ell(\mathcal{E})"] & & \mathbf{Spec}^1_{\mathfrak{L},\ell}(\mathcal{E}) \\ & \mathfrak{Sp}(\mathfrak{U}) \arrow[ul, "\varrho^0"] \arrow[ur, "\varrho^1"'] & \end{tikzcd} \qquad (1)$$

commutes. In particular, the canonical preorder morphism $\mathbf{Spec}^0_{\mathfrak{L},\ell}(\mathcal{E}) \xrightarrow{\theta_\ell(\mathcal{E})} \mathbf{Spec}^1_{\mathfrak{L},\ell}(\mathcal{E})$ is an isomorphism.

Proof. (a) Fix an infinite cardinal number α.

(i)$\Rightarrow$(ii). Let α be a non-limit cardinal. Then the union, $\widehat{\Sigma}_\alpha$ of all elements of the support of Σ_α (that is the union of all saturated left multiplicative system on $C_{\mathcal{E}}$ which do not contain Σ_α) coincides with $\Sigma_{\alpha-1*}$. This shows that Σ_α is an element of $\mathbf{Spec}^0_{\mathfrak{L},\ell}(\mathcal{E})$.

(ii)$\Rightarrow$(i). If α is a limit cardinal, then $\widehat{\Sigma}_\alpha = \bigcup\limits_{\beta<\alpha} \Sigma_{\beta*} \supseteq \bigcup\limits_{\beta<\alpha} \Sigma_\beta = \Sigma_\alpha$. This shows that the support, $\mathfrak{Supp}(\Sigma_\alpha)$, of Σ_α does not have the final object, i.e. the left multiplicative system Σ_α does not belong to $\mathbf{Spec}^0_{\mathfrak{L},\ell}(\mathcal{E})$.

(b) Let α be any cardinal of a set from $\mathfrak{U}$. The support of $\Sigma_{\alpha*}$ consists of all Σ_β and all $\Sigma_{\gamma*}$ with $\gamma < \alpha$. Thus $\widehat{\Sigma}_{\alpha*} = HomC_{\mathcal{E}^1} \bigcup \{\emptyset \longrightarrow N \mid Card(N) < \gamma,\ \gamma < \alpha\}$, where $C_{\mathcal{E}^1}$

is the full subcategory of $C_{\mathcal{E}}$ formed by all non-empty sets. It follows from this description that $\widehat{\Sigma}_{\alpha*}$ is not closed under composition, hence the support of $\Sigma_{\alpha*}$ does not have the final object, i.e. the left multiplicative system $\Sigma_{\alpha*}$ does not belong to $\mathbf{Spec}^0_{\mathfrak{L},\ell}(\mathcal{E})$. Since every saturated left multiplicative system on *Sets* is either Σ_α, or $\Sigma_{\alpha*}$, this proves that the map $\alpha \longmapsto \Sigma_\alpha$ is an isomorphism $\mathfrak{Sp}(\mathfrak{U}) \longrightarrow \mathbf{Spec}^0_{\mathfrak{L},\ell}(\mathcal{E})$.

(c) Let α be an infinite non-limit cardinal number. That $\Sigma_{\alpha-1*}$ is an element of $\mathbf{Spec}^1_{\mathfrak{L},\ell}(\mathcal{E})$ follows, together with the commutativity of the diagram (1), from the fact that $\Sigma_{\alpha-1*}$ is the final object of the support of Σ_α (see the argument (i)⇒(ii) above) and 1.4.6. Clearly the map

$$\mathfrak{Sp}(\mathfrak{U}) \xrightarrow{\varrho^1} \mathbf{Spec}^1_{\mathfrak{L},\ell}(\mathcal{E}), \quad \alpha \longmapsto \Sigma_{\alpha-1*}, \tag{2}$$

is a morphism of preorders. It remains to show that this map is bijective.

In fact, for any pair of infinite cardinal numbers β and γ such that $\beta < \gamma$, the system Σ_β is contained properly in Σ_γ and in $\Sigma_{\beta*}$. On the other hand, $\Sigma_\beta = \Sigma_\gamma \bigcap \Sigma_{\beta*}$. Therefore Σ_β does not belong to $\mathbf{Spec}^1_{\mathfrak{L},\ell}(\mathcal{E})$ for any infinite cardinal number β. Therefore elements of $\mathbf{Spec}^1_{\mathfrak{L},\ell}(\mathcal{E})$ are systems $\Sigma_{\beta*}$ for some β. Suppose $\Sigma_{\beta*}$ belongs to $\mathbf{Spec}^1_{\mathfrak{L},\ell}(\mathcal{E})$, i.e. there exists the smallest system $\Sigma^\star_{\beta*}$ in $\mathcal{S}^\ell\mathcal{M}_{\mathfrak{r}}(\mathcal{E})$ properly containing $\Sigma_{\beta*}$. Since $\Sigma_{\beta*}$ is not contained in Σ_σ for any σ, the system $\Sigma^\star_{\beta*}$ should coincide with $\Sigma_{\alpha*}$ for some α. But, $\Sigma_{\beta*} \subsetneqq \Sigma_{\gamma*}$ iff $\beta \lneqq \gamma$. Therefore, $\beta \lneqq \alpha$ and there are no intermediate cardinal numbers, i.e. α is a non-limit cardinal number and $\beta = \alpha - 1$. ■

3.4.2. Other spectra. Let $\Sigma \subseteq HomC_{\mathcal{E}}$. By definition, and object M of $C_{\mathcal{E}}$ is Σ-torsion free if every morphism $M \longrightarrow M'$ which belongs to Σ is a monomorphism. Suppose Σ is Σ_α or $\Sigma_{\alpha*}$ for some infinite cardinal number α. Then a set M is Σ-torsion free iff $Card(M) \le 1$ (that is either $M = \emptyset$, or M is a one-element set). It follows from the definitions of Σ_α and $\Sigma_{\alpha*}$ that objects of $C_{\mathcal{E}}$ having a morphism to a Σ-torsion free object are precisely sets N such that $Card(N) < \alpha$. This shows that the only closed saturated left multiplicative systems on $C_{\mathcal{E}}$ are $Iso(C_{\mathcal{E}})$ and $HomC_{\mathcal{E}}$. Since, by [R6, 5.2.2], every continuous saturated left multiplicative system is closed, there are no non-trivial continuous saturated left multiplicative systems either. Therefore, $\mathbf{Spec}^0_{\mathfrak{f},\ell} = \mathbf{Spec}^0_{\mathfrak{Fl},\ell} = \{HomC_{\mathcal{E}}\}$ and $\mathbf{Spec}^1_{\mathfrak{f},\ell} = \mathbf{Spec}^1_{\mathfrak{Fl},\ell} = \{Iso(C_{\mathcal{E}})\}$ (see notations in 3.2 and 3.3).

The relative continuous and closed left spectra (cf. 3.2, 3.3) are empty.

3.4.3. Sets without empty set. Let $C_{\mathcal{E}^1} = Sets^1_{\mathfrak{U}}$, where (in accordance with notations in Section 1), $Sets^1_{\mathfrak{U}}$ is the full subcategory of $Sets_{\mathfrak{U}}$ formed by non-empty sets which belong to the universe $\mathfrak{U}$. It follows that the preorder $\mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\mathfrak{r}}(\mathcal{E}^1)$ of saturated right multiplicative systems on $Sets^1_{\mathfrak{U}}$ is trivial and the set $\mathcal{S}^{\mathfrak{s}}\mathcal{M}_\ell(\mathcal{E}^1)$ of saturated left multiplicative systems consists of all systems $\Sigma^1_\alpha = \Sigma_\alpha \bigcap HomC_{\mathcal{E}^1}$, where α runs through infinite cardinal numbers (notice that $\Sigma_{\alpha *} \bigcap HomC_{\mathcal{E}^1} = \Sigma^1_\alpha$ for any α). There is a following analogue of 3.4.1:

3.4.4. Proposition. *(a) Let α be an infinite cardinal number. Then the following conditions are equivalent:*

(i) α is a non-limit cardinal,

(ii) Σ^1_α belongs to $\mathbf{Spec}^0_{\mathfrak{L},\ell}(\mathcal{E}^1)$.

(b) The map $\alpha \longmapsto \Sigma^1_\alpha$ defines an isomorphism of preorders $\mathfrak{Sp}(\mathfrak{U}) \xrightarrow{\nu^0} \mathbf{Spec}^0_{\mathfrak{L},\ell}(\mathcal{E}^1)$.

(c) If α is a non-limit infinite cardinal, then $\Sigma^1_{\alpha-1}$ belongs to $\mathbf{Spec}^1_{\mathfrak{L},\ell}(\mathcal{E}^1)$. The map $\alpha \longmapsto \Sigma^1_{\alpha-1}$ defines an isomorphism of preorders $\mathfrak{Sp}(\mathfrak{U}) \xrightarrow{\nu^1} \mathbf{Spec}^1_{\mathfrak{L},\ell}(\mathcal{E}^1)$ such that the diagram

$$\begin{tikzcd} \mathbf{Spec}^0_{\mathfrak{L},\ell}(\mathcal{E}^1) \arrow[rr, "\theta_\ell(\mathcal{E}^1)"] & & \mathbf{Spec}^1_{\mathfrak{L},\ell}(\mathcal{E}^1) \\ & \mathfrak{Sp}(\mathfrak{U}) \arrow[ul, "\nu^0"] \arrow[ur, "\nu^1"'] & \end{tikzcd} \qquad (1)$$

commutes. In particular, the canonical morphism $\mathbf{Spec}^0_{\mathfrak{L},\ell}(\mathcal{E}^1) \xrightarrow{\theta_\ell(\mathcal{E}^1)} \mathbf{Spec}^1_{\mathfrak{L},\ell}(\mathcal{E}^1)$ is an isomorphism.

Proof. The assertion follows from 3.4.1. Details are left to the reader. ■

The canonical map $\mathbf{Spec}^0_{\mathfrak{L},\ell}(\mathcal{E}^1) \xrightarrow{\theta_\ell(\mathcal{E}^1)} \mathbf{Spec}^1_{\mathfrak{L},\ell}(\mathcal{E}^1)$ (of 1.4.6) assigns to each element Σ^1_α of $\mathbf{Spec}^0_{\mathfrak{L},\ell}(\mathcal{E}^1)$ the system $\Sigma^1_{\alpha-1}$. Notice that the inverse map assigns to each element Σ of $\mathbf{Spec}^1_{\mathfrak{L},\ell}(\mathcal{E}^1)$ the smallest saturated left multiplicative system properly containing Σ.

3.5. Example: finite sets. Let $C_{\mathcal{E}_\mathfrak{f}}$ be the category $Set_\mathfrak{f}$ of finite sets. There are no non-trivial left or right multiplicative systems on $Sets_\mathfrak{f}$; so that the 'space' $\mathcal{E}_\mathfrak{f}$ can be viewed as an analog of a 'point' – the spectrum of a field.

4. Left exact multiplicative systems and injective spectra.

4.1. Left exact multiplicative systems. Fix a 'space' X. We call a saturated left multiplicative system Σ *left exact* if the localization functor $C_X \xrightarrow{q^*_\Sigma} \Sigma^{-1}C_X = C_{\Sigma^{-1}X}$ maps strict monomorphisms to strict monomorphisms. Let $\mathcal{S}^\mathfrak{s}_{\ell s}\mathcal{M}(X)$ denote the preorder of all left exact multiplicative systems.

If C_X is an abelian category, then every left exact multiplicative system is a right multiplicative system, i.e. $\mathcal{S}^\mathfrak{s}_{\ell s}\mathcal{M}(X) = \mathcal{S}^\mathfrak{s}\mathcal{M}(X)$.

4.2. Proposition. *Suppose that the category C_X has finite colimits and kernels of pairs of arrows. Let $C_X \xrightarrow{f^*} C_Y$ be a right exact functor (i.e. it preserves finite colimits)* *which maps strict monomorphisms to strict monomorphisms, and let* $\Sigma = \Sigma_{f^*} = \{s \in HomC_X \mid f^*(s) \in Iso(C_X)\}$. *Then both functors,* p_f^* *and* q_f^*, *in the canonical decomposition* $f^* = p_f^* q_f^*$ *(here* q_f^* *is the localization functor* $C_X \longrightarrow \Sigma^{-1}C_X$ *and* p_f^* *is a conservative functor) are right exact and map strict monomorphisms to strict monomorphisms. In particular,* $\Sigma \in \mathcal{S}^{\mathfrak{s}}_{\ell s}\mathcal{M}(X)$.

Proof. (a) By [GZ, I.3.4], the functors q_f^* and p_f^* are right exact and $\Sigma = \Sigma_{f^*}$ belongs to $\mathcal{S}^{\mathfrak{s}}_{\ell}\mathcal{M}(X)$. It remains to show that q_f^* and p_f^* map strict monomorphisms to strict monomorphisms.

(b) Let $L \xrightarrow{\mathfrak{j}} M \overset{p_1}{\underset{p_2}{\rightrightarrows}} M_1 = M \coprod_L M$ be an exact diagram. We claim that its image by the localization functor q_f^* is exact too.

In fact, let a morphism $q_f^*(N) \xrightarrow{g'} q_f^*(M)$ equalize the pair $q_f^*\big(M \overset{p_1}{\underset{p_2}{\rightrightarrows}} M_1\big)$. Since Σ is a left multiplicative system, there exist arrows $N \xrightarrow{g} M' \xleftarrow{s} M$ such that $s \in \Sigma$ and $g' = q_f^*(s)^{-1} q_f^*(g)$, and there exist commutative diagrams

$$\begin{array}{ccc} M & \xrightarrow{p_i} & M_1 \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s_i} \\ M' & \xrightarrow{p_i'} & N_i \end{array} \qquad i = 1, 2,$$

with $s_1, s_2 \in \Sigma$. By the same reason, there exists a commutative diagram

$$\begin{array}{ccc} M_1 & \xrightarrow{s_1} & N_1 \\ {\scriptstyle s_2}\downarrow & & \downarrow{\scriptstyle s_2'} \\ N_2 & \xrightarrow{s_1'} & N_1' \end{array}$$

with $s_2' \in \Sigma$. Since the system Σ is saturated and the arrows s_1, s_2, s_2' belong to Σ, the remaining arrow, s_1', belongs to Σ too. Thus, we obtain a commutative diagram

$$\begin{array}{ccccc} L & \xrightarrow{\mathfrak{j}} & M & \overset{p_1}{\underset{p_2}{\rightrightarrows}} & M_1 \\ & & {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle t} \\ N & \xrightarrow{g} & M' & \overset{\phi_1}{\underset{\phi_2}{\rightrightarrows}} & N_1' \end{array} \qquad (1)$$

where s and $t = s_2' s_1$ are arrows from Σ and $\phi_i = s_i' p_i'$. It follows from the definition of $N \xrightarrow{g} M'$ and the commutativity of the diagram (1) that $q_f^*(\phi_1 g) = q_f^*(\phi_2 g)$. Since Σ

is saturated, this means precisely that there exists an arrow $N'_1 \xrightarrow{u} M''_1$ in Σ such that $(u\phi_1)g = (u\phi_2)g$. Set $\phi'_i = u\phi_i$, $i = 1, 2$, and let $L' \xrightarrow{\mathrm{j}'} M'$ denote the kernel of the pair of arrows $M' \underset{\phi'_2}{\overset{\phi'_1}{\rightrightarrows}} M''_1$. Then we have the diagram $L' \xrightarrow{\mathrm{j}'} M' \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} M'_1 \xrightarrow{\sigma} M''_1$, where $M'_1 = M' \coprod_{L'} M'$, π_1, π_2 are coprojections, and σ is a morphism uniquely determined by the equalities $\sigma\pi_i = \phi'_i$, $i = 1, 2$. Combining these decompositions with (1), we obtain a commutative diagram

$$\begin{tikzcd}
 & L \arrow[r, "\mathrm{j}"] \arrow[d, "s'"'] & M \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] \arrow[d, "s"'] & M_1 \arrow[r, "t"] & N'_1 \arrow[d, "u"] \\
N \arrow[r, "g'"] & L' \arrow[r, "\mathrm{j}'"] & M' \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & M'_1 \arrow[r, "\sigma"] & M''_1
\end{tikzcd} \tag{2}$$

Here $N \xrightarrow{g'} L' \xrightarrow{\mathrm{j}'} M'$ is the unique decomposition of the morphism $N \xrightarrow{g} M'$ and the arrow $L \xrightarrow{s'} L'$ is uniquely determined by the commutativity of the diagram (2) and the fact that $L' \xrightarrow{\mathrm{j}'} M'$ is the kernel of the pair of arrows $M' \underset{\sigma\pi_2}{\overset{\sigma\pi_1}{\rightrightarrows}} M''_1$. Applying the localization functor q_f^* to (2) and using that arrows s, t, u, σ belong to Σ, we obtain a commutative diagram

$$\begin{tikzcd}
 & q_f^*(L) \arrow[r, "q_f^*(\mathrm{j})"] \arrow[d, "q_f^*(s')"'] & q_f^*(M) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\wr"'] & q_f^*(M_1) \arrow[d, "\wr"] \\
q_f^*(N) \arrow[r, "q_f^*(g')"] & q_f^*(L') \arrow[r, "q_f^*(\mathrm{j}')"] & q_f^*(M') \arrow[r, shift left] \arrow[r, shift right] & q_f^*(M'_1)
\end{tikzcd} \tag{3}$$

Since two of the three vertical arrows in (3) are isomorphisms and the diagrams

$$q_f^*(L) \xrightarrow{q_f^*(\mathrm{j})} q_f^*(M) \rightrightarrows q_f^*(M_1)$$

and

$$q_f^*(L') \xrightarrow{q_f^*(\mathrm{j}')} q_f^*(M') \rightrightarrows q_f^*(M'_1)$$

are exact, the third one, $q_f^*(s')$, is an isomorphism, or, equivalently, $s' \in \Sigma$. This shows that any morphism $q_f^*(N) \xrightarrow{g'} q_f^*(M)$ which equalizes the pair $q_f^*\big(M \underset{p_2}{\overset{p_1}{\rightrightarrows}} M_1\big)$ factors uniquely through the kernel of this pair. Notice that our argument shows the existence of this kernel.

(c) Let $q_f^*(L) \xrightarrow{\xi'} q_f^*(M')$ be a morphism in $\Sigma^{-1}C_X$ and $L \xrightarrow{\xi} M \xleftarrow{s} M'$ morphisms such that $s \in \Sigma$ and $\xi' = q_f^*(s)^{-1}q_f^*(\xi)$ (cf. (b) above). Consider the cokernel diagram

$$L \xrightarrow{\ \xi\ } M \overset{p_1}{\underset{p_2}{\rightrightarrows}} M_1 = M \coprod_L M$$

of the morphism $L \xrightarrow{\xi} M$. Since the localization functor q_f^* preserves finite colimits, the diagram

$$q_f^*\big(L \xrightarrow{\ \xi\ } M \overset{p_1}{\underset{p_2}{\rightrightarrows}} M_1 = M \coprod_L M\big)$$

is isomorphic to the cokernel diagram of the morphism $q_f^*(L) \xrightarrow{\xi'} q_f^*(M')$. Let $L' \xrightarrow{\mathfrak{j}} M$ denote the kernel of the pair of arrows $M \overset{p_1}{\underset{p_2}{\rightrightarrows}} M_1$, and let $L \xrightarrow{t} L' \xrightarrow{\mathfrak{j}} M$ be the canonical decomposition of $L \xrightarrow{\xi} M$.

It follows from the construction that $q_f^*(L) \xrightarrow{\xi'} q_f^*(M')$ is a strict monomorphism iff $t \in \Sigma$, hence the morphism ξ' is isomorphic to $q_f^*(L') \xrightarrow{q_f^*(\mathfrak{j})} q_f^*(M)$. Therefore $p_f^*(\xi')$ is isomorphic to $p_f^*\big(q_f^*(L') \xrightarrow{q_f^*(\mathfrak{j})} q_f^*(M)\big) = f^*(L') \xrightarrow{f^*(\mathfrak{j})} f^*(M)$. By hypothesis, the functor f^* preserves strict monomorphisms, hence $p_f^*(\xi')$ is a strict monomorphism. ■

The following assertion is suggested by (the part (b) of) the argument of 4.2.

4.2.1. Proposition. *Suppose the category C_X has kernels of pairs of arrows and for any arrow $L \longrightarrow M$, there exists a push-forward $M \coprod_L M$. Then the following conditions on a left saturated multiplicative system Σ are equivalent:*

(a) Σ is left exact.

(b) If in the commutative diagram

$$\begin{array}{ccccc} L & \xrightarrow{\ \mathfrak{j}\ } & M & \overset{p_1}{\underset{p_2}{\rightrightarrows}} & M_1 = M \coprod_L M \\ {\scriptstyle s'}\downarrow & & {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle t} \\ L' & \xrightarrow{\ \mathfrak{j}'\ } & M' & \overset{\phi_1}{\underset{\phi_2}{\rightrightarrows}} & M'' \end{array} \qquad (4)$$

the rows are exact and the vertical arrows s and t belong to Σ, then the left vertical arrow belongs to Σ too.

Proof. (a)$\Rightarrow$(b). The diagram (4) gives rise to the commutative diagram

$$
\begin{tikzcd}
L \arrow[r, "\mathrm{j}"] \arrow[d, "s'"'] & M \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] \arrow[d, "s"'] & M_1 \arrow[r, "t"] \arrow[d, "u"] & M'' \arrow[d, "id_{M''}"] \\
L' \arrow[r, "\mathrm{j}'"] & M' \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & M_1' \arrow[r, "\sigma"] & M''
\end{tikzcd}
\tag{5}
$$

where $M_1' = M' \coprod_{L'} M'$, $M_1 \xrightarrow{\sigma} M''$ is uniquely determined by $\sigma\pi_i = \phi_i$, $i = 1, 2$; and $M_1 \xrightarrow{u} M_1'$ is uniquely defined by the left square in (5) (due to the functoriality of coproducts). Applying the localization functor $C_X \xrightarrow{q^*} \Sigma^{-1}C_X$ to the diagram (5), we obtain the diagram

$$
\begin{tikzcd}
q^*(L) \arrow[r, "\widetilde{\mathrm{j}}"] \arrow[d, "q^*(s')"'] & q^*(M) \arrow[r, shift left, "p_1'"] \arrow[r, shift right, "p_2'"'] \arrow[d, "q^*(s)"'] & q^*(M_1) \arrow[d, shift right, "u'"'] \\
q^*(L') \arrow[r, "\widetilde{\mathrm{j}'}"] & q^*(M') \arrow[r, shift left, "\pi_1'"] \arrow[r, shift right, "\pi_2'"'] & q^*(M_1') \arrow[u, shift right, "\sigma'"']
\end{tikzcd}
\tag{6}
$$

where $\sigma' = q^*(t)^{-1}q^*(\sigma)$. Since the system Σ is left exact, the functor q^* maps the diagrams $L \longrightarrow M \rightrightarrows M_1$ and $L' \longrightarrow M' \rightrightarrows M_1'$ to exact diagrams. Therefore the commutative square

$$
\begin{tikzcd}
q^*(M) \arrow[r, shift left] \arrow[r, shift right] & q^*(M_1) \\
q^*(M) \arrow[r, shift left] \arrow[r, shift right] \arrow[u, "q^*(s)^{-1}"] & q^*(M_1) \arrow[u, "\sigma'"']
\end{tikzcd}
$$

and (6) yield a commutative square

$$
\begin{tikzcd}
q^*(L) \arrow[r, "\widetilde{\mathrm{j}}"] & q^*(M) \\
q^*(L') \arrow[r, "\widetilde{\mathrm{j}'}"] \arrow[u, "\widetilde{\sigma}"] & q^*(M') \arrow[u, "q^*(s)^{-1}"']
\end{tikzcd}
$$

It follows from the universal property of kernels that $\widetilde{\sigma} q^*(s') = id_{q^*(L)}$, hence $\widetilde{\sigma}$ is a strict epimorphism. On the other hand, the equality $\widetilde{\mathfrak{j}}\widetilde{\sigma} = q^*(s)^{-1}\widetilde{\mathfrak{j}'}$ (together with the fact that $\widetilde{\mathfrak{j}'}$ is a monomorphism) implies that $\widetilde{\sigma}$ is a monomorphism; hence it is an isomorphism.

(b)⇒(a). The argument is a repetition of the part (b) of the argument of 4.2. Details are left to the reader. ■

4.2.2. Example: left multiplicative systems in Sets. Let $C_{\mathcal{E}} = Sets$. Then $\mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\ell}(\mathcal{E}) = \mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\ell s}(\mathcal{E})$, i.e. every saturated left multiplicative system on $Sets$ is left exact.

In fact, $\mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\ell}(\mathcal{E})$ consists of systems Σ_α and $\Sigma_{\alpha*}$, where α is an infinite cardinal (which belongs to a given universe). Recall that $\Sigma_{\alpha*} = \Sigma_\alpha \bigcup \{\emptyset \longrightarrow N \mid Card(N) < \alpha\}$ and Σ_α is the family of all maps $M \xrightarrow{f} N$ (– morphisms of $C_{\mathcal{E}}$) such that

(a) $M \neq \emptyset$ if $N \neq \emptyset$,

(b) $Card(N - f(M)) < \alpha$,

(c) There exists a subset M' of M such that $Card(M - M') < \alpha$ and the restriction of the map f to M' is injective (see 3.4).

Notice that the conditions (b) and (c) (defining $\Sigma_{\alpha*}$) are equivalent to the following condition which explains the meaning of $\Sigma_{\alpha*}$ and is more convenient for our purposes:

(b') There exists a subset M' of M such that the restriction of the map f to M' is injective and $Card(M - M') < \alpha > Card(N - f(M'))$.

Let Σ be Σ_α or $\Sigma_{\alpha*}$. Consider a commutative square

$$\begin{array}{ccc} L & \xrightarrow{\mathfrak{j}} & M \\ s \downarrow & & \downarrow t \\ L' & \xrightarrow{\mathfrak{j}'} & M' \end{array}$$

in $C_{\mathcal{E}} = Sets$ whose horizontal arrows are monomorphisms and the right vertical arrow, $M \xrightarrow{t} M'$, belongs to Σ. Then $L \xrightarrow{s} L'$ belongs to Σ too.

Suppose first that $M = \emptyset$. Then $L = \emptyset$. If $\Sigma = \Sigma_\alpha$, then $L' = M' = \emptyset$. In particular, $L \xrightarrow{s} L'$ belongs to Σ.

If $M = \emptyset$ and $\Sigma = \Sigma_{\alpha*}$, then $Card(M') < \alpha$. Since $L' \xrightarrow{\mathfrak{j}'} M'$ is an injective map, and $Card(L') \leq Card(M') < \alpha$. Therefore $L \xrightarrow{s} L'$ belongs to $\Sigma_{\alpha*}$.

Suppose now that $M \neq \emptyset$ and L and L' are subsets of resp. M and M'. The map $M \xrightarrow{t} M'$ belongs to Σ iff there exists a subset M'' of M such that the restriction of t to M'' is injective and $Card(M - M'') < \alpha > Card(M' - t(M''))$ (see (b') above). Then the restriction of the map $L \xrightarrow{s} L'$ to $L \cap M''$ is injective and both $Card(L - L \cap M'')$ and $Card(L' - s(L \cap M''))$ are smaller than α. The latter means that $s \in \Sigma$.

Now it follows from 4.2.1 that Σ is a left exact multiplicative system.

4.3. Injective objects. Fix a 'space' X. Let C_{X_s} denote the subcategory of C_X formed by all objects of C_X and split monomorphisms.

An object E of the category C_X is called *injective* (or *strictly injective*) if the functor $C_X(-,E) : C^{op} \longrightarrow Sets$ maps strict epimorphisms (of C^{op}, i.e. strict monomorphisms of C_X) to epimorphisms. We denote by $C_{\mathfrak{I}(X)}$ the subcategory of C_X formed by injective objects and split monomorphisms (or, what is the same, strict monomorphisms, see 1.2.3.3) between them. For any object E of the category C_X, let Σ_E denote the family of all arrows of C_X which the functor $C_X(-,E)$ transforms into invertible morphisms.

4.3.1. Proposition. *(a) Suppose that the category C_X has finite colimits. Then the map $E \longmapsto \Sigma_E$ extends to a functor $C_{X_s}^{op} \longrightarrow \mathcal{S}_\ell^{\mathfrak{s}}\mathcal{M}(X)$.*

(b) If, in addition, C_X has kernels of pairs of arrows. Then the map $E \longmapsto \Sigma_E$ defines a functor $C_{\mathfrak{I}(X)}^{op} \longrightarrow \mathcal{S}_{\ell s}^{\mathfrak{s}}\mathcal{M}(X)$.

Proof. (a) Since the category C_X has finite colimits, it follows from [GZ, 1.3.4] that Σ_E belongs to $\mathcal{S}_\ell^{\mathfrak{s}}\mathcal{M}(X)$ for every object E.

Let $E_1 \xrightarrow{u} E$ be a split monomorphism; and let $L \xrightarrow{s} M$ be a morphism of C_X such that $C_X(s,E)$ is an isomorphism. Then $C_X(s,E_1)$ is an isomorphism. In fact, there exists a morphism $E \xrightarrow{v} E_1$ such that $v \circ u = id_{E_1}$. Thus, there are two commutative diagrams

$$\begin{array}{ccc} C_X(M,E) & \xrightarrow{C_X(s,E)} & C_X(L,E) \\ {\scriptstyle C_X(M,v)}\downarrow & & \downarrow{\scriptstyle C_X(L,v)} \\ C_X(M,E_1) & \xrightarrow{C_X(s,E_1)} & C_X(L,E_1) \end{array} \qquad \begin{array}{ccc} C_X(M,E_1) & \xrightarrow{C_X(s,E_1)} & C_X(L,E_1) \\ {\scriptstyle C_X(M,u)}\downarrow & & \downarrow{\scriptstyle C_X(L,u)} \\ C_X(M,E) & \xrightarrow{C_X(s,E)} & C_X(L,E) \end{array}$$

such that the vertical arrows and the upper horizontal arrow of the first diagram are surjective (hence the remaining arrow, $C_X(s,E_1)$ is surjective) and the vertical arrows and the lower horizontal arrow of the second diagram are injective, hence $C_X(s,E_1)$ is injective. Therefore, $C_X(s,E_1)$ is bijective, i.e. $s \in \Sigma_{E_1}$.

(b) If E is an injective object, then, by 4.2, the localization at Σ_E preserves strict monomorphisms, i.e. Σ_E belongs to $\mathcal{S}_{\ell s}^{\mathfrak{s}}\mathcal{M}(X)$. ∎

4.4. Proposition. *Let $X \xrightarrow{f} Y$ be a continuous morphism with an inverse image functor f^* and a direct image functor f_*. Suppose that the functor f^* maps strict monomorphisms to strict monomorphisms. Then f_* maps injective objects to injective objects.*

Proof. If E is an injective object in C_Y, then the functor $C_X(f^*(-),E)$ maps strict monomorphisms of the category C_Y to (strict) epimorphisms, because f^* preserves strict monomorphisms and, since E is injective, $C_X(-,E)$ maps strict monomorphisms to epimorphisms. But, $C_Y(-,f_*(E)) \simeq C_X(f^*(-),E)$, hence the assertion. ∎

4.5. Proposition. *(a) Let Σ be a left multiplicative system in C_X. Then $\Sigma \subseteq \Sigma_E$ for any Σ-torsion free injective object E.*

(b) Suppose that for every morphism $L \longrightarrow M$ in C_X, there exists a fibred coproduct $M \coprod_L M$ and the pair of coprojections $M \rightrightarrows M \coprod_L M$ has a kernel. Then for any $\Sigma \in \mathcal{S}^{\mathfrak{s}}_{\ell s}\mathcal{M}(X)$, the image of any injective Σ-torsion free object E in the quotient category $\Sigma^{-1}C_X = C_{\Sigma^{-1}X}$ is an injective object.

(c) Let $\Sigma \in \mathcal{S}^{\mathfrak{s}}_{\ell s}\mathcal{M}(X)$ be a continuous multiplicative system such that $\Sigma^{-1}C_X$ has a conservative family of injective objects. Then $\Sigma = \bigcap_{E \in \mathfrak{F}} \Sigma_E$, for some family $\mathfrak{F}$ of Σ-torsion free injective objects.

Proof. (a) Let Σ be a left multiplicative system and E a Σ-torsion free injective object. The claim is that for any arrow $L \xrightarrow{s} M$ in Σ, the map

$$C_X(M, E) \xrightarrow{C_X(s,E)} C_X(L, E), \quad f \longmapsto f \circ s, \tag{1}$$

is bijective. In fact, let $L \xrightarrow{f} E$ be an arbitrary morphism. Since $s \in \Sigma$ and Σ is a left multiplicative system, there exists a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{f'} & M' \\ {\scriptstyle s}\uparrow & & \uparrow{\scriptstyle t} \\ L & \xrightarrow{f} & E \end{array}$$

with $t \in \Sigma$. Since E is Σ-torsion free, the arrow $E \xrightarrow{t} M'$ is a strict monomorphism. Every strict monomorphism from an injective object splits, i.e. there exists a morphism $M' \xrightarrow{g} E$ such that $g \circ t = id_E$. Therefore $(g \circ f') \circ s = g \circ t \circ f = f$. This proves the surjectivity of the map (1). Suppose now that $M \overset{p_1}{\underset{p_2}{\rightrightarrows}} E$ is a pair of arrows such that $p_1 \circ s = p_2 \circ s$. Since Σ is a left multiplicative system and $s \in \Sigma$, there exists an arrow $E \xrightarrow{u} N$ in Σ such that $u \circ p_1 = u \circ p_2$. The arrow u is a (strict) monomorphism, because E is Σ-torsion free, hence $p_1 = p_2$. This shows that the map (1) is injective.

(b) Let q^* be the localization functor $C_X \longrightarrow \Sigma^{-1}C_X$. Consider a diagram

$$q^*(E) \xleftarrow{f} q^*(L) \xrightarrow{\mathfrak{j}'} q^*(M) \tag{1}$$

such that $\mathfrak{j}'$ is a strict monomorphism. Since Σ is a left multiplicative system, the diagram (1) corresponds to the diagram

$$E \xrightarrow{s} K \xleftarrow{f'} L \xrightarrow{\mathfrak{j}''} M' \xleftarrow{t} M. \tag{2}$$

Here $s, t \in \Sigma$ and $f = q^*(s)^{-1}q^*(f')$ and $q^*(\mathfrak{j}') = q^*(t)^{-1}q^*(\mathfrak{j}'')$. Since E is Σ-torsion free, the arrow s is a strict monomorphism. It is a split monomorphism, because E is injective; i.e. there exists an arrow $K \xrightarrow{h} E$ such that $hs = id_E$. In particular, $g^*(h) = g^*(s)^{-1}$. Thus, $f = q^*(\mathfrak{v})$, where $\mathfrak{v} = hf' : L \longrightarrow E$. Consider the decomposition of the arrow $L \xrightarrow{\mathfrak{j}''} M'$ into $L \xrightarrow{u'} L' \xrightarrow{\mathfrak{j}} M'$, where $L' \xrightarrow{\mathfrak{j}} M'$ is the kernel of the pair $M' \underset{p_2}{\overset{p_1}{\rightrightarrows}} M' \coprod_L M'$. Since $q^*(\mathfrak{j}') = q^*(t)^{-1}q^*(\mathfrak{j}'')$ is a strict monomorphism, $q^*(\mathfrak{j}'')$ is a strict monomorphism. The localization functor q^* is right exact [GZ, I.3.1]; in particular, the diagram

$$q^*\big(L \xrightarrow{\mathfrak{j}''} M' \underset{p_2}{\overset{p_1}{\rightrightarrows}} M' \coprod_L M'\big)$$

is isomorphic to

$$q^*(L) \xrightarrow{q^*(\mathfrak{j}'')} q^*(M') \rightrightarrows q^*(M') \coprod_{q^*(L)} q^*(M'). \tag{3}$$

Since $q^*(\mathfrak{j}'')$ is a strict monomorphism, the diagram (3) is exact. By hypothesis, q^* maps strict monomorphisms to strict monomorphisms, hence the diagram

$$q^*(L') \xrightarrow{q^*(\mathfrak{j})} q^*(M') \underset{q^*(p_2)}{\overset{q^*(p_1)}{\rightrightarrows}} q^*(M') \coprod_{q^*(L)} q^*(M'). \tag{3'}$$

is exact too. By the universal property of kernels, this implies that $q^*(L) \xrightarrow{q^*(u')} q^*(L')$ is an isomorphism. Since Σ is saturated, $u' \in \Sigma$. Therefore, there exists a commutative diagram

$$\begin{array}{ccc} L & \xrightarrow{\mathfrak{v}} & E \\ {\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle u''} \\ L' & \xrightarrow{\mathfrak{v}} & E' \end{array}$$

with $u'' \in \Sigma$. Since E is Σ-torsion free and injective, there exists $E' \xrightarrow{w} E$ such that $wu'' = id_E$. Thus, we obtain a diagram $E \xleftarrow{\mathfrak{v}'} L' \xrightarrow{\mathfrak{j}} M'$ in which $\mathfrak{j}$ is a strict monomorphism. Therefore, there exists a morphism $E \xrightarrow{g} M'$ such that $g\mathfrak{j} = \mathfrak{v}'$. But, then $\gamma f = \mathfrak{j}'$, where $\gamma = q^*(t)^{-1}q^*(g)$ (see notations above).

(c) Fix $\Sigma \in \mathcal{S}^{\mathfrak{s}}_{\ell s}\mathcal{M}(X)$ such that $\Sigma^{-1}C_X = C_{\Sigma^{-1}X}$ has a conservative family, $\mathfrak{F}$, of injective objects, i.e. $\{C_{\Sigma^{-1}X}(-, E) \mid E \in \mathfrak{F}\}$ is a conservative family of functors. Fix a direct image functor $C_{\Sigma^{-1}X} \xrightarrow{q_{\Sigma *}} C_X$ of the localization $\Sigma^{-1}X \xrightarrow{q_\Sigma} X$. Since the localization functor $C_X \xrightarrow{q^*_\Sigma} C_{\Sigma^{-1}X}$ maps strict monomorphisms to strict monomorphisms, the functor $q_{\Sigma *}$ maps injective objects to Σ-torsion free injective objects (see 4.4). By the (a), we have the inclusion $\Sigma \subseteq \bigcap_{E\in\mathfrak{F}} \Sigma_{q_{\Sigma *}(E)}$. The inverse inclusion follows from the conservativity of the family $\{C_{\Sigma^{-1}X}(-,E) \mid E \in \mathfrak{F}\}$. ∎

4.6. Example. Suppose that C_X is an abelian category with injective hulls (e.g. C_X is a Grothendieck category). Then the conditions of 4.5 hold. Moreover, every closed multiplicative system Σ is the intersection of the systems Σ_E, where E runs through a family of Σ-torsion free injective objects.

In fact, closed multiplicative systems are in bijective correspondence with coreflective thick subcategories, $\Sigma \longmapsto \mathbb{T}_\Sigma$, where $Ob\mathbb{T}_\Sigma = \{M \in ObC_X \mid (0 \longrightarrow M) \in \Sigma\}$. If E is an injective object, then $\mathbb{T}_{\Sigma_E} = {}^\perp E$, i.e. $Ob\mathbb{T}_{\Sigma_E} = \{M \in ObC_X \mid C_X(M,E) = 0\}$. Notice that $\Sigma_\mathbb{T}$-torsion free objects are precisely $\mathbb{T}$-torsion free objects, i.e. objects which have no nonzero subobjects from $\mathbb{T}$.

Fix a coreflective thick subcategory $\mathbb{T}$ and denote by $q^*_\mathbb{T}$ the localization functor $C_X \longrightarrow C_X/\mathbb{T}$, Let M be a nonzero $\mathbb{T}$-torsion free object. If $M \longrightarrow E$ is an essential monomorphism, then E is $\mathbb{T}$-torsion free object too. Let $\mathfrak{F}$ be a family of $\mathbb{T}$-torsion free objects such that $\{q^*_\mathbb{T}(M) \mid M \in \mathfrak{F}\}$ generates the quotient category $C_X/\mathbb{T} = C_{X/\mathbb{T}}$, i.e. $\{q^*_\mathbb{T}(M) \mid M \in \mathfrak{F}\}^\perp = 0$. For each $M \in \mathfrak{F}$ we chose an injective hull $E(M)$ of M. It follows from 4.5(a) that $\Sigma_\mathbb{T} \subseteq \bigcap_{M\in\mathfrak{F}} \Sigma_{E(M)}$, or, equivalently, $\mathbb{T} \subseteq \bigcap_{M\in\mathfrak{F}} {}^\perp E(M)$. We leave verifying the inverse inclusion to the reader.

Suppose that C_X is a Grothendieck category. Then every closed multiplicative system, Σ, is flat and the corresponding quotient category $C_{\Sigma^{-1}X}$, is a Grothendieck category too. In particular, it has a set of generators, $\mathfrak{F}$. By the argument above, $\Sigma = \bigcap_{M\in\mathfrak{F}} \Sigma_{E(M)} = \Sigma_{E_\mathfrak{F}}$, where $E_\mathfrak{F} = \prod_{M\in\mathfrak{F}} E(M)$. Here we use the fact that injective hulls and small products exist in a Grothendieck category [BD, 6.3.1, 6.3.2]. Thus, we have recovered a well known assertion: every Serre subcategory of a Grothendieck category is of the form ${}^\perp E$ for some injective object E.

4.7. Example. The conditions of 4.5 hold if C_X is an elementary (Lawvere-Tierney) topos, in particular if C_X is a Grothendieck topos. In fact, by a Lawvere-Tierney theorem [J, 1.26], *in a topos, all partial maps are representable.* This means that for any object M, there exists a monomorphism $M \xrightarrow{\eta_M} \widetilde{M}$ such that any diagram $L \xleftarrow{\mathfrak{j}} L' \xrightarrow{f} M$ with a monomorphic left arrow is uniquely completed to a commutative square

$$\begin{array}{ccc} L' & \xrightarrow{f} & M \\ \mathfrak{j}\Big\downarrow & & \Big\downarrow \eta_M \\ L & \xrightarrow{\widetilde{f}} & \widetilde{M} \end{array} \tag{1}$$

It follows from the uniqueness of $\widetilde{f}$ in (1) that the map $M \longmapsto \widetilde{M}$ defines a functor $C_X \xrightarrow{\mathfrak{I}_X} C_X$ and $\eta = \{\eta_M \mid M \in ObC_X\}$ is a functor morphism $Id_{C_X} \longrightarrow \mathfrak{I}_X$. Moreover, for every object M, the object $\mathfrak{I}_X(M) = \widetilde{M}$ is injective [J, 1.27].

Note that $\mathfrak{I}_X(\prod_{i\in I} M_i) \simeq \prod_{i\in I} \mathfrak{I}_X(M_i)$ provided the product $\prod_{i\in I} M_i$ exists, and if $Y \xrightarrow{f} X$ is a *geometric morphism* (that is f is continuous and f^* is (left) exact), then there is a functor isomorphism $f_*\mathfrak{I}_Y \simeq \mathfrak{I}_X f_*$.

Let Σ be a flat multiplicative system in C_X. Set $Y = \Sigma^{-1}X$ and denote by q the canonical morphism $Y \longrightarrow X$. The quotient category $C_Y = \Sigma^{-1}C_X$ is a topos too. Let $\mathfrak{F}$ be a family of generators in C_Y. Then $\bigcap_{M\in\mathfrak{F}} \Sigma_{\mathfrak{I}_Y(M)} = Iso(C_Y)$. Therefore $\Sigma = \bigcap_{M\in\mathfrak{F}} \Sigma_{q_*\mathfrak{I}_Y(M)} = \bigcap_{M\in\mathfrak{F}} \Sigma_{\mathfrak{I}_X q_*(M)}$.

Suppose now that C_X is a Grothendieck topos. Then $C_Y = \Sigma^{-1}C_X$ is a Grothendieck topos. In particular, C_Y has small products and a *set* of generators, $\mathfrak{F}$. Therefore $\Sigma = \bigcap_{M\in\mathfrak{F}} \Sigma_{\mathfrak{I}_X q_*(M)} = \Sigma_{\mathfrak{I}_X(M_{\mathfrak{F}})}$, where $M_{\mathfrak{F}} = \prod_{M\in\mathfrak{F}} q_*(M) \simeq q_*(\prod_{M\in\mathfrak{F}} M)$.

4.8. Note. The examples 4.6 and 4.7 suggest that abelian categories with injective hulls might be regarded as abelian versions of elementary toposes, and Grothendieck categories are abelian analogs of Grothendieck toposes.

4.9. Example. Let $C_{\mathcal{E}} = Sets$ (like in 4.2.2). Then all objects of $C_{\mathcal{E}}$ are injective (and projective). Fix an object E of $C_{\mathcal{E}}$. If E is a one-element set, or the empty set, then $\Sigma_E = HomC_{\mathcal{E}}$. If $Card(E) \geq 2$, then $\Sigma_E = Iso(C_{\mathcal{E}})$.

Evidently, the empty set and one-element sets are the only indecomposable injective objects.

Let $\Sigma \subseteq HomC_{\mathcal{E}}$. By definition, an object M of $C_{\mathcal{E}}$ is Σ-torsion free if every morphism $M \longrightarrow M'$ which belongs to Σ is a monomorphism. Suppose Σ is a saturated left multiplicative system, i.e. it coincides either with Σ_α or with $\Sigma_{\alpha*}$ for some infinite cardinal number α (cf. 4.2.2). Then a set M is Σ-torsion free iff $Card(M) \leq 1$; that is, again, either $M = \emptyset$, or M is a one-element set.

It follows from the definitions of Σ_α and $\Sigma_{\alpha*}$ that objects of $C_{\mathcal{E}}$ having a morphism to a Σ-torsion free object are precisely sets N such that $Card(N) < \alpha$. This shows that the only closed saturated left multiplicative systems on $C_{\mathcal{E}}$ are $Iso(C_{\mathcal{E}})$ and $HomC_{\mathcal{E}}$. Since, by [R6, 5.2.2], every continuous saturated left multiplicative system is closed, there are no non-trivial continuous saturated left multiplicative systems either.

4.10. The injective spectrum and the Gabriel spectrum.

4.10.1. Relatively maximal objects. Fix a functor $\mathfrak{G} \xrightarrow{F} \mathfrak{H}$. We call an object x of $\mathfrak{G}$ *relatively maximal*, or *F-maximal*, if F transforms any arrow $x \longrightarrow y$ into an isomorphism. We denote by $\mathfrak{Max}(\mathfrak{G}, F)$ the full subcategory of $\mathfrak{G}$ generated by relatively maximal objects.

4.10.2. Injective spectrum. Suppose that the category C_X has colimits of finite diagrams. Let F be the functor

$$C^{op}_{\mathfrak{I}(X)} \longrightarrow \mathcal{S}^{\mathfrak{s}}_{\ell s}\mathcal{M}(X), \quad E \longmapsto \Sigma_E$$

(see 4.3.1). We denote $\mathfrak{Max}(C^{op}_{\mathfrak{I}(X)}, F)$ by $ISpec(X)$. It follows that the objects of $ISpec(X)$ are injective objects E such that $\Sigma_E = \Sigma_{E_1}$ for every nontrivial split monomorphism $E_1 \longrightarrow E$. We denote by $\mathbf{ISpec}(X)$ the full subcategory of the $\mathcal{S}^{\mathfrak{s}}_{\ell s}\mathcal{M}(X)$ generated by the image of $ISpec(X)$ and call it the *injective spectrum of* X.

4.10.2.1. Note. The injective spectrum is introduced in [R, 6.5], in a slightly different way, in the case when C_X is an abelian category.

4.10.3. The Gabriel's spectrum. Recall that an object, E, of the category C_X is *indecomposable* if every nontrivial idempotent $E \xrightarrow{\mathfrak{p}} E$ is id_E (see 1.2.3.3).

We denote by $\widehat{ISpec}(X)$ the groupoid $\mathcal{M}in(C_{\mathfrak{I}(X)})$ formed by indecomposable injective objects and their isomorphisms. It follows that $\widehat{ISpec}(X) \subseteq ISpec(X)$.

The Gabriel's spectrum is the full subpreorder, $\widehat{\mathbf{ISpec}}(X)$, of the preorder $\mathcal{S}^{\mathfrak{s}}_{\ell s}\mathcal{M}(X)$ spanned by multiplicative systems Σ_E, where E runs through indecomposable injective objects of C_X. In particular, the Gabriel's spectrum is contained in the injective spectrum: $\widehat{\mathbf{ISpec}}(X) \subseteq \mathbf{ISpec}(X)$.

4.10.3.1. Note. The Gabriel's spectrum is introduced in [Gab] for a (locally noetherian) abelian category. Its elements are defined as isomorphism classes of indecomposable injective objects. The preorder inherited from $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$ is opposite to the *specialization* preorder.

4.11. Injective spectrum of an abelian category. Fix an abelian category C_X. Let $E \in ObC_X$, and let ${}^{\perp}E$ be the full subcategory of the category C_X generated by all objects M which are left orthogonal to E, i.e. $C_X(M,E)=0$. If E_1 is a subobject of the object E, then ${}^\perp E \subseteq {}^\perp E_1$.

If E is an injective object, then ${}^\perp E$ is a Serre subcategory of the category C_X and the map $E \longmapsto {}^\perp E$ is a functor $\mathfrak{J}(X)^{op} \longrightarrow \mathfrak{Se}(X)$. In particular, the spectrum $\mathbf{ISpec}(X)$ can be identified with the subpreorder of the preorder $\mathfrak{Se}(X)$ of Serre subcategories of C_X generated by the image of the map $ISpec(X) \longrightarrow \mathfrak{Se}(X)$, $E \longmapsto {}^\perp E$.

4.11.1. Proposition. *Let C_X be an abelian category with the property (sup).*
(a) If every object of C_X has an injective hull, then $\mathfrak{Spec}^1(\mathfrak{Se}(X)) \subseteq \mathbf{ISpec}(X)$. *In particular,* $\mathbf{Spec}^1_{\mathfrak{Se}}(X) \subseteq \mathbf{ISpec}(X)$.
(b) If C_X has a Gabriel-Krull dimension, then

$$\mathbf{Spec}^1_{\mathfrak{Se}}(X) = \mathfrak{Spec}^1(\mathfrak{Se}(X)) = \mathbf{ISpec}(X) = \widehat{\mathbf{ISpec}}(X).$$

Proof. (a) Let $\mathcal{P}$ be an object of $\mathfrak{Spec}^1(\mathfrak{Se}(X))$, i.e. there exists the smallest Serre subcategory, $\mathcal{P}^{\mathfrak{s}}$, properly containing $\mathcal{P}$. Let $M \in Ob\mathcal{P}^{\mathfrak{s}} - Ob\mathcal{P}$. Since, thanks to the property (sup), every Serre subcategory of C_X, in particular $\mathcal{P}$, is coreflective, we can and will assume that M is $\mathcal{P}$-torsion free. Let $E(M)$ be an injective hull of the object M. Then $E(M)$ is $\mathcal{P}$-torsion free, because M is $\mathcal{P}$-torsion free, hence $\mathcal{P} \subseteq {}^\perp E(M)$. Notice that ${}^\perp E(M)$ cannot contain $\mathcal{P}$ properly, because if $\mathcal{P} \neq {}^\perp E(M)$, then ${}^\perp E(M)$, being a Serre subcategory, would contain $\mathcal{P}^{\mathfrak{s}}$, in particular, it would contain the object M which is not the case. This verifies the inclusion $\mathfrak{Spec}^1(\mathfrak{Se}(X)) \subseteq \mathbf{ISpec}(X)$. By 2.4.1, $\mathbf{Spec}^1_{\mathfrak{Se}}(X) \subseteq \mathfrak{Spec}^1(\mathfrak{Se}(X))$, hence $\mathbf{Spec}^1_{\mathfrak{Se}}(X) \subseteq \mathbf{ISpec}(X)$.

(b) If C_X has a Gabriel-Krull dimension, then, by [R6, 7.9.1], $\mathbf{Spec}^1_{\mathfrak{s}}(X) = \mathbf{Spec}^-(X)$ and by [R, 6.6.1.1, 6.6.1.2], $\mathbf{Spec}^-(X) = \mathbf{ISpec}(X) = \widehat{\mathbf{ISpec}}(X)$. The assertion follows now from the inclusions $\mathbf{Spec}^-(X) \subseteq \mathbf{Spec}^1_{\mathfrak{Se}}(X) \subseteq \mathfrak{Spec}^1(\mathfrak{Se}(X)) \subseteq \mathbf{ISpec}(X)$. ■

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Glossary of notations

Chapter I

$|Cat|^o$ the category of 'spaces', 1.2

$\Sigma^{-1}Y$ the 'space' represented by the quotient category $\Sigma^{-1}C_Y$, 1.2

Σ_F the class of arrows which the functor F maps to isomorphisms, 1.2

$\mathbf{Sp}(R)$ the *categoric spectrum* of a ring R, 1.4

$\mathbf{Sp}_{\mathcal{G}}(R)$ the 'space' represented by the category of $\mathcal{G}$-graded R-modules, 1.5

$\mathbf{Cone}(R_+)$ the cone of a non-unital ring, 1.6

$\mathbf{Proj}_{\mathcal{G}}$ the Proj of $\mathcal{G}$-graded rings, 1.7

$\mathbf{Sp}(\mathcal{F}_f/Y)$ the categoric spectrum of the monad $\mathcal{F}_f$ on a 'space' Y, 2.1

Aff_X the category of affine schemes over X, 2.3.1

$\mathfrak{Mon}_{\mathfrak{c}}(X)$ the category of continuous monads of a 'space' X, 2.3.1

$|\mathcal{F}|$ the monoid of elements of a monad $\mathcal{F}$, 2.3.1

$|\mathcal{F}|^*$ the group of invertible elements of the monoid $|\mathcal{F}|$, 2.3.1

$\mathfrak{Mon}_{\mathfrak{c}}^{\mathfrak{r}}(X)$ 2.3.1

$S_w = \{k^* e^{\lambda}_{w\lambda} | \lambda \in \mathcal{G}_+\}$ the Ore set corresponding to an element of the Weyl group, 5.2.1

$Gr_{_{M,V}}$ Grassmannien presheaf, 5.3.1

$Qcoh(X)$ the category of quasi-coherent presheaves on a presheaf of sets X, 5.3.5

$Qcoh(X,\tau)$ the category of quasi-coherent sheaves on a presheaf of sets (X,τ), 5.3.7

Chapter II

$\mathfrak{T}(X)$ the preorder of topologizing subcategories of C_X, 1.1

$\mathfrak{T}(X)$ the preorder of coreflective topologizing subcategories of C_X, 1.1

$\mathbb{S} \bullet \mathbb{T}$ the Gabriel product of subcategories, 1.1.1

$\mathbb{T}^{(n+1)}$ he n^{th} infinitesimal neighborhood of a subcategory $\mathbb{T}$, 1.1.1

$\succ$ a preorder among objects, 1.2

$[S]$ the smallest topologizing subcategory containing S, 1.2.1

$\mathcal{T}^-$ the Serre subcategory corresponding to a subcategory $\mathcal{T}$, 1.4

$\mathfrak{Se}(X)$ the preorder of Serre subcategories of C_X, 1.4.3

$\mathbb{T}^\perp$ the right orthogonal to the subcategory $\mathbb{T}$, 1.4.4

$\mathbf{Spec}(X)$ the spectrum of a 'space' X, 2

$Supp(M) = \{\mathcal{Q} \in \mathbf{Spec}(X) \mid \mathcal{Q} \subseteq [M]\}$ the support of M in $\mathbf{Spec}(X)$, 2.2

$\tau_{\mathfrak{z}}$ Zariski topology on $\mathbf{Spec}(X)$, 2.4

$\mathbf{Spec}(X)$ 3.2

$\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$ 3.3

$\mathcal{P}^{\mathfrak{t}}$ the intersection of topologizing subcategories properly containing $\mathcal{P}$, 3.3

$\mathcal{P}_{\mathfrak{t}} = \mathcal{P}^{\mathfrak{t}} \cap \mathcal{P}^\perp$ 3.3.1

$\widehat{\mathcal{Q}}$ the union of all topologizing subcategories, which do not contain $\mathcal{Q}$, 3.3.1
$\langle L\rangle = \widehat{[L]}$ 3.3.1
$S \vee T$ the minimal Serre subcategory of C_X containing S and T, 4.2.1
$\tau_{\mathfrak{L}_{\mathfrak{s}}}$ the pretopology of Serre localizations, 4.4
$\mathbf{Spec}^1(X)$ the complete spectrum of X, 5
$S \sqcup T$ the smallest thick subcategory of C_X containing S and T, 5
$\mathcal{T}^\infty$ the smallest thick subcategory containing $\mathcal{T}$, 5.1
$\tau^{\mathfrak{f}}_{\mathfrak{L}_e}$ pretopology of exact localizations, 5.2
$\mathbf{Spec}^1_{\mathfrak{T}\mathfrak{h}}(X)$ 6
$\mathbf{Spec}^1_{\mathfrak{S}\mathfrak{e}}(X)$ 6
$\mathbf{Spec}^0_{\mathfrak{t}}(X)$ the counterpart of $\mathbf{Spec}^1_{\mathfrak{t}}(X)$, 7.1
$\mathcal{O}_X - \mathcal{M}od$ the category of sheaves of $\mathcal{O}_{\mathcal{X}}$-modules on a ringed space $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, 8.2
$Qcoh_{\mathbf{X}}$ the category of quasi-coherent sheaves on $\mathbf{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, 8.2
$\mathbf{Spec}^1_{\mathfrak{c}}(X)$ 9.1
$\mathcal{P}^{\mathfrak{c}}$ the intersection of coreflective topologizing subcategories properly containing $\mathcal{P}$, 9.1
$\mathbf{Spec}^0_{\mathfrak{c}}(X)$ 9.1
${}^{\mathfrak{c}}\widehat{\mathcal{Q}}$ the union of all coreflective subcategories of C_X which do not contain $\mathcal{Q}$, 9.1
$[\mathcal{Q}]_{\mathfrak{c}}$ the smallest coreflective topologizing subcategory containing $\mathcal{Q}$, 9.1
$\mathcal{P}_* = \mathcal{P}^{\mathfrak{c}} \cap \mathcal{P}^\perp$ 9.1.2
$\mathbf{Spec}^i_{\mathfrak{c}}(X)_\star$ and $\mathbf{Spec}^i_{\mathfrak{S}\mathfrak{e}}(X)_\star$, $i = 0, 1$, extended spectra, 9.2.2
$U^1_{\mathfrak{c}}(\mathbb{T}) = \{\mathcal{P} \in \mathbf{Spec}^1_{\mathfrak{c}}(X) \mid \mathbb{T} \subseteq \mathcal{P}\}$ 9.3
$V^1_{\mathfrak{c}}(\mathbb{T}) = \mathbf{Spec}^1_{\mathfrak{c}}(X) - U^1_{\mathfrak{c}}(\mathbb{T}) = \{\mathcal{P} \in \mathbf{Spec}^1_{\mathfrak{c}}(X) \mid \mathbb{T} \nsubseteq \mathcal{P}\}$ 9.3
$U^0_{\mathfrak{c}}(\mathbb{T}) = \{\mathcal{Q} \in \mathbf{Spec}^0_{\mathfrak{c}}(X) \mid \mathcal{Q} \nsubseteq [\mathbb{T}]_{\mathfrak{c}}\}$ 9.3
$V^0_{\mathfrak{c}}(\mathbb{T}) = \mathbf{Spec}^0_{\mathfrak{c}}(X) - U^0_{\mathfrak{c}}(\mathbb{T}) = \{\mathcal{Q} \in \mathbf{Spec}^0_{\mathfrak{c}}(X) \mid \mathcal{Q} \subseteq [\mathbb{T}]_{\mathfrak{c}}\}$ 9.3
$\tau_{\mathfrak{c}}$, $\tau^{\mathfrak{c}}$ topologies on the spectrum $\mathbf{Spec}^0_{\mathfrak{c}}(X)$, 9.6.3
$\mathfrak{T}^{\mathfrak{c}}(X)$ the preorder of reflective topologizing subcategories of C_X, C1.2
$\mathfrak{T}_{\mathfrak{z}}(X) = \mathfrak{T}_{\mathfrak{c}}(X) \bigcap \mathfrak{T}^{\mathfrak{c}}(X)$ bireflective topologizing subcategories of C_X, C1.2.4.1
τ^* a topology on $\mathbf{Spec}(X)$, C1.7.1
$\tau_{\mathfrak{s}}$ the topology on $\mathbf{Spec}(X)$ generated by $\mathbf{Spec}(X)$, C1.7.2
$\mathbf{Spec}^{0,0}_{\mathfrak{t}}(X) = \mathbf{Spec}^0_{\mathfrak{t}}(X) - \mathbf{Spec}(X)$ C2.1.1
$U^1_{\mathfrak{t}}(\mathbb{T}) = \{\mathcal{P} \in \mathbf{Spec}^1_{\mathfrak{t}}(X) \mid \mathbb{T} \subseteq \mathcal{P}\}$ C2.2
$U^0_{\mathfrak{t}}(\mathbb{T}) = \{\mathcal{Q} \in \mathbf{Spec}^0_{\mathfrak{t}}(X) \mid \mathcal{Q} \nsubseteq \mathbb{T}\}$ C2.2
$Supp^1(M) = \{\mathcal{P} \in \mathbf{Spec}^1(X) \mid M \notin Ob\mathcal{P}\}$ the support of M in $\mathbf{Spec}^1(X)$, C3.2
$Supp^-(M) = Supp^1(M) \bigcap \mathbf{Spec}^-(X)$ the support of M in $\mathbf{Spec}^-(X)$, C3.2

Chapter III

$\widetilde{End}(C_X) = (End(C_X), \circ, Id_{C_X})$ the monoidal category of endofunctors of C_X, 1.1
$\mathfrak{M}\mathfrak{F}(\widetilde{\mathcal{E}}, \widetilde{\mathcal{E}}')$ the category of monoidal functors from $\widetilde{\mathcal{E}}$ to $\widetilde{\mathcal{E}}'$, 1.3

$\mathbf{Spec}_{\mathfrak{c}}^{\mathcal{P}}(\mathfrak{A}_{\mathcal{P}})$ and $\mathbf{Spec}_{\mathfrak{c}}^{\mathcal{P}}(\mathfrak{A})$ 2.1.1
$\mathfrak{Der}_{\mathfrak{c}}(C_X)$ the subcategory of continuous exact differential endofunctors, 3.3
$\mathfrak{Ass}(M)$ the set of associated points of M, C3.1
$\mathfrak{Ass}_{\mathfrak{t}}^{1,1}(M)$ C3.1
$\mathfrak{Ass}_{\mathfrak{t}}^{-}(M)$ C3.1

Chapter IV

$\mathcal{CT}_{\mathfrak{X}} = (C_{\mathfrak{X}}, \gamma; \mathfrak{Tr}_{\mathfrak{X}})$ triangulated category representing a t-'space' $\mathfrak{X}$, 1.1
$\mathfrak{TrCat}_k$ the category of triangulated k-linear categories, 1.1
$\mathcal{S}^{\mathfrak{s}}\mathcal{M}(\mathfrak{X}))$ the preorder of saturated multiplicative systems of the t-'space' $\mathfrak{X}$, 1.3
$\mathfrak{Tht}(\mathfrak{X})$ the preorder of thick triangulated subcategories of $\mathcal{CT}_{\mathfrak{X}}$, 1.6
$\mathfrak{Esp}_{\mathfrak{Tr}}$ the category of t-'spaces', 1.9
$\mathfrak{F}_{\mathbb{Z}}\mathfrak{Cat}_k$ the category of svelte Frobenius k-linear abelian $\mathbb{Z}$-categories, 2.1
$\mathcal{C}_{\mathfrak{X}_{\mathfrak{a}}}$ the abelianization of the triangulated category $\mathcal{TC}_{\mathfrak{X}}$, 2.2
$\mathcal{T}^{\mathfrak{a}}$ the smallest thick subcategory of $\mathcal{C}_{\mathfrak{X}_{\mathfrak{a}}}$ generated by the image of $\mathcal{T}$ in $\mathcal{C}_{\mathfrak{X}_{\mathfrak{a}}}$, 4.1
$\mathcal{P}^{\star}$ the intersection of all thick triangulated subcategories properly containing $\mathcal{P}$, 5.1
$\mathbf{Spec}_{\mathfrak{L}}^{1}(\mathfrak{X}) = \{\mathcal{P} \in \mathfrak{Tht}(\mathfrak{X}) \mid \mathcal{P} \neq \mathcal{P}^{\star}\}$ 5.1
$\mathbf{Spec}_{\mathfrak{L}}^{1,1}(\mathfrak{X}) = \mathbf{Spec}_{\mathfrak{L}}^{1}(\mathfrak{X}) - \mathbf{Spec}_{\mathfrak{L}}^{1,1}(\mathfrak{X})$ 5.1
$\mathbf{Spec}_{\mathfrak{L}}^{1,1}(\mathfrak{X}) = \{\mathcal{P} \in \mathfrak{Tht}(\mathfrak{X}) \mid \mathcal{P}_{\star} = \mathcal{P}^{\star} \cap \mathcal{P}^{\perp} \neq 0\}$ 5.1
$\mathcal{P}_{\star} = \mathcal{P}^{\perp} \cap \mathcal{P}^{\star}$ 5.2.2
$\mathbf{Spec}_{\mathfrak{L}}^{1/2}(\mathfrak{X})$ 5.3
$Supp_{\mathfrak{L}}^{1}(M)$ the support of M in $\mathbf{Spec}_{\mathfrak{L}}^{1}(\mathfrak{X})$, 5.5.1
$Supp_{\mathfrak{L}}^{1,1}(M) = Supp_{\mathfrak{L}}^{1}(M) \bigcap \mathbf{Spec}_{\mathfrak{L}}^{1,1}(\mathfrak{X})$ the support of M in $\mathbf{Spec}_{\mathfrak{L}}^{1,1}(\mathfrak{X})$, 5.5.1
$\mathcal{V}_{\mathfrak{L}}^{1}(E) = \bigcap_{M \in E} Supp_{\mathfrak{L}}^{1}(M)$ 5,5,2
$\mathcal{V}_{\mathfrak{L}}^{1,1}(E)$ the intersection $\bigcap_{M \in E} Supp_{\mathfrak{L}}^{1,1}(M)$, 5.5.2
$\tau_{\mathfrak{z}}$ the compact topology on $\mathbf{Spec}_{\mathfrak{L}}^{1}(\mathfrak{X})$, 5.5.3
$\mathfrak{X}_{\mathcal{Q},\mathfrak{f}_*}$ the stabilizer of the morphism $\mathfrak{f}$ at the point $\mathcal{Q}$ of the spectrum, 6.1.1

Chapter V

$\mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\ell}(X)$ the family of all saturated left multiplicative systems in C_X, 1.3.4
$\mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\mathfrak{r}}(X)$ the family of all saturated right multiplicative systems in C_X, 1.3.4
$\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X) = \mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\mathfrak{r}}(X) \cap \mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\ell}(X)$ 1.3.4
$\widehat{\Sigma}$ the union of all saturated multiplicative systems of C_X which do not contain Σ, 2
$\mathbf{Spec}_{\mathfrak{L}}^{0}(X) = \{\Sigma \in \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X) \mid \widehat{\Sigma} \in \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)\}$ 2.1
$\mathbf{Spec}_{\mathfrak{L}}^{1}(X)$ 4
$\mathfrak{C}\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$ the preorder of closed saturated multiplicative systems on X, 5.2, 5.2.3

$\mathbf{Spec}^1_{\mathfrak{c}}(X) = \mathfrak{C}\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X) \bigcap \mathbf{Spec}^1_{\mathfrak{L}}(X)$ the complete closed spectrum of X, 5.2.3
$\mathbf{Spec}^0_{\mathfrak{c}}(X) = \{\Sigma \in \mathbf{Spec}^0_{\mathfrak{L}}(X) \mid \widehat{\Sigma} \in \mathbf{Spec}^1_{\mathfrak{c}}(X)\}$ the closed spectrum of X, 5.2.3
$\mathbf{Spec}^1_{\mathfrak{f}\mathfrak{L}}(X)$ the complete flat $\mathfrak{L}$-spectrum of X, 5.3.5
$\mathbf{Spec}^0_{\mathfrak{f}\mathfrak{L}}(X) = \{\Sigma \in \mathbf{Spec}^0_{\mathfrak{L}}(X) \mid \widehat{\Sigma} \in \mathbf{Spec}^1_{\mathfrak{f}\mathfrak{L}}(X)\}$ 5.3.5

Chapter VI

$\mathcal{S}^{\bar{\wedge}}$ the class of all arrows having a pull-back in $\mathcal{S}$, 1.6
$(C_X, \bar{\mathfrak{E}}_X) = (C_X, \mathfrak{E}_X, \mathcal{W}_X)$ a right exact category with weak equivalences, 1.9.2
$\mathfrak{E}^{\circledast}_X \stackrel{\mathrm{def}}{=} Iso(C_X)^{\bar{\wedge}} \cap \mathfrak{E}_X$ deflations with trivial kernels, 1.9.3
$Coim(f)$ the coimage of a morphism f, 1.11.1
$\mathfrak{E}^{\mathfrak{c}}_X$ the class of deflations which are isomorphic to their coimage, 1.11.2.1
$\mathfrak{S}(X, \bar{\mathfrak{E}}_X)$ the preorder of systems, 2.1
$\mathfrak{S}^{\bar{\wedge}}(X, \bar{\mathfrak{E}}_X)$ the subpreorder of $\mathfrak{S}(X, \bar{\mathfrak{E}}_X)$ formed by *stable* systems of deflations, 2.1
$\mathfrak{T}_\ell(X, \bar{\mathfrak{E}}_X)$ the preorder of left toplogizing systems, 2.4.3
$\mathfrak{T}(X, \bar{\mathfrak{E}}_X)$ the preorder of toplogizing systems, 2.4.3
$\mathfrak{E}_{X,\mathcal{T}} \stackrel{\mathrm{def}}{=} \Sigma_{\mathcal{T}} \cap \mathfrak{E}_X = \{\mathfrak{s} \in \mathfrak{E}_X \mid Ker(\mathfrak{s}) \in Ob\mathcal{T}\}$ 2.5.1
$\Sigma_1 \vee \Sigma_2$ the smallest Serre system containing Σ_1 and Σ_2, 2.6.5
$\mathfrak{M}(X, \bar{\mathfrak{E}}_X)$ the preorder of all thick systems of $(C_X, \bar{\mathfrak{E}}_X)$, 3
$\mathfrak{M}_{\mathfrak{T}}(X, \bar{\mathfrak{E}}_X) = \mathfrak{M}(X, \bar{\mathfrak{E}}_X) \cap \mathfrak{T}(X, \bar{\mathfrak{E}}_X$, 3
$\mathfrak{Se}_{\mathfrak{T}}(X, \bar{\mathfrak{E}}_X) = \mathfrak{Se}(X, \bar{\mathfrak{E}}_X) \cap \mathfrak{T}(X, \bar{\mathfrak{E}}_X)$, 3
$Supp_{\mathfrak{T}}(\mathcal{S})$ the subpreorder of topologizing systems, which do not contain $\mathcal{S}$, 3.1
$\widehat{\mathcal{S}}$ the union of all systems of $Supp_{\mathfrak{T}}(\mathcal{S})$, 3.1
$\mathbf{Spec}_{\mathfrak{t}}(X, \bar{\mathfrak{E}}_X)$ formed by topologizing systems $\mathcal{S}$ such that $\widehat{\mathcal{S}} = \widehat{\mathcal{S}}^-$, 3.2
$\mathcal{S}^*$ the intersection of all topologizing systems properly containing $\mathcal{S}$, 3.3
$\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X, \bar{\mathfrak{E}}_X) = \{\Sigma \in \mathfrak{T}(X, \bar{\mathfrak{E}}_X) \mid \Sigma = \Sigma^- \subsetneq \Sigma^*\}$, 3.3
$\mathbf{Spec}^{1,0}_{\mathfrak{t}}(X, \bar{\mathfrak{E}}_X) = \{\Sigma \in \mathfrak{M}_{\mathfrak{T}}(X, \bar{\mathfrak{E}}_X) \mid \Sigma \neq \Sigma^* \subseteq \Sigma^-\}$, 3.3
$\mathbf{Spec}^{1,1}_{\mathfrak{M}}(X, \bar{\mathfrak{E}}_X)$ is formed by Serre systems Σ such that $\Sigma_\star \stackrel{\mathrm{def}}{=} \Sigma^\star \cap \Sigma^\perp$ is non-trivial, 3.7
$\mathfrak{M}_{\mathfrak{s}}(X, \bar{\mathfrak{E}}_X)$ the preorder of all strongly thick systems of $(C_X, \bar{\mathfrak{E}}_X)$, 4.2
$\mathfrak{R}^{\mathfrak{s}}_{\mathcal{S}} = \{$systems $\mathcal{T}$ divisible in $\mathfrak{E}_X \mid \mathcal{T} = \mathfrak{E}_X \cap (\mathcal{T} \circ \mathcal{W}^{\bar{\wedge}}_X),\ \mathcal{T} \cap \mathcal{S}^\perp = \mathcal{W}_X\}$, 4.3.1
$\mathcal{S}^\dagger$ the union of all $\mathcal{T} \in \mathfrak{R}^{\mathfrak{s}}_{\mathcal{S}}$, 4.3.1
$\mathcal{S}^{\mathfrak{st}}$ the intersection of all semitopologizing systems properly containing $\mathcal{S}$, 4.4
$\mathcal{S}^{\mathfrak{sc}}$ the intersection of all *strongly stable* thick systems properly containing $\mathcal{S}$, 4.4
$\Sigma_{\mathfrak{sc}} \stackrel{\mathrm{def}}{=} \Sigma^{\mathfrak{sc}} \cap \Sigma^\perp$, 4.4
$\mathbf{Spec}^{1,1}_{\mathfrak{sc}}(X, \bar{\mathfrak{E}}_X) = \{$strongly closed systems of deflations $\Sigma \mid \Sigma_{\mathfrak{sc}}$ is non-trivial$\}$, 4,4
$\mathbf{Spec}^{1,1}_{\mathfrak{st}}(X, \bar{\mathfrak{E}}_X)$ is formed by strongly closed systems Σ for which $\Sigma^{\mathfrak{st}} \neq \Sigma$, 4.4

Chapter VII

$\mathfrak{Spec}^1(\mathfrak{H})$ the local spectrum of $\mathfrak{H}$, 1.2
$\mathcal{M}ax(\mathfrak{H})$ the full subcategory of $\mathfrak{H}$ generated by maximal proper objects, 1.2.2
$\mathfrak{Supp}_{\mathfrak{H}}(x)$ the support of x in $\mathfrak{H}$, 1.3
$\mathfrak{Spec}^0(\mathfrak{H})$ is generated by $x \in Ob\mathfrak{H}$ such that $\mathfrak{Supp}_{\mathfrak{H}}(x)$ has a final object, 1.4
$\mathfrak{Supp}^0_{\mathfrak{H}}(x)$ the support of x in $\mathfrak{Spec}^0(\mathfrak{H})$, 1.4.8
$\mathcal{U}_{\mathfrak{H}}(x)$ the full subcategory of $\mathfrak{H}$ generated by $y \in Ob\mathfrak{H}$ such that $\mathfrak{H}(y,x) = \emptyset$, 1.5.2, 1.5.3
$\mathfrak{Spec}^1(\mathfrak{G}, F)$ and $\mathfrak{Spec}^0(\mathfrak{G}, F)$ the spectra corresponding to a functor $\mathfrak{G} \xrightarrow{F} \mathfrak{H}$, 1.6
$\mathfrak{Supp}^{\mathfrak{s}}_{\mathfrak{H}}(x)$ the strict support of x, A1.2
$\mathfrak{Sup}^F(x)$ the relative strict support of x, A2.2
$\mathfrak{Spec}^{\vee}(\mathfrak{H})$ the full subcategory of $\mathfrak{Spec}(\mathfrak{H})$ is generated by all x such that $x \cap \widehat{x} = \mathfrak{x}$, A3.1
$\mathfrak{Spec}(\mathfrak{H})$ the spectrum of $\mathfrak{H}$, A3.1
$\mathfrak{Spec}_{\mathfrak{s}}(\mathfrak{H})$ the strict spectrum of $\mathfrak{H}$, A3.2
$\mathfrak{Spec}(\mathfrak{G}, F)$ and $\mathfrak{Spec}_{\mathfrak{s}}(\mathfrak{G}, F)$ relative spectra, A3.4
$Ass^1_{(\mathfrak{G},F)}(x)$ weakly associated points, C.2
$\mathfrak{Ass}^1_{(\mathfrak{G},F)}(x)$ associated points, C.3
$\mathbf{Spec}^i_{\mathfrak{L},\ell}(X) = \mathfrak{Spec}^i(\mathcal{S}^{\mathfrak{s}}\mathcal{M}_\ell(X)),\ i = 0, 2,$ left spectra, 3.2

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