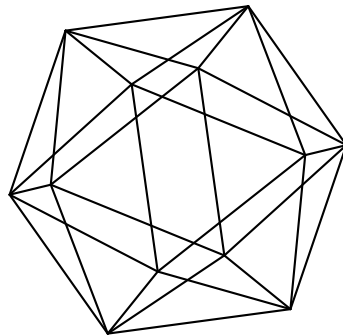


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Pullbacks of universal Brill-Noether classes via Abel-Jacobi morphisms

by

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PULLBACKS OF UNIVERSAL BRILL–NOETHER CLASSES VIA ABEL–JACOBI MORPHISMS

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ABSTRACT. Following Mumford and Chiodo, we compute the Chern character of the derived pushforward $\mathrm{ch}(R^\bullet \pi_* \mathcal{O}(D))$, for D an arbitrary element of the Picard group of the universal curve over the moduli stack of stable marked curves. This allows us to express the pullback of universal Brill–Noether classes via Abel–Jacobi sections to the compactified universal Jacobians, for all compactifications such that the section is a well-defined morphism.

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0. INTRODUCTION

Let $\pi: \overline{\mathcal{C}}_{g,P} \rightarrow \overline{\mathcal{M}}_{g,P}$ be the universal curve over the moduli stack of stable marked curves, where P is a nonempty ordered set of markings. The weak Franchetta conjecture, now a theorem due to Harer [7] and Arbarello–Cornalba [1], gives an explicit description of the Picard group of the universal curve. Every divisor on $\overline{\mathcal{C}}_{g,P}$, up to a divisor pulled back from $\overline{\mathcal{M}}_{g,P}$, is rationally equivalent to

$$(0.1) \quad D = \ell \tilde{K}_\pi + \sum_{p \in P} d_p \sigma_p + \sum_{h,S} a_{h,S} C_{h,S}$$

for some integers ℓ , d_p and $a_{h,S}$. Here $\tilde{K}_\pi = c_1(\omega_\pi)$ is the first Chern class of the relative dualising sheaf, σ_p is the class of the p -th section, and $C_{h,S}$ is the class of the irreducible component, *not* containing the moving point, in the inverse image of the boundary divisor $\Delta_{h,S}$ in the universal curve (more details in Section 1).

Our main result is an explicit formula for

$$\mathrm{ch}(R^\bullet \pi_* \mathcal{O}(D)),$$

in terms of the standard generators of the tautological ring (boundary strata classes decorated with κ classes and ψ classes). In Section 2 we prove:

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Theorem 1. *If D is as in (0.1), then after setting $e_p = d_p - \ell$, one has*

$$\begin{aligned} \text{ch}(R^\bullet \pi_* \mathcal{O}(D)) &= \sum_{\substack{t>0 \\ 0 \leq i \leq t}} \frac{B_{t-i}}{i!(t-i)!} \left(\ell^i \kappa_{t-1} + (-1)^{i-1} \sum_p e_p^i \psi_p^{t-1} \right) \\ &\quad + \sum_{t>0} \frac{B_{2t}}{(2t)!} \left(\frac{1}{2} \xi_{\text{irr}^*} \sum_{i=0}^{2t-2} (-\psi_q)^i \psi_r^{2t-2-i} + \sum_{h,S} \xi_{h,S^*} \sum_{i=0}^{2t-2} (-\psi_q)^i \otimes \psi_r^{2t-2-i} \right) \\ &\quad + \sum_{\substack{t>0 \\ 0 \leq i \leq t}} \frac{B_{t-i}}{i!(t-i)!} \sum_{h,S} \left(\sum_{\substack{0 \leq a < i \\ 0 \leq b < i-a}} (-1)^{i-a-1} \binom{i}{a} \binom{i-a-1}{b} \ell^a a_{h,S}^{i-a} \cdot \xi_{h,S^*} (\psi_q^b \otimes \kappa_{a+t-i-1} \psi_r^{i-a-1-b}) \right. \\ &\quad \left. + \sum_{\substack{p \leq S \\ 0 \leq b < i}} (-1)^{b-1} \binom{i}{b} e_p^b a_{h,S}^{i-b} \cdot \sum_{\Gamma \in \mathbb{G}(h,S)} \frac{(i-b)!}{|\text{Aut} \Gamma|} \cdot \left[\xi_{\Gamma^*} \left(\psi_p^{b+t-i-1} \cdot \prod_{(l,l') \in \mathbb{E}(\Gamma)} \frac{1 - e^{-\psi_l - \psi_{l'}}}{\psi_l + \psi_{l'}} \right) \right]_{t-1} \right). \end{aligned}$$

Here the coefficients B_k are the Bernoulli numbers defined by $\frac{x}{e^x-1} = \sum_{k \geq 0} \frac{B_k}{k!} x^k$.

The notation for tautological classes and for gluing maps (which we labelled ξ) is explained in Section 1. In the formula, we have adopted the conventions $\kappa_{-1} \cdot \psi^s = \psi^{s-1}$ for all $s \geq 0$, and $\psi^{-1} = 0$. Moreover, $\mathbb{G}(h, S) \subseteq \mathbb{G}_{g,p}$ is the subset consisting of stable graphs all of whose edges are separating and of type (h, S) . This is explained in Section 1.1.

Our formula expresses $\text{ch}_t(R^\bullet \pi_* \mathcal{O}(D))$ as a polynomial of degree $t+1$ in the unknowns ℓ , d_p , $a_{h,S}$ with coefficients in the tautological ring of $\mathcal{M}_{g,p}$. The special case where all $a_{h,S} = 0$ can be extracted from Chiodo's formula [3, Theorem 1.1.1]. Our formula shows that varying the coefficient $a_{h,S}$ modifies the result by a class supported on the boundary divisor $\delta_{h,S}$, unless $S = P$ and $h \geq g/2$, in which case the result is modified by a class supported deeper in the boundary, on strata of type $\mathbb{G}(h, P)$, see Section 1.1.

We prove Theorem 1 by applying the Grothendieck–Riemann–Roch formula to the universal curve π , as in Mumford's seminal calculation of the Chern classes of the Hodge bundle [12, Section 4].

Our main motivation is computing the pullback of (extended, cohomological) Brill–Noether classes w_d^r on the universal Jacobian via the Abel–Jacobi sections. Here we give a preview, full details are in Section 3.

Fix $0 \leq d \leq g-1$ and let $\mathcal{J}_{g,p}^d \rightarrow \mathcal{M}_{g,p}$ be the universal Jacobian parametrising line bundles of degree d over smooth P -pointed curves of genus g . Let \mathcal{L} denote the universal (or Poincaré) line bundle on the universal curve

$$p: \mathcal{J}_{g,p}^d \times_{\mathcal{M}_{g,p}} \mathcal{C}_{g,p} \rightarrow \mathcal{J}_{g,p}^d.$$

For $0 \leq r \leq d/2$, the universal Brill–Noether locus \mathcal{W}_d^r is set-theoretically defined by

$$\mathcal{W}_d^r := \{ (C, P, L) \mid L \in \text{Pic}^d C, h^0(C, L) > r \} \subset \mathcal{J}_{g,p}^d,$$

and can be endowed with the scheme structure of the $(g-d+r)$ -th Fitting ideal of $R^1 p_* \mathcal{L}$. Each \mathcal{W}_d^r is in general not equidimensional, and the dimension of its irreducible components is unknown. However a (cohomological) Brill–Noether cycle w_d^r supported on \mathcal{W}_d^r and of the correct dimension can be defined, via the Thom–Porteous formula, as the $(r+1) \times (r+1)$ determinant

$$(0.2) \quad w_d^r = \Delta_{g-d+r}^{(r+1)} c(-R^\bullet p_* \mathcal{L}) \in A^\bullet(\mathcal{J}_{g,p}^d).$$

The notation $\Delta_q^{(p)} c$ stands for the $p \times p$ determinant $|c_{q+j-i}|$, for $1 \leq i, j \leq p$ and a general series $c = \sum_k c_k$ (see Section 3.2 for more details).

The discussion of the previous paragraph extends verbatim over $\overline{\mathcal{M}}_{g,P}$. One constructs a compactified universal Jacobian

$$(0.3) \quad \overline{\mathcal{J}}_{g,P}(\phi) \rightarrow \overline{\mathcal{M}}_{g,P}$$

for all nondegenerate polarisations ϕ , and classes $w_d^r(\phi)$ also defined by Formula (0.2), *mutatis mutandis*. The compactified universal Jacobian (0.3) extends $\mathcal{J}_{g,P}^d \rightarrow \mathcal{M}_{g,P}$ and consists of torsion free sheaves of rank 1 on stable curves, whose multidegree is prescribed by ϕ . The rational sections of (0.3) are called *Abel–Jacobi sections*. By Franchetta’s conjecture, they are all of the form

$$(0.4) \quad s: (C, P) \rightarrow \omega_\pi^\ell \left(\sum_{p \in P} d_p \sigma_p + \sum_{h,S} a_{h,S} C_{h,S} \right),$$

for some integers ℓ , d_p and $a_{h,S}$. A natural question that has attracted lots of attention is computing the pullback of $w_d^r(\phi)$ via s . The result for $r = d = 0$ and certain ϕ ’s is known under the name of *double ramification cycle*, see [8].

This problem is complicated by the fact that s is, in general, only a rational map. Theorem 1 allows one to compute $s^*w_d^r(\phi)$ for every ϕ such that s is a morphism (these ϕ ’s are characterised in [11, Section 6.1]). Indeed, for every such ϕ , in Section 3 we will prove the equality

$$(0.5) \quad s^*w_d^r(\phi) = \Delta_{g-d+r}^{(r+1)} c(-R^\bullet \pi_* \mathcal{O}(D(\phi))),$$

where $D(\phi)$ is a modification of the divisor D of (0.1) obtained by replacing the coefficients $a_{h,S}$ with the unique coefficients $a_{h,S}(\phi)$ such that $D(\phi)$ is ϕ -stable on all curves with 1 node. Combining (0.5) with Theorem 1 and with the inversion Formula (3.9) for the Chern character, we obtain an explicit expression, for all ϕ such that s is a morphism, for the cohomology class $s^*w_d^r(\phi)$ in terms of the standard generators of the tautological ring.

Throughout we work over the field of complex numbers \mathbb{C} .

1. TAUTOLOGICAL CLASSES

Throughout we fix an integer $g \geq 1$ and an ordered set of markings $P \neq \emptyset$. We follow the exposition and the notation of [2, Section 17.4] to introduce the *tautological ring* of the moduli space $\overline{\mathcal{M}}_{g,P}$ of stable P -pointed curves of genus g .

We let

$$\pi: \overline{\mathcal{C}}_{g,P} \rightarrow \overline{\mathcal{M}}_{g,P}$$

be the universal stable P -pointed curve. For each marking $p \in P$, we let

$$\sigma_p \in A^1(\overline{\mathcal{C}}_{g,P})$$

denote the divisor class corresponding to the p -th section of π . Let ω_π be the relative dualising sheaf, and set

$$K_\pi = c_1 \left(\omega_\pi \left(\sum_p \sigma_p \right) \right), \quad \tilde{K}_\pi = c_1(\omega_\pi) = K_\pi - \sum_p \sigma_p.$$

We will often simply write K instead of K_π . We define the cotangent line classes, or *psi classes*, by

$$\psi_p := \sigma_p^* \tilde{K}_\pi \in A^1(\overline{\mathcal{M}}_{g,P}).$$

For $a \geq 0$, the *kappa classes*

$$\kappa_a := \pi_* K_\pi^{a+1}, \quad \tilde{\kappa}_a := \pi_* \tilde{K}_\pi^{a+1}$$

have codimension a and are related by the formula [2, Formula 3.4, p. 572]

$$(1.1) \quad \tilde{\kappa}_a = \kappa_a - \sum_{p \in P} \psi_p^a.$$

The *tautological ring* of the moduli space of stable marked curves

$$R^\bullet(\overline{\mathcal{M}}_{g,P}) \subset A_\mathbb{Q}^\bullet(\overline{\mathcal{M}}_{g,P})$$

was originally defined by Mumford in [12, Section 4] in the unmarked case $P = \emptyset$, and an elegant definition for all moduli spaces of stable marked curves at once was later given by C. Faber and R. Pandharipande [4]. We will give here an alternative definition to suit our purposes.

First we recall the notion of decorated boundary stratum class. For $\Gamma = (V(\Gamma), E(\Gamma))$ in the set $\mathcal{G}_{g,P}$ of (isomorphism classes of) stable P -pointed graphs of genus g , we let

$$\overline{\mathcal{M}}_\Gamma = \prod_{v \in V(\Gamma)} \overline{\mathcal{M}}_{g_v, P_v}$$

and denote by $\xi_\Gamma: \overline{\mathcal{M}}_\Gamma \rightarrow \overline{\mathcal{M}}_{g,P}$ the associated clutching morphism. Here, P_v is the set of edges and legs (half-edges) issuing from the vertex v , and the stability condition $2g_v - 2 + |P_v| > 0$ is fulfilled by all v . A “decoration” θ on the graph Γ is the datum of a monomial

$$\theta_v = \prod_{p \in P(v)} \psi_p^{a_p} \prod_j \kappa_j^{b_j} \in A^\bullet(\overline{\mathcal{M}}_{g_v, P_v})$$

for each vertex $v \in V(\Gamma)$. Classes of the form

$$\frac{1}{|\text{Aut}\Gamma|} \xi_{\Gamma*} \left(\prod_{v \in V(\Gamma)} \theta_v \right) \in A^\bullet(\overline{\mathcal{M}}_{g,P}),$$

for Γ and θ as above, are called *decorated boundary strata classes*. (Here and in the following, we omit writing the pullback via the projection map to the factor, and we omit writing the tensor product of classes, unless that helps identifying which factor they are pulled back from). We define $R^\bullet(\overline{\mathcal{M}}_{g,P})$ to be the vector subspace of $A_\mathbb{Q}^\bullet(\overline{\mathcal{M}}_{g,P})$ generated by these classes and then endow it with the intersection product. When θ_v is trivial for all v , we simply write $\delta_\Gamma := \xi_{\Gamma*}(\mathbb{1})/|\text{Aut}\Gamma|$.

The collection of decorated boundary strata classes can be made into a finite set (for fixed g and P) by only considering decorations θ that are not obviously vanishing for codimensional reasons. Even so, this collection is far from being a basis. All known relations among these generators belong to a vector space generated by the so-called *Pixton’s relations*, but whether or not these are all the existing relations is so far unknown.

In this paper, “calculating” an element of $R^\bullet(\overline{\mathcal{M}}_{g,P})$ will always mean expressing it as an explicit, non-unique, linear combination of decorated boundary strata classes.

Example 1.1. We define the set of *stable bipartitions* of (g, P) to be the collection of pairs (h, S) where $S \subset P$ contains a distinguished “first” marking and $0 \leq h \leq g$ is such that if $h = 0$ then $|S| \geq 2$ and if $h = g$ then $|S^c| \geq 2$ (where $S^c = P \setminus S$ denotes the complement). With this convention, there is a bijection between the set of stable bipartitions and the set of stable graphs $\Gamma_{h,S} \in \mathcal{G}_{g,P}$ with two vertices and one edge. The corresponding (codimension one) clutching morphism is denoted

$$\xi_{h,S}: \overline{\mathcal{M}}_{h,S \cup q} \times \overline{\mathcal{M}}_{g-h,S^c \cup r} \rightarrow \overline{\mathcal{M}}_{g,P}.$$

Its image is the boundary divisor $\Delta_{h,S}$ and its class $\delta_{\Gamma_{h,S}}$ will simply be denoted by $\delta_{h,S}$.

Example 1.2. There is one more boundary divisor of $\overline{\mathcal{M}}_{g,P}$, which parametrises irreducible singular curves. That divisor is the image of the clutching morphism ξ_{irr} attached to the stable graph Γ_{irr} consisting of one vertex of genus $g - 1$ with a loop, and carrying all markings P .

Now for a fixed stable bipartition (h, S) of (g, P) , the inverse image $\pi^{-1}(\Delta_{h,S})$ in $\overline{\mathcal{C}}_{g,P}$ consists of two irreducible components. We will denote by $C_{h,S}^+$ the class of the component that

contains the moving point on the universal curve, and by $C_{h,S}$ the class of the other component. We then have the obvious relation

$$\pi^*(\delta_{h,S}) = C_{h,S}^+ + C_{h,S} \in A^1(\overline{\mathcal{C}}_{g,P}).$$

1.1. Self-intersection of boundary divisors. For later use, we provide here an explicit expression for the intersections

$$\psi_p^a \cdot \delta_{h,S}^b \in R^{a+b}(\overline{\mathcal{M}}_{g,P})$$

in terms of decorated boundary strata classes. An edge of a stable graph $\Gamma \in \mathcal{G}_{g,P}$ is said to be *of type* (h, S) if it disconnects the graph into a component of genus h with markings S and another component of genus $g - h$ and markings S^c .

Let

$$\mathcal{G}(h, S) \subseteq \mathcal{G}_{g,P}$$

be the set of stable graphs all of whose edges are of type (h, S) . A direct calculation using the general formula to compute the intersection product of two decorated boundary strata classes (see [6, Appendix]) yields

$$(1.2) \quad \delta_{h,S}^b = \sum_{\Gamma \in \mathcal{G}(h,S)} \frac{b!}{|\text{Aut}\Gamma|} \left[\xi_{\Gamma^*} \left(\prod_{(l,l') \in \mathbb{E}(\Gamma)} \frac{1 - e^{-\psi_l - \psi_{l'}}}{\psi_l + \psi_{l'}} \right) \right]_b.$$

The notation $[\alpha]_c$ means taking the part of codimension c of a class α .

The self-intersection $\delta_{h,S}^b$ is easier to compute when $S \neq P$ or when $h < g/2$. Indeed, in this case the set $\mathcal{G}(h, S)$ contains only the graph $\Gamma_{h,S}$, so that the expansion

$$(1.3) \quad \frac{1 - e^{-x}}{x} = \sum_{i=0}^{\infty} \frac{(-x)^i}{(i+1)!}$$

yields the simple expression

$$\delta_{h,S}^b = \xi_{h,S^*} (-\psi_q - \psi_r)^{b-1}.$$

On the other hand, when $S = P$ and $h \geq g/2$, the graphs with a nonzero contribution in (1.2) may correspond to boundary strata of codimension bigger than one. They can be characterised as follows. Define, for all $t \geq 0$, the number $g_t = (t+1)h - tg$, and set

$$m_{g,h} = \min\{t \mid g_t < 0\}, \quad s = \min\{b, m_{g,h}\}.$$

Note that $g_{t-1} + t(g-h) = g$ for all $t > 0$. Consider, for $t > 0$, the *tree* Γ_t with a vertex v_t of genus g_{t-1} carrying all the markings P , and with t additional vertices, all of genus $g-h$, each connected to v_t by exactly one edge (see Figure 1). When $h \geq g/2$, we have $\mathcal{G}(h, P) = \{\Gamma_1, \dots, \Gamma_{m_{g,h}}\}$, where $\Gamma_1 = \Gamma_{h,P}$.

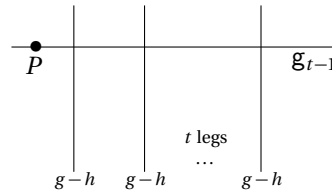


FIGURE 1. Curves of type Γ_t . Note that $|\text{Aut}\Gamma_t| = t!$.

The clutching morphism attached to Γ_t is

$$\xi_t: \overline{\mathcal{M}}_{g_{t-1}, P \cup \{q_1, \dots, q_t\}} \times \prod_{i=1}^t \overline{\mathcal{M}}_{g-h, r_i} \rightarrow \overline{\mathcal{M}}_{g,P}.$$

The product corresponding to $\Gamma = \Gamma_t$ in right hand side of (1.2) now consists of t factors. Expanding them via (1.3) again, one finds the explicit formula

$$\delta_{h,P}^b = \sum_{t=1}^s \frac{1}{t!} \sum_{\substack{\alpha_1 + \dots + \alpha_t = b \\ \alpha_j \geq 1}} \binom{b}{\alpha_1, \dots, \alpha_t} \cdot \xi_{t*} \left(\prod_{k=1}^t (-\psi_{q_k} - \psi_{r_k})^{\alpha_k - 1} \right).$$

Since ψ classes pull back along the clutching morphisms ξ_Γ , the projection formula yields, for each $a \geq 0$ and marking $p \in P$, the equality

$$(1.4) \quad \psi_p^a \cdot \delta_{h,S}^b = \sum_{\Gamma \in \mathcal{G}(h,S)} \frac{b!}{|\text{Aut}\Gamma|} \left[\xi_{\Gamma*} \left(\psi_p^a \cdot \prod_{(l,l') \in \mathbb{E}(\Gamma)} \frac{1 - e^{-\psi_l - \psi_{l'}}}{\psi_l + \psi_{l'}} \right) \right]_{a+b}.$$

2. PROOF OF MAIN THEOREM

This section provides a proof of our main result, Theorem 1, using the notation established in Section 1. We prove the theorem by following Mumford (and later Chiodo), namely by applying the Grothendieck–Riemann–Roch formula to the universal curve π . There are, in principle, different ways to approach the calculation. The main point is to compute the pushforward along π of products of divisors, and the main difficulty is to devise the computation so that all pushforwards that one actually has to deal with satisfy closed formulas with explicit coefficients in terms of decorated boundary strata classes (see (2.6) and (2.8)).

Consider the divisor class

$$D = \ell \tilde{K}_\pi + \sum_{p \in P} d_p \sigma_p + \sum_{h,S} a_{h,S} C_{h,S},$$

on the universal curve $\bar{\mathcal{C}}_{g,P}$, where the indices (h, S) run over the set of stable bipartitions of (g, P) and $\ell, d_p, a_{h,S} \in \mathbb{Z}$. Let

$$\Sigma \subset \bar{\mathcal{C}}_{g,P}$$

be the (smooth) closed codimension two substack parametrising the nodes in the fibers of the universal curve. Running the Grothendieck–Riemann–Roch formula we find

$$\text{ch}(R^\bullet \pi_* \mathcal{O}(D)) = \pi_*(\text{ch}(\mathcal{O}(D)) \cdot \text{Td}^\vee(\Omega_\pi^1)) = \pi_* \left(e^D \cdot \frac{\tilde{K}_\pi}{e^{\tilde{K}_\pi - 1}} \cdot \text{Td}^\vee(\mathcal{O}_\Sigma)^{-1} \right).$$

A classical argument first given by Mumford and described in [2, Chapter 17.5] shows that $\text{Td}^\vee(\mathcal{O}_\Sigma)^{-1} - 1$ intersects \tilde{K}_π trivially. Therefore

$$\text{ch}(R^\bullet \pi_* \mathcal{O}(D)) = \pi_* \left(e^D \cdot \frac{\tilde{K}_\pi}{e^{\tilde{K}_\pi - 1}} + \text{Td}^\vee(\mathcal{O}_\Sigma)^{-1} - 1 \right) = \pi_* \left(e^D \cdot \frac{\tilde{K}_\pi}{e^{\tilde{K}_\pi - 1}} \right) + \Psi,$$

where Ψ is the pushforward of $\text{Td}^\vee(\mathcal{O}_\Sigma)^{-1} - 1$ by π . The latter is also explicitly computed in loc. cit. as

$$(2.1) \quad \Psi = \sum_{t>0} \frac{B_{2t}}{(2t)!} \left(\frac{1}{2} \xi_{\text{irr}*} \sum_{i=0}^{2t-2} (-\psi_q)^i \psi_r^{2t-2-i} + \sum_{h,S} \xi_{h,S*} \sum_{i=0}^{2t-2} (-\psi_q)^i \otimes \psi_r^{2t-2-i} \right).$$

We now need a formula for

$$(2.2) \quad \text{ch}(R^\bullet \pi_* \mathcal{O}(D)) - \Psi = \pi_* \left(e^D \cdot \frac{\tilde{K}_\pi}{e^{\tilde{K}_\pi - 1}} \right) = \sum_{\substack{t>0 \\ 0 \leq i \leq t}} \frac{B_{t-i}}{i!(t-i)!} \pi_* (D^i \cdot \tilde{K}_\pi^{t-i}).$$

For fixed $t > 0$, its term of codimension $t - 1$ can be written as

$$(2.3) \quad \frac{B_t}{t!} \tilde{K}_{t-1} + \frac{1}{t!} \pi_* D^t + \sum_{0 < i < t} \frac{B_{t-i}}{i!(t-i)!} \pi_* (D^i \cdot \tilde{K}_\pi^{t-i}).$$

From now on our goal will be to work on each of the summands of (2.3) in order to express (2.2) in terms of decorated boundary strata classes. The first summand of (2.3) is already one of those generators.

We start out by setting $e_p = d_p - \ell$ and rewriting

$$D = \ell K + \sum_{p \in P} e_p \sigma_p + \sum_{h,S} a_{h,S} C_{h,S}$$

in terms of K instead of \tilde{K} . If $0 \leq i < t$ we can write

$$(2.4) \quad \tilde{K}^{t-i} = \left(K - \sum_p \sigma_p \right)^{t-i} = K^{t-i} + (-1)^{t-i} \sum_p \sigma_p^{t-i}.$$

Using the vanishing relations

- (1) $\sigma_p \cdot \sigma_q = 0$ for all $p \neq q$,
- (2) $K \cdot \sigma_p = 0$ for all $p \in P$,
- (3) $C_{h,S} \cdot C_{l,T} = 0$ for all $(h,S) \neq (l,T)$,
- (4) $\sigma_p \cdot C_{h,S} = 0$ for all $p \notin S$,

we obtain the formula

$$(2.5) \quad D^i \cdot \tilde{K}^{t-i} = \ell^i K^t + (-1)^{t-i} \sum_{p \in P} e_p^i \sigma_p^t + \sum_{h,S} \sum_{0 \leq a < i} \binom{i}{a} \ell^a a_{h,S}^{i-a} (K^{a+t-i} \cdot C_{h,S}^{i-a}) \\ + (-1)^{t-i} \sum_{\substack{h,S \\ p \in S}} \sum_{0 \leq b < i} \binom{i}{b} e_p^b a_{h,S}^{i-b} (\sigma_p^{b+t-i} \cdot C_{h,S}^{i-b}).$$

Note that when $i = 0$, the formula reduces to (2.4), and its pushforward via π reduces to (1.1). Substituting $i = t$ in the right hand side of (2.5) does *not* compute D^t (because (2.4) does not hold for $i = t$). Nevertheless, D^t satisfies the same formula, where the sum over $0 \leq b < i$ is replaced by a sum over $0 < b < t$. However, we are interested in the pushforward of this expression via π . So, for fixed $p \in S$, consider the identity

$$(2.6) \quad \pi_* (\sigma_p^c \cdot C_{h,S}^d) = (-\psi_p)^{c-1} \cdot \delta_{h,S}^d,$$

valid for all $d \geq 0$ and $c > 0$. If one sets $i = t$ and $b = 0$ in the right hand side of (2.5), the identity (2.6) would formally express the pushforward of $\sigma_p^0 \cdot C_{h,S}^t$ as $-\psi_p^{-1} \cdot \delta_{h,S}^t$. From now on we then adopt the convention

$$\psi_p^{-1} = 0,$$

for every $p \in P$. This allows us to view the pushforward $\pi_* D^t$ as a special case of the pushforward of (2.5), by substituting $i = t$.

Using (1.1) again along with (2.6) (and the convention $\psi^{-1} = 0$), we see that for fixed $t > 0$ and $i = 0, \dots, t$ the term from the right hand side of (2.2) that we need to compute is equal to $B_{t-i}/i!(t-i)!$ times the sum

$$(2.7) \quad \ell^i \kappa_{t-1} + (-1)^{i-1} \sum_{p \in P} e_p^i \psi_p^{t-1} + \sum_{h,S} \sum_{0 \leq a < i} \binom{i}{a} \ell^a a_{h,S}^{i-a} \pi_* (K^{a+t-i} \cdot C_{h,S}^{i-a}) \\ + \sum_{\substack{h,S \\ p \in S}} \sum_{0 \leq b < i} (-1)^{b-1} \binom{i}{b} e_p^b a_{h,S}^{i-b} (\psi_p^{b+t-i-1} \cdot \delta_{h,S}^{i-b}).$$

The powers $\psi^\alpha \cdot \delta^\beta$ are expressed in terms of boundary strata classes via (1.4). The pushforwards

$$\pi_* (K^\alpha \cdot C_{h,S}^\beta),$$

on the other hand, are taken care of by the following calculation.

Lemma 2.1. *Fix a stable bipartition (h, S) and integers $\alpha, \beta \geq 0$. Then*

$$(2.8) \quad \pi_* \left(K^\alpha \cdot C_{h,S}^\beta \right) = \begin{cases} \kappa_{\alpha-1} & \text{if } \beta = 0, \alpha > 0; \\ (-1)^{\beta-1} \sum_{b=0}^{\beta-2} \binom{\beta-1}{b} \xi_{h,S*} \left(\psi_q^b \otimes \psi_r^{\beta-2-b} \right) & \text{if } \beta > 0, \alpha = 0; \\ (-1)^{\beta-1} \sum_{b=0}^{\beta-1} \binom{\beta-1}{b} \xi_{h,S*} \left(\psi_q^b \otimes \kappa_{\alpha-1} \psi_r^{\beta-1-b} \right) & \text{if } \alpha, \beta > 0. \end{cases}$$

Proof. The case $\beta = 0$ gives the definition of the κ class (or 0, if $\alpha = 0$), so from now on we assume $\beta > 0$. We identify the universal curve $\bar{C}_{g,P}$ with the moduli stack $\bar{\mathcal{M}}_{g,P \cup x}$, mapping down to $\bar{\mathcal{M}}_{g,P}$ by forgetting the last marking x , so the class $C_{h,S}$ is identified with the class of the boundary divisor $\delta_{h,S} \in A^1(\bar{\mathcal{M}}_{g,P \cup x})$. We will use the commutative diagram

$$\begin{array}{ccccc} \bar{\mathcal{M}}_{g-h, S^c \cup \{x, r\}} & \xleftarrow{\eta_2} & \bar{\mathcal{M}}_{h, S \cup q} \times \bar{\mathcal{M}}_{g-h, S^c \cup \{x, r\}} & \xrightarrow{\iota} & \bar{\mathcal{M}}_{g, P \cup x} \\ \downarrow \tau & & \square & & \downarrow \bar{\tau} \\ \bar{\mathcal{M}}_{g-h, S^c \cup r} & \xleftarrow{\quad} & \bar{\mathcal{M}}_{h, S \cup q} \times \bar{\mathcal{M}}_{g-h, S^c \cup r} & \xrightarrow{\xi_{h,S}} & \bar{\mathcal{M}}_{g, P} \end{array}$$

where η_2 is the second projection, the clutching morphisms glue the markings q and r together, and $\bar{\tau}$ is shorthand for $\text{id} \times \tau$. By [2, p. 582], one has

$$(2.9) \quad \iota^* K_\pi = \eta_2^* K_\tau.$$

Furthermore, by [2, Lemma 4.36, p. 583] applied to $\bar{\mathcal{M}}_{g, P \cup x}$, and taking advantage of the convention $1 \in S$, one has

$$(2.10) \quad \iota^* C_{h,S} = \iota^* \delta_{h,S} = -\psi_q - \psi_r,$$

for every stable bipartition (h, S) . Using that $C_{h,S} = \iota_* \mathbb{1}$, we find

$$\begin{aligned} \pi_* \left(K^\alpha \cdot C_{h,S}^\beta \right) &= \pi_* \iota_* \iota^* \left(K^\alpha \cdot C_{h,S}^{\beta-1} \right) && \text{by projection formula} \\ &= \xi_{h,S*} \bar{\tau}_* \left(\eta_2^* K_\tau^\alpha \cdot (-\psi_q - \psi_r)^{\beta-1} \right) && \text{by (2.9) and (2.10)} \\ &= (-1)^{\beta-1} \sum_{b=0}^{\beta-1} \binom{\beta-1}{b} \xi_{h,S*} \left(\psi_q^b \otimes \tau_* \left(K_\tau^\alpha \cdot \psi_r^{\beta-1-b} \right) \right), \end{aligned}$$

where the last equality is obtained by expanding the binomial and by applying the Künneth decomposition. The result now follows from the string equation when $\alpha = 0$ and from the dilaton equation when $\alpha > 0$ (see [2, Proposition 4.9, p. 574]). To conclude the proof, we observe that if $\alpha = 0$ the term corresponding to $b = \beta - 1$ vanishes for dimension reasons. \square

We make the following convention, in order to level out the different formulas in (2.8). For all $s \geq 0$, put

$$\kappa_{-1} \cdot \psi^s = \psi^{s-1},$$

keeping the convention $\psi^{-1} = 0$. Then Formula (2.8) becomes, uniformly for all $\alpha, \beta \geq 0$,

$$(2.11) \quad \pi_* \left(K^\alpha \cdot C_{h,S}^\beta \right) = (-1)^{\beta-1} \sum_{b=0}^{\beta-1} \binom{\beta-1}{b} \xi_{h,S*} \left(\psi_q^b \otimes \kappa_{\alpha-1} \psi_r^{\beta-1-b} \right).$$

Substituting (1.4) and (2.11) in the corresponding expressions in (2.7), and summing over t , we can now write the right hand side of (2.2) as

(2.12)

$$\begin{aligned} \text{ch}(R^\bullet \pi_* \mathcal{O}(D)) - \Psi &= \sum_{\substack{t > 0 \\ 0 \leq i \leq t}} \frac{B_{t-i}}{i!(t-i)!} \left(\ell^i \kappa_{t-1} + (-1)^{i-1} \sum_p e_p^i \psi_p^{t-1} \right. \\ &+ \sum_{h,S} \sum_{0 \leq a < i} \binom{i}{a} \ell^a a_{h,S}^{i-a} (-1)^{i-a-1} \cdot \sum_{b=0}^{i-a-1} \binom{i-a-1}{b} \xi_{h,S*}(\psi_q^b \otimes \kappa_{a+t-i-1} \psi_r^{i-a-1-b}) \\ &\left. + \sum_{\substack{h,S \\ p \in S}} \sum_{0 \leq b < i} (-1)^{b-1} \binom{i}{b} e_p^b a_{h,S}^{i-b} \cdot \sum_{\Gamma \in \mathcal{G}(h,S)} \frac{(i-b)!}{|\text{Aut } \Gamma|} \left[\xi_{\Gamma*} \left(\psi_p^{b+t-i-1} \prod_{(l,l') \in \mathcal{E}(\Gamma)} \frac{1 - e^{-\psi_l - \psi_{l'}}}{\psi_l + \psi_{l'}} \right) \right]_{t-1} \right). \end{aligned}$$

Substituting the expression for Ψ from Formula (2.1) completes the proof of Theorem 1.

Example 2.2. Let us compute the coefficient of $\delta_{h,S}$ for $\text{ch}_1(R^\bullet \pi_* \mathcal{O}(D))$ in the basis of decorated boundary strata classes for the rational Chow group of codimension-1 classes of $\overline{\mathcal{M}}_{g,p}$, which consists of

$$\kappa_1, \{ \psi_p \}_{p \in P}, \{ \delta_{a,A} \}_{(a,A)}.$$

Such coefficient is extracted from the two double sums in Formula (2.12). Nonzero coefficients of $\delta_{h,S}$ appear in the last summand for $(i, b) = (1, 0)$ and $(i, b) = (2, 1)$ and in the previous summand (over stable bipartitions) for $(i, a) = (1, 0)$, $(2, 1)$ or $(2, 0)$, and $b = 0$. The former contributes

$$\frac{1}{2} a_{h,S} |S| + \frac{1}{2} a_{h,S} (2d_S - 2\ell |S|),$$

where $d_S = \sum_{p \in S} d_p$. The latter contributes

$$-\frac{1}{2} a_{h,S} (2h - 1 + |S|) + \frac{1}{2} a_{h,S} \cdot 2\ell (2h - 1 + |S|) - \frac{1}{2} a_{h,S}^2.$$

The coefficient of $\delta_{h,S}$ is therefore

$$\frac{1}{2} a_{h,S} ((2h - 1)(2\ell - 1) + 2d_S - a_{h,S}).$$

3. PULLBACK OF BRILL–NOETHER CLASSES VIA ABEL–JACOBI SECTIONS

In this section we review the definition of compactified universal Jacobians $\overline{\mathcal{J}}_{g,p}(\phi)$ and then define the cohomological, universal Brill–Noether classes

$$w_d^r(\phi) \in A^{g-\rho}(\overline{\mathcal{J}}_{g,p}(\phi)),$$

where $\rho = g - (r + 1)(g - d + r)$ is the Brill–Noether number. We always assume $r \geq 0$ and $d < g + r$ throughout.

Given integers ℓ and $d_p := \{d_p \mid p \in P\}$, and setting $d = \ell(2g - 2) + \sum_p d_p$, we define the pullbacks $Z_{\ell, d_p}^r(\phi) = s^* w_d^r(\phi)$, where $s = s_{\ell, d_p}$ is the Abel–Jacobi section defined in (0.4). Finally, we observe how the main result of the previous section allows one to explicitly compute $Z_{\ell, d_p}^r(\phi)$ in terms of decorated boundary strata classes, for all ϕ 's such that s is a well-defined morphism on $\overline{\mathcal{M}}_{g,p}$.

3.1. Compactified universal Jacobians. We first review the definition of the stability space $V_{g,P}^d$ from [11, Definition 3.2] and the notion of nondegenerate elements therein. An element

$$\phi \in V_{g,P}^d$$

is an assignment, for every stable P -pointed curve (C, P) of genus g and every irreducible component $C' \subseteq C$, of a real number $\phi(C, P)_{C'}$ such that

$$\sum_{C' \subseteq C} \phi(C, P)_{C'} = d,$$

and such that

(1) if $\alpha: (C, P) \rightarrow (C', P')$ is a homeomorphism of pointed curves, then

$$\phi(C', P') = \phi(\alpha(C, P));$$

(2) informally, the assignment ϕ is compatible with degenerations of pointed curves.

The notion of ϕ -(semi)stability was introduced in [11, Definitions 4.1 and 4.2]:

Definition 3.1. Given $\phi \in V_{g,P}^d$ we say that a family F of rank 1 torsion free sheaves of degree d on a family of stable curves is ϕ -stable if the inequality

$$(3.1) \quad \left| \deg_{C_0}(F) - \sum_{C' \subseteq C_0} \phi(C, P)_{C'} + \frac{\delta_{C_0}(F)}{2} \right| < \frac{\#(C_0 \cap \overline{C_0^c}) - \delta_{C_0}(F)}{2}$$

holds for every stable P -pointed curve (C, P) of genus g of the family, and for every subcurve (i.e. a union of irreducible components) $\emptyset \neq C_0 \subsetneq C$. Here $\delta_{C_0}(F)$ denotes the number of nodes $p \in C_0 \cap \overline{C_0^c}$ such that the stalk of F at p fails to be locally free. Semistability with respect to ϕ is defined by allowing equality in (3.1).

A stability parameter $\phi \in V_{g,P}^d$ is said to be *nondegenerate* when ϕ -semistability coincides with ϕ -stability for all stable P -pointed curves of genus g .

For all $\phi \in V_{g,P}^d$ there exists a moduli stack $\overline{\mathcal{J}}_{g,P}(\phi)$ of ϕ -semistable sheaves on stable curves, which comes with a forgetful morphism

$$p: \overline{\mathcal{J}}_{g,P}(\phi) \rightarrow \overline{\mathcal{M}}_{g,P}.$$

When ϕ is nondegenerate, by [11, Corollary 4.4] the stack $\overline{\mathcal{J}}_{g,P}(\phi)$ is a smooth Deligne–Mumford stack, and the morphism p is representable, proper and flat.

3.2. Universal Brill–Noether classes and their pullbacks. Let $\phi \in V_{g,P}^d$ be nondegenerate. Then by [10, Corollary 4.3] and [11, Lemma 3.35] combined with our general assumption $P \neq \emptyset$, there exists a tautological family $\mathcal{L}(\phi)$ of rank 1 torsion free sheaves of degree d on the total space of the universal curve

$$\tilde{\pi}: \overline{\mathcal{J}}_{g,P}(\phi) \times_{\overline{\mathcal{M}}_{g,P}} \overline{\mathcal{C}}_{g,P} \rightarrow \overline{\mathcal{J}}_{g,P}(\phi).$$

Recall the following notation from [5, Ch. 14]. Let $c = \sum_{k \in \mathbb{Z}} c_k$ be a formal sum of elements in a ring R . Then the $p \times p$ determinant $|c_{q+j-i}|$ in R is denoted

$$\Delta_q^{(p)} c = \begin{vmatrix} c_q & c_{q+1} & \cdots & c_{q+p-1} \\ c_{q-1} & c_q & \cdots & c_{q+p-2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{q-p+1} & c_{q-p+2} & \cdots & c_q \end{vmatrix}.$$

Generalising the idea of [10, Definition 3.38] (where the authors extended the universal theta divisor w_{g-1}^0), we define the (universal, cohomological) *Brill–Noether class* using the Thom–Porteous formula, namely by

$$(3.2) \quad w_d^r(\phi) := \Delta_{g-d+r}^{(r+1)} c(-R^* \tilde{\pi}_* \mathcal{L}(\phi)) \in A^{g-p}(\overline{\mathcal{J}}_{g,P}(\phi)),$$

for $\rho = g - (r + 1)(g - d + r)$ the Brill–Noether number. One can interpret the class (3.2) as follows. Define the *universal Brill–Noether scheme* as the closed subscheme

$$(3.3) \quad \mathcal{W}_d^r(\phi) = \text{Fit}_{g-d+r}(R^1\tilde{\pi}_*\mathcal{L}(\phi)) \subset \overline{\mathcal{J}}_{g,P}(\phi),$$

defined by the $(g - d + r)$ -th Fitting ideal of $R^1\tilde{\pi}_*\mathcal{L}(\phi)$ (see [2, Ch. 21] for the use of Fitting ideals in Brill–Noether theory). Then the Poincaré dual of (3.2) is the class that $\mathcal{W}_d^r(\phi)$ would have as its fundamental class if it were pure of the expected codimension $g - \rho$. The scheme (3.3) has an explicit description as a degeneracy scheme, which was already described in the proof of [8, Lemma 6] in the case $r = d = 0$. Fix a sufficiently $\tilde{\pi}$ -ample divisor H , and consider the short exact sequence

$$0 \rightarrow \mathcal{L}(\phi) \rightarrow \mathcal{L}(\phi)(H) \xrightarrow{u} \mathcal{L}(\phi) \otimes \mathcal{O}_H(H) \rightarrow 0.$$

Pushing this forward via $\tilde{\pi}$ yields a presentation

$$\mathcal{E}_0 \xrightarrow{\tilde{\pi}_*u} \mathcal{E}_1 \rightarrow R^1\tilde{\pi}_*\mathcal{L}(\phi) \rightarrow 0$$

of $R^1\tilde{\pi}_*\mathcal{L}(\phi)$, where $\tilde{\pi}_*u$ is a morphism of vector bundles whose virtual rank is

$$\text{rk } \mathcal{E}_0 - \text{rk } \mathcal{E}_1 = d - g + 1$$

by Riemann–Roch. The k -th degeneracy scheme of $\tilde{\pi}_*u$, where $k = \text{rk } \mathcal{E}_0 - (r + 1) = \text{rk } \mathcal{E}_1 - (g - d + r)$, is by definition the zero scheme

$$(3.4) \quad Z(\wedge^{k+1}\tilde{\pi}_*u) \subset \overline{\mathcal{J}}_{g,P}(\phi),$$

which agrees with (3.3) by the general theory of Fitting ideals. Note that, by this identification, the vanishing locus (3.4) is independent of the choice of H . Note that $\mathcal{W}_d^r(\phi)$ is set-theoretically supported on

$$\{(C, P, F) \mid h^0(C, F) > r\} \subset \overline{\mathcal{J}}_{g,P}(\phi).$$

The definition (3.2) is motivated by the following lemma.

Lemma 3.2. *The class $w_d^r(\phi)$ is supported on $\mathcal{W}_d^r(\phi)$. If the Brill–Noether scheme $\mathcal{W}_d^r(\phi)$ is pure of the expected codimension $g - \rho$, then $w_d^r(\phi)$ is its fundamental class.*

Proof. The first statement is proven in exactly the same manner as the first statement of [8, Lemma 6] (dealing with the case $r = d = 0$), namely by observing that the class (3.2) is by construction supported on the degeneracy scheme (3.4). The second statement follows from [5, Theorem 14.4]. \square

Example 3.3. For $r = 0$ we have

$$(3.5) \quad w_d^0(\phi) = (-1)^{g-d} c_{g-d}(R^\bullet\tilde{\pi}_*\mathcal{L}(\phi)).$$

These classes can therefore be seen, up to a sign, as the formal analogues of the λ -classes on $\overline{\mathcal{M}}_{g,P}$, where the (derived) pushforward of the tautological sheaf is taking the role of the pushforward of the relative dualizing sheaf. Note that for fixed d the classes (3.5) determine all classes $w_d^r(\phi)$ for arbitrary r .

Remark 3.4. While the restriction \mathcal{W}_d^0 of $\mathcal{W}_d^0(\phi)$ to $\mathcal{M}_{g,P}$ always has the expected dimension (being the image of the d -th symmetric product of the universal curve under the summation map), arguing as in [8, Remark 7] one sees that for each stable bipartition (h, S) there exists a nondegenerate ϕ such that $\mathcal{W}_d^0(\phi)$ contains the inverse image in $\overline{\mathcal{J}}_{g,P}(\phi)$ of the boundary divisor $\Delta_{h,S}$. In particular, $\mathcal{W}_d^0(\phi)$ is, in general, not even equidimensional.

Now fix integers $\ell \in \mathbb{Z}$ and $d_p := \{d_p\}_{p \in P}$ with $d = \ell(2g - 2) + \sum_p d_p$, and define the rational map

$$(3.6) \quad s = s_{\ell, d_p}: \overline{\mathcal{M}}_{g,n} \dashrightarrow \overline{\mathcal{J}}_{g,n}(\phi)$$

by Rule (0.4), for some choice of coefficients $a_{h,S}$. (This map is independent of the coefficients $a_{h,S}$ of $C_{h,S}$ as these divisors are zero on the open dense substack that parametrises line bundles over smooth pointed curves). Then define the pullback classes $Z_{\ell,d_p}^r(\phi)$ by the formula

$$(3.7) \quad Z_{\ell,d_p}^r(\phi) := s^* w_d^r(\phi) = p_*([\overline{\Sigma}(\phi)] \cdot w_d^r(\phi)),$$

where $\overline{\Sigma}(\phi)$ is the closure in $\overline{\mathcal{J}}_{g,p}(\phi)$ of the image of the section s . The second equality of Formula (3.7) follows from the definition of pullback of an algebraic class by a rational map, and it is well-defined because $\overline{\mathcal{J}}_{g,p}(\phi)$ is smooth and proper.

When ϕ is such that the line bundle D of (0.1) is ϕ -stable, the map (3.6) is a well-defined morphism on $\overline{\mathcal{M}}_{g,p}$, but the converse is not true. As explained in [11, Section 6.1], for all nondegenerate $\phi \in V_{g,p}^d$ there is a unique modification $D(\phi)$ of D that coincides with D on the locus parametrising smooth curves and that is ϕ -stable on all curves having 1 node. More explicitly, $D(\phi)$ is obtained from D by modifying the coefficients $a_{h,S}$ of $C_{h,S}$ into coefficients $a_{h,S}(\phi)$ in the unique way that makes the resulting $D(\phi)$ a divisor that is ϕ -stable on all curves with 1 node. By [11, Proposition 6.4], we have that s is a well-defined morphism on the open locus $U(\phi) = U_{\ell,d_p}(\phi)$ of $\overline{\mathcal{M}}_{g,p}$ where $D(\phi)$ is ϕ -stable.

We now show how Theorem 1 allows us to compute the restriction to $U(\phi)$ of the class $s^* w_d^r(\phi)$. Chiodo's formula recovers the particular case when $D(\phi)$ equals $\ell \tilde{K}_\pi + \sum_{p \in P} d_p \sigma_p$.

Corollary 3.5. *Let $\phi \in V_{g,p}^d$ be nondegenerate. Then the equality of classes*

$$(3.8) \quad Z_{\ell,d_p}^r(\phi) = \Delta_{g-d+r}^{(r+1)} c(-R^\bullet \pi_* \mathcal{O}(D(\phi)))$$

holds on the open substack $U(\phi)$ of $\overline{\mathcal{M}}_{g,p}$ where $D(\phi)$ is ϕ -stable.

Proof. Consider the Cartesian square

$$\begin{array}{ccc} \overline{\mathcal{C}}_{g,p} & \xrightarrow{\tilde{s}} & \overline{\mathcal{J}}_{g,p}(\phi) \times_{\overline{\mathcal{M}}_{g,p}} \overline{\mathcal{C}}_{g,p} \\ \downarrow \pi & \square & \downarrow \tilde{\pi} \\ \overline{\mathcal{M}}_{g,p} & \xrightarrow{s} & \overline{\mathcal{J}}_{g,p}(\phi) \end{array}$$

defining \tilde{s} . We have the following equalities in the Chow group of $U(\phi)$:

$$s^* c_k(R^\bullet \tilde{\pi}_* \mathcal{L}(\phi)) = c_k s^*(R^\bullet \tilde{\pi}_* \mathcal{L}(\phi)) = c_k(R^\bullet \pi_* \tilde{s}^* \mathcal{L}(\phi)) = c_k(R^\bullet \pi_* \mathcal{O}(D(\phi))).$$

All equalities require to restrict to the locus where s is a morphism. The first follows from the fact that Chern classes commute with pullbacks. The second is cohomology and base change [9, Theorem 8.3.2], using that $\tilde{\pi}$ is flat and $R^\bullet \tilde{\pi}_* \mathcal{L}(\phi)$ is represented by a two-term complex of vector bundles. The third and the last follows from the definition of a tautological sheaf and of \tilde{s} . Formula (3.8) now follows from the definition of $Z_{\ell,d_p}^r(\phi)$ and from the fact that the pullback along the morphism s is a ring homomorphism. \square

Combining Formula (3.8) with the formula

$$(3.9) \quad c_t(F) = \left[\exp \left(\sum_{s \geq 1} (-1)^{s-1} (s-1)! \text{ch}_s(F) \right) \right]_t$$

expressing the Chern classes of a K -theory element F in terms of the Chern character, and then applying Theorem 1, yields an explicit formula, in terms of decorated boundary strata classes, for the restriction of $Z_{\ell,d_p}^r(\phi)$ to the open locus of $\overline{\mathcal{M}}_{g,p}$ where $D(\phi)$ is ϕ -stable. In particular, this computes $Z_{\ell,d_p}^r(\phi)$ for all ϕ such that the corresponding Abel–Jacobi section (3.6) is a regular map.

4. OPEN PROBLEMS

4.1. Is $Z(\phi)$ tautological? Formula (3.8) implies that the restriction of each class $Z(\phi)$ to $U(\phi)$ is tautological on $U(\phi)$ – meaning that it is the restriction of a tautological class globally defined on $\overline{\mathcal{M}}_{g,P}$. That tautological class is explicitly expressed in terms of decorated boundary strata by combining Theorem 1 with Formulas (3.8) and (3.9). We do not know whether the class $Z(\phi)$ is, in general, itself tautological on $\overline{\mathcal{M}}_{g,P}$, although we do expect that this should be the case. Except for when $Z(\phi)$ has codimension 1 or 2 (when we know that the entire cohomology of $\overline{\mathcal{M}}_{g,P}$ is tautological), the only classes $Z(\phi)$ that we know to be tautological on $\overline{\mathcal{M}}_{g,P}$ for general g and P are those for $r = d = 0$ and ϕ a small perturbation of $\underline{0} \in V_{g,P}^0$. This follows from the main result of [8], showing that this class coincides with the *double ramification cycle*. The latter is shown to be tautological in [4].

4.2. Wall-crossing. For fixed $d \in \mathbb{Z}$ and for every choice of nondegenerate elements ϕ and ϕ' of $V_{g,P}^d$ one has classes $w_d^r(\phi) \in A^\bullet(\overline{\mathcal{J}}_{g,P}(\phi))$ and $w_d^r(\phi') \in A^\bullet(\overline{\mathcal{J}}_{g,P}(\phi'))$. A natural question is to “compute” (in terms of some natural classes) the difference

$$w_d^r(\phi) - \alpha^*(w_d^r(\phi')) \in A^{g-P}(\overline{\mathcal{J}}_{g,P}(\phi)),$$

where α is any birational isomorphism $\overline{\mathcal{J}}_{g,P}(\phi) \dashrightarrow \overline{\mathcal{J}}_{g,P}(\phi')$ that commutes with the forgetful morphisms to $\overline{\mathcal{M}}_{g,P}$ (such birational maps are explicitly characterised in [11, Section 6.2]). To the best of our knowledge, this question has been answered only for the case of the theta divisor, namely when $r = 0$ and $d = g - 1$, in [10].

Another natural question is to compute the difference of the pullbacks

$$(4.1) \quad Z_{\ell,d_P}^r(\phi) - Z_{\ell',d'_P}^r(\phi') \in A^{g-P}(\overline{\mathcal{M}}_{g,P})$$

for different assignments (ℓ, d_P) , (ℓ', d'_P) and different nondegenerate $\phi, \phi' \in V_{g,P}^d$. The case of the pullback of the theta divisor is again covered explicitly in [10]. Theorem 1 immediately allows us to generalise the result in loc. cit., in the sense that it computes explicitly, in terms of decorated boundary strata classes of $\overline{\mathcal{M}}_{g,P}$, the difference (4.1), whenever ϕ and ϕ' are such that the corresponding Abel–Jacobi sections s and s' extend to morphisms on $\overline{\mathcal{M}}_{g,P}$. Example 2.2 shows that the results of this paper match the earlier results of [10] for the case of the pullback of the theta divisor.

Acknowledgements. NP would like to thank Jesse Kass for his crucial insight on extending Brill–Noether classes to compactified universal Jacobians. He is also very grateful to Nicola Tarasca for informing him about Formula (1.2). AR wishes to thank Filippo Viviani for providing several useful insights on the topic of universal Brill–Noether theory, and Bashar Dudin for many fruitful conversations about compactified Jacobians.

NP acknowledges support from the EPSRC First Grant Scheme EP/P004881/1 with title “Wall-crossing on universal compactified Jacobians”. AR wishes to thank Max-Planck Institut für Mathematik for support.

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