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Abstract

We prove that the moduli space of compact genus three Riemann surfaces contains only finitely many algebraically primitive Teichmüller curves. For the stratum $\Omega\mathcal{M}_3(4)$, consisting of holomorphic one-forms with a single zero, our approach to finiteness uses the Harder-Narasimhan filtration of the Hodge bundle over a Teichmüller curve to obtain new information on the locations of the zeros of eigenforms. By passing to the boundary of moduli space, this gives explicit constraints on the cusps of Teichmüller curves in terms of cross-ratios of six points on $\mathbf{P}^1$.

These constraints are akin to those that appear in Zilber and Pink's conjectures on unlikely intersections in diophantine geometry. However, in our case one is lead naturally to the intersection of a surface with a family of codimension two algebraic subgroups of $\mathbf{G}_m^n \times \mathbf{G}_a^n$ (rather than the more standard $\mathbf{G}_m^n$). The ambient algebraic group lies outside the scope of Zilber's Conjecture but we are nonetheless able to prove a sufficiently strong height bound.

For the generic stratum $\Omega\mathcal{M}_3(1,1,1,1)$, we obtain global torsion order bounds through a computer search for subtori of a codimension-two subvariety of $\mathbf{G}^9$. These torsion bounds together with new bounds for the moduli of horizontal cylinders in terms of torsion orders yields finiteness in this stratum. The intermediate strata are handled with a mix of these techniques.

Contents

1 Introduction

A closed Riemann surface X of genus g equipped with a nonzero holomorphic quadratic differential q determines an isometrically immersed hyperbolic plane $\mathbf{H} \to \mathcal{M}_g$ in the moduli space of genus g Riemann surfaces. Occasionally this may cover an isometrically immersed algebraic curve $C = \mathbf{H}/\Gamma \to \mathcal{M}_g$. Such a curve is called an *Teichmüller curve*, and the pair (X, q) is called a *Veech surface.*

The trace field of $\mathbf{H}/\Gamma$ is the number field $F = \mathbf{Q}(\mathrm{Tr}(\gamma) : \gamma \in \Gamma)$. A Teichmüller curve is said to be *arithmetic* if $F = \mathbf{Q}$. It is said to be *algebraically primitive* if the generating quadratic differential q is the square of a holomorphic one-form ω and the degree of F attains its maximum, namely $[F : \mathbf{Q}] = g$. In this case, we call the pair (X, ω) an *algebraically primitive Veech surface.*

While arithmetic Teichmüller curves are dense in every $\mathcal{M}_g$, algebraically primitive Teichmüller curves seem to be much more rare. There are infinitely many examples of algebraically primitive Teichmüller curves in $\mathcal{M}_2$, discovered independently by Calta [Cal04] and McMullen [McM03], and it remains an open problem whether there are infinitely many such curves for any larger genus. The aim of this paper is the following partial solution to this problem.

Theorem 1.1. *There are only finitely many algebraically primitive Teichmüller curves in $\mathcal{M}_3$.*

The methods used here do not use any dynamics of the Teichmüller geodesic flow, with the exception of the hyperelliptic locus in the stratum $\Omega\mathcal{M}_3(2,2)^{\mathrm{odd}}$ (consisting of forms with two double zeros which are fixed by the hyperelliptic involution). We emphasize that our proofs of Theorem 1.1 are *effective*, in the sense that a reader keeping careful track of constants at every step should arrive at an explicit bound for the number of algebraically primitive Teichmüller curves in any stratum (except $\Omega\mathcal{M}_3(2,2)^{\mathrm{odd}}$). Good effective bounds would allow one to finish classifying Teichmüller curves in these strata with a computer search. Unfortunately, the bounds produced by our methods are so large that this is not feasible.

In parallel to our work, Matheus and Wright showed [MW] that for every fixed genus g which is an odd prime, there are only finitely many algebraically primitive Teichmüller curves generated by Veech surfaces with a single zero. These results rely on recent results of Eskin, Mirzakhani, and Mohammadi ([EM], [EMM]) on $\mathrm{SL}_2(\mathbf{R})$ orbit-closures in strata of holomorphic one-forms. In particular, these methods are not effective. We appeal to [MW] to obtain finiteness in the hyperelliptic locus of the stratum $\Omega\mathcal{M}_3(2,2)^{\mathrm{odd}}$, as none of our methods could handle this case. We give a summary of the known results on the classification of Teichmüller curves at the end of the introduction.

One essential ingredient of the proof of Theorem 1.1 is a height bound for the cusps of these Teichmüller curves. We obtain these bounds by applying methods used to attack conjectures on unlikely intersection in the multiplicative group $\mathbf{G}_m^n$ (whose complex points are just $(\mathbf{C}^*)^n$). In our case, we are lead to study similar unlikely intersection problems in the group $\mathbf{G}_m^n \times \mathbf{G}_a^n$.

The remaining techniques depend on the stratum the Teichmüller curve surface lies in. In the case of few zeros the main new ingredient is an application of the Harder-Narasimhan filtration of the Hodge bundle. In the case of many zeros we prove global torsion order bounds and we use conformal geometry to derive bounds for ratios of moduli. We now describe these techniques in more detail.

Harder-Narasimhan filtrations. Consider an algebraically primitive Veech surface (X, ω) with trace field F. One of the fundamental constraints on (X, ω), established in [Möl06b], is that the Jacobian of X has real multiplication by an order in F with ω an eigenform. This real multiplication in fact distinguishes g eigenforms $\omega_1 = \omega, \omega_2, \ldots, \omega_g$ (up to constant multiple). These other $g-1$ eigenforms are in general very mysterious from the point of view of the flat geometry of (X, ω); however, in §4, for Teichmüller curves in certain strata we obtain some information on the locations of the zeros of the other eigenforms.

More precisely, we denote by $\Omega\mathcal{M}_g(n_1, \ldots, n_k)$ the moduli space of genus g surfaces X equipped with a holomorphic one-form ω having k zeros of order given by the n_i. The minimal stratum $\Omega\mathcal{M}_g(2g-2)$ has as one connected component the hyperelliptic component $\Omega\mathcal{M}_g(2g-2)^{\mathrm{hyp}}$, consisting entirely of hyperelliptic curves. Here is one example of the type of control we obtain on the zeros of the other eigenforms. Similar statements are proved in §4 for all genus three strata except for the generic stratum $\Omega\mathcal{M}_3(1,1,1,1)$.

Theorem 1.2. *Suppose (X, ω) generates an algebraically primitive Teichmüller curve C in $\Omega\mathcal{M}_g(2g-2)^{\mathrm{hyp}}$, with $p \in X$ the unique zero of ω of order $2g-2$. Then the eigenforms ω_i, listed in an appropriate order, have a zero of order $2g-2i$ at p.*

The basic idea of the proof is to consider a canonical filtration of the Hodge bundle over C. Every vector bundle over a projective variety has a canonical filtration, the Harder-Narasimhan filtration. For Teichmüller curves in the strata under consideration, these filtrations were computed by Yu and Zuo [YZ13] in terms of the zero divisor of the family of one-forms generating the Teichmüller curve in the canonical family of curves over C of. Alternatively the decomposition of the Hodge bundle into eigenform bundles yields a second filtration. Uniqueness of the Harder-Narasimhan filtration implies that these two filtrations are in fact the same, and comparing them yields Theorem 1.2.

Finiteness in $\Omega\mathcal{M}_3(4)$. The real multiplication condition, together with Theorem 1.2 gives very strong constraints on algebraically primitive Teichmüller curves in $\Omega\mathcal{M}_3(4)^{\mathrm{hyp}}$. Unfortunately, these conditions are difficult to apply, as determining when the Jacobian of a given Riemann surface has real multiplication and understanding its eigenforms is generally very hard.

We bypass this difficulty by studying the cusps of Teichmüller curves. Veech [Vee89] established that every Teichmüller curve has at least one cusp. Exiting a cusp, the family of Riemann surfaces degenerates to a noded Riemann surface equipped with a meromorphic one-form (a *stable form*). By algebraic primitivity, this stable form has geometric genus zero, and because we are in the minimal stratum, it is in fact irreducible. More concretely, we may regard it as $\mathbf{P}^1$ with three pairs of distinct points (x_i, y_i), $i = 1, 2, 3$ each identified to form a node. In the hyperelliptic stratum we have moreover $y_i = -x_i$.

The advantage of passing to the boundary is that our constraints become completely explicit. More precisely, consider the cross-ratio

$$R_i = [x_{i+1}, y_{i+1}, x_{i+2}, y_{i+2}],$$

with indices taken modulo three. We established in [BM12] that the real multiplication condition is equivalent to

$$R_1^{b_1} R_2^{b_2} R_3^{b_3} = 1 \tag{1.1}$$

for some nonzero integers b_i. In §3, we show that the condition of Theorem 1.2 on the zeros of the second eigenform ω_2 is equivalent to

$$b_1 x_1 + b_2 x_2 + b_3 x_3 = 0, \tag{1.2}$$

where the b_i are the same integers as above. A similar constraint is established in the other component $\Omega\mathcal{M}_3(4)^{\mathrm{odd}}$.

By a theorem of [BM12], to prove finiteness of Teichmüller curves in these strata, it is enough to establish finiteness of cusps, thus these equations reduce the problem to an explicit problem in number theory.

Unlikely intersections. The cross-ratios R_i can be regarded as a diagonal embedding of a two-dimensional variety $\mathcal{Y}$ in the algebraic group $\mathbf{G}_m^3 \times \mathbf{G}_a^3$. Allowing all possible coefficients, equations (1.1) and (1.2) can then be interpreted as an intersection of the surface $\mathcal{Y}$ with a countable collection of codimension-two subgroups. Our problem is then most naturally regarded in the context of unlikely intersections in diophantine geometry. Unlikely intersections refer to vast conjectures due to Zilber [Zil02] and Pink [Pin05] and partially motivated by a theorem of Bombieri, Masser, and Zannier [BMZ99]. This last group of authors considered the problem of intersecting a curve $\mathcal{X} \subset \mathbf{G}_m^n$ with the infinite union $\mathcal{H} \subset \mathbf{G}_m^n$ of all proper algebraic subgroups of $\mathbf{G}_m^n$ and showed – as long as $\mathcal{X}$ itself is not contained in the translate of a proper subgroup – that $\mathcal{X} \cap \mathcal{H}$ is a set of bounded height. By height we mean the absolute logarithmic Weil height which we recall further down in §2.2. A non-empty intersection of a curve $\mathcal{X}$ with a subgroup of codimension one is more appropriately called "just likely", in contrast to what is studied by Zilber and Pink's conjectures, where the subgroups must have codimension at least two and the intersections are deemed "unlikely". Under this more stringent condition and when imposing an appropriate condition on $\mathcal{X}$, one expects finiteness instead of merely bounded height. However, boundedness of height in the "just likely" situation is often a gateway to proving finiteness in the "unlikely" case.

Zilber and Pink's conjectures are open in general. But several cases that incorporate classical results such as the Mordell or Manin-Mumford Conjectures are known. We provide a partial overview of state of this field in §2.1.

One aspect that sets our work apart from previous results is that it mixes the additive group of a field $\mathbf{G}_a$, which is unipotent, with the multiplicative group $\mathbf{G}_m$, which is not. The latter appears in the literature [BMZ99, Zil02] on these conjectures but the former seems to lie outside the general framework of the Zilber-Pink Conjectures. The following theorem is proved in §2. There we also provide all the necessary definitions used in the theorem's formulation.

Theorem 1.3. *Let $\mathcal{Y} \subset \mathbf{G}_m^n \times \mathbf{G}_a^n$ be an irreducible closed surface and let $\mathcal{Y}^{\mathbf{Q},\mathrm{ta}}$ denote the complement in $\mathcal{Y}$ of the union of all rational semi-torsion anomalous subvarieties of $\mathcal{Y}$ (for a definition see §2.4). There is a constant c with the following property. If $P = (x_1, \ldots, x_n, y_1, \ldots, y_n) \in \mathcal{Y}^{\mathbf{Q},\mathrm{ta}}$ such that there is $(b_1, \ldots, b_n) \in \mathbf{Z}^n \smallsetminus \{0\}$ with*

$$x_1^{b_1} \cdots x_n^{b_n} = 1 \quad \textit{and} \quad b_1 y_1 + \cdots + b_n y_n = 0, \tag{1.3}$$

then $h(P) \le c\log(2[\mathbf{Q}(P):\mathbf{Q}])$.

The equations (1.3) define an algebraic subgroup of $\mathbf{G}_m^n \times \mathbf{G}_a^n$ of codimension two. The fact that the coefficients on the additive side are coupled to the exponents on the multiplicative side is essential for the proof and for the application to Theorem 1.1.

We emphasize that this height bound only applies to points in the subset $\mathcal{Y}^{\mathbf{Q},\mathrm{ta}} \subset \mathcal{Y}$ which we define precisely in §2.4. This set arises by removing from $\mathcal{Y}$ certain subvarieties that have anomalously large intersection with certain translates of algebraic subgroups. It bears similarities to Bombieri, Masser, and Zannier's *open anomalous locus* $\mathcal{X}^{\mathrm{oa}}$ [BMZ07] of a subvariety of algebraic torus $\mathcal{X} \subset \mathbf{G}_m^n$. Indeed, the second named author [Hab09] proved a height bound on points in $\mathcal{X}^{\mathrm{oa}}$ that are contained in an algebraic subgroup of dimension at most $n - \dim \mathcal{X}$. However, $\mathcal{Y}^{\mathbf{Q},\mathrm{ta}}$ can have a delicate structure. We will see in Example 2.11 that it need not in general be Zariski open in $\mathcal{Y}$; its complement in $\mathcal{Y}$ can be a countable infinite union of curves. In general, it is difficult to determine the open anomalous locus. The analogous problem for a plane in $\mathbf{G}_m^n$ is already a difficult problem which was solved by Bombieri, Masser, and Zannier [BMZ08b]. For our application, $\mathcal{Y}^{\mathbf{Q},\mathrm{ta}}$ will cause additional difficulties. In §6.3, we compute $\mathcal{Y}^{\mathbf{Q},\mathrm{ta}}$ for the two cases arising from the two components of $\Omega\mathcal{M}_3(4)$. This is done via by amalgamating a theoretical analysis with the use of computer algebra software [S$^+$14].

A torus containment algorithm. At two places we rely on computer-assistance to establish the non-existence of tori in a given subvariety of $\mathbf{G}_m^n$. In §5 we provide an algorithm that deals with that problem effectively. The algorithm is designed so as to check only for tori whose character group is contained in a specified subgroup of $\mathbf{Z}^n$. For the application in §9 we can restrict to such a situation, and only with this restriction is the run-time of the algorithm reasonable.

Multiple zeros. The proof of finiteness for strata with multiple zeros is quite different and starts with the torsion condition of [Möl06a]. This states that if (X,ω) is an algebraically primitive Veech surface, and p, q are distinct zeros of ω, then the divisor $p - q$ represents a torsion point of $\mathrm{Jac}(X)$. As for the real multiplication condition, this torsion condition may be interpreted explicitly at the boundary, and we couple this with other conditions to obtain finiteness of cusps. We say that (X,ω) has *torsion dividing* N if the order of $p - q$ divides N for every two zeros p and q.

A fundamental difficulty is that the limiting stable forms arising from Teichmüller curves in these strata may have thrice-punctured sphere components (pairs of pants), and none of our conditions give any control on the one-form restricted to these components.

For our approach to work, we need to know that controlling all irreducible components of limiting stable curves, except for pants components, is enough to conclude finiteness of Teichmüller curves. To formalize this, we say that a collection of Teichmüller curves is *pantsless-finite* if the collection of all non-pants irreducible components of limiting stable forms is finite. In §7 and §8, we prove:

Theorem 1.4. *In any stratum of holomorphic one-forms, any pantsless-finite collection of algebraically primitive Teichmüller curves is in fact finite.*

In §9 and §10, we then show that for each remaining stratum in genus three, the set of algebraically primitive Teichmüller curves is pantsless-finite.

Torsion and moduli: the Abel metric. Its a simple observation that a pantsless-finite collection of algebraically primitive Teichmüller curves has uniform bounds on its torsion orders (though proving these uniform bounds is in fact the difficult step in establishing pantsless-finiteness, as we discuss below). A Veech surface (X, ω) has a canonical flat metric $|\omega|$. In §7, we study how this flat metric is controlled by these torsion order bounds. More precisely, (X, ω) has many *periodic directions* in which the surface decomposes as a union of parallel flat cylinders whose moduli have rational ratios, and the complement of these cylinders is a collection of parallel geodesic segments joining the zeros of ω (called the *spine* of this periodic direction). The main ingredient in the proof of Theorem 1.4 is new bounds on the moduli of these cylinders in terms of the torsion orders.

For a graph Γ we define its *blocks* to be maximal subgraphs which cannot be disconnected by removing any single vertex. Every graph has a canonical decomposition into blocks, with any two adjacent blocks meeting in a single vertex. This notion applies in particular to the dual graph of a periodic direction whose edges are cylinders and vertices are connected components of the spine (equivalently, the dual graph of a corresponding stable curve over a cusp of the Teichmüller curve). This induces a partition of the cylinders in any given periodic direction into blocks. In §7, we show that bounds for torsion orders yield bounds for ratios of moduli within any block.

Theorem 1.5 (Theorem 7.2)**.** *Let (X, ω) be an algebraically primitive Veech surface, with torsion dividing N. Then for any block of cylinders $C_1, \ldots, C_n$ of some periodic direction of (X, ω), the number of possibilities for the projectivized tuple $(\mathrm{mod}(C_1) : \ldots : \mathrm{mod}(C_n))$ is bounded in terms of N and n.*

We emphasize that this theorem is only useful for strata with multiple zeros. For strata with a single zero, torsion orders are bounded trivially, but each block consists of one cylinder, so Theorem 1.5 gives no information.

This theorem generalizes [Möl08, Theorem 2.4], which establishes a special case of this bound using properties of Néron models of the stable curves over the cusps of the Teichmüller curve. Our proof here uses instead conformal geometry and a new metric on (X, ω) coming from the torsion condition. The torsion condition determines via Abel's theorem a meromorphic function f whose divisor (f) is supported on the zeros of ω (in fact there are many possible functions f we can use here). Pulling back the flat metric dz/z on $\mathbf{P}^1$ via this map, we obtain a metric on (X, ω) with infinite cylinders at the zeros of ω, which we call the *Abel metric*.

We show that for (X, ω) close to the boundary of moduli space, the finite cylinders of (X, ω) have moduli close to corresponding cylinders in the Abel metric. Using the torsion condition, we obtain constraints on the moduli of these cylinders in the Abel metric, which then give corresponding constraints on the original cylinders of (X, ω). Combining these constraints for all of the possible choices of the initial function f, we obtain Theorem 1.5.

Geometry of cylinder widths. To complete the proof of Theorem 1.4, we need to be able to compare moduli of cylinders in a periodic direction which do not belong to the same block. An irreducible component of the limiting stable curve which joins two blocks must have at least four punctures, so the hypothesis of pantsless-finiteness means that we can control the geometry of a component connecting two blocks. In particular, for two cylinders which lie in different blocks but whose boundaries share a common connected component of the spine, the ratio of the circumference of these cylinders is an algebraic number λ which can be assumed to be known (in particular, all of its Galois conjugates are bounded). We would like to use this control on ratios of circumferences to bound the ratios of moduli of cylinders in these two blocks. Ordinarily knowing only the widths or moduli of cylinders gives no information about their moduli or widths, since their heights can be arbitrary. In §8, we see that on an algebraically primitive Veech surface the widths and moduli are intimately connected.

More precisely, consider a periodic direction of an algebraically primitive Veech surface (X, ω). The widths of cylinders are a collection of algebraic numbers (*a priori* only defined up to scale, but in §8, we define a nearly canonical way to normalize them). Considering their different Galois conjugates, we can regard these widths as a collection of vectors $\{v_1, \ldots, v_n\} \subset \mathbf{R}^g$. We group these vectors according to the blocks of cylinders defined above, and call $B_i \subset \mathbf{R}^g$ the span of the ith block.

Theorem 1.6. *The B_i are pairwise orthogonal subspaces of $\mathbf{R}^g$. In each B_i, the set of width vectors v_j it contains are determined by the moduli of the corresponding cylinders up to a similarity of B_i. The norm of the width vectors in B_i is inversely proportional to the moduli of the corresponding cylinders.*

Combining Theorem 1.5 and Theorem 1.6 (for simplicity in the case of a periodic direction with a single block), we see that given a bound on the torsion orders of (X, ω), there are finitely many possibilities for the collection of width vectors up to similarity of $\mathbf{R}^g$.

Theorem 1.5 and Theorem 1.6 imply that for any pantsless-finite collection of Teichmüller curves, there are uniform bounds for the ratios of widths and moduli of cylinders in any periodic direction, as we can use the width-ratio λ and Theorem 1.6 to control the scale of the moduli of adjacent blocks of cylinders. From this, we conclude using the Smillie-Weiss "no small triangles" condition from [SW10] that our collection of Teichmüller curves is in fact finite.

Bounding torsion orders in the generic stratum. In §9, we establish finiteness of algebraically primitive Teichmüller curves in the generic stratum $\Omega\mathcal{M}_3(1, 1, 1, 1)$. The key quantity governing the geometry of Teichmüller curves in the generic stratum is the torsion orders introduced above. Given an *a priori* bound for torsion orders in this stratum, we show using height estimates that the collection of algebraically primitive Teichmüller curves in this stratum is pantsless-finite in the sense of §8. From Theorem 1.4, we obtain:

Theorem 1.7. *Suppose that there is a uniform bound for the torsion orders of algebraically primitive Teichmüller curves in the generic stratum $\Omega\mathcal{M}_g(1^{2g-2})$. Then there are only finitely many algebraically primitive Teichmüller curves in this stratum.*

It remains to establish these torsion bounds. Consider an irreducible component of a cusp of a Teichmüller curve in this stratum which contains more than one zero of the stable form. Marking the zeros and poles of this stable form, we can regard it as a point in $\mathcal{M}_{0,n}$ for some n. A choice of two zeros and two poles determines a cross-ratio morphism $\mathcal{M}_{0,n} \to \mathbf{G}$. Choosing an appropriate collection of cross-ratios, we obtain a morphism $\mathcal{M}_{0,n} \to \mathbf{G}^N$, and the torsion condition may be interpreted as saying that our stable form maps to a torsion point of this torus. For example in the (most difficult) case of an irreducible stable form, we obtain an embedding $\mathcal{M}_{0,10} \to \mathbf{G}^9$.

The problem is then to show that a subvariety of a torus meets only finitely many torsion points. We appeal to Laurent's theorem [Lau84] which says that a subvariety of $\mathbf{G}^n$ contains only finitely many torsion points unless it contains a torsion-translate of a subtorus, so we are lead to study translates of tori in the seven-dimensional variety $\mathcal{M}_{0,10} \subset \mathbf{G}^9$.

We then use the torus-containment algorithm from §5 to study tori in this variety. As the algorithm is much too slow to try to classify all torus-translates in a nine-dimensional torus, in §9.2, we give a significant reduction that says that only subtori of one of three three-dimensional tori need to be considered. Applying the algorithm, we see that there are no subtori of $\mathcal{M}_{0,10}$ which could lead to Teichmüller curves with arbitrarily large torsion orders. Thus we obtain:

Theorem 1.8. *There is a uniform bound for the torsion orders of algebraically primitive Teichmüller curves in* $\Omega\mathcal{M}_3(1,1,1,1)$.

The final step using the torus-containment algorithm is heavily computer-aided and was only completed in genus three. Finding uniform torsion order bounds is all that remains to establish finiteness of algebraically primitive Teichmüller curves for the generic stratum in arbitrary genus.

Combining these two theorems yields Theorem 1.1 in the case of the generic stratum in genus three.

For the remaining strata with two or three zeros, we establish finiteness in §10 using a mix of these techniques.

Classification of Teichmüller curves: State of the art. Teichmüller curves with trace field $\mathbf{Q}$ all arise as branched covers of tori by [GJ00] and are dense in every stratum. More generally, any Teichmüller curve gives rise to many Teichmüller curves in higher genera by passing to branched covers. Teichmüller curves that do not arise from this branched covering construction are called *primitive*. The classification of primitive Teichmüller curves is an important guiding problem.

The first examples of primitive Teichmüller curves were constructed by Veech [Vee89]. The constructions in [BM10] subsumes his examples as well as those of Ward and provides all currently known examples of algebraically primitive Teichmüller curves for $g \geq 6$. Calta [Cal04] and McMullen [McM03] constructed an infinite sequence of primitive Teichmüller curves in $\mathcal{M}_2$, the first infinite sequence in any fixed genus. McMullen completed the classification of primitive Teichmüller curves in genus two [McM06b] and showed via a Prym construction [McM06a] that in genus three and four there are an infinite number of Teichmüller curves which are primitive but not algebraically primitive.

The torsion and real multiplication conditions are strong restrictions on the existence of algebraically primitive Teichmüller curves. After [McM06b], the torsion condition was used in [Möl08] to show finiteness of algebraically primitive Teichmüller curves in the hyperelliptic strata where ω has two zeros or order $g-1$. The real multiplication and torsion conditions were applied in [BM12] to prove finiteness in the case of the stratum $\Omega\mathcal{M}_3(3,1)$ and to give an effective algorithm showing the non-existence of Teichmüller curves for given discriminants. This used a classification of the stable forms in the boundary of the eigenform locus, and introduced the cross-ratio equation (1.1) which is essential in the present proof of finiteness in the minimal strata.

Independent of this work, Matheus and Wright showed in [MW] that for every fixed genus g which is an odd prime, there are only finitely many algebraically primitive Teichmüller curve generated by Veech surfaces with a single zero. Moreover, Nguyen and Wright showed [NW] that there are only finitely many primitive Teichmüller curves in genus $g=3$ generated by Veech surfaces with a single zero in the hyperelliptic stratum. Their methods are very different from ours, relying on the Teichmüller geodesic flow and recent work of Eskin-Mirzakhani [EM] and Eskin-Mirzakhani-Mohammadi [EMM] establishing a an analogue of Ratner's theorem for the $\mathrm{SL}_2(\mathbf{R})$ action on strata of one-forms.

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2 A theorem on height bounds

In this section, we discuss some necessary background in diophantine geometry and establish the "unlikely intersections" result, Theorem 1.3.

2.1 Zilber's Conjecture on Intersections with Tori

Zilber's Conjecture on Intersections with Tori [Zil02] governs the locus where a subvariety of $\mathbf{G}_m^n$ meets algebraic subgroups of sufficiently low dimension. Let us state a variant of the conjecture found in [Zil02].

Conjecture 2.1. *Let $\mathcal{Y}$ be an irreducible subvariety of $\mathbf{G}_m^n$ defined over $\mathbf{C}$. Let us suppose that the union*

$$\bigcup_{\substack{H\subset\mathbf{G}_m^n\\ \dim\mathcal{Y}+\dim H\leq n-1}} \mathcal{Y}\cap H \quad \textit{is Zariski dense in} \quad \mathcal{Y} \tag{2.1}$$

where H runs over algebraic subgroups with the prescribed restriction on the dimension. Then $\mathcal{Y}$ is contained in a proper algebraic subgroup of $\mathbf{G}_m^n$.

Zilber's Conjecture is stated more generally for semi-abelian varieties and Pink [Pin05] has a version for mixed Shimura varieties.

The algebraic subgroups of $\mathbf{G}_m^n$ can be characterized easily, they are in natural bijection with subgroups of $\mathbf{Z}^n$, cf. Chapter 3.2 [BG06].

The heuristics behind this conjecture are supported by the following basic observation. Two subvarieties of $\mathbf{G}_m^n$ in general position whose dimensions add up to something less than the dimension of the ambient group variety are unlikely to intersect; however, non-empty intersections are certainly possible. Unless we are in the trivial case $\mathcal{Y} = \mathbf{G}_m^n$, the union (2.1) is over a countable infinite set of algebraic subgroups. The content of the conjecture is just that any non-empty intersections that arise are contained in a sufficiently sparse subset of $\mathcal{Y}$ unless $\mathcal{Y}$ is itself inside a proper algebraic subgroup of $\mathbf{G}_m^n$.

Although the conjecture above is open, many partial results are known. We will now briefly mention several ones.

If $\mathcal{Y}$ is a hypersurface, i.e. $\dim \mathcal{Y} = n-1$, then the algebraic subgroups in question are finite. So the union (2.1) is precisely the set of points on $\mathcal{Y}$ whose coordinates are roots of unity. Describing the distribution of points of finite order on subvarieties of $\mathcal{Y}$ is a special case of the classical Manin-Mumford Conjecture. In general, the Manin-Mumford Conjecture states that a subvariety of a semi-abelian variety can only contain a Zariski dense set of torsion points if it is an irreducible component of an algebraic subgroup. The first proof in this generality is due to Hindry. In the important case of abelian varieties the Manin-Mumford Conjecture was proved earlier by Raynaud. Laurent's Theorem [Lau84] contains the Manin-Mumford Conjecture for $\mathbf{G}_m^n$.

Conjecture 2.1 is known also if $\dim \mathcal{Y} = n-2$ due to work of Bombieri, Masser, and Zannier [BMZ07].

In low dimension, Maurin [Mau08] proved the conjecture for curves defined over $\overline{\mathbf{Q}}$. Bombieri, Masser, and Zannier [BMZ08a] later generalized this to curves defined over $\mathbf{C}$.

A promising line of attack of Conjecture 2.1 is via the theory of heights, which we will review in the next section. It is this approach that motivates our Theorem 1.3. In some circumstances it is possible to prove instances of the conjecture by first studying the larger union over algebraic subgroups that satisfy the weaker dimension inequality $\dim \mathcal{Y} + \dim H \leq n$. It is no longer appropriate to call non-empty intersections $\mathcal{Y} \cap H$ unlikely and one cannot expect

$$\bigcup_{\substack{H \subset \mathbf{G}_m^n \\ \dim \mathcal{Y} + \dim H \leq n}} \mathcal{Y} \cap H$$

to be non-Zariski dense in $\mathcal{Y}$. We say that such a non-empty intersection $\mathcal{Y} \cap H$ is *just likely*. It is sometimes possible to show that the absolute logarithmic Weil height is bounded from above on this union. In fact, we will use such a bound which we state more precisely below in Theorem 2.2.

To ease notation we abbreviate

$$(\mathbf{G}_m^n)^{[m]} = \bigcup_{\substack{H \subset \mathbf{G}_m^n \\ \mathrm{codim} H \geq m}} H.$$

Two caveats are in order. First, in order to use the height in a meaningful way we need to work with subvarieties $\mathcal{Y}$ defined over $\overline{\mathbf{Q}}$, the field of algebraic numbers. Therefore, results on height bounds usually contain an additional hypothesis on the field of definition. Second, it is in general false that the elements of

$$(\mathbf{G}_m^n)^{[\dim \mathcal{Y}]} \cap \mathcal{Y}$$

have uniformly bounded height. Indeed, it is possible that $\mathcal{Y}$ has positive dimensional intersection with an algebraic subgroup of dimension $\dim \mathcal{Y}$. As the height is not bounded on a positive dimensional subvariety of $\mathbf{G}_m^n$ we must avoid such intersections.

We will see in moment that there are more delicate obstructions to boundedness of height. One must remove more as was pointed out by Bombieri, Masser, and Zannier [BMZ99]. They proved the following height-theoretic result for an irreducible algebraic curve $\mathcal{C}$ defined over $\overline{\mathbf{Q}}$ and contained in $\mathbf{G}_m^n$. A coset of $\mathbf{G}_m^n$ will mean the translate of an algebraic subgroup of $\mathbf{G}_m^n$. If $\mathcal{C}$ is not contained in a proper coset, then a point in $\mathcal{C}$ that is contained in a proper algebraic subgroup has height bounded in terms of $\mathcal{C}$ only. They also proved a converse in the second remark after their Theorem 1. If $\mathcal{C}$ is contained in a proper coset, then $\mathcal{C} \cap (\mathbf{G}_m^n)^{[1]}$ does not have bounded height. Observe that $\mathcal{C} \cap (\mathbf{G}_m^n)^{[1]}$ is always infinite.

The second named author later in [Hab09] proved a qualitative refinement of these height bounds for the intersection of a general subvariety $\mathcal{Y} \subset \mathbf{G}_m^n$ with algebraic subgroups of complementary dimension. An irreducible closed subvariety $\mathcal{Z} \subset \mathcal{Y}$ is called *anomalous* if there exists a coset $\mathcal{K} \subset \mathbf{G}_m^n$ with $\mathcal{Z} \subset \mathcal{K}$ and

$$\dim \mathcal{Z} \geq \max\{1, \dim \mathcal{Y} + \dim \mathcal{K} - n + 1\}.$$

Bombieri, Masser, and Zannier [BMZ07] showed that the (possibly infinite) union of all anomalous subvarieties is Zariski closed in $\mathcal{Y}$. We write $\mathcal{Y}^{\text{oa}}$ for its complement in $\mathcal{Y}$. The aforementioned result states that $\mathcal{Y}^{\text{oa}}$ is Zariski open in $\mathcal{Y}$.

In the case of a curve $\mathcal{C}$, we have $\mathcal{C}^{\text{oa}} = \mathcal{C}$ if and only if $\mathcal{C}$ is not contained in a proper coset. Otherwise we have $\mathcal{C}^{\text{oa}} = \emptyset$. Thus Bombieri, Masser, and Zannier's original height bound for curves [BMZ99] states that $\mathcal{C}^{\text{oa}} \cap (\mathbf{G}_m^n)^{[1]}$ has bounded height.

The following bound is the main theorem of [Hab09].

Theorem 2.2. *Let $\mathcal{Y} \subset \mathbf{G}_m^n$ be an irreducible closed subvariety defined over $\overline{\mathbf{Q}}$. There exists $B \in \mathbf{R}$ depending only on $\mathcal{Y}$ such that any point in $\mathcal{Y}^{\text{oa}} \cap (\mathbf{G}_m^n)^{[\dim \mathcal{Y}]}$ has absolute logarithmic Weil height bounded by B.*

After introducing more notation we will cite a quantitative version of Theorem 2.2 for curves in Theorem 2.10.

The height bound can be used to recover some cases of Zilber's Conjecture. The second named author later made this result completely explicit [Hab]. The height bound B is thus effective.

Before proceeding to the main result of this section we make, as promised, a brief detour to define the height function mentioned above and several others.

2.2 On Heights

We refer to the Chapter 1.5 [BG06] or Parts B.1 and B.2 [HS00] for proofs of many basic properties of the absolute logarithmic Weil height that we discuss in this section.

Every non-trivial absolute value $|\cdot|_v$ on a number field K is equivalent to one of the following type. If $|\cdot|_v$ is Archimedean, then there exists a field embedding $\sigma: K \to \mathbf{C}$, uniquely defined up to complex conjugation, such that $|x|_v = |\sigma(x)|$ for all $x \in K$, where $|\cdot|$ is the standard complex absolute value. In this case we call v infinite and write $v|\infty$. Depending on whether $\sigma(K) \subset \mathbf{R}$ or not we define the local degree of v as $d_v = 1$ or $d_v = 2$. If $|\cdot|_v$ is non-Archimedean, then its restriction to $\mathbf{Q}$ is the p-adic absolute value for some rational prime p. For fixed p, the set of extensions of the p-adic valuation to K is in bijection with the set of prime ideals in the ring of algebraic integers in K that contain the prime ideal $p\mathbf{Z}$. In this case we call v finite and write $v \nmid \infty$ or $v \mid p$. The local degree here is $d_v = [K_v : \mathbf{Q}_p]$ where K_v is a completion of K with respect to v. We write V_K for the set of all absolute values $|\cdot|_v$ on K as described above. This set is sometimes called the set of places of K.

We note that if $x \in K \smallsetminus \{0\}$, then $|x|_v = 1$ for all but finitely many $v \in V_K$. The choice of local degrees d_v facilitates the product formula

$$\prod_{v \in V_K} |x|_v^{d_v} = 1. \tag{2.2}$$

Now we are ready to defined the absolute logarithmic Weil height, or height for short, of a tuple $x = (x_1, \ldots, x_n) \in K^n$ as

$$h(x) = \frac{1}{[K : \mathbf{Q}]} \sum_{v \in V_K} d_v \log \max\{1, |x_1|_v, \ldots, |x_n|_v\} \geq 0. \tag{2.3}$$

The normalization constants $d_v/[K : \mathbf{Q}]$ guarantee that $h(x)$ does not change when replacing K by another number field containing all x_i. So we obtain a well-defined function $h: \overline{\mathbf{Q}}^n \to [0, \infty)$.

Northcott's Theorem, Theorem 1.6.8 [BG06], states that a subset of $\overline{\mathbf{Q}}^n$ whose elements have uniformly bounded height and degree over $\mathbf{Q}$ is finite. This basic result is an important tool for proving finiteness results in diophantine geometry. We will apply it in the proof of Theorem 1.1.

In the special case $n = 1$ the following estimates will prove useful. If $x, y \in \overline{\mathbf{Q}}$ then both inequalities

$$h(xy) \leq h(x) + h(y) \quad \text{and} \quad h(x + y) \leq h(x) + h(y) + \log 2,$$

follow from corresponding local inequalities applied to the definition (2.3). The height, taking no negative values, does not restrict to a group homomorphism $\overline{\mathbf{Q}} \smallsetminus \{0\} \to \mathbf{R}$. However, the definition and the product formula yield homogenity

$$h(x^k) = |k| h(x)$$

for any integer k if $x \neq 0$.

It is sometimes useful to work with the height of algebraic points in projective space. If $x = [x_0 : \cdots : x_n] \in \mathbf{P}^n$ is such a point with representatives $x_0, \ldots, x_n$ in K, we set

$$h(x) = \frac{1}{[K : \mathbf{Q}]} \sum_{v \in V_K} d_v \log \max\{|x_0|_v, \ldots, |x_n|_v\}.$$

The product formula (2.2) guarantees that $h(x)$ does not depend on the choice of projective coordinates of x.

If f is a non-zero polynomial in algebraic coefficients, we set $h(f)$ to be the height of the point in projective space whose coordinates are the non-zero coefficients of f.

We remark that different sources in the literature may employ different norms at the Archimedean places of K. For example, instead of taking the ℓ^∞-norm one can take the ℓ^2-norm at the infinite places. This leads to another height function $h_2(\cdot)$ on the algebraic points of $\mathbf{P}^n$ which differs from $h(\cdot)$ by a bounded function.

We will make use of a result of Silverman to control the behavior of the height function under rational maps between varieties.

Theorem 2.3. *Let $\mathcal{X} \subset \mathbf{A}^m$ and $\mathcal{Y} \subset \mathbf{A}^n$ be irreducible quasi-affine varieties defined over $\overline{\mathbf{Q}}$ with $\dim \mathcal{X} = \dim \mathcal{Y}$. Suppose that $\varphi: \mathcal{X} \to \mathcal{Y}$ is a dominant morphism. There exist constants $c_1 > 0$ and c_2 that depend only on $\mathcal{X}, \mathcal{Y}$ and a Zariski open and dense subset $U \subset \mathcal{X}$ such that*

$$h(\varphi(P)) \geq c_1 h(P) - c_2 \quad \textit{for all} \quad P \in U.$$

Moreover, this estimate holds true with

$$U = U_0 = \{P \in \mathcal{X} : \ P \textit{ is isolated in } \varphi^{-1}(\varphi(P))\},$$

which is Zariski open in $\mathcal{X}$.

Proof. The first statement follows from Silverman's Theorem 1 [Sil11].

The openness of U_0 from the last statement follows from Exercise II.3.22(d) [Har77]. By restricting to the irreducible components in the complement of the open set provided by Silverman's Theorem we may use Noetherian induction to prove the height inequality on U_0 with possibly worse constants. ∎

A reverse inequality, i.e.

$$h(\varphi(P)) \leq c_1^{-1} h(P) + c_2 \tag{2.4}$$

for any $P \in \mathcal{X}$ holds with possibly different constants. It requires neither φ being dominant or $\dim \mathcal{X} = \dim \mathcal{Y}$, and is more elementary, see e.g. [HS00, Theorem B.2.5].

It is also possible to assign a height to an irreducible closed subvariety $\mathcal{Y}$ of $\mathbf{P}^n$ defined over $\overline{\mathbf{Q}}$. The basic idea is to consider the Chow form of $\mathcal{Y}$, which is well-defined up-to scalar multiplication, as a point in some projective space. The height of this point with then be the height $h(\mathcal{Y})$ of $\mathcal{Y}$. In this setting it is common to use a norm at the Archimedean place which is related to the Mahler measure of a polynomial. The details of this definition are presented in Philippon's paper [Phi95].

With this normalization, the height of a singleton $\{P\}$ with P an algebraic point of $\mathbf{P}^n$ is the height of P with the ℓ^2-norm at the Archimedean places. Beware that the height of a projective variety is by no means an invariant of its isomorphism class. It depends heavily on the embedding $\mathcal{Y} \subset \mathbf{P}^n$.

Zhang's inequalities [Zha95] relate the height of $\mathcal{Y} \subset \mathbf{P}^n$, its degree, and the points of small height on $\mathcal{Y}$. In order to state them, we require the essential minimum

$$\mu^{\mathrm{ess}}(\mathcal{Y}) = \inf\Big\{x \geq 0 : \{P \in \mathcal{Y} : h_2(P) \leq x\} \text{ is Zariski dense in } \mathcal{Y}\Big\}$$

of $\mathcal{Y}$. The set in the infimum is non-empty and so $\mu^{\mathrm{ess}}(\mathcal{Y}) < +\infty$. In connection with the Bogomolov Conjecture Zhang proved

$$\mu^{\mathrm{ess}}(\mathcal{Y}) \leq \frac{h(\mathcal{Y})}{\deg \mathcal{Y}} \leq (1 + \dim \mathcal{Y})\mu^{\mathrm{ess}}(\mathcal{Y}). \tag{2.5}$$

The second inequality can be used to bound $h(\mathcal{Y})$ from above if one can exhibit a Zariski dense set of points on $\mathcal{Y}$ whose height is bounded from above by a fixed value.

The morphism $(x_1, \ldots, x_n) \mapsto [1 : x_1 : \cdots : x_n]$ allows us to consider $\mathbf{G}_m^n$ and $\mathbf{A}^n$ as open subvarieties of $\mathbf{P}^n$. The height of an irreducible closed subvariety of $\mathbf{G}_m^n$ or $\mathbf{A}^n$ defined over $\overline{\mathbf{Q}}$ is the height of its Zariski closure in $\mathbf{P}^n$.

We recall that $\deg \mathcal{Y}$ is the cardinality of the intersection of $\mathcal{Y}$ with a linear subvariety of $\mathbf{P}^n$ in general position with codimension $\dim \mathcal{Y}$. By taking the Zariski closure in $\mathbf{P}^n$ as in the previous paragraph we may speak of the degree of any irreducible closed subvariety of $\mathbf{G}_m^n$ or $\mathbf{A}^n$.

If $\mathcal{X}$ is a second irreducible closed subvariety of $\mathbf{P}^n$, then Bézout's Theorem states

$$\sum_{\mathcal{Z}} \deg \mathcal{Z} \leq (\deg \mathcal{X})(\deg \mathcal{Y})$$

where $\mathcal{Z}$ runs over all irreducible components $\mathcal{Z}$ of $\mathcal{X} \cap \mathcal{Y}$. For a proof we refer to Example 8.4.6 [Ful84].

We come to the arithmetic counterpart of this classical result. According to Arakelov theory, $h(\mathcal{Y})$ is the arithmetic counterpart of the geometric degree $\deg \mathcal{Y}$.

Theorem 2.4 (Arithmetic Bézout Theorem)**.** *There exists a positive and effective constant $c > 0$ that depends only on n and satisfies the following property. Let $\mathcal{X}$ and $\mathcal{Y}$ be irreducible closed subvarieties of $\mathbf{P}^n$, both defined over $\overline{\mathbf{Q}}$, then*

$$\sum_{\mathcal{Z}} h(\mathcal{Z}) \leq \deg(\mathcal{X})h(\mathcal{Y}) + \deg(\mathcal{Y})h(\mathcal{X}) + c\deg(\mathcal{X})\deg(\mathcal{Y})$$

where $\mathcal{Z}$ runs over all irreducible components of $\mathcal{X} \cap \mathcal{Y}$.

Proof. For a proof we refer to Philippon's Theorem 3 [Phi95]. ■

Not surprisingly, the height of a hypersurface is closely related to the height of a defining equation. For our purposes it suffices to have the following estimate.

Proposition 2.5. *There exists a positive and effective constant $c > 0$ that depends only on n and satisfies the following property. Let $f \in \overline{\mathbf{Q}}[X_0, \ldots, X_n]$ be a homogeneous, irreducible polynomial and suppose that $\mathcal{Y}$ is its zero set in $\mathbf{P}^n$. Then $h(\mathcal{Y}) \leq h(f) + c\deg f$.*

Proof. See page 347 of Philippon's paper [Phi95] for a more precise statement. ■

We will freely apply Zhang's inequalities and the Arithmetic Bézout Theorem to subvarieties $\mathbf{G}_m^n$ and $\mathbf{A}^n$, always keeping in mind the open immersions $\mathbf{G}_m^n \to \mathbf{P}^n$ and $\mathbf{A}^n \to \mathbf{P}^n$.

2.3 A Weak Height Bound

In this section, we will formulate and prove a height bound which is reminiscent of the result on just likely intersections in Theorem 2.2. But instead of working in the ambient group $\mathbf{G}_m^n$, we work instead in $\mathbf{G}_m^n \times \mathbf{G}_a^n$. We will also restrict to surfaces. The results of this section will be applied in the proof of Theorem 1.1. Our new height bound will only take a certain class of algebraic subgroups into account. It will also no longer be uniform, as it will depend logarithmically on the degree over $\mathbf{Q}$ of the point in question. However, the points in our application are known to have bounded degree over the rationals. Therefore, their height and degree are bounded from above. Northcott's Theorem will imply that the number of points under consideration is finite.

Let us consider an irreducible, quasi-affine surface $\mathcal{Y} \subset \mathbf{A}^k$ defined over $\overline{\mathbf{Q}}$ with two collections of functions. For $1 \leq i \leq n$ let $R_i: \mathbf{A}^k \dashrightarrow \mathbf{G}_m$ and let $\ell_i: \mathbf{A}^k \dashrightarrow \mathbf{G}_a$ be rational maps, defined on Zariski open and dense subsets of $\mathbf{A}^k$. We also suppose that their restrictions to $\mathcal{Y}$ (denoted by the same letter) are regular. The main theorem of this section is a height bound for points on $\mathcal{Y}$ that satisfy both a multiplicative relation among the R_i and a linear relation among the ℓ_i, with the same coefficients. We write $R: \mathcal{Y} \to \mathbf{G}_m^n$ and $\ell: \mathcal{Y} \to \mathbf{G}_a^n$ for the product maps.

In the theorem below, we will suppose that $R: \mathcal{Y} \to \mathbf{G}_m^n$ has finite fibers. Then $\mathcal{S} = \overline{R(\mathcal{Y})} \subset \mathbf{G}_m^n$ is a surface. Here and below $\overline{\cdot}$ refers to closure with respect to the Zariski topology.

Theorem 2.6. *Let us keep the assumptions introduced before. There is an effective constant $c > 0$ depending only on $\mathcal{Y}$, the ℓ_i, and the R_i with the following property. Suppose $y \in \mathcal{Y}$ is such that $R(y) \in \mathcal{S}^{\mathrm{oa}}$. If there is $(b_1, \ldots, b_n) \in \mathbf{Z}^n \smallsetminus \{0\}$*

(i) with $b_1\ell_1 + \cdots + b_n\ell_n \neq 0$ in the function field of $\mathcal{Y}$,

(ii) such that y is contained in an irreducible curve $\mathcal{C}_b$ cut out on $\mathcal{Y}$ by

$$b_1\ell_1 + \cdots + b_n\ell_n = 0$$

with $\overline{R(\mathcal{C}_b)}^{\mathrm{oa}} = \overline{R(\mathcal{C}_b)}$,

(iii) and

$$R_1(y)^{b_1} \cdots R_n(y)^{b_n} = 1, \tag{2.6}$$

then

$$h(y) \leq c \log(2[\mathbf{Q}(y) : \mathbf{Q}]).$$

Recall that the condition on $\overline{R(\mathcal{C}_b)}$ in (ii) stipulates that the said curve is not contained in a proper coset of $\mathbf{G}_m^n$.

At the end of this section we will provide another formulation for this theorem which is more in line with known results towards Zilber's Conjecture. The formulation at hand was chosen with our application to Teichmüller curves in mind.

The theorem is effective in the sense that one can explicitly express c in terms of $\mathcal{Y}$.

The proof splits up into two cases.

1. In the first case we forget about the additive relation in (2.6) but assume that there is an additional multiplicative relation. This will lead to a bound for the height that is independent of $[\mathbf{Q}(y) : \mathbf{Q}]$.

2. Second, we assume that there is precisely one multiplicative relation up-to scalars. This time we need the additive equation in (2.6) and we will obtain a height bound that depends on $[\mathbf{Q}(y) : \mathbf{Q}]$.

We remark that Theorem 1.1 uses Theorem 2.6 applied to $n = 3$. The latter relies on height bound in Theorem 2.2. In this low dimension, $\mathcal{S} \setminus \mathcal{S}^{\mathrm{oa}}$ coincides with the union of all positive dimensional cosets contained completely in $\mathcal{S}$. It turns out that the result of Bombieri-Zannier, cf. Appendix of [Sch00], can be used instead of Theorem 2.2. One could also use Theorem 1 [Hab08] in the case $n = 3$, $s = 2$, and $m = 1$ to obtain a completely explicit height bound while avoiding the crude bound of [Hab]. For general n it seems that Theorem 2.2 is indispensable.

Lemma 2.7. *There exist effective constants c_1, c_2 depending only on $\mathcal{Y}$ with $c_1 > 0$ such that if $y \in \mathcal{Y}$ then*

$$h(y) \leq c_1 h(R(y)) + c_2. \tag{2.7}$$

Proof. This statement follows from Theorem 2.3 as R has finite fibers on $\mathcal{Y}$. One checks that readily that Silverman's second proof is effective. ∎

We use $|\cdot|$ to denote the ℓ^∞-norm on any power of $\mathbf{R}$. Let us recall the following basic result called Dirichlet's Theorem on Simultaneous Approximation.

Lemma 2.8. *Let $\theta \in \mathbf{R}^n$ and suppose $Q > 1$ is an integer. There exist $q \in \mathbf{Z}$ and $p \in \mathbf{Z}^n$ with $1 \leq q < Q^n$ and $|q\theta - p| \leq 1/Q$.*

Proof. See Theorem 1A in Chapter II, [Sch80]. ∎

If $r = (r_1, \ldots, r_n) \in \mathbf{G}_m^n$ is any point and $b = (b_1, \ldots, b_n)$ then we abbreviate $r_1^{b_1} \cdots r_n^{b_n}$ by r^b.

Lemma 2.9. *There is an effective constant $c > 0$ depending only on n with the following property. Let $d \geq 1$ and suppose $r \in (\mathbf{G}_m^n)^{[1]}$ is algebraic with $[\mathbf{Q}(r) : \mathbf{Q}] \leq d$. There exists $b \in \mathbf{Z}^n$ with $|b| \leq cd^{2n} \max\{1, h(r)\}^n$ such that r^b is a root of unity.*

Proof. Let $Q > 1$ be a sufficiently large integer to be fixed later on. Since r is contained in a proper algebraic subgroup of $\mathbf{G}_m^n$ there is $b' \in \mathbf{Z}^n \setminus \{0\}$ with $r^{b'} = 1$.

By Dirichlet's Theorem, Lemma 2.8, there exists $b \in \mathbf{Z}^n$ and an integer q with $1 \leq q < Q^n$ such that $|qb'/|b'| - b| \leq Q^{-1}$. We remark that $b \neq 0$ since $Q > 1$. Moreover, $|b| \leq q + Q^{-1} < Q^n + 1$ by the triangle inequality. Hence $|b| \leq Q^n$ since $|b|$ and Q^n are integers.

With $\delta = |b'|b - qb' \in \mathbf{Z}^n$ we have

$$r^{|b'|b} = r^{\delta + qb'} = r^\delta.$$

The height estimates mentioned above yield

$$|b'|h(z) \leq |\delta|(h(r_1) + \cdots + h(r_n)) \leq n|\delta|h(r),$$

where $z = r^b$. We divide by $|b'|$ and find $h(z) \le nQ^{-1}h(r)$.

We note that $z \in \mathbf{Q}(r) \setminus \{0\}$ and recall $[\mathbf{Q}(r) : \mathbf{Q}] \le d$. By Dobrowolski's Theorem [Dob79], which is effective, we have either $h(z) = 0$ or $h(z) \ge c'd^{-2}$ for some absolute constant $c' \in (0, 1]$. Observe that we do not need the full strength of Dobrowolski's bound. The choice $Q = [2nd^2 \max\{1, h(r)\}/c']$ forces z to be a root of unity. The lemma follows with $c = (2n/c')^n$. ■

We are now almost ready to prove our main result. It relies on the following explicit height bound.

Theorem 2.10 ([Hab08]). *Suppose $\mathcal{C} \subset \mathbf{G}_m^n$ is an irreducible algebraic curve defined over $\overline{\mathbf{Q}}$ that is not contained in a coset of $\mathbf{G}_m^n$. Any point in $\mathcal{C} \cap (\mathbf{G}_m^n)^{[1]}$ has height at most*

$$c(\deg \mathcal{C})^{n-1}(\deg \mathcal{C} + h(\mathcal{C}))$$

where $c > 0$ is effective and depends only on n.

Proof of Theorem 2.6. Suppose $y \in \mathcal{Y}$ is as in the hypothesis and $d = [\mathbf{Q}(y) : \mathbf{Q}]$. In particular, (2.6) holds for some $b \in \mathbf{Z}^n \setminus \{0\}$. As discussed in the introduction, we split up into two cases.

In the first case, suppose the point $r = R(y)$ satisfies two independent multiplicative relations. Then Theorem 2.2 applies because $r \in \mathcal{S}^{\mathrm{oa}}$ by hypothesis. Since R has finite fibers Lemma 2.7 implies that the height of y is bounded from above solely in terms of $\mathcal{Y}$. This is stronger than the conclusion of the theorem.

In the second case, we will assume that the coordinates of r satisfy precisely one multiplicative relation up-to scalar multiple. Here we shall make use of the additive relation in (2.6). By assumption, the group

$$\{a \in \mathbf{Z}^n : r^a = 1\}$$

is free abelian of rank 1. It certainly contains b from the multiplicative relation in (2.6). However, it also contains a positive multiple of a vector $b' \in \mathbf{Z}^n \setminus \{0\}$ coming from Lemma 2.9. Thus b and b' are linearly dependent and hence the additive relation (2.6) holds with b replaced by $b' = (b'_1, \ldots, b'_n)$. By hypothesis (i) our point y lies on an irreducible curve $\mathcal{C} \subset \mathcal{Y}$ on which

$$b'_1\ell_1 + \cdots + b'_n\ell_n \tag{2.8}$$

vanishes identically with $\mathcal{C}^{\mathrm{oa}} = \mathcal{C}$.

Recall that the curve $\mathcal{C}$ is an irreducible component of the zero set of (2.8) on $\mathcal{Y}$. Each ℓ_i can be expressed by a quotient of polynomials mappings. From this point of view, $b'_1\ell_1 + \cdots + b'_n\ell_n$ is a quotient of polynomials whose degrees are bounded by a quantity that is independent of $(b'_1, \ldots, b'_n)$. So Bézout's Theorem implies that the degree of the Zariski closure of $\mathcal{C}$ in $\mathbf{A}^k$ is bounded from above in terms of $\mathcal{Y}$ only. We observe that $\overline{R(\mathcal{C})}$ is an irreducible curve. As $\deg \overline{R(\mathcal{C})}$ equals the generic number of intersection points of $\overline{R(\mathcal{C})}$ with a hyperplane, we conclude, again using Bézout's Theorem, that

$$\deg \overline{R(\mathcal{C})} \ll 1, \tag{2.9}$$

where here and below $\ll$ signifies Vinogradov's notation with a constant that depends only on $\mathcal{Y}$, the ℓ_i, and the R_i. These constants are effective.

We also require a bound for the height of the curve $\overline{R(\mathcal{C})}$. This we can deduce with the help of Zhang's inequalities (2.5). Indeed, the numerator of (2.8) is a polynomial whose height is $\ll \log(2|b'|)$ by elementary height inequalities. Any irreducible component of its zero set has height $\ll \log(2|b'|)$ by Proposition 2.5 and degree $\ll 1$. The Arithmetic Bézout Theorem implies $h(\overline{\mathcal{C}}) \ll \log(2|b'|)$. Using the degree bound we deduced above and the first inequality in (2.5) we conclude that $\overline{\mathcal{C}}$ contains a Zariski dense set of points P with $h(P) \le h_2(P) \ll \log(2|b'|)$. The height bound (2.4) just below Silverman's result yields $h(R(P)) \ll \log(2|b'|)$. So the second bound in (2.5) and $\deg \overline{R}(\mathcal{C}) \ll 1$ give

$$h(\overline{R(\mathcal{C})}) \ll \log|b'|. \tag{2.10}$$

Now $r = R(y) \in \overline{R(\mathcal{C})}$ and $r \in (\mathbf{G}_m^n)^{[1]}$ by the original multiplicative relation (2.6). We insert (2.9) and (2.10) into Theorem 2.10 and use the upper bound for $|b'|$ to find

$$h(r) \ll \log(2d \max\{1, h(r)\}).$$

Linear beats logarithmic, so $h(r) \ll \log(2d)$. Finally, we use Lemma 2.7 again to deduce $h(y) \ll \log(2d)$. This completes the proof. ∎

2.4 Intersecting with algebraic subgroups of $\mathbf{G}_m^n \times \mathbf{G}_a^n$

The unipotent group $\mathbf{G}_a^n$ is not covered by Conjecture 2.1 or Zilber's more general formulation for semi-abelian varieties. Indeed, a verbatim translation of the statement of Conjecture 2.1 to $\mathbf{G}_a^n$ fails badly. Any point of $\mathbf{G}_a^n$ is contained in a line passing through the origin, and is thus in a 1-dimensional algebraic subgroup.

Motivated by Theorems 2.2 and 2.6 we will deduce a height bound for points on a surface inside $\mathbf{G}_m^n \times \mathbf{G}_a^n$ which are contained in a *restricted* class of algebraic subgroups of codimension 2. Our aim is to formulate a result that is comparable to the more well-known case of the algebraic torus. The reader whose main interest lies in proof of Theorem 1.1 may safely skip this section.

Any algebraic subgroup of $\mathbf{G}_m^n \times \mathbf{G}_a^n$ splits into the product of an algebraic subgroup of $\mathbf{G}_m^n$ and of $\mathbf{G}_a^n$. We call the translate of an algebraic subgroup of $\mathbf{G}_m^n \times \mathbf{G}_a^n$ by a point in $\mathbf{G}_m^n \times \{0\}$ a semi-torsion coset. We call it rational if it is the translate of an algebraic subgroup of $\mathbf{G}_m^n \times \mathbf{G}_a^n$ defined over $\mathbf{Q}$ by any point of $\mathbf{G}_m^n \times \{0\}$. A rational semi-torsion coset need not be defined over $\mathbf{Q}$, but its associated algebraic subgroup of $\mathbf{G}_a^n$ is defined by linear equations with rational coefficients.

Let $\mathcal{Y}$ be an irreducible subvariety of $\mathbf{G}_m^n \times \mathbf{G}_a^n$ defined over $\mathbf{C}$. We single out an exceptional class of subvarieties of $\mathcal{Y}$ related to Bombieri, Masser, and Zannier's anomalous subvarieties [BMZ07].

We say that an irreducible closed subvariety $\mathcal{Z}$ of $\mathcal{Y}$ is *rational semi-torsion anomalous* if it is contained in a rational semi-torsion coset $\mathcal{K} \subset \mathbf{G}_m^n \times \mathbf{G}_a^n$ with

$$\dim \mathcal{Z} \ge \max\{1, \dim \mathcal{Y} + \dim \mathcal{K} - 2n + 1\}. \tag{2.11}$$

We let $\mathcal{Y}^{\mathbf{Q},\mathrm{ta}}$ denote the complement in $\mathcal{Y}$ of the union of all rational semi-torsion anomalous subvarieties of $\mathcal{Y}$.

Bombieri, Masser, and Zannier's $\mathcal{Y}^{\mathrm{oa}}$ for $\mathcal{Y} \subset \mathbf{G}_m^n$ is always Zariski open. In the example below we show that this is not necessarily the case for $\mathcal{Y}^{\mathbf{Q},\mathrm{ta}}$ if $\mathcal{Y} \subset \mathbf{G}_m^n \times \mathbf{G}_m^a$ is a surface.

Example 2.11. Let us consider the case $n = 2$ and let $\mathcal{Y}$ be the irreducible surface given by

$$\begin{aligned} x_1 y_1 + (x_1 + 1) y_2 &= 0, \\ x_1 y_1 + x_2 y_2 &= 1 \end{aligned} \tag{2.12}$$

where $(x_1, x_2, y_1, y_2) \in \mathbf{G}_m^2 \times \mathbf{G}_a^2$. Observe that the projection of $\mathcal{Y}$ to $\mathbf{G}_m^2$ is dominant. We will try to understand some features of $\mathcal{Y}^{\mathbf{Q},\mathrm{ta}}$.

Let $\mathcal{K}$ be a rational semi-torsion anomalous subvariety of $\mathcal{Y}$. Then a certain number of additive and multiplicative relations hold on the coordinates of $\mathcal{K}$ and the dimension of $\mathcal{K}$ cannot be below the threshold determined by (2.11).

Suppose first that a relation $b_1 y_1 + b_2 y_2 = 0$ holds on $\mathcal{K}$ where $(b_1, b_2) \in \mathbf{Z}^2 \smallsetminus \{0\}$. The second equality in (2.12) yields $(y_1, y_2) \neq 0$, and the first one yields $0 = x_1 b_2 - (x_1 + 1) b_1$, so the projection of $\mathcal{K}$ to $\mathbf{G}_m^2$ maps to one of countably many algebraic curves. In particular, $\mathcal{K}$ is a curve and there are at most countably many possibilities for $\mathcal{K}$.

Second, let us assume that no linear relation as above holds on $\mathcal{K}$. Then a certain number of multiplicative relations $x_1^{b_1} x_2^{b_2} = \lambda$ hold on $\mathcal{K}$. We cannot have $\mathcal{K} = \mathcal{Y}$, as $\mathcal{Y}$ has dense image in $\mathbf{G}_m^2$. So there must be two multiplicative relations with independent exponent vectors for $\mathcal{K}$ to be anomalous. In particular, x_1 and x_2 are constant on $\mathcal{K}$. But for a fixed choice of (x_1, x_2) the two linear equations (2.12) are linear in (y_1, y_2) and have at most one solution in these unknowns. This contradicts $\dim \mathcal{K} \geq 1$.

Now we know that any rational semi-torsion anomalous subvariety of $\mathcal{Y}$ is a curve and that their cardinality is at most countable.

Finally, let us exhibit such curves. For a given $\xi \in \mathbf{Q} \smallsetminus \{0\}$ the equation $x_1 = \xi$ cuts out an irreducible curve in $\mathcal{Y}$. This equation and the one obtained by substituting ξ for x_1 in first line of (2.12) establishes that this curve is rational semi-torsion anomalous.

Thus $\mathcal{Y} \smallsetminus \mathcal{Y}^{\mathbf{Q},\mathrm{ta}}$ is a countable, infinite union of curves. In particular, $\mathcal{Y}^{\mathbf{Q},\mathrm{ta}}$ is not Zariski open in $\mathcal{Y}$; it is also not open with respect to the Euclidean topology.

Above we introduced the notation x^b for a point $x \in \mathbf{G}_m^n$ and $b \in \mathbf{Z}^n$. If $y = (y_1, \ldots, y_n) \in \mathbf{G}_a^n$ and $b = (b_1, \ldots, b_n)$ we set

$$\langle y, b \rangle = y_1 b_1 + \cdots + y_n b_n.$$

An algebraic subgroup $G \subset \mathbf{G}_m^n \times \mathbf{G}_a^n$ is called *coupled* if there exists a subgroup $\Lambda \subset \mathbf{Z}^n$ with

$$G = \left\{ (x, y) \in \mathbf{G}_m^n \times \mathbf{G}_a^n : x^b = 1 \quad \text{and} \quad \langle y, b \rangle = 0 \quad \text{for all} \quad b \in \Lambda \right\}.$$

The dimension of G is $2(n - \operatorname{rank} \Lambda)$.

We define $(\mathbf{G}_m^n \times \mathbf{G}_a^n)^{[s]}$ to be the union of all coupled algebraic subgroups of $\mathbf{G}_m^n \times \mathbf{G}_a^n$ whose codimension is at least s.

Using this notation we have the following variant of Theorem 2.6.

Theorem 2.12. *Let $\mathcal{Y} \subset \mathbf{G}_m^n \times \mathbf{G}_a^n$ be an irreducible, closed algebraic surface defined over $\overline{\mathbf{Q}}$. There exists a constant $c > 0$ with the following property. If $P \in \mathcal{Y}^{\mathbf{Q},\mathrm{ta}} \cap (\mathbf{G}_m^n \times \mathbf{G}_a^n)^{[2]}$, then*

$$h(P) \leq c \log(2[\mathbf{Q}(P) : \mathbf{Q}]).$$

Proof. The current theorem resembles Theorem 2.6 but it is not a direct consequence. However, we will invoke Theorem 2.6 below. Indeed, we take $\mathcal{Y}$ as a quasi-affine subvariety of $\mathbf{G}_m^n \times \mathbf{G}_a^n \subset \mathbf{A}^{2n}$. The rational maps R and ℓ are the two projections $\mathbf{G}_m^n \times \mathbf{G}_a^n \to \mathbf{G}_m^n$ and $\mathbf{G}_m^n \times \mathbf{G}_a^n \to \mathbf{G}_a^n$, respectively.

Say $P = (x, y) \in \mathcal{Y}^{\mathbf{Q},\mathrm{ta}} \cap (\mathbf{G}_m^n \times \mathbf{G}_a^n)^{[2]}$. We begin a study of various cases.

Suppose first that $\overline{R(\mathcal{Y})} \subset \mathbf{G}_m^n$ is a point. The existence of P implies that $\mathcal{Y}$ meets a proper coupled subgroup. So $R(\mathcal{Y})$ is in a proper algebraic subgroup of $\mathbf{G}_m^n$ which means that $\mathcal{Y}$ is in the product of this subgroup with $\mathbf{G}_a^n$. In this case $\mathcal{Y}^{\mathbf{Q},\mathrm{ta}}$ is empty, a contradiction.

So we have $\dim \overline{R(\mathcal{Y})} \geq 1$. Let us assume that P is not isolated in its fiber of $R|_{\mathcal{Y}}$. Here we can argue much as in the proof of Theorem 2.6. The point P is contained in some irreducible component $\mathcal{D}_x$ of $R|_{\mathcal{Y}}^{-1}(x)$ with $\dim \mathcal{D}_x \geq 1$. But $\dim \mathcal{D}_x = 1$ since $R|_{\mathcal{Y}}$ is non-constant. Observe that $\mathcal{D}_x$ is an irreducible component in the intersection of $\mathcal{Y}$ and the rational semi-torsion coset $\{x\} \times \mathbf{G}_a^n$. We now split-up into 2 subcases.

Suppose first that $\overline{R(\mathcal{Y})}$ has dimension 1. Then $\overline{R(\mathcal{Y})}$ cannot be contained in a proper coset of $\mathbf{G}_m^n$ as $\mathcal{Y}^{\mathbf{Q},\mathrm{ta}} \neq \emptyset$. So $\overline{R(\mathcal{Y})} \cap (\mathbf{G}_m^n)^{[1]}$ has bounded height by Theorem 1 of Bombieri, Masser, and Zannier [BMZ99]. As x lies in this intersection we have $h(x) \ll 1$; here and below the constant implied in Vinogradov's notation depends only on $\mathcal{Y}$.

In the second subcase we suppose that $\overline{R(\mathcal{Y})}$ has dimension 2. The set of points in $\mathcal{Y}$ that are contained in a positive dimensional fiber of $R|_{\mathcal{Y}}$ is a Zariski closed proper subset of $\mathcal{Y}$, cf. Exercise II.3.22(d) [Har77] already used above. Hence $\mathcal{D}_x$ is a member of a finite set of curves depending only on $\mathcal{Y}$. So x, being the image of $\mathcal{D}_x$ under R, is member of a finite set depending only on $\mathcal{Y}$. In particular, $h(x) \ll 1$ holds trivially.

In both subcases we have $h(x) \ll 1$. The Arithmetic Bézout Theorem yields the height bound

$$h(\mathcal{D}_x) \ll 1 \quad \text{and} \quad \deg \mathcal{D}_x \ll 1 \tag{2.13}$$

the degree bound follows from the classical Bézout Theorem.

Let us abbreviate $d = [\mathbf{Q}(P) : \mathbf{Q}]$. The coordinates of x are multiplicatively dependent. But there cannot be 2 independent relations as $\mathcal{D}_x$ would otherwise be contained in rational semi-torsion anomalous subvariety of $\mathbf{G}_m^n \times \mathbf{G}_a^n$. Lemma 2.9 and $h(x) \ll 1$ implies $\langle y, b\rangle = 0$ for some $b \in \mathbf{Z}^n$ with $|b| \ll d^{2n}$.

The vanishing locus of the linear form

$$y \mapsto \langle y, b\rangle$$

determines a linear subvariety of $\mathbf{G}_m^n \times \mathbf{G}_a^n$ with height $\ll \log(2|b|)$. The point $P = (x, y)$ is isolated in its intersection with $\mathcal{D}_x$ as $P \in \mathcal{Y}^{\mathbf{Q},\mathrm{ta}}$. The Arithmetic Bézout Theorem and (2.13) yield $h(P) \ll \log(2|b|)$. We combine this bound with the upper bound for $|b|$ to establish the theorem if P is not isolated in the corresponding fiber of $R|_{\mathcal{Y}}$.

From now on we assume that P is isolated in $R|_{\mathcal{Y}}^{-1}(x)$. The set of all such points of $\mathcal{Y}$ is a Zariski open subset $\mathcal{Y}'$ of $\mathcal{Y}$. The restriction $R|_{\mathcal{Y}'}$ has finite fibers and hence the hypothesis leading up to Theorem 2.6 is fulfilled for $\mathcal{Y}'$ where the ℓ_i run over the n the projection morphisms to $\mathbf{G}_a$. We write $\mathcal{S}$ for the Zariski closure of $R(\mathcal{Y}')$; this is an irreducible surface.

Say $b \in \mathbf{Z}^n \smallsetminus \{0\}$ with $x^b = 1$ and $\langle y, b\rangle = 0$. The conditions (i), (ii), and (iii) in Theorem 2.6 are met; for the first two we need $P \in \mathcal{Y}^{\mathbf{Q},\mathrm{ta}}$. If $x \in \mathcal{S}^{\mathrm{oa}}$ holds, then the height bound from the said theorem completes the proof.

So it remains to treat the case $x \notin \mathcal{S}^{\mathrm{oa}}$. By definition there is a coset $\mathcal{K} \subset \mathbf{G}_m^n$ and an irreducible component $\mathcal{Z}$ of $\mathcal{S} \cap \mathcal{K}$ containing x with

$$\dim \mathcal{Z} \geq \max\{1, 3 + \dim \mathcal{K} - n\}. \tag{2.14}$$

This inequality implies $\dim \mathcal{K} \leq n-1$ because $\dim \mathcal{Z} \leq 2$.

Observe that $\dim \mathcal{Z} = 1$. Indeed, otherwise $\mathcal{Z} = \mathcal{S}$ would be contained in $\mathcal{K}$. Then $\mathcal{Y}$ would be contained in the rational semi-torsion coset $\mathcal{K} \times \mathbf{G}_a^n$ which would contradict $P \in \mathcal{Y}^{\mathbf{Q},\mathrm{ta}}$.

Since $\mathcal{Z}$ is a curve we find

$$\dim \mathcal{K} \leq n - 2 \tag{2.15}$$

from (2.14).

Of course $P = (x, y) \in R|_{\mathcal{Y}}^{-1}(\mathcal{Z})$. Let $\mathcal{Z}'$ be an irreducible component $R|_{\mathcal{Y}}^{-1}(\mathcal{Z})$ containing P with largest dimension. Now $\mathcal{Z}'$ is in the rational semi-torsion coset $\mathcal{K} \times \mathbf{G}_a^n$ already used above. If $\mathcal{Z}'$ has positive dimension, then $\dim \mathcal{Z}' \geq 2+\dim \mathcal{K}\times\mathbf{G}_a^n - 2n+1$ because of (2.15). But then $\mathcal{Z}'$ is a rational semi-torsion anomalous subvariety of $\mathcal{Y}$. This is again a contradiction to $P \in \mathcal{Y}^{\mathbf{Q},\mathrm{ta}}$.

We conclude that $\mathcal{Z}' = \{P\}$. This is an awkward situation as one would expect that the pre-image of a curve under the dominant morphism $R|_{\mathcal{Y}}\colon \mathcal{Y} \to \mathcal{S}$ between surfaces to be again a curve. So we can hope to extract useful information. We are in characteristic 0, so by Lemma III.10.5 [Har77] there is a Zariski open and non-empty set $U \subset \mathcal{Y}$ such that $R|_U\colon U \to \mathcal{S}$ is a smooth morphism. This restriction is in particular open. It has the property that the preimage of any irreducible curve in $R(U)$ is a finite union of irreducible curves. We claim that P does not lie in U. Indeed, otherwise P would be an isolated point of a fiber of $R|_U$. This contradicts smoothness of $R|_U$ as $R(U) \cap \mathcal{Z}$ is an irreducible curve containing $R(P)$.

The complement $\mathcal{Y} \smallsetminus U$ has dimension at most 1 and does not depend on P. It contains P by the previous paragraph. After omitting the finitely many isolated points in $\mathcal{Y} \smallsetminus U$ we may suppose that P is in a curve $\mathcal{C} \subset \mathcal{Y} \smallsetminus U$. Thus $\mathcal{C}$ arises from a finite set depending only on $\mathcal{Y}$.

The restriction $R|_{\mathcal{C}}\colon \mathcal{C} \to \mathbf{G}_m^n$ is non-constant because we already reduced to the case where P is isolated in the fiber of $R|_{\mathcal{Y}}$. So $\overline{R(\mathcal{C})}$, the Zariski closure of $R(\mathcal{C})$ in $\mathbf{G}_m^n$, is a curve. By Theorem 2.3 we have

$$h(P) \ll \max\{1, h(x)\}, \tag{2.16}$$

where the constant implicit in $\ll$ depends only on $\mathcal{C}$ and thus only on $\mathcal{Y}$.

If $\overline{R(\mathcal{C})}$ is not contained in a proper coset, then $h(x) \ll 1$ by Theorem 2.10 or by Bombieri, Masser, and Zannier's original height bound [BMZ99]. So (2.16) yields $h(P) \ll 1$ and this is better than what the theorem claims.

But what if $R(\mathcal{C})$ is contained in a proper coset of $\mathbf{G}_m^n$? As we have already pointed out, there is no hope that $\overline{R(\mathcal{C})} \cap (\mathbf{G}_m^n)^{[1]}$ has bounded height. But we know that $\overline{R(\mathcal{C})}$ is not contained in a coset of codimension at least two since $P \in \mathcal{Y}^{\mathbf{Q},\mathrm{ta}}$. The projection of $\mathcal{C}$ to a suitable choice of $n-1$ coordinates of $\mathbf{G}_m^{n-1}$ is a curve that is not in a proper coset. So if the coordinates of x happen to satisfy two independent multiplicative relations, then these $n-1$ coordinates will be multiplicatively dependent and thus have bounded height by Theorem 1 in [BMZ99]. Using Theorem 2.3, applied now to the projection, we can bound the height of the remaining coordinates. So $h(x) \ll 1$ and even $h(P) \ll 1$ by (2.16). Therefore, we may assume that the coordinates of x satisfies only one multiplicative relation, up to scalars. From here we proceed in a similar fashion as we have done several times before. We use Lemma 2.9 to deduce that b is linearly dependent to some $b' \in \mathbf{Z}^n \smallsetminus \{0\}$ with $|b'| \ll d^{2n} \max\{1, h(x)\}^n$. We certainly have $\langle y, b\rangle = \langle y, b'\rangle = 0$. The morphism $(x', y') \mapsto \langle y', b'\rangle$ does not vanish identically on $\mathcal{C}$ because $R(\mathcal{C})$ is already assumed to lie in a proper coset and since $P \in \mathcal{Y}^{\mathbf{Q},\mathrm{ta}}$. So P is an isolated point of $\mathcal{C} \cap \{(x', y') : \langle y', b'\rangle = 0\}$. A final application of the Arithmetic Bézout Theorem and (2.16) yield

$$h(P) \ll \log(2|b'|) \ll \log(2d \max\{1, h(P)\}). \tag{2.17}$$

The inequality (2.17) marks the final subcase in this proof and so the theorem is established. ■

3 Background on Teichmüller curves

We recall here some necessary background on flat surfaces and the stratification of $\Omega\mathcal{M}_g$. We also recall the cross-ratio equation for the cusps of algebraically primitive Teichmüller curves and the defining equation for the exponents appearing in this equation. These two equations were the motivation for the height bound theorem (Theorem 2.6) in the previous section.

Flat surfaces. A *flat surface* is a pair (X, ω), where X is a closed Riemann surface and ω a nonzero holomorphic one-form on X. The one-form ω gives X a flat metric $|\omega|$ which has cone points of cone angle $2\pi(n+1)$ at zeros of order n of ω. The metric $|\omega|$ is obtained by pulling back the metric $|dz|$ on $\mathbf{C}$ by local charts $\phi\colon U \to \mathbf{C}$ defined by integrating ω, defining an atlas on $X \setminus Z(\omega)$ (where $Z(\omega)$ is the set of zeros of ω) whose transition functions are translations of $\mathbf{C}$. There is an action of $\mathrm{SL}_2(\mathbf{R})$ on the moduli space of genus g flat surfaces $\Omega\mathcal{M}_g$, defined by postcomposing these charts with the standard linear action of $\mathrm{SL}_2(\mathbf{R})$ on $\mathbf{C} = \mathbf{R}^2$.

Let $\mathrm{Aff}^+(X, \omega)$ be the group of locally affine homeomorphisms of (X, ω), and let $D\colon \mathrm{Aff}^+(X, \omega) \to \mathrm{SL}_2(\mathbf{R})$ the homomorphism sending an affine map to its derivative. The *Veech group* of (X, ω) is the image $D\,\mathrm{Aff}^+(X, \omega)$, denoted by $\mathrm{SL}(X, \omega)$. The group $\mathrm{SL}(X, \omega)$ is always discrete in $\mathrm{SL}_2(\mathbf{R})$. If it is a lattice, we call (X, ω) a *Veech surface*. The $\mathrm{SL}_2(\mathbf{R})$ orbit of a Veech surface is closed in $\Omega\mathcal{M}_g$ and called a Teichmüller curve (we also refer to the projection of this orbit to $\mathbf{P}\Omega\mathcal{M}_g$ or $\mathcal{M}_g$ as a Teichmüller curve).

A *saddle connection* on a flat surface (X, ω) is an embedded geodesic segment connecting two zeros of ω. The foliation $\mathcal{F}_\theta$ of slope θ is said to be *periodic* if every leaf of $\mathcal{F}_\theta$ is either closed (i.e. a circle) or a saddle connection. In this case, we call θ a periodic direction. A periodic direction θ yields a decomposition of (X, ω) into finitely many maximal cylinders foliated by closed geodesics of slope θ. We refer to the length of the waist curve of the cylinder C as its *width* $w(C)$. The ratio of height over width is called the *modulus* $m(C) = h(C)/w(C)$. The complement of these cylinders is a finite collection of saddle connections.

We constantly use Veech's dichotomy [Vee89] stating that if (X, ω) is a Veech surface with either a closed geodesic or a saddle connection of slope θ, then the foliation $\mathcal{F}_\theta$ is periodic and the moduli of the cylinders in the direction θ are commensurable.

Given a Veech surface (X, ω) generating a Teichmüller curve $C \subset \mathbf{P}\Omega\mathcal{M}_g$, there is a natural bijection between the cusps of C and the periodic directions on (X, ω), up to the action of $\mathrm{SL}(X, \omega)$. The cusp associated to a periodic direction θ is the limit of the Teichmüller geodesic given by applying to (X, ω) the one-parameter subgroup of $\mathrm{SL}_2(\mathbf{R})$ contracting the direction θ and expanding the perpendicular direction. The stable form in $\mathbf{P}\Omega\overline{\mathcal{M}}_g$ which is the limit of this cusp is obtained by cutting each cylinder of slope θ along a closed geodesic and gluing a half-infinite cylinder to each resulting boundary component (see [Mas75]). These infinite cylinders are the poles of the resulting stable form, and the two poles resulting from a single infinite cylinder are glued to form a node.

The *spine* of (X, ω) is the union of all horizontal saddle connection, whose complement is a union of cylinders. The *dual graph* Γ is the graph whose vertices correspond to components of the spine, and edges correspond to complementary cylinders. Equivalently, Γ is the dual graph of the stable curve associated to this periodic direction. Its vertices correspond to irreducible components, and its edges correspond to nodes.

Strata of $\Omega\mathcal{M}_g$. The moduli space of flat surfaces $\Omega\mathcal{M}_g$ is stratified according to the number and multiplicities of zeros of ω. The stratum $\Omega\mathcal{M}_g(1^{2g-2})$ parameterizing flat surfaces with only simple zeros is open and dense. It is called the *principal stratum*.

Connected components of the strata were classified in [KZ03]. In genus three all the strata but $\Omega\mathcal{M}_3(4)$ and $\Omega\mathcal{M}_3(2,2)$ are connected. These two strata have two connected components, distinguished by the *parity of the spin structure* $h^0(X, \mathrm{div}(\omega)/2)$. We write $\Omega\mathcal{M}_3(4)^{\mathrm{hyp}}$ and $\Omega\mathcal{M}_3(4)^{\mathrm{odd}}$ for the hyperelliptic and odd components respectively. The case of even spin parity coincides in genus three with the hyperelliptic components, the component in which all the curves are hyperelliptic. Note that also $\Omega\mathcal{M}_3(2,2)^{\mathrm{odd}}$ contains hyperelliptic curves, forming a divisor in this stratum. In this case, the hyperelliptic involution fixes the zeros, while for a flat surface in $\Omega\mathcal{M}_3(2,2)^{\mathrm{hyp}}$ the hyperelliptic involution swaps the two zeros.

The family over a Teichmüller curve. Let $f\colon \mathcal{X} \to C$ be the universal family over a Teichmüller curve generated by a Veech surface (X, ω) (or possibly the pullback to an unramified covering of C). We also denote by $f\colon \overline{\mathcal{X}} \to \overline{C}$ the corresponding extension to a family of stable curves. Again passing to an unramified covering of C we may suppose that the zeros of ω define sections $s_j\colon \overline{C} \to \overline{\mathcal{X}}$ and we let $D_j = s_j(\overline{C})$ denote their images. In a fiber X of f we write z_j for the zeros of ω, that is is for the intersection of X with D_j. We also write just D for the section and z for the zero, if $k = 1$.

The eigenform locus and its degeneration. Let $\mathcal{RM}_\mathcal{O} \subset \mathcal{M}_g$ denote the locus of Riemann surfaces that admit real multiplication by an order $\mathcal{O}$ in a totally real number field F with $[F : \mathbf{Q}] = g$. In the bundle of one-forms over $\mathcal{RM}_\mathcal{O}$ there is the locus of *eigenforms* $\mathcal{E}_\mathcal{O} \subset \Omega\mathcal{M}_g$ consisting of pairs (X, ω), where $[X] \in \mathcal{RM}_\mathcal{O}$ and where ω is an eigenform for real multiplication. The intersection of the closure of $\mathcal{RM}_\mathcal{O}$ in $\overline{\mathcal{M}}_g$ with the boundary was described in [BM12]. We gave a necessary and sufficient condition for a stable form to lie in the boundary of $\mathcal{RM}_\mathcal{O}$ in genus three where there is no Schottky problem involved. We summarize these results here.

The irreducible stable curves in the boundary of $\mathcal{RM}_\mathcal{O}$ are *trinodal curves*, rational curves with three pairs of points x_i and y_i, $i = 1, 2, 3$ identified. Coordinates on the boundary component of trinodal curves in $\overline{\mathcal{M}}_3$ are given by the cross-ratios defined by

$$R_{jk} = [x_j, y_j, x_k, y_k]. \tag{3.1}$$

where for $z_1, \ldots z_4 \in \mathbf{C}$,

$$[z_1, z_2, z_3, z_4] = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}.$$

We often use complementary index notation, writing R_k for R_{ij} where $\{i, j, k\} = \{1, 2, 3\}$.

In general, cusps of Hilbert modular varieties are determined by the ideal class of a nonzero module $\mathcal{I}$ for the order $\mathcal{O}$ and an extension class E. In [BM12] we called a triple $(r_1, r_2, r_3) \in F$ *admissible*, if $\{r_1, r_2, r_3\}$ is a basis for such an ideal $\mathcal{I}$ and if $\left\{\frac{N(r_1)}{r_1}, \frac{N(r_2)}{r_2}, \frac{N(r_3)}{r_3}\right\}$ is $\mathbf{Q}^+$-linear dependent.

The boundary components of $\mathcal{RM}_\mathcal{O}$ intersected with the locus of trinodal curves are in bijection with projectivized admissible triples up to permutation and sign change. On the component given by an admissible triple (r_1, r_2, r_3), the closure of $\mathcal{RM}_\mathcal{O}$ is cut out by the cross-ratio equation

$$R_1^{a_1} R_2^{a_2} R_3^{a_3} = \zeta_E, \tag{3.2}$$

where ζ_E is a root of unity determined by the extension class E (see [BM12] for the precise definition, which is irrelevant in this paper) and where the exponents a_i are defined as follows.

Let (s_1, s_2, s_3) be dual to (r_1, r_2, r_3) with respect to the trace pairing on F, and let $(b_1, b_2, b_3) \in \mathbf{Z}^3$ be such that (indices read mod 3)

$$\sum_{i=1}^{1} b_i s_{i+1} s_{i+2} = 0,$$

as stated in [BM12, Proof of Theorem 8.5]. The existence of such b_i and the fact that $b_i \neq 0$ for all i is a consequence of admissibility. We let (a_1, a_2, a_3) be a tuple proportional to (b_1, b_2, b_3) that is relatively prime. This only determines the a_i up to sign. A more precise description of the a_i including the sign can be found in [BM12], but that choice of sign is not relevant in this paper. The defining condition for the cross-ratio exponents can equivalently be stated as

$$\sum_{i=1}^{3} b_i / s_i = 0. \tag{3.3}$$

Applications of algebraic primitivity. By [Möl06b] an algebraically primitive Teichmüller curve lies in the real multiplication locus $\mathcal{RM}_\mathcal{O}$ for some order $\mathcal{O}$ in the trace field F. Consequently, cusps of Teichmüller curves, if they correspond to an irreducible stable curve, have an associated admissible triple (r_1, r_2, r_3), which is a basis of F over $\mathbf{Q}$, and the corresponding stable curve satisfies the cross-ratio equation (3.2), with exponents defined by (3.3). We constantly use the fact from [BM12] that this triple is, up to a suitable rescaling and permutation of indices, the triple of widths of cylinders in a periodic direction that has this associated cusp.

4 Harder-Narasimhan filtrations

In this section, we recall the notion of a Harder-Narasimhan filtration of a vector bundle, and following Yu-Zho [YZ13, YZ], we describe the Harder-Narasimhan filtration of the Hodge bundle for most strata in genus three. We apply this in Proposition 4.3 to obtain information on the zeros of the other eigenforms of Veech surfaces.

Let $\mathcal{V}$ be a vector bundle on a compact curve $\overline{C}$. The *slope* of a vector bundle is defined as $\mu(\mathcal{V}) = \deg(\mathcal{V})/\mathrm{rank}(\mathcal{V})$. A bundle is called *semistable* if it contains no subbundle of strictly larger slope. A filtration

$$0 = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \mathcal{V}_2 \cdots \subset \mathcal{V}_g = \mathcal{V}$$

is called a *Harder-Narasimhan filtration* if the successive quotients $\mathcal{V}_i/\mathcal{V}_{i-1}$ are semi-stable and the slopes are strictly decreasing, i.e.

$$\mu_i := \mathcal{V}_i/\mathcal{V}_{i-1} > \mu_{i+1} := \mathcal{V}_{i+1}/\mathcal{V}_i.$$

The Harder-Narasimhan filtration is the unique filtration with these properties (see e.g. [HL10]).

We now study the Harder-Narasimhan filtration in the case of a family $f\colon \mathcal{X} \to C$ over a Teichmüller curve in any nonprincipal stratum in genus threem with the Hodge bundle $\mathcal{V} = f_*\overline{\mathcal{X}}/\overline{\mathcal{C}}$ over $\overline{C}$. Here $\omega_{\mathcal{X}/\mathcal{C}}$ denotes the relative dualizing sheaf of f, whose sections are fibrewise stable differentials.

Proposition 4.1 ([YZ13])**.** *For C a Teichmüller curve in any nonprincipal stratum in genus three, the Harder-Narasimhan filtrations of $f_*\overline{\mathcal{X}}/\overline{\mathcal{C}}$ are given by the direct image sheaves*

$$\begin{array}{llll}
\Omega\mathcal{M}_3(4)^{\mathrm{hyp}}: & f_*\overline{\mathcal{X}}/\overline{\mathcal{C}}(-4D_1) & \subset f_*\overline{\mathcal{X}}/\overline{\mathcal{C}}(-2D_1) & \subset f_*\overline{\mathcal{X}}/\overline{\mathcal{C}}. \\
\Omega\mathcal{M}_3(4)^{\mathrm{odd}}: & f_*\overline{\mathcal{X}}/\overline{\mathcal{C}}(-4D_1) & \subset f_*\overline{\mathcal{X}}/\overline{\mathcal{C}}(-D_1) & \subset f_*\overline{\mathcal{X}}/\overline{\mathcal{C}}. \\
\Omega\mathcal{M}_3(3,1): & f_*\overline{\mathcal{X}}/\overline{\mathcal{C}}(-3D_1 - D_2) & \subset f_*\overline{\mathcal{X}}/\overline{\mathcal{C}}(-D_1) & \subset f_*\overline{\mathcal{X}}/\overline{\mathcal{C}}. \\
\Omega\mathcal{M}_3(2,2)^{\mathrm{hyp}}: & f_*\overline{\mathcal{X}}/\overline{\mathcal{C}}(-2D_1 - 2D_2) & \subset f_*\overline{\mathcal{X}}/\overline{\mathcal{C}}(-D_1) & \subset f_*\overline{\mathcal{X}}/\overline{\mathcal{C}}. \\
\Omega\mathcal{M}_3(2,2)^{\mathrm{odd}} & f_*\overline{\mathcal{X}}/\overline{\mathcal{C}}(-2D_1 - 2D_2) & \subset & f_*\overline{\mathcal{X}}/\overline{\mathcal{C}}. \\
\Omega\mathcal{M}_3(2,1,1): & f_*\overline{\mathcal{X}}/\overline{\mathcal{C}}(-2D_1 - D_2 - D_3) & \subset f_*\overline{\mathcal{X}}/\overline{\mathcal{C}}(-D_1) & \subset f_*\overline{\mathcal{X}}/\overline{\mathcal{C}}.
\end{array}$$

For a Teichmuller curre that is a family of hyperelliptic curves in the stratum $\Omega\mathcal{M}_3(2,2)^{\mathrm{odd}}$ the bundle $f_*\overline{\mathcal{X}}/\overline{\mathcal{C}}/f_*\overline{\mathcal{X}}/\overline{\mathcal{C}}(-2D_1 - 2D_2)$ is the direct sum of the two line bundles $f_*\overline{\mathcal{X}}/\overline{\mathcal{C}}(-2D_i)/f_*\overline{\mathcal{X}}/\overline{\mathcal{C}}(-2D_1 - 2D_2) \cong f_*\overline{\mathcal{X}}/\overline{\mathcal{C}}(-4D_i)$, $i = 1, 2$, of the same slope. That is,

$$f_*\overline{\mathcal{X}}/\overline{\mathcal{C}} = f_*\overline{\mathcal{X}}/\overline{\mathcal{C}}(-2D_1 - 2D_2) \oplus f_*\overline{\mathcal{X}}/\overline{\mathcal{C}}(-4D_1) \oplus f_*\overline{\mathcal{X}}/\overline{\mathcal{C}}(-4D_2).$$

In each case the bottom term $\mathcal{V}_1$ is the maximal Higgs subbundle, its fibres are the generating one-form of the Teichmüller curve and

$$\deg \mathcal{V}_1 = \tfrac{1}{2} \deg \Omega^1_{\overline{C}}(\partial C),$$

where $\partial C = \overline{C} \setminus C$. In the strata $\Omega\mathcal{M}_3(2,1,1)$ and $\Omega\mathcal{M}_3(3,1)$ the filtration is split as well, i.e. a direct sum of line bundles, see [YZ13, Section 5.3].

The first case is a special case of the following the more general statement for the strata $\Omega\mathcal{M}_g(2g-2)^{\mathrm{hyp}}$.

Proposition 4.2. *For a Teichmüller curve generated by a Veech surface in the stratum $\Omega\mathcal{M}_g(2g-2)^{\mathrm{hyp}}$ the bundle $\mathcal{V}_j = f_*\overline{\mathcal{X}}/\overline{\mathcal{C}}(-(2g-2j)D)$ is a vector bundle of rank j. The quotient line bundles $\mathcal{V}_j/\mathcal{V}_{j-1}$ have degree*

$$\deg(\mathcal{V}_j/\mathcal{V}_{j-1}) = \frac{2g+1-2j}{2g-1} \deg(\mathcal{L}). \tag{4.1}$$

In particular, the filtration by the $\mathcal{V}_j$ is the HN-filtration of $f_*\omega_{\overline{\mathcal{X}}/\overline{C}}$.

Proof. We reproduce the argument of [YZ13] for convenience. For all fibres X of f the dimensions of the following cohomology spaces are the same, namely for j odd $h^0(X, \mathcal{O}_X(jz)) = (j+1)/2$ and for j even $h^0(X, \mathcal{O}_X(jz)) = (j+2)/2$. Consequently, the direct image sheaves $f_*\mathcal{O}_\mathcal{X}(jD)$ and $R^1 f_*\mathcal{O}_\mathcal{X}(jD)$ are vector bundles.

Suppose first that j is odd. Since every section of $f_*\mathcal{O}_\mathcal{X}(jD)$ is also a section of $f_*\mathcal{O}_\mathcal{X}((j-1)D)$, these bundles are then isomorphic.

The long exact sequence associated to

$$0 \to \mathcal{O}_\mathcal{X}((j-1)D) \to \mathcal{O}_\mathcal{X}(jD) \to \mathcal{O}_D(jD) \to 0$$

is

$$\begin{aligned} 0 \to f_*\mathcal{O}_\mathcal{X}((j-1)D) \to f_*\mathcal{O}_\mathcal{X}(jD) \to f_*\mathcal{O}_{jD}(jD) \\ \to R^1 f_*\mathcal{O}_\mathcal{X}((j-1)D) \to R^1 f_*\mathcal{O}_\mathcal{X}(jD) \to 0. \end{aligned}$$

Since $f|_D$ is an isomorphism, the middle term is a line bundle. Its degree is

$$\deg f_*\mathcal{O}_{jD}(jD) = jD^2 = \frac{j}{2g-1} \deg \mathcal{L}$$

by [Möl11a, Lemma 4.11]. Here $\mathcal{L} = \mathcal{V}_1 \subset f_*\overline{\mathcal{X}}/\overline{\mathcal{C}}$ is the ("maximal Higgs") line bundle whose fibers are the generating one-forms of the Teichmüller curve.

Now supposes j is even. By Serre duality and the same argument as in the odd case, the last map of the long exact sequence is an isomorphism. Hence the degree of $f_*\mathcal{O}_\mathcal{X}((j-1)D)$ and $f_*\mathcal{O}_\mathcal{X}(jD)$ differs by $\deg \mathcal{O}_D(jD)$.

To obtain (4.1) from $\deg \mathcal{O}_D(jD)$, note that note that

$$f_*\overline{\mathcal{X}}/\overline{\mathcal{C}}((2j-2g+2)D) = \mathcal{L} \otimes f_*\mathcal{O}_\mathcal{X}(2jD).$$

The last statement is always true for a filtration, whose successive quotients are line bundles with strictly decreasing degrees. ■

Proof of Proposition 4.1. In all the cases the direct images are vector bundles, since the dimensions of their fibers are constant by Riemann-Roch and by definition of the parity of the spin structure. The degrees of the successive quotients can be computed by the same method as in the last section. The values appear in [YZ13, Table 1] (rescaled dividing by $\deg \mathcal{V}_1$) and the degrees are decreasing. Consequently the filtation is the Harder-Narasimhan filtration.

We provide full details in the case $\Omega\mathcal{M}_3(2,2)^{\mathrm{odd}}$ and prove the last statement of the proposition. Note that by the parity of the spin structure implies $f_*\overline{\mathcal{X}}/\overline{\mathcal{C}}(-2D_1-2D_2) = f_*\overline{\mathcal{X}}/\overline{\mathcal{C}}(-D_1-D_2)$. By [YZ13], if

$$h^0(\mathcal{O}_X(\sum_{i=1}^{n} d_i z_i)) = h^0(\mathcal{O}_X(\sum_{i=1}^{n}(d_i-a_i)z_i)) + \sum_{i=1}^{n} a_i$$

then

$$f_*\mathcal{O}_X(\sum_{i=1}^{n} a_i D_i)/f_*\mathcal{O}_X(\sum_{i=1}^{n}(a_i-d_i)D_i) \cong \oplus_{i=1}^{n} f_*\mathcal{O}_{D_i}(a_i D_i).$$

In this stratum we apply this to the case $a_1 = a_2 = 2$ and $d_1 = d_2 = 1$. Since the self-intersection number of D_i only depends on the order of the corresponding zero z_i we obtain that the rank two second step of the filtration is a direct sum of two line bundles of the same slope.

In the hyperelliptic case the bundles $f_*\overline{\mathcal{X}}/\overline{\mathcal{C}}(-2D_i)$ are of rank two for $i = 1,2$, each of the being the direct sum of $f_*\overline{\mathcal{X}}/\overline{\mathcal{C}}(-4D_i)$ and $f_*\overline{\mathcal{X}}/\overline{\mathcal{C}}(-2D_1-2D_2)$. By Riemann-Roch and the preceding remark on the self-intersection number of D_i we conclude $\deg f_*\overline{\mathcal{X}}/\overline{\mathcal{C}}(4D_1) = \deg f_*\overline{\mathcal{X}}/\overline{\mathcal{C}}(4D_2)$ and all the claims follows. ■

Consequences for the algebraically primitive case. Suppose from now on that (X,ω) is algebraically primitive. Then

$$f_*\overline{\mathcal{X}}/\overline{\mathcal{C}} = \oplus_{j=1}^{g}\mathcal{L}_j,$$

where the $\mathcal{L}_j$ are the bundles generated by the eigenforms for real multiplication by the trace field F. We may enumerate them such that $\deg(\mathcal{L}_j)$ is non-increasing with j. We let $\omega^{(j)} \in H^0(X,\Omega^1_X)$ be a one-form on the generating Veech surface X that spans $\mathcal{L}_j$. More generally, we write $\omega_c^{(j)}$ for a generator of $\mathcal{L}_j$ in the fiber over $c \in \overline{C}$. For $g=3$, we also label the bundles and their generators with field embeddings $\omega = \omega^{(1)}$, $\omega^\sigma = \omega^{(2)}$ and $\omega^\tau = \omega^{(3)}$.

Proposition 4.3. *In each of the strata $\Omega\mathcal{M}_3(4)^{\mathrm{hyp}}$, $\Omega\mathcal{M}_3(4)^{\mathrm{odd}}$, $\Omega\mathcal{M}_3(3,1)$, $\Omega\mathcal{M}_3(2,1,1)$, and $\Omega\mathcal{M}_3(2,2)^{\mathrm{hyp}}$ the second step of the HN-filtration consists of the first two eigenform bundles, i.e.*

$$\mathcal{L}_1 \oplus \mathcal{L}_2 = \mathcal{V}_2 \subset \mathcal{V}_3 = f_*\overline{\mathcal{X}}/\overline{\mathcal{C}}.$$

Consequently, $\mathcal{L}_2$ is generated for all $c \in \overline{C}$ by a one-form $\omega_c^{(2)}$ with a zero at D_1. This zero is necessarily a double zero in the case $\Omega\mathcal{M}_3(4)^{\mathrm{hyp}}$.

This is based on the following principle.

Lemma 4.4. *If $\mathcal{V} = \oplus_{j=1}^{g}\mathcal{L}_j$ is a direct sum of line bundles, ordered with non-increasing degrees, and if the successive quotients $\mathcal{V}_j/\mathcal{V}_{j-1}$ of the Harder-Narasimhan filtration are line bundles, then $\mathcal{V}_i = \oplus_{j=1}^{i}\mathcal{L}_j$.*

Proof. Since $\mathcal{V}_1$ is the unique line subbundle of $\mathcal{V}$ of maximal degree, only one of the projection maps $\mathcal{V}_1 \to \mathcal{V} \to \mathcal{L}_j$ is non-zero, by the non-increasing ordering necessarily for $j = 1$. By maximality of the degree, this map is an isomorphism. We may consequently consider $\mathcal{V}/\mathcal{V}_1$ and proceed inductively. ∎

Proof of Proposition 4.3. This follows from Lemma 4.4 and Proposition 4.1 ∎

Proof of Theorem 1.2. This is a direct consequence of Lemma 4.4 together with Proposition 4.2. ∎

The fixed part in genus three. As a final application of the Harder-Narasimhan filtration, we discuss the fixed part of the family of Jacobians over a Teichmüller curve, which is one source of zero Lyapunov exponents of the Kontsevich-Zorich cocycle.

Let $h : \mathrm{Jac}(\mathcal{X}/C) \to C$ be the family of Jacobian varieties over a Teichmüller curve. The family h is said to have a *fixed part* of dimension d, if there is an abelian variety A of dimension d and an inclusion $A \times C \to \mathrm{Jac}(\mathcal{X}/C)$ of the constant family with fiber A into the family of Jacobians. It was shown in [Möl11a], that the only Teichmüller curve in genus three with a fixed part of rank two is the family $y^4 = x(x-1)(x-t)$, generated by a square tiled surface in the stratum $\Omega\mathcal{M}_3(1,1,1,1)$. Studying the fixed part of Teichmüller curves is motivated from dynamics, since a Teichmüller curve with a Forni-subspace of rank $2d$ has a fixed part of dimension d, see [Aul] for definitions and background.

Proposition 4.5. *There does not exist a Teichmüller curve C generated by a genus three Veech surface in a stratum other than $\Omega\mathcal{M}_3(1,1,1,1)$ with a positive-dimensional fixed part.*

Proof. By the preceding remark we may restrict to a one-dimensional fixed part. The variation of Hodge structures over C decomposes into rank-two summands $R^1 f_* \mathbf{C} = \mathbb{L} \oplus \mathbb{U} \oplus \mathbb{M}$, where $\mathbb{L}$ is maximal Higgs, $\mathbb{U}$ is the unitary summand stemming from the fixed part and $\mathbb{M}$ is the rest. The $(1,0)$-pieces of these summands form a decomposition of $f_*\overline{\mathcal{X}}/\overline{\mathcal{C}}$ into line bundles with $\deg(\mathbb{M}^{1,0}) > 0 = \deg(\mathbb{U}^{(1,0)})$. For all but the stratum $\Omega\mathcal{M}_3(2,2)^{\mathrm{odd}}$ the claim follows from Lemma 4.4 and the fact that the lowest degree quotient of the filtration in Proposition 4.1 does not have degree zero, as calculated in [YZ13, Table 1].

In the remaining case, the inequality of degrees in the non-maximal Higgs part implies that the Harder-Narasimhan filtration has three terms, contradicting Proposition 4.1, which says that the filtration has only two terms. ∎

5 Tori contained in subvarieties of $\mathbf{G}_m^n$

At two occasions in this paper, for the principal stratum $\Omega\mathcal{M}_3(1,1,1,1)$ and for the stratum $\Omega\mathcal{M}_3(4)^{\mathrm{odd}}$, we are facing the task of enumerating torus translates in an algebraic subvariety Y of $\mathbf{G}_m^n$. In this section we describe an algorithm that we have implemented in both cases to check that all the tori contained in the subvariety are irrelevant for the finiteness statements we are aiming for.

When we apply this algorithm in §9, we will actually need only to find torus-translates contained in Y which are parallel to a subtorus of a fixed torus $T \subset \mathbf{G}_m^9$ of large codimension. This is a useful reduction, since the running time of the algorithm increases exponentially in n and is only useful in practice for very small dimensions.

To this end, given a rank r subgroup $M \subset \mathbf{Q}^n$, determining a subtorus $T_M \subset \mathbf{G}_m^n$, let $V_M \subset \mathbf{G}_m^n$ denote the subvariety of $\boldsymbol{a} \in \mathbf{G}_m^n$ such that $\boldsymbol{a}T_M \subset Y$. We wish to enumerate those codimension-one subspaces $N \subset M$ such that the V_N potentially strictly contains V_M. Applying this procedure inductively then produces a list of subspaces $N \subset M$ and varieties V_N which account for all of the torus-translates contained in Y.

Consider a r-by-n matrix E whose rows vectors span M. A coefficient vector $\boldsymbol{a} \in \mathbf{G}_m^n$ determines a parametrization $f_{E,\boldsymbol{a}} \colon \mathbf{G}_m^r \to \boldsymbol{a}T_M$ given by

$$f_{E,\boldsymbol{a}}(\boldsymbol{t}) = (a_1 \boldsymbol{t}^{E_1}, \ldots, a_s \boldsymbol{t}^{E_s}),$$

where the E_i are the column vectors of E. Suppose first that Y is defined by the single polynomial $h(\boldsymbol{z}) = \sum b_I \boldsymbol{z}^I$ in the variables $z_1, \ldots, z_n$ with coefficients in a number field. The vanishing of the composition

$$h \circ f_{E,\boldsymbol{a}}(\boldsymbol{t}) = \sum_I b_I \boldsymbol{a}^I \boldsymbol{t}^{E \cdot I}$$

is then equivalent to the vanishing of the system of rational functions

$$p_J(\boldsymbol{a}) = \sum_{E \cdot I = J} b_I a^I$$

obtained by partitioning the coefficients according to the images of the exponent vectors $I \in \mathrm{Supp}(h)$ under E, or equivalently according to their images under the orthogonal projection to M. We take the numerator of each p_J, yielding an ideal in $\overline{\mathbf{Q}}[a_1, \ldots a_s]$[1].

Now suppose Y is defined by an ideal $I = (h_1, \ldots, h_s)$. Applying this construction to each h_i, the collection of resulting polynomials defines an ideal $I_M \subset \overline{\mathbf{Q}}[a_1, \ldots, a_s]$ which cuts out the desired variety V_M. We call I_M the *coefficient ideal* of I.

Now, for a generic subspace $N \subset M$, the partition of each $\mathrm{Supp}(h_i)$ induced by orthogonal projection to N will be unchanged, so $V_N = V_M$. If $v, w \in \mathrm{Supp}(h_i)$ are identified by projection to N, but not by projection to M, then N must be the orthogonal complement of $p_M(v - w)$ in M. Enumerating all N arising in this way then yields all N for which V_N potentially strictly contains V_M.

Applying this algorithm inductively to the list of subspaces obtained in each dimension, we obtain a list of subspaces $N \subset M$ together with varieties V_N, which together account for all torus-translates contained in Y.

The complete algorithm builds in this way the possible subspaces N with $1 \leq \mathrm{rank}(N) \leq r$ from the difference of the projections of monomials onto any subspace of larger rank that is already in the list of candidate subspaces. In the next step it assembles the ideals defining V_N, as summarized in Algorithm 1.

[1]Although we are only interested in solutions in $\mathbf{G}_m^n$, it is computationally convenient to work over the polynomial ring and later discard components contained in the coordinate hyperplanes

Data: List of polynomials $h_1, \ldots, h_s \in \overline{\mathbf{Q}}[z_1, \ldots, z_n]$, a matrix $M \in \mathbf{Z}^{r \times n}$
Result: Pairs (N, V) of matrices N and subvarieties $V \subset \mathbf{G}_m^n$

`/* Build the set subspaces` N `that may occur recursively */`
Subspaces $= \{M\}$;
for $S \in$ Subspaces **do**
  for $i = 1$ **to** s **do**
    ProjectionList $\leftarrow$ `OrthProjection`(Supp$(h_i), S$);
    for $(j, k) \in \{1, \ldots, |$ProjectionList$|\}$ **do**
      $v \leftarrow$ ProjectionList$[j]$ $-$ ProjectionList$[k]$;
      `AddTo` (Subspaces, `OrthComplement` (v,S));
    end
  end
end
`/* Check subspaces for the existence of a torus translate */`
TorusTranslates $= \{\}$;
for $N \in$ Subspaces **do**
  $I_N \leftarrow \langle\rangle$;
  for $i = 1$ **to** s **do**
    foreach $j \in$ `OrthProjection`(Supp(h_i),N) **do**
      $I_N = I_N +$ `Extract` (h_i, j) ; `/* Extracts` p_j `*/`
      `/*` $P_{i,j}$ `as in` (5) `*/`
    end
  end
  if $I_N \neq \langle 1 \rangle$; `/* Discard if` I_N `is the unit ideal */`
  then
    `AddTo` (TorusTranslates,(N,$V(I_N)$))
  end
end
return TorusTranslates

Algorithm 1: Torus containment

6 Finiteness in the minimal strata

The aim of this section is to prove the finiteness result Theorem 1.1 in the cases when the torsion condition is void (i.e. in the strata $\Omega\mathcal{M}_3(4)^{\mathrm{hyp}}$ and $\Omega\mathcal{M}_3(4)^{\mathrm{odd}}$) or of limited use, as in the hyperelliptic locus of the stratum $\Omega\mathcal{M}_3(2,2)^{\mathrm{odd}}$ where it is automatically satisfied when both the points are Weierstrass points.

In all three cases we study the degenerate fibers of the universal family $f\colon \mathcal{X} \to C$ over (a cover of) an algebraically primitive Teichmüller curve. We first make explicit the conditions arising from real multiplication and the Harder-Narasimhan filtration, which puts us in the situation considered in Theorem 2.6. We then check that the hypothesis of Theorem 2.6 are met.

The case $\Omega\mathcal{M}_3(4)^{\mathrm{hyp}}$ is quickly dealt with and shows all the essential features. The case $\Omega\mathcal{M}_3(4)^{\mathrm{odd}}$ requires a long detour to check the hypothesis (i) and (ii) of Theorem 2.6. Note for comparison that in [MW] the odd stratum is of no more complexity than the hyperelliptic.

The case of the hyperelliptic locus in $\Omega\mathcal{M}_3(2,2)^{\mathrm{odd}}$ is quite different, as the Harder-Narasimhan filtration has a rank-two piece. The information this yields seems less useful, and unfortunately our methods fail completely for this locus. To handle this case, we instead appeal to [MW].

6.1 The stratum $\Omega\mathcal{M}_3(4)^{\mathrm{hyp}}$

Let X_∞ be a degenerate fiber of $f\colon \mathcal{X} \to C$. It is necessarily irreducible, since the generating form ω has a single zero. It is geometric genus zero by real multiplication. Hence X_∞ is a trinodal curve. We let $\mathbf{P}^1$ with coordinate z be the normalization of X_∞. We may choose the coordinate z such that the hyperelliptic involution is $z \mapsto -z$ and that the zero section D specializes to $z = 0$. The three pairs of points on the normalization of the trinodal curve are thus x_i and $y_i = -x_i$, $i = 1, 2, 3$. This normalization still leaves one parameter for scaling the x_i multiplicatively. Due to this choice of z the generating eigenform specializes to

$$\omega_\infty = \sum_{i=1}^{3} \left(\frac{r_i}{z - x_i} - \frac{r_i}{z + x_i} \right) dz = \frac{Cz^4}{\prod_{i=1}^{3}(z^2 - x_i^2)} dz. \tag{6.1}$$

The condition that ω has a four-fold zero amounts to the equations

$$\sum_{i=1}^{3} r_i x_{i+1} x_{i+2} = 0 \tag{6.2}$$

$$\sum_{i=1}^{3} r_i x_i (x_{i+1}^2 + x_{i+2}^2) = 0, \tag{6.3}$$

where indices are to be read mod 3.

Let ω^σ be one of the two Galois conjugate eigenforms, the one generating the eigenform bundle of second largest degree. From Proposition 4.3 we deduce the following information.

Corollary 6.1. *The form ω^σ has a double zero along D. In particular*

$$\omega_\infty^\sigma = \sum_{i=1}^{3} \left(\frac{r_i^\sigma}{z - x_i} - \frac{r_i^\sigma}{z + x_i} \right) dz = \frac{C_1 z^4 + C_2 z^2}{\prod_{i=1}^{3}(z^2 - x_i^2)} dz, \tag{6.4}$$

which can also be expressed by the condition

$$\sum_{i=1}^{3} r_i^\sigma x_{i+1} x_{i+2} = 0. \tag{6.5}$$

Note that for the third embedding, denoted by τ, the analogous condition $\sum_{i=1}^3 r_i^\tau x_{i+1}x_{i+2} = 0$ should not hold since $f_*\overline{\mathcal{X}}/\overline{\mathcal{C}}(-D)$ has just rank two. Indeed this relation does not hold for the degenerate fiber of the 7-gon. The values are given in [BM12, Example 14.4].

We may normalize to $r_3 = 1$ and to $x_3 = 1$.

Corollary 6.2. *The points x_i scaled such that $x_3 = 1$ lie in the Galois closure of trace field F. Moreover, the tuple $1/x_i^\sigma$ is proportional to the dual basis (s_1, s_2, s_3) of (r_1, r_2, r_3). In particular, the x_i are real.*

Proof. We can rewrite (6.2) and (6.5) as $\sum_{i=1}^3 r_i/x_i = 0$ and $\sum_{i=1}^3 r_i^\sigma/x_i = 0$. This implies the second statement and the first follows.

Alternatively, we can deduce from (6.2) and (6.5) the equality

$$(r_1 - r_1^\sigma)x_2 + (r_2 - r_2^\sigma)x_1 = 0, \tag{6.6}$$

i.e. the ratio $x_1/x_2 \in F$. Plugging this information back into (6.2) implies the first statement. ∎

Proposition 6.3. *In $\mathbf{A}^2$ with coordinates (x_1, x_2) we define*

$$\mathcal{Y} = \mathbf{A}^2 \smallsetminus \{(x_1, x_2) : x_1 x_2 (x_1 \pm 1)(x_2 \pm 1)(x_1 \pm x_2) = 0\}.$$

If we define $R\colon \mathcal{Y} \to \mathbf{G}_m^3$ by

$$R(x_1, x_2) = \left(\frac{x_2 - 1}{x_2 + 1}, \frac{1 - x_1}{1 + x_1}, \frac{x_1 - x_2}{x_1 + x_2} \right).$$

and $\ell(x_1, x_2) = (x_1, x_2, 1)$, then the boundary points of Teichmüller curves in the normalization of (6.1) and $x_3 = 1$ are points $y = (x_1, x_2) \in \mathcal{Y}$ with $[\mathbf{Q}(y) : \mathbf{Q}] \le 3$ which satisfy (2.6) for some $b = (b_1, b_2, b_3) \in (\mathbf{Z} \smallsetminus \{0\})^3$.

Proof. By definition of a stable curve no two poles coincide. Moreover on the degenerate fibers of a Teichmüller curve zeros of the generating one-form are disjoint from the poles (see e.g. [Möl11b]). This implies that the boundary point has coordinates in $\mathcal{Y}$.

The x_i lie in F^σ by Corollary 6.2, and the degree bound follows.

Since $y_i = -x_i$ the cross-ratios can be simplified to

$$R_{ij} = \left(\frac{x_i + x_j}{x_i - x_j} \right)^2. \tag{6.7}$$

so that $R^{-2} = (R_{23}, R_{13}, R_{12})$. Since the x_i are real, the root of unity on the right of (3.2) is ± 1. Possibly multiplying the b_i by 2, we can take it to be 1.

By the argument preceding the proposition, the tuples $(1/x_1^\sigma, 1/x_2^\sigma, 1/x_3^\sigma)$ and (s_1, s_2, s_3) are proportional. Let $a = (a_1, a_2, a_3)$ be the cross-ratio exponents, as defined along with (3.2). Since the a_i are integers, (3.3) is restated as $\sum_{i=1}^3 a_i x_i = 0$. If we define $b_i = 2a_i$, then both conditions in (2.6) hold as a consequence of (3.2) and (3.3). ∎

We next check that $\mathcal{Y}$, R and ℓ match the hypothesis of (i) and (ii) of Theorem 2.6. The map R is in fact injective and the closure of its image $\mathcal{S} = \overline{R(Y)} \subset \mathbf{G}_m^3$ is defined by the cubic

$$r_1 r_2 r_3 + r_1 + r_2 + r_3 \in \mathbf{Q}[r_1, r_2, r_3].$$

The following lemma is the first instance of the problem discussed in Section 5. In this case we give the complete discussion without computer assistance.

We recall that $\mathcal{S} \smallsetminus \mathcal{S}^{\mathrm{oa}}$ is the union of all positive dimensional cosets that are contained in $\mathcal{S}$.

Lemma 6.4. *The complement $\mathcal{S} \smallsetminus \mathcal{S}^{\mathrm{oa}}$ is the union of the 6 lines obtained by permuting the coordinates of*

$$\{(1, -1)\} \times \mathbf{G}_m.$$

Proof. Let $(r_1, r_2, r_3) \in \mathcal{S}$ be in a positive dimensional coset contained entirely in $\mathcal{S}$. Then there exist $e_1, e_2, e_3 \in \mathbf{Z}$, not all zero, with

$$r_1 r_2 r_3 t^{e_1+e_2+e_3} + r_1 t^{e_1} + r_2 t^{e_2} + r_3 t^{e_3} = 0 \tag{6.8}$$

for all $t \in \mathbf{G}_m$. There cannot be any exponent in this equation which appears by itself, so without loss of generality (permuting the coordinates if necessary) $e_1 + e_2 + e_3 = e_1$ and $e_2 = e_3$, from which we obtain $e_2 = e_3 = 0$. Then (6.8) implies $r_1 r_2 r_3 + r_1 = 0$ and $r_2 + r_3 = 0$, so $r_2 = \pm 1$ and $r_3 = \mp 1$, leading to the two lines inside $\mathcal{S}$:

$$\{(t, 1, -1);\ t \in \mathbf{G}_m\} \quad \text{and} \quad \{(t, -1, 1);\ t \in \mathbf{G}_m\} \qquad \blacksquare$$

The next lemma checks the hypothesis on the image of $\mathcal{C}_b$, needed for Theorem 2.6 (ii). Recall that $\mathcal{C}_b \subset \mathcal{Y}$ is the curve cut out by the linear equation $\sum b_i \ell_i = 0$. In our case, it this is more explicitly $b_1 x_1 + b_2 x_2 + b_3 = 0$.

Lemma 6.5. *Suppose $b = (b_1, b_2, b_3) \in (\mathbf{C}^*)^3$. Then $\overline{R(\mathcal{C}_b)}$ is not contained in the translate of a proper algebraic subgroup of $\mathbf{G}_m^3$.*

Proof. If the contrary holds then there exists $(a_1, a_2, a_3) \in \mathbf{Z}^3 \smallsetminus \{0\}$ with

$$\left(\frac{x_2 - 1}{x_2 + 1}\right)^{a_1} \left(\frac{1 - x_1}{1 + x_1}\right)^{a_2} \left(\frac{x_1 - x_2}{x_1 + x_2}\right)^{a_3}, \tag{6.9}$$

constant on $\mathcal{C}_b$. We evaluate the order of this function at certain points of $\mathcal{C}_b$ to derive a contradiction. Without loss of generality we assume $a_1 \geq 0$.

Suppose for the moment that $a_1 > 0$. The first factor of (6.9) has a pole when $x_2 = -1$. Therefore, $x_2 = -1$ must imply $x_1 = \pm 1$ and we have

$$b_1 - b_2 + b_3 = 0 \quad \text{or} \quad -b_1 - b_2 + b_3 = 0. \tag{6.10}$$

Now (6.9) vanishes at $x_2 = 1$, so $x_1 = \pm 1$ as well. As before we find

$$b_1 + b_2 + b_3 = 0 \quad \text{or} \quad -b_1 + b_2 + b_3 = 0. \tag{6.11}$$

We immediately observe that any pair of linear equations, one coming from (6.10) and the other from (6.11), is linearly independent. Moreover, a common zero of any of these four systems satisfies $b_1b_2b_3 = 0$. This contradicts our hypothesis.

We must also treat the case $a_1 = 0$. Then $a_2 \neq 0$, since $(x_1-x_2)/(x_1+x_2)$ is non-constant. This time we consider the points on $\mathcal{C}_b$ with $x_1 = 1$ and $x_1 = -1$. In either case we must have $x_2 = 1$ or $x_2 = -1$. We find the same linear equations as above, but paired up differently. Again, any solution of any pair must have a vanishing coordinate. ■

Proof of Theorem 1.1, case $\Omega\mathcal{M}_3(4)^{\mathrm{hyp}}$. By Lemma 6.4 the image of any $y \in \mathcal{Y}$ lies in $\mathcal{S}^{\mathrm{oa}}$. All the hypothesis of Theorem 2.6 are satisfied, so that the height of any pair (x_1, x_2) possibly arising from a cusp of an algebraically primitive Teichmüller curve in this stratum is bounded. From the degree bound in Proposition 6.3 and Northcott's theorem, we deduce that the number of such pairs is finite. By [BM12, Proposition 13.10], there are only finitely many algebraically primitive Teichmüller curves in the stratum $\Omega\mathcal{M}_3(4)^{\mathrm{hyp}}$. ■

6.2 The stratum $\Omega\mathcal{M}_3(4)^{\mathrm{odd}}$

Let now X_∞ be a degenerate fiber of a family $f : \mathcal{X} \to C$ over an algebraically primitive Teichmüller curve generated by a Veech surface (X, ω) in $\Omega\mathcal{M}_3(4)^{\mathrm{odd}}$. Let $\mathbf{P}^1$ with coordinate z be the normalization of X_∞. It turns out to be convenient to normalize the zero to be at $z = \infty$ (as opposed to the previous section where we had $z = 0$), so that the stable form on the limit curve is

$$\omega_\infty = \sum_{i=1}^{3} \left(\frac{r_i}{z - x_i} - \frac{r_i}{z - y_i} \right) dz = \frac{C}{\prod_{i=1}^{3}(z - x_i)(z - y_i)} dz. \tag{6.12}$$

We may suppose that $y_3 = -x_3$, leaving still a global scalar multiplication as degree of freedom and further down we will moreover let $x_3 = 1$, hence $y_3 = -1$.

The surface $\mathcal{Y}$. We parameterize stable forms by points $(x_1, y_1, x_2, y_2) \in \mathbf{A}^4$. For the form defined by the fraction on the right of (6.12) to be stable, we need

$$\frac{1}{(x_i - y_i)\prod_{i\neq j}(x_i - x_j)(x_i - y_j)} = -\frac{1}{(y_i - x_i)\prod_{i\neq j}(y_i - x_j)(y_i - y_j)}$$

for $i = 1, 2, 3$. Checking it for all but one i is sufficient by the residue theorem. This conditions are equivalent to the vanishing of the two polynomials

$$P_1 = (y_1 - x_2)(y_1 - y_2)(y_1^2 - 1) - (x_1 - x_2)(x_1 - y_2)(x_1^2 - 1) \tag{6.13}$$
$$P_2 = (y_2 - x_1)(y_2 - y_1)(y_2^2 - 1) - (x_2 - x_1)(x_2 - y_1)(x_2^2 - 1). \tag{6.14}$$

The stability conditions also contain the hyperelliptic locus $x_i = -y_i$ dealt with above and we want to get rid of this locus.[2]

Lemma 6.6. *The polynomials*

$$\left.\begin{array}{ll} f_1 & = x_1x_2 + y_1x_2 - x_2^2 + x_1y_2 + y_1y_2 - y_2^2 + 2 \\ f_2 & = x_1^2 + y_1^2 - x_2^2 - y_2^2 \end{array}\right\} \tag{6.15}$$

[2]Starting here, several lemmas are based heavily on computations made using `sage`. The code required in the proof of Lemmas 6.6, 6.8, 6.11, 6.12, 6.13, 6.14, 6.16 can be found in `g3fin_ch6.sage`.

generate a prime ideal I of $\mathbf{Q}[x_1, y_1, x_2, y_2]$. The set of common zeros of I is a geometrically irreducible affine variety $\overline{\mathcal{Y}} \subset \mathbf{A}^4$ of dimension 2 on which P_1 and P_2 vanish. Moreover, a point at which P_1 and P_2 vanish but at which $(x_1 - y_1)(y_1 - y_2)(x_1 - y_2)(x_2 - y_2)(x_1 + y_1)$ does not, lies on $\overline{\mathcal{Y}}$.

Proof. Assisted by a computer algebra system one verifies that f_1 and f_2 generate a prime ideal of $\mathbf{Q}[x_1, y_1, x_2, y_2]$, i.e. that $\overline{\mathcal{Y}}$ is irreducible over $\mathbf{Q}$. In a similar manner we check that $P_{1,2} \in I$ and, using the Jacobian criterion, we find that

$$(x_1, y_1, x_2, y_2) = (1, -1, 1, -1)$$

is a smooth point of $\overline{\mathcal{Y}}$. So it lies on precisely one geometric component of $\overline{\mathcal{Y}}$. This component is defined over $\mathbf{Q}$ as the said point is rational. Therefore, $\overline{\mathcal{Y}}$ is geometrically irreducible. ■

We define

$$\mathcal{Y} = \overline{\mathcal{Y}} \smallsetminus \left\{ \prod_{i,j=1;i\neq j}^{3} (x_i - x_j)(y_i - y_j) \prod_{i,j=1}^{3} (x_i - y_j) = 0 \right\}. \tag{6.16}$$

As a consequence of the definition of a stable form, the forms ω_∞ normalized as in (6.12) have coordinates in $\mathcal{Y}$.

Using the Harder-Narasimhan filtration and the function ℓ. Let ω^σ be one of the two Galois conjugate eigenform, generating the eigenform bundle of second largest degree. From Proposition 4.3 we deduce

$$\omega_\infty^\sigma = \sum_{i=1}^{3} \left(\frac{r_i^\sigma}{z - x_i} - \frac{r_i^\sigma}{z - y_i} \right) dz = \frac{P_3(z)}{\prod_{i=1}^{3} (z - x_i)(z - y_i)} dz, \tag{6.17}$$

where $P_3(z)$ is some polynomial of degree (less or equal to) three. This condition is equivalent to

$$\sum_{i=1}^{3} r_i^\sigma (x_i - y_i) = 0.$$

The same argument as above gives that the tuple of $(x_i - y_i)^{-1}$ is up to scale a dual basis to $(r_1^\tau, r_2^\tau, r_3^\tau)$. With the normalization $x_3 = 1$ and $y_3 = -1$, this implies that the set $(x_i - y_i)$, $i = 1, 2, 3$, lies in the Galois closure of the trace field F, in particular is its real.

In this stratum the tuple (b_1, b_2, b_3) is proportional to the tuple of cross-ratio exponents if and only if

$$\sum_{i=1}^{3} b_i \frac{1}{x_i - y_i} = 0.$$

Here we let $R = (R_1, R_2, R_3)$ be the three cross-ratios as defined in (3.1). For a given boundary point of an algebraically primitive Teichmüller curve we take $c = \mathrm{ord}(\zeta_E)$, the multiplicative order of the root of unity. If $a = (a_1, a_2, a_3)$ is the tuple of cross-ratio exponents, we let $b = (b_1, b_2, b_3) = ca$. This discussion is then summarized in the following statement.

Proposition 6.7. *With $\ell_i = 1/(x_i - y_i)$, boundary points on algebraically primitive Teichmüller curve in the stratum $\Omega\mathcal{M}_3(4)^{\mathrm{odd}}$ correspond, in the normalization of* (6.12)*, to points in $y \in \mathcal{Y}$ with $[\mathbf{Q}(y) : \mathbf{Q}] \leq 72$ that satisfy* (2.6) *for some $b = (b_1, b_2, b_3) \in (\mathbf{Z} \smallsetminus \{0\})^3$.*

Proof. After the preceding discussion, we only need to justify the field degree bound. For a given triple of $r_i \in F$, with the normalization $x_3 = 1$ and $y_3 = -1$ the numerator of (6.12) gives 4 equations for the unknowns x_1, x_2, y_2, y_3 of total degree $1, 2, 3$ and 4 respectively. Period coordinates imply that a stable form with a 4-fold zero is locally uniquely determined by its residues. Consequently, the set of solution to this system of equations is finite and each solution is of degree at worst 24 over F. ∎

Determining $\mathcal{S}^{\mathrm{oa}}$ for $\mathcal{S} = \overline{R(\mathcal{Y})}$. We next start checking that $\mathcal{Y}$, R and ℓ match the hypothesis (i) and (ii) of Theorem 2.6. The map R is in fact two-to-one on $\mathcal{Y}$ [BM12, Corollary 8.4]. With the help of a computer algebra system we find that the closure of its image $\mathcal{S} = \overline{R(\mathcal{Y})} \subset \mathbf{G}_m^3$ is cut out by the equation

$$\begin{aligned}
h = {} & X^6Y^6Z^2 - 4X^6Y^5Z^3 - 4X^5Y^6Z^3 \\
& + 6X^6Y^4Z^4 - 124X^5Y^5Z^4 + 6X^4Y^6Z^4 \\
& - 4X^6Y^3Z^5 - 124X^5Y^4Z^5 \\
& - 124X^4Y^5Z^5 - 4X^3Y^6Z^5 \\
& + X^6Y^2Z^6 - 4X^5Y^3Z^6 + 6X^4Y^4Z^6 \\
& - 4X^3Y^5Z^6 + X^2Y^6Z^6 + 336X^5Y^5Z^3 \\
& + 864X^5Y^4Z^4 + 864X^4Y^5Z^4 \\
& + 336X^5Y^3Z^5 + 864X^4Y^4Z^5 \\
& + 336X^3Y^5Z^5 - 2X^6Y^5Z - 2X^5Y^6Z \\
& + 8X^6Y^4Z^2 - 246X^5Y^5Z^2 + 8X^4Y^6Z^2 \\
& - 12X^6Y^3Z^3 - 1672X^5Y^4Z^3 \\
& - 1672X^4Y^5Z^3 - 12X^3Y^6Z^3 \\
& + 8X^6Y^2Z^4 - 1672X^5Y^3Z^4 \\
& - 4800X^4Y^4Z^4 - 1672X^3Y^5Z^4 \\
& + 8X^2Y^6Z^4 - 2X^6YZ^5 - 246X^5Y^2Z^5 \\
& - 1672X^4Y^3Z^5 - 1672X^3Y^4Z^5 \\
& - 246X^2Y^5Z^5 - 2XY^6Z^5 - 2X^5YZ^6 \\
& + 8X^4Y^2Z^6 - 12X^3Y^3Z^6 + 8X^2Y^4Z^6 \\
& - 2XY^5Z^6 + 72X^5Y^5Z + 1088X^5Y^4Z^2 \\
& + 1088X^4Y^5Z^2 + 2800X^5Y^3Z^3 \\
& + 8288X^4Y^4Z^3 + 2800X^3Y^5Z^3 \\
& + 1088X^5Y^2Z^4 + 8288X^4Y^3Z^4 \\
& + 8288X^3Y^4Z^4 + 1088X^2Y^5Z^4 \\
& + 72X^5YZ^5 + 1088X^4Y^2Z^5 \\
& + 2800X^3Y^3Z^5 + 1088X^2Y^4Z^5 \\
& + 72XY^5Z^5 + X^6Y^4 - 2X^5Y^5 \\
& + X^4Y^6 - 4X^6Y^3Z - 246X^5Y^4Z \\
& - 246X^4Y^5Z - 4X^3Y^6Z + 6X^6Y^2Z^2 \\
& - 1672X^5Y^3Z^2 - 5229X^4Y^4Z^2 \\
& - 1672X^3Y^5Z^2 + 6X^2Y^6Z^2 - 4X^6YZ^3 \\
& - 1672X^5Y^2Z^3 - 13532X^4Y^3Z^3 \\
& - 13532X^3Y^4Z^3 - 1672X^2Y^5Z^3 \\
& - 4XY^6Z^3 + X^6Z^4 - 246X^5YZ^4 \\
& - 5229X^4Y^2Z^4 - 13532X^3Y^3Z^4 \\
& - 5229X^2Y^4Z^4 - 246XY^5Z^4 + Y^6Z^4 \\
& - 2X^5Z^5 - 246X^4YZ^5 - 1672X^3Y^2Z^5 \\
& - 1672X^2Y^3Z^5 - 246XY^4Z^5 - 2Y^5Z^5 \\
& + X^4Z^6 - 4X^3YZ^6 + 6X^2Y^2Z^6 \\
& - 4XY^3Z^6 + Y^4Z^6 + 336X^5Y^3Z \\
& + 1088X^4Y^4Z + 336X^3Y^5Z \\
& + 864X^5Y^2Z^2 + 8288X^4Y^3Z^2 \\
& + 8288X^3Y^4Z^2 + 864X^2Y^5Z^2 \\
& + 336X^5YZ^3 + 8288X^4Y^2Z^3 \\
& + 21888X^3Y^3Z^3 + 8288X^2Y^4Z^3 \\
& + 336XY^5Z^3 + 1088X^4YZ^4 \\
& + 8288X^3Y^2Z^4 + 8288X^2Y^3Z^4 \\
& + 1088XY^4Z^4 + 336X^3YZ^5 \\
& + 864X^2Y^2Z^5 + 336XY^3Z^5 - 4X^5Y^3 \\
& + 8X^4Y^4 - 4X^3Y^5 - 124X^5Y^2Z \\
& - 1672X^4Y^3Z - 1672X^3Y^4Z
\end{aligned}$$

The polynomial h has total degree 14 and 199 non-zero terms. It is symmetric under permutation of coordinates.

Lemma 6.8. *For the vanishing locus $\mathcal{S}$ of h, we have*

$$\mathcal{S}^{\mathrm{oa}} = \mathcal{S} \smallsetminus \{(t,1,1),(1,t,1),(1,1,t);\ t \in \mathbf{G}_m\}.$$

Proof. We observe first that $\mathcal{S}$ itself is no coset. Indeed, $h(1,1,1) = 0$ and a computation shows that h is irreducible over $\mathbf{Q}$. But a coset that is irreducible over the rationals and contains the unit element is an absolutely irreducible algebraic subgroup of $\mathbf{G}_m^3$. If this were true for the zero set of h, then h would consist of 2 monomials. This is obviously not the case.

Any anomalous subvariety of $\mathcal{S}$ must be a 1-dimensional coset contained completely in $\mathcal{S}$; we recall that anomalous subvarieties were defined on page 11. It is thus the image of

$$t \mapsto (u_1 t^{e_1}, u_2 t^{e_2}, u_3 t^{e_3})$$

where $u_1, u_2, u_3 \in \mathbf{G}_m$ and $E = (e_1, e_2, e_3) \in \mathbf{Z}^3 \smallsetminus \{0\}$ are fixed.

This is the situation where the algorithm of Section 5 applies. The exponents here are unconstrained, M is the identity matrix. However, since we are interested in torus translates with $u_i \neq 0$ we may discard immediately subspaces defined by a matrix M' where a part of the partition induced by projecting the support of h onto M' consists of a single element.

Therefore, any λ in $\mathrm{Supp}(h)$ must have a *friend*, that is there has to exist $\lambda' \in \mathrm{Supp}(h)$ with $\lambda \neq \lambda'$ and

$$\langle \lambda - \lambda', E \rangle = 0.$$

For convenience, we fix $\lambda_1 = (6,6,2) \in \mathrm{Supp}(h)$ and the possible E contained in the subspaces $\langle \lambda_1 - \lambda_1' \rangle^{\perp}$ for $\lambda_1' \in \mathrm{Supp}(h) \smallsetminus \{\lambda_1\}$.

Tier 1: The list of possible torus translates is reduced to the consideration of one-dimensional subspaces by the following fact about h that is proven by a (computer-assisted) check of all possibilities.

For any $\lambda_1' \in \mathrm{Supp}(h) \smallsetminus \{\lambda_1\}$ there exists $\lambda_2 \in \mathrm{Supp}(h)$ such that

$$\lambda_1 - \lambda_1' \quad \text{and} \quad \lambda_2 - \lambda_2'$$

are linearly independent for all $\lambda_2' \in \mathrm{Supp}(h) \smallsetminus \{\lambda_2\}$. In other words, given a potential friend λ_1' of λ_1, there is some λ_2 in the support of h whose potential friends set-up a system of linear equations

$$\langle \lambda_1 - \lambda_1', E \rangle = \langle \lambda_2 - \lambda_2', E \rangle = 0$$

that has a unique solution E up-to scalar multiplication.

Any E coming from an anomalous curve in X must be a solution of one of these systems. In particular, only finitely many E are possible. But we can use `sage` to create a list of possibilities for E. In total there are 8796 and we will not reproduce them here. This is well beyond the 3 possibilities that appear in the conclusion of this lemma. In the next tier we will reduce the number of possibilities dramatically.

Tier 2: Given one of the 8796 candidates E from tier 1 we use `sage` to check that any element in the support of f has a friend with respect to E. As

f has 199 non-zero terms this seems quite a strong restriction. However, 51 of candidates pass this test. They are

$$\Big\{ (1,2,1)\,, (4,-1,-1)\,, (1,8,-1)\,, (1,-8,1)\,, (1,-1,8)\,, (1,1,6)\,,\\ (2,1,1)\,, (0,1,0)\,, (8,-1,-1)\,, (6,1,1)\,, (1,-6,1)\,, (1,0,0)\,, (1,1,2)\,,\\ (6,1,-1)\,, (6,-1,-1)\,, (1,1,4)\,, (1,-6,-1)\,, (1,-1,4)\,, (1,2,-1)\,,\\ (1,-8,-1)\,, (1,-1,-6)\,, (1,-1,-8)\,, (1,-1,-4)\,, (1,8,1)\,, (4,-1,1)\,,\\ (1,-4,-1)\,, (1,-1,2)\,, (1,4,-1)\,, (1,1,-4)\,, (1,-2,-1)\,, (2,-1,1)\,,\\ (8,1,1)\,, (2,-1,-1)\,, (8,-1,1)\,, (1,1,-6)\,, (1,1,-2)\,, (1,6,-1)\,, (4,1,1)\,,\\ (1,1,-8)\,, (1,4,1)\,, (1,-1,6)\,, (0,0,1)\,, (1,6,1)\,, (2,1,-1)\,, (4,1,-1)\,,\\ (1,-4,1)\,, (1,-2,1)\,, (6,-1,1)\,, (8,1,-1)\,, (1,1,8)\,, (1,-1,-2)\Big\}.$$

Tier 3: In the final tier we will reduce the 51 candidates to 3 using the second part of the algorithm of Section 5. A candidate (e_1, e_2, e_3) leads to a coset contained in the zero set of h only if the following property is true. The polynomial $h(u_1 t^{e_1}, u_2 t^{e_2}, u_3 t^{e_3})$, where u_1, u_2, u_3 are independents and the coefficient ring is $\mathbf{C}[t]$, vanishes at some complex point $\mathbf{G}_m^3$. This is a strong restriction because each power of t yields a polynomial in complex coefficients and all of these need to vanish at the same point. In a matter of seconds, `sage` eliminates 48 of the 51 candidates above; indeed, the said polynomials yield the unit ideal in $\mathbf{C}[u_1^{\pm 1}, u_2^{\pm 1}, u_3^{\pm 1}]$.

The 3 remaining candidates are

$$(1,0,0), (0,1,0), (0,0,1).$$

Here, `sage` tells us that we must have

$$u_2 = u_3 = 1, \quad u_1 = u_3 = 1, \quad \text{or} \quad u_1 = u_2 = 1$$

respectively. Not only are these three curves anomalous but they are even torsion anomalous.

Conversely, we easily find

$$h(t,1,1) = h(1,t,1) = h(1,1,t) = 0$$

and so the candidates are indeed torsion anomalous curves. ■

The curve $\mathcal{C}_c$. We now study the locus cut out by the equation $\sum_i b_i \ell_i = 0$. Recall that the b_i are non-zero. We divide by b_3 and let c_1 and c_2 be new independent variables that take the role of $c_i = b_i/b_3$. Chasing denominators and if $c = (\widetilde{c_1}, \widetilde{c_2}) = (b_1/b_3, b_2/b_3)$ we write $\mathcal{C}_c$ for the algebraic subset of $\mathbf{A}^4$ cut out in $\mathcal{Y}$ by

$$2\widetilde{c_1}(x_2 - y_2) + 2\widetilde{c_2}(x_1 - y_1) + (x_1 - y_1)(x_2 - y_2) \in \mathbf{Q}[x_1, y_1, x_2, y_2]. \quad (6.18)$$

Indeed, it will be useful to consider c_1 and c_2 as independent variables and let $\widetilde{c_1}$ and $\widetilde{c_2}$ denote their specializations to the coordinates of a given $c \in \mathbf{Q}^2$. We set

$$f = 2c_1(x_2 - y_2) + 2c_2(x_1 - y_1) + (x_1 - y_1)(x_2 - y_2) \in \mathbf{Q}[c_1, c_2, x_1, y_1, x_2, y_2].$$

Although $\mathcal{C}_c$ is defined by polynomials in rational coefficients it is sometimes useful to think of it as an algebraic set over $\mathbf{C}$. For any scheme $\mathcal{C}$ over $\operatorname{Spec}\mathbf{Q}$ we write $\mathcal{C}\otimes\mathbf{C}$ for its base change to $\operatorname{Spec}\mathbf{C}$.

It is not difficult to show that $\mathcal{C}_c$ is an algebraic curve. We state this result in the next lemma, which is proved further down and which justifies the title of this subsection.

Lemma 6.9. *Say $c\in\mathbf{Q}^2$ has non-zero coordinates. Then $\mathcal{C}_c\neq\emptyset$. Let $\mathcal{C}$ be an irreducible component of $\mathcal{C}_c\otimes\mathbf{C}$.*

(i) Say $i\in\{1,2\}$. The functions $x_i\pm 1$, $y_i\pm 1$, x_1-y_2, x_2-y_1, x_2-x_1, and y_2-y_1 are non-zero elements of the function field of $\mathcal{C}$.

(ii) The component $\mathcal{C}$ is a curve.

Much of this section deals with the more intricate question of the irreducibility of $\mathcal{C}_c$. In fact, we believe that $\mathcal{C}_c$ is irreducible for all $c\in(\mathbf{Q}\smallsetminus\{0\})^2$. We are only able to prove a weaker statement which is ultimately sufficient for our needs.

The irreducibility of $\mathcal{C}_c$ is merely an ingredient in the study of multiplicative relations among the 3 cross-ratio maps

$$\begin{aligned} R_1=R_{[23]}&=\frac{(x_2-1)(y_2+1)}{(x_2+1)(y_2-1)},\\ R_2=R_{[13]}&=\frac{(x_1-1)(y_1+1)}{(x_1+1)(y_1-1)},\\ R_3=R_{[12]}&=\frac{(x_2-x_1)(y_1-y_2)}{(x_1-y_2)(x_2-y_1)}\end{aligned}\tag{6.19}$$

which are non-zero rational maps on irreducible components of $\mathcal{C}_c\otimes\mathbf{C}$ by the previous lemma. The following proposition is the main technical result of this section. It will be crucial in verifying the hypothesis of Theorem 2.6 in order to obtain a height bound.

Proposition 6.10. *There is a finite subset $\Sigma\subset\mathbf{Q}^2$ with the following property. Suppose $c\in\mathbf{Q}^2$ has non-zero coordinates and that $\mathcal{C}$ is an irreducible component of $\mathcal{C}_c\otimes\mathbf{C}$.*

(i) If $c\notin\Sigma$ then R_1,R_2,R_3 are multiplicatively independent on $\mathcal{C}$.

(ii) If $c\in\Sigma$ and if $(b_1,b_2,b_3)\in\mathbf{Z}^3\smallsetminus\{0\}$ with $R_1^{b_1}R_2^{b_2}R_3^{b_3}$ a constant, then $(b_1,b_2,b_3)\notin(\widetilde{c_1},\widetilde{c_2},1)\mathbf{Q}$.

The proofs of both the lemma and the proposition will require some preparation as well as computational support from `sage`.

It will prove convenient to introduce

$$t=\frac{y_1+1}{x_1+1}.\tag{6.20}$$

Using this new variable and the residue condition (6.13) we have $\frac{y_1-y_2}{x_1-y_2}=$

$\frac{x_2-x_1}{x_2-y_1}\frac{x_1^2-1}{y_1^2-1} = \frac{x_2-x_1}{x_2-y_1}\frac{x_1-1}{y_1-1}t^{-1}$. In terms of t we have

$$\begin{aligned} R_1 &= \frac{(x_2-1)(y_2+1)}{(x_2+1)(y_2-1)}, \\ R_2 &= \frac{x_1-1}{y_1-1}t, \\ R_3 &= \left(\frac{x_2-x_1}{x_2-y_1}\right)^2 \frac{x_1-1}{y_1-1}t^{-1}. \end{aligned} \tag{6.21}$$

For brevity we set $B = \mathbf{Q}[c_1^{\pm1}, c_2^{\pm1}, x_1, y_1, x_2, y_2]$. If $c \in (\mathbf{Q} \smallsetminus \{0\})^2$ we specialize an element $g \in B$ to $\widetilde{g} \in \mathbf{Q}[x_1, y_1, x_2, y_2]$ by substituting c_1, c_2 by the coordinates of c.

Lemma 6.11. *Let $c \in \mathbf{Q}^2$ satisfy*

$$\widetilde{c_1}\widetilde{c_2}(\widetilde{c_1}+\widetilde{c_2}+1)(-\widetilde{c_1}+\widetilde{c_2}+1)(\widetilde{c_1}-\widetilde{c_2}+1)(-\widetilde{c_1}-\widetilde{c_2}+1) \neq 0 \tag{6.22}$$

and suppose that $\mathcal{C}_c$ is non-empty. Then $\mathcal{C}_c \otimes \mathbf{C}$ is smooth and irreducible. In particular, $\mathcal{C}_c$ is irreducible.

Proof. We may consider $\mathbf{A}^4$ as a Zariski open subset of $\mathbf{P}^4$ using the open immersion $(x_1, y_1, x_2, y_2) \mapsto [x_1 : y_1 : x_2 : y_2 : 1]$. Recall that $\overline{\mathcal{Y}} \subset \mathbf{A}^4$ is geometrically irreducible, cf. Lemma 6.6. We let $\overline{\overline{\mathcal{Y}}}$ denote the Zariski closure of $\overline{\mathcal{Y}}$ in $\mathbf{P}^4$ and consider it as an irreducible projective variety over $\operatorname{Spec}\mathbf{C}$.

Homogenizing the polynomial $\widetilde{f}$ given in (6.18) yields a hypersurface $\mathcal{H}$ of $\mathbf{P}^4$. A direct calculation by hand and using the fact that c has non-zero coordinates shows that f is absolutely irreducible. Thus $\mathcal{H}$ is geometrically irreducible.

Assisted by computer algebra one can show that (6.22) is enough to ensure that $\mathcal{C}_c \otimes \mathbf{C}$ is smooth. This is done by first homogenizing the defining equations (6.15) and f and treating c_1 and c_2 as independent varieties. The Jacobian criterion in connection with elimination restricts the possibilities for c in presence of a non-smooth point. For c satisfying (6.22) the only restriction is the vanishing to a certain degree 12 polynomial q in integer coefficients. The homogenization of this polynomial is

$$\begin{aligned} &c_1^{12} - 3c_1^{10}c_2^2 + 6c_1^8c_2^4 - 7c_1^6c_2^6 + 6c_1^4c_2^8 - 3c_1^2c_2^{10} + c_2^{12} - 3c_1^{10}c_3^2 - 51c_1^8c_2^2c_3^2 \\ &+ 78c_1^6c_2^4c_3^2 + 78c_1^4c_2^6c_3^2 - 51c_1^2c_2^8c_3^2 - 3c_2^{10}c_3^2 + 6c_1^8c_3^4 + 78c_1^6c_2^2c_3^4 + 414c_1^4c_2^4c_3^4 \\ &+ 78c_1^2c_2^6c_3^4 + 6c_2^8c_3^4 - 7c_1^6c_3^6 + 78c_1^4c_2^2c_3^6 + 78c_1^2c_2^4c_3^6 - 7c_2^6c_3^6 + 6c_1^4c_3^8 - 51c_1^2c_2^2c_3^8 \\ &+ 6c_2^4c_3^8 - 3c_1^2c_3^{10} - 3c_2^2c_3^{10} + c_3^{12}. \end{aligned} \tag{6.23}$$

If q were to have a rational zero, then its homogenization would have a non-trivial integral zero with coprime coefficients. Even a human can check that (6.23) has no non-trivial zeros modulo 2. Therefore, q does not vanish at the given c.[3]

[3] Over $\mathbf{Q}(\sqrt{-3})$ the polynomial (6.23) factors into

$$(c_1^6-3c_1^2c_2^4+c_2^6-3c_1^4c_3^2-21c_1^2c_2^2c_3^2-3c_2^2c_3^4+c_3^6)+3(c_2-c_3)(c_2+c_3)(-c_1+c_2)(c_1-c_3)(c_1+c_3)(c_1+c_2)\omega$$

times its conjugate where $\omega = (\sqrt{-3}+1)/2$. Using this presentation it is not hard to see that (6.23) has only finitely many zeros in $\mathbf{P}^3(\mathbf{R})$.

We have actually verified that all points of $\overline{\overline{\mathcal{Y}}} \cap \mathcal{H}$ are smooth for c as in the hypothesis. Fulton and Hansen's Corollary 1 [FH79] implies that $\overline{\overline{\mathcal{Y}}} \cap \mathcal{H}$ is connected. A variety that is connected and smooth must be irreducible. Since $\mathcal{C}_c \otimes \mathbf{C}$ is Zariski open in $\overline{\overline{\mathcal{Y}}} \cap \mathcal{H}$ it is either empty or irreducible. In the latter case $\mathcal{C}_c$ is irreducible. ■

Let us define
$$J = IB + fB = (f_1, f_2, f)$$
which is an ideal of B.

We observe that the ring automorphisms
$$(c_1, c_2, x_1, y_1, x_2, y_2) \mapsto (c_2, c_1, x_2, y_2, x_1, y_1) \tag{6.24}$$
and
$$(c_1, c_2, x_1, y_1, x_2, y_2) \mapsto (-c_1, -c_2, y_1, x_1, y_2, x_2) \tag{6.25}$$
map the ideal $J = (f_1, f_2, f) \subset B = \mathbf{Q}[c_1^{\pm 1}, c_2^{\pm 1}, x_1, y_1, x_2, y_2]$ to itself. Indeed, (6.24) maps f_1 to $f_1 - f_2$, f_2 to $-f_2$, and fixes f. Moreover, (6.25) leaves f_1, f_2, and f invariant.

The following lemmas subsumes elimination-theoretic properties of J that are necessary for our application. The elements can be found by computer algebra assisted elimination of variables. The computer assisted computations are usually done in the polynomial ring $\mathbf{Q}[c_1, c_2, x_1, y_1, x_2, y_2]$, where $c_{1,2}$ are not units. For example, we find that B/J is an integral domain since J is a prime ideal.

Lemma 6.12. *For $i, j \in \{1, 2\}$ there exist $f_{x_i y_j} \in J \cap \mathbf{Q}[c_1, c_2, x_i, y_j]$ that satisfy*
$$\begin{array}{rclrcl} f_{x_i y_i} & = x_i^6 & + \quad O(x_i^5), & f_{x_i y_i} & = y_i^6 & + \quad O(y_i^5), \\ f_{x_i y_i}(x_i, \pm 1) & = x_i^6 & + \quad O(x_i^5), & f_{x_i y_i}(\pm 1, y_i) & = y_i^6 & + \quad O(y_i^5), \\ f_{x_1 y_2}(x_1, x_1) & = x_1^8 & + \quad O(x_1^7), & f_{x_2 y_1}(x_2, x_2) & = x_2^8 & + \quad O(x_2^7) \end{array}$$
and
$$f_{x_2 y_1}(1, 1) = -64 c_1 c_2 (-c_1 + c_2 + 1), \tag{6.26}$$
$$f_{x_2 y_2}(1, 1) = 64 c_2^2. \tag{6.27}$$

Proof. We verify that $f_{x_1 y_1}$ exists and satisfies the first two equalities. The symmetry (6.24) implies the existence of $f_{x_2 y_2}$ and the first two equalities.

The same reasoning applies to the third and forth equalities. Moreover, the sixth equality follows from the fifth one by the same symmetry as before.

We verify final two inequalities directly. ■

Lemma 6.13. *There exist polynomials $f_{x_1 x_2} \in J \cap \mathbf{Q}[c_1, c_2, x_1, x_2]$ and $f_{y_1 y_2} \in J \cap \mathbf{Q}[c_1, c_2, y_1, y_2]$ that satisfy*
$$\begin{aligned} f_{x_1 x_2} &= -2(x_2 + c_2)^2 x_1^6 && + O(x_1^5), \\ f_{x_1 x_2} &= -2(x_1 + c_1)^2 x_2^6 && + O(x_2^5), \\ f_{y_1 y_2} &= -2(y_2 - c_2)^2 y_1^6 && + O(y_1^5), \\ f_{y_1 y_2} &= -2(y_1 - c_1)^2 y_2^6 && + O(y_2^5), \\ f_{x_1 x_2}(x_1, x_1) &= x_1^8 && + O(x_1^7), \\ f_{y_1 y_2}(y_1, y_1) &= y_1^8 && + O(y_1^7), \\ f_{x_1 x_2}(1, 1) &= 64 c_1 c_2 (c_1 + c_2 + 1). \end{aligned} \tag{6.28}$$

Proof. As in the previous lemma we use symmetry, here (6.25), to reduce the statements on $f_{y_1y_2}$ to the corresponding statements on $f_{x_1x_2}$. We then check the claims on $f_{x_1x_2}$ directly. ∎

We recall the variable t introduced near (6.20). Using the ideal

$$K = JB[t] + ((x_1+1)t - (y_1+1))B[t] \subset B[t]$$

we will eliminate all variables but c_1, c_2, x_2, and t. We observe that $B[t]/K$ equals $(B/J)[(y_1+1)/(x_1+1)]$ inside the field of fractions of B/J. Using computer algebra we can verify that c_1 is a member of the ideal in $\mathbf{Q}[c_1, c_2, x_1, y_1, x_2, y_2]$ generated by $f_1, f_2, f, x_1+1, y_1+1$. So we can write 1 as a B-linear combination of these generators. After dividing by x_1+1 we see that $B[t]/K$ equals the localization of B/J at x_1+1.

Note that the scheme $\mathcal{C}_c$, whose irreducible components we want to show are curves, is just the specialization of $B[t]/K$ to the corresponding value of c.

If $g \in B[t]$ and $c \in (\mathbf{Q} \smallsetminus \{0\})^2$ then as usual $\widetilde{g}$ denotes the specialization of g in $\mathbf{Q}[x_1, y_1, x_2, y_2, t]$.

Lemma 6.14. *The following hold true.*

(i) The intersection $K \cap \mathbf{Q}[c_1^{\pm 1}, c_2^{\pm 1}, x_2, t]$ is generated as an ideal by $f_{x_2t} \in \mathbf{Q}[c_1, c_2, x_2, t]$ which is an irreducible element of $\mathbf{Q}[c_1, c_2, x_2, t]$ with $\deg_t f_{x_2t} = 6$,

$$\begin{aligned} f_{x_2t}(1,t) &= 32c_1^3t^2\left((-c_1+c_2+1)t^2 - (c_1+c_2+1)\right), \quad \textit{and} \\ f_{x_2t}(-1,t) &= 32c_1(t^2-(c_1+c_2+1)t+c_2)\cdot \\ &\quad \cdot (c_2t^2+(c_1-c_2-1)t+1)((c_1+c_2-1)t^2+c_1-c_2+1). \end{aligned} \tag{6.29}$$

(ii) Say $\varphi : \mathbf{Q}[c_1, c_2, x_2, t] \to \mathbf{Q}[c_1, x_2, t]$ is the ring homomorphism that maps c_2 to $\pm c_1 + 1$ and c_1, x_2, t to themselves. Then $\varphi(f_{x_2t})$ is absolutely irreducible as an element of $\mathbf{C}(c_1)[x_2, t]$.

(iii) If $c \in \mathbf{Q}^2$ has non-zero coordinates, then $\mathcal{C}_c$ is non-empty. If we assume further that $\mathcal{C}_c$ is irreducible, the projection of $\mathcal{C}_c$ to $\mathbf{A}^2$ by taking the coordinates x_2 and t is Zariski dense in the curve cut out by $\widetilde{f_{x_2t}}$. Moreover, $\widetilde{f_{x_2t}}$ is irreducible in $\mathbf{Q}[x_2, t]$.

Proof. The existence of f_{x_2t} and an explicit presentation follows from computer algebra assisted elimination. We will not reproduce this polynomial here as it has 453 non-zero terms and total degree 12. In this way we also check the properties (6.29) and that f_{x_2t} is irreducible as an element of $\mathbf{Q}[c_1, c_2, x_2, t]$. This settles part (i) and we move to part (ii).

Using computer algebra it is possible to check that a given polynomial is irreducible over a given number field. We find that $\varphi(f_{x_2t})$ is irreducible as an element of $\mathbf{Q}(\sqrt{3})[c_1, x_2, t]$. It is irreducible as an element of $\mathbf{Q}(\sqrt{3}, c_1)[x_2, t]$ as it depends on t. To deduce geometric irreducibility, we apply the following trick. Let $g \in \mathbf{Q}(\sqrt{3}, c_1)[x_2, t, u]$ denote the homogenization of $\varphi(f_{x_2t})$ with respect to x_2 and t and where u is the new projective variable. Then g is irreducible in $\mathbf{Q}(\sqrt{3}, c_1)[c_2, t, u]$. Next we claim that it is irreducible in $L[c_2, t, u]$ where L is an algebraic closure of $\mathbf{Q}(c_1)$. Indeed, we remark that $g(1/\sqrt{3}, 1, 1) = 0$ and

$\frac{\partial}{\partial x_2} g(1/\sqrt{3}, 1, 1) \neq 0$. In other words, $[1/\sqrt{3} : 1 : 1]$ is contained in precisely one geometric component of the vanishing locus in $\mathbf{P}^2$ of g. If h were reducible over L, then some element of the Galois group of $L/\mathbf{Q}(\sqrt{3}, c_1)$ would map said component to some other component. But this is impossible as the Galois group leaves $[1/\sqrt{3} : 1 : 1]$ invariant. So g is irreducible in $L[x_2, t, u]$. It follows that $\varphi(f_{x_2 t})$ is irreducible in $L[x_2, t]$. So this polynomial must be irreducible over any base field containing $\mathbf{Q}(\sqrt{3}, c_1)$ as L is algebraically closed. This yields part (ii) and we now prove part (iii).

The natural ring homomorphism $\mathbf{Q}[c_1^{\pm 1}, c_2^{\pm 1}, x_2, t]/(f_{x_2 t}) \to B[t]/K$ is injective by (i). Thus so is

$$\mathbf{Q}[c_1^{\pm 1}, c_2^{\pm 1}, x_2, t, (t^2+1)^{-1}]/(f_{x_2 t}) \to B[t, (t^2+1)^{-1}]/K \tag{6.30}$$

as localizing is an exact functor. Therefore, the corresponding morphism of affine schemes $\pi : \mathcal{Z} \to \mathcal{W}$ is dominant.

Using computer algebra we may verify that the classes of $(t^2+1)x_1, (t^2+1)y_1$, and y_2 in the quotient ring $B[t]/K$ are integral over $\mathbf{Q}[c_1, c_2, x_2, t]$. Indeed, to check this for $(t^2+1)x_1$ we first produce a list of generators of the ideal $(f_1, f_2, f, (x_1+1)t - (y_1+1)) \subset \mathbf{Q}[c_1, c_2, x_1, y_1, x_2, y_2, t]$ intersected with $\mathbf{Q}[c_1, c_2, x_1, x_2, t]$. For each member of this list we extract the leading term as a polynomial in the variable x_1. Next we show that these leading terms generate an ideal containing t^2+1. A similar procedure yields our claims for $(t^2+1)y_1$ and y_2.

Thus any element on right of (6.30) is integral over the ring on the left. In other words, π is a finite morphism. Finite morphisms are closed and since π is dominant we conclude that π is surjective.

After doing a base change we may view π as a family of maps $\pi_c : \mathcal{Z}_c \to \mathcal{W}_c$ parameterized by $c \in (\mathbf{Q} \smallsetminus \{0\})^2$. Surjectivity is stable under base change, so each π_c is surjective. The target $\mathcal{W}_c$ of the map π_c is the affine scheme attached to $\mathbf{Q}[x_2, t, (t^2+1)^{-1}]/(\widetilde{f_{x_2 t}})$. It is non-empty since $\widetilde{f_{x_2 t}} \in \mathbf{Q}[x_2, t, (t^2+1)^{-1}]$ is not a unit; indeed, taking $f_{x_2 t}(\pm 1, t)$ from (i) into account we find that x_2 must appear in $\widetilde{f_{x_2 t}}$.

Now $\mathcal{Z}_c$ is homeomorphic to an open subspace of $\mathcal{C}_c$, obtained by specialization to c of a localization at $1/(x_1+1)$ and t^2+1. Surjectivity of $\pi_c : \mathcal{Z}_c \to \mathcal{W}_c$ implies $\mathcal{Z}_c \neq \emptyset$. We conclude that $\mathcal{C}_c \neq \emptyset$, so $\mathcal{C}_c$ is irreducible by hypothesis and therefore $\mathcal{Z}_c$ is irreducible too. Thus $\mathcal{W}_c$, being the continuous image of $\mathcal{Z}_c$, is also irreducible. This is equivalent to the fact that the nilradical of $\mathbf{Q}[x_2, t, (t^2+1)^{-1}]/(\widetilde{f_{x_2 t}})$ is a prime ideal. As $\mathbf{Q}[x_2, t, (t^2+1)^{-1}]$ is a factorial domain we conclude that $\widetilde{f_{x_2 t}}$ is the power of an irreducible element of $\mathbf{Q}[x_2, t, (t^2+1)^{-1}]$.

Next we show that up-to association, $\widetilde{f_{x_2 t}}$ has at most one prime divisor in $\mathbf{Q}[x_2, t]$. Otherwise it would be divisible by t^2+1. Then the right-hand sides of both equations in (6.29) would be divisible by t^2+1 leaving us with the contradictory

$$-\widetilde{c_1} + \widetilde{c_2} + 1 = -(\widetilde{c_1} + \widetilde{c_2} + 1) \quad \text{and} \quad \widetilde{c_1} + \widetilde{c_2} - 1 = \widetilde{c_1} - \widetilde{c_2} + 1.$$

So $\widetilde{f_{x_2 t}}$ is the e-th power of an element in $\mathbf{Q}[x_2, t]$. We must now prove that $e = 1$. By specialization we see that $\widetilde{f_{x_2 t}}(\pm 1, t)$ are e-th powers in $\mathbf{Q}[t]$. If $e > 1$

then the first equation in the conclusion of part (i) implies $(-\widetilde{c_1}+\widetilde{c_2}+1)(\widetilde{c_1}+\widetilde{c_2}+1)=0$. If $\widetilde{c_1}=\widetilde{c_2}+1$, then the second one simplifies to

$$64\widetilde{c_1}(t^2-2\widetilde{c_1}t+\widetilde{c_1}-1)((\widetilde{c_1}-1)t^2+1)^2$$

by (6.29). The only rational value $\widetilde{c_1}$ for which the displayed expression is an e-th power with $e>1$ is 0, which we excluded. If $\widetilde{c_1}=-\widetilde{c_2}-1$, then the second equation in (6.29) simplifies to

$$64\widetilde{c_1}(-t^2+\widetilde{c_1}+1)^2((\widetilde{c_1}+1)t^2-2\widetilde{c_1}t-1).$$

By a similar argument as before and using $\widetilde{c_1}\neq 0$, this yields $e=1$, as desired. ∎

Let $\mathcal{C}$ be an irreducible component of $\mathcal{C}_c\otimes\mathbf{C}$. This is an irreducible variety over $\operatorname{Spec}\mathbf{C}$. By abuse of notation $x_{1,2}$ and $y_{1,2}$ as elements of $\mathbf{C}(\mathcal{C})$, the function field of $\mathcal{C}$.

Lemma 6.15. *Suppose $c\in\mathbf{Q}^2$ has non-zero coordinates and let $\mathcal{C}$ be an irreducible component of $\mathcal{C}_c\otimes\mathbf{C}$.*

(i) The functions x_1,y_1,x_2 and y_2 are non-constant when considered as elements of $\mathbf{C}(\mathcal{C})$.

(ii) Let v be a valuation of $\mathbf{C}(\mathcal{C})$ that is constant on $\mathbf{C}$. Then x_i is regular at v if and only if y_i is regular at v.

Proof. Since $\mathcal{C}$ is a component of the intersection of a surface with a hypersurface we have $\dim\mathcal{C}\geq 1$. So at least one among x_1,y_1,x_2, and y_2 is non-constant, as these elements generate the function field of $\mathcal{C}$. By the first equality in Lemma 6.12 we see that $\widetilde{f_{x_iy_i}}$ is non-zero. So x_i is constant if and only if y_i is constant. Therefore, x_1 or x_2 is non-constant

If we are in the first case and if x_2 or y_2 are constant, then x_2 and y_2 are constant. So $x_2=-\widetilde{c_2}$ by the first equality and $y_2=\widetilde{c_2}$ by the third equality of Lemma 6.13. So $x_2-y_2=-2\widetilde{c_2}$. Now $\widetilde{f}(x_1,y_1,x_2,y_2)=0$ as an element of $\mathbf{C}(\mathcal{C})$ and (6.18) implies $\widetilde{c_1}\widetilde{c_2}=0$, a contradiction. By using the second and fourth inequality of Lemma 6.13 we arrive at a similar contradiction if x_2 is non-constant and x_1 is constant.

Thus all 4 functions are non-constant and part (i) follows.

Part (ii) follows from the first two equalities of Lemma 6.12. Indeed, x_i is integral over $\mathbf{Q}[y_i]$ and vice versa. ∎

Proof of Lemma 6.9. Lemma 6.14(iii) implies that $\mathcal{C}_c$ is non-empty. By Lemma 6.15(i) the coordinates x_1,y_1,x_2,y_2 are non-constant on $\mathcal{C}$. We have $x_i\pm 1\neq 0$ by the fourth equality of Lemma 6.12 and $y_i\pm 1\neq 0$ by the third equality. The statements $x_1-y_2\neq 0$ and $x_2-y_1\neq 0$ are a consequence of the fifth and sixth equalities. Finally, $x_2-x_1\neq 0$ and $y_2-y_1\neq 0$ by the fifth and sixth equalities of Lemma 6.13. We conclude part (i).

We now compute the dimension of $\mathcal{C}$. By the first equality in Lemma 6.12 we see $\widetilde{f_{x_iy_i}}\neq 0$. So x_i,y_i are algebraically dependent over $\mathbf{C}$. Now x_1 and x_2 are also algebraically dependent due to the first equality in Lemma 6.13. Hence x_1,y_1,x_2,y_2 are pairwise algebraically dependent. Therefore, $\mathbf{C}(\mathcal{C})/\mathbf{C}$ has transcendence degree 1 and thus $\mathcal{C}$ is a curve. ∎

The coset condition. We write $\deg f$ for the degree of an element $f \neq 0$ of the function field of an irreducible component of $\mathcal{C}_c \otimes \mathbf{C}$. We recall some facts, to be used further down, which we call basic degree properties. If g lies in the same function field, then it is well-known that

$$\deg(f+g) \leq \deg f + \deg g \quad \text{and} \quad \deg fg \leq \deg f + \deg g$$

assuming the relevant quantities are well-defined. If $n \in \mathbf{Z}$, then

$$\deg f^n = |n| \deg f. \tag{6.31}$$

Finally,

$$\deg f = 0 \quad \text{or} \quad \deg f \geq 1$$

and the first case happens if and only if f is constant.

In the proof of the next lemma we use $\deg \mathcal{Z}$ to denote the degree of the Zariski closure of the affine variety $\mathcal{Z}$ in $\mathbf{P}^n \supset \mathbf{A}^n$.

Lemma 6.16. *Say $c \in \mathbf{Q}^2$ has non-zero coordinates and let $\mathcal{C}$ be an irreducible component of $\mathcal{C}_c \otimes \mathbf{C}$. We consider R_1, R_2, and R_3 presented in (6.19) as non-zero elements of the function field $\mathbf{C}(\mathcal{C})$.*

(i) We have $\deg R_2 \leq 32$ and $\deg R_3 \leq 64$.

(ii) Say $\pm\widetilde{c_1} - \widetilde{c_2} + 1 = 0$ with $|\widetilde{c_1}| \geq 1/2$ and suppose there is a multiple $(b_1, b_2, b_3) \in \mathbf{Z}^3 \smallsetminus \{0\}$ of $(\widetilde{c_1}, \widetilde{c_2}, 1)$ with $R_1^{b_1} R_2^{b_2} R_3^{b_3}$ a constant. Then $\widetilde{f_{x_2 t}}$ is irreducible in $\mathbf{Q}[x_2, t]$.

Proof. We recall that R is two-to-one on $\mathcal{Y}$. So at least one among R_1, R_2, R_3 is non-constant. The quadratic polynomials f_1 and f_2 from (6.15) are irreducible. Their set of common zeros is $\overline{\mathcal{Y}}$ by Lemma 6.6. By Bézout's Theorem we find $\deg \overline{\mathcal{Y}} \leq 4$. The curve $\mathcal{C}$ is then an irreducible component of the intersection of $\overline{\mathcal{Y}}$ with a hypersurface of degree 2. Thus $\deg \mathcal{C} \leq 8$. Hence each coordinate function x_1, y_1, x_2, y_2 on $\mathcal{C}$ has degree at most 8.

We apply the basic degree properties and (6.19) to bound $\deg R_2 \leq 32$ and $\deg R_3 \leq 64$. This yields part (i).

To prove (ii) let $(b_1, b_2, b_3) \in \mathbf{Z}^3$ be a non-zero multiple of $(\widetilde{c_1}, \widetilde{c_2}, 1)$ with

$$R_1^{b_1} R_2^{b_2} R_3^{b_3} \text{ a constant.} \tag{6.32}$$

We observe that $b_1 b_2 b_3 \neq 0$. Let $Q > 1$ be an integer to be specified later on. By Lemma 2.8 with $n = 1$ and $\theta = 1/\widetilde{c_1}$ there are integers p and q with $1 \leq q < Q$ and

$$\left| q \frac{1}{\widetilde{c_1}} - p \right| \leq \frac{1}{Q}. \tag{6.33}$$

Without loss of generality, p and q are coprime. The fact that $R_1^{b_1} R_2^{b_2} R_3^{b_3}$ is constant implies

$$|b_1| \deg (R_1^q R_2^{\pm q + p} R_3^p) = \deg (R_2^{b_1(\pm q + p) - b_2 q} R_3^{b_1 p - b_3 q}).$$

We use basic degree properties and the bounds from (i) to estimate

$$\deg (R_1^q R_2^{\pm q + p} R_3^p) \leq 32 \left| \pm q + p - \frac{b_2}{b_1} q \right| + 64 \left| p - \frac{b_3}{b_1} q \right| = 96 \left| q \frac{1}{\widetilde{c_1}} - p \right| \leq \frac{96}{Q}$$

since $\widetilde{c_1} = b_1/b_3$ and $\pm b_1 - b_2 + b_3 = 0$.

We fix $Q = 97$, then

$$R_1^q R_2^{\pm q+p} R_3^p \text{ must be a constant.} \tag{6.34}$$

We recall $|\widetilde{c_1}|\geq 1/2$ and use (6.33) again to bound $|p|\leq 2q + 1/Q < 2q + 1$. Hence $|p|\leq 2q$.

We know from Lemma 6.8 that there are exactly 3 cosets of dimension 1 in the cross-ratio domain $\mathcal{S}$. As $b_1 b_2 b_3 \neq 0$ there is up-to scalars at most one multiplicative relation among the R_i. So (6.34) implies that the vectors $(\widetilde{c_1}, \widetilde{c_2}, 1)$ and $(q, \pm q + p, p)$ are linearly dependent. In particular, $p \neq 0$ and $p \neq \mp q$.

Using `sage` we run over all coprime integers p, q with $1 \leq q \leq Q - 1 = 96$, $1 \leq |p|\leq 2q$, and $p \neq \mp q$ to verify that $f_{x_2 t}$ is irreducible as an element of $\mathbf{Q}[x_2, t]$ when specializing c to $(q/p, 1 \pm q/p)$ As our $\widetilde{f_{x_2 t}}$ is among these we conclude part (ii). ■

Lemma 6.17. *Let $c \in \mathbf{Q}^2$ have non-zero coefficients with $(\widetilde{c_1}+\widetilde{c_2}+1)(-\widetilde{c_1}+\widetilde{c_2}+1) \neq 0$ and suppose that $\widetilde{f_{x_2 t}}$ is irreducible in $\mathbf{Q}[x_2, t]$. Let $\mathcal{C}$ be an irreducible component of $\mathcal{C}_c \otimes \mathbf{C}$ and $(b_1, b_2, b_3) \in \mathbf{Z}^3$ with $R_1^{b_1} R_2^{b_3} R_3^{b_3}$ a constant in the function field of $\mathcal{C}$. If $|b_2|\leq |b_1|$ then $b_1 = b_2 = b_3 = 0$.*

Proof. We will repeatedly use Lemma 6.9 and write $K = \mathbf{C}(\mathcal{C})$ for the function field of $\mathcal{C}$. First, let us fix an irreducible factor $p \in \mathbf{C}[x_2, t]$ of $\widetilde{f_{x_2 t}}$ that vanishes when taken as a function on $\mathcal{C}$. As $\widetilde{f_{x_2 t}}$ has rational coefficients we may suppose that the coefficients of p are in a finite extension of $\mathbf{Q}$. Let us suppose that $(b_1, b_2, b_3) \in \mathbf{Z}^3$ is as in the hypothesis.

We consider F, the function field of the plane curve defined by the vanishing locus of p, as a subfield of K containing x_2 and t. So K/F is a finite extension of function fields.

By (6.29) of Lemma 6.14 and the hypothesis on c we see that $\widetilde{f_{x_2 t}}(1, t)$ has a non-zero root. As $\widetilde{f_{x_2 t}}$ is a product of conjugates of p over $\mathbf{Q}$ up-to a factor in $\mathbf{Q}^*$ we conclude that $p(1, t)$ also has a non-zero root $t_0 \in \mathbf{C}$. So there is a valuation v of F, constant on $\mathbf{C}$, that corresponds to the point $(1, t_0)$ on the vanishing locus of f. In other words, $v(x_2 - 1) > 0$ and $v(t) = 0$. We extend v to $K \supset F$ and remark

$$b_1 v(R_1) + b_2 v(R_2) + b_3 v(R_3) = 0 \tag{6.35}$$

We also have $v(x_2) = v(x_2 + 1) = 0$ by the ultrametric triangle inequality. Moreover, y_2 is regular at v by Lemma 6.15(ii). By (6.27) and $\widetilde{c_2} \neq 0$ we must have $v(y_2 - 1) = 0$. Using these facts together with (6.19) yields

$$v(R_1) = v(x_2 - 1) + v(y_2 + 1) - v(x_2 + 1) - v(y_2 - 1) = v(x_2 - 1) + v(y_2 + 1) > 0.$$

We proceed to show that R_2 and R_3 have valuation 0. This will imply $b_1 = 0$ and then $b_2 = 0$, since $|b_2|\leq |b_1|$.

If $v(x_1) < 0$ or $v(y_1) < 0$ then both are negative according to Lemma 6.15(ii). In this case $v(R_2) = v(R_3) = 0$ is an immediate consequence of the ultrametric triangle inequality and (6.19). So we may assume that x_1 and y_1 are regular at v.

Since x_2 specializes to 1 at v we may use (6.26) and (6.28) to deduce $v(y_1 - 1) = 0$ and $v(x_1 - 1) = 0$, respectively. The expression for R_2 in (6.21) yields

$$v(R_2) = v(t). \tag{6.36}$$

So $v(R_2) = 0$ by the construction of v.

Next we use R_3 as given in (6.21). As x_2 specializes to 1 but x_1 and y_1 do not, we have $v((x_2 - x_1)/(x_2 - y_1)) = 0$. Therefore,

$$v(R_3) = v(t^{-1}) \tag{6.37}$$

and so $v(R_3) = 0$.

We have established $b_1 = b_2 = 0$. To conclude that b_3 also vanishes we proceed similarly. By (6.29) we find $\widetilde{f_{x_2t}}(1, 0) = 0$ and thus $p(1, 0) = 0$. We again fix a valuation w of F with $w(x_2 - 1) > 0$. But this time we impose $w(t) > 0$.

As above we first conclude that y_2 is regular with respect to w.

But what if $w(x_1) < 0$ or $w(y_1) < 0$? The expression f_2 from (6.15) is identically 0 on $\mathcal{C}$, so $w(x_1^2 + y_1^2) = w(x_2^2 + y_2^2) \geq 0$. The ultrametric triangle inequality implies $w(x_1) = w(y_1) < 0$. We recall (6.20) to find $w(t) = w(y_1 + 1) - w(x_1 + 1) = 0$ and this contradicts our choice of t. Hence $w(x_1) \geq 0$ and $w(y_1) \geq 0$.

To complete the proof we use again (6.26) and (6.28) and conclude $w(y_1 - 1) = w(x_1 - 1) = 0$. As above we have $w(R_3) = w(t^{-1}) \neq 0$. So $b_3 = 0$ because (6.35) holds with v replaced by w. ■

Proof of Proposition 6.10. Suppose $(b_1, b_2, b_3) \in \mathbf{Z}^3$ such that $R_1^{b_1} R_2^{b_2} R_3^{b_3}$ is constant on an irreducible component $\mathcal{C}$ of $\mathcal{C}_c \otimes \mathbf{C}$.

Swapping $\widetilde{c_1}$ with $\widetilde{c_2}$ corresponds to swapping (x_1, y_1) with (x_2, y_2) by (6.24). This has the effect of swapping R_1 with R_2 and thus swapping b_1 with b_2. So without loss of generality we may suppose $|b_2| \leq |b_1|$.

Multiplying c by -1 corresponds to swapping (x_1, x_2) with (y_1, y_2) by (6.25). This induces the transformation $(R_1, R_2, R_3) \mapsto (R_1^{-1}, R_2^{-1}, R_3)$ and replaces (b_1, b_2, b_3) by $(-b_1, -b_2, b_3)$. We also observe that

$$(c_1 + c_2 + 1)(-c_1 + c_2 + 1) - (c_1 - c_2 + 1)(-c_1 - c_2 + 1) = 4c_2.$$

So

$$(\widetilde{c_1} + \widetilde{c_2} + 1)(-\widetilde{c_1} + \widetilde{c_2} + 1) \neq 0 \tag{6.38}$$

or $(\widetilde{c_1} - \widetilde{c_2} + 1)(-\widetilde{c_1} - \widetilde{c_2} + 1) \neq 0$. Hence after possibly replacing c by $-c$ we may suppose that (6.38) holds.

If we suppose that $\widetilde{f_{x_2t}}$ is irreducible in $\mathbf{Q}[x_2, t]$, then Lemma 6.17 applies and we conclude $b_1 = b_2 = b_3 = 0$, as is desired in (i).

So say $\widetilde{f_{x_2t}}$ is not irreducible in $\mathbf{Q}[x_2, t]$. Then $\pm\widetilde{c_1} - \widetilde{c_2} + 1 = 0$ by Lemmas 6.11 and 6.14(iii). Now $\widetilde{f_{x_2t}}$ equals the specialization $\widetilde{\varphi(f_{x_2t})}$ with φ as in Lemma 6.14(ii). But this lemma implies that $\varphi(f_{x_2t})$ is absolutely irreducible when considered as a polynomial in x_2 and t and coefficients in $\mathbf{C}(c_1)$.

We can thus apply a variant of the Bertini-Noether Theorem, cf. Proposition VIII.7 [Lan62]. It states that $\varphi(f_{x_2t})$ remains irreducible in $\mathbf{C}[x_2, t]$ for all but at most finitely many complex specializations of c_1. Therefore, our $\widetilde{c_1}$ is contained in a finite set Σ of exceptions which is accounted for by part (ii) of the proposition's conclusion. The reference in Lang's book provides an effective way to determine this exceptional set provided one has access to an effective version of the Nullstellensatz of which there are many variants.

Let us suppose, as in (ii), that c is among such an exception and that $(b_1, b_2, b_3) \in \mathbf{Z}^3 \smallsetminus \{0\}$ is a multiple of $(\widetilde{c_1}, \widetilde{c_2}, 1)$. Then $|\widetilde{c_2}| \leq |\widetilde{c_1}|$ since $|b_2| \leq |b_1|$ and so $\widetilde{c_2} = \pm\widetilde{c_1} + 1$ implies $|\widetilde{c_1}| \geq 1/2$. The conclusion of Lemma 6.16(ii) contradicts the fact that $\widetilde{f_{x_2 t}}$ is not irreducible. Therefore, $(\widetilde{c_1}, \widetilde{c_2}, 1)$ and (b_1, b_2, b_3) are linearly independent. This completes the proof. ∎

Proof of Theorem 1.1, case $\Omega\mathcal{M}_3(4)^{\text{odd}}$. By Lemma 6.8 the image of $\mathcal{Y}$ lies in $\mathcal{S}^{\text{oa}}$, hence condition (i) of Theorem 2.6 is met. By Proposition 6.10 either condition (ii) of Theorem 2.6 is met (for $c \notin \Sigma$) or we are led to one of finitely many curves coming from the c in Σ. In this case, let $(c_1, c_2, c_3) \in \mathbf{Z}^3 \smallsetminus \{0\}$ be a primitive multiple of $(\widetilde{c_1}, \widetilde{c_2}, 1)$; it is not a multiple of a hypothetical vector (b_1, b_2, b_3) as in Proposition 6.10(ii). Now $R_1(y)^{c_1} R_2(y)^{c_2} R_3(y)^{c_3}$ is a root of unity and this relation is not constant on the whole curve. So $R(y)$ has bounded height, use for example Theorem 2.3. In any case, together with the degree bound in Proposition 6.7, Northcott's theorem shows that the number of points that could appear as cusps of an algebraically primitive Teichmüller curve in the stratum $\Omega\mathcal{M}_3(4)^{\text{odd}}$ is finite. By [BM12, Proposition 13.10] there are only finitely many Teichmüller curves in the stratum $\Omega\mathcal{M}_3(4)^{\text{hyp}}$. ∎

6.3 The hyperelliptic locus in $\Omega\mathcal{M}_3(2,2)^{\text{odd}}$

In this section we rely on the methods of [MW].

Proof of Theorem 1.1. Suppose there was an infinite sequence of algebraically primitive Teichmüller curves in this locus. We claim that there is no linear manifold M strictly contained in the hyperelliptic locus in $\Omega\mathcal{M}_3(2,2)^{\text{odd}}$ and containing an algebraically primitive Teichmüller curve. This is the analog of Theorem 1.5 in [MW].

In order to prove the claim we rely on [Wri, Theorem 1.5]. Since in this stratum there are no relative periods, the theorem reads $\dim(M) \cdot \deg_{\mathbf{Q}}(k(M)) \leq 6$, where $k(M)$ is the affine field of definition. Since M properly contains a Teichmüller curve, $\dim(M) > 2$. Since $k(M)$ is contained in F, the only possibility is $k(M) = \mathbf{Q}$. This can only happen if M equal the hyperelliptic locus in $\Omega\mathcal{M}_3(2,2)^{\text{odd}}$ by the argument given in [Wri, Corollary 8.1]. This contradiction completes the proof.

Next we claim that in this locus there exists a square-tiled surface, whose monodromy representation on the complement of $\langle \omega, \overline{\omega} \rangle$ is Zariski-dense in the 4-dimensional symplectic group. This is the analog of Theorem 1.3 in [MW], which does to directly apply, since we are in a codimension one subvariety of a stratum. Given this claim, we can apply Theorem 1.2 and Theorem 1.4 in loc. cit to conclude.

There are several methods to prove Zariski-density. This is shown using Lie algebra calculations in [MW]. Here, alternatively, we invoke a criterion of Prasad and Rapinchuk ([PR14]): If the representation contains two matrices M_1 and M_2 that do not commute, with M_2 of infinite order and the Galois group of the characteristic polynomial of M_1 is as large as possible for a symplectic

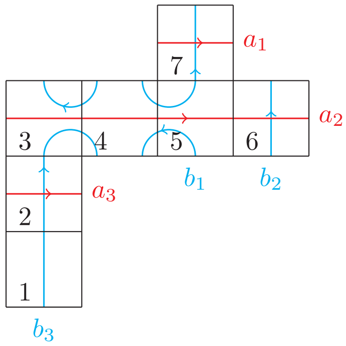

Figure 1: A square-tiled surface in the hyperelliptic locus of $\Omega\mathcal{M}_3(2,2)^{\mathrm{odd}}$

matrix, i.e. here the dihedral group with 8 elements, then the representation is Zariski dense or a product of $\mathrm{SL}_2(\mathbf{C}) \times \mathrm{SL}_2(\mathbf{C})$. Consequently, if we give three pairwise non-commuting elements of infinite order, one having the required Galois group and such that the common 1-eigenspaces of their second exterior power representation is just one-dimensional (generated by the symplectic form), then we have shown Zariski-density in $\mathrm{Sp}_4(\mathbf{C})$.

We use the square-tiled surface given in Figure 1 with side gluings horizontally by the permutation (1)(2)(3456)(7) and vertically by (123)(4)(57)(6). The representation of the Veech group elements $\left(\begin{smallmatrix} 1 & 4 \\ 0 & 1 \end{smallmatrix}\right)$ resp. $\left(\begin{smallmatrix} 1 & 0 \\ 6 & 1 \end{smallmatrix}\right)$ resp. $\left(\begin{smallmatrix} 13 & -6 \\ 24 & -11 \end{smallmatrix}\right)$ on the complement of $\langle \omega, \overline{\omega} \rangle$ in $H^1(X, \mathbf{Q})$ is given in the basis

$$\{4a_1 - a_2, -a_2 + 4a_3, -b_1 + b_2, -2b_2 + b_3\}$$

by

$$A = \left(\begin{smallmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right) \quad \text{resp.} \quad B = \left(\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -9 & 3 & 1 & 0 \\ -2 & 6 & 0 & 1 \end{smallmatrix}\right) \quad \text{resp.} \quad C = \left(\begin{smallmatrix} 11/2 & -3/2 & 3/2 & 0 \\ -3/2 & 3/2 & -1/2 & 0 \\ -15 & 5 & -4 & 0 \\ -6 & 2 & -2 & 1 \end{smallmatrix}\right).$$

Let $M_1 = AB$. Its characteristic polynomial is $x^4 - 25x^3 + 144x^2 - 25x + 1$ and has the required Galois group. With $M_2 = B$ and $M_3 = C$ the remaining conditions are easily checked. ■

7 Torsion and moduli

The following theorem from [Möl06a] gives strong constraints on the possible algebraically primitive Veech surfaces with multiple zeros. Recall that the Abel-Jacobi map is a homomorphism $\mathrm{Div}^0(X) \to \mathrm{Jac}(X)$. A *torsion divisor* on X is one whose image under the Abel-Jacobi map is a torsion point of $\mathrm{Jac}(X)$.

Theorem 7.1 ([Möl06a]). *If (X, ω) is an algebraically primitive Veech surface with zeros p and q, then $p - q$ is a torsion divisor.*

We say that (X, ω) has *torsion dividing* N, if for any pair of zeros p and q of ω, the order of $p - q$ divides N.

In this section, we show that this torsion condition gives strong control over the moduli of the cylinders of (X, ω) in any periodic direction. More precisely, consider a periodic direction of (X, ω), and let Γ be its dual graph. The *blocks* of Γ are the maximal subgraphs of Γ which cannot be disconnected by removing a single vertex. As the edges of Γ correspond to cylinders in our direction, this gives a partition of this set of cylinders, which we will also call blocks of cylinders.

Theorem 7.2. *Let (X, ω) be an algebraically primitive Veech surface, with torsion dividing N. Then for any block of cylinders $C_1, \ldots, C_n$ of some periodic direction of (X, ω), we have*

$$h[\mathrm{mod}(C_1) : \ldots : \mathrm{mod}(C_n)] \leq (n-1) \log N + \log(n-1)!\,.$$

In particular, there are only finitely many choices up to scale for the tuple of moduli in any block of cylinders.

Remark. Note that if ω has only one zero, the torsion condition is trivial. Likewise, the conclusion of Theorem 7.2 is trivial, as in this case the dual graph has only one vertex, so each block consists of a single cylinder.

Notation and definitions. We establish some basic notation and definitions which we will use throughout this section.

Given a hyperbolic Riemann surface X, we write ρ_X for its Poincaré metric, the unique conformal metric of constant curvature -1. We write $\ell_X(\gamma)$ for the length of a closed curve γ, and $\ell_X([\gamma])$ for the length of the shortest curve in the homotopy class $[\gamma]$.

Recall that a Riemann surface A is an *annulus* if $\pi_1(A) \cong \mathbf{Z}$. Every hyperbolic annulus is conformally equivalent to a unique (up to scale) *round annulus*, a planar domain bounded by concentric circles. The *modulus* of A is $\mathrm{mod}(A) = \frac{1}{2\pi} \log(R)$, where $R > 1$ is determined by $A \cong \{z : 1 < |z| < R\}$. If A has a flat, conformal metric with geodesic boundary, then $\mathrm{mod}(A) = h/w$, where h and w are its height and width respectively.

Suppose $\gamma \subset X$ is a simple closed geodesic, and let $\widetilde{X} \to X$ be the corresponding annular cover. The hyperbolic length of γ is related to the modulus of $\widetilde{X}$ by

$$\ell_X(\gamma) = \frac{\pi}{\mathrm{mod}(\widetilde{X})}. \tag{7.1}$$

We will use the notion of a flat family of stable curves (in the analytic category) also in the case that the fibers are of finite type. That is, a *family of stable curves* over a Riemann surface $\mathcal{C}$ is a two-dimensional analytic space $\mathcal{X}$ together with a holomorphic function $f \colon \mathcal{X} \to \mathcal{C}$ whose fibers are stable curves, that is connected one-dimensional analytic spaces whose only singularities are nodes, and with each component of the complement of the nodes a hyperbolic Riemann surface. A model of a family of nonsingular curves degenerating to a node is given by the family $\pi_k \colon V_k \to \Delta$, where

$$V_k = \{(x, y, t) \in \Delta^3 : xy = t^k\},$$

and $\pi_k(x, y, t) = t$. Roughly speaking, the family $f \colon \mathcal{X} \to \mathcal{C}$ is *flat* if near every singularity of a fiber of f, there is a change of coordinates where the family is $\pi_k \colon V_k \to \Delta$. We refer to [HK] for a more precise definition.

Given a family of stable curves $f\colon \mathcal{X} \to \mathcal{C}$, for $t \in \mathcal{C}$, we will write X_t for the fiber $f^{-1}(t)$. A subscript t will denote the restriction of various objects to X_t, for example if ω is a section of $\omega_{\mathcal{X}/\mathcal{C}}$, then ω_t is the restriction to X_t.

If p is a node of a fiber X_{t_0} for every t close to t_0, there is a homotopy class of a simple closed curve $[\gamma_t]$ which degenerate to p as $t \to t_0$, and such that the monodromy around t_0 preserves each homotopy class $[\gamma_t]$. We call these curves γ_t the *vanishing curves* of p.

Tall cylinders. Consider a family of flat surfaces (X_t, ω_t) which is degenerating to a stable curve as $t \to 0$, where the period of ω_t around a vanishing curve γ_t is real and independent of t. The following theorem makes precise the intuition that as $t \to 0$, the surfaces (X_t, ω_t) are developing cylinders of large modulus in this homotopy class.

Theorem 7.3. *Consider a proper flat family of stable curves $f\colon \mathcal{X} \to \mathcal{C}$ with p a node of a singular fiber X_{t_0}. Let ω be a meromorphic section of $\omega_{\mathcal{X}/\mathcal{C}}$ such that*

1. *p is not contained in any zero or polar divisor of (ω), and*
2. *the periods $\int_{\gamma_t} \omega_t$ around the vanishing curves of p are a real constant.*

Then for t sufficiently close to t_0, there is on (X_t, ω_t) a unique maximal, horizontal, flat cylinder C_t homotopic to γ_t.

Moreover, we have as $t \to t_0$,

$$\operatorname{mod}(C_t) \sim \frac{\pi}{\ell_{X_t}([\gamma_t])}.$$

Remark. Similar statements have appeared in the literature, see for example [Mas75] or [Bai07].

If ω is a one-form defined on a neighborhood of 0 in $\mathbf{C}$, with a simple pole at 0 of residue $1/2\pi i$, it is well-known that there is a change of coordinates ϕ such that $\phi^*\omega = \frac{1}{2\pi i}\frac{dz}{z}$. It follows that in the flat metric ω, a neighborhood of 0 is an infinite cylinder circumference 1. The proof of Theorem 7.3 will be based on the following relative version of this change of coordinates, giving a standard form for a section of $\omega_{\mathcal{X}/\mathcal{C}}$ near an isolated node, where the family is modeled by $\pi_k\colon V_k \to \Delta$.

Lemma 7.4. *Let ω be a holomorphic section of $\omega_{V_k/\Delta}$ such that for each vanishing curve γ_t, we have*

$$\int_{\gamma_t} \omega_t = 1.$$

Then there is a holomorphic change of coordinates Φ of V_k which fixes 0 and commutes with the projection π_k, such that

$$\Phi^*\left(\frac{1}{2\pi i}\frac{dx}{x}\right) = \omega. \tag{7.2}$$

Proof. The form ω may be written as the restriction to the fibers of

$$\omega = \frac{1}{2\pi i}\left(\frac{dx}{x} + f\,dx + g\,dy\right).$$

for some holomorphic functions f and g on Δ^3.

Consider now the first order PDE

$$xu_x - yu_y = xf - yg + h(xy,t), \tag{7.3}$$

where $h(s,t)$ is a to-be-determined holomorphic function defined near 0. We claim that there is a unique choice of h so that (7.3) has a holomorphic solution $u(x,y,t)$ defined near 0. To see this, consider the Taylor series representations,

$$f = \sum a_{ijl}x^iy^jt^l, \quad g = \sum b_{ijl}x^iy^jt^l, \quad \text{and} \quad u = \sum c_{ijl}x^iy^jt^l.$$

We define h by

$$h(s,t) = -\sum_{i,k}(a_{i-1,i,l} - b_{i,i-1,l})s^it^l,$$

so that $xf - yg + h(xy,t)$ has no $x^iy^it^l$ terms. We then obtain a solution to (7.3) by taking for $i \neq j$,

$$c_{i,j,l} = \frac{a_{i-1,j,l} - b_{i,j-1,l}}{i-j},$$

and making an arbitrary choice if $i = j$.

Now suppose $u(x,y,t)$ is a solution to (7.3), and define

$$\alpha = xe^u, \quad \beta = ye^{-u}, \quad \text{and} \quad \Phi = (\alpha, \beta).$$

If $xy = t^k$, then $\alpha\beta = xy = t^k$, whence Φ preserves the variety $V(xy - t^k)$.

We now compute,

$$\Phi^*\left(\frac{dx}{x}\right) = d(\log\alpha) = \frac{dx}{x} + u_x\,dx + u_y\,dy. \tag{7.4}$$

The equality $xy = t^k$ implies

$$y\,dx + x\,dy = 0. \tag{7.5}$$

(interpreting equality of forms as equality of the restrictions to the fibers). Combining (7.3) with (7.5) implies $u_x\,dx + u_y\,dy = f\,dx + g\,dy + h(xy,t)$, which when combined with (7.4) yields

$$\frac{1}{2\pi i}\Phi^*\left(\frac{dx}{x}\right) = \omega + \frac{1}{2\pi i}h(t^k,t)\frac{dx}{x}.$$

Since ω has by assumption unit periods on the vanishing curves, as does $\frac{1}{2\pi i}\frac{dx}{x}$, we must have $h(t^k,t) \equiv 0$, so Φ is the desired change of coordinates. ■

The previous lemma tells us that our flat family of curves is developing flat cylinders C_t of growing modulus. We now need a way to show that the hyperbolic length of the core curve of C_t in its intrinsic hyperbolic metric is nearly as small as its length as a curve on X. We will now show that it is enough for the boundary of C_t to lie in the "thick part" of X.

Recall that the injectivity radius of X at x is the length of the shortest essential loop through x. Given a curve γ on X, the γ-injectivity radius of X at x is the length of the shortest essential loop through x which is homotopic to γ. The *ϵ-thick part* (resp. γ,ϵ-thick part) of X is the locus of points with injectivity radius (resp. γ-injectivity radius) at least ϵ. We denote by $\mathrm{Thick}_\epsilon(X)$ and $\mathrm{Thick}_{\gamma,\epsilon}(X)$ the ϵ-thick and γ,ϵ-thick parts respectively.

Lemma 7.5. *Let X_n be a sequence of hyperbolic Riemann surfaces, each containing an essential annulus A_n, with $\alpha_n \subset A_n$ the unique primitive ρ_{A_n}-geodesic, and $\gamma_n \subset X_n$ the homotopic simple closed ρ_{X_n}-geodesic. Suppose that $\partial A_n \subset \mathrm{Thick}_{\gamma_n,\epsilon}(X)$, and $\ell_{A_n}(\alpha_n) \to 0$. Then $\ell_{X_n}(\gamma_n)/\ell_{A_n}(\alpha_n) \to 1$.*

Proof. Consider a single essential annulus $A \subset X$. Passing to the annular cover determined by the homotopy class of A, we may take X to be a hyperbolic annulus with $\partial A \subset \mathrm{Thick}_\epsilon(X)$. We write ℓ_A and ℓ_X for the lengths of the lengths of the corresponding simple closed geodesics in their respective Poincaré metrics, which are related to their respective moduli by (7.1).

If $\mathrm{mod}(X)$ is sufficiently large compared to $1/\epsilon$, then $\mathrm{Thick}_\epsilon(X)$ is the union of two round annuli (that is, bounded by concentric circles) T_1, T_2 of modulus

$$\mathrm{mod}(T_i) = \frac{1}{\ell_X} \sin^{-1}\left(\frac{\ell_X}{\epsilon}\right).$$

(This is a straightforward calculation in the band model of the hyperbolic plane.) If ℓ_A is sufficiently small, then A is not contained in the thick part, and it follows that $X = A \cup T_1 \cup T_2$. Let $B = A \setminus (T_1 \cup T_2)$. We then have

$$\mathrm{mod}(A) \leq \mathrm{mod}(X) = \mathrm{mod}(B) + \mathrm{mod}(T_1) + \mathrm{mod}(T_2) \leq \mathrm{mod}(A) + 2\,\mathrm{mod}(T_1),$$

where the inequalities follow from monotonicity of moduli. That is, $A \subset B$ implies $\mathrm{mod}(A) \leq \mathrm{mod}(B)$ (see for example [McM94]). In terms of lengths,

$$\frac{1}{\ell_A} \leq \frac{1}{\ell_X} \leq \frac{2}{\ell_X} \sin^{-1}\left(\frac{\ell_X}{\epsilon}\right) + \frac{1}{\ell_A}.$$

Letting $\ell_A \to 0$ with ϵ fixed, the claim follows. ■

Proof of Theorem 7.3. Passing to an open subset of $\mathcal{C}$, we may take our family of curves to be of the form $f\colon \mathcal{X} \to \Delta$, with f_0 the only singular fiber. The homotopy class $[\gamma_t]$ of the vanishing curve is then well-defined for every $t \neq 0$. We define a function $\iota\colon \mathcal{X} \to \mathbf{R}$, by taking $\iota(x)$ to be the γ_t-injectivity radius of X_t at x. We may define ι even on the singular fiber by taking it to be 0 at the node p, and $\iota \equiv \infty$ on an irreducible component of X_0 which does not contain p.

Let $\mathcal{X}'$ denote be the complement of the nodes of $\mathcal{X}$. The vertical hyperbolic metric ρ_{X_t} is continuous as a function on $T^*_{\mathcal{X}'/\Delta}$, as is shown in [HK] or [Wol90]. It follows that ι is continuous on $\mathcal{X}$.

Applying Lemma 7.4, we may take a compact neighborhood S of p whose intersection with each fiber X_t is either empty or a flat annulus C'_t which is contained in a unique maximal flat annulus C_t. As ∂S is compact, ι is bounded below on ∂S, so each $\partial C'_t$ is contained in $\mathrm{Thick}_{\gamma_t,\epsilon}(X_t)$ for some $\epsilon > 0$. Applying Lemma 7.5, we obtain $\mathrm{mod}(C') \sim \pi/\ell_{X_t}([\gamma_t])$ as $t \to 0$. As

$$\mathrm{mod}(C'_t) \leq \mathrm{mod}(C_t) \leq \frac{\pi}{\ell_{X_t}([\gamma_t])},$$

the same holds for $\mathrm{mod}(C_t)$. ■

The Abel metric. Let X be a Riemann surface and $D = \sum n_i z_i$ a degree zero divisor whose Abel-Jacobi image in $\mathrm{Jac}(X)$ is 0. By Abel's theorem, there is a meromorphic function $h_D\colon X \to \mathbf{P}^1$ with $(h_D) = D$. Let $\tau_D = h_D^*(dz/z)$, a meromorphic one-form having integral periods and a simple pole at each z_j with residue $n_j/2\pi i$. We will often simply write τ when we do not wish to emphasize the divisor D.

We call the associated flat metric $|\tau|$ the *Abel metric* associated to D. The horizontal direction of τ is periodic, as it comes from pulling back the flat metric on a cylinder. In this metric, a neighborhood of each z_i is a half-infinite cylinder of width n_i.

Now fix an algebraically primitive Veech surface with periodic horizontal direction. We identify the punctured unit disk Δ^* with the quotient of the hyperbolic plane by the corresponding parabolic element of the Veech group. Given $t \in \Delta^*$, we write (X_t, ω_t) for the corresponding flat surface with ω_t normalized so that the horizontal direction is periodic with periods independent of t. Let $f\colon \mathcal{X} \to \Delta^*$ be the associated universal curve, and $f\colon \overline{\mathcal{X}} \to \Delta$ the proper flat family of stable curves. The forms ω_t yield a section ω of $\omega_{\overline{\mathcal{X}}/\Delta}$.

The dual graphs of the (X_t, ω_t) may be canonically identified as the monodromy is composed of Dehn twists in the cylinders. We denote the dual graph of each by Γ, and write E and V for the set of vertices and edges. Given an edge e of Γ, we denote by $C_e(\omega_t)$ the corresponding cylinder of ω_t and let $[\gamma_e] \subset X_t \setminus Z(\omega_t)$ denote the homotopy class of a core curve of $C_e(\omega_t)$, oriented so that its period is positive.

We give each edge of Γ an orientation as follows. The bottom and top boundary components of $C_e(\omega_t)$ correspond to vertices v_1 and v_2 of Γ. If $v_1 \neq v_2$, give e the orientation pointing from v_1 to v_2. Otherwise choose an arbitrary orientation.

Let $(m_e)_{e\in E}$ be the tuple of moduli of the cylinders of some (X_t, ω_t). As the m_e have rational ratios, we may scale them uniquely so that they are relatively prime positive integers. We regard the tuple (m_e) as weights assigned to the edges of the graph Γ.

Possibly passing to a cover of Δ^*, we make take the zeros of ω to be sections z_i of the universal curve over Δ. Let $Z \subset \mathcal{X}$ denote the divisor of zeros of ω.

We denote by $\mathrm{Div}^0(Z)$ the group of divisors of degree 0 supported on D. Let K be the kernel of the Abel-Jacobi map $\mathrm{Div}^0(Z) \to \mathrm{Jac}(\mathcal{X}/\Delta^*)$, a finite index subgroup by the torsion condition.

A divisor $D = \sum n_i z_i \in K$ defines a meromorphic section τ_D of $\omega_{\mathcal{X}/\Delta}$ which has a simple pole along each z_j with residue $n_j/2\pi i$. The restriction to each X_t is the Abel metric defined above, which we denote $\tau_{D,t}$ or just τ_t.

Given $e \in \Gamma$ and $D \in K$, let

$$w_e(D) = \int_{\gamma_e} \tau_{D,t}.$$

Equivalently, $w_e(D)$ is the winding number of $h_D(\gamma_e)$ around 0.

Proposition 7.6. *If $w_e \neq 0$, then for t sufficiently large, $(X_t \setminus Z_t, \tau_t)$ has a unique maximal horizontal cylinder $C_t(\tau_t)$ homotopic to γ_e. Moreover, as $t \to 0$, we have*

$$\mathrm{mod}(C_t(\omega_t)) \sim \mathrm{mod}(C_t(\tau_t)). \tag{7.6}$$

If $w_e = 0$, then there is no such cylinder for any $t > 0$.

Proof. If $w_e \neq 0$, then the cylinder is provided by Theorem 7.3, and the moduli are asymptotic as both are asymptotic to the hyperbolic length of γ_t.

If $w_e = 0$, then there can be no such cylinder as a closed geodesic on a translation surface never has zero period. ■

Corollary 7.7. *Given z_1, z_2 distinct zeros of ω, let $D = N(z_1 - z_2) \in K$. Then for each edge e of Γ, we have $w_e(D) \in [-N, N] \cap \mathbf{Z}$.*

Proof. If $w_e \neq 0$, then for large t the surface $(X_t \setminus Z_t, \tau_t)$ has a flat cylinder C in the homotopy class γ_e. Under the meromorphic function $f_D \colon X_t \to \mathbf{P}^1$, the cylinder C is a degree $w_e(D)$ covering of a flat cylinder $C' \subset \mathbf{C}^*$. As the degree of f is N, we must have $|w_e(D)| \leq N$. ■

Since the cylinders of τ_t have nearly the same modulus as the corresponding cylinders of ω_t, we are able to use the geometry of τ_t to obtain the following strong restriction on the moduli.

Proposition 7.8. *Suppose that γ is a closed circuit of Γ which crosses the edges $e_{i_1}, \ldots, e_{i_n}$. Let $\sigma_i = \pm 1$ depending on whether γ crosses e_i respecting its orientation. Then*

$$\sum_i \sigma_i w_{e_i} m_{e_i} = 0. \tag{7.7}$$

Proof. Take a closed disk $0 \in \overline{\Delta}' \subset \Delta$ such that for each edge e with $w_e(D) \neq 0$, the cylinder $C_e(\tau_t)$ exists, and let $\overline{\mathcal{X}}' \to \overline{\Delta}'$ be the restriction of our family of curves to $\overline{\Delta}'$. If $w_e(D) \neq 0$, let $\mathcal{C}_e \subset \overline{\mathcal{X}}'$ be the open set consisting of the cylinders $C_e(\tau_t)$. We let J be the complement of these $\mathcal{C}_e$.

Since J is compact, the vertical metric $|\tau|$ is continuous away from the nodes, and τ has no poles at the nodes remaining in J, there is a uniform constant C such that for any distinct e and f and any $t \in \overline{\Delta}'$, if the boundaries of $C_e(\tau_t)$ and $C_f(\tau_t)$ lie in the same component of J, then any pair of points in these boundaries may be joined by a curve of length at most C.

For each $t \in \overline{\Delta}'$, we choose a lift of γ to a closed curve $\widetilde{\gamma}$ on X_t as follows. For each e_{i_k} with nonzero weight, let $\widetilde{\gamma}_k$ be a segment joining the boundary circles of $C_e(\tau_t)$ with the orientation indicated by γ. For e_{i_k} with zero weight, we let $\widetilde{\gamma}_k$ be the constant curve at the corresponding node. We close the curve by choosing for each k a curve in J of length at most C which joins the endpoints of γ_k and γ_{k+1}.

For each e with nonzero weight, we define $h_e(t)$ to be the height of the corresponding cylinder in the τ-metric, taken with the same sign as $w_e(D)$. In the ω-metric, the corresponding cylinder has modulus $cm_e \log|t|$ for some non-zero constant c, so by (7.6), we have

$$h_e(t) \sim c \log|t| w_e(D) m_e.$$

Now since τ_t has integral periods, the form $\operatorname{Im}\tau$ is exact, so all of its periods are 0. In particular, for some contribution D_t bounded by $|D_t| \leq nC$ stemming from the part of the path in J, we obtain in the limit $t \to 0$

$$0 = (\log|t|)^{-1} \int_{\widetilde{\gamma}} \operatorname{Im}\tau_t = (\log|t|)^{-1} \sum_i \sigma_i h_{e_i}(t) + (\log|t|)^{-1} D_t \to c \sum_i \sigma_i w_{e_i} m_{e_i},$$

implying (7.7). ■

The vertices of Γ determine a partition of the zeros of each ω_t, assigning to each vertex the set S_v of zeros which lie on the corresponding component of the spine of ω_t. We define for each vertex v and divisor $D = \sum n_i z_i$ the weight

$$c_v(D) = \sum_{z_i \in S_v} n_i.$$

For each v, let v_{in} and v_{out} be the set of incoming and outgoing edges.

Proposition 7.9. *For each vertex v of Γ,*

$$c_v(D) = \sum_{w \in v_{\text{in}}} w_e(D) - \sum_{w \in v_{\text{out}}} w_e(D) \tag{7.8}$$

Proof. On the component of X_0 corresponding to v, the form τ_0 has simple poles at each of the z_i of residue n_i. The nodes of this component correspond to the edges of Γ adjacent to v, and the residues of τ_0 there are the weights $w_e(D)$, with the sign determined by the orientation of e. Since the sum of residues is 0, (7.8) follows. ∎

Electrical networks. Propositions 7.8 and 7.9 have a physical interpretation in terms of electrical networks. We regard the graph Γ as an electrical network whose edges have resistances m_e, with incoming or outgoing current c_v at each vertex v, and with current w_e across any edge. Then (7.8) reflects Kirchhoff's current law "what goes in must go out." , and (7.6) says, using Ohm's law, that the potential drop around any closed loop is zero. So our w_e and m_e indeed satisfy all the axioms of an electrical network.

In this language, our task in the remainder of the section is the following: Given a natural number N and an electrical network with the property that passing a current of N through any two vertices results in integral currents bounded by N along any edge (Corollary 7.7), show that there is a finite number (depending on N) of possibilities for the resistances.

The idea of studying Riemann surfaces through electrical networks has appeared elsewhere. See for example [Dil01].

Torsion determines moduli. Equation (7.7) and Corollary 7.7 together put strong constraints on the moduli m_e, which we will now show determines these moduli up to finitely many choices.

For each $D \in K$, the weights $w_e(D)$ define a relative homology class

$$\widetilde{w}_e(D) = \sum_e w_e(D)[e] \in H_1(\Gamma, V),$$

with V the vertex set of Γ, using the previously chosen orientation of the edges of Γ.

Let $\widetilde{H}_0(V)$ denote the kernel of the natural homomorphism $H_0(V) \to H_0(\Gamma)$. The assignment $D \mapsto c_D(v)$ is a group homomorphism $\rho \colon \operatorname{Div}^0(Z) \to \widetilde{H}_0(V)$.

The tuple of moduli $\boldsymbol{m} = (m_e)_{e \in E}$ determines a group homomorphism $\phi_{\boldsymbol{m}} \colon H_1(\Gamma, V) \to H^1(\Gamma, V)$ defined by $\phi_{\boldsymbol{m}}([e]) = m_e[e^*]$, where $[e^*]$ is the corresponding cocycle killing the other edges.

Proposition 7.10. *The map $\widetilde{w} \colon K \to H_1(\Gamma, Q)$ is a group homomorphism.*

Proof. For any two divisors, we have $\tau_{D_1+D_2} = \tau_{D_1} + \tau_{D_2}$. As the $w_e(D)$ are periods of τ_D, it follows that $\widetilde{w}$ is a homomorphism. ■

As $K \subset \mathrm{Div}^0(Z)$ is finite index, there is an induced homomorphism (abusing notation) $\widetilde{w}\colon \mathrm{Div}^0(Z) \otimes \mathbf{Q} \to H_1(\Gamma, V; \mathbf{Q})$.

Below we write $\partial\colon H_1(\Gamma, V) \to \widetilde{H}_0(V)$ and $j\colon H^1(\Gamma, P) \to H^1(\Gamma)$ for the homomorphisms of from the respective long exact sequences.

Proposition 7.11. *The homomorphism $\widetilde{w}$ satisfies $\partial \circ \widetilde{w} = \rho$ and descends to a splitting $w\colon \widetilde{H}_0(V; \mathbf{Q}) \to H_1(\Gamma, V; \mathbf{Q})$ of the homology exact sequence of the pair (Γ, V).*

The composition $j \circ \phi_{\boldsymbol{m}} \circ w = 0$ is trivial, and moreover

$$\mathrm{Im}(\phi_{\boldsymbol{m}} \circ w) = \mathrm{Ker}(j). \tag{7.9}$$

To summarize this discussion, these maps fit into the following commutative diagram with the composed map $\widetilde{H}_0(V) \to H^1(\Gamma)$ trivial and all (co)homology having $\mathbf{Q}$-coefficients.

$$\begin{tikzcd}
 & & & \mathrm{Div}^0(Z)\otimes\mathbf{Q} \arrow[dl, "\widetilde{w}"'] \arrow[d, "\rho"] & \\
0 \arrow[r] & H_1(\Gamma) \arrow[r] & H_1(\Gamma, V) \arrow[r, "\partial"] \arrow[d, "\phi_{\boldsymbol{m}}"] & \widetilde{H}_0(V) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^0(V)/H^0(\Gamma) \arrow[r, "\delta"] & H^1(\Gamma, V) \arrow[r, "j"] & H^1(\Gamma) \arrow[r] & 0
\end{tikzcd}$$

Proof. That $\partial \circ \widetilde{w} = \rho$ is exactly the content of Proposition 7.9.

That $j \circ \phi_{\boldsymbol{m}} \circ w = 0$ means that $\phi_{\boldsymbol{m}} \circ w$ kills every cycle of Γ. This is exactly the content of Proposition 7.8.

We now claim that

$$\mathrm{Im}(\phi_{\boldsymbol{m}} \circ \widetilde{w}) = \mathrm{Ker}(j). \tag{7.10}$$

Since $\partial \circ \widetilde{w} = \rho$, we have $\mathrm{Ker}(\widetilde{w}) \subset \mathrm{Ker}(\rho)$, so $\dim \mathrm{Ker}(\widetilde{w}) \le |Z| - |V|$, and $\dim \mathrm{Im}(\widetilde{w}) \ge |V| - 1$. Then since $\phi_{\boldsymbol{m}}$ is injective, $\dim \mathrm{Im}(\phi_{\boldsymbol{m}} \circ \widetilde{w}) \ge |V| - 1 = \dim \mathrm{Ker}(j)$. Since $\mathrm{Im}(\phi_{\boldsymbol{m}} \circ \widetilde{w}) \subset \mathrm{Ker}(j)$, they must be equal.

Now (7.10) implies $\dim \mathrm{Ker}(\widetilde{w}) = |Z| - |V|$, so $\mathrm{Ker}(\widetilde{w}) = \mathrm{Ker}(\rho)$. Thus $\widetilde{w}$ descends to the desired splitting w. The remaining claims about w then follow from the corresponding properties of $\widetilde{w}$ which we have already proved. ■

Consider now the group $\mathbf{G}(\mathbf{Q})^{|E|}$, acting diagonally on $H^1(\Gamma, V; \mathbf{Q})$ by $(q_e)_{e \in E} \cdot [e_0^*] = q_{e_0}[e_0^*]$. Let $B \subset \mathbf{G}(\mathbf{Q})^{|E|}$ be the subgroup of $(q_e)_{e \in E}$ for which $q_e = q_f$ whenever the edges e and f lie in the same block of Γ.

Lemma 7.12. *B is exactly the subgroup of $\mathbf{G}(\mathbf{Q})^{|E|}$ which stabilizes $\mathrm{Im}(\delta)$.*

Proof. That B stabilizes $\mathrm{Ker}(j)$ is clear, since any circuit γ can be written as a sum of circuits, each contained in only one block.

For the converse, suppose $\boldsymbol{q} = (q_e)_{e\in E}$ stabilizes $\mathrm{Im}(\delta)$. Given a vertex v, we have $\delta v = \sum_{e\in E(v)}[e^*]$, where the sum is over the edges incident to v, with the outward-pointing orientation. As $\boldsymbol{q}$ stabilizes $\mathrm{Im}(\delta)$, the cocycle

$$\boldsymbol{q}\cdot\delta v = \sum_{e\in E(V)} q_e[e^*]$$

lies in the kernel of j. Suppose that e_1 and e_2 are two edges incident to v which lie in the same block B. As B is 2-connected, by Menger's theorem (see [BM76]) there is a circuit $\gamma \subset B$ which passes through e_1 and e_2 but through no other edge incident to v. As $(\boldsymbol{q}\cdot\delta v)(\gamma) = 0$, we obtain $q_{e_1} = q_{e_2}$. Applying this repeatedly, it follows that $q_{e_1} = q_{e_2}$ whenever e_1 and e_2 lie in the same block, so $\boldsymbol{q} \in B$. ■

Proof of Theorem 7.2. Consider a divisor $D = N(z_1 - z_2) \in K$. By Corollary 7.7, we have $w_e(D) \in [-N, N] \cap \mathbf{Z}$ for each edge e. In particular, given N there are only finitely many possibilities for the splitting w, so we may take w to be known.

Now any possible tuple of moduli $\boldsymbol{m}$ satisfies $\mathrm{Im}(\phi_{\boldsymbol{m}} \circ w) = \mathrm{Ker}(\delta)$ by Proposition 7.11. By Lemma 7.12, if $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$ are two such tuples, we must have $\phi_{\boldsymbol{m}_1} \circ \phi_{\boldsymbol{m}_2}^{-1} \in B$, which means exactly that in each block the m_i are determined up to scaling.

To make this effective, consider a tuple $(m_1, \ldots, m_n)$ of moduli from a single block B of Γ. Given D as above together with a circuit $\gamma \subset B$, we obtain a linear function f in the m_i expressing the condition that $\phi_{\boldsymbol{m}}(w(D))(\gamma) = 0$. Each such f has integral coefficients of absolute value at most N, so $h(f) \leq \log(N)$. Among such linear equations, we may choose $f_1, \ldots, f_{n-1}$ which determine (m_i) up to scale. From Cramer's rule, one may deduce that

$$h(m_1, \ldots, m_n) \leq (n-1)\log N + \log(n-1)!$$ ■

8 A general finiteness criterion

We say that two stable forms are *pantsless-equivalent* if the underlying stable curves are isomorphic by a map which identifies the one-forms on any irreducible component which is not a pair of pants. A collection of Teichmüller curves in a stratum $\mathcal{S}$ is *pantsless-finite* if among their cusps there are only finitely many pantsless-equivalence classes of stable forms.

In this section, we prove:

Theorem 8.1. *In a fixed stratum $\mathcal{S}$, any pantsless-finite collection of algebraically primitive Teichmüller curves is finite.*

Theorem 8.1 is an easy consequence of the methods in [BM12] if the Teichmüller curves in the collection are generated by Veech surfaces with just one zero (and thus have only irreducible degenerations). All the results on torsion orders are of course void in this case – and the final proof works without them.

The cusps control the torsion orders. We first observe that a pantsless-finite collection of algebraically primitive Teichmüller curves has uniform torsion order bounds, allowing us to apply Theorem 7.2 to control the moduli.

Proposition 8.2. *For any pantsless-finite collection of algebraically primitive Teichmüller curves, there is a uniform bound on the torsion orders of $D_1 - D_2$ for any two zero-sections D_i.*

Half of the argument is isolated in the following lemma for later use. The converse statement implicitly appears also in [Möl08] at the end of Section 2. We refer to [HM98] for the notion of admissible coverings.

Lemma 8.3. *Let $f: \overline{\mathcal{X}} \to \overline{C}$ be a family of curves, smooth over $C \subset \overline{C}$. Let $s_1, s_2: \overline{C} \to \overline{\mathcal{X}}$ be two sections with image D_1, D_2, whose difference $D_1 - D_2$ is N-torsion in the family of Jacobians $\mathrm{Jac}(\mathcal{X}/C)$. Then for each fiber X of f there exists an admissible cover h from X to a tree of projective lines of degree N.*

Let X_∞ be a singular fiber of f and suppose that both sections pass through the same component Y of X_∞. Then there exists a partition $S = \{S_j, j \in J\}$ of the nodes $p_1, \ldots, p_n$ of Y and $h: Y \to \mathbf{P}^1$ of degree N with $h^{-1}(\{0\}) = D_1 \cap X_\infty$, with $h^{-1}(\{\infty\}) = D_2 \cap X_\infty$ and such that h is constant on each part S_j of the partition.

Conversely, suppose that s_1, s_2 are two sections, whose difference is torsion and suppose that there is a component Y and a map h of degree N, as above. Then $s_1 - s_2$ is N-torsion.

Proof. For smooth fibers, an admissible cover is just a morphism $h: X \to \mathbf{P}^1$, and the existence of such a morphism follows from the definition of the Jacobian and Abel's theorem. The space of admissible covers is compact, so such an admissible cover exists also for an appropriate semistable model of the singular fiber.

Remove the vertex corresponding to Y from the dual graph of the semistable curve X_∞. The remaining graph decomposes into connected components and we partition the nodes of Y according the the component they are adjacent to. Let z_1 and z_2 be two nodes in the same partition S_j. Since the image of h is a tree T, the components of X_∞ corresponding to S_j map to a branch of $T \setminus h(Y)$. This branch intersects $h(Y)$ in a single point b and $h(z_i) = b$ for all $z_i \in S_j$.

For the converse statement we claim that in a neighborhood U of a singular fiber with D_1 and D_2 limiting to the same component, $D_1 - D_2$ defines a section of the relative Picard scheme $\mathrm{Pic}^0(\overline{\mathcal{X}}/U)$. This scheme parameterizes line bundles on $f^{-1}(\mathcal{X})$ that are of total degree zero, when restricted to any fiber, up to bundles pulled back from the base 0. In fact, any section can be extended to the Néron model of the family of Jacobians, by the universal property of the Néron model ([BLR90, Theorem 9.5.4 b)]). Since s_1 and s_2 pass through the same component Y of X_∞, their difference maps to the connected component of the identity of the Néron model, which is $\mathrm{Pic}^0(\overline{\mathcal{X}}/U)$ ([BLR90, Definition 1.2.1]).

The existence of the map h shows that the bundle $\mathcal{O}_{X_\infty}(N(D_1 - D_2)|_{X_\infty})$ is isomorphic to the trivial bundle, i.e. $(D_1 - D_2)|_{X_\infty}$ is of order N in $\mathrm{Pic}^0(\overline{\mathcal{X}}/U)$. Since the torsion order is constant in families, this implies the claim. ■

Proof of Proposition 8.2. Suppose s_1 and s_2 are zero sections of the universal curve of some Teichmüller curve in a pantsless-finite collection of Teichmüller curves. If in some smooth fiber there is a saddle connection joining s_1 to s_2, then rotating the form so that this direction is horizontal and following the Teichmüller geodesic flow to the boundary, we obtain a singular fiber where s_1 and

s_2 intersect the same irreducible component (Y, η). Since η has more than one zero, Y is not pants, so by assumption, there are only finitely many possibilities for (Y, η). By Lemma 8.3, there are only finitely many possibilities for the order of $s_1 - s_2$, since it is determined by Y. In particular, $N(s_1 - s_2) = 0$ for some N depending only on $\mathcal{S}$.

It is possible that two zeros z, z' of a translation surface are not joined by a saddle connection; however, taking any curve joining z to z, the shortest curve in its homotopy class is a union of saddle connections (see [FLP79]). Taking $z = z_1, z_2, \ldots, z_n = z'$, to be the successive endpoints of these saddle connections, we obtain $z - z' = \sum(z_i - z_{i+1})$ with $N(z_i - z_{i+1}) = 0$ for each i, so $N(z - z') = 0$. ■

Geometry of cylinder widths. Given an algebraically primitive Veech surface (X, ω) with periodic horizontal direction, the widths $(r_1, \ldots, r_n)$ and heights $(s_1, \ldots, s_n)$ of its cylinders may be regarded as vectors in $\mathbf{R}^g$ via the g embeddings of the trace field F. We now show that the geometry of this set of vectors nearly determines – and is determined by – the tuple of moduli (m_i) of the cylinders. This will give us more control over the moduli then provided by Theorem 7.2.

Applying the Teichmüller flow to (X, ω) scales the widths (r_i) by a constant multiple, so generally they should be regarded as a point of $\mathbf{P}^{g-1}(F)$; however, there is an almost canonical choice for their scale.

We say that (X, ω) is *normalized* if its cylinder heights and widths satisfy the following properties:

- each r_i and s_j either belong to F, or for some real quadratic extension $K = F(\alpha)$, they belong to $W = \operatorname{Ker} \operatorname{Tr}_{K/F}$ (using in either case an implicit real embedding);
- $s_i/r_i \in \mathbf{Q}^+$ for each i; and
- for every pair of oriented closed curves α, β on X, with α horizontal, we have
$$\alpha \cdot \beta = \left\langle \int_\alpha \omega, \operatorname{Im} \int_\beta \omega \right\rangle, \tag{8.1}$$
where $\alpha \cdot \beta$ is the intersection pairing on homology, and
$$\langle x, y \rangle = \operatorname{Tr}_{F/\mathbf{Q}}(xy).$$
(Note that $xy \in F$ whenever $x, y \in W$.)

Proposition 8.4. *There is a diagonal matrix $D \in \mathrm{GL}_2^+(\mathbf{R})$ such that $D \cdot (X, \omega)$ is normalized. Such a D is unique up to multiplication by $\left(\begin{smallmatrix} c & 0 \\ 0 & c^{-1} \end{smallmatrix}\right)$ for any c with $c^2 \in \mathbf{Q}$.*

Proof. Since (X, ω) is an eigenform for real multiplication, $H_1(X; \mathbf{Q})$ is equipped with the structure of an F vector space. We let $M \subset H_1(X; \mathbf{Q})$ be the subspace spanned by the horizontal cylinders of (X, ω), and $N = H_1(X; \mathbf{Q})/M$. As real multiplication preserves the span of short curves, M and N are F vector spaces (see for example [BM12, Proposition 5.5]). As $\omega \colon M \to \mathbf{R}$ and $\operatorname{Im} \omega \colon N \to \mathbf{R}$ are F-linear and spanned by the r_i and s_j respectively, we may suppose they belong to F after applying an appropriate diagonal element.

We now claim we may choose uniquely a diagonal matrix $\left(\begin{smallmatrix}\mu & 0\\ 0 & 1\end{smallmatrix}\right)$ so that (8.1) holds. To see this, note that both pairings in (8.1) are bilinear pairings $M \times N \to \mathbf{Q}$ compatible with the F vector space structure in the sense that $(\lambda \cdot x, y) = (x, \lambda y)$. The space of such pairings is a one-dimensional vector space over F, so a unique such μ exists.

Making this normalization, we still have the freedom to apply a diagonal matrix $\left(\begin{smallmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{smallmatrix}\right)$. Even if $\alpha \notin F$, the trace pairing in (8.1) is still defined and independent of α as $(\alpha x)(\alpha^{-1} y) = xy \in F$. We take $\alpha = \sqrt{s_1/r_1}$. Note that as (X, ω) is Veech, the moduli of its cylinders have rational ratios, so

$$\frac{\alpha^{-1} s_i}{\alpha r_i} = \frac{r_1 s_i}{r_i s_1} = \frac{\operatorname{mod}(C_i)}{\operatorname{mod}(C_1)} \in \mathbf{Q}^+,$$

which is the second required property of our normalization. Finally note that the αr_i and $\alpha^{-1} s_i$ either lie in F or $W \subset K = F(\alpha)$ depending on whether or not s_1/r_1 has a square root in F.

Uniqueness follows, since applying $\left(\begin{smallmatrix}c & 0\\ 0 & c^{-1}\end{smallmatrix}\right)$ to (X, ω) preserves the rationality condition if and only if $c^2 \in \mathbf{Q}$. ■

Remark. Proposition 8.4 is a generalization of some basic observations from [BM12] in the case where (X, ω) has g horizontal cylinders, namely that (X, ω) may be normalized so that the heights of the cylinders are dual to the widths with respect to the trace pairing, and the vectors $\theta(r_i)$ are orthogonal in $\mathbf{R}^g$.

Taking such a normalization, let $\iota_1, \ldots, \iota_g$ be the g real embeddings of F, and if we have taken an extension K, extend each ι_i to the real embedding of K for which $\iota_i(\alpha) > 0$. For $x \in F$ or K, we write $x^{(i)}$ for $\iota_i(x)$ and θ for the associated embedding of F or W in $\mathbf{R}^g$. The vectors $\theta(r_i)$ are then a basis of $\mathbf{R}^g$. While our normalization was not unique, a different choice of normalization only changes this basis by a constant multiple, so the shape of this basis of $\mathbf{R}^g$ does not depend on any choices.

In the sequel, it will not matter whether we have passed for a quadratic extension of F or not, so we will simply write W for either F or $\operatorname{Ker} \operatorname{Tr}_{K/F}$.

Now suppose that (X, ω) is normalized as above. Let Γ be the corresponding dual graph. We orient the edges of Γ as in §7. Assign to each edge e of Γ the weight $r_e \in W$ to be the width of the corresponding cylinder of (X, ω), as well as the weight $m_e = s_e/r_e \in \mathbf{Q}$, the modulus of this cylinder. At each vertex of Γ, the sum of the incoming weights equals the sum of the outgoing weights (we called such an object a W-weighted graph in [BM12]), so we may define

$$\rho_{\boldsymbol{r}} = \sum_e r_e[e] \in H_1(\Gamma; W) \quad \text{and} \quad \sigma_{\boldsymbol{r}} = \sum_e m_e r_e [e^*] \in H^1(\Gamma; W). \tag{8.2}$$

Since the r_e span W, ρ defines an isomorphism $\rho^*_{\boldsymbol{r}} \colon H^1(\Gamma; \mathbf{Q}) \to W$. From this point of view, we may interpret (8.1) as saying for all $x \in H_1(\Gamma; \mathbf{Q})$ and $y \in H^1(\Gamma; \mathbf{Q})$,

$$(x, y) = \langle \sigma^*_{\boldsymbol{r}}(x), \rho^*_{\boldsymbol{r}}(y) \rangle, \tag{8.3}$$

where the right pairing is the trace pairing on W defined above and the left is the canonical pairing between homology and cohomology. Since $\rho^*_{\boldsymbol{r}}$ is an isomorphism, it follows that $\sigma^*_{\boldsymbol{r}}$ is an isomorphism as well.

Now let $\Gamma = \Gamma_1 \cup \ldots \cup \Gamma_k$ be its decomposition into blocks, and define $B_i \subset W$ to be the span of the vectors r_e with e in Γ_i.

Proposition 8.5. *The subspaces $B_1, \dots, B_k$ span W and are orthogonal.*

Proof. First, since $H^1(\Gamma; \mathbf{Q}) = \oplus H^1(\Gamma_i; \mathbf{Q})$, $\rho^*_{\boldsymbol{r}}$ is an isomorphism, and B_i is the image of $\rho^*_{\boldsymbol{r}}$ restricted to $H^1(\Gamma_i; \mathbf{Q})$, we see that the B_i span W.

Now let e be an edge in Γ_i and $s \in B_j$ for $i \neq j$. We claim that $s = \sigma^*_{\boldsymbol{r}}(\gamma)$ for some $\gamma \in H_1(\Gamma_j; \mathbf{Q})$. Assuming this claim, we obtain

$$\langle r_e, s\rangle = ([e^*], \gamma) = 0,$$

since γ_i and e are in different blocks.

To see the claim, note that $\sigma^*_{\boldsymbol{r}} H_1(\Gamma_j; \mathbf{Q}) \subset B_j$, since the m_e are rational. As they have the same dimension, these spaces are in fact equal, so we may find the desired γ. ∎

Consider now a single block Γ_0 in Γ. We label its edges $e_1, \dots, e_n$, with edge e_i having weight r_i and modulus m_i. We define Q to be its matrix of traces

$$(Q_{ij}) = (\langle r_i, r_j\rangle),$$

which determines the collection of vectors $\theta(r_i)$ up to orthogonal rotation of $\mathbf{R}^g$.

Proposition 8.6. *The matrix of traces Q is determined by the moduli $m_1, \dots, m_n$ and the graph γ_0. Moreover, Q is inversely proportional to the m_i in that if the m_i are multiplied by some $q \in \mathbf{Q}$, the matrix Q is divided by q.*

Moreover, $h(Q) \leq C_1 h(m_1, \dots, m_n) + C_2$ for constants C_1 and C_2 which depend only on the graph Γ_0.

Proof. Choose a spanning tree $T \subset \Gamma_0$, and reorder the edges so that $e_1, \dots, e_m$ belong to $\Gamma_0 \setminus T$ and $e_{m+1}, \dots, e_n$ belong to T.

Suppose we are given $r_1, \dots, r_m \in W$. There is then a unique choice of $r_{m+1}, \dots . r_n$ such that $\partial \sum r_i[e_i] = 0$. Namely, let $\partial \colon H_1(\Gamma_0; V) \to \widetilde{H}_0(\Gamma_0; V)$ (where V is the set of vertices of Γ_0) be given by the block matrix (A, B) (using the basis $[e_i]$ of $H_1(\Gamma_0; W)$). Since T is a spanning tree, the $n - m \times n - m$ matrix B is invertible. Let $K = (k_{ij}) = -B^{-1}A$ and $L = \left(\begin{smallmatrix} I \\ K \end{smallmatrix}\right)$. Then given $r_1, \dots, r_m$ in W, we set

$$r_i = \sum_{j=1}^{n} l_{ij} r_j.$$

With $\boldsymbol{r} = (r_1, \dots, r_n)$, this is the unique vector extending $(r_1, \dots, r_m)$ such that $\partial \rho_{\boldsymbol{r}} = 0$.

Let M be the diagonal matrix with entries $m_1, \dots, m_n$ on the diagonal.

For $1 \leq i \leq m$, let $\gamma_i \in H_1(\Gamma_0)$ be the unique circuit which crosses e_i once, positively oriented, then completes the circuit by the unique path in T joining the endpoints. We write $\gamma_i = \sum_i n_{ij} e_j$, and let $N = (n_{ij})$. Finally, let $P = NML$. This matrix P has been constructed so that if $s_i = \sigma_{\boldsymbol{r}}(\gamma_i)$ are the periods of the γ_i, R is the $m \times g$ matrix defined by $(R_{ij}) = (r_i^{(j)})$, and S is the matrix defined by $(S_{ij}) = (s_i^{(j)})$, then $S = PR$.

Now, for $1 \leq i, j \leq m$, each $([e_i^*], \gamma_j) = \delta_{ij}$. Then by (8.3), we have $\langle r_i, s_j\rangle = \delta_{ij}$, which means that $SR^t = I$, so $RR^t = P^{-1}$. The full matrix of traces is then

$$Q = \begin{pmatrix} P^{-1} & P^{-1}L^t \\ LP^{-1} & LP^{-1}L^t \end{pmatrix},$$

which depends only on Γ_0 and the m_i, as desired.

To prove the height bound, we use the inequalities,

$$h(MN) \le h(M) + h(N) + \log n,$$
$$h(M^{-1}) \le (n-1)h(M) + \log(n-1)!\,,$$

where in either case n is the number of columns of M. Since, A, B, and N consists only of zeros and ones, they have height 0, and the bound for $h(Q)$ follows. ■

Remark. Keeping careful track of constants, one obtains the explicit bound,

$$h(Q) \le (e-v)h(m_1,\dots,m_e) + (e-v)\log e^2(v-e)! + \log(e-v)!\,(v-1)!^2\,(e-v+1)^2,$$

where v and e are the number of vertices and edges of Γ_0.

Taken together, Theorem 7.2 and Propositions 8.5 and 8.6 gives strong constraints for the geometry of the cylinder widths $v_i = \theta(r_i)$ in a normalized periodic direction in terms of a given N which is a multiple of the torsion orders of (X, ω). By Proposition 8.5, the v_i lie in orthogonal blocks $B_i \subset \mathbf{R}^g$ corresponding to the partition of the dual graph Γ into blocks. For each block Γ_i of Γ, the torsion order N determines its vector of moduli $[m_1 : \dots : m_{k_i}]$ up to finitely many choices by Theorem 7.2. Proposition 8.6 then tells us that this vector of moduli determines the vectors v_j in B_i up to scaling and orthogonal rotation.

While Theorem 7.2 tells that a bound for torsion orders gives the tuple of moduli of a periodic direction up to scaling in each block, these torsion orders don't give any information about the scale of these tuples of moduli in different blocks. Proposition 8.6 allows us to control this scale using knowledge of the residues of the stable form where two blocks intersect.

Proposition 8.7. *Given a pantsless-finite collection of algebraically primitive Teichmüller curves, there is a constant M such that for any periodic direction of a surface in this collection, the ratio of moduli of any two cylinders is at most M.*

Proof. By Proposition 8.2, the torsion orders of any curve in this collections divide some constant N.

Let (X, ω) be a Veech surface with periodic horizontal direction which lies on one of the Teichmüller curves in our collection, let Γ be its dual graph, and let $B, B' \subset \Gamma$ be adjacent blocks containing edges $e_1, \dots, e_k$ and $e'_1, \dots, e'_l$, ordered so that e_1 and e'_1 meet at the common separating vertex v.

We take (X, ω) to be normalized as above, which means in particular that the horizontal cylinders have rational moduli. We write $qm_1, \dots, qm_k$ and $q'm'_1, \dots, q'm'_l$ for the moduli of the cylinders corresponding to e_i and e'_i, with $q, q' \in \mathbf{Q}$ chosen so that the tuples of m_i and m'_i are both relatively prime integers. We let r_i and r'_i denote the corresponding cylinder widths.

By Theorem 7.2, the m_i and m'_i are determined up to finitely many choices by N, so we may take them to be fixed. It then suffices to bound q'/q.

Since Γ has no separating edges, there must be at least four edges incident to v, so the corresponding component of the limiting stable form is not pants, so it is known up to finitely many choices. In particular, we may take $\lambda = r_1/r_1'$ to be fixed. It follows that

$$\mathrm{Tr}(r_1^2) = \mathrm{Tr}(\lambda^2 {r_1'}^2) \leq \|\lambda^2\|_\infty \mathrm{Tr}({r_1'}^2), \tag{8.4}$$

Where $\|\lambda\|_\infty = \max_i |\lambda^{(i)}|$.

By Proposition 8.6, we have

$$\mathrm{Tr}(r_1^2) = \frac{t}{q} \quad \text{and} \quad \mathrm{Tr}({r_1'}^2) = \frac{t'}{q'}, \tag{8.5}$$

where $t \in \mathbf{Q}$ is determined by the corresponding moduli m_i and graph B, and likewise for t'. Combining (8.4) and (8.5), we obtain

$$\frac{q'}{q} \leq \|\lambda^2\|_\infty \frac{t'}{t}. \qquad \blacksquare$$

Proposition 8.8. *Given a pantsless-finite collection of algebraically primitive Teichmüller curves, there is a constant K such that for any periodic direction of a surface in this collection, the ratio of widths of any two cylinders or saddle connections is at most K.*

Proof. Consider a surface (X, ω) on one of these Teichmüller curves, and let C_1 and C_2 be adjacent horizontal cylinders, in the sense that their boundaries share a saddle connection. Suppose that one of the cylinders, say C_1, has distinct zeros of ω on its upper and lower boundaries (which is true if C_1 does not correspond to a loop in the dual graph Γ). We claim that there is a universal constant L_1 such that $w(C_2) \leq L_1 w(C_1)$.

To see this, first shear the surface so that a vertical saddle connection γ joins the zeros in ∂C_1. Let δ be a parallel geodesic which crosses C_1 and C_2. Since the surface is Veech, δ is closed. These curves limit to curves on a stable curve at infinity which is not pants, since it contains more than one zero. By pantsless finiteness, $\ell(\delta) \leq C\ell(\gamma)$ for a universal constant C. We then have,

$$\mathrm{mod}(C_2)w(C_2) = h(C_2) \leq \ell(\delta) \leq C\ell(\gamma) = Ch(C_1) = C\,\mathrm{mod}(C_1)w(C_1).$$

Since the ratios of moduli are bounded by M from Proposition 8.7, we obtain $w(C_2) \leq MCw(C_1)$.

If one of the cylinders, say C_1, actually corresponds to a loop, then its adjacent vertex is a component of the limiting stable curve which is not a pair of pants. Pantsless finiteness then implies that there exists a universal constant L_2 such that $w(C_2) \leq L_2 w(C_1)$ and $w(C_1) \leq L_2 w(C_2)$.

Now consider the graph whose vertices are horizontal cylinders of (X, ω) and whose edges are cylinders that share a saddle connection. Given any two vertices C, C' of this graph, we may find a path $C = C_1, \dots, C_m = C'$, since the surface X is connected. We can find such a path of length at most the number of horizontal saddle connections, a constant n depending only on the stratum. Using the previous bounds we conclude that $w(C_m) \leq \max\{L_1, L_2\}^{n-1} w(C_1)$.

It then follows from pantsless finiteness that ratios of lengths of saddle connections are bounded as well, since any saddle connection has a nearby closed geodesic, and the ratio of their lengths is bounded above and below uniformly. $\blacksquare$

Recall from [SW10] that a *triangle* in (X, ω) is the image of a triangle in the plane under an affine map to (X, ω) which sends the vertices to zeros of ω, and which is injective except possibly at the vertices. They showed that (X, ω) is Veech if the set of areas of its triangles is bounded away from 0.

Let $\mathcal{S}_1$ be the locus of unit area forms in our stratum, and $\mathrm{NST}(\alpha) \subset \mathcal{S}_1$ to be the set of such surfaces which have no triangles of area less than α. Our proof of finiteness will use the following theorem.

Theorem 8.9 ([SW10])**.** *The set* $\mathrm{NST}(\alpha)$ *consists of finitely many Teichmüller curves.*

Remark. Smillie and Weiss actually proved a much stronger finiteness theorem that we do not need, namely that the union of these $\mathrm{NST}(\alpha)$ over all strata in any genus is finite.

Proof of Theorem 8.1. Let (X, ω) be a surface in our collection of Teichmüller curves, normalized to have unit area, and let $T \subset (X, \omega)$ be a triangle. We may rotate ω so that the base of T is horizontal, hence this horizontal direction is periodic. Then apply a diagonal matrix so that (X, ω) has a horizontal cylinder of unit width. By Propositions 8.7 and 8.8, all horizontal cylinders have moduli and widths bounded above and below, as do the horizontal saddle connections. Their heights are then bounded as well. It follows that the area of T is uniformly bounded below, and by Theorem 8.9, our collection of Teichmüller curves is finite. ■

9 Bounding torsion and the principal stratum

In this section, we aim to apply the finiteness criterion, Theorem 8.1, to obtain finiteness for the principal stratum in genus three and prepare the grounds for the discussion of the remaining strata in the next section. In order to verify the pantsless-finite hypothesis, we will use the torsion condition together with *a priori* bounds on the torsion orders and height bounds from diophantine geometry to control the possible stable forms arising as cusps of Teichmüller curves generated by forms with many zeros.

More precisely, given an irreducible component $(X_\infty, \omega_\infty)$ of the stable form over a cusp of an algebraically primitive Teichmüller curve, if ω_∞ has multiple zeros, then the torsion condition may be interpreted as saying that certain cross-ratios involving these zeros and the cusps of X_∞ are roots of unity. Torsion order bounds, height bounds and information about the degrees of the residues of ω_∞ often allow us to conclude that there are only finitely many possibilities for $(X_\infty, \omega_\infty)$. To formalize this, we introduce now the notion of a form which is determined by torsion.

We say that a meromorphic one-form on $\mathbf{P}^1$ is *a form of type* $(n; m_1, \dots, m_k)$ if it has n poles, all of which are simple, and k zeros of orders $m_1, \dots, m_k$. We denote the poles $x_1, \dots, x_n$ and the zeros $z_1, \dots, z_k$. For example, a pair of pants is a form of type $(3; 1)$.

Definition 9.1. A one-form on $\mathbf{P}^1$ of type $(n; m_1, \dots, m_k)$ *is determined by torsion* if for every $N \in \mathbf{N}$ and every $g \in \mathbf{N}$ there are only a finite number of forms ω of this type, up to the action of $\mathrm{Aut}(\mathbf{P}^1)$ and constant multiple, that satisfy the following condition. There exists a partition of $\{1, \ldots, n\}$ into subsets $S_j, j \in J$, none of which consists of a single element, such that

i) for each part S_j of the partition $\sum_{i \in S_j} \mathrm{Res}_{x_i}(\omega) = 0$,

ii) the ratios of residues $\mathrm{Res}_{x_i}(\omega)/\mathrm{Res}_{x_1}(\omega)$ for $i = 1, \ldots, n$ are elements of a totally real number field of degree g whose $\mathbf{Q}$-span has dimension $n - |J|$, and

iii) for all $a \neq b$ the cross-ratio $[z_a, z_b, x_{i_1}, x_{i_2}]$ is a root of unity of order dividing N whenever there exists a part S_j containing both i_1 and i_2.

These conditions are motivated by the constraints imposed on a component of a limiting stable form of a Teichmüller curve.

Proposition 9.2. *Suppose $(X_\infty, \omega_\infty)$ is a stable form lying over a boundary point of an algebraically primitive Teichmüller curve, and Y is an irreducible component of X_∞. Then there is a partition of the nodes of Y satisfying the above conditions.*

Proof. Consider the dual graph Γ of X_∞ with edges labeled by the corresponding residues of ω_∞ as in §8, and consider the vertex v corresponding to Y. We partition the set of edges $E(v)$ incident to v according to the component of $\Gamma \setminus \{v\}$ in which they lie. As X_∞ has no separating edges, each part of this partition has at least two elements.

Condition i) is then a consequence of the residue theorem. The condition iii) rephrases that the difference between any two zeros is a torsion section with torsion order divisible by N, as stated in Lemma 8.3.

To obtain ii), consider the set S of edges of Γ obtained by deleting from $E(v)$ one edge from each component of $\Gamma \setminus \{v\}$. We wish to show that the corresponding residues are linearly independent over $\mathbf{Q}$. Recall from §8 that assigning the residues of ω_∞ to the edges of Γ determines a homology class

$$\rho_{\boldsymbol{r}} = \sum_e r_e[e] \in H_1(\Gamma; F)$$

(after normalizing the form by dividing by $\mathrm{Res}_{x_1}(\omega_\infty)$ so that the residues belong to the trace field F). The induced map $\rho_{\boldsymbol{r}}^*\colon H^1(\Gamma; \mathbf{Q}) \to F$ is an isomorphism, so it suffices to show that the cohomology classes $\{[e^*] : e \in S\}$ are linearly independent. This follows from the fact that for each $e \in S$ there is a circuit of Γ which crosses e and no other edges in S. ∎

Forms with sufficiently many zeros are determined by torsion. We state below the relevant results, which we will prove in §9.1 and §9.2.

Proposition 9.3. *A form of type $(n; m_1, \ldots, m_k)$ having three or more zeros, i.e. $k \geq 3$, is determined by torsion.*

Proposition 9.4. *A form of type $(4; 1, 1)$ is determined by torsion.*

Proof of Theorem 1.7. Suppose we have an *a priori* bound for torsion orders of algebraically primitive Teichmüller curves in $\Omega\mathcal{M}_g(1^{2g-2})$. Propositions 9.3 and 9.4 imply that there are only finitely many possibilities for any non-pants component of a stable curve at a cusp of an algebraically primitive Teichmüller curve in this stratum. Theorem 8.1 then implies finiteness. ∎

To apply these results, we need *a priori* bounds on torsion orders. When an irreducible component $(X_\infty, \omega_\infty)$ of the stable form over a cusp contains two zeros, one can often bound the torsion order of the difference of these two zeros by appealing to Laurent's theorem [Lau84] on torsion points in a subvariety of $\mathbf{G}$. Whether this works depends on the combinatorial type of $(X_\infty, \omega_\infty)$. To formalize this, we introduce the notion of prescribing torsion.

Definition 9.5. A form of type $(n; m_1, \ldots, m_k)$ *prescribes torsion* if for every $g \in \mathbf{N}$ there exists $N_0 \in \mathbf{N}$ such that for every partition of $\{1, \ldots, n\}$ into subsets $S_j, j \in J$, none of which consists of a single element, and for every form ω on $\mathbf{P}^1$, such that

- i) for each part S_j of the partition $\sum_{i \in S_j} \mathrm{Res}_{x_i}(\omega) = 0$,
- ii) the ratios of residues $\mathrm{Res}_{x_i}(\omega)/\mathrm{Res}_{x_1}(\omega)$ for $i = 1, \ldots, n$ are elements of a totally real number field of degree g whose $\mathbf{Q}$-span has dimension $n - |J|$, and
- iii) for all $a \neq b$ the cross-ratio $[z_a, z_b, x_{i_1}, x_{i_2}]$ is a root of unity whenever i_1 and i_2 belong to the same part of the partition,

then all of the roots of unity appearing in iii) have order dividing N_0.

We will prove the following results on which types of forms prescribe torsion.

Proposition 9.6. *A form of type* $(6; 1, 1, 1, 1)$ *prescribes torsion.*

It might well be true – and would suffice to prove finiteness in the principal stratum in all genera – that this proposition holds for all components of type $(2k; 1^{2k-2})$. The following lemma is also part of checking the hypothesis for pantsless-finiteness and works for all k.

Lemma 9.7. *If a form of type* $(n; m_1, \ldots, m_k)$ *with three or more zeros, i.e.* $k \geq 3$*, occurs as an irreducible component of a stable curve lying over a boundary point of an algebraically primitive Teichmüller curve, then* n *is even. Moreover, there is only a finite number (depending on* n*) of tuples of residues (up to constant multiple) that occur for such boundary points.*

Proposition 9.8. *A form of type* $(4; 1, 1)$ *prescribes torsion.*

From these statements we will deduce the main theorem in the case of the principal stratum.

Proof of Theorem 1.1, case $\Omega\mathcal{M}_3(1, 1, 1, 1)$. Consider a Veech surface generating an algebraically primitive Teichmüller curve in this stratum, and suppose that two of its zeros z_i and z_j can be joined by a saddle connection of slope θ. Consider the stable curve X_∞ obtained by applying the Teichmüller geodesic flow in the direction θ. The two zeros z_i and z_j will limit to the same component of X_∞. By Lemma 9.7, there are two possibilities: either X_∞ is irreducible, or it has two components, each containing two zeros. The Propositions 9.6 and 9.8 and Lemma 8.3 then bound in either case the torsion order of the corresponding difference of sections $s_i - s_j$ by a universal constant.

More generally, any two zeros can be joined by a finite chain of saddle connections, where the length of the chain is at most the number of zeros. Consequently, the torsion order of any difference of sections $s_i - s_j$ is bounded by a universal constant.

Theorem 1.7 then implies finiteness of algebraically primitive Teichmüller curves in this stratum. ∎

9.1 Forms with few zeros

Proof of Proposition 9.4 and Proposition 9.8. We consider a form ω_∞ of type $(4;1,1)$ and use $\mathrm{Aut}(\mathbf{P}^1)$ to normalize the form ω_∞ to have its two zeros at 0 and ∞. The partition of $\{1,2,3,4\}$ may have one or two parts.

In the case of two parts,

$$\omega_\infty = \left(\frac{r_1}{z-x_1} - \frac{r_1}{z-x_2} + \frac{r_2}{z-u_1} - \frac{r_2}{z-u_2}\right) dz = \frac{C z\, dz}{\prod_{i=1}^2 (z-x_i) \prod_{i=1}^2 (z-u_i)},$$

where $\{x_1, x_2\}, \{u_1, u_2\}$ is the partition of the nodes and where $C \in \mathbf{C}$ is some constant.

The torsion conditions imply that there exist two roots of unity ζ_x and ζ_u such that

$$x_2 = \zeta_x^2 x_1, \quad u_2 = \zeta_u^2 u_1.$$

The resulting equation for the z^2 and constant terms of the numerator of ω_∞ to vanish imply, with the normalization $x = 1$ and $r_1 = 1$, that

$$r_2 = \frac{\zeta_u}{\zeta_x} \frac{1-\zeta_x^2}{1-\zeta_u^2} = \frac{\overline{\zeta_x} - \zeta_x}{\overline{\zeta_u} - \zeta_u} \quad \text{and} \quad u_1 = -\frac{\zeta_x}{\zeta_u}.$$

This is the situation of where McMullen's theorem on ratio of sines [McM06b] applies. As a consequence, since the r_i belong to a cubic field, there are only a finite number of choices for the torsion orders of the roots of unity and a finite number of possibilities for the r_i. This concludes the proof of both propositions in this case.

In the case of one part

$$\omega_\infty = \left(\sum_{i=1}^4 \frac{r_i}{z-u_i}\right) dz = \frac{C z\, dz}{\prod_{i=1}^4 (z-u_i)}, \tag{9.1}$$

where $r_4 = -(r_1 + r_2 + r_3)$, and r_1, r_2, r_3 are linearly independent over $\mathbf{Q}$. The torsion condition now implies that the pairwise ratios of the u_i are roots of unity. We may normalize $u_1 = 1$ and $u_i = \zeta_i$ for some Nth roots of unity ζ_i. There are rational functions f_i such that $r_i = f_i(u_1, \ldots, u_4)$, so if the roots of unity ζ_i are known, then so is ω_∞. This proves Proposition 9.4.

To prove Proposition 9.8, we need to show that in the same situation, there are only finitely many choices for the roots of unity ζ_i for which the residues r_i belong to a field of bounded degree.

The degrees of the f_i are independent of g. The image of a tuple of roots of unity, which are of height zero, is thus of bounded height by (2.4). Since the ratios of residues moreover lie in a field of degree bounded by g, Northcott's theorem implies that for each g there are only a finite number of possible residue tuples up to scale.

For each such tuple, we have to check that there are only finitely many roots of unity that give rise to this residue. For fixed residues r_i, the possible ζ_i belong to the curve cut out by the equations,

$$\sum_{i=1}^4 r_i \zeta_i = 0 \quad \text{and} \quad \sum_{i=1}^4 r_i \zeta_i^{-1} = 0, \tag{9.2}$$

with the normalization $\zeta_1 = 1$. If there were an infinite number of solutions in roots of unity, they would lie in a translate of a torus by a torsion point. That is there exist $a_2, a_3, a_4 \in \mathbf{Z}$, not all zero, and roots of unity η_i, such that $\zeta_i = \eta_i t^{a_i}$ for $i = 2, 3, 4$ is a solution to (9.2) for all t. Considering the limit $t \to 0$ and $t \to \infty$ implies that the highest and lowest exponent must not be isolated. We may thus renumber the terms such that $a_2 = 0$ and $a_3 = a_4 \neq 0$. Substitution into (9.2) implies that $r_3\eta_3 + r_4\eta_4 = 0$, hence $r_3/r_4 = \pm 1$. This contradicts $\mathbf{Q}$-linear independence. ■

9.2 Forms with many zeros

For forms with more zeros, the basic idea of the proofs is similar, but the extra dimensions involved create significant difficulties.

Lemma 9.9. *Let ω_∞ be a form of type $(n; m_1, \ldots, m_k)$ with $k \geq 3$. Suppose that ω_∞ satisfies the conditions i), ii) and iii) in Definition 9.5. Then n is even and $|S_j| = 2$ for all parts of the partition.*

Moreover, if we normalize $z_1 = \infty$ and $z_2 = 0$ and renumber the poles such that $S_j = \{j, \ell + j\}$, where $\ell = n/2$, then x_j is the complex conjugate of $x_{\ell+j}$.

We will write subsequently y_j instead of $x_{\ell+j}$. We use the normalization $z_1 = \infty$ and $z_2 = 0$ and $z_3 = 1$ and we call this *standard normalization* in the rest of this section.

Proof. Fix a part S_j. By condition iii) all the points x_i for $i \in S_j$ lie on the same circle around zero and they also lie on the same circle around one. Since the two circles intersect in precisely two points that are complex conjugate, all of the claims follow. ■

Such a form ω_∞ is determined, up to scale, by the location of its zeros and poles, i.e. by a point $P = (z_1, \ldots, z_k, x_1, \ldots, x_\ell, y_1, \ldots, y_\ell)$ in $\mathcal{M}_{0,k+n}$. Coordinates on this moduli space are cross-ratios, among which we select

$$R_{abj} = [z_a, z_b, y_j, x_j], \quad 1 \leq a < b \leq k, \quad 1 \leq j \leq \ell,$$

since these will be roots of unity for a form satisfying iii).

We define $\mathbf{A} = \mathbf{C}^{\ell k(k-1)/2}$ with coordinates t_{abj} $(a, b, j$ as above), and we define $\mathbf{A}' \subset \mathbf{A}$ to be the complement of the hyperplanes of the form $V(t_{abj})$, $V(t_{abk} - 1)$, $V(t_{abj} - t_{a'bj})$, and $V(t_{abj} - t_{ab'j})$. The cross-ratios R_{abj} then define a morphism $\mathrm{CR}\colon \mathcal{M}_{0,k+n} \to \mathbf{A}'$.

Since $R_{abj}R_{bcj} = R_{acj}$, we may without loss of information restrict our attention to the cross-ratios where the first zero is fixed to be z_1, which we denote by

$$R_{ij} = [z_1, z_i, y_j, x_j], \quad 1 < i \leq k, \quad 1 \leq j \leq \ell.$$

We define $\mathbf{A}_{\mathrm{red}} = \mathbf{C}^{\ell(k-1)}$ with coordinates t_{ij} (with i, j as above), and we define $\mathbf{A}'_{\mathrm{red}} \subset \mathbf{A}_{\mathrm{red}}$ to be complement of the hyperplanes of the form $V(t_{ij})$, $V(t_{ij} - 1)$, and $V(t_{ij} - t_{i'j})$. The cross-ratios R_{ij} then define a morphism $\mathrm{CR}^{\mathrm{red}}\colon \mathcal{M}_{0,k+n} \to \mathbf{A}'_{\mathrm{red}}$.

Finally, we define $\mathbf{A}_{\min} = \mathbf{C}^{n+k-3}$ with coordinates t_{ij} for $i = 2, 3$ and $1 \leq j \leq \ell$ together with t_{i1} for $4 \leq i \leq k$, and we define $\mathbf{A}'_{\min}$ to be the complement of the hyperplanes of the form $V(t_{ij})$, $V(t_{ij} - 1)$, and $V(t_{ij} - t_{i'j})$.

The cross-ratios R_{ij} then define a morphism $\mathrm{CR}^{\min}\colon \mathcal{M}_{0,k+n} \to \mathbf{A}_{\min}$. There is a canonical projection $p_{\min}\colon \mathbf{A}_{\mathrm{red}} \to \mathbf{A}_{\min}$, forgetting the indices which do not appear for $\mathbf{A}_{\min}$.

Lemma 9.10. *The morphism* $\mathrm{CR}^{\min}$ *is injective and dominant. The morphism* $\mathrm{CR}^{\mathrm{red}}$ *is injective and dominant onto the subvariety* $\mathcal{Y}$ *of* $\mathbf{A}'_{\mathrm{red}}$ *cut out by the equations*

$$\mathrm{Eq}(i,j,j')\colon \quad [1,t_{2j},t_{3j},t_{ij}] - [1,t_{2j'},t_{3j'},t_{ij'}] = 0$$

for $i \geq 4$ *and* $1 \leq j < j' < \ell$.

Obviously, the same subvariety is also cut out by all the equations $\mathrm{Eq}(i,1,j')$ for $i \geq 4$ and $2 \leq j' < \ell$.

Proof. Again, normalize so that $z_1 = \infty$ and $z_2 = 0$ and $z_3 = 1$. Then $R_{2j} = x_j/y_j$ and $R_{3j} = (1-x_j)/(1-y_j)$. Since $R_{2j} \neq R_{3j}$, the knowledge of these two cross-ratios thus determines the location of x_j and y_j. Since $R_{i1} = (z_i - x_1)/(z_i - y_i)$, and $R_{i1} \neq 1$, this cross-ratio determines the location of z_i. This proves injectivity. Dominance of $\mathrm{CR}^{\min}$ follows since the dimensions agree.

This argument states more precisely that for any fixed j the knowledge of R_{2j}, R_{3j} and R_{ij} determines the location of z_i. In fact, a straightforward calculation yields

$$z_i = [1, R_{2j}, R_{3j}, R_{ij}].$$

These of course have to agree for any pair of indices j and j', which is expressed by $\mathrm{Eq}(i,j,j')$. Consequently, the image of $\mathrm{CR}^{\mathrm{red}}$ is contained in the subvariety cut out by all the $\mathrm{Eq}(i,j,j')$. Given $R_{2j'}$, $R_{3j'}$, R_{21},R_{31}, and R_{j1}, one can solve $\mathrm{Eq}(i,1,j')$ uniquely for $R_{1j'}$. Since $\mathrm{CR}^{\min}$ is dominant, it follows that $\mathrm{CR}^{\mathrm{red}}$ is dominant to $\mathcal{Y}$. ∎

Proof of Proposition 9.3 and Lemma 9.7. Consider a form ω_∞ satisfying the conditions i), ii) and iii) in Definition 9.1 or Definition 9.5. Its image under CR and hence also $\mathrm{CR}^{\min}$ is then a torsion point. By Lemma 9.10 the map $\mathrm{CR}^{\min}$ has an inverse rational map, hence ω_∞ is determined up to scale and finitely many choices by a bound on its torsion orders. This proves Proposition 9.3.

Consider now the rational map

$$\mathrm{Res}\colon \mathcal{M}_{0,k+n} \dashrightarrow \mathbf{P}^n$$

that associates to a point in $\mathcal{M}_{0,k+n}$ the projective tuple of residues of the corresponding one-form

$$\omega_\infty = \frac{\prod_{j=1}^{k}(z-z_j)^{m_j}\, dz}{\prod_{i=1}^{\ell}(z-x_i)(z-y_i)}.$$

The rational map $\mathrm{Res} \circ (\mathrm{CR}^{\min})^{-1}$ depends only on the type of the stratum. Consequently, by (2.4) the $\mathrm{Res} \circ (\mathrm{CR}^{\min})^{-1}$-image of the set of torsion points has bounded height. By Northcott's theorem, there are at most finitely many possible residue tuples of degree at most g, proving Lemma 9.7. ∎

We prepare now for the proof of Proposition 9.6. Suppose that its statement was false for a stratum of some fixed type $(n; m_1, \ldots, m_k)$ with $k \geq 3$, not specializing to $n = 6$ and $m_i = 1$ yet. By Laurent's Theorem [Lau84], this means that there exists a subvariety $T \subset \mathbf{A}'$ which is

a) the translate of a positive-dimensional torus by a torsion point,
b) generically contained in the image of CR,
c) generically contained in the image of the locus of stable forms (meaning $\operatorname{Res}_{x_i} \omega_\infty = -\operatorname{Res}_{y_i} \omega_\infty$), and
d) contained in a fiber of Res' (since there are only finitely many possible residue tuples by Lemma 9.7).

Here Res' is the morphism $\operatorname{Res}' = \operatorname{Res} \circ (\mathrm{CR}^{\min})^{-1} \circ p_{\min}$.

In a proof by contradiction we may restrict to the case $\dim T = 1$, i.e.

$$T = \{(c_{abj} t^{e_{abj}}),\ 1 \leq a < b \leq k,\ 1 \leq j \leq \ell, \quad t \in \mathbf{C}^*\}.$$

We next discuss the possibilities for the limit point $T(0)$ of the $(\mathrm{CR}^{\min})^{-1} \circ p_{\min}$-image of T as $t \to 0$. This is a well-defined point in the Deligne-Mumford compactification $\overline{\mathcal{M}}_{0,k+n}$ and corresponds to a stable curve Y_0 in $\partial\overline{\mathcal{M}}_{0,k+n}$ together with the limiting stable form $\eta = \lim_{t \to 0} \omega_\infty(t)$.

We represent the dual graph of the stable curve Y_0 determined by T by a tree $\mathcal{T}_T$, whose vertices are decorated by the zeros and poles that limit in the corresponding component.

Lemma 9.11. *The limiting stable form (Y_0, η) associated with $T(0)$ has the following properties.*

i) Whenever a curve γ is pinched as $t \to 0$, no pair of poles x_i and y_i lies on one side of γ such that two zeros lie on the other side.
ii) None of the pinched curves has η-period equal to zero, in particular η has a pole at each of the nodes of Y_0.
iii) Each component Y of Y_0 has at least one zero.

Proof. Since T is generically contained in $\mathrm{CR}(\mathcal{M}_{k+n})$, for each index (a, b, j) either $e_{abi} \neq 0$ or $c_{abi} \neq 1$. In particular the limit $t \to 0$ of $c_{abi} t^{e_{abi}}$ is not 1. This implies the first statement.

Since the residues $\operatorname{Res}_{x_i} \omega_\infty(t)$ are $\mathbf{Q}$-linearly independent and constant, the period of a curve could only be zero if it does not separate any pair of poles $\{x_i, y_i\}$. As there are at least four zeros, such a curve must violate i).

Now, since Y_0 is a stable curve, each component must have at least three nodes and poles. Since each node is a simple pole of the stable form by ii), each component must have at least one zero as well. ∎

Corollary 9.12. *In the case of a stratum of type $(6; 1, 1, 1, 1)$ a complete list of decorated trees arising as $\mathcal{T}_T$ is given by the three possibilities in Figure 2 up to renumbering zeros and poles, together with trees obtained by collapsing one or more edges of these three trees.*

Proof. The number of vertices is bounded by four by Lemma 9.11 iii) and there are only two possible trees with four vertices, as listed in Figure 2, each with one zero that we label as in the figure. Denote by x_1 one of the two poles on the component of z_1. Then y_1 lies on the component of z_3 or of z_4 by Lemma 9.11 i). The remaining case distinction is now easily completed. ∎

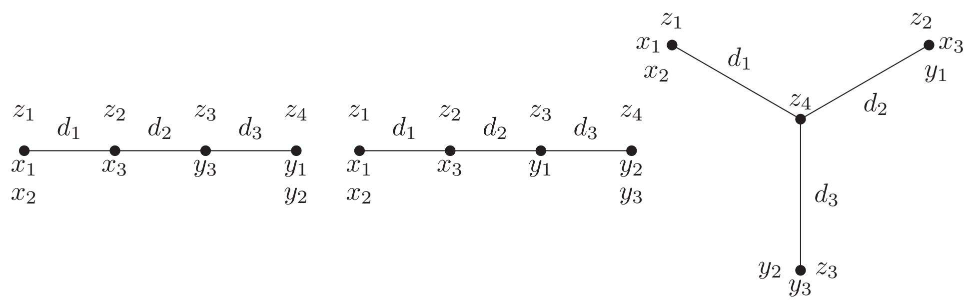

Figure 2: Possible stable limits of T

We attach to every edge e of $\mathcal{T}_T$, equivalently to every curve that is pinched when degenerating to Y_0, the number d_e of right Dehn twists performed by the mododromy of a simple loop around $t = 0$ in T. As the monodromy must perform a nontrivial twist around each pinched curve, we interpret $d_e = 0$ as indicating that the edge e has been deleted from T.

Proposition 9.13. *Given two zeros z_a and z_b let S be the oriented segment in $\mathcal{T}_T$ joining z_a to z_b. For any $j \in \{1, \ldots, \ell\}$ let S_j be the oriented subsegment of S joining the projection $\bar{y}_j$ of y_j onto S to the projection $\bar{x}_j$ of x_j onto S.*

Then the exponent in T of the coordinate R_{abj} is $e_{abj} = \pm \sum_{e \in S_j} d_e$, where the sign is positive if S and S_j have the same orientation, and negative otherwise.

Proof. We assume that S and S_j have the same orientation, as swapping x_j and y_j has the effect of inverting R_{abj}.

Normalize the zeros and poles of $\omega_\infty(t)$ so that $z_a = 0$, $z_b = \infty$ and $x_j = 1$. Then $y_j = c_{abj} t^{e_{abj}}$ by definition of the torus. Let γ be a path joining y_j to x_j. Then the monodromy around $t = 0$ sends γ to $\gamma + e_{abj}\delta$, where δ is a loop winding once around 0.

On the other hand, the edges S_j correspond the pinching loops which separates z_a and y_j from z_b and x_j. As each loop intersects γ once, performing d_e Dehn twists about each loop sends γ to $\gamma + \sum_{e \in S_j} d_e$. Comparing these two computations of the monodromy, the formula for e_{abj} follows. ∎

The torus-translate T defines via the projection $p_{\mathrm{red}} \colon \mathbf{A} \to \mathbf{A}_{\mathrm{red}}$ a torus translate T_{red} in $\mathbf{A}_{\mathrm{red}}$. Conversely every translate of a torus $T_{\mathrm{red}} \subset \mathbf{A}_{\mathrm{red}}$ determines a torus-translate $T \subset \mathbf{A}$, since $R_{abj} = R_{bj}/R_{aj}$. We will thus subsequently work with tori in $\mathbf{A}_{\mathrm{red}}$ only. Such a torus is parameterized as

$$T_{\mathrm{red}} = \{(c_{ij} t^{e_{ij}}) \colon 1 < i \leq k,\ 1 \leq j \leq \ell, \quad t \in \mathbf{C}^*\}.$$

From now on we restrict our attention to $n = 6$ and $m_i = 1$, i.e. to a stratum of type $(6; 1, 1, 1, 1)$.

Lemma 9.14. *Suppose that $T_{\mathrm{red}} = p_{\mathrm{red}}(T)$ for some torus translate T satisfying conditions a), b), c), and d) above. Then the tuple of exponents e_{ij} is in the row span of one of the following three matrices*

$$M_1 = \begin{pmatrix} 1&1&1&1&1&1&0&0&0 \\ 0&1&1&0&1&1&0&1&1 \\ 0&0&1&0&0&1&0&0&0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1&1&1&1&1&1&0&0&0 \\ 0&1&1&0&1&1&0&1&1 \\ 0&0&0&0&0&1&0&0&1 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 1&0&0&-1&0&0&0&0&0 \\ 0&-1&0&0&0&0&0&1&0 \\ 0&0&0&0&0&1&0&0&-1 \end{pmatrix}.$$

The columns appear in the order $(R_{21}, R_{31}, R_{41}, R_{22}, R_{32}, R_{42}, R_{23}, R_{33}, R_{43})$.

Proof. The rows correspond to the exponents associated with

$$(d_1, d_2, d_3) \in \{(1,0,0), (0,1,0), (0,0,1)\},$$

calculated according to Proposition 9.13, in each of the three cases in Figure 2. ■

Proof of Proposition 9.6. [4] By the above discussion and Lemma 9.14, we must show that there is no vector N contained in the row-span of one of the matrices M_i and corresponding torus-translate $\boldsymbol{a}T_N \subset \mathbf{A}'_{\mathrm{red}}$ satisfying these properties:

- $\boldsymbol{a}T_N$ is generically contained in the image $\mathrm{CR}^{\mathrm{red}}(\mathcal{M}_{0,10}) \subset \mathcal{Y} \subset \mathbf{A}'_{\mathrm{red}}$.
- The opposite-residue condition $\mathrm{Res}_{x_i}\,\omega_\infty = -\,\mathrm{Res}_{y_i}$ is satisfied along $\boldsymbol{a}T_N$.

For computational convenience, we work in the full affine plane $\mathbf{A}_{\mathrm{red}} \supset \mathbf{A}'_{\mathrm{red}}$. Let $h_1, h_2, h_3 \in \mathbf{Q}[t_{ij}]$ be the numerators of the rational functions $\mathrm{Eq}(4,1,2)$, $\mathrm{Eq}(4,1,3)$, and $\mathrm{Eq}(4,2,3)$, which cut out $\mathcal{Y}$ by Lemma 9.10. The ideal $I = (h_1, h_2, h_3) \subset \mathbf{Q}[t_{ij}]$ is prime, so it cuts out the $\mathbf{Q}$-Zariski-closure $\overline{\mathcal{Y}}$ of $\mathcal{Y}$ in $\mathbf{A}_{\mathrm{red}}$ (by the same argument as in Lemma 6.6, $\overline{\mathcal{Y}}$ is the Zariski closure of $\mathcal{Y}$ as $\mathcal{Y}$ has smooth rational points, though we do not need this). The rational map $\mathrm{CR}^{\mathrm{red}}$ is a birational equivalence between $\mathcal{M}_{0,10}$ and $\overline{\mathcal{Y}}$.

As we are only interested in torus-translates contained in the image of $\mathcal{M}_{0,10}$, we define the *peripheral divisor* $D_0 = \mathbf{A}_{\min} \setminus \mathrm{CR}^{\min}(\mathcal{M}_{0,10})$, which we compute as follows. We identify $\mathcal{M}_{0,10}$ with an open subset of $\mathbf{C}^7$ with coordinates z_4, x_i, y_i (with $i = 1,2,3$) using the standard normalization of the zeros, $z_1 = \infty$, $z_2 = 0$, and $z_3 = 1$. The divisor $D_1 = \mathbf{C}^7 \setminus \mathcal{M}_{0,10}$ consists of 21 hyperplanes, each corresponding to the collision of two of the marked points. Given a hyperplane $H \subset D_1$ cut out by an affine polynomial h, the numerator of the rational function $h \circ (\mathrm{CR}^{\min})^{-1}$ cuts out a divisor in $\mathbf{A}_{\min} \setminus \mathrm{CR}^{\min}(\mathcal{M}_{0,10})$. We collect the irreducible factors of the numerators of these 21 rational functions, as well as the irreducible factors of the denominators of the coordinates of $(\mathrm{CR}^{\min})^{-1}$ (which express the condition that a marked point is colliding with $z_1 = \infty$). Together these polynomials cut out 36 $\mathbf{Q}$-irreducible divisors in $\mathrm{CR}^{\min}$ whose union is the peripheral divisor D_0.

We also define the peripheral divisor $D = p_{\min}^{-1}(D_0) \subset \mathbf{A}_{\mathrm{red}}$, and $J \subset \mathbf{Q}[t_{ij}]$ the corresponding ideal. The rational map $\mathrm{CR}^{\mathrm{red}}$ then induces an isomorphism $\mathrm{CR}^{\mathrm{red}} \colon \mathcal{M}_{0,10} \to \overline{\mathcal{Y}} \setminus D$.

We now apply the torus-containment algorithm to the variety $\overline{\mathcal{Y}}$ with initial subspaces M_1, M_2, M_3. The result is a list of 554 subspaces of $\mathbf{Z}^9$ (3 of rank three, 97 of rank two, and 454 of rank one), each contained in the row span of one M_i. For each subspace N, there is a subvariety $V_N \subset \mathbf{C}^9$ cut out by an ideal $I_N \subset \mathbf{Q}[a_{ij}]$ (with indices $i = 2,3,4$, $j = 1,2,3$) such that the torus-translate $\boldsymbol{a}T_N$ is contained in $\overline{\mathcal{Y}}$ if and only if $\boldsymbol{a} \in V_N$. We now check that none of these varieties yield a torus-translate satisfying all of our constraints.

First, we rule out those subspaces N which do not in fact yield any torus-translates, that is, when V_N is contained in a coordinate hyperplane. In terms

[4] This proof is heavily computer-based. All of the assertions below were verified by `sage` [S+14]. The file `g3fin_ch9.sage` contains all of the calculations, and is available on the authors' web pages.

of the torus-containment algorithm, this occurs when for one of the polynomials h_i defining $\overline{\mathcal{Y}}$, the partition of a set of exponent-vectors induced by N has a singleton. This condition rules out all but 78 of the 554 subspaces.

We then rule out subspaces N parameterizing only peripheral torus-translates, that is, contained in the peripheral divisor D. There is a peripheral divisor $D_N \subset \mathbf{C}^9$ parameterizing coefficients $\boldsymbol{a}$ such that $\boldsymbol{a}T_N \subset D$, cut out by the coefficient ideal (defined in §5) $J_N \subset \mathbf{Q}[a_{ij}]$. Nonperipheral torus-translates are then parameterized by $V_N \setminus D_N$, whose Zariski-closure is cut out by the saturation ideal $K_N = \bigcup_{i=1}^{\infty} I_N : J_N^i$, which we compute for each remaining N (see for example [GP08][§1.8.9] for information on saturation ideals). In particular, if $K_N = (1)$, then $V_N \subset D_N$, and this N yields only peripheral torus-translates. This condition rules out all but 17 subspaces, and the ones which remain are only rank-one.

We now apply the opposite-residue condition to each of the remaining vectors N. The numerators of the rational functions $\operatorname{Res}_{x_i} \omega_\infty / \operatorname{Res}_{y_i} \omega_\infty + 1$ $(i = 1, 2, 3)$ generate an ideal $L \subset \mathbf{Q}[t_{ij}]$ with coefficient ideal $L_N \subset \mathbf{Q}[a_{ij}]$, which cuts out the variety parameterizing those $\boldsymbol{a}$ such that $\boldsymbol{a}T_N$ satisfies the opposite-residue condition. We then compute the saturation ideals $\mathfrak{I}_N = \bigcup_{i=1}^{\infty}(K_N + L_N) : J_N^i$. For all but one of the remaining vectors N, we have $\mathfrak{I}_N = (1)$, meaning there is no nonperipheral torus-translate corresponding to N which satisfies the opposite-residue condition.

For the final remaining vector N, we were not able to compute the saturation ideal $\mathfrak{I}_N$ directly. In this case, we instead compute the primary decomposition of K_N. For each associated prime P_i, we compute that $\bigcup_{i=1}^{\infty}(P_i + L_N) : J_N^i = (1)$, which implies that $\mathfrak{I}_N = (1)$. ■

10 Intermediate strata: Using the torsion condition

In this section we prove Theorem 1.1 for all remaining strata. We maintain the general hypothesis that $f\colon \mathcal{X} \to C$ is the family over a Teichmüller curve, generated by an algebraically primitive Veech surface (X, ω).

10.1 The stratum $\Omega\mathcal{M}_3(2, 1, 1)$

Consider a stable form $(X_\infty, \omega_\infty)$ which is the limit of a cusp of an algebraically primitive Teichmüller curve in $\Omega\mathcal{M}_3(2, 1, 1)$. In order to apply Theorem 8.1, we need to check that there are finitely many possibilities for the non-pants components of $(X_\infty, \omega_\infty)$. There are three cases to consider:

- $(X_\infty, \omega_\infty)$ has a component of type $(4; 2)$ and either two pants components or a component of type $(4; 1, 1)$.
- $(X_\infty, \omega_\infty)$ has a component of type $(5; 2, 1)$ and one pants component.
- $(X_\infty, \omega_\infty)$ is irreducible.

We now establish finiteness for each of these cases in turn.

Proposition 10.1. *A limiting stable form of an algebraically primitive Teichmüller curve in* $\Omega\mathcal{M}_3(2, 1, 1)$ *has no irreducible components of type* $(4; 2)$.

Proof. Consider an irreducible component (Y, η) of a limiting stable form of type $(4; 2)$. In either of the configurations described above, the four poles of (Y, η) come in two pairs (x_i, y_i), with $i = 1, 2$, such that $\mathrm{Res}_{x_i} \eta = -\mathrm{Res}_{y_i} \eta$ for each i. The involution J of Y swapping each pair (x_i, y_i) then satisfies $J^*\eta = -\eta$, in particular the unique zero p of η is fixed by J.

By Proposition 4.3, the form η^σ with Galois-conjugate residues also vanishes at p. This must be a simple zero, since η^σ is not a constant multiple of η. But η can only vanish to even order at a fixed point of J, as $J^*\eta = -\eta$, a contradiction. ■

Proposition 10.2. *A stable curve of type* $(5; 2, 1)$ *prescribes torsion and is determined by torsion.*

Proof. We compare with the case of type $(4; 1, 1)$. Here, too, there are at most two connected components of the complement of Y in the dual graph. The case that there is only one connected component follows exactly along the same lines as for $(4; 1, 1)$.

In the case of two connected components we may suppose that the one-form is

$$\omega_\infty|_Y = \left(\sum_{i=1}^{3} \frac{r_i}{z - u_i} + \frac{r_4}{z - x_1} - \frac{r_4}{z - x_2} \right) dz = \frac{z^2 dz}{\prod_{i=1}^{3}(z - u_i) \prod_{i=1}^{2}(z - x_i)}. \tag{10.1}$$

We may normalize $u_1 = 1$ and by the torsion condition $u_i = \eta_i$ is a root of unity for $i = 2, 3$ and $x_2 = \zeta x_1$ for some root of unity $\zeta \neq 1$. The residues at x_1 and x_2 add up to zero and this amounts to the condition

$$\zeta^2 \prod_{i=1}^{3}(x_1 - u_i) - \prod_{i=1}^{3}(\zeta x_1 - u_i) = 0. \tag{10.2}$$

Since $\zeta \neq 1$, this polynomial (with coefficients in $\mathbf{Q}(\zeta, \eta_i)$) has degree exactly three in x_1. For fixed torsion order, there is a finite number of choices for the roots of unity and for each of them there are at most three possibilities for x_1. This shows that forms of this type are determined by torsion.

We may view (10.2) as defining a hypersurface H in $\mathbf{A}^4$ with coordinates x_1, ζ, u_2 and u_3. Over the open set $\zeta \neq 1$ the projection Q to $\mathbf{A}^3$ forgetting x_1 is finite, in fact degree three. Since we are interested in points on H whose image consists of roots of unity, all the possibilities for x_1 have bounded height by Lemma 2.7. Being residues, the r_i are images of a rational map Res on H, and consequently their heights are bounded, too. Since the ratios r_i/r_j lie in a field of degree three, we conclude that there is only a finite number of possibilities for the ratios r_i/r_j.

We now normalize $r_1 = 1$ and fix one of the finitely many choices for the other r_i. We still have to show that there is only a finite number of roots of unity that give rise to such a tuple of residues. We follow the argument of [BM12, Proposition 13.9]. Since there is only one relative period, and since Q is finite, $Q(\mathrm{Res}^{-1}(r_1 : r_2 : r_3 : r_4))$ is a curve inside $(\mathbf{C}^*)^3$. If the claim was false, this curve has to be a translate of a subtorus, in fact a subtorus as explained in loc. cit. If this were true, we could find roots of unity η_2, η_3 and ζ and a function

$x_1(a)$ such that (defining $\eta_1 = 1$)

$$\omega_\infty|_Y = \left(\sum_{i=1}^{3} \frac{r_i}{z-\eta_i^a} + \frac{r_4}{z - x_1(a)} - \frac{r_4}{z - \zeta^a x_1(a)} \right) dz \qquad (10.3)$$

has a double zero at $z = 0$ and a simple zero at $z = \infty$. Clearing denominators, we can either use the z^3-term or the linear term to solve for $x_1(a)$ and take the limit $a \to 0$. We obtain

$$x_1(0) = \frac{(q_2 - q_3)r_2 - q_3 r_1}{q_1 r_4} \quad \text{and} \quad x_1(0) = \frac{q_1 r_4}{(q_2 - q_3)r_2 - q_3 r_1},$$

where $\eta_j = e^{2\pi i q_j}$ and $\zeta = e^{2\pi i q_1}$, which implies that

$$((q_2 - q_3)r_2 - q_3 r_1 - q_1 r_4)((q_2 - q_3)r_2 - q_3 r_1 + q_1 r_4) = 0.$$

The vanishing of either factor contradicts the fact that $\{r_1, r_2, r_4\}$ is a $\mathbf{Q}$-basis of F and this completes the proof that such a stratum prescribes torsion. ■

Proposition 10.3. *A stable curve of type* $(6; 2, 1, 1)$ *at the cusp of a Teichmüller curve generated by* $(X, \omega) \in \Omega\mathcal{M}_3(2, 1, 1)$ *prescribes torsion and is determined by torsion.*

The proof combines the fact that the residues are determined up to finitely many choices by height bounds as in Section 9 and properties of the Harder-Narasimhan filtration.

Proof. The stable form is determined by the location of the 6 poles and the three zeros, these we may assume to be at $z_1 = \infty$, $z_2 = 0$ and $z_3 = 1$ with z_1 corresponding to the double zero. As in the cases in the principal stratum we obtain a rational map

$$\mathrm{Res}\colon \mathcal{M}_{0,9} \dashrightarrow \mathbf{P}^3$$

that associates to a point in $\mathcal{M}_{0,9}$ the projective tuple of residues of the corresponding one-form

$$\omega_\infty = \frac{z(z-1)\,dz}{\prod_{j=1}^{3}(z - x_j)(z - y_j)}.$$

The cross ratios $R_{2j} = y_j/x_j$ and $R_{3j} = (y_j - 1)/(x_j - 1)$ are roots of unity by the torsion condition. By Lemma 9.7, the fact that roots of unity have bounded height implies there are at most finitely many residue tuples $(r_1 : r_2 : r_3)$ lying in a field of degree three.

Fix one of these residue tuples. If the statement of the proposition was wrong, then there is a translate of a torus contained in the fibers of Res over such a tuple. We will rule out that there is a one-dimensional such torus T, given by

$$R_{2j} = c_j t^{e_j} \quad \text{and} \quad R_{3j} = d_j t^{f_j}.$$

We now use the conditions imposed by the Harder-Narasimhan filtration to conclude that on the one hand

$$\omega_\infty = \left(\sum_{j=1}^{3} \frac{r_j}{z - x_j} - \frac{r_j}{z - y_j} \right) dz$$

has a double zero at ∞, and that on the other hand by Proposition 4.1

$$\omega_\infty^\sigma = \left(\sum_{j=1}^{3} \frac{r_j^\sigma}{z - x_j} - \frac{r_j^\sigma}{z - y_j} \right) dz$$

also has a zero at $z_1 = \infty$. Equating for the top degree terms in the numerator we find that $\sum_{j=1}^{3} r_j(x_j - y_j) = 0$ and $\sum_{j=1}^{3} r_j^\sigma(x_j - y_j) = 0$. This implies that

$$(x_1 - y_1 : x_2 - y_2 : x_3 - y_3) = (s_1^\tau : s_2^\tau : s_3^\tau)$$

where the s_i are the dual basis of (r_1, r_2, r_3). From this we deduce that the ratios $(x_i - y_i)/(x_j - y_j)$ have to be constant along T. Now this differences are expressed in cross-ratios as $x_j - y_j = (1 - R_{2j})(1 - R_{3j})/(R_{2j} - R_{3j})$, so that we have to rule out that

$$\frac{(1 - c_1 t^{e_1})(1 - d_1 t^{f_1})\,(c_2 t^{e_2} - d_2 t^{f_2})}{(1 - c_2 t^{e_2})(1 - d_2 t^{f_2})\,(c_1 t^{e_1} - d_1 t^{f_1})} = \text{const.}$$

Using the valuation at $t = 0$ of the numerator and denominator we deduce that one of $e_1 = e_2$, $e_1 = f_1$, $e_2 = f_2$ or $f_1 = f_2$ holds. Switching the roles of x_1 and y_1 or x_2 and y_2 we may assume that in fact $e_1 = e_2$ together with $e_1 < f_1$ and $e_2 < f_2$ hold. Degree considerations now imply that $f_1 = f_2$. If $e_1 = 0$ then both $d_1 = d_2$ and $d_2/c_2 = d_1/c_1$ hold or $d_1 = d_1/c_1$ and $d_2 = d_2/c_2$. The first case is a contradiction since $x_1 = y_1$ and the second case is a contradiction since $R_{21} = 1$, i.e. the pole of ω_∞ coincides with the zero along T. If $e_1 > 0$ then t^{f_i} is the largest t-power appearing in one of the linear factors. We deduce that $d_1 = d_2$ and hence $R_{31} = R_{32}$ along T. The case $e_1 < 0$ is ruled out the same way. ■

Proof of Theorem 1.1, case $\Omega\mathcal{M}_3(2,1,1)$. By Propositions 10.1, 10.2, and 10.3, the collection of algebraically primitive Teichmüller curves in this stratum is pantsless-finite, so finiteness follows by Theorem 8.1. ■

10.2 The stratum $\Omega\mathcal{M}_3(2,2)^{\text{hyp}}$

Proof of Theorem 1.1, case $\Omega\mathcal{M}_3(2,2)^{\text{hyp}}$. See [Möl08, Theorem 3.1] ■

In the language introduced above, the main ingredient of this paper (besides a special case of Theorem 7.2 using Néron models) is that on a hyperelliptic curve, a stable form of type $(2g; g-1, g-1)$ prescribes torsion and is determined by torsion.

10.3 The stratum $\Omega\mathcal{M}_3(3,1)$

Proof of Theorem 1.1, case $\Omega\mathcal{M}_3(3,1)$. See [BM12, Theorem 13.1] ■

Part of this proof is [BM12, Lemma 13.5] which states in the language introduced above, that an irreducible stable curve of type $(6;3,1)$ is *not* determined by torsion. This lemma states that this holds only if we exclude three-torsion.

We will see another instance of this phenomenon in the next case.

10.4 The stratum $\Omega\mathcal{M}_3(2,2)^{\mathrm{odd}}$

The hyperelliptic locus in this stratum has been dealt with in §6.3 using [MW]. It remains to establish finiteness in the nonhyperelliptic locus. The proof in this case parallels [BM12, Section 13].

We will see that a component of type $(6;2,2)$ is determined by torsion only if we exclude 2-torsion. Using the hyperelliptic open-up of [CM12], these excluded forms only arise as cusps of Teichmüller curves in the hyperelliptic locus.

Let $\mathcal{M}_{0,8}$ be the moduli space of 8 distinct labeled points on $\mathbf{P}^1$, corresponding to two points z_1 an z_2 (zeros) and three pairs of points x_i,y_i, $i=1,2,3$ (poles). We associate to such a point the one-form

$$\omega_P = \frac{z^2 dz}{\prod_{j=1}^3 (z-x_j)(z-y_j)}.$$

We normalize usually the two zeros to be at $z_1 = 0$ and $z_2 = \infty$. With this normalization, $\mathcal{M}_{0,8}$ is naturally a subset of $\mathbf{P}^5$. Let $S(2,2) \subset \mathcal{M}_{0,8}$ be the locus where ω_P satisfies the opposite residue condition $\mathrm{Res}_{x_j}\,\omega_P = -\mathrm{Res}_{y_j}\,\omega_P$ for $j=1,2,3$. The variety $S(2,2)$ is locally parameterized by the projective 4-tuple consisting of the three residues and one relative period, so $S(2,2)$ is three-dimensional.

Define the cross-ratio morphisms $Q_i \colon S(2,2) \to \mathbf{G}$ and $R_i \colon S(2,2) \to \mathbf{G}$ by

$$Q_i = [z_1, z_2, y_i, x_i] \quad \text{and} \quad R_i = [x_{i+1}, y_{i+1}, x_{i+2}, y_{i+2}],$$

with indices taken mod 3. In the above normalization $Q_i = y_i/x_i$. We define $Q, \mathrm{CR}\colon S(2,2) \to \mathbf{G}^3$ by $Q = (Q_1,Q_2,Q_3)$ and $\mathrm{CR} = (R_1,R_2,R_3)$, and define $\mathrm{Res}\colon S(2,2) \to \mathbf{P}^2$ by $\mathrm{Res}(P) = (\mathrm{Res}_{x_i}\,\omega_P)_{i=1}^3$. Finally, given $\zeta = (\zeta_1,\zeta_2,\zeta_3) \in \mathbf{G}^3$, we define $S_\zeta(2,2) \subset S(2,2)$ to be the locus where $Q_i = \zeta_i$ for each i.

Lemma 10.4. *Any irreducible stable form $(X,\omega) \in \Omega\mathcal{M}_3(2,2)^{\mathrm{odd}}$ lying over a cusp of an algebraically primitive Teichmüller curve C generated by $(X,\omega) \in \Omega\mathcal{M}_3(2,2)^{\mathrm{odd}}$ is equal to ω_P for some $P \in S_{(\zeta_1,\zeta_2,\zeta_3)}(2,2) \in \mathrm{CR}^{-1}(T)$, where the ζ_i are non-identity roots of unity and where $T \subset \mathbf{G}^3$ is a proper algebraic subgroup. Moreover, if we normalize the components $(r_1 : r_2 : r_3)$ of $\mathrm{Res}(P)$ such that $r_1 \in \mathbf{Q}$, then $\{r_1, r_2, r_3\}$ is a basis of some totally real cubic number field.*

Proof. The proof of [BM12, Lemma 13.4], using only the description of boundary points and the torsion condition, applies verbatim. ∎

Lemma 10.5. *Let ζ_i be roots of unity, all different from one. Unless $\zeta_i = -1$ for all i, the variety $S_{(\zeta_1,\zeta_2,\zeta_3)}(2,2)$ is zero-dimensional. If $\zeta_i = -1$ for all i, then $S_{(-1,-1,-1)}(2,2) = S(2,2)$ is two-dimensional.*

Proof. $S_{(\zeta_1,\zeta_2,\zeta_3)}(2,2)$ is cut out by the equations $y_i = \zeta_i x_i$ and

$$D_i = \zeta_i^2 \prod_{j\neq i}(x_i - x_j)(x_i - \zeta_j x_j) - \prod_{j \neq i}(\zeta_i x_i - x_j)(\zeta_i x_i - \zeta_j x_j), \tag{10.4}$$

which expresses the opposite-residue condition.

Suppose that $S_{(\zeta_1,\zeta_2,\zeta_3)}(2,2)$ has a positive dimensional component. Then there is a homogeneous polynomial P of some degree $d < 4$ which divides D_k for all k. Expanding D_k, we obtain

$$D_k = x_k^4\zeta_k^2(1-\zeta_k)(1-\zeta_k^2) + \cdots + \zeta_{k+1}x_{k+1}^2\zeta_{k+2}x_{k+2}^2(1-\zeta_k^2)(1-\zeta_k),$$

with indices taken mod 3. Because each D_k contains x_k^4 with non-zero coefficient, each monomial x_k^d appears in P with non-zero coefficient. We have

$$P(0,x_2,x_3) = \alpha_2 x_2^d + \alpha_3 x_3^d + \ldots \quad | \quad D_1(0,x_2,x_3) = \zeta_2 x_2^2 \zeta_3 x_3^2(1-\zeta_1^2)(1-\zeta_1).$$

This is not possible since the α_i are nonzero since we may suppose $\zeta_1^2 \neq \pm 1$, possibly after swapping the indices.

■

Proposition 10.6. *There is a finite number of projectivized triples of real cubic numbers $(r_1 : r_2 : r_3)$ such that for any irreducible periodic direction on any $(X,\omega) \in \Omega\mathcal{M}_3(2,2)^{\mathrm{odd}}$ generating an algebraically primitive Teichmüller curve, the projectivized widths of the cylinders in that direction is one of the triples $(r_1 : r_2 : r_3)$.*

In particular, there are only a finite number of trace fields F of algebraically primitive Teichmüller curves in $\Omega\mathcal{M}_3(2,2)$.

Proof. By Northcott's Theorem, we need only to give a uniform bound for the heights of the triples $(r_1 : r_2 : r_3)$ of widths of cylinders, or equivalently of residues of limiting irreducible stable forms satisfying the conditions of Lemma 10.4.

If $\zeta_i = -1$ for $i = 1,2,3$ the resulting stable form over any cusp of the Teichmüller curve is hyperelliptic. By the hyperelliptic open-up [CM12] then the whole Teichmüller curve parameterizes a family of hyperelliptic curves. This case has been dealt with in Section 6.3.

In the remaining cases $S_{(\zeta_1,\zeta_2,\zeta_3)}(2,2)$ is zero-dimensional by Lemma 10.5 and the proof of [BM12, Proposition 13.7] can be copied. ■

Proposition 10.7. *Given a basis (r_1,r_2,r_3) over $\mathbf{Q}$ of a totally real cubic number field, there are only finitely many stable forms over cusps of algebraically primitive Teichmüller curves in $\Omega\mathcal{M}_3(2,2)^{\mathrm{odd}}$ having residues (r_1,r_2,r_3).*

Proof. Consider the variety $C = \mathrm{Res}^{-1}(r_1 : r_2 : r_3) \subset S(2,2)$ of forms having residues $\pm r_i$ and two zeros of order 2. A dimension count shows that C is at least one-dimensional. In fact, C is exactly one-dimensional, as C is locally parameterized by the single relative period of the forms ω_P. Let C_0 be a component of C. We suppose that C_0 contains infinitely many cusps of algebraically primitive Teichmüller curves and derive a contradiction. Consider the image $Q(C_0) \subset (\mathbf{C}^*)^3$. We claim that $Q(C_0)$ is a curve. If not, and $Q(C_0) = (\zeta_1,\zeta_2,\zeta_3)$, then C_0 is a component of $S_{(\zeta_1,\zeta_2,\zeta_3)}$, hence $\zeta_i = -1$ for all i and we are in the hyperelliptic case that has already been dealt with.

Now since C_0 contains infinitely many cusps of Teichmüller curves, $Q(C_0)$ must contain infinitely many torsion points of $(\mathbf{C}^*)^3$ by Lemma 10.4. From this it follows that $Q(C_0)$ is a translate of a subtorus of $(\mathbf{C}^*)^3$ by a torsion point. As in [BM12, Proposition 13.9] one checks that $Q(C_0)$ is in fact a subtorus of $(\mathbf{C}^*)^3$, rather than a translate.

It remains to show that $Q(C)$ is not a subtorus of $(\mathbf{C}^*)^3$. If this were true, we could find roots of unity ζ_i and a projective triple $(x_1(a) : x_2(a) : x_3(a))$ depending on a parameter a, such that for all $a \in \mathbf{C}$ the differential

$$\omega_\infty = \left(\sum_{i=1}^{3} \frac{r_i}{z - x_i(a)} - \frac{r_i}{z - \zeta_i^a x_i(a)} \right) dz = \frac{p(z)dz}{\prod_i (z - x_i(a))(z - \zeta_i^a x_i(a))}$$

has double zeros at $z = 0$ and at $z = \infty$. The vanishing of the z^4-term of $p(z)$ implies

$$\sum r_i x_i (1 - \zeta_i^a) = 0$$

and the constant term (divided by $x_1 x_2 x_3$) also yields a linear equation. Using the normalization $x_1 = 1$ we may solve the two linear equations for x_2 and x_3. We then take the limit of x_2 and x_3 as $a \to 0$, applying l'Hôpital's rule twice. If we let $\zeta_i = e^{2\pi i q_i}$ for some $q_i \in \mathbf{Q}$, we obtain

$$x_2(0) = \frac{q_3 r_3 - q_1 r_1}{q_2 r_2 - q_3 r_3} \quad \text{and} \quad x_3(0) = \frac{q_2 r_2 - q_1 r_1}{q_3 r_3 - q_2 r_2}. \tag{10.5}$$

We normalize $r_1 = 1$ and write $\widetilde{r_i} = q_i r_i$ as shorthand. Taking the derivative of the z^3-term of $p(z)$ with respect to a at $a = 0$ and making the substitution (10.5), we obtain

$$\sum_{i=1}^{3} \widetilde{r}_i^3 + 3\widetilde{r}_1 \widetilde{r}_2 \widetilde{r}_3 - \sum_{i \neq j} \widetilde{r}_i \widetilde{r}_j^2 = 0.$$

and from the constant terms with this substitution the limit $a = 0$ is

$$\widetilde{r}_3 \left(-6\widetilde{r}_1 \widetilde{r}_2 \widetilde{r}_3 + \sum_{i \neq j} \widetilde{r}_i \widetilde{r}_j^2 \right) = 0$$

Taking the resultant with respect to r_2 we obtain

$$q_1 r_2^6 q_3 r_3 \left(q_1 - q_3 r_3\right) \left(q_1 + q_3 r_3\right) = 0.$$

Since $\{r_1 = 1, r_2, r_3\}$ is a $\mathbf{Q}$-basis of F, this gives the contradiction we are aiming for. ■

Proof of Theorem 1.1, case $\Omega\mathcal{M}_3(2,2)^{\text{odd}}$. This is now a consequence of Proposition 10.7 and [BM12, Proposition 13.10]. ■

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