ON LINEAR FORMS WITH COEFFICIENTS IN $\mathbb{N}\zeta(1+\mathbb{N})$

The detailed version, **Part 1 (INTRODUCTION)**

Leonid A. Gutnik

In memory of Professor Alexander Buchstab

Foreword

This is the first part of the detailed version of my work "On linear forms with coefficients in $\mathbb{N}\zeta(1+\mathbb{N})$," [8]. The goal of this work is to give a complete proof of the following theorem.

Theorem. *Let*

$$\phi_i(x_1, x_2) = \sum_{k=1}^{2} (2ik - i - k + 2)\,\zeta(i + k + 1)\,x_k, \qquad i = 1, 2.$$

For any $d \in \mathbb{R}$, let $\|d\|$ be the distance between d and $\mathbb{Z}$; let $\gamma = 43,464412$. There is a positive constant c such that

$$\|\phi_1(x_1, x_2)\| + \|\phi_2(x_1, x_2)\| \geq c(|x_1| + |x_2|)^{-\gamma},$$

where $x_1 \in \mathbb{Z},\ x_2 \in \mathbb{Z},\ |x_1| + |x_2| > 0$.

`Remark.` The lattices $L_1,\ L_2$ in $\mathbb{R}^n$ are said to be incommensurable, if $L_1 \cap L_2 = \{0\}$. The qualitative part of the theorem asserts that lattice generated by the vectors $f_1 = \big(2\,\zeta(3),\, 3\,\zeta(4)\big)$ and $f_2 = \big(3\,\zeta(4),\, 6\,\zeta(5)\big)$ and the lattice $\mathbb{Z}^2$ (generated by the vectors $(1,0)$ and $(0,1)$) are incommensurable.

Corollary. *If $p \in \mathbb{Q},\ q \in \mathbb{Q}$, and $p^2 + q^2 > 0$, then*

$$\{2p\,\zeta(3) + 3q\,\zeta(4),\ 3p\,\zeta(4) + 6q\,\zeta(5)\} \not\subseteq \mathbb{Q},$$

and therefore

$$\left\{\frac{\zeta(3+2k)}{\zeta(4)},\ \frac{12\,\zeta(3)\zeta(5) - 9\,\zeta(4)^2}{\zeta(4)}\right\} \not\subseteq \mathbb{Q}, \quad k = 0, 1.$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

In the first part, I give a short survey of the C. S. Mejer functions, and then define and compute my auxiliary functions. In what follows, given $a \in \mathbb{C}$ and $M \subseteq \mathbb{C}$, $M \neq \emptyset$, let

$$a + M = M + a = \{w \in \mathbb{C}\colon\, w = a + z,\, z \in M\}.$$

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§ *1. Short survey of the Mejer functions*

In this work, I use the functions introduced and studied by C. S. Mejer in a long series of papers published during the decade 1936 – 1946. The functions of Mejer can be defined as follows ([9], ch. 5). Let

(1.1)

$$G_{p,q}^{(m,n)}\left(z\left|\begin{matrix} a_1, & \dots, & a_p \\ a_1, & \dots, & a_p\end{matrix}\right.\right) = \frac{1}{2\pi i}\int\limits_L \frac{z^s \prod\limits_{j=1}^{m}\Gamma(b_j - s)\prod\limits_{j=1}^{n}\Gamma(1 - a_j + s)}{\prod\limits_{j=m+1}^{q}\Gamma(1 - b_j + s)\prod\limits_{j=n+1}^{p}\Gamma(a_j - s)}\,ds,$$

where an empty product is, by definition, equal to 1, $0 \le m \le q$, $0 \le n \le p$; the parameters $a_j \in \mathbb{C}$, $j = 1, \dots, p$, and $b_k \in \mathbb{C}$, $k = 1, \dots, q$, are chosen in such a way that none of the poles of the function $\Gamma(b_k - s)$, $k = 1, \dots, m$, is equal to a pole of one of the functions $\Gamma(1 - a_j + s)$, $j = 1, \dots, n$. There are 3 possibilities to choose the curve L.

(A) First, the curve $L = L_0$ may be chosen to pass from $-i\infty$ to $+i\infty$ in such way that for any $k = 1, \dots, m$ all the poles of the function $\Gamma(b_k - s)$ lie to the right of it and for any $j = 1, \dots, n$ all the poles of the function $\Gamma(1 - a_j + s)$ lie to the left of it. The integral (1.1) is convergent in either of the following two cases:

($A1$) $|\arg(z)| < (m + n - \frac{p}{2} - \frac{q}{2})\pi$;

($A2$) $|\arg(z)| \le (m + n - \frac{p}{2} - \frac{q}{2})\pi$ and $\frac{p-q}{2} + \mathrm{Re}\,\Delta^* < -1$, where

$$\Delta^* = \sum_{k=1}^{q} b_k - \sum_{k=1}^{q} a_j. \tag{1.2}$$

(B) Second, the curve $L = L_1$ may be chosen to pass from $+i\infty$ to $+i\infty$, encircling each of the poles of the functions $\Gamma(b_k - s)$, $k = 1, \dots, m$, in the negative direction, but not including any of the poles of the functions $\Gamma(1 - a_j + s)$, $j = 1, \dots, n$. The integral (1.1) is convergent in each of the following three cases:

$(B1)$ $p < q$;

$(B2)$ $1 \le p \le q$ and $|z| < 1$;

$(B3)$ $1 \le p \le q$, $|z| \le 1$ and $\mathrm{Re}\ \Delta^* < -1$.

(C) Third, the curve $L = L_2$ may be chosen to pass from $-\infty$ to $-\infty$, encircling each of the poles of the functions $\Gamma(1-a_j+s)$, $j = 1, \dots, n$, in the positive direction, but not including any of the poles of the functions $\Gamma(b_k - s)$, $k = 1, \dots, m$. The integral (1.1) is convergent in each of the following 3 cases:

$(C1)$ $q < p$;

$(C2)$ $1 \le q \le p$ and $|z| > 1$;

$(C3)$ $1 \le q \le p$, $|z| \ge 1$ and $\mathrm{Re}\ \Delta^* < -1$.

If both conditions (A) and (B) (respectively (A) and (C)) are satisfied, then the result does not depend on whether the curve L is defined as in (A) or as in (B) (respectively in (C)).

Let g denote the integrand of the integral (1.1), let G denote the integral (1.1) with $L = L_k$, where $k = 1, 2$, and let S_k be the set of all the unremovable singularities of g encircled by L_k. If one of the conditions (B1) – (B3) (respectively (C1) – (C3)) holds for $k = 1$ (respectively for $k = 2$), then

$$G = (-1)^k \sum_{s \in S_k} \mathrm{Res}\,(g; s), \tag{1.3}$$

where $\mathrm{Res}\,(g; s)$ stands for the residue of the function g at the point s. Let us prove these assertions. Let $\sigma_0 > 0$, $\tau_0 > 0$, and let

$$\min(\sigma_0, \tau_0) \ge 1 + \sum_{j=0}^{p} 2(|a_j| + 1) + \sum_{k=0}^{q} 2(|b_k| + 1). \tag{1.4}$$

The curve L_0 may be chosen in such a way that, except for a compact piece, it coincides with a part of the imaginary axis $|\mathrm{Im}\ s| \ge \tau_0$; the curves L_1 and L_2 may be chosen so that, except for a compact part, the curve L_1 coincides with the union of the two rays $\sigma \pm i\tau_0$, $\sigma \ge \sigma_0$, and the curve L_2, except for a compact part, coincides with the union of the two rays $\sigma \pm i\tau_0$, $\sigma \le -\sigma_0$.

Let $r_0 \ge \sigma_0 + \tau_0$ be chosen big enough, so that the specified compact parts of L_0, L_1 and L_2 lie inside the disk $K_0 = \{s \in \mathbb{C}\colon |s| \le r_0\}$.

Let $u \ge 0$, $\eta(u) = \int\limits_0^u ([t] - t + \frac{1}{2})dt$; then $0 \le \eta(u) \le 1/8$. Let

$$\overline{D}_0 = \{s \in \mathbb{C}\colon |s| \ge r_0\} \backslash \{s \in \mathbb{C}\colon \mathrm{Re}\ s < 0,\ |\mathrm{Im}\ s| < \tau_0\}.$$

If $z = x + iy,\ |y| \geq \tau_0$, then the integral in the complex Stirling formula ([10], ch. 4)

$$\log\Gamma(z) = (z - \frac{1}{2})\log z - z + \frac{1}{2}\log(2\pi) + \int\limits_0^\infty \frac{\eta(u)}{(u+z)^2}du$$

can be estimated as follows:

$$\Big|\int\limits_0^\infty \frac{\eta(u)}{(u+z)^2}du\Big| \leq \int\limits_0^\infty \Big|\frac{\eta(u)}{(u+z)^2}\Big|du =$$

$$\int\limits_0^\infty \Big|\frac{\eta(u)}{(u+x)^2+y^2}\Big|du < \int\limits_{-\infty}^{+\infty} \Big|\frac{\eta(u)}{(u+x)^2+y^2}\Big|du < \frac{\pi}{8\tau_0} = O(1).$$

If $z = x + iy$ and $x \geq \sigma_0$, then

$$\Big|\int\limits_0^\infty \frac{\eta(u)}{(u+z)^2}du\Big| \leq \int\limits_0^\infty \Big|\frac{\eta(u)}{(u+z)^2}\Big|du =$$

$$\int\limits_0^\infty \Big|\frac{\eta(u)}{(u+x)^2+y^2}\Big|du \leq \int\limits_0^\infty \Big|\frac{\eta(u)}{(u+x)^2}\Big|du < \frac{1}{8\sigma_0} = O(1).$$

Therefore

$$\log\Gamma(s+c) = (s + c - \frac{1}{2})(\log s - 1) + O(1) \tag{1.5}$$

in $\overline{D}_0$, for any constant c with $2|c| \leq \min\{\sigma_0, \tau_0\}$.

Let $s = \sigma + i\tau = |s|\,e^{i\psi}$ with $|\psi| \leq \pi,\ \phi = \arg z$, then

$$\mathrm{Re}\,\log\Gamma(s+c) = \big(\sigma + \mathrm{Re}\,(c) - \frac{1}{2}\big)(\log|s| - 1) - \tau\psi + O(1). \tag{1.6}$$

One may write $g(s) = g_1(s)g_2(s)$ with

$$g_1(s) = z^s \prod_{j=1}^p \Gamma(1 - a_j + s)\Big/\prod_{j=1}^q \Gamma(1 - b_j + s), \tag{1.7}$$

$$g_2(s) = \prod_{j=n+1}^p \frac{\sin(\pi(a_j - s))}{\pi} \prod_{j=1}^m \frac{\pi}{\sin(\pi(b_j - s))}. \tag{1.8}$$

Since $\tau\psi \geq 0$, it follows from (1.6) that

$$g_1(s) = O(1) \cdot |z|^\sigma e^{|\phi||\tau|} (|s|/e)^{((p-q)(\sigma+\frac{1}{2})+\mathrm{Re}\ \Delta^*)} e^{-\tau(p-q)\psi} = \tag{1.9}$$

$$= O(1) \cdot |z|^\sigma (|s|/e)^{((p-q)(\sigma+\frac{1}{2})+\mathrm{Re}\ \Delta^*)} e^{|\tau|(p-q)(\frac{\pi}{2}-|\psi|)} e^{|\tau|(\phi-(p-q)\frac{\pi}{2})}$$

in $\overline{D}_0$. In view of our choice of τ_0, it follows that

$$|\sin(\pi(a_j - s))| = O(1) \cdot e^{\pi|\tau|}, \quad |\sin(\pi(b_k - s))|^{-1} = O(1) \cdot e^{-\pi|\tau|}$$

for $s \in \overline{D}_0$, $j = 1, \dots, p$ and $k = 1, \dots, q$, and consequently

$$g_2(s) = O(1) \cdot e^{\pi(p-m-n)|\tau|}. \tag{1.10}$$

Therefore, according to (1.9) and (1.10) the following equality

$$|g(s)| = O(1) \cdot |z|^\sigma e^{|\phi||\tau|} (|s|/e)^{((p-q)(\sigma+\frac{1}{2})+\mathrm{Re}\ \Delta^*)} e^{-|\tau|(p-q)|\psi|} e^{\pi(p-m-n)|\tau|} = \tag{1.11}$$

$$= O(1) \cdot |z|^\sigma (|s|/e)^{((p-q)(\sigma+\frac{1}{2})+\mathrm{Re}\ \Delta^*)} \cdot e^{((p-q)(\frac{\pi}{2}-|\psi|)+|\phi|+\pi(\frac{p+q}{2}-m-n))|\tau|}$$

holds in $\overline{D}_0$. According to (1.11), on the part of the curve L_0 lying on the rays $\sigma = 0$, $|\tau| \geq \tau_0$ we have the equality

$$|g(s)| = O(1) \cdot (|\tau|/e)^{(\frac{p-q}{2}+\mathrm{Re}\ \Delta^*)} e^{|\tau|(|\phi|+\pi(\frac{p+q}{2}-m-n))};$$

if the condition (A1) holds, then

$$|g(s)| = |\tau|^{O(1)} e^{|\tau|(|\phi|+\pi(\frac{p+q}{2}-m-n))},$$

where $|\tau| \geq \tau_0$, and the integral (1.1) is convergent; if the condition (A2) holds, then

$$|g(s)| = O(1) \cdot (|\tau|/e)^{(\frac{p-q}{2}+\mathrm{Re}\ \Delta^*)},$$

where $|\tau| \geq \tau_0$, and the integral (1.1) is convergent. Let the number σ_0 be chosen so that

$$\sigma_0 > 3 + \left| \frac{p-q}{2} + \mathrm{Re}\ \Delta^* \right|. \tag{1.12}$$

If one of the condition (B), (C) holds, then

$$(p-q)\sigma = -|(p-q)||\sigma|, \tag{1.13}$$

$$(p-q)(\frac{\pi}{2}-|\psi|)=-|p-q||\frac{\pi}{2}-|\psi||, \tag{1.14}$$

$$\sigma(\log|z|)=-|\sigma||\log|z|| \tag{1.15}$$

in $\overline{D}_0$, and (1.9), (1.11) takes respectively the form

$$|g_1(s)|=O(1)\cdot e^{-|\sigma||\log|z||}(|s|/e)^{-|\sigma||p-q|+\frac{p-q}{2}+\mathrm{Re}\ \Delta^*}\times \\ e^{-|p-q||\frac{\pi}{2}-|\psi|||\tau|}e^{(|\phi|-(p-q)\frac{\pi}{2})|\tau|} \tag{1.16}$$

$$|g(s)|=O(1)\cdot e^{-|\sigma||\log|z||}(|s|/e)^{-|\sigma||p-q|+\frac{p-q}{2}+\mathrm{Re}\ \Delta^*}\times \\ e^{(-|p-q||\frac{\pi}{2}-|\psi||+|\phi|+\pi(\frac{p+q}{2}-m-n))|\tau|}. \tag{1.17}$$

If one of conditions (B1), (C1) holds and $|\sigma|\geq\sigma_0$, then $|p-q|\geq 1$ and, in view of (1.12) and (1.17), we have

$$|g_1(s)|=O(1)\cdot e^{-|\sigma||\log|z||}(|s|/e)^{-3}e^{-|p-q||\frac{\pi}{2}-|\psi||}e^{(|\phi|-(p-q)\frac{\pi}{2})|\tau|}, \tag{1.18}$$

$$|g(s)|=O(1)\cdot e^{-|\sigma||\log|z||}(|s|/e)^{-3}e^{(-|p-q||\frac{\pi}{2}-|\psi||+|\phi|+\pi(\frac{p+q}{2}-m-n))|\tau|} \tag{1.19}$$

in $\overline{D}_0$. Therefore if one of conditions (B1), (C1) holds, then on the rays $\tau=\tau_0$, $|\sigma|\geq\sigma_0$ we have the equality

$$|g(s)|=O(1)\cdot(|\sigma|/e)^{-3}, \tag{1.20}$$

and the integral (1.1) is convergent; if one of conditions (B2), (C2) holds, then on the rays $\tau=\tau_0,\ |\sigma|\geq\sigma_0$ with $|\log|z||>0$ we have the equality

$$|g(s)|=|\sigma|^{O(1)}\cdot e^{-|\sigma||\log|z||}, \tag{1.21}$$

and the integral (1.1) is convergent. Finally, if one of condition (B3), (C3) holds, then, in view of (1.17),

$$|g(s)|=O(1)\cdot|\sigma|^{\mathrm{Re}\ \Delta^*} \tag{1.22}$$

on the rays $\tau=\tau_0,\ |\sigma|>\sigma_0$, and the integral (1.1) is convergent. For any $r\geq r_0$ and $k=0,1,2$ let us denote by $L_{r,k}$ the part of the curve L_k contained in the disk $K_r=\{s\in\mathbb{C}\colon\ |s|\leq r\}$ and directed as the curve L_k. Let further $C_{r,k,j}$, where

$k = 1, 2$, $j = 0, 1$, be the smallest counter-clockwise directed arc of the circle $|s| = r$ connecting the point $ir(-1)^j$ with the point $i\tau_0(-1)^j - (-1)^k\sqrt{r^2 - \tau_0^2}$. Finally, let $L^*_{r,k}$, where $k = 1, 2$, be the positively directed curve consisting of the curves $L_{0,r}$, $L_{k,r}$, $C_{r,k,0}$ and $C_{r,k,1}$. Clearly,

$$\int\limits_{L_k} g(s) = \lim_{r\to\infty} \int\limits_{L_{r,k}} g(s)ds \tag{1.23}$$

for $k = 0, 1, 2$. The function $g(s)$ has no singular points in the domain bounded by the curve L^*_k. Therefore

$$\int\limits_{L_{r,k}} g(s)ds - (-1)^k \int\limits_{C_{r,k,0}} g(s)ds - \int\limits_{L_{r,0}} g(s)ds - (-1)^k \int\limits_{C_{r,k,1}} g(s)ds = \int\limits_{L^*_k} g(s)ds = 0\,. \tag{1.24}$$

For $k = 1, 2$ and $j = 0, 1$, let $\check{C}_{r,k,j}$ be the counter-clockwise directed part of the arc $C_{r,k,j}$ lying in the strip $|\sigma| \leq \sigma_0$, and let $\hat{C}_{r,k,j}$ be the counter-clockwise directed part of the arc $C_{r,k,j}$ lying outside this strip. If $p \neq q$ and one of conditions (B1), (C1) holds, then according to (1.19)

$$|g(s)| = O(1) \cdot (r/e)^{-3} \tag{1.25}$$

on the $\hat{C}_{r,k,j}$ for $k = 1, 2$ and $j = 0, 1$, and, because the length of the arc $\hat{C}_{r,k,j}$ is not bigger than $\pi r/2$, it follows that

$$\lim_{r\to\infty} \int\limits_{\hat{C}_{r,k,j}} g(s)ds = 0. \tag{1.26}$$

Let $\gamma = 4^{-1}\min\{||\phi| + \pi(p\ +\ q)/2 - m - n|, |\log|z||\}$. In view of (1.17), if $p = q$ and one of conditions (B), (C) holds, then

$$|g(s)| = O(1) \cdot e^{-\gamma r}(r/e)^{\mathrm{Re}\ \Delta^*} \tag{1.27}$$

on the $\hat{C}_{r,k,j}$ for $k = 1, 2$ and $j = 0, 1$; now, if the condition (A) is satisfied, then the equality $\gamma = 0$ implies the inequality $\mathrm{Re}\,(\Delta^*) < -1$, and because the length of the arc $\hat{C}_{r,k,j}$ is not bigger than $\pi r/2$, relation (1.26) still holds. Moreover,

$$|\tau| = \sqrt{r^2 - \sigma_0^2} = r - O(1), \quad |\psi| = \frac{\pi}{2} - O(\arcsin(\sigma_0/r)) = \frac{\pi}{2} - O(1/r)$$

on the $\check{C}_{r,k,j}$ for $k = 1, 2,\ j = 0, 1$. Therefore if one of conditions (B), (C) holds, then according to (1.17)

$$|g(s)| = O(1) \cdot (r/e)^{\frac{p-q}{2} + \mathrm{Re}\ \Delta^*} e^{(|\phi| + \pi(\frac{p+q}{2} - m - n))|\tau|} = \tag{1.28}$$

$$O(1) \cdot (r/e)^{\frac{p-q}{2} + \mathrm{Re}\ \Delta^*} e^{(|\phi| + \pi(\frac{p+q}{2} - m - n))r}$$

on the curve $\check{C}_{r,k,j}$. Consequently, if the condition (A1) and one of the condition (B), (C) hold, then

$$\lim_{r\to\infty} \int\limits_{\check{C}_{r,k,j}} g(s)ds = \lim_{r\to\infty} O(1) \cdot e^{(|\phi| + \pi(\frac{p+q}{2} - m - n))r} r^{O(1)} = 0 \tag{1.29}$$

because in this case $|\phi| + \pi(\frac{p+q}{2} - m - n) < 0$. If the condition (A2) holds, then

$$\lim_{r\to\infty} \int\limits_{\check{C}_{r,k,j}} g(s)ds = \lim_{r\to\infty} O(1) \cdot r^{\frac{p-q}{2} + \mathrm{Re}\ \Delta^* + 1} = 0. \tag{1.30}$$

We have thus proved that if both of the convergence conditions (A) and (B) hold, then

$$\int\limits_{\check{L}_0} g(s)ds = \int\limits_{L_1} g(s)ds;$$

and if both of the convergence conditions (A) and (C) hold, then

$$\int\limits_{\check{L}_0} g(s)ds = \int\limits_{L_2} g(s)ds.$$

Now I shall prove the equality (1.3). Let us choose σ_1 so that it is bigger than r_0 and lies outside of the countable set

$$\bigcup_{\mu=0}^{1} \bigcup_{j=1}^{p} \left((-1)^{\mu}\,\mathrm{Re}\,(a_j) + \mathbb{Z}\right) \bigcup \bigcup_{\nu=0}^{1} \bigcup_{k=1}^{q} \left((-1)^{\nu}\,\mathrm{Re}\,(b_k) + \mathbb{Z}\right).$$

For $k = 1, 2$ and $n \in \mathbb{N} - 1$, let $r(n) = \sqrt{\tau_0^2 + (\sigma_1 + n)^2}$, let $c_{n,k}$ denote the segment

$$[-(-1)^k(\sigma_1 + n) - i\tau_0, -(-1)^k(\sigma_1 + n) + i\tau_0],$$

and let $\widetilde{L}_{n,k}$ be the counter-clockwise directed curve consisting of the curve $L_{r(n),k}$ and the segment $c_{n,k}$; finally, let $\widetilde{S}_{n,k}$ denote the set of all the unremovable singularities encircled by the curve $\widetilde{L}_{n,k}$. It is clear that

$$\frac{1}{2\pi i}\int\limits_{\widetilde{S}_{n,k}} g(s)ds = \sum_{s\in\widetilde{S}_{n,k}} \mathrm{Res}\,(g;s),$$

where $k=1,2$. The equality (1.3) will be proved if we establish that for each $k=1,2$ the equality

$$\lim_{n\to\infty}\int\limits_{c_{n,k}} g(s)ds = 0 \tag{1.31}$$

holds. First, consider the case $k=1$. In view of the choice of σ_1, the function $g_2(s)$ is continuous on the segment $c_{0,k}$, and therefore there is a constant M_0 with $|g_2(s)|\le M_0$ for $s\in c_{0,k}$, where $k=1,2$. Since the period of the function $|g_2(s)|$ is equal to one, we have

$$|g_2(s)|\le M_0 \tag{1.32}$$

for $s\in c_{n,k},\ k=1,2$, and $n\in\mathbb{N}$. Let us first consider the case $k=1$. If condition (B) holds, then (1.31) follows from (1.16). Suppose now that condition (C) is satisfied. Write

$$g_1(s) = g_3(s)g_4(s) \tag{1.33}$$

with

$$g_3(s) = z^s\prod_{j=1}^{q}\Gamma(b_j-s)\Big/\prod_{j=1}^{p}\Gamma(a_j-s), \tag{1.34}$$

$$g_4(s) = \left(\prod_{j=1}^{p}\frac{\sin(\pi(a_j-s))}{\pi}\prod_{j=1}^{q}\frac{\pi}{\sin(\pi(b_j-s))}\right)^{-1}. \tag{1.35}$$

Since the period of the function $|g_4(s)|$ is equal to one, there is a constant M_2 such that

$$|g_4(s)|\le M_2 \tag{1.36}$$

for $s\in c_{n,2},\ n\in\mathbb{N}-1$, in view of the choice of σ_1. Because $|\tau|\le\tau_0$ if $s\in c_{n,2}$ and $-c_{n,2}$ lies in $\overline{D}_0,\ n\in\mathbb{N}-1$, it follows from (1.6) that

$$g_1(s) = O(1)\cdot|z|^\sigma\big(|s|/e\big)^{\sum\limits_{j=1}^{q}\mathrm{Re}\,(b_j-\sigma-\frac12)-\sum\limits_{j=1}^{p}\mathrm{Re}\,(b_j-\sigma-\frac12)} =$$
$$= O(1)\cdot|z|^\sigma\big(|s|/e\big)^{((p-q)(\sigma+\frac12)+\mathrm{Re}\ \Delta^*)}$$

with $|s|\ge-\sigma=\sigma_1+n$. Therefore the equality (1.31) holds also for $k=2$. This proves (1.3).

§ 2. *My auxiliary functions*

I shall work with the set Ω^* consisting of all the points $z \in \mathbb{C}$ for which

$$|z| \geq 1 \quad \text{and} \quad -\frac{3\pi}{2} < \arg(z) \leq \frac{\pi}{2}\,. \tag{2.1}$$

Thus $\log(-z) = \log z - i\pi$ for $\mathrm{Re}\,(z) > 0$. Let $\Delta \in \mathbb{N},\ 1 < \Delta,\ \delta_0 = 1/\Delta$,

$$\gamma_1 = (1-\delta_0)/(1+\delta_0), \quad d_l = \Delta + (-1)^l, \quad l = 1, 2.$$

To introduce the first of my auxiliary function $f_1(z,\nu)$, I use the auxiliary set

$$\Omega^{(h)} = \{z \in \mathbb{C}\colon\ |z| \leq 1,\ -\frac{3\pi}{2} < \phi = \arg(z) \leq \frac{\pi}{2}\}.$$

I shall prove that, for each $\nu \in \mathbb{N}$, the function $f_1(z,\nu)$ belongs to $\mathbb{Q}[z]$; therefore using the principle of analytic continuation we may regard it as being defined in $\mathbb{C}$, and consequently, in Ω^*. For $\nu \in \mathbb{N}$, let

$$f_1(z,\nu) = \tag{2.2}$$

$$= -(-1)^{\nu(\Delta+1)} G^{(1,3)}_{6,6}\left(z\left|\begin{matrix} -\nu d_1, & -\nu d_1, & -\nu d_1, & 1+\nu d_2, & 1+\nu d_2, & 1+\nu d_2 \\ 0, & 0, & 0, & \nu, & \nu, & \nu \end{matrix}\right.\right)$$

$$= -(-1)^{\nu(\Delta+1)} \frac{1}{2\pi i} \int\limits_{L_1} g^{(1,3)}_{6,6}(s)ds,$$

where

$$g^{(1,3)}_{6,6}(s) = z^s \Gamma(-s)\Gamma(1+s)^{-2}\big(\Gamma(1+\nu d_1+s)/(\Gamma(1-\nu+s)\Gamma(1+\nu d_2-s))\big)^3,$$

and the curve L_1 passes from $+\infty$ to $+\infty$ encircling the set $\mathbb{N}-1$ in the negative direction, but not including any point of the set $-\mathbb{N}$. Here $p = q = 6$, $m = 1$, $n = 3$, $a_1 = a_2 = a_3 = -\nu d_1$, $a_4 = a_5 = a_6 = 1 + \nu d_2$, $b_1 = b_2 = b_3 = 0$, $b_4 = b_5 = b_6 = \nu$, $\Delta^* = -3\nu - 3$ and, since we take $|z| \leq 1$, convergence conditions (B2) and (B3) hold. To compute the function $f_1(z,\nu)$, we use formula (1.3) and the well-known formula

$$\Gamma(s) = \Gamma(s+l)\prod_{k=1}^{l}(s+l-k)^{-1} \tag{2.3}$$

with $l \in \mathbb{N}$. The set of unremovable singular points of the function $g_{6,6}^{(1,3)}(s)$, which are encircled by the curve L_1, consists of the points $s = \nu, \dots, \nu d_2$, all these points are poles of the first order, and, for $k = 0, \dots, \nu\Delta$, the following equality holds:

$$\mathrm{Res}\,(g_{6,6}^{(1,3)}; \nu + k) = (-z)^{\nu+k}((\nu + k)!)^{-3}((\nu\Delta + k)!)^3(k!)^{-3}((\nu\Delta - k)!)^{-3} =$$

$$= (-z)^{\nu+k}\big((\nu d_1)!/(\nu\Delta)!\big)^3 \binom{\nu\Delta}{k}^3 \binom{\nu\Delta + k}{\nu d_1}^3.$$

The function $f_1(z, \nu)$ is equal to a finite sum

$$f_1(z, \nu) = \big((\nu d_1)!/(\nu\Delta)!\big)^3 z^\nu (-1)^{\nu\Delta} \sum_{k=0}^{\nu\Delta} (-z)^k \binom{\nu\Delta}{k}^3 \binom{\nu\Delta + k}{\nu d_1}^3. \tag{2.4}$$

Therefore, as it has been already remarked, using the principle of analytic continuation we may regard it as being defined in $\mathbb{C}$ and consequently in Ω^*.

Let $\Omega_0^* = \{z \in \Omega^* : \mathrm{Re}\,(z) > 0\}$. If $z \in \Omega_0^*$, then

$$-\frac{3\pi}{2} < \phi = \arg(-z) = \arg(z) - \pi < -\frac{\pi}{2},$$

and therefore $-z \in \Omega^*$. Now, let me introduce my second auxiliary function defined for $z \in \Omega_0^*$.

Let

$$f_2(z, \nu) = -(-1)^{\nu\Delta} \frac{1}{2\pi i} \int\limits_{L_2} g_{6,6}^{(4,3)}(s) ds =$$

$$= -(-1)^{\nu\Delta} G_{6,6}^{(4,3)} \left(-z \middle| \begin{matrix} -\nu d_1, & -\nu d_1, & -\nu d_1, & 1 + \nu d_2, & 1 + \nu d_2, & 1 + \nu d_2 \\ 0, & 0, & 0, & \nu, & nu, & \nu \end{matrix} \right),$$

where $z \in \Omega_0^*$, $\nu \in \mathbb{N}$, and

$$g_{6,6}^{(4,3)} = g_{6,6}^{4,3)}(s) = (-z)^s \Gamma(-s)^3 \Gamma(-s+\nu) \Gamma(1-\nu+s)^{-2} \Gamma(1+\nu d_1+s)^3 \Gamma(1+\nu d_2-s)^{-3};$$

and the curve L_2 passes from $-\infty$ to $-\infty$, encircling the set $-\mathbb{N}$ in the positive direction but no point in the set $\mathbb{N} - 1$. Here $p = q = 6$, $m = 4$, $n = 3$, $a_1 = a_2 = a_3 = -\nu d_1$, $a_4 = a_5 = a_6 = 1 + \nu d_2$, $b_1 = b_2 = b_3 = 0$, $b_4 = b_5 = b_6 = \nu$, $\Delta^* = -3\nu - 3$; since now $|-z| \geq 1$, convergence conditions (C2) and (C3) are satisfied. To compute the function $f_2(z, \nu)$, we use formula (1.3). The set of all the unremovable singular points of the function $g_{6,6}^{(4,3)}(s)$, encircled by the curve L_2, consists of the points $s=-1-\nu d_1-k$ with $k\in\mathbb{N}-1$; each of these points is a pole of the first order. Therefore making use of (2.3) one obtains

$$\operatorname{Res}(g_{6,6}^{(4,3)};-1-\nu d_1-k)=$$

$$=(-z)^{-1-\nu d_1-k}((\nu d_1+k)!)^3((\nu\Delta+k)!)^3(-1)^k(k!)^{-3}((1+2\nu\Delta+k)!)^{-3}=$$

$$=(-1)^{1+\nu d_1}z^{-(1+\nu d_1+k)}\left(\frac{\prod\limits_{j=1}^{\nu\Delta-\nu}(1+\nu\Delta-\nu+k-j)}{\prod\limits_{j=0}^{\nu\Delta}(1+\nu\Delta+k+j)}\right)^3,$$

$$f_2(z,\nu)=\sum_{k=0}^{+\infty}z^{-(1+\nu d_1+k)}\left(\frac{\prod\limits_{j=1}^{\nu\Delta-\nu}(1+\nu\Delta-\nu+k-j)}{\prod\limits_{j=0}^{\nu\Delta}(1+\nu\Delta+k+j)}\right)^3. \tag{2.5}$$

Let $a\in\mathbb{N}-1$, $b\in\mathbb{N}+a$, and

$$R(a;b;t)=\frac{b!}{(b-a)!}\prod_{\kappa=a+1}^{b}(t-\kappa)\prod_{\kappa=0}^{b}\frac{1}{(t+\kappa)},\qquad R_0(t;\nu)=R(\nu;\nu\Delta;t). \tag{2.6}$$

Let $t=1+\nu\Delta+k$ with $k\in\mathbb{N}-1$; in view of (2.5), it follows that

$$f_2(z,\nu)\big((\nu\Delta)!/(\nu d_1)!\big)^3=(-1)^\nu\sum_{t=\nu\Delta+1}^{\infty}R_0(t;\nu)^3z^{-t+\nu}. \tag{2.7}$$

Since $R_0(t;\nu)=0$ for $t=\nu+1,\dots,\nu\Delta$, we have

$$f_2(z,\nu)\big((\nu\Delta)!/(\nu d_1)!\big)^3=(-1)^\nu\sum_{t=\nu+1}^{\infty}R_0(t;\nu)^3z^{-t+\nu}. \tag{2.8}$$

Let

$$f_3=(-1)^{\nu(\Delta+1)}\frac{1}{2\pi i}\int\limits_{L_2}g_{6,6}^{(5,3)}(s)ds=$$

$$(-1)^{\nu(\Delta+1)}G_{6,6}^{(5,3)}\left(z\left|\begin{matrix}-\nu d_1, & -\nu d_1, & -\nu d_1, & 1+\nu d_2, & 1+\nu d_2, & 1+\nu d_2\\ 0, & 0, & 0, & \nu, & \nu, & \nu\end{matrix}\right.\right),$$

where $z \in \Omega_0^*$, $\nu \in \mathbb{N}$, and

$$g_{6,6}^{(5,3)} = g_{6,6}^{(5,3)}(s) = z^s \Gamma(-s)^3 \Gamma(\nu - s)^2 \Gamma(1-\nu+s)^{-1} \Gamma(1+\nu d_1 + s)^3 \Gamma(1+\nu d_2 - s)^{-3}.$$

Here $p = q = 6$, $m = 5$, $n = 3$, $a_1 = a_2 = a_3 = -\nu d_1$, $a_4 = a_5 = a_6 = 1 + \nu d_2$, $b_1 = b_2 = b_3 = 0$, $b_4 = b_5 = b_6 = \nu$, $\Delta^* = -3\nu - 3$; convergence conditions (C2) and (C3) are satisfied since now $|z| \geq 1$. The set of all the unremovable singular points of the function $g_{6,6}^{(5,3)}(s)$, encircled by the curve L_2, consists of the points $s = -1 - \nu d_1 - k$ with $k \in \mathbb{N} - 1$; each of these points is a pole of the second order. Therefore

$$\operatorname{Res}\,(g_{6,6}^{(5,3)}; -\nu d_1 - 1 - k) = \lim_{s \to -\nu d_1 - 1 - k} \frac{\partial}{\partial s}\big((s + \nu d_1 + 1 + k)^2 g_{6,6}^{(5,3)}\big),$$

where $k \in \mathbb{N} - 1$.

Let $s = -\nu d_1 - 1 - k + u$ and

$$H_1(u) = g_{6,6}^{(5,3)}(-\nu d_1 - 1 - k + u) = z^{-\nu d_1 - 1 - k + u}\, \Gamma(\nu d_1 + 1 + k - u)^3 \times$$

$$\Gamma(\nu\Delta + 1 + k - u)^2 \Gamma(-\nu\Delta - k + u)^{-1} \Gamma(-k+u)^3 \Gamma(2\nu\Delta + 2 + k - u)^{-3} =$$

$$= z^{-\nu d_1 - 1 - k + u}\, \Gamma(\nu d_1 + 1 + k - u)^3 \Gamma(\nu\Delta + 1 + k - u)^3 \Gamma(\nu\Delta + 1 + k - u)^{-1} \times$$

$$\Gamma(-\nu\Delta - k + u)^{-1} \Gamma(1 + k - u)^3 \Gamma(-k+u)^3 \Gamma(1+k-u)^{-3} \Gamma(2 + 2\nu\Delta + k - u)^{-3} =$$

$$= \big(\pi/\sin(\pi u)\big)^2 (-1)^{\nu\Delta} z^{-\nu d_1 - 1 - k + u}\, \Gamma(\nu d_1 + 1 + k - u)^3 \times$$

$$\Gamma(\nu\Delta + 1 + k - u)^3 \Gamma(1 + k - u)^{-3} \Gamma(2 + 2\nu\Delta + k - u)^{-3} = \big(\pi/\sin(\pi u)\big)^2 H^*(u),$$

where

$$H^*(u) = (-1)^{\nu\Delta}\, z^{-\nu d_1 - 1 - k + u} \times$$

$$\Gamma(\nu d_1 + 1 + k - u)^3 \Gamma(1 + k - u)^{-3} \Gamma(1 + \nu\Delta + k - u)^3 \Gamma(2 + 2\nu\Delta + k - u)^{-3} =$$

$$= (-1)^{\nu\Delta} z^{-T+\nu} \Gamma(T)^3 \Gamma(T + 1 + \nu\Delta)^{-3} \Gamma(T - \nu\Delta)^{-3} \Gamma(T - \nu)^3 =$$

$$= (-1)^{\nu\Delta} z^{-T+\nu} \Bigg(\prod_{\kappa=\nu+1}^{\nu\Delta} (T - \kappa)\Bigg)^3 \Bigg(\prod_{\kappa=0}^{k+\nu\Delta} (T + \kappa)^{-1}\Bigg)^3 =$$

$$= (-1)^{\nu\Delta} z^{-T+\nu} R_0(T;\nu)^3 ((\nu\Delta)!/(\nu d_1)!)^{-3}$$

and $T = \nu\Delta + 1 + k - u$. Therefore

$$(-1)^{\nu d_2} ((\nu\Delta)!/(\nu d_1)!)^3 \operatorname{Res}\,(g_{6,6}^{(5,3)}; -1 - \nu d_1 - k) =$$

$$= (-1)^\nu \left(z^{-T+\nu} R_0(T;\nu)^3 \log z - \frac{\partial}{\partial T} R_0(T;\nu)^3 \right) \Bigg|_{T=1+\nu\Delta+k}$$

because $(\pi u/(\sin(\pi u))^2$ is an even function. Thus

$$f_3(z,\nu) = f_2(z,\nu)\log z - (-1)^\nu ((\nu\Delta)!/(\nu d_1)!)^{-3} \cdot \sum_{t=\nu\Delta+1}^{\infty} z^{-t+\nu} \frac{\partial}{\partial t} R_0(t;\nu)^3;$$

since $R_0(t;\nu)^3$ has zeros of the third order in the points $t = 1, \dots, \nu\Delta$, it follows that

$$f_3(z,\nu) = f_2(z,\nu)\log z - (-1)^\nu ((\nu\Delta)!/(\nu d_1)!)^{-3} \sum_{t=1+\nu}^{\infty} z^{-t+\nu} \frac{\partial}{\partial t} R_0(t;\nu)^3. \tag{2.9}$$

Let

$$f_4(z,\nu) = -((\nu\Delta)!/(\nu d_1)!)^{-3} (-z)^\nu \sum_{t=\nu\Delta+1}^{\infty} z^{-t} \frac{\partial}{\partial t} R_0(t;\nu)^3; \tag{2.10}$$

then

$$f_3(z,\nu) = f_2(z,\nu)\log z + f_4(z,\nu). \tag{2.11}$$

Let

$$f_5^\vee(z,\nu) = -\frac{(-1)^{\nu\Delta}}{2\pi i} \int_{L_2} g_{6,6}^{(6,3)}(s)ds =$$

$$= -(-1)^{\nu\Delta} G_{6,6}^{(6,3)} \left(-z \middle| \begin{array}{cccccc} -\nu d_1, & -\nu d_1, & -\nu d_1, & 1+\nu d_2, & 1+\nu d_2, & 1+\nu d_2 \\ 0, & 0, & 0, & \nu, & \nu, & \nu \end{array} \right),$$

where $z \in \Omega_0^*$, $\nu \in \mathbb{N}$, and

$$g_{6,6}^{(6,3)} = g_{6,6}^{(6,3)}(s) = (-z)^s \Gamma(-s)^3 \Gamma(\nu - s)^3 \Gamma(1 + \nu d_1 + s)^3 \Gamma(1 + \nu d_2 - s)^{-3}.$$

Here $p = q = 6$, $m = 6$, $n = 3$, $a_1 = a_2 = a_3 = -\nu d_1$, $a_4 = a_5 = a_6 = 1 + \nu d_2$, $b_1 = b_2 = b_3 = 0$, $b_4 = b_5 = b_6 = \nu$, $\Delta^* = -3\nu - 3$; since $-z \in \Omega^*$, each of convergence conditions (C2) and (C3) is satisfied. The set of all the unremovable singular points of the function $g_{6,6}^{(5,3)}(s)$, encircled by the curve L_2, consists of the points $s = -1 - \nu d_1 - k$ with $k \in \mathbb{N} - 1$; each of these points is a pole of the third order. Therefore

$$\operatorname{Res}(g_{6,6}^{(6,3)}; -\nu d_1 - 1 - k) = \lim_{s \to -\nu d_1 - 1 - k} \frac{1}{2} \left(\frac{\partial}{\partial s}\right)^2 \left((s + \nu d_1 + 1 + k)^2 g_{6,6}^{(5,3)}\right),$$

where $k \in \mathbb{N}-1$.

Let $s = -\nu d_1 - 1 - k + u$, and

$$H_2(u) = g_{6,6}^{(6,3)}(-\nu d_1 - 1 - k + u) =$$

$$= (-z)^{-\nu d_1 - 1 - k + u}\Gamma(-k+u)^3\Gamma(\nu d_1+1+k-u)^3\Gamma(\nu\Delta+1+k-u)^3\Gamma(2+2\nu\Delta+k-u)^{-3}$$

$$= (-1)^k(-z)^{-\nu d_1 - 1 - k + u}\Gamma(1+\nu\Delta+k-u)^3\Gamma(2+2\nu\Delta+k-u)^{-3}\times$$

$$\times\,\Gamma(\nu d_1+1+k-u)^3\Gamma(1+k-u)^{-3}(\pi/\sin(\pi u))^3 = (\pi/\sin(\pi u))^3 H_2^*(u),$$

where

$$H_2^*(u) = (-1)^k(-z)^{-T+\nu}R_0(T;\nu)^3((\nu\Delta)!/(\nu d_1)!)^{-3}$$

and $T = \nu\Delta + 1 + k - u,\ t = \nu\Delta + 1 + k$, as before.

Therefore

$$-(-1)^{\nu\Delta}((\nu\Delta)!/(\nu d_1)!)^3 \operatorname{Res}(g_{6,6}^{(6,3)}; -1-\nu d_1 - k) =$$

$$= -(-1)^{\nu\Delta+k}\biggl(\frac{1}{2}((\log(-z))^2(-z)^{-t+\nu}R_0(t;\nu)^3 - (-z)^{-t+\nu}\log(-z)\frac{\partial}{\partial T}R_0(t;\nu)^3 +$$

$$+\frac{1}{2}\bigl((-z)^{-t+\nu}\bigl(\frac{\partial}{\partial T}\bigr)^2R_0(t;\nu)^3 + \pi^2(-z)^{-t+\nu}R_0(t;\nu)^3\bigr)\biggr)$$

because $(\pi u/\sin(\pi u))^3 = 1 + \frac{1}{2}(\pi u)^2 + \dots$.

Consequently,

$$f_5^{\vee}(z,\nu) = \frac{1}{2}f_2(z,\nu)\bigl((\log(-z))^2+\pi^2\bigr) + f_4(z,\nu)\log(-z)+ \tag{2.12}$$

$$+\frac{1}{2}(-z)^{\nu}((\nu\Delta)!/(\nu d_1)!)^{-3}\sum_{t=\nu\Delta+1}^{\infty} z^{-t}\bigl(\frac{\partial}{\partial T}\bigr)^2R_0(t;\nu)^3 =$$

$$= \frac{1}{2}\,f_2(z,\nu)(\log z)^2 + f_4(z,\nu)\log z +$$

$$+\frac{1}{2}(-z)^{\nu}((\nu\Delta)!/(\nu d_1)!)^{-3}\sum_{t=\nu\Delta+1}^{\infty} z^{-t}\bigl(\frac{\partial}{\partial T}\bigr)^2R_0(t;\nu)^3 - \pi i f_3(z,\nu),$$

where $z \in \Omega_0^*$.

Since $R_0(t;\nu)^3$ has zeros of the third order at the points $t = 1+\nu,\dots,\nu\Delta$, it follows that

$$f_5^{\vee}(z,\nu) = \frac{1}{2}f_2(z,\nu)(\log z)^2 + f_4(z,\nu)\log z +$$

$$+\frac{1}{2}(-z)^{\nu}((\nu\Delta)!/(\nu d_1)!)^{-3}\sum_{t=\nu+1}^{\infty} z^{-t}\big(\frac{\partial}{\partial T}\big)^2 R_0(t;\nu)^3 - i\pi f_3(z,\nu).$$

Let

$$f_6(z,\nu) = \frac{1}{2}(-z)^{\nu}((\nu\Delta)!/(\nu d_1)!)^{-3}\sum_{t=1}^{\infty} z^{-t}\big(\frac{\partial}{\partial T}\big)^2 R_0(t;\nu)^3, \tag{2.13}$$

$$f_5(z,\nu) = f_6(z,\nu) + \frac{1}{2}f_2(z,\nu)(\log z)^2 + f_4(z,\nu)\log z,$$

where $z \in \Omega_0^*$. Then

$$f_5(z,\nu) = f_5^{\vee}(z,\nu) + i\pi f_3(z,\nu), \tag{2.14}$$

where $z \in \Omega_0^*$. Let further

$$f_j^*(z,\nu) = \big((\nu\Delta)!/(\nu d_1)!\big)^3 f_j(z,\nu), \qquad j = 1,\dots,6. \tag{2.15}$$

Expanding function $R_0(t;\nu)^3$ into partial fractions, we obtain

$$R_0(t;\nu)^3 = \sum_{k=0}^{\nu\Delta}\alpha_{\nu,k}^*(t+k)^{-3} + \sum_{k=0}^{\nu\Delta}\beta_{\nu,k}^*(t+k)^{-2} + \sum_{k=0}^{\nu\Delta}\gamma_{\nu,k}^*(t+k)^{-3}$$

with

$$\alpha_{\nu,k}^* = (-1)^{\nu+\nu\Delta+k}\binom{\nu\Delta}{k}^3\binom{\nu\Delta+k}{\nu\Delta-\nu}^3, \tag{2.16}$$

(2.17)

$$\beta_{\nu,k}^* = \lim_{t\to -k}\frac{\partial}{\partial t}\big(R_0(t;\nu)^3(t+k)^3\big) = \alpha_{\nu,k}^*\big(3(-\sum_{\kappa=\nu+k+1}^{\nu\Delta+k}\kappa^{-1} - \sum_{\kappa=1}^{\nu\Delta-k}\kappa^{-1} + \sum_{\kappa=1}^{k}\kappa^{-1})\big),$$

$$2\gamma_{\nu,k}^* = \alpha_{\nu,k}^*\big(3(-\sum_{\kappa=\nu+k+1}^{\nu\Delta+k}\kappa^{-1} - \sum_{\kappa=1}^{\nu\Delta-k}\kappa^{-1} + \sum_{\kappa=1}^{k}\kappa^{-1})\big)^2 - \tag{2.18}$$

$$-\alpha_{\nu,k}^*\big(3(\sum_{\kappa=\nu+k+1}^{\nu\Delta+k}\kappa^{-2} - \sum_{\kappa=1}^{\nu\Delta-k}\kappa^{-2} - \sum_{\kappa=1}^{k}\kappa^{-2})\big),$$

where $k=0,\dots,\nu\Delta$. It follows from (2.4), (2.7), (2.10), (2.13) and (2.15) - (2.18) that

$$f_2^*(z,\nu)=(-z)^\nu\sum_{t=1+\nu}^{+\infty} z^{-t}R_0(t;\nu)^3= \tag{2.19}$$

$$=(-z)^\nu\sum_{t=1+\nu}^{+\infty} z^{-t}\sum_{k=0}^{\nu\Delta}\alpha^*_{\nu,k}(t+k)^{-3}+(-z)^\nu\sum_{t=1+\nu}^{+\infty} z^{-t}\sum_{k=0}^{\nu\Delta}\beta^*_{\nu,k}(t+k)^{-2}+$$

$$+(-z)^\nu\sum_{t=1+\nu}^{+\infty} z^{-t}\sum_{k=0}^{\nu\Delta}\gamma^*_{\nu,k}(t+k)^{-1}=(-z)^\nu\sum_{k=0}^{\nu\Delta}\alpha^*_{\nu,k}z^k\sum_{t=1+\nu+k}^{+\infty} z^{-t}t^{-3}+$$

$$+(-z)^\nu\sum_{k=0}^{\nu\Delta}\beta^*_{\nu,k}z^k\sum_{t=1+\nu+k}^{+\infty} z^{-t}t^{-2}+(-z)^\nu\sum_{k=0}^{\nu\Delta}\gamma^*_{\nu,k}z^k\sum_{t=1+\nu+k}^{+\infty} z^{-t}t^{-1}=$$

$$=\alpha^*(z;\nu)L_3(z^{-1})+\beta^*(z;\nu)L_2(z^{-1})+\gamma^*(z;\nu)(-\log(1-1/z))-\phi^*(z;\nu)$$

with

$$L_n(z)=\sum_{t=1}^{+\infty} z^t t^{-n}, \tag{2.20}$$

$$\alpha^*(z;\nu)=(-z)^\nu\sum_{k=0}^{\nu\Delta}\alpha^*_{\nu,k}z^k=f_1^*(z;\nu), \tag{2.21}$$

$$\beta^*(z;\nu)=(-z)^\nu\sum_{k=0}^{\nu\Delta}\beta^*_{\nu,k}z^k \tag{2.22}$$

$$\gamma^*(z;\nu)=(-z)^\nu\sum_{k=0}^{\nu\Delta}\gamma^*_{\nu,k}z^k, \tag{2.23}$$

$$\phi^*(z;\nu)=(-z)^\nu\sum_{k=0}^{\nu\Delta}\alpha^*_{\nu,k}z^k\sum_{t=1}^{k+\nu} z^{-t}t^{-3}+ \tag{2.24}$$

$$+(-z)^\nu\sum_{k=0}^{\nu\Delta}\beta^*_{\nu,k}z^k\sum_{t=1}^{k+\nu} z^{-t}t^{-2}+(-z)^\nu\sum_{k=0}^{\nu\Delta}\gamma^*_{\nu,k}z^k\sum_{t=1}^{k+\nu} z^{-t}t^{-1}=$$

$$= (-z)^{\nu}\sum_{t=1}^{\nu} z^{-t}\bigl(t^{-3}\alpha^*(z;\nu)+t^{-2}\beta^*(z;\nu)+t^{-1}\gamma^*(z;\nu)\bigr)+$$

$$+(-z)^{\nu}\sum_{k=0}^{\nu\Delta}\alpha^*_{\nu,k}z^k\sum_{t=1+\nu}^{k+\nu} z^{-t}t^{-3}+(-z)^{\nu}\sum_{k=0}^{\nu\Delta}\beta^*_{\nu,k}z^k\sum_{t=1+\nu}^{k+\nu} z^{-t}t^{-2}+$$

$$+(-z)^{\nu}\sum_{k=0}^{\nu\Delta}\gamma^*_{\nu,k}z^k\sum_{t=1+\nu}^{k+\nu} z^{-t}t^{-1};$$

$$f_4^*(z,\nu) = -(-z)^{\nu}\sum_{t=1+\nu}^{+\infty} z^{-t}\frac{\partial}{\partial t}R_0(t;\nu)^3 = \tag{2.25}$$

$$= (-z)^{\nu}\sum_{t=1+\nu}^{+\infty} z^{-t}\sum_{k=0}^{\nu\Delta}3\alpha^*_{\nu,k}(t+k)^{-4}+(-z)^{\nu}\sum_{t=1+\nu}^{+\infty} z^{-t}\sum_{k=0}^{\nu\Delta}2\beta^*_{\nu,k}(t+k)^{-3}+$$

$$+(-z)^{\nu}\sum_{t=1+\nu}^{+\infty} z^{-t}\sum_{k=0}^{\nu\Delta}\gamma^*_{\nu,k}(t+k)^{-2} = (-z)^{\nu}\sum_{k=0}^{\nu\Delta}3\alpha^*_{\nu,k}z^k\sum_{t=1+\nu+k}^{+\infty} z^{-t}t^{-4}+$$

$$+(-z)^{\nu}\sum_{k=0}^{\nu\Delta}2\beta^*_{\nu,k}z^k\sum_{t=1+\nu+k}^{+\infty} z^{-t}t^{-3}+(-z)^{\nu}\sum_{k=0}^{\nu\Delta}\gamma^*_{\nu,k}z^k\sum_{t=1+\nu+k}^{+\infty} z^{-t}t^{-2} =$$

$$= 3\alpha^*(z;\nu)L_4(z^{-1})+2\beta^*(z;\nu)L_3(z^{-1})+\gamma^*(z;\nu)L_2(z)^{-1}-\psi^*(z;\nu),$$

where

$$\psi^*(z;\nu) = (-z)^{\nu}\sum_{k=0}^{\nu\Delta}3\alpha^*_{\nu,k}\,z^k\sum_{t=1}^{k+\nu} z^{-t}t^{-4}+ \tag{2.26}$$

$$+(-z)^{\nu}\sum_{k=0}^{\nu\Delta}2\beta^*_{\nu,k}\,z^k\sum_{t=1}^{k+\nu} z^{-t}t^{-3}+(-z)^{\nu}\sum_{k=0}^{\nu\Delta}\gamma^*_{\nu,k}\,z^k\sum_{t=1}^{k+\nu} z^{-t}t^{-2} =$$

$$= (-z)^{\nu}\sum_{t=1}^{\nu} z^{-t}\bigl(3t^{-4}\alpha^*(z;\nu)+2t^{-3}\beta^*(z;\nu)+t^{-2}\gamma^*(z;\nu)\bigr)+$$

$$+(-z)^{\nu}\sum_{k=0}^{\nu\Delta}3\alpha^*_{\nu,k}z^k\sum_{t=1+\nu}^{k+\nu} z^{-t}t^{-4}+(-z)^{\nu}\sum_{k=0}^{\nu\Delta}2\beta^*_{\nu,k}z^k\sum_{t=1+\nu}^{k+\nu} z^{-t}t^{-3}+$$

$$+(-z)^{\nu}\sum_{k=0}^{\nu\Delta}\gamma^*_{\nu,k}z^k\sum_{t=1+\nu}^{k+\nu} z^{-t}t^{-2};$$

$$f_6^*(z,\nu) = \frac{1}{2}(-z)^\nu \sum_{t=1+\nu}^{+\infty} z^{-t}(\frac{\partial}{\partial t})^2 R_0(t;\nu)^3 = \tag{2.27}$$

$$= (-z)^\nu \sum_{t=1+\nu}^{+\infty} z^{-t} \sum_{k=0}^{\nu\Delta} 6\alpha^*_{\nu,k}(t+k)^{-5} + (-z)^\nu \sum_{t=1+\nu}^{+\infty} z^{-t} \sum_{k=0}^{\nu\Delta} 3\beta^*_{\nu,k}(t+k)^{-4} +$$

$$+ (-z)^\nu \sum_{t=1+\nu}^{+\infty} z^{-t} \sum_{k=0}^{\nu\Delta} \gamma^*_{\nu,k}(t+k)^{-3} = (-z)^\nu \sum_{k=0}^{\nu\Delta} 6\alpha^*_{\nu,k} z^k \sum_{t=1+\nu+k}^{+\infty} z^{-t}t^{-5} +$$

$$+ (-z)^\nu \sum_{k=0}^{\nu\Delta} 3\beta^*_{\nu,k} z^k \sum_{t=1+\nu+k}^{+\infty} z^{-t}t^{-4} + (-z)^\nu \sum_{k=0}^{\nu\Delta} \gamma^*_{\nu,k} z^k \sum_{t=1+\nu+k}^{+\infty} z^{-t}t^{-3} =$$

$$= 6\alpha^*(z;\nu)L_5(z^{-1}) + 3\beta^*(z;\nu)L_4(z^{-1}) + \gamma^*(z;\nu)L_3(z^{-1}) - \xi^*(z;\nu),$$

and

$$\xi^*(z;\nu) = (-z)^\nu \sum_{k=0}^{\nu\Delta} 6\alpha^*_{\nu,k} z^k \sum_{t=1}^{k+\nu} z^{-t}t^{-5} + \tag{2.28}$$

$$+ (-z)^\nu \sum_{k=0}^{\nu\Delta} 3\beta^*_{\nu,k} z^k \sum_{t=1}^{k+\nu} z^{-t}t^{-4} + (-z)^\nu \sum_{k=0}^{\nu\Delta} \gamma^*_{\nu,k} z^k \sum_{t=1}^{k+\nu} z^{-t}t^{-3} =$$

$$= (-z)^\nu \sum_{t=1}^{\nu} z^{-t}\big(6t^{-5}\alpha^*(z;\nu) + 3t^{-4}\beta^*(z;\nu) + t^{-3}\gamma^*(z;\nu)\big) +$$

$$+ (-z)^\nu \sum_{k=0}^{\nu\Delta} 6\alpha^*_{\nu,k} z^k \sum_{t=1+\nu}^{k+\nu} z^{-t}t^{-5} + (-z)^\nu \sum_{k=0}^{\nu\Delta} 3\beta^*_{\nu,k} z^k \sum_{t=1+\nu}^{k+\nu} z^{-t}t^{-4} +$$

$$+ (-z)^\nu \sum_{k=0}^{\nu\Delta} \gamma^*_{\nu,k} z^k \sum_{t=1+\nu}^{k+\nu} z^{-t}t^{-3}.$$

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Moscow State Institute
of Electronics and Mathematics
Bolshoi Trekhsvyatitelskij pereulok, 1-3/12, str. 8
Moscow 109028
Russian Federation
E-mail: gutnik@gutnik.mccme.ru