LOCAL TYPES OF CONFORMALLY-EINSTEIN KÄHLER METRICS IN HIGHER DIMENSIONS

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Abstract. The requirement that a Kähler metric in any given complex dimension $m > 2$ be conformally Einstein (but not Einstein) turns out to be much more restrictive, even locally, than in the case of complex surfaces. The local-isometry types of such metrics depend, for each $m > 2$, on three real parameters along with an arbitrary Kähler-Einstein metric h in complex dimension $m-1$. We provide a complete classification of these local-isometry types (for any given h) via an explicit description in terms of suitable rational functions of the conformal-factor function.

§0. Introduction

The present paper deals with the local structure of Kähler metrics in complex dimensions $m > 2$ which are conformally Einstein, but not Einstein. We provide a complete local classification of the local-isometry types of such metrics at points "in general position". (See Theorem 20.1 in §20.) As it turns out, for each $m > 2$ they depend on three real constants plus an additional parameter in the form of an arbitrary local-isometry type of a Kähler-Einstein metric in complex dimension $m-1$.

The result in question is derived from a more general local-classification theorem (Theorem 13.2) for Kähler manifolds admitting a nonconstant special ρ-ω potential, that is, a function τ whose second covariant derivative is a suitable functional combination of the Ricci tensor and the metric; see (5.1), (5.3) and (5.4) in §5.

Our classification procedure works in dimensions $m > 2$, but not for Kähler *surfaces*. The reason is that, when $m = 2$, the requirement that a given Kähler metric be conformally Einstein is (in its local, almost-everywhere version) much less restrictive than in higher dimensions. This is illustrated by the fact [14] that, at points where the scalar curvature s of a Kähler-surface metric g is nonzero, the conformally related metric $\tilde{g} = g/\mathrm{s}^2$ already satisfies a "part" of the Einstein condition (namely, vanishing of the divergence of the self-dual Weyl tensor) and is, up to a constant factor, the only metric conformal to g with this property. Nothing similar seems to occur in higher dimensions.

Various aspects of this question have been studied, especially in the compact case. Here belong, for instance, LeBrun's structure theorem [25] for Hermitian Einstein metrics on compact complex surfaces, as well as a family of (globally) conformally-Kähler compact Einstein manifolds representing all complex dimensions $m > 2$, which was found by Bérard Bergery (see [2], or Chapter 9, Section K in [3]). Bérard

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Bergery's construction is a higher-dimensional analogue of the conformally-Kähler Einstein metric on $M = \mathbf{C}\mathrm{P}^2 \,\#\, \overline{\mathbf{C}\mathrm{P}^2}$ discovered by Page [28], whose conformally related Kähler metric was among those independently found by Calabi ([7], [10]).

There are also known examples of compact Kähler-surface metrics which are almost-everywhere (but not globally) conformally Einstein ([17], [32]). Further papers on this and related topics are listed in the bibliography.

In a forthcoming paper [13] we use the results presented here to construct various examples of *compact* non-Einstein Kähler manifolds in dimensions $m > 2$ which are almost-everywhere conformally Einstein, and prove a classification theorem for the compact case.

§1. Preliminaries

To save space, we sometimes display several equalities as a single block of expressions (such as (6.2) in §6). When referring to each individual formula in such a block, we will label it by a lower-case roman numeral; thus, the equations forming (6.2) are (6.2)i) – (6.2)v).

We will denote ∇, R, r and s the Levi-Civita connection, curvature tensor, Ricci tensor and scalar curvature of a given Riemannian metric g on a manifold M. (See also (1.18) and (1.19) below.) In addition, the symbol $\nabla\tau$ will stand for the g-gradient of a C^1 function $\tau : M \to \mathbf{R}$.

For a C^1 vector field v on a Riemannian manifold (M, g) we will write

$$\nabla v : TM \to TM \qquad \text{with} \quad (\nabla v)w = \nabla_w v\,, \tag{1.1}$$

treating the covariant derivative ∇v as a vector-bundle morphism $TM \to TM$ sening each $w \in T_xM$, $x \in M$, to $\nabla_w v \in T_xM$. Note that, since ∇ is torsion-free,

$$\nabla_v w \,-\, \nabla_w v \,=\, [v, w] \tag{1.2}$$

for any two C^1 vector fields v, w.

For a multiply-covariant tensor b and a vector (field) v, we use the standard notation

$$\imath_v b \,=\, b(v, \cdot, \ldots, \cdot)\,. \tag{1.3}$$

This applies in particular to the case where b is a 1-form; a twice-covariant symmetric tensor (such as g, the second covariant derivative $\nabla d\tau$ of a C^2 function τ, or the Ricci tensor r); or, a differential 2-form. For instance,

$$\imath_v g \,=\, d\tau \qquad \text{if} \quad v = \nabla\tau\,. \tag{1.4}$$

The divergence operator δ acts on such C^1 tensor fields b by

$$\delta b \,=\, \sum_u [\nabla_u b](u, \cdot, \ldots, \cdot)\,, \tag{1.5}$$

the summation being over the vectors of any orthonormal basis of the tangent space at the given point. For instance, we have the Bianchi identity for the Ricci tensor:

$$2\delta\mathrm{r} \,=\, d\mathrm{s}\,, \tag{1.6}$$

whose local-coordinate version is $2\mathrm{r}_{jl,}{}^{j} = \mathrm{s}_{,l}$ (see, e.g., [15, (5.2)]). Another example is

$$\Delta\tau \,=\, \delta(d\tau) \tag{1.7}$$

for any C^1 function τ on the Riemannian manifold in question.

We will also write

$$\delta v \,=\, \mathrm{Trace}\,\nabla v \tag{1.8}$$

for C^1 vector fields v, which is consistent with (1.3) as $\delta v = \delta\xi$, where $\xi = g(v,\,\cdot\,) = \imath_v g$ is the 1-form corresponding to v via g. Thus, (1.7) can be rewritten as $\Delta\tau = \delta(\nabla\tau)$.

We will also use the formula

$$(\nabla d\tau)(u,w) \,=\, g(u,\nabla_w v) \,=\, g(w,\nabla_u v)\,, \qquad \text{where} \quad v \,=\, \nabla\tau\,, \tag{1.9}$$

for C^2 functions τ and vector fields u, w in any Riemannian manifold, as well as the obvious relations

$$\delta[fw] \,=\, d_w f \,+\, f\,\delta w\,, \quad \delta[fb] \,=\, \imath_v b \,+\, f\,\delta b\,, \quad \text{with} \quad v = \nabla f\,, \tag{1.10}$$

for functions f, C^1 vector fields w, and twice-covariant symmetric tensor fields b, all of class C^1. Here d_w is the directional derivative corresponding to the tangent vector (field) w. For instance, any C^1 function τ in a Riemannian manifold has

$$d_v\tau \,=\, Q\,, \quad \text{with} \quad v \,=\, \nabla\tau \quad \text{and} \quad Q \,=\, g(\nabla\tau,\nabla\tau)\,. \tag{1.11}$$

For any C^2 function τ in a Riemannian manifold, we have

$$2\imath_v b \,=\, dQ\,, \quad \text{where} \quad v = \nabla\tau, \;\; b = \nabla d\tau \;\text{ and }\; Q = g(\nabla\tau,\nabla\tau)\,, \tag{1.12}$$

$$\imath_w b \,=\, \nabla_w d\tau \qquad \text{for} \quad b = \nabla d\tau\,, \tag{1.13}$$

with an arbitrary vector (field) w, and

$$\delta b \,=\, \imath_v \mathrm{r} \,+\, d\Delta\tau \quad \text{for} \quad v = \nabla\tau \;\text{ and }\; b = \nabla d\tau\,. \tag{1.14}$$

In fact, (1.13) is immediate from (1.3), while (1.12) is obvious in view of the local-coordinate expression $2\tau^{,k}\tau_{,jk} = [\tau^{,k}\tau_{,k}]_{,j}$. As for (1.14), it follows from the "contracted" Ricci-Weitzenböck formula

$$\imath_v \mathrm{r} \,=\, \delta([\nabla v]^*) \,-\, d\delta v \tag{1.15}$$

applied to $v = \nabla\tau$ (so that $[\nabla v]^* = \nabla v = \nabla d\tau$, with ∇v as in (1.1), and $\delta v = \Delta\tau$). Formula (1.15) is valid for every C^2 vector field v in any Riemannian manifold and has the local-coordinate form $\mathrm{r}_{jk}v^k = v^k{}_{,jk} - v^k{}_{,kj}$.

As for the exterior product and exterior derivative of 1-forms ξ, ξ' on a manifold M, they are given by

$$(\xi\wedge\xi')(u,v) \,=\, \xi(u)\xi'(v) \,-\, \xi(v)\xi'(u)\,, \tag{1.16}$$

$$(d\xi)(u,v) \,=\, d_u[\xi(v)] \,-\, d_v[\xi(u)] \,-\, \xi([u,v]) \tag{1.17}$$

for any C^1 vector fields u, v on M.

Finally, let us note that the Levi-Civita connection ∇ of a given Riemannian metric g has the property that, for any C^1 vector fields v, w, the covariant derivative $\nabla_w v$ is uniquely characterized by the requirement that, for all C^1 vector fields u,

$$\begin{aligned} 2g(\nabla_w v, u) \,&=\, d_w[g(v,u)] \,+\, d_v[g(w,u)] \,-\, d_u[g(w,v)] \\ &\quad+\, g(v,[u,w]) \,+\, g(u,[w,v]) \,-\, g(w,[v,u])\,. \end{aligned} \tag{1.18}$$

Our sign convention for the curvature tensor R of any connection ∇ in a real/complex vector bundle $\mathcal{E}$ over a manifold M is

$$R(u,v)w \,=\, \nabla_v\nabla_u w \,-\, \nabla_u\nabla_v w \,+\, \nabla_{[u,v]}w \tag{1.19}$$

for any C^2 (local) sections u, v of TM and w of $\mathcal{E}$. Denoting m the real/complex fibre dimension of $\mathcal{E}$ and choosing C^∞ sections w_a, $a = 1,\dots,m$, which trivialize the bundle $\mathcal{E}$ over some open set $U \subset M$, we have

$$\nabla_v w_a = \Gamma_a^b(v)\, w_b \qquad \text{(summed over} \quad b = 1,\dots,m) \tag{1.20}$$

for any vector field v on U, with some real/complex valued 1-forms Γ_a^b. From (1.19) and (1.16) – (1.17), we now obtain

$$R(u,v)w_a \,=\, R_a^b(u,v)\,w_b \quad \text{with} \quad R_a^b \,=\, -\,d\Gamma_a^b \,+\, \Gamma_a^c \wedge \Gamma_c^b\,. \tag{1.21}$$

Remark 1.2. Given a connection ∇ in a complex line bundle $\mathcal{E}$ over a manifold M, let ζ denote the *curvature form* of ∇, that is, the 2-form characterized by $R(u,v)w = i\zeta(u,v)w$ for u, v, w and $R(u,v)w$ as in (1.19). For any local C^∞ section w without zeros, the *connection* 1*-form* Γ characterized by $\nabla_v w = \Gamma(v)w$ (i.e., (1.20) with $m = 1$) then satisfies

$$\zeta \,=\, i\,d\Gamma\,. \tag{1.22}$$

In fact, this is immediate from (1.21) for $m = 1$.

§2. Kähler metrics and the Ricci form

We discuss Kähler manifolds primarily in real terms, using real-valued functions and differential forms, and real vector fields (rather than complexified ones). The few exceptions, such as Lemma 2.2 below, will be clearly identified as such. All components of tensors refer to *real* coordinate systems.

For any complex manifold M we will denote J its complex-structure tensor, treated (depending on the circumstances) either as a real vector-bundle morphism $J : TM \to TM$ with $J^2 = -1$, or as the multiplication by i in the complex vector bundle TM.

Let (M, g) now be a Kähler manifold, that is, a complex manifold M with a Riemannian metric g that makes $J : TM \to TM$ both skew-adjoint and parallel. We will use the generic symbols ω and ρ for its complex-structure tensor, Kähler form and, respectively, Ricci form. Our notational conventions are such that ω and ρ are related to g, J and the Ricci tensor r by

$$\omega(u,v) = g(Ju,v)\,, \qquad \rho(u,v) = \mathrm{r}(Ju,v) \tag{2.1}$$

for any $u,v\in T_xM$, $x\in M$. Skew-symmetry of ω and ρ reflects the fact that the symmetric twice-covariant tensors $b=g$ and $b=\mathrm{r}$ are both Hermitian, in the sense that

$$b(Jw,w') = -\,b(w,Jw') \quad \text{for all} \quad w,w'\in T_xM \quad \text{and} \quad x\in M\,. \tag{2.2}$$

The components of ρ in any local coordinates thus are

$$\rho_{jk} = -\,\mathrm{r}_{jl}J^l_k\,. \tag{2.3}$$

The following lemma describes a well-known fact that the Ricci form of any Kähler manifold (M,g) is, essentially, the curvature form of the natural connection in the highest complex exterior power of TM.

Lemma 2.1. *For any C^2 vector fields u,v on a Kähler manifold (M,g),*

$$\mathrm{Trace}_{\mathbf{C}}[R(u,v)] = i\,\rho(u,v)\,, \tag{2.4}$$

$R(u,v): TM\to TM$ being the complex-linear bundle morphism sending any C^2 vector field w to $R(u,v)w$ given by (1.19).

Proof. Both J and the unique Hermitian fibre metric $\langle\,,\rangle$ in TM whose real part is g are ∇-parallel. Thus, ∇ is a connection in the *complex* bundle TM, and it is compatible with $\langle\,,\rangle$. Therefore, the morphisms $R(u,v)$ and J commute, and are both skew-adjoint (relative to $\langle\,,\rangle$); consequently, their composite is self-adjoint. Denoting that composite F (for fixed vector fields u,v), we now have $2i\,\mathrm{Trace}_{\mathbf{C}}[R(u,v)] = 2\,\mathrm{Trace}_{\mathbf{C}}F = \mathrm{Trace}_{\mathbf{R}}F$ (the last equality being clear when one evaluates both traces using a complex basis $e_1,\dots,e_m$ of any given tangent space which diagonalizes F and, respectively, the real basis $e_1,\, ie_1,\,\dots,\,e_m,\, ie_m$). On the other hand, in terms of components (relative to any real local-coordinate system), symmetry of F means that $\mathrm{r}_{jpkq}J^q_l$ is symmetric in k,l. Thus, by (2.3),

$$\rho_{jk} = -\,\mathrm{r}_{jq}J^q_k = g^{pl}\mathrm{r}_{jplq}J^q_k = g^{pl}\mathrm{r}_{jpkq}J^q_l = \mathrm{r}_{jpkq}\alpha^{pq}\,, \tag{2.5}$$

α being the bivector field corresponding to J via g, so that $\alpha^{jk}=g^{jl}J^k_l$. The Bianchi identity gives $(\mathrm{r}_{jpkq}-\mathrm{r}_{jkpq}-\mathrm{r}_{jqkp})\alpha^{pq}=0$, i.e., $2\mathrm{r}_{jpkq}\alpha^{pq}=\mathrm{r}_{jkpq}\alpha^{pq}= -\,\mathrm{r}_{jkq}{}^pJ^q_p$. Summing (2.5) against u^jv^k we thus obtain $2\,\rho(u,v) = -\,\mathrm{Trace}_{\mathbf{R}}F = -\,2i\,\mathrm{Trace}_{\mathbf{C}}[R(u,v)]$, with F as above, which completes the proof. ■

Lemma 2.2. *Let (M,g) be a Kähler manifold of complex dimension m and let w_a, $a=1,\dots,m$, be C^∞ vector fields on an open set $U\subset M$ which trivialize the tangent bundle TU treated as a complex vector bundle. Furthermore, let Γ^b_a be the complex-valued* 1*-forms with* (1.20) *for the Levi-Civita connection ∇ of g. The Ricci form ρ of (M,g), restricted to U, then is given by*

$$\rho = i\,d\Gamma^a_a \qquad (\text{summed over} \quad a=1,\dots,m)\,. \tag{2.6}$$

In fact, the matrix $[R^b_a(u,v)]$ (see (1.21)) represents the operator $R(u,v)$ (cf. Lemma 2.1) in the complex basis $w_a(x)$ at any point $x\in U$. Now (2.4) and (1.21) give $i\,\rho = R^a_a = -\,d\Gamma^a_a$, as required. (Note that $\Gamma^c_a\wedge\Gamma^a_c = 0$ due to obvious pairwise cancelations.) ■

§3. Killing potentials

A (real) C^∞ vector field v on a complex manifold M is called *holomorphic* if it satisfies the Lie-derivative relation $\boldsymbol{L}_v J = 0$. Thus, since $\boldsymbol{L}_v(Jw) = (\boldsymbol{L}_v J)w + J(\boldsymbol{L}_v w)$ and $\boldsymbol{L}_v w = [v,w]$ for C^1 vector fields v,w, we have

$$[v,u] = 0 \qquad \text{if} \quad \boldsymbol{L}_v J = 0 \text{ and } u = Jv\,. \tag{3.1}$$

Let (M,g) be a Kähler manifold. A real-valued C^∞ function τ on M is said to be a *Killing potential* if $u = J(\nabla\tau)$ is a Killing vector field on (M,g). On the other hand, a (real) C^∞ vector field v on (M,g) then is holomorphic if and only if it satisfies the condition

$$[J,\nabla v] = 0\,, \tag{3.2}$$

whose local-coordinate form is $J_k^p v^l{}_{,p} = J_p^l v^p{}_{,k}$. (In fact, as $\nabla J = 0$, we have $\mathcal{L}_v J = [J,\nabla v]$ for any C^1 vector field v, as one sees applying both sides to an arbitrary C^∞ vector field and using (1.2), (11).)

Remark 3.1. For a C^∞ function τ on a Kähler manifold (M,g), the following three conditions are equivalent:

(a) τ is a Killing potential.
(b) Its gradient $v = \nabla\tau$ is a holomorphic vector field.
(c) $b = \nabla d\tau$ is Hermitian, i.e., satisfies (2.2) for all $v,v' \in T_xM$ and $x \in M$.

In fact, let $v = \nabla\tau$ and $u = Jv$. Then $\nabla_w u = J(\nabla_w v)$ for every tangent vector w, so that $\nabla u = J\circ(\nabla v)$, while $[\nabla v]^* = \nabla v$, that is, ∇v is self-adjoint at every point. Thus, $\nabla u + [\nabla u]^* = J\circ(\nabla v) + [\nabla v]^*\circ J^* = [J,\,\nabla v]$, and so (a) is equivalent to (b). On the other hand, in view of (2.2) for $b = g$ and (1.9), Hermitian symmetry of $\nabla d\tau$ is equivalent to $[J,\,\nabla v] = 0$, i.e., to (a).

For a nonconstant Killing potential $\tau : M \to \mathbf{R}$, Remark 3.1 yields

$$d\tau \neq 0 \qquad \text{on a dense open subset of } M. \tag{3.3}$$

Remark 3.2. Using Remark 3.1 we now arrive at a (well-known) complete characterization of Killing potentials: locally, up to an additive constant, a Killing potential τ may be represented by $u = J(\nabla\tau)$, which is a completely arbitrary *holomorphic Killing vector field* on the Kähler manifold in question. In fact, if τ is a Killing potential, $u = J(\nabla\tau)$ is not only Killing, but also holomorphic as, for $v = \nabla\tau$, $\nabla u = J\circ(\nabla v)$ commutes with J since so does ∇v (see Remark 3.1(b) and (3.2)). Conversely, given a holomorphic Killing vector field u, setting $v = -\,Ju$ we get $\nabla v = -\,J\circ(\nabla u)$ and, as J and ∇u commute and are skew-adjoint, their composite ∇v must be self-adjoint, so that, locally, $v = \nabla\tau$ for some function τ.

A function τ satisfying $u = J(\nabla\tau)$ with a given a holomorphic Killing field u on a Kähler manifold (M,g) will be referred to as a (local) *Killing potential for* u.

Lemma 3.3. *Let $\tau : M \to \mathbf{R}$ be a Killing potential on a Kähler manifold (M,g). Setting $v = \nabla\tau$, $u = Jv = J(\nabla\tau)$ and $\xi = g(Jv,\,\cdot\,)$ we then have*

$$\xi = \imath_v\omega = \imath_u g\,. \tag{3.4}$$

Moreover, setting $b = \nabla d\tau$ *and*

$$\beta(w,w') \,=\, b(Jw,w') \tag{3.5}$$

for $w, w' \in T_xM$, $x \in M$, *we obtain a differential* 2*-form* β *on* M *with*

$$d\xi \,=\, 2\beta\,. \tag{3.6}$$

Proof. Relation (3.4) is obvious from (1.3) and (2.1). Since $b = \nabla d\tau$ is Hermitian (Remark 3.1(c)), (3.5) is skew-symmetric in w, w'. Finally, by (1.17), $(d\xi)(w,w') = g(Aw,w')$ with $A = \nabla u - [\nabla u]^*$, for vectors $w, w' \in T_xM$ and $x \in M$, whenever $\xi = \imath_u g = g(u,\,\cdot\,)$ for a C^1 vector field u. In our case u is a Killing field (as τ is a Killing potential), so that $A = 2\nabla u$ and, as $u = Jv$ with $v = \nabla\tau$, $(d\xi)(w,w') = 2g(\nabla_w u, w') \,=\, 2g(J\nabla_w v, w') \,=\, -\,2g(\nabla_w v, Jw') \,=\, -\,2b(w, Jw') \,=\, 2\beta(w,w')$ in view of (1.9) with $b = \nabla d\tau$ and (3.5). This completes the proof. ∎

The coordinate form of the identity (1.15) gives

$$\mathrm{r}_{jk}u^k \,=\, u^k{}_{,jk} \quad \text{for a } C^2 \text{ vector field } u \text{ with } \delta u \,=\, 0\,, \tag{3.7}$$

e.g., in the case where u is a Killing field. Applied to the Killing field $u = J(\nabla\tau)$, where τ is a Killing potential on a Kähler manifold (M,g), (3.7) easily yields

$$2\imath_v\mathrm{r} \,=\, -\,d\Delta\tau\,, \qquad \text{where} \quad v \,=\, \nabla\tau\,. \tag{3.8}$$

In fact, $J_k^p u^k \,=\, -\,v^p$ (i.e., $Ju \,=\, -\,v$, as $u \,=\, Jv$), while $J_l^p u_p \,=\, \tau_{,l}$ since, for any vector w, $w^l J_l^p u_p = g(Jw,u) = -\,g(w,Ju) = g(v,w) = d_w\tau = w^l\tau_{,l}$. Thus, the relations $\nabla J = 0$, $u_{p,k} + u_{k,p} \,=\, 0$ along with (3.7) and (2.3) give $\tau_{,lk}{}^k \,=\, (J_l^p u_p)_{,k}{}^k \,=\, J_l^p u_{p,k}{}^k \,=\, -\,J_l^p u_{k,p}{}^k \,=\, -\,J_l^p \mathrm{r}_{pk}u^k \,=\, \rho_{kl}u^k \,=\, -\,\rho_{lk}u^k \,=\, \mathrm{r}_{lp}J_k^p u^k \,=\, -\,\mathrm{r}_{lp}v^p$. Thus, $\delta b \,=\, -\,\imath_v\mathrm{r}$, with v and b as in (1.14). Combined with (1.14), this yields (3.8).

Finally, applying to (3.8) the divergence operator δ and using (1.6), we obtain the formula ([26], [6], [7]): for any Killing potential τ in a Kähler manifold,

$$\Delta^2\tau \,+\, 2\langle \mathrm{r}\,,\, \nabla d\tau\rangle \,+\, g(\nabla\mathrm{s}, \nabla\tau) \,=\, 0\,. \tag{3.9}$$

§4. Conformally-Einstein Kähler metrics

For the Ricci tensors $\tilde{\mathrm{r}}$, r and scalar curvatures $\tilde{\mathrm{s}}$, s of two conformally related pseudo-Riemannian metrics g and $\tilde g \,=\, g/\tau^2$ in any real dimension n, we have, with ∇ and $\Delta = g^{jk}\nabla_j\nabla_k$ corresponding to g,

$$\tilde{\mathrm{r}} \,=\, \mathrm{r} \,+\, (n-2)\,\tau^{-1}\nabla d\tau \,+\, \big[\tau^{-1}\Delta\tau \,-\, (n-1)\tau^{-2}Q\big]\,g\,, \tag{4.1}$$

$$\tilde{\mathrm{s}} \,=\, \mathrm{s}\tau^2 \,+\, 2(n-1)\,\tau\Delta\tau \,-\, n(n-1)Q\,, \tag{4.2}$$

where

$$Q \,=\, g(\nabla\tau, \nabla\tau)\,. \tag{4.3}$$

(See, for instance, [15, (16.13)].) In view of (4.1), $\tilde g$ is Einstein if and only if

$$(n-2)\,\nabla d\tau \,+\, \tau\mathrm{r} \,=\, \Phi g\,, \qquad \text{where} \quad \Phi \,=\, (1-2/n)\Delta\tau \,+\, \mathrm{s}\tau/n\,. \tag{4.4}$$

Remark 4.1. Although relation (4.4) (as well as (4.6) below) are directly established only in the open set with $\tau \neq 0$, they are automatically valid *everywhere in* M. In fact, the left-hand sides are zero both in the set where $\tau \neq 0$ and in the interior of the zero level $\tau^{-1}(0)$, while the union of these two open sets is clearly dense in M. Consequently, equalities (4.5), (4.7), (4.8) below then hold everywhere in M as well.

We now proceed to study Kähler metrics which are locally conformally Einstein.

Lemma 4.2. *Let (M,g) be a pseudo-Riemannian Kähler manifold of any real dimension $n\ge 4$ with a nonconstant C^∞ function $\tau:M\to\mathbf{R}$, and let $\tilde g$ be the conformally related metric defined, wherever $\tau\ne 0$, by $\tilde g = g/\tau^2$. If $\tilde g$ is Einstein, then τ is a Killing potential on M.*

In fact, this is clear from Remark 3.1 as, by (4.4), the Hermitian symmetry of $\nabla d\tau$ follows from that of g and r (cf. (2.1) – (2.2)). (The final phrase 'on M' is added to emphasize that (3.2) with $v=\nabla\tau$ must hold everywhere in M even in the case where $\tau:M\to\mathbf{R}$ is allowed to vanish somewhere; see Remark 4.1.) ■

Suppose now that (M,g), n and τ are as in Lemma 4.2, and the metric $\tilde g = g/\tau^2$ is Einstein on $M\smallsetminus\tau^{-1}(0)$. Applying to both sides of (4.4) the operator $\imath_v$ for $v=\nabla\tau$ and using (1.12), (1.4) and (3.8), we then obtain, with $Q=g(\nabla\tau,\nabla\tau)$,

$$2\mathrm{s}\tau d\tau \,+\, 2(n-2)(\Delta\tau)\,d\tau \,+\, n\tau\,d\Delta\tau \,-\, n(n-2)\,dQ \,=\, 0\,. \tag{4.5}$$

On the other hand, since $\tilde g$ is Einstein, $\tilde{\mathrm{s}}$ in (4.2) must be constant, and taking the differential of (4.2) we get

$$2\mathrm{s}\tau d\tau \,+\, \tau^2 d\mathrm{s} \,+\, 2(n-1)[(\Delta\tau)\,d\tau \,+\, \tau\,d\Delta\tau] \,-\, n(n-1)\,dQ \,=\, 0\,. \tag{4.6}$$

Subtracting (4.5) from (4.6) or, respectively, adding $(1-n)$ times (4.5) to $(n-2)$ times (4.6), we now see that, whenever $\tilde g = g/\tau^2$ is Einstein, we have

$$\tau^2 d\mathrm{s} \,+\, 2(\Delta\tau)\,d\tau \,+\, (n-2)\tau\,d\Delta\tau \,-\, n\,dQ \,=\, 0\,,\quad \text{with } Q=g(\nabla\tau,\nabla\tau)\,, \tag{4.7}$$

$$(n-1)(n-4)\,d\Delta\tau \,+\, (n-2)\tau\,d\mathrm{s} \,-\, 2\mathrm{s}\,d\tau \,=\, 0\,. \tag{4.8}$$

Note that we canceled a τ factor to obtain (4.8), which is possible for reasons stated in Remark 4.1.

Proposition 4.3. *Suppose that $\tau:M\to\mathbf{R}$ is a nonconstant C^∞ function on a pseudo-Riemannian Kähler manifold (M,g), of any real dimension $n>4$, such that the conformally related metric $\tilde g = g/\tau^2$, defined wherever $\tau\ne 0$, is Einstein. Then, locally in the open set where $d\tau\ne 0$, each of the functions*

$$\mathrm{s}\,,\qquad g(\nabla\tau,\nabla\tau)\,,\qquad \Delta\tau$$

is a function of τ. The symbol s stands, as usual, for the scalar curvature of g.

Proof. Applying d to both sides of (4.8), we obtain $n\,d\tau\wedge d\mathrm{s}=0$, which proves our assertion for the function s. Similarly, taking the exterior derivative of of (4.7) and using our assertion for s, we see that $(n-4)\,d\tau\wedge d\Delta\tau=0$. Our assertion has thus been established also for the functions s and $\Delta\tau$. Now (4.2) (with a *constant* $\tilde{\mathrm{s}}$) yields our assertion for $Q=g(\nabla\tau,\nabla\tau)$ as well. This completes the proof. ■

Remark 4.4. The part of the above assertion concerning s clearly remains valid when $n=4$. More precisely, (4.8) with $n=4$ gives $\tau\,d\mathrm{s}\,-\,\mathrm{s}\,d\tau=0$, so that, in every connected component of the open set where $\mathrm{s}\ne 0$, τ is a constant multiple of s (see also [14]).

§5. One particular type of Killing potentials

By a *ρ-ω potential* on a Riemannian manifold (M,g) we mean any C^∞ function $\tau: M \to \mathbf{R}$ such that

$$\nabla d\tau \,+\, \chi \mathrm{r} \,=\, \sigma g \tag{5.1}$$

for some C^∞ functions $\sigma, \chi : M \to \mathbf{R}$. For instance, by (4.4), condition (5.1) holds for suitable σ and χ whenever the metric $\tilde{g} = g/\tau^2$ on $M \smallsetminus \tau^{-1}(0)$ is Einstein. (See also Example 5.2 below.)

Lemma 5.1. *Let a C^∞ function τ on a Kähler manifold (M,g) satisfy* (5.1) *with some C^∞ functions χ and σ. Then τ is a Killing potential, i.e., $u = J(\nabla\tau)$ is a Killing field on (M,g). Furthermore,*

$$d\xi \,=\, 2\,[\sigma\omega \,-\, \chi\rho]\,, \qquad \text{with} \quad \xi \,=\, \imath_v\omega \quad \text{and} \quad v \,=\, \nabla\tau\,. \tag{5.2}$$

In fact, τ is a Killing potential in view of Remark 3.1(a), (c) along with (2.2) for $b = g$ and $b = \mathrm{r}$. As for (5.2), it is nothing else than (3.6). (Note that $\xi = g(J(\nabla\tau),\,\cdot\,) \,=\, \omega(\nabla\tau,\,\cdot\,)$, cf. (2.1) and (1.3)).) ■

A C^∞ function $\tau : M \to \mathbf{R}$ on a Riemannian manifold (M,g) will be called a *special ρ-ω potential* if it satisfies condition (5.1) with some σ, χ of class C^∞ such that

$$d\sigma \wedge d\tau \,=\, d\chi \wedge d\tau \,=\, 0 \qquad \text{everywhere in } M\,, \tag{5.3}$$

$$d\chi \neq 0 \qquad \text{on a dense open subset of } M\,. \tag{5.4}$$

Condition (5.3) amounts to requiring that, locally, at points with $d\tau \neq 0$, both σ and χ be functions of τ. As for (5.4), it does not follow from (5.1) and (5.3). (See §21.)

Example 5.2. Let $\tau : M \to \mathbf{R}$ be a nonconstant C^∞ function on a Kähler manifold (M,g) of real dimension $n > 4$ such that the metric $\tilde{g} = g/\tau^2$, defined on the open set U where $\tau \neq 0$, is Einstein. Then τ is a special ρ-ω potential on (M,g). (This is immediate from (4.4), Proposition 4.3 and (3.3).)

Although it is Example 5.2 that makes special ρ-ω potentials interesting for us, it will nevertheless be convenient to discuss them in full generality (on Kähler manifolds). Such general examples will be presented in §12, and preceded by some simpler examples (see §9) on product manifolds. We start here by discussing a class 2-dimensional examples which will provide one of the factors for such products.

In dimension 2, equation (5.1) clearly amounts to

$$\nabla d\tau \,=\, [\Delta\tau/n]\,g \tag{5.5}$$

with $n = 2$. Assuming that the Riemannian surface in question is *oriented*, we may treat it as a Kähler manifold of complex dimension 1, with the obvious complex structure tensor J. The following well-known lemma states that, on oriented Riemannian surfaces, functions τ with (5.5) for $n = 2$ are nothing else than Killing potentials for Killing fields (as defined in §3).

Lemma 5.3. *Let (M,g) be an oriented Riemannian surface, and let us set $n=2$. For every C^∞ function $\tau : M \to \mathbf{R}$ satisfying* (5.5), *formula $u = J(\nabla\tau)$ defines a holomorphic Killing vector field on (M,g). Conversely, every Killing field u on (M,g) is holomorphic and, locally, there exists a C^∞ function τ, unique up to an additive constant, with $u = J(\nabla\tau)$ and* (5.5).

In fact, a function τ with (5.5) must be a Killing potential in view of Lemma 5.1, and so $u = J(\nabla\tau)$ is holomorphic and Killing by Remark 3.2. On the other hand, let u be a Killing field. Since the vector bundle morphism $\nabla u : TM \to TM$ (cf, (1.1)) is skew-adjoint, it must, for dimensional reasons, coincide with σJ for some function σ. In particular, ∇u commutes with J, i.e., u is holomorphic. Setting $v = -Ju$ we now obtain $\nabla_w v = -J(\nabla_w u) = -J\sigma J w = \sigma w$ for all vectors w, which, according to Remark 3.2 and (1.9), is nothing else than (5.5) for τ chosen to be a (local) Killing potential for u. ∎

Remark 5.4. Equation (5.1) becomes (5.5) when the Riemannian manifold in question is Einstein or 2-dimensional. This can be again rewritten as (5.1) with a completely arbitrary choice of χ (and a suitable σ). Hence, in this case, whether (5.3) or (5.4) (or both) are satisfied may depend the choice of χ. Choosing $\chi = 0$ we then automatically have (5.3) (but not (5.4)). More precisely, $d\psi \wedge d\tau = 0$ for $\psi = \Delta\tau/n$ whenever a C^∞ function τ satisfies (5.5) on a Riemannian manifold of any dimension n. In fact, applying the operator $\imath_v$ for $v = \nabla\tau$ to the equality $\nabla d\tau = \psi g$ and using (1.12) and (1.4), we obtain $dQ = 2\psi\, d\tau$ and so $0 = ddQ/2 = d\psi \wedge d\tau$, as required.

Riemannian (and pseudo-Riemannian) manifolds admitting (nonconstant) C^∞ functions τ with (5.5) have been studied extensively; see, e.g., [24] and the references therein. Note that (5.5) means that $v = \nabla\tau$ is both a gradient and a conformal vector field. A special case of (5.5) is also briefly discussed in §21.

§6. Some ordinary differential equations

Let $m \ge 2$ be a fixed integer. In the space $\mathbf{R}^7$ with the Cartesian coordinates $(Q, Y, \phi, \psi, \lambda, \mu, \mathrm{s})$, we denote Z the function of $(Q, Y, \phi, \psi, \lambda, \mu, \mathrm{s})$ given by

$$Z = \frac{2(m-1)(\psi-\phi)}{\lambda-\mu} \qquad \text{wherever} \quad \lambda \ne \mu\,. \tag{6.1}$$

We will study the system

$$\begin{aligned} dQ &= 2\psi\, d\tau\,, \qquad dY = -2\mu\, d\tau\,, \qquad Q\, d\phi = 2(\psi-\phi)\phi\, d\tau\,, \\ Q\, d\psi &= [2(m-1)(\phi-\psi)\phi - \mu Q]\, d\tau\,, \qquad Q\, d\lambda = 2(\mu-\lambda)\phi\, d\tau\,, \end{aligned} \tag{6.2}$$

$$\begin{aligned} QZ\, d\mu &= \big[2(m-1)\mu Q + (\lambda-\mu)Q + 4(m-1)^2(\psi-\phi)\phi\big]\, d\tau\,, \\ Z\, d\mathrm{s} &= 2\,[(2m-3)\mu + \lambda]\, d\tau\,, \end{aligned} \tag{6.3}$$

of seven ordinary differential equations, imposed on the septuple $(Q, Y, \phi, \psi, \lambda, \mu, \mathrm{s})$, the constituents of which now are treated as unknown C^1 functions of the real variable τ, with

$$(\lambda-\mu)(\phi-\psi)Q \ne 0 \quad \text{everywhere} \tag{6.4}$$

in the open interval of the variable τ on which such a septuple $(Q, Y, \phi, \psi, \lambda, \mu, \mathrm{s})$ is defined. In other words, we are studying the vector field corresponding to (6.2) – (6.3) in the open subset of $\mathbf{R}^7$ defined by the inequality (6.4).

We will often assume that, besides (6.2) – (6.3), the septuple $(Q, Y, \phi, \psi, \lambda, \mu, \mathrm{s})$ also satisfies the following constraints (see (6.9) and Corollary 7.3 below):

$$Y = 2\psi + 2(m-1)\phi, \quad \mathrm{s} = 2\mu + 2(m-1)\lambda, \quad Z = \tau. \tag{6.5}$$

The system (6.2) – (6.5) arises in our classification (in §4 and §7) of conformally-Einstein Kähler manifolds in complex dimensions $m > 2$. If such a Kähler metric happens to be a Riemannian product (see §15), we obtain a "special" solution to (6.2) – (6.5), namely, one in which λ is constant and $\phi = 0$ identically. These special solutions are easy to classify (see §14); it is the remaining case, in which

$$\phi \neq 0 \quad \text{everywhere} \tag{6.6}$$

in the open interval, that will require further discussion. Among solutions to (6.2) – (6.5) which satisfy (6.6), we will separately deal with those having

$$p \neq 0, \tag{6.7}$$

where p is the constant appearing in (6.11) below, as they are quite different from solutions to (6.2) – (6.6) with $p = 0$. (The latter are classified in §17.)

Lemma 6.1. *Let Z be as in* (6.1)*. Each of the following six functions of the octuple $(Q, Y, \phi, \psi, \lambda, \mu, \mathrm{s}, \tau)$, defined wherever the inequality* (6.4) *holds, namely: p and κ, given by*

$$2p = 2Z - Q/\phi, \qquad \kappa = (\phi Y + \lambda Q)/|\phi|, \tag{6.8}$$

the functions

$$2\psi + 2(m-1)\phi - Y, \quad 2\mu + 2(m-1)\lambda - \mathrm{s}, \quad Z - \tau, \tag{6.9}$$

and the function γ, with

$$m\gamma = [\mu + (m-1)\lambda]Z^2 + 2(2m-1)[\psi + (m-1)\phi]Z - m(2m-1)Q, \tag{6.10}$$

then is an integral of the system (6.2) – (6.3)*, i.e., is constant along any solution to* (6.2) – (6.3)*.*

Furthermore, the three functions of τ given by

$$2\psi + 2(m-1)\phi - Y, \qquad 2p = 2\tau - Q/\phi, \qquad \kappa = (\phi Y + \lambda Q)/|\phi| \tag{6.11}$$

are constant whenever $(Q, Y, \phi, \psi, \lambda, \mu, \mathrm{s})$ satisfy (6.2) *alone.*

Proof. From (6.2) we have $d[Q/\phi]/d\tau = 2$ and

$$d[2\psi + 2(m-1)\phi]/d\tau = -2\mu, \tag{6.12}$$

which clearly implies the clause about (6.11). Similarly, $Z-\tau$ and $2\mu+2(m-1)\lambda-\mathrm{s}$ are constant (cf. the last equation in (6.3)) since, by by (6.1) – (6.3),

$$\frac{dZ}{d\tau}=1 \qquad \text{and} \qquad \frac{d}{d\tau}\,[2\mu+2(m-1)\lambda]\,=\,\frac{2}{Z}\,[(2m-3)\mu\,+\,\lambda]\,. \tag{6.13}$$

Also, evaluating $m\,d\gamma/d\tau$ (for $m\gamma$ given by (6.10)) from (6.13), (6.12) and the equation $dQ/d\tau=2\psi$ in (6.2), and then replacing Z with the right-hand side of (6.1), we now find that $d\gamma/d\tau=0$. Finally, constancy of the functions (6.8) and $2\psi+2(m-1)\phi-Y$ is clear from that of (6.11). This completes the proof. ■

The solutions $(Q,Y,\phi,\psi,\lambda,\mu,\mathrm{s})$ to equations (6.2) – (6.3) that are of particular geometric interest are those which, in addition, satisfy (6.5), i.e., for which the integrals (6.9) are all zero. (See Corollary 7.3 in §7.) A solution $(Q,Y,\phi,\psi,\lambda,\mu,\mathrm{s})$ to (6.2) – (6.5) must also satisfy the relations

$$2(\lambda-\mu)\,=\,2m\lambda\,-\,\mathrm{s}\,, \tag{6.14}$$

$$\tau\,d\mathrm{s}/d\tau\,=\,(2m-3)\mathrm{s}\,-\,(2m-1)(2m-4)\lambda\,. \tag{6.15}$$

In fact, the second equality in (6.5) gives (6.14) and $2\mu=\mathrm{s}+(2-2m)\lambda$, so that $2(2m-3)\mu=(2m-3)[\mathrm{s}+(2-2m)\lambda]$. Consequently, the last equation in (6.3) with $Z=\tau$ (cf. (6.5)) is nothing else than (6.15).

Suppose that $(Q,Y,\phi,\psi,\lambda,\mu,\mathrm{s})$ form a C^1 solution to equations (6.2) – (6.5), defined on some interval of the real variable τ. Denoting p the first constant in (6.8), we then have

$$Q\,=\,2(\tau-p)\phi \tag{6.16}$$

(by (6.8) with $Z=\tau$) which, for reasons of continuity, holds in the whole interval.

§7. More on ρ-ω potentials

We now proceed to prove several lemmas which later, in §13, will enable us to classify special ρ-ω potentials (locally, at points "in general position").

Lemma 7.1. *Let a nonconstant C^∞ function $\tau:M\to\mathbf{R}$ on a Kähler manifold (M,g) of any complex dimension $m\geq 2$ satisfy (5.1) and (5.3), (5.4) with some C^∞ functions χ and σ. If v and u are the vector fields on M given by $v=\nabla\tau$ and $u=Jv$, and $M'\subset M$ denotes the open set where $d\tau\neq 0$, then $\mathcal{V}=\mathrm{Span}\,(v,u)$ is a 2-dimensional distribution on M', and*

$$TM'\,=\,\mathcal{V}\oplus\mathcal{H}\,, \qquad \text{with} \quad \mathcal{H}\,=\,\mathcal{V}^\perp\,, \tag{7.1}$$

while, setting $Q=g(\nabla\tau,\nabla\tau)$ we have

$$g(v,v)\,=\,g(u,u)\,=\,Q\,, \qquad g(v,u)\,=\,0\,. \tag{7.2}$$

Moreover,

(i) *The vector fields v,u on M are holomorphic and $[v,u]=0$.*
(ii) *The distribution $\mathcal{V}$ on M' is integrable and its leaves are totally geodesic.*

(iii) *The distribution* $\mathcal{H}$ *is invariant under the flows of both* v *and* u.

(iv) *The distributions* $\mathcal{V}$ *and* $\mathcal{H}$ *are both* J*-invariant, as well as* g*-orthogonal, and*

$$\mathrm{r}(\mathcal{V},\mathcal{H}) = (\nabla d\tau)(\mathcal{V},\mathcal{H}) = \{0\}\,, \tag{7.3}$$

that is, $\mathcal{V}$ *and* $\mathcal{H}$ *are both* r*-orthogonal and* $\nabla d\tau$*-orthogonal to each other. Furthermore,*

$$\begin{aligned} \mathrm{r} &= \mu g \quad \text{on} \quad \mathcal{V}\,, \qquad & \mathrm{r} &= \lambda g \quad \text{on} \quad \mathcal{H}\,,\\ \nabla d\tau &= \psi g \quad \text{on} \quad \mathcal{V}\,, \qquad & \nabla d\tau &= \phi g \quad \text{on} \quad \mathcal{H}\,, \end{aligned} \tag{7.4}$$

with the "eigenvalue functions" λ, μ, ϕ, ψ *characterized by*

$$\begin{aligned} \lambda &= d\sigma/d\chi\,, \qquad & 2\mu &= \mathrm{s} + (2-2m)\lambda\,,\\ \phi &= \sigma - \chi\lambda\,, \qquad & 2\psi &= 2\sigma - \mathrm{s}\,\chi + 2(m-1)\chi\lambda\,, \end{aligned} \tag{7.5}$$

in the open dense set where $d\chi \neq 0$, *and, at points with* $\lambda \neq \mu$,

$$\chi = \frac{\psi-\phi}{\lambda-\mu}\,, \qquad \sigma = \frac{\psi\lambda-\phi\mu}{\lambda-\mu}\,. \tag{7.6}$$

(iv) *For* λ, μ, ϕ, ψ *as in* (iv) *and* $v = \nabla\tau$, $u = Jv$,

$$\Delta\tau = 2\psi + 2(m-1)\phi\,, \qquad \mathrm{s} = 2\mu + 2(m-1)\lambda\,, \tag{7.7}$$

$$\nabla_v v = -\nabla_u u = \psi v\,, \quad \nabla_v u = \nabla_u v = \psi u\,. \tag{7.8}$$

Proof. Assertion (i) is immediate from Lemma 5.1, Remark 3.1(b), (3.2) (which implies $[J,\nabla u] = 0$ as $\nabla u = J\circ(\nabla v)$), and (3.1).

Furthermore, at points with $d\tau \neq 0$, we have, for some function G,

$$(d\chi/d\tau)\,\rho - (d\sigma/d\tau)\,\omega = G\,\xi\wedge d\tau\,, \quad \text{with} \quad \xi = \imath_v\omega\,. \tag{7.9}$$

In fact, applying d to (5.2) we obtain $0 = dd\xi/2 = \beta'\wedge d\tau$, where β' denotes the left-hand side of (7.9). Thus, wherever $d\tau\neq 0$, $\beta' = \zeta\wedge d\tau$ for some 1-form ζ. In the open set where $d\tau \neq 0$, the nullspace distribution $\mathcal{H} = \operatorname{Ker}\beta'$ of β', formed by all tangent vectors w with $\imath_w\beta' = 0$, is (by (1.3) and (1.16)) nothing else than the intersection of $\operatorname{Ker}\zeta$ and $\operatorname{Ker} d\tau$. In other words, $\mathcal{H} = w^\perp\cap v^\perp$, where $v = \nabla\tau$ and w is the vector field with $\zeta = \imath_w g$. On the other hand, $b' = (d\chi/d\tau)\,\mathrm{r} - (d\sigma/d\tau)g$ is a Hermitian tensor, related to β' as in (3.5) (cf. (2.1), (2.2)), so that $\operatorname{Ker}\beta' = \operatorname{Ker} b'$ and $\operatorname{Ker} b'$ is J-invariant. Hence $[\operatorname{Ker} b']^\perp = \operatorname{Span}(v,w)$ is J-invariant as well. Therefore, at every point with $d\tau\neq 0$, our w is a combination of v and $u = Jv$, and so $\zeta = \imath_w g$ is a combination of $d\tau = \imath_v g$ and $\xi = \imath_u g$ (cf. (1.4), (3.4)). Thus, $\zeta\wedge d\tau = G\,\xi\wedge d\tau$ for some G, which proves (7.9).

Since $\mathcal{H} = u^\perp\cap v^\perp$, we obviously have $\mathcal{H}^\perp = \mathcal{V}$ with $\mathcal{V} = \operatorname{Span}(v,u)$. Furthermore, as $\mathcal{H} = \operatorname{Ker}\xi\cap\operatorname{Ker} d\tau$, the left-hand side of of (7.9) annihilates $\mathcal{H}$, i.e., nonzero vectors in $\mathcal{H}$ are, at every point with $d\tau\neq 0\neq d\chi$, eigenvectors of r for the eigenvalue $(d\sigma/d\tau)/(d\chi/d\tau) = d\sigma/d\chi$. Now (5.1) (along with the fact that $\mathcal{V}$ is J-invariant) yields assertion (iv). Also, (7.7) follows from (7.4) and the relations $\mathrm{Trace}\,\mathrm{r} = \mathrm{s}$, $\Delta\tau = \mathrm{Trace}\,\nabla d\tau$. On the other hand, we easily obtain (7.8) using the fact that $\nabla_v v = \psi v$ by (7.4), while the other three relations in (7.8) follow from the first one since J is parallel and we have $[v,u] = 0$ and (1.2). This in turn yields (ii).

Let w now be any vector field which commutes with v and u, so that $\nabla_v w = \nabla_w v$, $\nabla_u w = \nabla_w u$. (Such fields w exist, locally, and realize all vectors at any given point x, as one verifies by prescribing w along a submanifold N with $x \in N$ and $T_x N = \mathcal{H}_x$, and then spreading it over a neighborhood of x using the flows of v and u, cf. (i).) Therefore, setting $\vartheta = g(v,w)$ and $\vartheta' = g(u,w)$ and using the Leibniz rule along with (7.8), (1.9), (7.4) and the fact that ∇v is self-adjoint (as $v = \nabla\tau$), while ∇u is skew-adjoint (as u is Killing), we now obtain $d_v\vartheta = 2\psi\vartheta$, $d_v\vartheta' = 2\psi\vartheta'$, $d_u\vartheta = d_u\vartheta' = 0$. (For instance, $d_v\vartheta = 2\psi\vartheta$ since $d_v\vartheta = g(\nabla_v v, w) + g(v, \nabla_v w)$, while $g(v,\nabla_v w) = g(v, \nabla_w v) = g(\nabla_v v, w)$.) If $\vartheta = \vartheta' = 0$ at one point of a leaf of $\mathcal{V}$, it follows now that $\vartheta = \vartheta' = 0$ along the whole leaf. This implies (iii), and completes the proof. ∎

Lemma 7.2. *Let a nonconstant C^∞ function $\tau : M \to \mathbf{R}$ be a special ρ-ω potential on a Kähler manifold (M,g) of some complex dimension $m \geq 2$, i.e., satisfy conditions* (5.1), (5.3), (5.4) *with suitable C^∞ functions χ and σ. If*

$$Q\,,\ Y\,,\ \phi\,,\ \psi\,,\ \lambda\,,\ \mu\,,\ \mathrm{s} \tag{7.10}$$

is the septuple of functions defined wherever $d\tau \neq 0$ and consisting of $Q = g(\nabla\tau,\nabla\tau)$, $Y = \Delta\tau$, the eigenvalue functions ϕ,ψ,λ,μ characterized by (7.4), *and the scalar curvature s of g, then*

(a) *In a neighborhood of any point with $d\tau \neq 0$, each of the seven functions* (7.10) *is a function of τ.*
(b) *The septuple* (7.10) *satisfies equations* (6.2) *at points with $d\tau \neq 0$.*

Proof. Let us refer to the five equations forming (6.2) as (6.2)i) – (6.2)v) (cf. §1). We will successively prove them in the open subset M' of M on which $d\tau \neq 0$.

The first relation (6.2)i) is, in our case, immediate from (1.12) and (1.3), along with (7.4) and (1.4). On the other hand, using (3.8) we see that $d\Delta\tau = -2\imath_v\mathrm{r}$ which, by (7.4), implies (6.2)ii) with $Y = \Delta\tau$. Thus, by (7.7),

$$2\mu\, d\tau \;=\; (2-2m)\, d\phi \;-\; 2\, d\psi\,. \tag{7.11}$$

Also, from (1.14) and (3.8), $\delta(\nabla d\tau) = d\Delta\tau/2$ and so, by (6.2)ii),

$$\delta(\nabla d\tau) \;=\; -\,\mu\, d\tau\,. \tag{7.12}$$

Hence, applying the divergence operator δ to (5.1) and using (1.10), (1.6) and (1.4), we see that $-\mu\, d\tau + (d\chi/d\tau)\imath_v\mathrm{r} + \chi\, d\mathrm{s}/2 = d\sigma$, where $v = \nabla\tau$. As $\imath_v\mathrm{r} = \mu\, d\tau$ by (7.4) and (1.4), this gives $\chi\, d\mathrm{s} = 2[(d\sigma/d\tau) + \mu - (d\chi/d\tau)\mu]\, d\tau$. Consequently, $\chi\, d\mathrm{s}\wedge d\tau = 0$. However, $\chi \neq 0$ on a dense subset of M'. (In fact, any open set on which $\chi = 0$ must be empty by (5.4).) Thus, $d\mathrm{s}\wedge d\tau = 0$, i.e., $d\mathrm{s}$ equals a function times $d\tau$, which proves that s is, locally, a function of τ. The same is true for Q and $Y = \Delta\tau$ (since we already proved (6.2)i), (6.2)ii)), as well as for χ

and σ (since we assume (5.3)) and hence, by (7.5), also for λ, μ, ϕ and ψ. This proves (a).

Furthermore, applying the divergence operator δ to the equality $\nabla_v v = \psi v$ (see (7.8)), we obtain $\imath_v(\delta b) + \langle b, b\rangle = d_v\psi + \psi\Delta\tau$, where $v = \nabla\tau$ and $b = \nabla d\tau$. (In fact, the two left-hand sides coincide due to the obvious local-coordinate equality $[\tau^{,k}\tau_{,jk}]^{,j} = \tau^{,k}\tau_{,jk}{}^{j} + \tau_{,jk}\tau^{,jk}$, while the right-hand sides do so in view of (1.10).) However, by (7.12) and (1.11), $\imath_v(\delta b) = -\,\mu\, d_v\tau = -\,\mu Q$. Thus, in view of Lemma 7.1(iv), the preceding relation can be rewritten as $-\mu Q + 2\psi^2 + 2(m-1)\phi^2 = d_v\psi + [2\psi + 2(m-1)\phi]\psi$, i.e., $d_v\psi = 2(m-1)(\phi-\psi)\phi - \mu Q$. Both sides of (6.2)iv) thus yield the same value under $\imath_v$ and, since they both are functional multiples of $d\tau$ (by (a)), while $d_v\tau \neq 0$ by (1.11), they must be equal. We thus have shown (6.2)iv). Replacing $d\psi$ in (7.11) with the expression resulting from (6.2)iv), we now also obtain (6.2)iii).

To establish (6.2)v), we will separately prove it in two open sets whose union is dense in M', namely: the interior U of the set given by $\phi = 0$, and the open subset on which $\phi \neq 0$. To prove (6.2)v) in U, it suffices to show it holds everywhere in the open set $U' \subset U$ characterized by the additional requirement that $\chi\, d\chi \neq 0$. (That U is dense in U is clear from (5.4), cf. the argument following (7.12).) By (7.5) we have, in U', $\sigma = \chi\lambda$ and $d\sigma = \lambda\, d\chi$, so that $\chi\, d\lambda = 0$, i.e., $d\lambda = 0$. This, with $\phi = 0$, yields (6.2)v) in U', and hence in U.

Finally, to prove (6.2)v) at points where $\phi \neq 0$, note that combining (6.2)iii) with (6.2)i) we obtain $d(Q/\phi) = 2\, d\tau$. On the other hand, we have, by (7.4), $\langle \mathrm{r}, \nabla d\tau\rangle = 2\psi\mu + 2(m-1)\phi\lambda$, while, as a consequence of relations (6.2)ii), (1.10) and (7.7), $-\Delta^2\tau = -\,\delta\,(d\Delta\tau) = 2\,\delta\,(\mu\, d\tau) = 2\, d_v\mu + 2\mu\Delta\tau = 2\, d_v\mu + 4\psi\mu + 4(m-1)\phi\mu$, with $v = \nabla\tau$. This, along with (7.4), allows os to rewrite (3.9) as $-\,[2\, d_v\mu + 4\psi\mu + 4(m-1)\phi\mu] + 2[2\psi\mu + 2(m-1)\phi\lambda] + d_v\mathrm{s} = 0$. In other words, as $\mathrm{s} = 2\mu + 2(m-1)\lambda$ (by (7.7)), $2(m-1)\, d_v\lambda = d_v(\mathrm{s} - 2\mu) = 4(m-1)(\mu-\lambda)\phi$. This in turn implies (6.2)v) since, as we just saw, both sides of (6.2)v) give the same value under $\imath_v$ (by (1.11)) and, by (a), they both are functional multiples of $d\tau$ (cf. the argument above that led to (6.2)iv)). This completes the proof. ■

Corollary 7.3. *Let $\tau : M \to \mathbf{R}$ be a nonconstant C^∞ function on a Kähler manifold (M, g) of complex dimension $m > 2$ such that the metric $\tilde g = g/\tau^2$, defined wherever $\tau \neq 0$, is Einstein. Then, in the open set of all points at which $\tau\, d\tau \neq 0$ and the Ricci tensor is not a multiple of g, the functions $Q, Y, \phi, \psi, \lambda, \mu, \mathrm{s}$ with $Q = g(\nabla\tau, \nabla\tau)$, $Y = \Delta\tau$, s denoting the scalar curvature of g, and ϕ, ψ, λ, μ as in* (7.4)*, satisfy conditions* (6.2) – (6.5).

Proof. Since τ is a special ρ-ω potential on (M, g) (Example 5.2), Lemma 7.2(b) and (7.7) give (6.2) and the first two equalities in (6.5). Furthermore, wherever $(\lambda - \mu)\, d\tau \neq 0$, we have $Z = \tau$, with Z given by (6.1). In fact, (4.4) is a special case of (5.1), with $\chi = \tau/(2m-2)$, while χ is simultaneously given by (7.6) (see Lemma 7.1(iv)). This yields (6.5).

Applying d to both sides of the equality $Z = \tau$ (with Z given by (6.1)), and using the expressions for $d\phi$, $d\psi$ and $d\lambda$ provided by (6.2), we now easily obtain the first relation in (6.3). The second one is in turn immediate from the first along with the formula for $d\lambda$ in (6.2) and the second equality in (6.5).

Finally, we have (6.4) at the points in question since $Q \neq 0$ (as $d\tau \neq 0$), $\lambda \neq \mu$ (as r is not a multiple of g) and $\phi \neq \psi$ (for otherwise $\nabla d\tau$ would be a multiple of g at the given point, which, by (4.4) with $\tau \neq 0$, would imply the same for r).

This completes the proof. ∎

§8. Reducibility and ρ-ω potentials

Recall that special ρ-ω potentials are those functions τ which satisfy (5.1) and (5.3), (5.4) for some χ and σ.

Lemma 8.1. *Let a nonconstant C^∞ function $\tau : M \to \mathbf{R}$ be a special ρ-ω potential on a Kähler manifold (M, g) of any complex dimension $m \geq 2$, and let $(Q, Y, \phi, \psi, \lambda, \mu, \mathrm{s})$ be the septuple appearing in (7.10). Then*

(i) *Locally, at points with $\phi\, d\tau \neq 0$, there exist constants p and κ with*

$$\frac{Q}{\phi} = 2(\tau - p) \qquad \text{and} \qquad \frac{\lambda Q + \phi\Delta\tau}{|\phi|} = \kappa\,. \tag{8.1}$$

(ii) *For any two C^1 vector fields w, w', defined in the open set M' where $d\tau \neq 0$ and orthogonal to $v = \nabla\tau$ and $u = Jv$,*

a) *If, in addition, w, w' commute with both v and u, then*

$$\nabla_v w = \nabla_w v = \phi w\,, \quad \nabla_u w = \nabla_w u = \phi J w\,, \tag{8.2}$$

$$d_v \varPsi = d_u \varPsi = 0 \qquad \text{for} \quad \varPsi = \frac{\phi}{Q}\, g(w, w')\,. \tag{8.3}$$

b) *The $\mathcal{V}$ components of $\nabla_w w'$ and the Lie bracket $[w, w']$ relative to the decomposition (7.1) with $\mathcal{V} = \mathrm{Span}\,(v, u)$ and $\mathcal{H} = \mathcal{V}^\perp$ are*

$$[\nabla_w w']^{\mathrm{vrt}} = -\frac{\phi}{Q}\,[g(w, w')v + \omega(w, w')u]\,, \tag{8.4}$$

$$[w, w']^{\mathrm{vrt}} = -\frac{2\phi}{Q}\,\omega(w, w')\,u\,. \tag{8.5}$$

Proof. (i) is immediate from Lemma 7.2(b) and the final clause of Lemma 6.1. Let w, w' now satisfy the assumptions of (ii). Thus, the expression $g(\nabla_w v, w') = (\nabla d\tau)(w, w') = \phi g(w, w')$ (cf. (1.9) and (7.4)) is symmetric in w, w'. Since $g(\nabla_w v, w') = g(\nabla_v w, w')$ whenever w, w' commute with v, this gives $d_v \varUpsilon = 2\phi\varUpsilon$ for the function $\varUpsilon = g(w, w')$. Similarly, $g(\nabla_w u, w') = g(\nabla_w (Jv), w') = g(J(\nabla_w v), w') = -\,g(\nabla_w v, Jw') = -\,(\nabla d\tau)(w, Jw') = -\,\phi g(w, Jw') = \phi\,\omega(w, w')$ (cf. (1.9), (7.4), (2.1)) is skew-symmetric in w, w'. As $g(\nabla_w u, w') = g(\nabla_u w, w')$ whenever w, w' commute with u, its skew-symmetry implies $d_u \varUpsilon = 0$ for $\varUpsilon = g(w, w')$. On the other hand, by (8.1), $d(\phi/Q) = -(\phi/Q)^2 d(Q/\phi) = -2(\phi/Q)^2 d\tau$ and, as $d_v\tau = Q$ (see (1.11)) and $d_u\tau = g(u, v) = 0$, we have $d_v(\phi/Q) = -2\phi^2/Q$ and $d_u(\phi/Q) = 0$. Since $\varPsi$ in (8.3) equals $(\phi/Q)\varUpsilon$, the above formulae for $d_v\varUpsilon$ and $d_u\varUpsilon$ now yield (8.3).

Moreover, $g(\nabla_w v, w') = -\,g(v, \nabla_w w')$ (and similarly for u instead of v). This proves (8.4); in fact, the above formulae for $g(\nabla_w v, w')$ and $g(\nabla_w u, w')$ now show that both sides of (8.4) have the same inner product with u and v (cf. (7.2)), while $\mathcal{V} = \mathrm{Span}\,(v, u)$. Now (8.5) is immediate from (8.4) and (1.2).

Finally, for w, w' as in (ii)a), the covariant derivatives listed in (8.2) are all orthogonal to v and u (by (7.8) and the Leibniz rule), while their inner products with any horizontal vector field w' are given by the relations $g(\nabla_w v, w') = \phi g(w, w')$ and $g(\nabla_w u, w') = \phi\,\omega(w, w')$ established above. Now (8.2) follows, as both sides of each of the four equalities in (8.2) yield the same inner products with v, u and any such w'. This completes the proof. ∎

Corollary 8.2. *Let a nonconstant C^∞ function $\tau : M \to \mathbf{R}$ on a Kähler manifold (M, g) of some complex dimension $m \ge 2$ satisfy conditions* (5.1), (5.3), (5.4) *with some C^∞ functions χ and σ, and let $x \in M$ be a point at which $d\tau \ne 0$ and the Ricci tensor $\mathrm{r}(x)$ is not a multiple of $g(x)$. Then the following two conditions are equivalent:*

(a) *$\phi = 0$ identically in a neighborhood of x, where ϕ is the function appearing in* (7.4).

(b) *g restricted to a neighborhood of x is decomposable, as a Kähler metric, into a Riemannian product of two Kähler metrics in lower dimensions.*

Moreover, if a decomposition as in (b), *exists, then there exists one for which $\mathcal{V}$ and $\mathcal{H} = \mathcal{V}^\perp$, defined in* Lemma 7.1, *are the factor distributions.*

Proof. Assuming (a), let us choose a connected neighborhood U of x with $\phi = 0$ and $d\tau \ne 0$ everywhere on U. The distribution $\mathcal{V} = \mathrm{Span}\,(v, u)$ is parallel on U since, by (7.8) and (8.2), the covariant derivatives of v and u in all directions lie in $\mathcal{V}$. Hence, making U smaller if necessary, we conclude that $\mathcal{V}$ and $\mathcal{H} = \mathcal{V}^\perp$ are the factor distributions for a Riemannian-product decomposition of g on U which (as $\mathcal{V}$ and $\mathcal{H}$ are J-invariant, cf. Lemma 7.1(iv)) is also a product decomposition of the *Kähler* manifold (U, g). Thus, (a) implies (b).

Conversely, let us assume (b). Since, near x, $\lambda \ne \mu$ (notation of (7.4)), for any Riemannian-Kähler product decomposition of a neighborhood of x the distribution $\mathcal{V}$ defined in Lemma 7.1 is contained in the tangent bundle of one of the factors. Applying (8.4) to a nonzero vector $w = w'$ tangent to the other factor, we now obtain $\phi = 0$. This completes the proof. ■

§9. Riemannian-product examples

Lemma 9.1. *Suppose that $\tau : M \to \mathbf{R}$ is a C^∞ function on a Riemannian manifold (M, g) and $\mathcal{V}$ is a distribution on M such that, setting $\mathcal{H} = \mathcal{V}^\perp$, we have* (7.3) *and* (7.4) *for some functions λ, μ, ϕ, ψ on M. Furthermore, let $U \subset M$ be the open set of all points at which $\lambda \ne \mu$, i.e., r is not a multiple of g. Then equation* (5.1) *holds everywhere in U with σ and χ given by* (7.6).

In fact, (7.6) gives $\psi + \chi\mu = \sigma$ and $\phi + \chi\lambda = \sigma$, so that (7.4) yields (5.1). ■

Example 9.2. We will now construct C^∞ functions $\tau : M \to \mathbf{R}$ on suitable Riemannian-product manifolds (M, g) of any real dimension $n \ge 4$ which are special ρ-ω potentials, i.e, satisfy (5.1) and (5.3), (5.4) for some χ and σ while, in addition, $d\tau \ne 0$ and $\mathrm{r} - \mathrm{s}g/n \ne 0$ everywhere in M, and $\phi = 0$ identically, ϕ being the function appearing in (7.4). (Cf. also Corollary 8.2.)

To this end, we choose oriented Riemannian manifolds (N, h) and (P, η) with $\dim P = 2$, and let $M = N \times P$ be endowed with the metric g which is the product of h and η. In addition, we assume that (P, η) admits a function τ satisfying (5.5) (with $n = 2$, and η instead of g) and having $d\tau \ne 0$ everywhere; according to Lemma 5.3, this amounts, locally, to assuming that (P, η) admits a nontrivial Killing field u (namely, $u = J(\nabla\tau)$) and then replacing P with an open subset in which u has no zeros. Let us treat τ as a function $M \to \mathbf{R}$ constant in the direction of the N factor. A further assumption we make is that (N, h) is Einstein, so that its Ricci tensor equals $\mathrm{r}^{(h)} = \lambda h$ for a constant λ, and that the Gaussian curvature μ of (P, η) has $\mu \ne \lambda$ everywhere in P. Let $\mathcal{V}$ and $\mathcal{H}$ now denote the P and N factor distributions on M. Conditions (7.3) and (7.4) are obviously satisfied by λ, μ just described, $\phi = 0$, and $\psi = \Delta\tau/2$ (cf. (5.5)). By Lemma 9.1, we thus have (5.1) everywhere in (M, g), with σ and χ defined by (7.6). Thus, we have (5.3) as well, since ϕ and λ are constant, while ψ and μ are, locally, functions of τ in view of Remark 5.4 and the fact that the Gaussian curvature μ is invariant under the flow of the Kiling field $u = J(\nabla\tau)$ in (P, η).

As for condition (5.4), it will be satisfied once we replace P with an open subset on which $(\lambda - \mu)\, d\psi + \psi\, d\mu \neq 0$. (This is clear from (7.6) with $\phi = 0$ and $d\lambda = 0$.)

If, in addition, (N, h) is a *Kähler*-Einstein manifold, our (M, g) is a product Kähler manifold.

§10. Submersions and horizontal vectors

For any submersion $\pi : M \to N$ we will denote $\mathcal{V} = \operatorname{Ker} d\pi$ its vertical distribution.

Throughout this section we assume that $\pi : M \to N$ is a C^∞ submersion with $\pi(M) = N$, while M is endowed with a fixed Riemannian metric g. This gives rise to the horizontal distribution $\mathcal{H} = \mathcal{V}^\perp$, with $TM = \mathcal{V} \oplus \mathcal{H}$. The symbols w^{vrt} and w^{hrz} will stand for the $\mathcal{V}$ and $\mathcal{H}$ components of any vector or vector field w tangent to M.

Furthermore, g and its Levi-Civita connection ∇ naturally induce a connections ∇^{vrt} and ∇^{hrz} in $\mathcal{V}$ and $\mathcal{H}$ (treated as vector bundles over M), with

$$\nabla^{\mathrm{vrt}}_w v = [\nabla_w v]^{\mathrm{vrt}}, \qquad \nabla^{\mathrm{hrz}}_w v = [\nabla_w v]^{\mathrm{hrz}} \tag{10.1}$$

for vector fields w and C^1 vertical/horizontal vector fields v on M. On the other hand, since $\pi(M) = N$, we can now identify C^∞ vector fields on N with those C^∞ vector fields w on M which are both horizontal and projectable under π. The identification in question is achieved by the horizontal lift operation.

Remark 10.1. In the situation just described, we will always treat vector fields on N as nothing else than projectable horizontal vector fields w on M. Most of the time this will not lead to ambiguities in the notation; one important exception is, however, the Lie bracket. Although the Lie-bracket operation preserves projectability (in fact, in a "homomorphic" way, see [19]), it usually does not preserve horizontality. However, all vertical vector fields are projectable (onto 0). These facts, combined, show that, for any projectable horizontal C^∞ vector fields w, w' on M, if $[w, w']$ denotes their Lie bracket in M, then, under our identification,

$$[w, w']^{\mathrm{hrz}} \quad \text{is the Lie bracket of } w \text{ and } w' \text{ in } N\,. \tag{10.2}$$

Remark 10.2. Let (M, g) be a Riemannian manifold with a surjective submersion $\pi : M \to N$. Later (in §11 and §13) we will encounter situations such that

(a) $[w, w']^{\mathrm{vrt}} = \beta(w, w')u$ for all horizontal vector fields w, w' of class C^1, and

(b) $[\nabla_u w]^{\mathrm{hrz}} = \Theta w$ for all projectable horizontal vector fields w of class C^1, where

(c) u is a specific vertical vector field on M, β is a differential 2-form on M, and $\Theta : \mathcal{H} \to \mathcal{H}$ is a fixed vector-bundle morphism.

Conditions (a) – (c) are perfectly plausible. In (a), $[w,w']^{\mathrm{vrt}}$ is the "horizontal curvature form" of the "nonlinear connection" $\mathcal{H}$ and its dependence on w,w' is pointwise, i.e., the value of $[w,w']^{\mathrm{vrt}}$ at any $x\in M$ depends only on $w(x)$ and $w'(x)$. (To see this, multiply w or w' by a function.) In (b), projectable horizontal vector fields form a trivializing space of sections in the vector bundle $\mathcal{H}$ restricted to any of the submanifolds $\pi^{-1}(y)$, $y\in N$, and this is why, for ∇^{hrz} as in (10.1), ∇^{hrz}_u makes sense as a *bundle morphism*. (See Remark 10.3 below.)

Remark 10.3. Let V be a *trivializing space of* C^∞ *sections* of a vector bundle $\mathcal{E}$ over a manifold P, that is, a vector space such that for every $y\in P$ and $\varPhi\in\mathcal{E}_y$ there exists a unique $\varPsi\in V$ with $\varPsi(y)=\varPhi$. For any fixed connection ∇ in $\mathcal{E}$ and any vector field u on P, we then may treat ∇_u as a bundle morphism $\mathcal{E}\to\mathcal{E}$, sending any $\varPhi\in\mathcal{E}_y$, $y\in P$, to $(\nabla_u\varPsi)(y)$, with $\varPsi$ chosen as above for y and $\varPhi$.

§11. Horizontally homothetic submersions

Let (M,g) and (N,h) be Riemannian manifolds. We will call a given C^∞ submersion $\pi:M\to N$ *horizontally homothetic* if, for some function $f:M\to(0,\infty)$,

$$\pi^*h\,=\,fg\quad\text{on}\quad\mathcal{H}\,,\quad\text{with}\quad[\nabla f]^{\mathrm{hrz}}=0\,. \tag{11.1}$$

In other words, the pullback tensor π^*h agrees with fg on the horizontal distribution $\mathcal{H}$, while f has a vertical gradient, i.e., is constant along all horizontal curves.

Lemma 11.1. *Let $\pi:M\to N$ be a horizontally homothetic, surjective submersion between Riemannian manifolds (M,g) and (N,h), and let ∇ and D stand for the Levi-Civita connections of g and h, respectively. If w,w' are projectable horizontal C^∞ vector fields on M and we treat them, according to* Remark 10.1, *as vector fields on N, then $[\nabla_w w']^{\mathrm{hrz}}$ is projectable and*

$$\mathrm{D}_w w'\,=\,[\nabla_w w']^{\mathrm{hrz}}\,. \tag{11.2}$$

Proof. Equality (1.18) holds, as written, for g, ∇ and any projectable horizontal C^∞ vector fields u,v,w on M. Due to the inner products involved, it will obviously remain valid when all Lie brackets in (1.18), as well as $\nabla_w v$ on the left-hand side, are replaced by their horizontal components. Since the function f in (11.1) is constant along u, v and w, (1.18) will still hold if, in addition, we replace every occurrence of g by fg. Finally, using (10.2) we see that this last modified version of (1.18) is also valid after $[\nabla_w v]^{\mathrm{hrz}}$ has been replaced by $\mathrm{D}_w v$ (since (1.18) holds for (N,h) as well). Both sides of (11.2) thus are horizontal vector fields on M having the same inner product with every horizontal field, and hence must coincide. This completes the proof. ■

Remark 11.2. A horizontally homothetic submersion π with $\pi^*h=fg$ on $\mathcal{H}$ obviously becomes a *Riemannian submersion* if one replaces g with the conformally related metric fg. The assertion of Lemma 11.1 thus can also be derived from its well-known special case for Riemannian submersions (see [4]), combined with the obvious expression for the Levi-Civita connections of fg in terms of ∇ and df.

Lemma 11.3. *Let $\mathcal{V} = \operatorname{Ker} d\pi$ and $\mathcal{H} = \mathcal{V}^\perp$ be, as usual, the vertical and horizontal distributions of a holomorphic, surjective, horizontally homothetic submersion $\pi : M \to N$ between Kähler manifolds (M, g) and (N, h). Furthermore, let conditions* (a) – (c) *in* Remark 10.2 *hold for some u, β and a complex-linear morphism $\Theta : \mathcal{H} \to \mathcal{H}$. Denoting ρ, $\rho^{(h)}$ and ζ the Ricci forms of g, h and, respectively, the curvature form, defined as in* Remark 1.2*, of the connection in the highest complex exterior power of $\mathcal{V}$, induced by the connection ∇^{vrt} with* (10.1)*, and letting $\mathrm{Trace}_{\mathbf{C}}\Theta : M \to \mathbf{C}$ stand for the pointwise trace of Θ, we then have*

$$\rho = \zeta + \pi^*\rho^{(h)} - i[\mathrm{Trace}_{\mathbf{C}}\Theta]\beta \qquad \text{on} \quad \mathcal{H} = \mathcal{V}^\perp . \tag{11.3}$$

Proof. Let m and q be the complex dimensions of M and N, and let us fix the following ranges for indices: $a, b \in \{1, \ldots, m\}$, $j, k \in \{1, \ldots, q\}$, $\lambda, \mu \in \{q+1, \ldots, m\}$. Locally, we can choose projectable horizontal vector fields w_j and vertical vector fields w_λ, which together trivialize the complex vector bundle TM on an open set $U \subset M$. Choosing the corresponding 1-forms Γ_a^b as in (1.20), we now have $[\nabla_v w_j]^{\mathrm{hrz}} = \Gamma_j^k(v)\, w_k$ and $[\nabla_v w_\lambda]^{\mathrm{vrt}} = \Gamma_\lambda^\mu(v)\, w_\mu$ for any vector field v on U (due to (1.20) with $[w_j]^{\mathrm{vrt}} = [w_\lambda]^{\mathrm{hrz}} = 0$). Now (10.1) gives $\nabla_v^{\mathrm{vrt}} w_\lambda = \Gamma_\lambda^\mu(v)\, w_\mu$, and so, for $w = w_{q+1} \wedge \ldots \wedge w_m$ we have $\nabla_v^{\mathrm{vrt}} w = \Gamma(v)\, w$ with $\Gamma = \Gamma_\lambda^\lambda$. Thus, (1.22) yields $\zeta = i\, d\Gamma_\lambda^\lambda$. (As usual, we sum over repeated indices.)

Aside from being 1-forms in M, the Γ_j^k (or, more precisely, their restrictions to the horizontal distribution $\mathcal{H}$) may be viewed as 1-forms in N, sending any vector field in N (that is, a projectable horizontal vector field v in M) to the function $\Gamma_j^k(v)$. Relation $[\nabla_v w_j]^{\mathrm{hrz}} = \Gamma_j^k(v)\, w_k$ established above, combined with the projectability clause of Lemma 11.1, shows that $\Gamma_j^k(v)$ then is indeed projectable, i.e., descends to a function in N, and at the same time it implies, by (11.2), that $\mathrm{D}_v w_j = \Gamma_j^k(v)\, w_k$. From Lemma 2.2 applied to (N, h) we obtain $\rho^{(h)} = i\, d_N \Gamma_j^j$. Here d_N is the exterior derivative in N and it differs from d (the exterior derivative in M) due to the Lie-bracket term in (1.17) and the difference between the M and N Lie brackets of two projectable horizontal vector fields w, w'. By (10.2), the latter difference equals $[w, w']^{\mathrm{vrt}}$, i.e., $\beta(w, w')\, u$ (cf. Remark 10.2(a)). Therefore, on $\mathcal{H}$, $d\Gamma_j^j - d_N \Gamma_j^j = -\Gamma_j^j(u)\beta$. However, $\Gamma_j^j(u) = \mathrm{Trace}_{\mathbf{C}}\Theta$, since we have $\Theta w_j = [\nabla_u w_j]^{\mathrm{hrz}} = \Gamma_j^k(u)\, w_k$. Thus, $d\Gamma_j^j = d_N \Gamma_j^j - [\mathrm{Trace}_{\mathbf{C}}\Theta]\beta$. Since, by Lemma 2.2 for (M, g), $\rho = i\, d\Gamma_a^a = i\, d\Gamma_j^j + i\, d\Gamma_\lambda^\lambda$, combining the last relation with the previously established equalities $\zeta = i\, d\Gamma_\lambda^\lambda$ and $\rho^{(h)} = i\, d_N \Gamma_j^j$, we obtain (11.3). This completes the proof. ∎

Corollary 11.4. *Let $\mathcal{V}$ and $\mathcal{H}$ again denote vertical and horizontal distributions of a holomorphic, surjective, horizontally homothetic submersion $\pi : M \to N$ be between Kähler manifolds (M, g) and (N, h), and let the complex dimensions of M and N be m and $m - 1$, with $m \geq 2$. Furthermore, suppose that we are given a C^∞ vertical vector field v without zeros on M and functions $Q, \phi, \psi : M \to \mathbf{R}$ such that $Q|\phi| > 0$ everywhere and, setting $u = Jv$, we have, for all C^∞ horizontal vector fields w, w' on M,*

(a) *$[\pi^* h](w, w') = |\phi|\, g(w, w')/Q$ with $[\nabla(\phi/Q)]^{\mathrm{hrz}} = 0$, i.e.,* (11.1) *for $f = |\phi|/Q$.*

(b) *$\nabla_v v = \psi v$, $\nabla_u v = \psi u$ and $\nabla_w v = \phi w$, $\nabla_u w = \phi J w$ if, in addition, w is projectable under π.*

(c) *Relation* (8.5) *with* ω *standing for the Kähler form of* (M,g).

Under these assumptions, and with r *and* $\mathrm{r}^{(h)}$ *denoting the Ricci tensors of* g *and* h,

(i) *If* $\mathrm{r}^{(h)} = \kappa h$ *for a function* $\kappa : N \to \mathbf{R}$, *then* $\mathrm{r} = \lambda g$ *on* $\mathcal{H}$, *with* $\lambda = (\kappa|\phi| - \phi Y)/Q$ *and* $Y = 2\psi + 2(m-1)\phi$.

(ii) *If* $\mathrm{r} = \lambda g$ *on* $\mathcal{H}$ *for a function* $\lambda : M \to \mathbf{R}$, *then* $\mathrm{r}^{(h)} = \kappa h$ *with* $\kappa : N \to \mathbf{R}$ *given by* $\kappa = (\lambda Q + \phi Y)/|\phi|$ *for* Y *as in* (i).

Proof. Due to our assumption on the dimensions, $\mathcal{V}$ is a complex line bundle and v is a global trivializing section of $\mathcal{V}$. Now, for all vectors w, $\nabla^{\mathrm{vrt}}_w v = \varGamma(w)v$ with the complex-valued 1-form $\varGamma$ given by $\varGamma(v) = \psi$, $\varGamma(u) = i\psi$, and $\varGamma(w) = 0$ if w is horizontal. (This is clear from (b) and (10.1) plus the fact that the dependence of $\nabla^{\mathrm{vrt}}_w v$ on w is pointwise.) Using (1.17) for $\xi = \varGamma$ and (c) we find that $d\varGamma(w,w') = [2i\psi\phi/Q]\,\omega(w,w')$ for all horizontal vectors w, w'. Since the 'highest complex exterior power' of $\mathcal{V}$ appearing in Lemma 11.3 now is $\mathcal{V}$ itself, the curvature form ζ in (11.3) is given by (1.22), so that $\zeta = -[2\psi\phi/Q]\,\omega$ on $\mathcal{H}$.

Lemma 11.3 now yields, on $\mathcal{H}$, $\rho - \pi^*\rho^{(h)} = -[\phi Y/Q]\,\omega = -\varepsilon Y\,\pi^*\varOmega$, where ρ, $\rho^{(h)}$ stand for the Ricci forms of g, h, while ω, $\varOmega$ are their Kähler forms, and $\varepsilon = \operatorname{sgn}\phi = \pm 1$. In fact, we can use Lemma 11.3, since conditions (a) – (c) of Remark 10.2 are satisfied by $u = Jv$ along with $\beta = -[2\phi/Q]\,\omega$ and $\varTheta = \phi J$ (that is, $i\phi$ times the identity). This choice of $\varTheta$ gives $\mathrm{Trace}_{\mathbf{C}}\varTheta = (m-1)i\phi$ and $-i[\mathrm{Trace}_{\mathbf{C}}\varTheta]\beta = -2(m-1)[\phi^2/Q]\,\omega$ (while, on $\mathcal{H}$, $\pi^*\varOmega = |\phi|\omega/Q$ due to (2.1) and our assumption (a)).

By (a), assertions (i), (ii) now follow from (2.1). This completes the proof. ■

§12. Irreducible examples

Suppose that we are given the following objects.

(a) An open interval $\mathcal{I} \subset \mathbf{R}$ and a positive C^∞ function Q of the real variable $\tau \in \mathcal{I}$.

(b) An integer $m \ge 1$ and real constants $a, p, \kappa, \varepsilon$ with $a \ne 0$, $p \notin \mathcal{I}$ and $\varepsilon = \pm 1$, such that ε is characterized by $\varepsilon = \operatorname{sgn}(\tau - p)$ for all $\tau \in \mathcal{I}$, i.e., $\varepsilon(\tau - p) > 0$ whenever $\tau \in \mathcal{I}$.

(c) A Kähler-Einstein manifold (N,h) of complex dimension $m-1$, with the Ricci tensor $\mathrm{r}^{(h)} = \kappa h$.

(d) A holomorphic line bundle $\mathcal{L}$ over N with a Hermitian fibre metric $\langle\,,\rangle$.

(e) A connection in $\mathcal{L}$ compatible with $\langle\,,\rangle$ whose curvature $R^{\mathcal{L}}$ is a specific multiple of the Kähler form $\varOmega$ of (N,h), namely $R^{\mathcal{L}} = -2\varepsilon a i\varOmega$.

Note that when $m = 1$ the manifold N is a one-point set and conditions (c) – (e) are devoid of geometric content. The reason we allow this case at all is that Lemma 12.1 with $m = 1$ will be needed later in §15.

Let us denote $\mathcal{H}$ the horizontal distribution of the connection chosen in $\mathcal{L}$. Thus, $\mathcal{H}$ is a subbundle of the tangent bundle $T\mathcal{L}$ of the total space (also denoted $\mathcal{L}$, and treated as the set of all pairs (y,z) with $y \in N$ and $z \in \mathcal{L}_y$). Hence $T\mathcal{L} = \mathcal{H} \oplus \mathcal{V}$, with the vertical distribution $\mathcal{V}$ characterized by $\mathcal{V}_{(y,z)} = T_z\mathcal{L}_y$ (so that $\mathcal{V}_{(y,z)}$ may be viewed as the fibre $\mathcal{L}_y$ identified with a subspace of $T_{(y,z)}\mathcal{L}$). Using this identification, we will denote v and u the vertical vector fields on $\mathcal{L}$ given by

$$v(y,z) = az\,, \qquad u(y,z) = iaz\,, \tag{12.1}$$

with the same constant $a \neq 0$ as in (e) above.

More generally, denoting $R^{\mathcal{L}}$ the curvature tensor of any connection in any real or complex vector bundle $\mathcal{L}$, we have the formula

$$[w, w']^{\mathrm{vrt}} = R^{\mathcal{L}}(w, w')z \tag{12.2}$$

(with $...^{\mathrm{vrt}}$ standing for 'the vertical component'), valid at any point $(y, z) \in \mathcal{L}$, whenever w, w' are C^∞ vector fields defined near y in the base manifold (and identified with their horizontal lifts to $\mathcal{L}$).

Let us again assume (a) – (e). Condition $Q\, d[\log r]/d\tau = a$ determines, uniquely up to a positive constant factor, a function $r : \mathcal{I} \to \mathcal{I}'$ which is a C^∞ diffeomorphism onto some open interval $\mathcal{I}' \subset (0, \infty)$. Fixing such a function r, we can use its inverse $\mathcal{I}' \to \mathcal{I}$ to treat τ as a function of r.

Let $\mathcal{L} \smallsetminus N$ be the m-dimensional complex manifold obtained by removing the zero section from the total space of $\mathcal{L}$. We will at the same time use the symbol r for the function $\mathcal{L} \smallsetminus N \to (0, \infty)$ assigning to each (y, z) the norm $|z| = \langle z, z\rangle^{1/2}$. In this way τ, as a function of r, will from now on be treated as a function

$$\tau : U \to \mathbf{R} \tag{12.3}$$

on a suitable open set $U \subset \mathcal{L} \smallsetminus N$. Any C^1 function F of the real variable τ defined on $\mathcal{I}$ now becomes a function $F : U \to \mathbf{R}$ (the composite of (12.3) and F), and the vector fields defined in (12.1) then obviously satisfy

$$d_v F = ar\,\frac{dF}{dr}\,, \qquad d_u F = 0\,. \tag{12.4}$$

Using the data (a) – (e), we now proceed to define a Riemannian metric g on the m-dimensional complex manifold U by requiring that

$$\begin{aligned} g &= (ar)^{-2} Q\,\mathrm{Re}\langle\,,\rangle \quad \text{on } \mathcal{V}\,, \qquad g(\mathcal{V}, \mathcal{H}) = \{0\}\,,\\ g &= 2|\tau - p|\,\pi^* h \quad \text{on } \mathcal{H}\,, \end{aligned} \tag{12.5}$$

with $\pi : \mathcal{L} \to N$ standing for the bundle projection. (Here $g(\mathcal{V}, \mathcal{H}) = \{0\}$ means that $\mathcal{H}$ and $\mathcal{V}$ are g-orthogonal.) In other words, using the the same symbols w, w' both for any given C^∞ vector fields in N, and for their horizontal lifts to $\mathcal{L}$, we obtain, with v, u given by (12.1),

$$\begin{aligned} &g(v, v) = g(u, u) = Q\,, \quad g(w, w') = 2|\tau - p|\,h(w, w')\,,\\ &g(v, u) = g(v, w) = g(u, w) = 0\,. \end{aligned} \tag{12.6}$$

Lemma 12.1. *Let g, $\mathcal{V}$, $\mathcal{H}$ and U be as in* (12.5), (12.3) *with* (a) – (e), *and let ∇ denote the Levi-Civita connection of the metric g on the complex manifold U. Furthermore, let the functions $\psi, \phi, Y, \lambda, \mu : U \to \mathbf{R}$ be given by*

$$\psi = \frac{1}{2}\,\frac{dQ}{d\tau}\,, \qquad \phi = \frac{Q}{2(\tau - p)}\,, \qquad Y = 2\psi + 2(m-1)\phi\,, \tag{12.7}$$

$$\lambda = \frac{\kappa|\phi| - \phi Y}{Q}\,, \qquad \mu = -\frac{1}{2}\,\frac{dY}{d\tau}\,. \tag{12.8}$$

Then g is a Kähler metric on U and τ is a Killing potential on (U,g), while the g-gradient and g-Laplacian of τ are $\nabla\tau = v$, $\Delta\tau = Y$, and $\nabla d\tau$ along with the Ricci tensor r of g, and the distributions $\mathcal{V}$, $\mathcal{H}$ defined above, satisfy relations (7.3) – (7.4) *with ψ,ϕ,λ,μ given by* (12.7) – (12.8). *Finally, with ε as in* (b),

$$\frac{2(\lambda-\mu)}{\tau-p} = \frac{d}{d\tau}\,\frac{Y-\varepsilon\kappa}{\tau-p}\,. \tag{12.9}$$

Proof. Throughout this argument, w, w' will stand for any two C^∞ vector fields in N and, simultaneously, for their horizontal lifts to $\mathcal{L}$.

Combined with (12.1) and the assumption $R^{\mathcal{L}} = -\,2\varepsilon a i\varOmega$ in (e), formula (12.2) gives

$$[w,w']^{\mathrm{vrt}} = -\,2\varepsilon\,\varOmega(w,w')u\,. \tag{12.10}$$

Denoting D the Levi-Civita connection of h we have, with ψ, ϕ as in (12.7),

$$\begin{aligned}
&\nabla_v v = -\nabla_u u = \psi v\,, \quad \nabla_v u = \nabla_u v = \psi u\,,\\
&\nabla_v w = \nabla_w v = \phi w\,, \quad \nabla_u w = \nabla_w u = \phi J w\,,\\
&\nabla_w w' = \mathrm{D}_w w' - \varepsilon[h(w,w')v + \varOmega(w,w')u]\,.
\end{aligned} \tag{12.11}$$

To verify (12.11), note that the connection ∇ *defined* by (12.11) makes g parallel (by (12.6)) and is torsion-free (by (12.10)), so that it must coincide with the Levi-Civita connection of g.

Using the same symbol J for the complex structure tensors of N and $\mathcal{L}$, we now have $\nabla J = 0$ in U in view of (12.11) along with the obvious relations $Jv = u$, $Ju = -\,v$ and the fact that the meaning of Jw does not change whether w is treated as a vector field in N, or as a projectable horizontal field in $U \subset \mathcal{L} \smallsetminus N$. Thus, g is a Kähler metric. Note that the Kähler forms $\varOmega$ of h and ω of g satisfy

$$\omega(w,w') = 2|\tau-p|\,\varOmega(w,w') \tag{12.12}$$

as a consequence of (2.1) and the expression for $g(w,w')$ in (12.6).

The fact that the g-gradient of τ is $\nabla\tau = v$ now is clear since, by (12.4) and (12.56), $v - \nabla\tau$ is g-orthogonal to v, u and all horizontal vectors w.

By (1.9), the eigenvalues and eigenvectors of the symmetric 2-tensor $\nabla d\tau$ are the same as those of ∇v (with (1.1)). This implies the part of (7.3) – (7.4) involving $\nabla d\tau$, with ψ,ϕ as in (12.7). As an obvious consequence, we now have $\Delta\tau = Y$, with Y defined in (12.7).

Assumptions (a) – (c) of Corollary 11.4 are now satisfied. In fact, (a) holds in view of (12.6), (12.7), and $f = |\phi|/Q = [2|\tau-p|]^{-1}$ has a vertical gradient as $\nabla\tau = v$ with (12.1); (b) is immediate from (12.11); and (c) is clear from (12.10), (12.12) and (12.7). Since $\mathrm{r}^{(h)} = \kappa h$ (see (c) earlier in this section), we may apply Corollary 11.4(i), obtaining $\mathrm{r} = \lambda g$ on $\mathcal{H}$, with λ as in (12.8).

By (12.11), the bundle morphism ∇u (cf. (1.1)) is skew-adjoint (by (12.6), as it sends v to ψu and u to ψv, and equals ϕJ on $\mathcal{H}$). Since $u = Jv = J(\nabla\tau)$, this means that τ is a Killing potential (§3), so that (3.8) with (1.3) and $\Delta\tau = Y$ give $2\,\mathrm{r}(v,\cdot\,) = -\,[dY/d\tau]\,d\tau$. Thus, v is an eigenvector of r at every point, with the eigenvalue μ given by (12.8), and by Hermitian symmetry of r the same is true for $u = Jv$. Along with the relation $\mathrm{r} = \lambda g$ on $\mathcal{H}$, this proves the part of (7.3) – (7.4) dealing with r. Finally, (12.9) is immediate from (12.7) – (12.8), which completes the proof. ■

For (a) – (e) as at the beginning of this section, let us consider the additional requirement that

$$\frac{d}{d\tau}\,\frac{Y - \varepsilon\kappa}{\tau - p} \neq 0 \qquad \text{everywhere in } \mathcal{I}\,, \tag{12.13}$$

with Y as in (12.7). Note that *vanishing* of the derivative in (12.13) is a second-order linear differential equation imposed on Q as a function of τ. Thus, as long as Q is chosen not to be a solution to that equation, we will have (12.13) with $\mathcal{I}$ replaced by a suitable (nonempty) open subinterval.

Proposition 12.2. *Given any set of data* (a) – (e) *as above, with $m \ge 2$, which also satisfies* (12.13)*, let g be defined by* (12.5)*. Then g is a Kähler metric on the complex manifold U of the complex dimension m appearing in* (12.3)*, and the nonconstant function τ with* (12.3) *satisfies conditions* (5.1) *and* (5.3) *for some C^∞ functions χ and σ.*

In fact, this is immediate from Lemma 12.1 combined with Lemma 9.1; as for (5.3), it holds since λ, μ, ϕ, ψ all are functions of τ (cf. (12.7) – (12.8)), and hence so are χ and σ with (7.6). ■

Remark 12.3. The function τ obtained in Proposition 12.2 is a special ρ-ω potential, i.e, satisfies (5.4) (in addition to (5.1) and (5.3)), if and only if $d\chi \neq 0$ everywhere for χ given by (7.6) (with λ, μ, ϕ, ψ as in (12.7) – (12.8)). Requiring $d\chi$ to vanish identically now amounts to imposing a specific third-order ordinary differential equation on Q (treated as a function of the variable τ). We now obtain (5.4) by choosing Q which is not a solution to that equation, and suitably restricting the interval $\mathcal{I}$ in (a).

Remark 12.4. Since our discussion is local, i.e., N in (c) is not assumed compact, a in (e) may be any real number; in fact, as $\varOmega$ is closed, a connection with $R^{\mathcal{L}} = -\,2\varepsilon a i\varOmega$ may be obtained, locally, from any given Hermitian connection via an obvious gauge transformation. On the other hand, if $\kappa \neq 0$, (c) and (e) together yield

$$R^{\mathcal{L}} = -\,\frac{2\varepsilon a}{\kappa}\,i\rho^{(h)}\,.$$

Thus, if N is compact, the number $q = 2a/\kappa$ is necessarily rational (unless $\kappa = 0$); in fact, $c_1(\mathcal{L})$ and $c_1(N)$ then lie in $H^2(N, \mathbf{Z})$.

§13. A local classification theorem

Any set of data with the properties (a) – (e) of §12 and (12.12) gives rise, by the construction described in Proposition 12.2 (with Remark 12.3) to a Kähler manifold (U, g) of the complex dimension m on which the function τ is a nonconstant special ρ-ω potential. Another family of special ρ-ω potentials (this time on product Kähler manifolds) has been described in Example 9.2.

We will now show that these examples are, locally, the only ones that may exist. More precisely, we establish that fact at points x which are "in general position", that is, satisfy (a) or (b) in Theorem 13.2 below; such points form (in view of Lemma 5.1, (3.3) and (5.4)) an open dense subset of the manifold in question.

Lemma 13.1. *Let a real vector field v on a complex manifold M of complex dimension $m \geq 2$ be holomorphic, i.e., satisfy* (3.2)*, and let $u = Jv$. Given a real constant $a \neq 0$ and a point $x \in M$ with $v(x) \neq 0$, there exists a biholomorphic identification of a neighborhood of x in M with an open set in $\mathcal{L} \smallsetminus N$ for some holomorphic line bundle $\mathcal{L}$ over a complex manifold N which makes v and u appear as the vector fields defined by* (12.1)*. Furthermore, let $\mathcal{H}$ be a C^∞ distribution on a neighborhood U of x in M such that*

(i) *$\mathcal{H}$ is invariant under the flows of both v and u,*
(ii) *$TU = \mathcal{H} \oplus \mathcal{V}$, where $\mathcal{V} = \mathrm{Span}\,(v, u)$,*
(iii) *The real span of $\mathcal{H}$ and u is an integrable distribution on U,*

then, under a biholomorphic identification as above, $\mathcal{H}$ becomes the horizontal distribution of some linear connection in $\mathcal{L}$ which makes a Hermitian fibre metric parallel.

Proof. Using the flow of v and the holomorphic inverse mapping theorem we can find holomorphic local coordinates $y_1, \ldots, y_m$ for M at x for which v and u are the coordinate vector fields for the real coordinate directions of $\mathrm{Re}\, y_m$ and $\mathrm{Im}\, y_m$. A new coordinate system $y_1, \ldots, y_{m-1}, z$ obtained by replacing y_m with $z = \exp a y_m$ now identifies points near x with pairs (y, z), where $y = (y_1, \ldots, y_{m-1})$. We have thus biholomorphically identified a neighborhood of x in M with $N \times D$ for some open sets $N \subset \mathbf{C}^{m-1}$ and $D \subset \mathbf{C}$. Regarding $N \times D$ as an open subset of the product line bundle $\mathcal{L} = N \times \mathbf{C}$, we easily see (for instance, by expressing the flows of v and u in terms of (y, y_m), and then (y, z)) that v and u now have the form (12.1), and the flows of v and u consist of multiplications by complex scalars in the line bundle $\mathcal{L} = N \times \mathbf{C}$.

Let us now define a complex-valued 1-form Γ on N by assigning, to any vector $w \in T_y N$, $y \in N$, the complex number $\Gamma(w)$ such that for some (or any) $z \in \mathbf{C}$ with $(y, z) \in N \times D$, $(0, \Gamma(w) z)$ is the $\mathcal{V}$ component of the vector $(w, 0) \in T_{(y,z)}\mathcal{L}$ or, equivalently, $(w, -\Gamma(w) z)$ is its $\mathcal{H}$ component. Assumption (i) guaranteeing the invariance of $\mathcal{H}$ (and $\mathcal{V}$, cf. (3.1)) under multiplications by complex scalars in the line bundle $\mathcal{L}$ now implies that this connection is well-defined, i.e., independent of the choice of z. Furthermore, $\mathcal{H}$ coincides, on $N \times D$, with the horizontal distribution of the linear connection in $\mathcal{L} = N \times \mathbf{C}$ which associates with a C^1 section of $\mathcal{L}$, i.e., a $\mathbf{C}$-valued function $y \mapsto z$ on N, and with a vector field w on N, the covariant derivative $d_w z + \Gamma(w) z$. (In fact, both lead to the same horizontal-lift operation $T_y N \ni w \mapsto (w, -\Gamma(w) z) \in T_{(y,z)}\mathcal{L}$.)

Finally, by (iii), there exists (locally in $N \times D$) a C^∞ function φ with $d_v \varphi > 0$ and $d_u \varphi = d_w \varphi = 0$ for all w in $\mathcal{H}$. In view of (i) and (3.1), $d_v \varphi$ is constant along u and $\mathcal{H}$ as well, so that $d_v \varphi = H(\varphi)$ for some C^∞ function H of a real variable. Choosing a positive function r of the variable φ with $H(\varphi)\, d[\log r]/d\varphi = a$, and treating it as a function on (an open subset of) $N \times D$, we now get $d_v r = ar$ and $d_u r = 0$, so that r is the restriction to an open subset of the norm function $\mathcal{L} \smallsetminus N \to (0, \infty)$ corresponding to some Hermitian fibre metric in $\mathcal{L}$. Since r is also constant along $\mathcal{H}$, the horizontal distribution of the linear connection defined above, that connection makes the fibre metric parallel (as its parallel transport is norm-preserving). This completes the proof. ■

Theorem 13.2. *Let (M, g) be a Kähler manifold of complex dimension $m \geq 2$, and let a nonconstant C^∞ function $\tau : M \to \mathbf{R}$ be a special ρ-ω potential on M,* *that is, satisfy* (5.1) *and* (5.3), (5.4) *for some* C^∞ *functions* χ *and* σ*. Suppose that* $x \in M$ *is a point at which* $d\tau \neq 0$*,* $d\chi \neq 0$*, and the Ricci tensor* $\mathrm{r}(x)$ *is not a multiple of* $g(x)$*. Finally, for* ϕ *appearing in* (7.4) *with* $\mathcal{H}$ *defined in* Lemma 7.1*, let either*

(a) $\phi = 0$ *identically in a neighborhood of* x*, or*
(b) $\phi(x) \neq 0$*.*

Then x *has a neighborhood* U *such that, up to a biholomorphic isometry,* g *and* τ *restricted to* U *are obtained, in case* (a)*, as in* Example 9.2 *or, in case* (b)*, as in* Proposition 12.2*. In the latter case, the parameters* a*,* p*,* κ*,* ε *needed in* (b) *of* §12 *include an arbitrary prescribed real number* $a \neq 0$ *and the constants* p*,* κ *given by* (6.11)*.*

Proof. Let $v = \nabla\tau$ and $u = Jv$. In case (a), Corollary 8.2 allows us to choose a connected neighborhood U of x such that (U, g) is a product Kähler manifold for which $\mathcal{V}$ and $\mathcal{H} = \mathcal{V}^\perp$ are the factor distributions. The fact that (U, g) and $\tau : U \to \mathbf{R}$ are obtained as in Example 9.2 is now immediate since λ in (7.4) is constant (by Lemma 7.2(b) and the last formula in (6.2) with $\phi = 0$), so that, as $\mathrm{r} = \lambda g$ on $\mathcal{H}$ by (7.4), the factor Kähler manifold (N, h) tangent to $\mathcal{H}$ is Einstein; while, as $u = J(\nabla\tau)$ is tangent to the factor surface (P, η) corresponding to $\mathcal{V}$ and is a Killing field (by Lemma 5.1), τ treated as a function on P is a Killing potential for u treated as a Killing field on (P, η). This proves our assertion in case (a).

Let us now assume (b). Using the notations of Lemma 7.1, we may consider a connected neighborhood U of any given point in M at which $\phi\, d\tau \neq 0$, such that $\phi\, d\tau$, $d\chi$ and $2m\,\mathrm{r} - \mathrm{s}g$ are all nonzero throughout U and U forms a bundle over a manifold N with the fibres that are the leaves of $\mathcal{V}$ restricted to U. Thus, N is the quotient (leaf space) $U/\mathcal{V}$ and points of N may be thought of as leaves y of $\mathcal{V}$ in U. The tangent space T_yN at any leaf y is clearly nothing else than the set of those sections of $\mathcal{H}$ restricted to y (i.e., those normal vector fields on the submanifold y of (M, g)) which are invariant under the local action of $\mathbf{C}$ in U with the infinitesimal generators v and u. Consequently, C^∞ vector fields on N may be (and from now on will be) identified with those C^∞ vector fields w on U which commute with, and are orthogonal to, both v and u.

Since the leaves of $\mathcal{V}$ are the orbits of the action of $\mathbf{C}$ in U just mentioned, and that action is holomorphic (cf. Lemma 7.1(i)), it follows that N has a natural structure of a complex manifold such that the projection mapping $\pi : U \to N$ is holomorphic. We will use the same symbol J for the complex structure tensors of both M and N. Thus, for a vector field w as in the last paragraph, both meanings of J give rise to the same meaning of Jw (where Jw stands both for a vector field on U with the special properties just listed, and for a vector field on N).

For vector fields w, w' on N interpreted as before, the formula

$$h(w, w') = \frac{|\phi|}{Q}\, g(w, w') \tag{13.1}$$

defines a Riemannian metric h on N, which is clearly Hermitian relative to J. In fact, for *vectors* $w, w' \in T_yN$, the left-hand side of (13.1) is a function on the leaf y and, by (8.3), it is *constant* on y. Since the Levi-Civita connection D of the metric h on N is given by (11.2), and $\mathcal{H}$ is J-invariant (Lemma 7.1(iv)), it now follows easily that J (on N) is D-parallel, i.e., (N,h) is a Kähler manifold. Denoting $\varOmega$ its Kähler form we have, by (2.1), $\varOmega(w,w')=h(Jw,w')$, that is, for w,w' as in (13.1),

$$\varOmega(w,w') \;=\; \frac{|\phi|}{Q}\,\omega(w,w')\,. \tag{13.2}$$

By Lemma 7.2(a), as $v=\nabla\tau$ lies in $\mathcal{V}$, condition (a) of Corollary 11.4 is satisfied. So are conditions (b) and (c) (by (7.8), (8.2), (8.5)), and the premise of Corollary 11.4(ii) (by (7.4)). The conclusion of Corollary 11.4(ii), along with (7.7) and Lemma 8.1(i), now states that (N,h) is a Kähler-Einstein manifold with the Ricci tensor

$$\mathrm{r}^{(h)} \;=\; \kappa h\,, \tag{13.3}$$

where κ is the constant appearing in (8.1).

Furthermore, conditions (i) – (iii) in Lemma 13.1 are satisfied, on U, by our v, u, $\mathcal{H}$ and $\mathcal{V}$. (This is clear from Lemma 7.1(iii), (7.1) and (8.5), (8.2) along with (1.2).) Applying the conclusion of Lemma 13.1 to a smaller version of U we now see that (U,g) and τ are obtained as in Proposition 12.2. Specifically, Lemma 7.2(a) for the function Q gives (a) in §12, the interval $\mathcal{I}$ being the range of τ on U; since $Q\neq 0$ on U by (7.2), the first equality in (8.1) shows that $\tau\neq p$ on U (i.e., we have (b) in §12) and, at the same time, yields (12.6) as a consequence of (13.1); furthermore, (c) in §12 is clear from (13.3); (d) is provided by Lemma 13.1; (e) is a consequence of Lemma 13.1, (13.2), (8.5), (12.2) and (12.1); and, finally, since Lemma 12.1 now applies, we have (12.13) in view of (12.9) and our assumption $2m\,\mathrm{r}\neq\mathrm{s}g$ (i.e., $\lambda\neq\mu$, cf. (7.3) – (7.4). This completes the proof. ■

§14. Solutions to (6.2) – (6.5) with $\phi = 0$

Let a solution $(Q,Y,\phi,\psi,\lambda,\mu,\mathrm{s})$ to the system (6.2) – (6.5) be, in addition, "special" in the sense that $\phi=0$. By (6.2), λ then is constant and, setting $K=\lambda/(3-2m)$ we obtain the explicit expressions

$$(2m-1)Q \;=\; (1-2m)K\tau^2 \,+\, \alpha\tau^{2m-1} \,-\, \gamma\,, \tag{14.1}$$

$$Y \;=\; -\,2K\tau \,+\, \alpha\tau^{2m-2}\,,\quad \phi \;=\; 0\,,\quad \psi \;=\; -\,K\tau \,+\, \alpha\tau^{2m-2}/2\,, \tag{14.2}$$

$$\lambda \;=\; (3-2m)K\,,\qquad \mu \;=\; K \,-\, (m-1)\alpha\tau^{2m-3}\,, \tag{14.3}$$

and

$$\mathrm{s} \;=\; -\,(2m-1)(2m-4)K \,-\, 2(m-1)\alpha\tau^{2m-3}\,, \tag{14.4}$$

with arbitrary constants $K,\ \alpha,\ \gamma$.

In fact, replacing λ in (6.15) with $(3-2m)K$ and multiplying the resulting equality by τ^{2-2m}, we obtain (14.4), with α representing a constant of integration. On the other hand, using (6.14) with $\lambda=(3-2m)K$ and the relation $\psi-\phi=\psi=Y/2$ (due to (6.5) with $\phi=0$), we can rewrite (6.1) with $Z=\tau$ as $\tau=(m-1)Y/[m(3-2m)K-\mathrm{s}/2]$ which, combined with (14.4), gives (14.2). Finally, the first equation in (6.3) with $\psi=Y/2$ becomes $dQ/d\tau=Y\,d\tau$, which implies (14.1) for some constant γ which is easily seen to be the same as in (6.10).

§15. Conformally-Einstein, product Kähler metrics

Lemma 15.1. *There exist Riemannian metrics g on oriented real surfaces U along with C^∞ surjective functions $\tau : U \to \mathcal{I}$, where $\mathcal{I} \subset \mathbf{R}$ is an open interval, for which $\nabla d\tau = [\Delta\tau/2]g$ and $g(\nabla\tau, \nabla\tau)$ coincides with any prescribed positive C^∞ function Q of τ defined on $\mathcal{I}$.*

This is a special case of Lemma 12.1 (with $m = 1$ in the assumption (b)); the fact that τ is a Killing potential, which is a part of the conclusion of Lemma 12.1, now yields our assertion in view of Lemma 5.3. ∎

Example 15.2. Let (M, g) be a product Kähler manifold of complex dimension $m \ge 2$ with a C^∞ function $\tau : M \to \mathbf{R}$ obtained as in Example 9.2. In addition, let the oriented Riemannian surface (P, η) used in the construction and the function τ (originally defined on P) be such that, on P, $g(\nabla\tau, \nabla\tau) = Q$ for a fixed function Q of the variable τ given by (14.1) with arbitrary constants K, α, γ such that $|K| + |\alpha| > 0$. By (14.3), the latter assumption allows us to assume that $\lambda \ne \mu$ everywhere in the given interval of the variable τ and, restricting the interval further and imposing additional restrictions on K, α and γ, we may assume that Q with (14.1) is positive on the whole interval. Note that the existence of (P, η) with such a function τ and $g(\nabla\tau, \nabla\tau) = Q$ is a consequence of Lemma 15.1.

It follows now that the metric $\tilde{g} = g/\tau^2$, defined wherever $\tau \ne 0$, is Einstein. In fact, as stated in Example 9.2, we now have (5.1) everywhere in (M, g), with σ and χ defined by (7.6). Combining (7.6) with (14.2) – (14.3) we thus obtain $\chi = \tau/(2m - 2)$. In other words, relation (5.1), generally valid in Example 9.2, states in our case that $(2m - 2)\,\nabla d\tau \,+\, \tau\,\mathrm{r}$ equals a function times the metric g, and so we have (4.4), as required.

Example 15.3. As a special case of Example 15.2 we obtain *compact* product Kähler manifolds (M, g) of any complex dimension $m \ge 2$ which admit C^∞ functions $\tau : M \to \mathbf{R}$ such that $\tilde{g} = g/\tau^2$ is an Einstein metric on he open set where $\tau \ne 0$. (Since τ in these examples must vanish somewhere, (M, g) thus is conformally Einstein "almost everywhere", but not globally.)

To obtain such manifolds, it suffices to choose $\alpha = 0$, $K > 0$ and $\gamma < 0$ in (14.1). The formula for ψ in (14.2), combined with (7.3) – (7.4) for the product manifold, now gives, on the surface (P, η), $\nabla d\tau = \psi\eta = -K\eta$, which is satisfied when we choose (P, η) to be a 2-sphere S^2 of constant Gaussian curvature K. As for the other product factor (N, h), we may let it be any compact Kähler-Einstein manifold of complex dimension $m - 1 \ge 1$ whose constant Ricci curvature (Einstein constant) is the negative number $\lambda = (3 - 2m)K$.

The function τ leading to the required conformal change is constant in the direction of N, i.e., is a function on S^2. Specifically, it is an eigenfunction of the spherical Laplacian for the first nonzero eigenvalue. When S^2 is realized as a sphere of radius $1/\sqrt{K}$ about 0 in a Euclidean 3-space V, τ is just the restriction to S^2 of any nonzero linear functional on V.

§16. A single equation replacing (6.2) – (6.6)

As stated at the beginning of §1, we will refer to individual equations in a multi-formula expression (e.g., (6.2)) by using tags such as (6.2)i) – (6.2)v). We will also write $F' = dF/d\tau$.

For a fixed integer $m \geq 2$, let $(Q, Y, \phi, \psi, \lambda, \mu, \mathrm{s})$ be a solution to (6.2) – (6.6), and let p be the constant in (6.11). Then

$$(16.1) \qquad Q = 2(\tau - p)\phi\,, \quad \psi = \phi + (\tau - p)\phi'\,, \quad Y = 2m\phi + 2(\tau - p)\phi'\,,$$

In fact, (16.1)i) is just (6.16); applying $d/d\tau$ to (16.1)i) and using (6.2)i), we obtain $2\psi = Q' = 2(\tau - p)\phi' + 2\phi$, i.e., (16.1)ii); and (16.1)iii) is immediate from (6.5)i) and (16.1)ii). Similarly,

$$(16.2) \qquad \mu = -(m+1)\phi' - (\tau - p)\phi''\,, \qquad \lambda - \mu = 2(m-1)(\tau - p)\,\phi'/\tau\,.$$

To see this, we may apply $d/d\tau$ to (16.1)iii) obtaining, by (6.2)ii), $-2\mu = Y' = 2(m+1)\phi' + 2(\tau - p)\phi''$, i.e., (16.2)i); as for (16.2)ii), it is obvious from (6.1) with $Z = \tau$ (cf. (6.5)), as $\psi - \phi = (\tau - p)\phi$ by (16.1)ii).

Remark 16.1. A solution $(Q, Y, \phi, \psi, \lambda, \mu, \mathrm{s})$ to (6.2) – (6.6) with a fixed integer $m \geq 2$ is uniquely determined by its constituent function ϕ and the constant p given by (6.11). In fact, explicit formulae for Q, Y, ψ, μ and λ in terms of ϕ and p are provided by (16.1) and (16.2), while a similar formula for s is immediate from (6.5)ii) combined with (16.2).

Lemma 16.2. *Given an integer $m \geq 2$ and a solution $(Q, Y, \phi, \psi, \lambda, \mu, \mathrm{s})$ to* (6.2) *–* (6.6)*, let p be the constant appearing in* (6.11)*. Then*

$$(16.3) \quad (\tau - p)\tau^2 \frac{d^3\phi}{d\tau^3} = [(m-4)\tau^2 - 2(m-1)p\tau]\frac{d^2\phi}{d\tau^2} + 2(m-1)(\tau + p)\frac{d\phi}{d\tau}\,.$$

Proof. By (16.1)ii), $\psi - \phi = (\tau - p)\phi'$, and so, from (16.1)i), $(\psi - \phi)\phi = Q\phi'/2$. Hence (6.3)i) with $Z = \tau$ (cf. (6.5)), divided by Q, becomes $\tau\mu' = (2m-3)\mu + \lambda + 2(m-1)^2\phi'$, that is, $\tau\mu' = \lambda - \mu + 2(m-1)\mu + 2(m-1)^2\phi'$. If we rewrite this, replacing $\lambda - \mu$ by the right-hand side of (16.2)ii), and $2(m-1)\mu$ by $2(m-1)$ times the right-hand side of (16.2)i), we obtain $\tau\mu' = 2(m-1)(\tau - p)\phi'/\tau - 4(m-1)\phi' - 2(m-1)(\tau - p)\phi''$. On the other hand, applying $d/d\tau$ to (16.2)i) and multiplying the result by τ we get $\tau\mu' = -(m+2)\tau\phi'' - (\tau - p)\tau\phi'''$. Equating the last two expressions for $\tau\mu'$ we arrive at (16.3). This completes the proof. ■

Proposition 16.3. *Let us fix an integer $m \geq 2$, a real number p and an open interval of the variable τ, not containing 0 or p. Then there exists a bijective correspondence between solutions $(Q, Y, \phi, \psi, \lambda, \mu, \mathrm{s})$ to* (6.2) *–* (6.6)*, defined on our interval, for which p is the constant in* (6.11)*, and functions ϕ on this interval which satisfy equation* (16.3) *and have $\phi\, d\phi/d\tau \neq 0$ everywhere. The correspondence in question associates with $(Q, Y, \phi, \psi, \lambda, \mu, \mathrm{s})$ the constituent function ϕ.*

Proof. The fact that $\phi\phi' \neq 0$ for any solution $(Q, Y, \phi, \psi, \lambda, \mu, \mathrm{s})$ to (6.2) – (6.6) is clear from (6.6) and (6.4) (since $\phi' = 0$ somewhere would, by (16.1)ii), give $\psi = \phi$ somewhere, contrary to (6.4)). Also, the assignment of ϕ to $(Q, Y, \phi, \psi, \lambda, \mu, \mathrm{s})$ is injective according to Remark 16.1.

As for surjectivity, it follows since the other constituent functions, determined by ϕ as in Remark 16.1, are easily verified to satisfy (6.2) – (6.6). Specifically, (6.2)i) follows from (16.1)i), (16.1)ii); (6.2)ii) from (16.1)iii), (16.2)i); (6.2)iv) from (16.1)i), (16.1)ii), (16.2)i); and (6.2)iii) is clear since, by (16.1)ii) and (16.1)i),

$\phi' = (\psi - \phi)/(\tau - p) = 2(\psi - \phi)\phi/Q$. Similarly, (6.5)i) is immediate from (16.1)ii) and (16.1)iii); (6.5)iii) from (6.1), (16.1)ii), (16.2)ii); and (6.5)ii) is the equality defining s in Remark 16.1. Furthermore, (6.3)i) is a direct consequence of (16.1)i), (6.5)iii), (16.2)i), (16.1)ii) and (16.3). To obtain (6.2)v), we first rewrite it, using (6.5)iii) and (6.3)i), in the form $Q\tau(\lambda' - \mu') = 2(\mu - \lambda)\phi\tau - 2(m-1)\mu Q - (\lambda - \mu)Q - 4(m-1)^2(\psi - \phi)\phi$, which in turn is easily verified using (16.1)i), (16.1)ii) and (16.2). Now (6.3)ii) is obvious from (6.5)ii), (6.2)v) and (6.3)i). Finally, (6.4) follows since our assumption $\phi\phi' \neq 0$ leads, by (16.1)i), (16.1)ii) and (16.2)ii) to $Q \neq 0$, $\psi \neq \phi$ and $\lambda \neq \mu$. This completes the proof. ∎

Remark 16.4. For an integer $m \geq 2$, let $(Q, Y, \phi, \psi, \lambda, \mu, \mathrm{s})$ and p satisfy (6.2) – (6.6) and (6.11) on some interval of the variable τ. Then $\lambda - \mu$ cannot be identically zero on the interval. In fact, if it were, (16.2)ii) would imply that ϕ is constant, contradicting the conclusion $\phi\phi' \neq 0$ obtained in Proposition 16.3.

§17. Solutions to (6.2) – (6.6) with $p = 0$

For any fixed integer $m \geq 2$, equation (16.3) with $p = 0$ is a special case of an Euler equation imposed on $d\phi/d\tau$. As a result, we can easily find a basis of the solution space for (16.3) formed by power functions (with exponents m, 0 and -1). Thus, setting $\varepsilon = \operatorname{sgn}\tau$, we have

$$\phi \,=\, \frac{\varepsilon\kappa}{2m} \,+\, \alpha\tau^m \,-\, \frac{\gamma}{(m+1)\tau}\,, \tag{17.1}$$

where α is a constant of integration (and the other two coefficients have been expressed in terms of m and the constants κ and γ given by (6.11)). As $Q = 2\phi\tau$ by (6.16), this gives

$$Q \,=\, \frac{\kappa}{m}\,|\tau| \,+\, 2\alpha\tau^{m+1} \,-\, \frac{2\gamma}{m+1}\,. \tag{17.2}$$

Using Remark 16.1, it is now easy to write explicit formulae for all the constituent functions. For instance,

$$\mathrm{s} \,=\, \frac{2(m-1)(m-2)\gamma}{(m+1)\tau^2} \,-\, 4m(2m-1)\alpha\tau^{m-1}\,, \tag{17.3}$$

$$2(n-1)\lambda \,=\, \mathrm{s} \,+\, 2(m-1)\gamma/\tau^2\,, \tag{17.4}$$

and (as $2\psi = dQ/d\tau$),

$$\psi \,=\, \frac{\varepsilon\kappa}{2m} \,+\, (m+1)\alpha\tau^m\,. \tag{17.5}$$

§18. Solutions with $p \neq 0$

For any fixed integer $m \geq 2$, let us introduce the rational functions

$$F(t) \,=\, \frac{(t-2)t^{2m-1}}{(t-1)^m}\,, \qquad E(t) \,=\, (t-1)\sum_{k=1}^{m}\frac{k}{m}\binom{2m-k-1}{m-1}t^{k-1} \tag{18.1}$$

of the real variable t. Using the dot symbol for the derivative d/dt (as in $\dot{F} = dF/dt$), we now clearly have

$$t(t-1)(t-2)\dot{F}(t) \;=\; \Lambda(t)F(t)\,, \tag{18.2}$$

where

$$\Lambda(t) \;=\; mt^2 - 2(2m-1)t + 2(2m-1) \;=\; m(t-1)^2 - 2(m-1)(t-1) + m\,. \tag{18.3}$$

Also, we find that

$$t(t-1)(t-2)\dot{E}(t) \;=\; \Lambda(t)E(t) \,+\, 2(2m-1)(t-1)E(0) \tag{18.4}$$

by expanding the difference of both sides into powers of t via (18.1).

Lemma 18.1. *For a fixed integer $m \geq 2$ and any real constant $p \neq 0$, the vector space of all solutions ϕ to the linear equation* (16.3) *has a basis consisting of the constant function* 1 *along with $E(t)$ and $F(t)$, treated as functions of the variable τ via the substitution $t = \tau/p$.*

Proof. Setting $\Xi(t) = \Lambda(t)/(t-1)$ we can rewrite (18.2), (18.4) as the equation $t(t-2)\dot{\varphi}(t) = \Xi(t)\varphi(t) + 2(2m-1)\varphi(0)$ satisfied both by $\varphi = E$ and $\varphi = F$. Differentiating it, we obtain

$$t(t-2)\ddot{\varphi} \,+\, 2(t-1)\dot{\varphi} \;=\; \dot{\Xi}\varphi \,+\, \Xi\dot{\varphi}\,, \tag{18.5}$$

where we let φ, $\dot{\varphi}$ stand for $\varphi(t)$, $\dot{\varphi}(t)$, etc. Differentiating (18.5) again we obtain an equation stating that a combination, with specific functional coefficients, of φ and its derivatives up to order three vanishes. If we now use (18.5) to express φ as a combination of $\dot{\varphi}$ and $\ddot{\varphi}$ and substitute that expression for φ (but not its derivatives) in the latter third-order equation, we will obtain (16.3) with ϕ, τ and p replaced everywhere by φ, t and, respectively, 1. (This is easily verified using (18.3).)

However, the modified version of (16.3) to which, as we just found, both $\varphi = E$ and $\varphi = F$ are solutions, is nothing else than the result of applying to (16.3) the variable change $\tau = pt$. This completes the proof. ■

As a consequence of Lemma 18.1, any solution ϕ to (16.3) is a constant-coefficient combination of $E(t)$, $F(t)$ and 1, treated as functions of τ by setting $t = \tau/p$. In terms of the function Q related to ϕ by (16.1)i), this becomes

$$Q \;=\; (t-1)\,[\,AE(t) \,+\, BF(t) \,+\, C\,] \tag{18.6}$$

with constants A, B, C. It is easy to verify that these constants are in turn related to p, κ and γ in (6.11) and (6.10) by

$$m\binom{2m-1}{m-1}A \;=\; (2m-1)p\varepsilon\kappa \,-\, m\gamma\,, \qquad C \;=\; \frac{p\varepsilon\kappa}{m}\,, \tag{18.7}$$

with $\varepsilon = \operatorname{sgn}(\tau - p)$ on the given interval.

§19. Irreducible conformally-Einstein Kähler metrics

We will now apply the construction of §12 to obtain examples of nonconstant functions τ on Kähler manifolds (M,g) for which the conformally related metric $\tilde{g} = g/\tau^2$, defined wherever $\tau \neq 0$, is Einstein. In fact, as we already observed (Example 5.2) the requirement that $\tilde{g} = g/\tau^2$ be Einstein implies conditions (5.1) and (5.3) – (5.4) (for suitable C^∞ functions χ and σ), while the examples described in Proposition 12.2 and Remark 12.3 all satisfy (5.1) and (5.3), (5.4).

All we need to do is choose the data listed in (a) – (e) of §12 in such a way that the function $Q > 0$ in (a), rather than being completely arbitrary, now has the form (17.2) (when p in (b) is zero) or (18.6) with $t = \tau/p$ and (18.7) (when $p \neq 0$). In fact, Q must be of this form if $\tilde{g} = g/\tau^2$ is to be Einstein (by Corollary 7.3 and Proposition 16.3).

Proposition 19.1. *Let there be given a set of data* (a) – (e) *as in* §12*, such that* Q *as a function of* τ *is given either by* (17.2)*, with* $p = 0$ *and* $|\alpha| + |\gamma| > 0$*, or by* (18.6)*, with* $p \neq 0$*,* $t = \tau/p$*,* $|A| + |B| > 0$ *and* (18.7)*. Then* g *given by* (12.5) *is a Kähler metric on the complex manifold* U *of complex dimension* $m \geq 2$*, appearing in* (12.3)*, while* τ *with* (12.3) *is a nonconstant* C^∞ *function for which the metric* $\tilde{g} = g/\tau^2$*, defined in the open set where* $\tau \neq 0$*, is Einstein.*

Proof. Our "positivity assumption" ($|\alpha| + |\gamma| > 0$ or $|A| + |B| > 0$) simply guarantees, by (16.1)i) and (17.2) or (18.6), that ϕ is not constant, i.e., (cf. also Remark 16.4), $\lambda \neq \mu$ somewhere in the interval $\mathcal{I}$ of the variable τ. Since $\lambda - \mu$ is a rational function of τ wherever $(\tau - p)\tau \neq 0$ (by (17.2) or (18.6) along with (16.2)ii)), this means that $\tau \neq p$ on a dense subset $\mathcal{I}_0$ of $\mathcal{I}$. By (12.9), we thus have (12.13) on $\mathcal{I}_0$, and the conclusion of Proposition 12.2 holds with $\mathcal{I}$ replaced by $\mathcal{I}_0$. However, formula (7.6)i) for χ becomes in our case $\chi = \tau/(2m-2)$ (as one sees using (16.1)ii) and (16.2)ii)). By Proposition 12.2 we have (5.1) on U_0, the part of U corresponding to $\mathcal{I}_0$, so that $(n-2)\,\nabla d\tau + \tau\mathrm{r}$ equals, on U_0, a function times g, i.e., (4.4) holds on U_0. Since U_0 is dense in U, this completes the proof. ■

§20. Local types of conformally-Einstein Kähler metrics

Our main result now is an immediate consequence of Theorem 13.2, Corollary 7.3, Proposition 16.3, and the explicit formulae for solutions to (6.2) – (6.5) given in §14, §17, and §18.

Theorem 20.1. *Let* $\tau : M \to \mathbf{R}$ *be a nonconstant* C^∞ *function on a non-Einstein Kähler manifold* (M,g) *of complex dimension* $m > 2$ *such that the conformally related metric* $\tilde{g} = g/\tau^2$*, defined wherever* $\tau \neq 0$*, is Einstein, and let* $x \in M$ *be a point at which* $d\tau \neq 0$ *and the Ricci tensor* $\mathrm{r}(x)$ *is not a multiple of* $g(x)$*. Furthermore, let us assume that the function* ϕ *appearing in* (7.4)*, with* $\mathcal{H}$ *defined in* Lemma 7.1*, has either*

(a) $\phi = 0$ *identically in a neighborhood of* x*, or*
(b) $\phi(x) \neq 0$.

Then there exists a neighborhood U *of* x *in* M *such that, up to a biholomorphic isometry,* g *and* τ *restricted to* U *are obtained, in case* (a)*, as in* Example 15.2 *or, in case* (b)*, as in* Proposition 19.1. ■

§21. On ρ-ω potentials satisfying (5.1) and (5.3), but not (5.4)

Conditions (5.1) and (5.3) do not imply (5.4); as the other extreme, let us consider the case where a function τ on a Kähler manifold (M, g) satisfies (5.1) with a *constant* χ and some function σ. Applying d to (5.2) we now obtain $d\sigma \wedge \omega = 0$ and so, if M is of some complex dimension $m \geq 2$, σ must be constant as well. (In fact, in real dimensions $n \geq 4$ the nondegenerate form ω is also nondecomposable.)

Thus, let us consider a C^∞ function τ satisfying (5.1) with some constants χ, σ on a Kähler manifold (M, g) of complex dimension $m \geq 2$.

(a) If $\chi\sigma > 0$, one calls τ a *Kähler-Ricci soliton*. This case has been studied extensively (see [8], [9], [31], as well as [11], [12], [16], [29], [34]).

(b) If $\chi = 0$, assuming compactness of M we conclude that $\sigma = 0$ and τ is constant; in fact, the 2-form $\sigma\omega$ then is exact. An obvious noncompact example with a nonconstant function τ is, on the other hand, the standard $\mathbf{C}^m$ with the norm-squared function.

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