Towards Iversen's formula for the second Chern classes of regular surfaces in any characteristic

Igor Zhukov

December 6, 2005

Introduction

Let $h : T \to S$ be a finite separable morphism of complete non-singular surfaces over an algebraically closed field k of any characteristic. We establish a formula that expresses the Euler characteristic χ_T of T (understood as the degree of the second Chern class $\int c_{2,T}$) of T via the Euler characteristic of S and some local terms associated with components of the branch divisor B_h of h and certain points on B_h.

Let $B_h = \sum_i b_i B_i$ (here B_i are prime divisors on S). Then

$$\chi_T - n\chi_S = \sum_i b_i \chi_{B_i} + \sum_Q \lambda_f(\widehat{\mathcal{O}_{T,Q}}/\widehat{\mathcal{O}_{S,h(Q)}}).$$

Here Q runs over closed points of T, $\lambda_f(A'/A)$ is a certain invariant defined explicitly for an extension of complete 2-dimensional regular local rings A'/A and a (sufficiently general) element f of the maximal ideal of A which is a local equation of the curve at $h(Q)$ in any fixed sufficiently good pencil of curves on S. This invariant is defined in terms of the differents of all $A'/\mathfrak{q}$ over $A'/(\mathfrak{q} \cap A)$, the invariants of singularity of arcs corresponding to $A'/\mathfrak{q}$ and $A/(\mathfrak{q} \cap A)$, and the invariants of intersection of the latter arc with the branch divisor, where $\mathfrak{q}$ runs over prime divisors of f in A'. (For an exact statment, see definitions in Sections 1, 3, 4 and Theorem 7.4.)

The term $\lambda_f(\widehat{\mathcal{O}_{T,Q}}/\widehat{\mathcal{O}_{S,h(Q)}})$ is non-vanishing only for a finite number of points Q, all of them lying on the ramification divisor of f.

What is also important, this term depends on infinitesimal (rather than merely local) behavior of h, i. e., on the properties of extensions of *completed* local rings, and this reduces the further analysis to some questions related only to complete regular local rings.

This formula is a 2-dimensional analog of Riemann-Hurwitz formula. In characteristic 0 it was established by Iversen in [Iv].

0.0.1 Remark. We could not avoid the dependence on f in the definition of the term λ that describes the ramification in codimension 2. However, we expect that λ_f is independent of f (and, therefore, the formula is in its final form) in case there is no ferocious ramification. (This condition means that all morphisms of curves induced by the given finite morphism of surfaces are separable.)

In the good (non-ferocious) case we can show directly that in all "non-exceptional" points λ_f does not depend on the choice of a pencil of curves if pencils are "sufficiently general". (At the moment this is completed under some mild restrictions.) Then the exceptional points can be managed presumably by a local-global argument like that in [L].

This research was accomplished during the autor's stay at Humboldt Univerität zu Berlin under support of Alexander von Humboldt Foundation and at Max-Planck-Institut für Mathematik. I am very much grateful to these organizations. I would like also to thank A. N. Parshin and H. Kurke for fruitful discussions.

1 Definitions, notation and preliminary facts

For arbitrary domain A, denote by $\tilde{A}$ the integral closure of A; $\delta(A) = l_A(\tilde{A}/A)$; $\nu(A)$ is the number of maximal prime ideals in $\tilde{A}$; $\mathfrak{q}_A$ is the conductor of A, i. e., $\{c \in \tilde{A} | c\tilde{A} \subset A\}$; $Q(A)$ is the field of fractions.

If C is a reduced irreducible curve, and P is a closed point on it, we denote $\nu_P(C) = \nu(\mathcal{O}_{C,P})$.

If A is a 1-dimensional domain, $a \in A$, and $a \neq 0$, we denote $\operatorname{ord}_A a = l_A(A/aA)$.

If A is a 1-dimensional local domain, $\omega \in \Omega_A$, and v is the valuation in $\tilde{A}$, we denote $v(\omega) = v(g)$, where $\omega = g\,dt$, t is any prime element of $\tilde{A}$.

If A is a local ring, we denote by $\mathfrak{m}_A$ the maximal ideal of A, and by $\widehat{A}$ the completion of A.

S_i denotes the set of i-dimensional points of a scheme S.

$k(S)$ denotes the field of rational functions on an integral scheme S.

Let C be a divisor on a complete regular surface S over a perfect field k. Its arithmetic genus is defined as

$$p_a(C) = 1 + \frac{1}{2}(C + K_S.C).$$

1.1 Lemma. *Let A be a 1-dimensional local domain such that $\tilde{A}$ is finite over A. Let $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ be all maximal ideals of $\tilde{A}$. Then for any $a \in A$, $a \neq 0$ we have*

$$\operatorname{ord}_A a = \sum_{i=1}^{n} \operatorname{ord}_{\tilde{A}_{\mathfrak{m}_i}} a \cdot [\tilde{A}_{\mathfrak{m}_i}/\mathfrak{m}_i\tilde{A}_{\mathfrak{m}_i} : A/\mathfrak{m}_A].$$

Proof. See [F, Example A.3.1].

1.1.1 Corollary. *In the setting of Lemma assume that A is a k-algebra, where k is a field. Then*

$$\dim_k(A/a) = \sum_{i=1}^{n} \dim_k(\widetilde{A_{\mathfrak{m}_i}}/a).$$

Wild different. Let B/A be a finite extension of complete discrete valuation rings. The order of the different $\mathfrak{D}_{B/A}$ can be written in the form

$$v_B(\mathfrak{D}_{B/A}) = e_{B/A} - 1 + d(B|A);$$

this $d(B|A)$ is said to be the *wild different* of B/A; we assume $d(B|A) = \infty$, if B/A is inseparable.

Let $\mathcal{O}$ be a complete 2-dimensional regular local ring with a coefficient subfield k. Let $b \in \mathcal{O}$, $a \in \mathfrak{m} \setminus \{0\}$, where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}$. In this situation we introduce $d(a,b) \in \mathbb{N} \cup \{+\infty\}$.

If b is an irredusible element, denote by $\overline{a}$ the image of a in $B = \widetilde{\mathcal{O}/b}$. Then

$$d(a,b) = d(B|k[[\overline{a}]]),$$

if $\overline{a} \neq 0$, and $d(a,b) = +\infty$ otherwise. In the general case, if $b = \varepsilon \prod_i p_i^{r_i}$ is a canonical factorization, we put

$$d(a,b) = \sum_i r_i d(a, p_i).$$

Let A be a complete discrete valuation ring with a coefficient subfield k and valuation v. For $\omega \in \Omega_{A/k}$ define $v(\omega) = v(g) \leq \infty$, where $\omega = g\,dt$, t is any local parameter in A.

1.2 Lemma. *For any $f \in A$ we have $v(df) = v(f) + 1 + d(A|k[[f]])$.*

2 Analytic adjunction formula

Let A be a 1-dimensional complete local domain with a perefect coefficient subfield k. Assume that the embedding dimension of A does not exceed 2. In other words, A is isomorphic to $k[[X,Y]]/(f)$, where f is an irreducible element of the maximal ideal of $k[[X,Y]]$.

2.1 Proposition. *There exists an irreducible polynomial $f \in Xk[X,Y] + Yk[X,Y]$ such that A is isomorphic to $k[[X,Y]]/(f)$.*

2.2 Proposition. *We have*

$$\dim_k(\tilde{A}/\mathfrak{q}_A) = 2\delta(A).$$

Proof. Let $A_0 = (k[X,Y]/(f))_{(X,Y)}$, where f is as in Prop. 2.1. Then A is the completion of A_0; $\dim_k(\tilde{A}/\mathfrak{q}_A) = \dim_k(\widetilde{A_0}/\mathfrak{q}_{A_0})$; $\delta(A) = \delta(A_0)$. It remains to show that $\dim_k(\widetilde{A_0}/\mathfrak{q}_{A_0}) = 2\delta(A_0)$, but this is [Sa, Th. 5]. $\square$

Let s,t be generatots of the maximal ideal of A such that $f \nmid \frac{\partial f}{\partial t}$; Let v be the valuation in $\tilde{A}$.

2.3 Proposition. *Assume that $f \nmid \frac{\partial f}{\partial t}$. Then*

$$v_i(\mathfrak{q}_A) + v(d\overline{s}) = v\big(\overline{\tfrac{\partial f}{\partial t}}\big).$$

Proof. Indeed, $\dim_k(\tilde{A}/\mathfrak{q}_A) = \dim_k(\widetilde{A_0}/\mathfrak{q}_{A_0})$ as in the previous proof; we may assume that $s,t \in A_0$. It remains to apply [Sa, Th. 3bis]. $\square$

2.4 Theorem. *Let s,t be as above. Then*

$$2\delta(A) + v(d\overline{s}) = v\big(\overline{\tfrac{\partial f}{\partial t}}\big).$$

Proof. From Propositions 2.2 and 2.3. $\square$

3 Tame and wild singularity

In this section k is an infinite perfect field, $\mathcal{O}$ is a 2-dimensional complete regular local ring with the coefficient field k, K is the fraction field of $\mathcal{O}$. For $f, g \in \mathcal{O}$ such that (f, g) is an ideal of definition, we denote $(f.g) = \dim_k \mathcal{O}/(f, g)$.

Let $\pi_1, \dots, \pi_r$ be pairwise non-associated prime elements of $\mathcal{O}$, and $f = \pi_1 \dots \pi_r$. Introduce the *tame singularity of f*

$$\operatorname{sing}^t_{\mathcal{O}} f = 2\sum_i \delta(\mathcal{O}/\pi_i) - r + \sum_{i \neq j} (\pi_i.\pi_j) + 1.$$

Next, introduce the *wild singularity* $\operatorname{sing}^w_{\mathcal{O}} f$ of f. Assume that there exist regular local parameters s, t of $\mathcal{O}$ such that

(i) $d(f, s)$, $d(f, s)$, $d(f, \frac{\partial f}{\partial t})$, $d(s, \frac{\partial f}{\partial t})$ are all finite.

(ii) $\pi_i \nmid \frac{\partial \pi_i}{\partial t}$ for any i; $s \nmid \frac{\partial f}{\partial t}$.

Then

$$\operatorname{sing}^w_{\mathcal{O}} f = -d(f, s) + d(s, f) + d\big(f, \tfrac{\partial f}{\partial t}\big) - d\big(s, \tfrac{\partial f}{\partial t}\big).$$

(We shall see in Cor. 3.3.1 that this value is independent of the choice of s, t.) If no required s, t exist, we assume $\operatorname{sing}^w_{\mathcal{O}} f = \infty$.

Our plan is to express $\big(\frac{\partial f}{\partial s} \cdot \frac{\partial f}{\partial t}\big)$ as the sum of tame and wild singularity. It is easy to see that this value is independent of the choice of s and t.

3.1 Lemma (Generalization of Weierstraß preparation lemma). *Let $f \in k[[X, Y]]$ be irreducible. Then, after a possible exchange of X and Y, we have $f = uf_0$, where $u \in k[[X, Y]]^*$, and*

$$f_0 = Y^n + a_1 Y^{n-1} + \dots + a_{n-1} Y + a_n$$

is a separable polynomial in Y, $a_i \in Xk[[X]]$.

Proof. This is exactly [Iv, Lemma 2.4]. □

3.1.1 Corollary. *Let π be an irreducible element of $\mathcal{O}$. Then $d\pi \notin \pi\Omega_{\mathcal{O}/k}$.*

Proof. Since $\Omega_{\mathcal{O}/k}$ is a free $\mathcal{O}$ with a basis ds, dt, an equivalent statement is that either $\pi \nmid \frac{\partial \pi}{\partial s}$, or $\pi \nmid \frac{\partial \pi}{\partial t}$. But this follows from Lemma 3.1. □

3.2 Lemma. *One can choose local parameters s, t in $\mathcal{O}$ so that*

1) $\pi_i \nmid \frac{\partial \pi_i}{\partial t}$ for every i;

2) $s \nmid \frac{\partial f}{\partial t}$.

Proof. Let s_0 and t_0 be arbitrary regular local parameters in $\mathcal{O}$. Put

$$s = s_0 + \alpha t_0 + \beta t_0^p,$$
$$t = t_0,$$

where $p = \max(\operatorname{char} k, 2)$, α, β are some elements of k. It is easy to see that

$$\frac{\partial}{\partial s} = \frac{\partial}{\partial s_0},$$
$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t_0} - \alpha\frac{\partial}{\partial s_0} - \beta p t^{p-1}\frac{\partial}{\partial s_0}.$$

In view of Corollary 3.1.1, we have $\pi_i \nmid d\pi_i$ for any i, i. e., $\pi_i \nmid \frac{\partial \pi_i}{\partial s_0}$ or $\pi_i \nmid \frac{\partial \pi_i}{\partial t_0}$. It follows that the set M of all pairs (α, β) satisfying the condition 1) is non-empty (since k is infinite) and open on the plane.

After a possible intermediate change of variables, we may assume that $(0,0) \in M$. Suppose that the assertion to prove is false; this means in particular that

$$s_0 + \alpha t_0 \mid \frac{\partial f}{\partial s_0} - \alpha \frac{\partial f}{\partial t_0}$$

for all α such that $(\alpha, 0) \in M$, i. e., for infinitely many values of α. From here it can be easily deduced using the completion that

$$s_0 \frac{\partial f}{\partial s_0} = -t_0 d \frac{\partial f}{\partial t_0}.$$

It follows from this and Lemma 3.1.1 that $\frac{\partial f}{\partial s_0} \neq 0$. Next, for arbitrary β such that $(0, \beta) \in M$ we have

$$s_0 + \beta t_0^p \mid \frac{\partial f}{\partial t_0} - \beta p t_0^{p-1} \frac{\partial f}{\partial s_0},$$

whence

$$s_0 + \beta t_0^p \mid -t_0 \frac{\partial f}{\partial t_0} + \beta p t_0^p \frac{\partial f}{\partial s_0}$$

and

$$s_0 + \beta t_0^p \mid \frac{\partial f}{\partial s_0}(s_0 + \beta p t_0^p).$$

Since $\frac{\partial f}{\partial s_0} \neq 0$, we conclude that

$$s_0 + \beta t_0^p \mid s_0 + \beta p t_0^p$$

for infinitely many values of β, and this is impossible. $\square$

3.3 Proposition. *Let s, t be any regular local parameters of $\mathcal{O}$ satisfying the conditions in Lemma 3.2. Assume that $\left(\frac{\partial f}{\partial s} \cdot \frac{\partial f}{\partial t}\right) < \infty$. Then $d(f,s)$, $d(s,f)$, $d\left(f, \frac{\partial f}{\partial t}\right)$, $d\left(s, \frac{\partial f}{\partial t}\right)$ are all finite, and*

$$\left(\frac{\partial f}{\partial s} \cdot \frac{\partial f}{\partial t}\right) = \operatorname{sing}^t_{\mathcal{O}} f - d(f,s) + d(s,f) + d\left(f, \frac{\partial f}{\partial t}\right) - d\left(s, \frac{\partial f}{\partial t}\right).$$

Proof. First, we show that $d\left(f, \frac{\partial f}{\partial t}\right) < \infty$, $d\left(s, \frac{\partial f}{\partial t}\right) < \infty$, and

$$\left(\frac{\partial f}{\partial s} \cdot \frac{\partial f}{\partial t}\right) = \left(f \cdot \frac{\partial f}{\partial t}\right) + d\left(f, \frac{\partial f}{\partial t}\right) - \left(s \cdot \frac{\partial f}{\partial t}\right) - d\left(s, \frac{\partial f}{\partial t}\right). \quad (1)$$

It is sufficient to check that $d(f,q) < \infty$, $d(s,q) < \infty$, and

$$(f.q) = \left(\frac{\partial f}{\partial s} \cdot q\right) - d(f,q) + (s.q) + d(s,q),$$

where q is any irreducible divisor of $\frac{\partial f}{\partial t}$. Note that all other terms in both sides of the relation are known to be finite.

According to Corollary 1.1.1, for any $a \in \mathcal{O}$ such that $q \nmid a$ we have $(a.q) = v(\overline{a})$, where $\overline{a}$ is the class of a in $\mathcal{O}/q$, and v is the valuation in $\widetilde{\mathcal{O}/q}$. It remains to note that the equation $df = \frac{\partial f}{\partial s}\, ds + \frac{\partial f}{\partial t}\, dt$ yields $d\bar{f} = \overline{\frac{\partial f}{\partial s}}\, d\bar{s}$ in $\Omega_{\mathcal{O}/q}$, whence $v(ds) < \infty$. Applying Lemma 1.2 twice, we obtain finiteness of $d(f,q)$ and $d(s,q)$, and

$$\begin{aligned} v(\bar{f}) &= v(d\bar{f}) + 1 - d(f,q) \\ &= v\left(\overline{\frac{\partial f}{\partial s}}\right) + v(d\bar{s}) + 1 - d(f,q) \\ &= v\left(\overline{\frac{\partial f}{\partial s}}\right) + v(\bar{s}) - d(f,q) + d(s,q). \end{aligned}$$

Similarly, considering $\Omega_{\mathcal{O}/s}$ instead of $\Omega_{\mathcal{O}/q}$, we obtain

$$(f.s) - 1 = \left(\tfrac{\partial f}{\partial t} \,.\, s\right) - d(f, s). \tag{2}$$

(Note that $(f.s) < \infty$ since $\pi_i \nmid \frac{\partial \pi_i}{\partial t}$ for any i.)

Combine (1) and (2):

$$\left(\tfrac{\partial f}{\partial s} \,.\, \tfrac{\partial f}{\partial t}\right) = \left(f \,.\, \tfrac{\partial f}{\partial t}\right) - ((f.s) - 1) + d\left(f, \tfrac{\partial f}{\partial t}\right) - d\left(s, \tfrac{\partial f}{\partial t}\right) - d(f, s).$$

It remains to show that $d(s, f) < \infty$, and

$$\left(f \,.\, \tfrac{\partial f}{\partial t}\right) - ((f.s) - 1) = d(s, f) + \operatorname{sing}^t_{\mathcal{O}} f.$$

It is sufficient to show that for every i:

$$\left(\pi_i \,.\, \tfrac{\partial \pi_i}{\partial t}\right) - (\pi_i.s) = 2\delta(\mathcal{O}/\pi_i) - 1 + d(s, \pi_i).$$

In view of Corollary 1.1.1, this is equivalent to

$$v\left(\overline{\tfrac{\partial \pi_i}{\partial t}}\right) - v(\bar{s}) + 1 - d(s, \pi_i) = 2\delta(\mathcal{O}/\pi_i), \tag{3}$$

where v is the valuation in $\widetilde{\mathcal{O}/\pi_i}$. By Lemma 1.2, $d\bar{s} \neq 0$ implies $d(s, \pi_i) < \infty$, and (3) can be rewritten as

$$v\left(\overline{\tfrac{\partial \pi_i}{\partial t}}\right) - v(d\bar{s}) = 2\delta(\mathcal{O}/\pi_i)$$

However, the last equality is nothing else but the analytic adjunction formula 2.4 for the ring $\mathcal{O}/\pi_i$. □

3.3.1 Corollary. $\operatorname{sing}^w_{\mathcal{O}} f$ *is independent of the choice of* s, t.

In what follows, we shall write $\operatorname{sing}_{\mathcal{O}} f = \operatorname{sing}^t_{\mathcal{O}} f + \operatorname{sing}^w_{\mathcal{O}} f$.

Examples. Let x, y be any system of regular local parameters in $\mathcal{O}$.

1. Let $f = x^l - y^m$, $p \nmid l$, $p \nmid m$. Then $\operatorname{sing}^w_{\mathcal{O}} f = 0$.
2. Let $f = y^p + y^M - x^2$, $p \nmid M$. Taking $s = x, t = y$, we compute

$$d(f, s) = d(s, f) = d(s, \tfrac{\partial f}{\partial t}) = 0,$$

and

$$d(f, \tfrac{\partial f}{\partial t}) = M - p,$$

whence $\operatorname{sing}^w_{\mathcal{O}} f = M - p$.

3. Let $f = y^p - x^2$. Then $d(f, x) = \infty$, whence for any choice of regular local parameters s, t we have $d(f, \frac{\partial f}{\partial t}) = \infty$, and $\operatorname{sing}^w_{\mathcal{O}} f = \infty$.

4 Extensions of 2-dimensional complete regular local rings

In this section $\mathcal{O}'/\mathcal{O}$ is a finite separable extension of complete 2-dimensional regular local rings of some degree n, both having an infinite prefect coefficient subfield k.

4.1 Proposition. *Let π be a prime element of $\mathcal{O}$ which does not divide a local equation β of the branch divisor of $\mathcal{O}'/\mathcal{O}$; let $\pi = \pi_1 \dots \pi_r$, where $\pi_1, \dots, \pi_r$ are irreducible elements of $\mathcal{O}'$. Then we have*

$$\operatorname{sing}^t_{\mathcal{O}'} \pi - n \operatorname{sing}^t_{\mathcal{O}} \pi = (\beta.\pi) - (n-1) - \sum_{i=1}^{r} d((\mathcal{O}'/\pi_i)|(\mathcal{O}/\pi)).$$

Proof. Choose such regular local parameters s, t in $\mathcal{O}$ that $\pi \nmid \frac{\partial \pi}{\partial t}$. This can be done by Lemma 3.2. Fix j and choose such regular local parameters u, v in $\mathcal{O}'$ that $\pi_j \nmid \frac{\partial \pi_j}{\partial v}$. Denote $A_j = \widetilde{\mathcal{O}'/\pi_j}$.

Now we make some computations in $\Omega_{Q(A_j)/k}$; the elements of $\mathcal{O}'$ (resp., of $\Omega_{\mathcal{O}'/k}$) will be denoted by the same letters as their images in A_j (resp., in $\Omega_{Q(A_j)/k}$). First of all,

$$\tfrac{\partial \pi_j}{\partial u} du + \tfrac{\partial \pi_j}{\partial v} dv = d\pi_j = 0.$$

Therefore,

$$\begin{aligned}\big(\tfrac{\partial \pi}{\partial t}\big)^{-1} ds &= \big(\tfrac{\partial \pi}{\partial t}\big)^{-1} \big(\tfrac{\partial s}{\partial u} du + \tfrac{\partial s}{\partial v} dv\big) \\ &= \big(\tfrac{\partial \pi}{\partial t}\big)^{-1} \big(\tfrac{\partial s}{\partial u} \cdot \tfrac{\partial \pi_j}{\partial v} - \tfrac{\partial s}{\partial v} \tfrac{\partial \pi_j}{\partial u}\big) \big(\tfrac{\partial \pi_j}{\partial v}\big)^{-1} du\end{aligned}$$

The derivations $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial t}$ of $Q(\mathcal{O})$ can be uniquely prolonged to continuous derivations of $Q(\mathcal{O}')$ also denoted by $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial t}$. We have

$$\tfrac{\partial}{\partial u} = \tfrac{\partial s}{\partial u} \cdot \tfrac{\partial}{\partial s} + \tfrac{\partial t}{\partial u} \cdot \tfrac{\partial}{\partial t},$$

since these derivations coincide on the elements s and t, and $Q(\mathcal{O}')$ is separable over $Q(\mathcal{O}) = Q(k[[s,t]])$. In view of this the equality in $\Omega_{Q(A_j)/k}$

$$\begin{aligned}ds &= \big(\tfrac{\partial s}{\partial u} \tfrac{\partial \pi_j}{\partial v} du + \tfrac{\partial s}{\partial v} \tfrac{\partial \pi_j}{\partial v} dv\big) \big(\tfrac{\partial \pi_j}{\partial v}\big)^{-1} \\ &= \big(\tfrac{\partial s}{\partial u} \tfrac{\partial \pi_j}{\partial v} du - \tfrac{\partial s}{\partial v} \tfrac{\partial \pi_j}{\partial u} du\big) \big(\tfrac{\partial \pi_j}{\partial v}\big)^{-1} \\ &= \begin{vmatrix} \frac{\partial s}{\partial u} & \frac{\partial s}{\partial v} \\ \frac{\partial \pi_j}{\partial u} & \frac{\partial \pi_j}{\partial v} \end{vmatrix} \big(\tfrac{\partial \pi_j}{\partial v}\big)^{-1} du\end{aligned}$$

implies

$$\begin{aligned}\big(\tfrac{\partial \pi}{\partial t}\big)^{-1} ds &= \big(\tfrac{\partial \pi}{\partial t}\big)^{-1} \begin{vmatrix} \frac{\partial s}{\partial s} & \frac{\partial s}{\partial t} \\ \frac{\partial \pi_j}{\partial s} & \frac{\partial \pi_j}{\partial t} \end{vmatrix} \begin{vmatrix} \frac{\partial s}{\partial u} & \frac{\partial s}{\partial v} \\ \frac{\partial t}{\partial u} & \frac{\partial t}{\partial v} \end{vmatrix} \big(\tfrac{\partial \pi_j}{\partial v}\big)^{-1} du \\ &= \big(\tfrac{\partial \pi}{\partial t}\big)^{-1} \tfrac{\partial \pi_j}{\partial t} \begin{vmatrix} \frac{\partial s}{\partial u} & \frac{\partial s}{\partial v} \\ \frac{\partial t}{\partial u} & \frac{\partial t}{\partial v} \end{vmatrix} \big(\tfrac{\partial \pi_j}{\partial v}\big)^{-1} du \\ &= \prod_{i \neq j} \pi_i^{-1} \begin{vmatrix} \frac{\partial s}{\partial u} & \frac{\partial s}{\partial v} \\ \frac{\partial t}{\partial u} & \frac{\partial t}{\partial v} \end{vmatrix} \big(\tfrac{\partial \pi_j}{\partial v}\big)^{-1} du.\end{aligned} \tag{4}$$

We shall apply the analytic adjunction formula 2.4. Let w_j and w be the valuations in A_j and $A = \widetilde{\mathcal{O}/\pi}$ respectively, $e_j = e(A_j/A)$, $d_j = d(A_j|A)$. Denote by ρ_j and ρ any local parameters in A_j and A respectively. We write the left hand side and right hand side of (4) in the form $L d\rho_j = L_0 d\rho$ and $R d\rho_j$ respectively and compute the valuations of L and R.

It is clear from the exact sequence

$$\Omega_{A/k} \otimes_A A_j \to \Omega_{A_j/k} \to \Omega_{A_j/A} \to 0$$

that $d\rho = ad\rho_j$, where a is a generator of the different of A_j/A, i. e. $w_j(a) = e_j - 1 + d_j$. Next, $w(L_0) = -w\big(\frac{\partial \pi}{\partial t}\big) + w(ds)$. It follows $w_j(L) = \big(-w\big(\frac{\partial \pi}{\partial t}\big) + w(ds)\big)e_j + e_j - 1 + d_j$. Next,

$$w_j(R) = -\sum_{i \neq j} w_j(\pi_i) + w_j\left(\begin{vmatrix} \frac{\partial s}{\partial u} & \frac{\partial s}{\partial v} \\ \frac{\partial t}{\partial u} & \frac{\partial t}{\partial v} \end{vmatrix}\right) - w_j\big(\tfrac{\partial \pi_j}{\partial v}\big) + w_j(du).$$

Thus, we have

$$\begin{aligned} &\big(-w\big(\tfrac{\partial \pi}{\partial t}\big) + w(ds)\big)e_j + e_j - 1 + d_j = \\ &\qquad = -\sum_{i \neq j} w_j(\pi_i) + w_j\left(\begin{vmatrix} \frac{\partial s}{\partial u} & \frac{\partial s}{\partial v} \\ \frac{\partial t}{\partial u} & \frac{\partial t}{\partial v} \end{vmatrix}\right) - w_j\big(\tfrac{\partial \pi_j}{\partial v}\big) + w_j(du). \end{aligned}$$

Apply the analytic adjunction formula in the both sides, and apply Corollary 1.1.1 in the right hand side:

$$(-2\delta(\mathcal{O}/\pi) + 1)e_j + d_j = 1 - \sum_{i \neq j} (\pi_i.\pi_j) + (\beta'.\pi_j) - 2\delta(\mathcal{O}'/\pi_j),$$

where β' is an equation of the ramification divisor in $\mathcal{O}'/\mathcal{O}$. Finally, take the sum over j:

$$-n(\operatorname{sing}^t_{\mathcal{O}} \pi - 1) + \sum_j d_j = -(\operatorname{sing}^t_{\mathcal{O}'} \pi - 1) + (\beta.\pi). \quad \square$$

Definition of λ_f. For the branch divisor of $\mathcal{O}'/\mathcal{O}$ (as a closed subscheme in $\operatorname{Spec} \mathcal{O}'$), denote by β_i the equations of its prime components, and by b_i their multiplicities. Take an element $f \in \mathcal{O}$ which is a product of pairwise non-associated prime elements in $\mathcal{O}$, such that $\operatorname{sing}^w_{\mathcal{O}} f < \infty$ and none of β_i divides f. Let $f = \pi'_1 \ldots \pi'_r$ be a factorization of f in $\mathcal{O}'$. Every π'_i divides exactly one prime divisor π_i of f in $\mathcal{O}$, and we denote by A'_i and A_i the integral closures of $\mathcal{O}'/\pi'_i$ and $\mathcal{O}/\pi_i$ respectively. We define

$$\lambda_f(\mathcal{O}'/\mathcal{O}) = \sum_i b_i(1 - d(f, \beta_i)) - (n-1) - \sum_{i=1}^r d(A'_i|A_i) + \operatorname{sing}^w_{\mathcal{O}'} f - n \operatorname{sing}^w_{\mathcal{O}} f.$$

4.2 Conjecture. *Assume that there is no ferocious ramification in $\mathcal{O}'/\mathcal{O}$. Let $f, f' \in \mathcal{O}$ be elements such that $\lambda_f(\mathcal{O}'/\mathcal{O})$ and $\lambda_{f'}(\mathcal{O}'/\mathcal{O})$ are defined. Then $\lambda_f(\mathcal{O}'/\mathcal{O}) = \lambda_{f'}(\mathcal{O}'/\mathcal{O})$.*

Example. One can construct an ample series of examples, taking for $\mathcal{O}'$ the integral closure of $\mathcal{O}$ in the extension of the fraction field of $\mathcal{O}$ given by equation $x^p - x = t^{-pm+1}u^{-pn}$, where t, u are fixed regular local parameters in $\mathcal{O}$, m, n are non-negative integers. In is not difficulty to see that such $\mathcal{O}'$ is always regular.

In this version of the text we consider only an example of a non-exceptional point on a component of the branch divisor without ferocious ramification. For this, take $m > 0$, $n = 0$.

The notion of an exceptional point is due to Deligne [D], see also Brylinski [Br]. Roughly speaking, these are the points on the ramification divisor where the codimension 2 ramification invariants take their non-generic values.

Let $y = t^m x$. Then $y^p - t^{(p-1)m} y = t$, whence $\mathcal{O}' \supset \mathcal{O}[y] = k[[y, u]]$. Since $\mathcal{O}[y]$ is regular, we have $\mathcal{O}[y] = \mathcal{O}'$, and

$$t = y^p - y^{p+(p-1)(pm-1)} + O(y^{p+2(p-1)(pm-1)}).$$

The branch divisor B consists of one component $t = 0$, and $b_1 = (p-1)pm$.

Let $f = t - u^i$, $i > 0$, $(i, p) = 1$. Then the expansion of f in $k[[y, u]]$ is

$$f = F(y, u) = y^p - y^{p+(p-1)(pm-1)} - u^i + O(y^{p+2(p-1)(pm-1)}).$$

We have

$$\begin{aligned} \operatorname{sing}^w_{\mathcal{O}'} f &= d(f, y) - d(y, f) - d\big(f, \tfrac{\partial f}{\partial u}\big) + d\big(y, \tfrac{\partial f}{\partial u}\big) \\ &= 0 - 0 - (i-1) d\big(k[[y]] | k[[F(y, 0)]]\big) + 0 \\ &= (i-1)(p-1)(pm-1). \end{aligned}$$

It follows

$$\begin{aligned} \lambda_f(\mathcal{O}'/\mathcal{O}) &= (p-1)pm - (p-1) - d\big(k[[y, u]]/f | k[[u]]\big) + (i-1)(p-1)(pm-1) \\ &= (p-1)(pm-1) - i(p-1)(pm-1) + (i-1)(p-1)(pm-1) = 0. \end{aligned}$$

5 Severi's formula

In this section S is a regular geometrically irreducible complete surface over a perfect field k; K is the field of functions on S.

5.1 Lemma. *Let s, t be regular local parameters in $P \in S$. Then for any P' in some neighborhood of P the functions $s - s(P')$, $t - t(P')$ are regular local parameters at P'.*

Proof. Let U be a neighborhood of P such that s and t are regular functions on U. Consider morphism $f : U \to \mathbb{A}^2_k$ determined by a pair of functions s, t. Obviously, f is unramified at P, whence f is also unramified in all points of some neighborhood of P. □

Let ω be a non-zero rational 1-differential on S, i. e. $\omega \in \Omega_{K/k}$, and $\omega \neq 0$. We define a divisor (ω) and a 0-cycle $\langle \omega \rangle$ on S.

Let $P \in S$, and let s, t be regular local parameters at P. Then $\omega = f_P \cdot (a_P\, ds + b_P\, dt)$, where a_P and b_P are coprime in $\mathcal{O}_{S,P}$, $f_P \in K$. Then (ω) is defined as the divisor on S such that in a neighborhood of any closed point P it coincides with the divisor of the function f_P; this can be done in view of Lemma 5.1. Finally, by definition,

$$\langle \omega \rangle = \sum_{P \in S_0} \dim_k \mathcal{O}_{S,P}/(a_P, b_P) \cdot P.$$

5.2 Theorem. *In the group $A_0(S)$ we have*

$$c_{2,S} = \langle\omega\rangle + (\omega) \cdot K_S - (\omega) \cdot (\omega).$$

Proof. See [Y] (Corollary to Theorem 2') or [K]. $\square$

6 Computation of the second Chern class by means of a pencil of curves

Let S be as in the previous section. Here a *pencil of curves on S* is treated as a dominant rational map of S in $\mathbb{P}^1_k$. In other words, this is a surjective morphism

$$\mathcal{C} : S \setminus B \to \mathbb{P}^1_k,$$

which cannot be extended onto any point of B; B is a closed subfield of S, the so-called set of base points of a given pencil of curves. [Sh, Ch. II, §3, Th.3] implies that $\dim B = 0$.

We shall consider only pencils of curves $\mathcal{C}$ satisfying the following condition:

(*) the fiber of $\mathcal{C}$ over any point $s \in \mathbb{P}^1_k$ is a reduced subscheme.

The closure of $\mathcal{C}^{-1}(s) \subset S \setminus B$ in S will be denoted by $\mathcal{C}_s$. The theorem on dimension of fibers of a morphism (see, e. g., [H, Ch. II, Ex. 3.22]) implies that $\mathcal{C}_s$ is of pure dimension 1. The schemes $\mathcal{C}_s$ are said to be *curves in $\mathcal{C}$*.

It is easy to see that all curves in $\mathcal{C}$ belong to the same divisor class, on S. Therefore, for arbitrary divisor D on S the intersection multiplicity $(\mathcal{C}_s.D)$ is independent of s and will be denoted merely by $(\mathcal{C}.D)$.

We require also the following condition to be satisfied:

(**) for any base point b there exist two curves in $\mathcal{C}$ that meet in b transversally.

6.1 Lemma. *Let $\mathcal{C}$ be a pencil of curves on S, b a base point. Then the following two conditions are equivalent:*

1. *There exist two curves in $\mathcal{C}$ that meet in b transversally.*
2. *All curves in $\mathcal{C}$ contain b and meet in b transversally.*

If $s = \mathcal{C}(P)$, the curve $\mathcal{C}_s$ will be denoted also by $\mathcal{C}_P$.

For a reduced curve C on S and a point $P \in C$ we put $\operatorname{sing}^w_P C = \operatorname{sing}^w_{\widehat{\mathcal{O}_{S,P}}} f$ and $\operatorname{sing}^t_P C = \operatorname{sing}^t_{\widehat{\mathcal{O}_{S,P}}} f$, where f is a local equation of C at P. Obviously, $\operatorname{sing}^t_P C$ and $\operatorname{sing}^w_P C$ are well defined.

In this situation we say that C has an inseparable singularity at P, if $(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}) = \infty$. Our trird requirement is as follows.

(***) None of the curves in $\mathcal{C}$ has inseparable singularities.

We say that a pencil $\mathcal{C}$ on S is *separable* if it satisfies the conditions (*), (**) and (***).

6.2 Proposition. *Let S be a regular projective surface; $C_1, \ldots, C_n$ prime divisors. Then there exists a pencil of curves $\mathcal{C}$ on S with the set of base points B such that*

– *for any i, $B \cap C_i = \emptyset$;*
– *for any i, $\mathcal{C}$ induces a separable morphism $C_i \to \mathbb{P}^1_k$;*
– *$\mathcal{C}$ is separable.*

Proof. From the following two lemmas. $\square$

6.3 Lemma. *In the setting of Prop. 6.2 S admits a finite separable morphism g onto $\mathbb{P}^2_k$ such that $C_1, \dots, C_n$ are not components of its ramification divisor R_g.*

6.4 Lemma. *Let $h : T \to S$ be a finite separable morphism of surfaces, and let $\mathcal{C}$ be a separable pencil on S such that the curves in $\mathcal{C}$ have no common components with the branch divisor B_h. Then the pencil $\mathcal{D} = \mathcal{C} \circ h$ on T is also separable.*

Proof. Let b be any base point of $\mathcal{D}$. Since h is unramified at b, regular local parameters at $h(b)$ generate the maximal ideal of $\mathcal{O}_{T,b}$. In other words, if two curves are regular at $h(b)$ and meet transversally at this point, the same is true for their preimages with respect to the point b. This proves that $\mathcal{D}$ satisfies (**). The condition (*) can be verified by means of a similar argument applied to any regular point on a given curve from $\mathcal{D}$ such that h is unramified at that point.

To verify the condition (***), it is sufficient to note that the Jacobi matrix of h is of rank 2 at any point of T. $\square$

For a separable pencil $\mathcal{C}$ on S introduce a 0-cycle

$$\operatorname{sing} \mathcal{C} = \sum_{P \in S \setminus B} (\operatorname{sing}^t_P \mathcal{C}_P + \operatorname{sing}^w_P \mathcal{C}_P) P.$$

Since all curves in $\mathcal{C}$ belong to the same divisor class, their arithmetic genera are the same; this common value will be denoted by $p_a(\mathcal{C})$.

6.5 Proposition. *Let $\mathcal{C}$ be a separable pencil of curves on S with the set of base points B. Then*

$$\int c_{2,S} = \int \operatorname{sing} \mathcal{C} - \#B - 4(p_a(\mathcal{C}) - 1).$$

Proof. Pick any rational function on the projective line such that its divisor of zeroes as well as its divisor of poles has degree 1. (If we choose the infinite point suitably, any such function is of the form $\lambda \frac{X-\alpha}{X-\beta}$ for some $\alpha, \beta, \lambda \in k$.) Denote by f the inverse image of this function $f \circ \mathcal{C} \in k(S)$. The idea of proof is to apply Theorem 5.2 to df.

The above explicit description of the original function implies that any other choice of such a function would change f into $\frac{af+b}{cf+d}$, where $a, b, c, d \in k$, $ad - bc \neq 0$. It follows that $\langle df \rangle = \sum_{P \in S_0} n_P P$ is independent of this choice.

Now we compute n_P for $P \notin B$. Denote by D any curve in $\mathcal{C}$ which is distinct from $\mathcal{C}_P$. Choose the function f such that its divisor is $\mathcal{C}_P - D$. Then f is a local equation of $\mathcal{C}_P$ at P. If s and t are local parameters at P, then $df = \frac{\partial f}{\partial s} ds + \frac{\partial f}{\partial t} dt$. Since $\mathcal{C}_p$ has no inseparable singularities, by Proposition 3.3 we have $\left(\frac{\partial f}{\partial s} . \frac{\partial f}{\partial t}\right) = \operatorname{sing}_{\widehat{\mathcal{O}_{S,P}}} f < \infty$; thence

$$n_P = \left(\tfrac{\partial f}{\partial s} . \tfrac{\partial f}{\partial t}\right) = \operatorname{sing}_{\widehat{\mathcal{O}_{S,P}}} f,$$

i. e., the 0-cycles $\langle df \rangle$ and $\operatorname{sing} \mathcal{C}$ have equal coefficients at P.

Let now $P \in B$. Choose f so that its divisor is $T - D$, where C and D are curves in $\mathcal{C}$ which meet transversally at P. Then f can be written as g/q, where g and q are local equations of C and D at P. We have

$$df = \frac{g\,dq - q\,dg}{g^2},$$

whence $n_P = 1$.

Thus, we obtained

$$\langle df \rangle = \operatorname{sing} \mathcal{C} + \sum_{P \in B} P. \tag{5}$$

Next, we compute (df). The divisor of f is $C - D$, where C and D are some curves in $\mathcal{C}$; we shall prove that

$$(df) = -2D. \tag{6}$$

Take any $P \notin B$; let s and t be regular local parameters. If $P \in C$, we saw in the calculation of $\langle df \rangle$, that here $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$ are coprime, i. e. $P \notin \operatorname{Supp}(df)$. If $P \in D$, note that $df = f^2 \cdot d(f^{-1})$ and, similarly to the previous case, $P \notin \operatorname{Supp}(d(f^{-1}))$. Finally, if $P \notin C \cup D$, then P belongs to the divisors of zeroes of $f - f(P)$, and this case can be reduced to the case $P \in C$ by replacing f with $f - f(P)$.

Substituting (5) and (6) into the formula from Theorem 5.2, we obtain the following equality in $A_0(S)$:

$$c_{2,S} = \operatorname{sing} \mathcal{C} + \sum_{P \in B} B - 2(C.K_S) - 4(C.C),$$

where C is an arbitrary curve from $\mathcal{C}$. It is clear from Lemma 6.1 that $(C.C) = \sum_{P \in B} B$. Therefore,

$$c_{2,S} = \operatorname{sing} \mathcal{C} - \sum_{P \in B} B - 2(C.K_S) - 2(C.C),$$

Now it is sufficient to calculate the degrees and to apply the definition of arithmetic genus. □

We prove another property of pencils.

6.6 Proposition (generalized Plücker equation). *Let $\mathcal{C}$ be a pencil of curves on S with the set of base points B, D a reduced irreducible curve on S which is not a component of any curve in $\mathcal{C}$, and $D \cap B = \emptyset$. Assume that the restriction φ of the morphism $\mathcal{C}$ on D is a separable morphism of D onto a projective line. Then*

$$\sum_{P \in D} \Big((\mathcal{C}_P.D)_P - \nu_P(D) + \sum_{\pi} d_P \Big) = 2(\mathcal{C}.D) + 2p_g(D) - 2,$$

where d_P is the sum of wild differents of φ at all points of the normalization of D over P.

Proof. Denote by $\lambda : \tilde{D} \to D$ the normalization morphism. Let $P \in D$. For $P' \in \lambda^{-1}(D)$ denote by $e_{P'}$ the ramification index of $\varphi \circ \lambda$ at P'. Corollary 1.1.1 implies

$$\sum_{P' \in \lambda^{-1}(P)} e_{P'} = (\mathcal{C}_P.D)_P.$$

It is easy to see that $\deg(\varphi \circ \lambda) = (\mathcal{C}.D)$, and it remains to apply Riemann-Hurwitz formula to the morphism $\varphi \circ \lambda$. $\square$

7 Morphisms of surfaces

In this section we consider a finite separable morphism of surfaces $h : T \to S$ of degree n. We consider the corresponding ramification divisor

$$R_h = \sum_{\eta \in T_1} l_{\mathcal{O}_{T,\eta}}(\Omega_{T/S,\eta}) \cdot D_\eta,$$

where D_η is a prime divisor divisor with the generic point η, and the branch divisor

$$B_h = f_* R_h.$$

It is known (see, e. g., [Ii]) that in $A_1(T)$ we have

$$K_T = h^* K_S + R_f. \tag{7}$$

A local equation of the ramification divisor can be determined as follows.

7.1 Lemma. *Let u, v be regular local parameters at $Q \in T$; s, t regular local parameters at $h(Q)$. Then* $\begin{vmatrix} \frac{\partial s}{\partial u} & \frac{\partial s}{\partial v} \\ \frac{\partial t}{\partial u} & \frac{\partial t}{\partial v} \end{vmatrix}$ *is a local equation of R_f at Q.*

Proof. In the exact sequence

$$\Omega_{\mathcal{O}_{S,h(Q)}/k} \otimes_{\mathcal{O}_{S,h(Q)}} \mathcal{O}_{T,Q} \to \Omega_{\mathcal{O}_{T,Q}/k} \to \Omega_{\mathcal{O}_{T,Q}/\mathcal{O}_{S,h(Q)}} \to 0$$

the first arrow is a homomorphism of two free $\mathcal{O}_{T,Q}$-modules with bases ds, dt and du, dv respectively; ds and dt are mapped respectively to $\frac{\partial s}{\partial u} du + \frac{\partial s}{\partial v} dv$ and $\frac{\partial t}{\partial u} du + \frac{\partial t}{\partial v} dv$. Localizing with respect to all prime ideals of $\mathcal{O}_{T,Q}$ of height 1, we obtain that for any prime divisor containing Q with generic point η we have

$$l_{\mathcal{O}_{T,\eta}}(\Omega_{T/S,\eta}) = v_\eta\Bigl(\begin{vmatrix} \frac{\partial s}{\partial u} & \frac{\partial s}{\partial v} \\ \frac{\partial t}{\partial u} & \frac{\partial t}{\partial v} \end{vmatrix}\Bigr),$$

where v_η is the valuation of $k(T)$ associated with η. $\square$

Fix a pencil $\mathcal{C}$ as in Prop. 6.2. Denote by $\mathcal{D}$ the pencil of curves $\mathcal{C} \circ h$ on T, and by B and B' the sets of base points of $\mathcal{C}$ and $\mathcal{D}$ respectively.

It is clear that $\#B' = n\#B$. Therefore, Proposition 6.5 implies

$$\chi_T - n\chi_S = \int \operatorname{sing} \mathcal{D} - n \int \operatorname{sing} \mathcal{C} - 2((2p_a(\mathcal{D}) - 2) - n(2p_a(\mathcal{C}) - 2)). \tag{8}$$

7.2 Lemma (Zeuthen). *Let C be a divisor on S, $D = h^* C$. Then*

$$(2p_a(D) - 2) - n(2p_a(C) - 2) = (C.B_h).$$

Proof. Using (7) and the projection formula, we obtain

$$\begin{aligned}2p_a(D)-2 &= (D.D)+(K_T.D)\\ &= n(C.C)+(h^*K_S.D)+(R_h.D)\\ &= n(C.C)+n(K_S.C)+(B_h.C)\\ &= n(2p_a(C)-2)+(B_h.C)\end{aligned}$$

□

By the definition, we have

$$\int \operatorname{sing}\mathcal{D} - n\int \operatorname{sing}\mathcal{C} = \sum_{P\in S\setminus B}\Big(\sum_{Q\in h^{-1}(P)} \operatorname{sing}^t_Q \mathcal{D}_Q - n\operatorname{sing}^t_P \mathcal{C}_P + \sum_{Q\in h^{-1}(P)} \operatorname{sing}^w_Q \mathcal{D}_Q - n\operatorname{sing}^w_P \mathcal{C}_P\Big).$$

For an effective divisor C on S without multiple components and without common components with B_h and a point Q on T, introduce the notation

$$d_Q(C)=\sum_{\pi,\pi'} d\big((\widehat{\mathcal{O}_{T,Q}}/\pi')\,\big|\,(\widehat{\mathcal{O}_{S,h(Q)}}/\pi)\big),$$

where π runs over non-associated prime divisors in $\widehat{\mathcal{O}_{S,h(Q)}}$ of a local equation of C at $h(Q)$, and π' runs over non-associated prime divisors of π in $\widehat{\mathcal{O}_{T,Q}}$.

7.3 Proposition. *Let C be an effective divisor on S without multiple components and without common components with B_h; P any point of* $\operatorname{Supp} C$. *Then we have*

$$\sum_{Q\mapsto P} \operatorname{sing}^t_Q h^*C - n\operatorname{sing}^t_P C = (C.B_h)_P - (n-\#h^{-1}(P)) - \sum_{Q\mapsto P} d_Q(C).$$

Proof. We can immediately reduce Proposition to a similar statement, where S is the spectrum of a complete 2-dimensional regular local ring. Next, it can be reduced to the case, when T is connected, i. e., is also the spectrum of a complete 2-dimensional regular local ring.

Let $C=\sum C_i$, where C_i are prime divisors. For $i\neq j$ we have

$$n(C_i.C_j)_P=\sum_{Q\in h^{-1}(P)}(h^*C_i.h^*C_j)_Q.$$

In view of this formula, Proposition can be immediately reduced to the case when C is a prime divisor and this case is nothing else but Prop. 4.1. □

7.4 Theorem. *Let $B_h=\sum b_iB_i$ be the branch divisor of h. Let $\mathcal{C}$ be any separable pencil on S such that none of B_i is a component of any curve in $\mathcal{C}$. Then*

$$\chi_T - n\chi_S = \sum_i b_i(2p_g(B_i)-2)+\sum_Q \lambda_f(\widehat{\mathcal{O}_{T,Q}}/\widehat{\mathcal{O}_{S,h(Q)}}),\tag{9}$$

where f is a local equation of $\mathcal{C}_{h(Q)}$ at $h(Q)$.

Proof. Proposition 7.3 implies

$$\int \operatorname{sing}\mathcal{D} - n\int \operatorname{sing}\mathcal{C} = \sum_{P\in S\setminus B}\Big((\mathcal{C}_P.B_h)_P - (n-\#h^{-1}(P)) - \sum_{Q\mapsto P} d_Q(\mathcal{C}_P) + \sum_{Q\mapsto P}\operatorname{sing}^w_Q \mathcal{D}_Q - n\operatorname{sing}^w_P \mathcal{C}_P\Big)P.$$

Together with (8) and Lemma 7.2, this implies:

$$\chi_T - n\chi_S = -2(\mathcal{C}.B_h) + \sum_{P\in S\setminus B}\big((\mathcal{C}_P.B_h)_P - (n-\#h^{-1}(P)) - \sum_{Q\mapsto P} d_Q(\mathcal{C}_P) + \sum_{Q\mapsto P}\operatorname{sing}^w_Q \mathcal{D}_Q - n\operatorname{sing}^w_P \mathcal{C}_P\big).$$

Writing B_h as $\sum b_i B_i$, where B_i are prime divisors, we obtain

$$\begin{aligned}\chi_T - n\chi_S = &-2\sum_i b_i(\mathcal{C}.B_i) + \sum_i\sum_{P\in S} b_i((\mathcal{C}_P.B_i)_P - \nu_P(B_i)) \\ &+\sum_P\Big(\sum_i b_i\nu_P(B_i) - (n-\#h^{-1}(P)) - \sum_{Q\mapsto P} d_Q(\mathcal{C}_P) \\ &\qquad + \sum_{Q\mapsto P}\operatorname{sing}^w_Q \mathcal{D}_Q - n\operatorname{sing}^w_P \mathcal{C}_P\Big).\end{aligned}$$

Note that for any i the morphism $\varphi_i : B_i \to \mathbb{P}^1_k$ determined by $\mathcal{C}$ is separable. Indeed, if $P \in B_i$ is any point of B_i, regular on B_h and such that $\mathcal{C}_P$ meets B_h at P transversally, then a local parameter at $\varphi_i(P)$ on $\mathbb{P}^1_k$ is mapped to a local parameter at P on B_i. Thus, we can apply Proposition 6.6, and this gives

$$\begin{aligned}\chi_T - n\chi_S = &\sum_i b_i(2p_g(B_i)-2)+ \\ &+\sum_P\Big(\sum_i\Big(-b_i\sum_{P'} d\big(\widehat{\mathcal{O}_{\widetilde{B_i},P'}}\,|\,\widehat{\mathcal{O}_{\mathbb{P}^1_k,\varphi_i(P)}}\big) + b_i\nu_P(B_i)\Big) \\ &\qquad - (n-\#h^{-1}(P)) - \sum_{Q\mapsto P} d_Q(\mathcal{C}_P) \\ &\qquad + \sum_{Q\mapsto P}\operatorname{sing}^w_Q \mathcal{D}_Q - n\operatorname{sing}^w_P \mathcal{C}_P\Big),\end{aligned}$$

where P' runs over the points over P of the normalization $\widetilde{B_i}$ of B_i. Obviously,

$$n - \#h^{-1}(P) = \sum_{Q\mapsto P}(n_Q - 1),$$

where n_Q is the degree of $\widehat{\mathcal{O}_{T,Q}}$ over $\widehat{\mathcal{O}_{S,P}}$. It remains to rewrite b_i as a sum over $Q \mapsto P$ of similar infinitesimal terms. This completes the proof of Theorem 7.4. □

References

[Br] J.-L. Brylinski, *Théorie du corps de classes de Kato et revêtements abéliens de surfaces*, Ann. Inst. Fourier, Grenoble **33** (1983), 23–38.

[D] P. Deligne, *Letter to L. Illusie of 28.11.76*, to appear in Proceedings of the conference "Ramification theory of arithmetic schemes" (Luminy, 1999).

[F] W. Fulton, *Intersection theory.* Springer-Verlag, Berlin, 1998.

[H] R. Hartshorn, *Algebraic geometry.* Springer-Verlag, New York-Heidelberg, 1977.

[L] G. Laumon, *Semi-continuité du conducteur de Swan (d'après P. Deligne)*, Astérisque 83 (1981), 173–219.

[K] H. Kurke, *Vorlesungen über algebraische Flchen.* BSB B. G. Teubner Verlagsgesellschaft, Leipzig, 1982.

[Ii] S. Iitaka, *Algebraic geometry. An introduction to birational geometry of algebraic varieties.* Springer-Verlag, New York-Berlin, 1982.

[Iv] B. Iversen, *Numerical invariants and multiple plains.* Amer. J. Math. 92 (1970), 968–996.

[Sa] P. Samuel, *Singularités des variétés algébriques*, Bull. Soc. Math. France 79 (1951), 121–129.

[Sh] I. R. Shafarevich, *Basic algebraic geometry.* Springer-Verlag, New York-Heidelberg, 1974.

[Y] H. Yamada, *Rational sections and Chern classes of vector bundles*, J. Math. Univ. Kyoto 6 (1967), no. 2, 295–312.