CURVATURE DEFORMATION

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1. Introduction.

In [H], Hamilton introduced an important evolution equation for Riemannian metrics in his study of three dimensional manifolds with positive Ricci curvature. In the present note we study this equation with emphasis not so much on the metric but more on the connections involved. The evolution equation for the connection is the gradient flow $\dot{\omega} = -\delta^{\omega}\Omega$, where $\Omega = d\omega + [\omega,\omega]$ is the curvature form and $\dot{\omega}$ indicates the infinitesimal change in the connection. The Lagrangean of this flow is the well known Yang-Mills integral $\int|\Omega|^2$. The connection to be used is a Cartan connection of hyperbolic type. The deformation is closely related to our previous work [MR] on non-compact almost symmetric spaces. Even for the study of metrics with positive curvature the hyperbolic model seems to be the appropriate one.

In the next section we give a derivation of Hamilton's equations without explicitly using the notion of Cartan connections. This section also motivates the computations for the deformation equations for Cartan connections which we derive in the last section. While the effect on the deformation of metrics is the same in both approaches the definition of the control function of the process is not. This provides additional flexibility in choosing the quantities to be estimated.

2. Deformation of the Levi-Civita connection.

In this section we reformulate Hamilton's deformation in a different set-up so that the corresponding evolution equations for the Levi-Civita

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connection and the Riemannian curvature can be derived in a natural and easy fashion.

Let (M^n,g) be a compact Riemannian manifold. First of all, we deform metrics not through tensors of type (0;2) directly but by using gauge transformations, i.e., tensors of type (1,1): $\theta: TM \longrightarrow TM$. A metric deformation is therefore a curve g_t of metrics defined by:

(2.1) $$g_t(X,Y) = g(\theta_t X, \theta_t Y),$$
where θ_t is a 1-parameter family of invertible maps
$\theta_t: TM \longrightarrow TM$ with $\theta_o = id$.

In order to be able to calculate the infinitesimal changes in the Levi-Civita connection and the curvature tensor caused by such a metric deformation in an efficient manner, we introduce the bundle $Aff(M) = TM \oplus TM^* \otimes TM$.

An infinitesimal gauge transformation $\dot{\theta} = \frac{d}{dt}\theta_t\Big|_{t=o}$ can now be considered as a 1-form with values in TM or sometimes as a o-form with values in $TM^* \otimes TM$. An infinitesimal change in the connection is a 1-form with values in $TM^* \otimes TM$ and we interpret curvature as a 2-form with values in $TM^* \otimes TM$.

The Levi-Civita connection ∇ of the metric g induces a natural direct sum connection in Aff(M) and we denote the corresponding exterior covariant derivative on p-forms with values in Aff(M) by d^∇.

(2.2) $$(d^\nabla\alpha)(X_o \ldots X_p) = \sum_{i=o}^{p} (-1)^i \nabla_{X_i} (\alpha(\ldots \hat{X}_i \ldots)) + \sum_{i<j} (-1)^{i+j} \alpha([X_i,X_j], \ldots \hat{X}_i \ldots \hat{X}_j \ldots) .$$

We also introduce an algebraic operator d_2 acting on a p-form α by the formula:

(2.3) $$(d_2\alpha)(X_o \ldots X_p) = \sum_{i=o}^{p} (-1)^i [X_i, \alpha(\ldots \hat{X}_i \ldots)],$$
where the bracket [,] is defined to be:

$[X,A] = - AX \in TM$ for $X \in TM$, $A \in TM^* \otimes TM$
$[X,Y] = X \wedge Y \in TM^* \otimes TM$ for $X,Y \in TM$,

where $X \wedge Y$ is the map $Z \longmapsto g(Z,Y)X - g(Z,X)Y$.

For $A,B \in TM^* \otimes TM$, we have of course the usual definition for [A,B].

If R is the Riemannian curvature tensor of g, interpreted as an Aff(M)-valued 2-form, then the Bianchi identities can be stated as follows.

Lemma 1. $d_2 R = o$ (1^{st} Bianchi identity)

$d^\nabla R = o$ (2^{nd} Bianchi identity)

We will make use of the adjoint operator of d^∇:

(2.4) $$(\delta^\nabla \alpha)(X_2 \ldots X_p) = - \sum_{k=1}^{n} (\nabla_{e_k} \alpha)(e_k, X_2 \ldots X_p),$$

where $\{e_k\}$ is an orthonormal basis for the metric g.

We also define an algebraic operator δ_2 by the formula:

(2.5) $$(\delta_2 \alpha)(X_2 \ldots X_p) = \sum_{k=1}^{n} [e_k, \alpha(e_k, X_2 \ldots X_p)]$$

The Ricci tensor, interpreted as a TM-valued 1-form can now be defined as $\delta_2 R$. Using this notation, we derive a consequence of the Bianchi identities which is fundamental for our deformation equations.

Lemma 2. $d^\nabla \delta_2 R + d_2 \delta^\nabla R = 0$.

Proof. $$\begin{aligned} d^\nabla \delta_2 R(X,Y) &= (\nabla_X \delta_2 R)(Y) - (\nabla_Y \delta_2 R)(X) \\ &= (\nabla_X R)(Y, e_k, e_k) - (\nabla_Y R)(X, e_k, e_k) \\ &= (\nabla_{e_k} R)(Y, X, e_k), \end{aligned}$$

where we sum over k from 1 to n and used the 2^{nd} Bianchi identity. On the other hand,

$$\begin{aligned} d_2 \delta^\nabla R(X,Y) &= - (\nabla_{e_k} R)(e_k, X, Y) + (\nabla_{e_k} R)(e_k, Y, X) \\ &= (\nabla_{e_k} R)(X, e_k, Y) + (\nabla_{e_k} R)(e_k, Y, X) \\ &= - (\nabla_{e_k} R)(Y, X, e_k), \end{aligned}$$

where the 1^{st} Bianchi identity is used. The sum of the two terms is zero.

Let $\dot\gamma$ be the infinitesimal change in the Levi-Civita connection caused by an infinitesimal gauge transformation $\dot\theta$: TM $\longrightarrow$ TM, which we assume from now on, without loss of generality, to be symmetric with respect to g. $\dot\gamma$ is a 1-form with values in TM* ⊗ TM and can be decomposed as:

(2.6) $$\dot\gamma = \dot\eta + \dot\sigma,$$

where $\dot\eta$ has values in the skew-symmetric, and $\dot\sigma$ in the symmetric component of TM* ⊗ TM.

Differentiating the condition $\nabla g = 0$, we get $\dot{\gamma} \cdot g + \nabla\dot{g} = 0$. It is easy to see that $\dot{\gamma} \cdot g = -2\dot{\sigma}$ and that $\nabla\dot{g} = 2\nabla\theta$, where θ is considered as a o-form with values in the symmetric part of TM* ⊗ TM. Hence $\dot{\sigma}$ is given by

(2.7) $\dot{\sigma} = \nabla\dot{\theta}$.

Differentiating the condition that the Levi-Civita connection is torsion-free, we obtain:

$\dot{\gamma}(X)Y - \dot{\gamma}(Y)X = 0$, which we write as

$d_2\dot{\gamma} = d_2\dot{\eta} + d_2\dot{\sigma} = d_2\dot{\eta} + d_2\nabla\dot{\theta} = 0.$

Now, $d_2\nabla\dot{\theta} = d^\nabla\dot{\theta}$, where $\dot{\theta}$ on the left is viewed as a section in TM* ⊗ TM and on the right as a 1-form with values in TM. Moreover, d_2 resticted to 1-forms with values in the skew symmetric endomorphisms is well know to be an isomorphism onto the 2-forms with values in TM. Incidentally, this fact is responsible for the uniqueness of the Levi-Civita connection among metric connections. Hence $\dot{\eta}$ is determined uniquely by the equation:

(2.8) $d_2\dot{\eta} + d^\nabla\dot{\theta} = 0$, and the change in the Levi-Civita connection is given by

(2.9) $\dot{\gamma} = \dot{\eta} + \nabla\dot{\theta}$.

By Lemma 2, the relation (2.8) is satisfied if we set

(2.10) $\dot{\theta} = -\delta_2 R$, $\dot{\eta} = -\delta^\nabla R$, and we have proved the following result.

<u>Lemma 3</u>. The infinitesimal change in the Levi-Civita connection caused by the infinitesimal gauge transformation $\dot{\theta} = -\delta_2 R$ is given by

(2.11) $\dot{\gamma} = -\delta^\nabla R - \nabla\delta_2 R$.

The corresponding metric deformation is computed to be $\dot{g}(X,Y) = g(\dot{\theta}X,Y) + g(X,\dot{\theta}Y) = -2\,\mathrm{Ric}(X,Y)$, which is exactly Hamilton's deformation without the normalizing term. Hamilton [H] proved that the deformation exists at least for a short time and the above infinitesimal computations are not just formal. Introduction of a normalization

(2.12) $\dot{\theta} = -\delta_2 R + c\ \mathrm{id}$, c any constant, does not alter the formula $\dot{\gamma} = -\delta^\nabla R - \nabla\delta_2 R$.

The infinitesimal change of the curvature tensor, considered as a 2-form with values in $TM^* \otimes TM$ is then given by (compare also [M]):

<u>Lemma 4</u>.

$$(2.13) \qquad \dot{R} = - d^\nabla \delta^\nabla R - R.\delta_2 R = - \Delta^\nabla R - R.\delta_2 R, \text{ where } \Delta^\nabla = d^\nabla\delta^\nabla + \delta^\nabla d^\nabla$$

is the Laplacian and $(R.\delta_2 R)(X,Y) = [R(X,Y), \delta_2 R] \in TM^* \otimes TM$.

<u>Proof</u>. $\dot{R} = d^\nabla \dot{\gamma} = - d^\nabla \delta^\nabla R - d^\nabla \nabla \delta_2 R$
$= - \Delta^\nabla R - d^\nabla \nabla \delta_2 R$, (2^{nd} Bianchi id.)

and $d^\nabla \nabla$ applied to a zero-form is by definition of curvature, the curvature applied to this zero-form. In our notation, $(d^\nabla \nabla \delta_2 R)(X,Y) = [R(X,Y), \delta_2 R]$.

In order to compare (2.11) to Hamilton's formula [H, Theorem 7.1] for the evolution of the curvature, we need to recall first the following Weitzenböck formula for the Laplacian $\Delta^\nabla R$. (compare eg. [B])

$$(2.14) \qquad (\Delta^\nabla R)(X,Y) = (\bar{\Delta} R)(X,Y) + R(\mathrm{Ric}\, X,Y) + R(X,\mathrm{Ric}\, Y) - R(R(X,Y)) + 2\sum_{p=1}^{n} [R(e_p,X), R(e_p,Y)],$$

where $\bar{\Delta}$ is the rough Laplacian and $\{e_p\}$ is an orthonormal basis.

Using index notation, we can write the last two terms of the algebraic expression on the right hand side of (2.14) as:

$$(2.15) \qquad R_{ij}{}^{pq} R_{pqk}{}^{l} + 2 R_{piq}{}^{l} R_{pjk}{}^{q} - 2 R_{pjq}{}^{l} R_{pik}{}^{q} ,$$ where $R_{ijkl} = g(R(e_i,e_j)e_k,e_l)$ and $R_i{}^j$ denotes the Ricci tensor.

As in [H] we define $B_{ijkl} = R_{piqj} R_{pkql}$, where an orthonormal frame is used. Now,

$$\begin{aligned} B_{ijkl} - B_{ijlk} &= R_{piqj}(R_{pkql} - R_{plqk}) \\ &= R_{piqj} R_{pqkl} = (- R_{pqji} - R_{pjiq}) R_{pqkl} \\ &= R_{ijpq} R_{pqkl} - R_{piqj} R_{pqkl}, \text{ and hence} \end{aligned}$$

$$B_{ijkl} - B_{ijlk} \doteq R_{piqj} R_{pqkl} = \frac{1}{2} R_{ijpq} R_{pqkl} .$$

By definition, we have

$$\begin{aligned} B_{ijkl} - B_{iljk} &= R_{piqk} R_{pjql} - R_{piql} R_{pjqk} \\ &= - R_{pjql} R_{pikq} + R_{piql} R_{pjkq} , \end{aligned}$$

and we accounted for all the terms involving the whole curvature tensor in Hamilton's formula and in our expression (2.15).

The four terms containing the Ricci curvature in Hamilton's formula can be written as follows:

$$R(\mathrm{Ric}\,X,Y)Z + R(X,\mathrm{Ric}\,Y)Z + R(X,Y)\mathrm{Ric}\,Z + \mathrm{Ric}\,(R(X,Y)Z)\,.$$

The first two terms coincide with the corresponding terms in (2.14), but instead of the last two terms our formula (2.13) gives us $R(X,Y)\,\mathrm{Ric}\,Z - \mathrm{Ric}\,(R(X,Y)Z)$. The difference $2\,\mathrm{Ric}\,(R(X,Y)Z)$ is due to the fact that Hamilton's equation is for the curvature of type (0,4) and Lemma 4 treats the curvature as a tensor of type (1,3). We have

$$\dot{R}_{ijkl} = g_{ml}\,\dot{R}_{ijk}{}^{m} + \dot{g}_{ml}\,R_{ijk}{}^{m}\,, \quad \text{and}$$
$$\dot{g} = -\,2\,\mathrm{Ric}\,.$$

This proves Hamilton's Theorem 7.1:

$$\dot{R}_{ijkl} + (\bar{\Delta}R)_{ijkl} = 2(B_{ijkl} - B_{ijlk} + B_{ikjl} - B_{iljk}) - R_{pi}\,R_{pjkl} - R_{pj}\,R_{ipkl} - R_{pk}\,R_{ijpl} - R_{pl}\,R_{ijkp}\,.$$

3. Deformation of Cartan connections.

Let (M,g) as before, denote a compact Riemannian manifold. In this section we identify $X \wedge Y \in TM \wedge TM$ with the skew-symmetric endomorphism $Z \longrightarrow g(Z,Y)X - g(Z,X)Y$, and define $E = TM \oplus TM \wedge TM$, a vector bundle over M with fibre metric $\langle\,,\rangle$ defined by g. For skew-symmetric maps $A,B\colon TM \longrightarrow TM$ we have $\langle A, X\wedge Y\rangle = -\langle AX,Y\rangle$ and $\langle A,B\rangle = -\frac{1}{2}\,\mathrm{tr}\,AB$.

Let ∇ be a metric connection for TM, not necessarily torsion-free. A gauge transformation θ is simply an invertible map $\theta\colon TM \longrightarrow TM$. The image, $\mathrm{im}\,\theta$, should be viewed as the subspace $TM \subset E$. We define a Cartan connection on E by:

$$D_X Y = \nabla_X Y + \theta X \wedge Y$$
$$D_X A = -\,A\theta X + \nabla_X Y, \qquad X,Y \in TM, \quad A \in TM \wedge TM\,.$$

D defines a Cartan connection of <u>hyperbolic type</u> for M. Note that the metric is not invariant under D. To simplify the above formula we define the structure of a Lie algebra, isomorphic to the Lie algebra of the isometry group of hyperbolic space $o(n,1)$, on the fibres by

$$[(X,A),(Y,B)] = (AY - BX, [A,B] + X \wedge Y)\,.$$

D leaves this bracket invariant and is expressed as follows:

(3.1) $D_X s = \nabla_X s + [\theta X, s]$, s a section in E.

Let R^D denote the curvature tensor of the connection D, i.e., $R^D(X,Y)s = (D_X D_Y - D_Y D_X - D_{[X,Y]})s$. We define the Cartan curvature form Ω by the formula

(3.2) $[\Omega(X,Y),s] = R^D(X,Y)s$.

It will be convenient to split the curvature tensor R^D, and accordingly Ω, into the components $TM \wedge TM$ and TM of E. We write

$$R^D(X,Y) = R_1(X,Y) + R_2(X,Y),$$
$$\Omega(X,Y) = \Omega_1(X,Y) + \Omega_2(X,Y),$$

where $R_1(X,Y) = R^\nabla(X,Y) + \theta X \wedge \theta Y$ with R^∇ the curvature of the connection ∇, and $R_2 = T^{(\nabla,\theta)}$, with $T^{(\nabla,\theta)}(X,Y) = \nabla_X(\theta Y) - \nabla_Y(\theta X) - \theta[X,Y]$ the Cartan torsion.

The connection D defines an exterior covariant derivative for E-valued p-forms on M.

(3.3) $$(d^D\alpha)(X_o \dots X_p) = \sum_{i=o}^{p} (-1)^i D_{X_i}(\alpha(\dots \hat{X}_i \dots)) + \sum_{i<j} (-1)^{i+j} \alpha([X_i,X_j], \dots \hat{X}_i \dots \hat{X}_j \dots),$$

which we also write as $d^D\alpha = d_1\alpha + d_2\alpha$, where

$$(d_2\alpha)(X_o \dots X_p) = \sum_{i=o}^{p} (-1)^i [\theta X_i, \alpha(\dots \hat{X}_i \dots)].$$

In this notation the Bianchi identities take the form $d^D\Omega = 0$. Corresponding to Lemma 1 of the previous section we have

<u>Lemma 5</u>. If $\Omega_2 = 0$, then $d_1\Omega = 0$, and $d_2\Omega = 0$.

The purpose of this section is to study deformations of metrics on M via deformations of Cartan connections. We will start with ∇^o equal to the Levi-Civita connection of the initial metric $g_o = g$. In the course of the deformation the metric g on M of course will change but we will keep the fixed metric <,> as well as the fixed Lie algebra bracket defined by g_o in the fibres. A 1-parameter family $D = D^t$ of Cartan connections determines a family (∇^t, θ_t) of connections and gauge transformations on TM. D_t and (∇^t, θ_t) are related by the formula $D^t_X s = \nabla^t_X s + [\theta_t X, s]$ of (3.1). The changing metric g_t on M is related to the fixed metric <,> on E by the formula $g_t = \theta_t^* g_o$, where $(\theta_t^* g_o)(X,Y) = g_o(\theta_t X, \theta_t Y)$ as in (2.1).

We will mainly be interested in deformations of Cartan connections with the property $\nabla^t g_o = 0$, and $T^{(\nabla^t,\theta_t)} = 0$ at all times t. The reason is the following

Lemma 6. If $\nabla g_o = 0$, and $T^{(\nabla,\theta)} = 0$, then θ-gauge transform $\theta^{-1}\nabla\theta$ is the Levi-Civita connection of the metric $g = \theta^* g_o$.

So, a metric deformation is equivalent to a deformation of Cartan connections with vanishing Cartan torsion on the vector bundle $E = TM \oplus TM \wedge TM$ with fiexed metric g_o and fixed Lie algebra structure $o(n,1)$ on the fibres. The curvature R of the Levi-Civita connection of Lemma 6, of course, is $R(X,Y)Z = \theta^{-1}(R^\nabla(X,Y)\theta Z)$.

In order to define a suitable deformation of Cartan connections we introduce the adjoint δ^D of the exterior derivative d^D, defined by $D = D_t$, with respect to the variable metric $g = g_t$ on the base, and the fixed metric $\langle , \rangle$ (defined by g_o) on the fibres of E, compare (2.4).

$$(3.4) \quad (\delta^D\alpha)(X_2 \ldots X_p) = -\sum_{k=1}^{n} (\nabla_{e_k}\alpha)(e_k, X_2 \ldots X_p) + \sum_{k=1}^{n} [\theta e_k, \alpha(e_k, X_2 \ldots X_p)],$$

where $\{e_k\}$ is an orthonormal basis with respect to the variable metric $g = g_t$. We also write $\delta^D\alpha = \delta_1\alpha + \delta_2\alpha$, where

$$(3.5) \quad (\delta_2\alpha)(X_2 \ldots X_p) = \sum_{k=1}^{n} [\theta e_k, \alpha(e_k, X_2 \ldots X_p)].$$

This explains the definition (2.5). Note that the positive sign occurs because we are using the non-compact Lie algebra $o(n,1)$ as typical fibre in E. For spherical Cartan connections the sign would be negative.

We consider the evolution equation

$$(3.6) \quad \dot{\omega} = -\delta^D\Omega,$$

where $\dot{\omega}$ is an E-valued 1-form on M and defines an infinitesimal deformation of the Cartan connection D by $\dot{D}_X s = [\dot{\omega}(X), s]$.

The equation (3.6) is not just formal. The integrability condition $\delta^D(\delta^D\Omega) = 0$ makes it parabolic and by [H, Theorem 5.1] the equation has a solution for $o \leq t < \varepsilon$ and some $\varepsilon > o$. The equation (3.6) yields the following evolution equation for the Riemannian metric $g = g_t$ on M.

(3.7) $\dot{g} = -2\,\mathrm{Ric}(g) - 2(n-1)g$, which coincides with (2.12) and is Hamilton's equation except for a normalization. The purpose of section 2 was to study this evolution emphasizing the evolution of the Levi-Civita connection. In the present section we stay with Cartan connections.

The evolution equation for the Cartan curvature form Ω is

(3.8) $\dot{\Omega} = -\Delta^D\Omega.$

Next we prove that the evolution (3.6) is tangent to the space of Cartan connections with vanishing Cartan torsion, see definition (3.2). The following result corresponds to (2.8).

Lemma 7. Let $\dot{\omega} = \dot{\eta} + \dot{\theta}$ denote the splitting of the infinitesimal connection form $\dot{\omega}$ of (3.6) into $TM \wedge TM$ and TM components respectively. If the Cartan torsion vanishes, then $d_2\dot{\eta} + d_1\dot{\theta} = 0$.

Proof. Vanishing Cartan torsion means $\Omega_2 = 0$ and Lemma 5 applies. Lemma 6 applies also, i.e., the Levi-Civita connection is the gauge transform by θ of the connection defined by D. Since we are working with the Levi-Civita connection on the base M we can choose vector fields X, Y, Z, e_k in TM which at a given point have vanishing covariant derivative and vanishing (vector fields) bracket. Since the connection ∇ in the fibre is related by the gauge transformation θ to the Levi-Civita connection on the base M, the sections θX, θY, θZ, θe_k in $TM \subset E$ have vanishing covariant derivative with respect to ∇ as well. This simplifies the following computation. Now, $d_1\dot{\theta} = -d_1\delta_2\Omega$, and $d_2\dot{\eta} = -d_2\delta_1\Omega$, and

$$
\begin{aligned}
-(d_1\delta_2\Omega)(X,Y) &= -\nabla_X[\theta e_k, \Omega(e_k,Y)] + \nabla_Y[\theta e_k, \Omega(e_k,X)] \\
&= -[\theta e_k, \nabla_X\Omega(e_k,Y)] + [\theta e_k, \nabla_Y\Omega(e_k,X)] \\
&= -[\theta e_k, \nabla_{e_k}\Omega(X,Y)], \text{ since } d_1\Omega = 0.
\end{aligned}
$$

$$
\begin{aligned}
-(d_2\delta_1\Omega)(X,Y) &= [\theta X, \nabla_{e_k}\Omega(e_k,Y)] - [\theta Y, \nabla_{e_k}\Omega)(e_k,X)] \\
&= [\theta e_k, \nabla_{e_k}\Omega(X,Y)], \quad \text{since } d_2\Omega = 0.
\end{aligned}
$$

The two terms add up to zero.

The following Lemma states that the evolution defined by (3.6) is tangent to the space of Cartan connections with vanishing Cartan torsion.

<u>Lemma 8</u>. Let ω evolve according to (3.6). If $\Omega_2 = 0$ for a given time t, then $\dot{\Omega}_2 = 0$ at that time.

<u>Proof</u>. $\Omega_2(X,Y) = \nabla_X(\theta Y) - \nabla_Y(\theta X) - \theta[X,Y]$.

$$\dot{\Omega}_2(X,Y) = \dot{\eta}(X)\theta Y + \nabla_X(\dot{\theta}Y) - \dot{\eta}(Y)\theta X - \nabla_Y(\dot{\theta}X) - \dot{\theta}[X,Y]$$

$$= d_2\dot{\eta}(X,Y) + d_1\dot{\theta}(X,Y) = 0 \quad \text{by Lemma 7.}$$

Since the original Cartan connection $D = D^0$ is constructed from the Levi-Civita connection of the original metric g_o, the Cartan torsion remains zero at all times. For this reason, the Laplacian $\Delta^D = d^D\delta^D + \delta^D d^D$ can be conveniently written as the sum of two non-negative operators as the following Lemma states.

<u>Lemma 9</u>. If the Cartan torsion Ω_2 vanishes, then $\Delta^D = \Delta_1 + \Delta_2$, where $\Delta_1 = d_1\delta_1 + \delta_1 d_1$, and $\Delta_2 = d_2\delta_2 + \delta_2 d_2$.

<u>Proof</u>. To simplify the computation we choose vector fields as in the proof of Lemma 7. We prove it for a $TM \wedge TM$-valued 2-form α only to keep things simple. This is the case we need anyway. We have to prove: $(d_1\delta_2 + d_2\delta_1 + \delta_1 d_2 + \delta_2 d_1)\alpha = 0$.

$$(d_1\delta_2\alpha)(X,Y) = \nabla_X[\theta e_k,\alpha(e_k,Y)] - \nabla_Y[\theta e_k,\alpha(e_k,X)]$$

$$(d_2\delta_1\alpha)(X,y) = -[\theta X,\nabla_{e_k}\alpha(e_k,Y)] + [\theta Y,\nabla_{e_k}\alpha(e_k,X)]$$

$$(\delta_1 d_2\alpha)(X,Y) = -\nabla_{e_k}[\theta e_k,\alpha(X,Y)] - \nabla_{e_k}[\theta X,\alpha(Y,e_k)] - \nabla_{e_k}[\theta Y,\alpha(e_k,X)]$$

$$(\delta_2 d_1\alpha)(X,Y) = [\theta e_k,\nabla_{e_k}\alpha(X,Y)] + [\theta e_k,\nabla_X\alpha(Y,e_k)] + [\theta e_k,\nabla_Y\alpha(e_k,X)]$$

Each summand occurs twice with opposite signs if we take the choice of the vector fields into consideration.

The main result of this section in the following simple form of the evolution equation for the Cartan curvature.

<u>Theorem</u>. Assume that the family of Cartan connections $D = D^t$ evolves according to $\dot{\omega} = -\delta^D\Omega$. Then, the Cartan curvature satisfies the parabolic equation

$$\dot{\Omega} = -\Delta^D\Omega.$$

If in addition the initial Cartan connection is torsion free, i.e., $\Omega_2 = 0$ initially, then the torsion remains zero at all times and we have $\dot{\Omega}_1 = -\Delta_1\Omega_1 - \Delta_2\Omega_1$.

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