THE RIGIDITY OF SPHERE PACKINGS

by

Robert Connelly

Department of Mathematics
Cornell University
Ithaca, N.Y., 14853
U.S.A.

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26
D-5300 Bonn 3

SFB/MPI 84-30

1. INTRODUCTION

Imagine a box filled with ball bearings. How does one arrange the position of these "spheres" to obtain the greatest possible density? What properties do the positions of the centers have? If the density is at least locally maximal (i.e. a small change in the positions does not allow us to shrink the box), are the ball bearings rigidly held in place?

We address problems such as the above. The goal first is to clarify what the problems are and then to provide some tools for handling them. For example in Tarnai and Gàspàr [25] they consider the problem of finding the most dense circle packing of the 2-sphere for n circles for a few (mostly small) values of n. They found that the most dense positions had to be "Danzer rigid" (to be explained later), for if not, they could be improved. Here we clarify what the relevant notions are and put the problem in the context of what has already been done in the rigidity of frameworks.

Our results are of two types. First the local analysis of the problem and the linearization of it leads to the notion of infinitessimal rigidity. Roughly speaking when the ambient space is of constant non-positive curvature and the container is "concave", then we provide a guarantee that infintessimal rigidity must occur for a locally maximally dense packing. Ironically this does not include the case of circles in a sphere. However, one would "expect" infinitessimal rigidity in all but a few pathological cases. We have shown that no such case exists for some situations.

Second a global analyis of the rigid configurations shows that the graph of the packing has at most one rigid realization under the same conditions as above. The graph of a packing is obtained by regarding the centers of the spheres as the vertices and putting an edge between two vertices when the corresponding spheres intersect. In principle this global information should be useful for determining maximal densities.

Danzer [11] observed that if a circle packing in the sphere was maximally dense, then the graph of the packing was usually rigid. However, it was apparant that something a bit more than the rigidity of such "rod" framework was needed. Thus Tarnai and Gáspár [25] called such graphs "Danzer rigid". It seems clear that what is needed is the notion of a "strut" framework, roughly graphs that are not allowed to move as to decrease their edge lengths. (Rods must stay the same length).

Also related to these ideas is L. Fejes Toth's definition of stability and solidity of a packing. He defined a packing as stable if each sphere is fixed by its neighbors. The problem here is that not enough of the packing is allowed to move. We provide several alternate notions of stability. Roughly, they say that a packing is stable if the whole packing, or at most a finite part of it, cannot move away from the rest, fixing some boundary or is in a container. Naturally this is clearly related to the rigidity of the graph (with struts). A packing is solid if no finite subset of the disks can be rearranged with the rest of the disks so as to form a packing not congruent to the original. Our notion for a packing being finitely stable may be regarded as being "locally" solid.

We prefer to use the word disk for the set that is all the points that are within a certain distance from some point, its center; we call a sphere, of one dimension lower, the boundary of the disk. It is more precise to use packings of disks rather than spheres.

In section 2 we provide most of the basic defintions and background.

In section 3 we use the definitons by themselves to show that a general packing is "made up" of stable packings.

In section 4 we discuss the basic rigidity theorems in the context of general frameworks.

In section 5 we apply these results to packings and show several examples and calculations.

In section 6 we discuss conjectures and related questions.

The author thanks, I.H.E.S. at Bures-sur-Yvette, The University of Dijon, Savoie University at Chambery, in France, the Max-Planck-Institute für Mathematik at Bonn, West Germany, and the N.S.F., grant number MCS-790251, U.S.A. all for theirkind support and encouragement. This paper is dedicated to Gail and Abram for their patience and understanding.

2. DEFINITIONS

2.1 Packings

In order to define a packing we start with an ambient space X, a Riemaniann manifold, of constant (sectional) curvature, complete as a metric space. In X a <u>disk of radius</u> r about a point p, the center, is

$$D_r(p) = \{q \mid d(p,q) \leq r\} ,$$

where $D_r(p)$ is topologically homeomorphic to the standard $D_1(0)$ for the case when X is euclidean space. $d(p,q)$ is the Riemannian distance from p to q. The <u>interior</u> of $D_r(p)$ is the topological interior, where the inequality above is strict. A <u>packing</u> P is a collection of disks with pairwise disjoint interiors. Unless otherwise stated, all of our packings will have disks of the same radius $r(P)$.

Let K be some compact subset of X. Of course if X is compact we can take K to be all of X. For any packing P of X we consider the disks that are contained in K. Let

$$D_K = \frac{\text{volume of the disks of } P \text{ in } K}{\text{volume of } K}$$

be the density of P in K. Then the <u>density</u> of P in X, if it exists, is the limit

$$D = \lim_{K_n} D_{K_n} ,$$

where $K_1 \subset K_2 \subset \ldots$ and $\bigcup_{n=1}^{\infty} K_n = X$. See Rogers [20] .
Note that when X = hyperbolic space, then D may very well depend on the choice of the sequence of K_n's. See Böröczky [5] for instance.

2.2. Locally maximally dense packings.

One thinks of K as a container where one puts in a finite number of disks. Of one "shrinks" the container keeping each disk with its same radius, then allowing the disks to move and bump into each other ,as well as the boundary of K , one would expect D_K to increase untill finally it is "maximal".

We, however, will adopt a dual point of view. We regard the container as fixed and consider what happens as the radius $r(P)$ is increased. For $X = K = S^2$, the 2-dimensional sphere, this is the idea of the "heating" algorithms of Tarnai and Gáspár [25]. See also Bernal [3].

We say P is locally maximally dense in K if there is an $\epsilon > 0$ such that for every corresponding packing Q , where each (center of a) disk of Q is within ϵ of the (center of the) corresponding disk of P, then $r(Q) \leq r(P)$. In other words, for packings close enough to P , the packing density cannot be improved.

2.3. Stability of Packings

L. Fejes Tóth in [12]p 47 calls a packing stable if each disk is fixed by its neighbours. We wish to change this definition slightly and generalize it. We say P is 1-<u>unstable</u> if there is a disk $D_r(p)$ of P such that for every $\epsilon > 0$ there is a new position $D_r(q)$, $d(p,q) < \epsilon$, where $D_r(q)$ is completely disjoint from the rest of P. If P is not 1-unstable, it is called 1-<u>stable</u>. In other words, even if some disk can move fixing its neighbours, it is still 1-stable (but not stable in Feges-Toth's sense) if it must intersect them. In a sense, however, the container is the complement of all but one of the given disks of P. This is not to be confused with other notions called n-stability defined by L. Fejes Tóth in [15] .

Consider the following example where $X = S^2$, the unit 2-dimensional sphere. Here 1-stability and Fejes Tóth stability will not be the same. P consists of 5 equal disks of radius $\pi/4$, where the centers are placed at 5 of the 6 vertices of the inscribed regular octahedron . Here any one of 4 of 5 disks can slide between two others loosing contact with a third. This packing is also the densest packing of 5 disks in S^2.

In general for S^n, the unit n-dimensional sphere, we can find a similar packing with $n + 3$ disks of radius $\pi/4$. It is easy to see that for S^n the disks must have radius $\pi/4$ for this type of behavior.

For our first extension of the idea of stability, we say P is <u>finitely unstable</u> if for some non-zero finite subset

Q of the disks of P for every $\epsilon > 0$ there is a new position for Q, ϵ near to the origonal position in P, such that Q is completely disjoint from the rest of P. If P is not finitely unstable it is called finitely stable.

Thus if P finitely stable, it is 1-stable.But the converse is not always true. Consider the following packing in the plane $\mathbb{R}^2$.

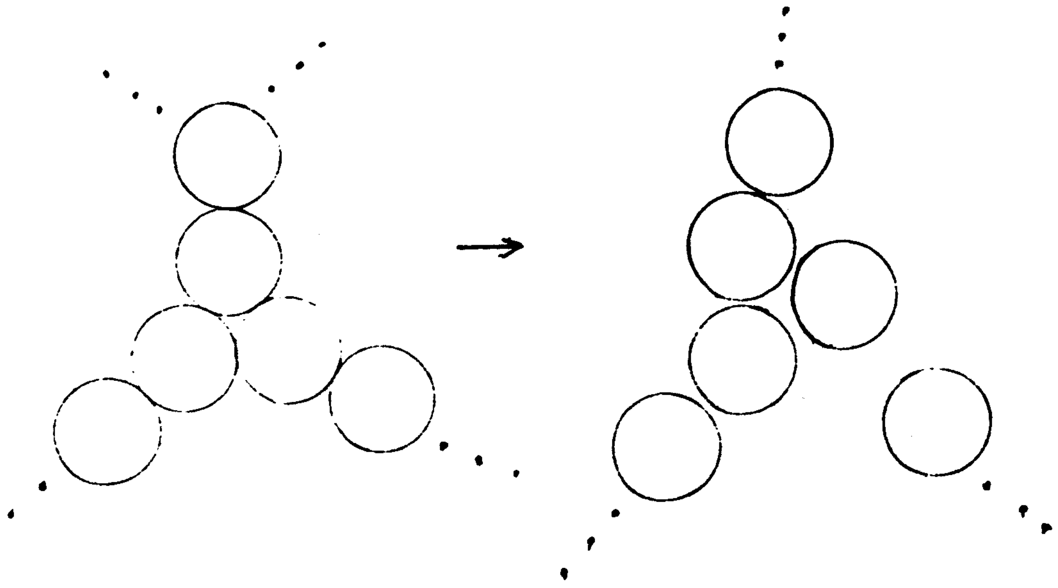


Figure 2.1

A group of 3 disks can rotate and then expand slightly. This shows that the packing is not finitely stable. But it is clearly Fejes Toth stable and in the plane that is equivalent to being 1-stable.

A packing being finitely stable is closely related to what L. Fejes Tóth in [68] calls being "solid". A packing P is solid if no finite subset can be rearranged to give, with the rest of P, another packing not congruent to P. If P is solid, it is certainly finitely stable, but not conversely. In [14] and [16] L. Fejes Tóth and and G. Fejes Tóth found many examples of solid packings. In the following we shall see many examples of 1-stable packings that are not solid. See Figure 5.5 or any of the finitely stable packings of Figure 5.2 except the first maximally dense triangular lattice example. In the plane if P is not maximally dense but still has a density, P can be rearangend into a more dense triangular packing and put back into the same space more densely. See L. Fejes Tóth's comments in [13] .

As an even further extension, we consider the case when we have a container K. Suppose P is a packing of K. If P

has a single element we say P is unstable in K if it can be moved away from the boundary of K by a small motion. If P has more than one element we say P is unstable in K if there is some non-empty subpacking Q of P $Q \neq P$ and a small motio of P in K such that Q is disjoint from the rest of P after the motion. If P is not unstable in K, we say P is stable in K. Note that if P is finitely unstable it is unstable in K for an appropriate chosen K, a large bounded subset of the complement of the interior of all but finite number disks of P.

The following is an example of an unstable packing of the triangle.

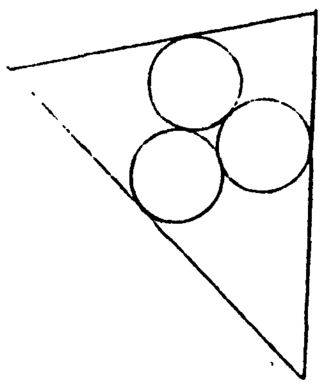


Figure 2.2

Later our results will deal with only packings in a container K, since this case "includes" the previous types of stability.

2.4. The graph of a packing

To each packing P we associate a graph G_P as follows. The disks of P correspond to vertices of G_P and we have an edge between two vertices if the corresponding disks intersect. Let $p_1,\dots,p_k$ be the centers of the corresponding disks. Then we regard $G_P(p)$ as a realization of G_P, where $p=(p_1,\dots,p_k)$ is as in the case of the rigidity of graphs, see Asimow and Roth [1], [2], or [21],or Connelly[10] . $G_P(p)$ is called a framework.

Suppose P is in a container K . We say K is concave if K is the complement in X of a finite number of disks and, in the case X is flat, the compliment of a finite number of half spaces. We shall almost always assume K is concave.

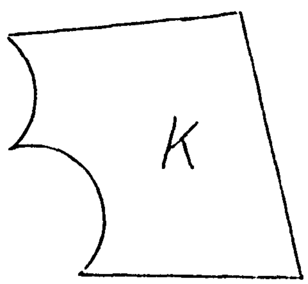


Figure 2.3.

We regard each portion of the boundary of K as corresponding to another fixed vertex of G_P . The vertices corresponding to the disks of P we call the variable vertices of G_P . If some portion of the boundary of K corresponds to the boundary of $D_r(p_j)$, we regard p_j as the realization of the fixed vertex of G_P . If that part of the boundary is a hyperplane, then we say the realization of this vertex is <u>at infinity</u>. Again intersections determine the edges of G_p .

The example in the picture below we denote a variable vertex by o , fixed vertices by ●, edges connected to vertices at infinity by ═══> , and the other edges by ═══ .

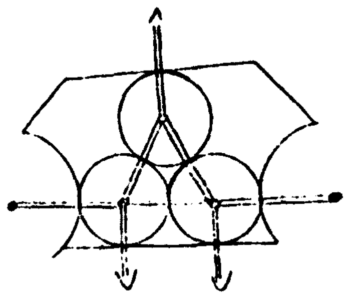


Figure 2.4

2.5. Rigidity of Packings

We can now discuss the rigidity of $G_P(p)$ in K, as with the case of a more general framework G(p). In this more general case each edge or member of G is one of three types, a cable, a rod (or a bar), or strut. For G_P each member is a strut.

Let $p(t) = (p_1(t), p_2(t), \ldots, p_k(t))$, where p(t) is continuous for $0 \leq t \leq 1$, $p(0) = p$ and $p_i(t) = p_i$ for the fixed vertices i in G. We say p(t) is a flex of G(p) if

$$\begin{Bmatrix} \text{cables} \\ \text{rods} \\ \text{struts} \end{Bmatrix} \text{ are not } \begin{Bmatrix} \text{increased} \\ \text{changed} \\ \text{decreased} \end{Bmatrix} \text{ in length.}$$

If each p(t) is obtained by restricting a rigid motion of K to $G_P(p)$, then we say p(t) is a trivial flex. If G(p) admits only trivial flexes, then we say G(p) is rigid. The members defined above for a flex of G(p) are regarded as geodesics in X, and the notions of increasing on decreasing lengths are regarded in terms of these geodesics. In the case of G_P, the geodesic is the one between the centers of the disks through their point of common intersection. In the case of fixed vertices at infinity, if a variable vertex j is adjacent to a fixed vertex i at infinity, we assume $p_j(t)$ is not any closer to any point on the geodesic ray from p_j perpendicular to the hyperplane defining p_i. This is the case of a strut.

For example $G_P(p)$ is rigid in Figure 2.4. With all of the above in mind we say P is rigid in K if $G_P(p)$ is rigid.

It is clear that if P is rigid in K, then it is stable in K. Unfortunately, the converse is not always true as we have seen with the example of the maximally dense packings of 5 disks in S^2. This example should be an anomoly, but we only provide a converse when X has non-positive curvature, and K is concave.

.6. Examples

We shall be mostly interested in the following cases.

(a) $X = \mathbb{R}^n$, n-dimensional euclidean space, $K = [-1,1]^n$.

(b) $X = \mathbb{R}^n$ and finitely stable packings.

(c) $X = K = S^n$, the unit sphere of dimension n ,

$$S^n = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\} .$$

(d) $X = \mathbb{H}^n$, and finitely stable packings, where $\mathbb{H}^n$ is n-dimensional hyperbolic space.

(e) $X = K = T^n$, an n-dimensional torus obtained as the quotient of $\mathbb{R}^n$ by a lattice.

(f) $X = K = \mathbb{H}^n/\Gamma$, where Γ is a discrete group of hyperbolic rigid motions. Here we need only that K have finite volume. K need not be compact.

It is interesting to compare cases (e) and (f) to cases (b) and (d) respectively. For every packing of the quotient space T^n or H^n/Γ (called the quotient packing) we can find the inverse image, a periodic packing in $\mathbb{R}^n$ or H^n. Of course if a packing in the total space, $\mathbb{R}^n$ or $\mathbb{H}^n$, is finitely stable, the quotient packing in the quotient space may be unstable, since the motion of unstability may have to lift to an infinite motion in the total space.

In case (b) or (d) when P is the lift (the inverse image in $\mathbb{R}^n$ or H^n) of some packing in a space of finite volume, we say P is periodicly stable if the quotient of P is stable for all possible periods.

For example the square lattice packing in $\mathbb{R}^2$ is not periodicly stable. However, it is finitely stable in $\mathbb{R}^2$.

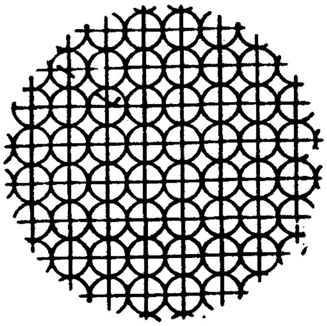


Figure 2.5

On the other hand, if the quotient packing is stable for all possible finite volume quotients, (given that the packing is periodic in $\mathbb{R}^n$), then the packing in the total space must be finitely stable.

In the spirit of L. Fejes Tóth's idea of solid packings, we say that a periodic packing P (of $\mathbb{R}^n$ or H^n) is periodicly solid if for every finite volume quotient, the quotient of P cannot be rearranged to give another packing (with the same radius) that is not congruent to the origonal. It is also clear that if P is a periodic packing and is periodicly solid, then P is solid. Any finite rearrangement can be performed inside some large bounded fundamental domain. Nevertheless, A. Bezdek [4] and L. Fejes Tóth[14]have many examples of circle packings that are not periodic, but are still solid.

One should be aware that due to some examples of Böröczky [5] it is not reasonable to define the density of a packing in H^n . Thus perhaps the density is more appropriately defined using quotient packings as above when possible. In fact it seems that the density is the same for all possible quotients. Any of the regular or trihedral Archimedean packings have compact quotients. The regular case follows from a result of Brown and Connelly, [7].

3. GENERAL PRINCIPLES

Suppose a packing P is locally maximally dense in K. If P is unstable, then there is another packing Q, close to P, where some subpacking of Q does not intersect the others and $r_1 = r(Q) = r(P)$. (A subpacking of a packing is just a subset of the packing.) Let P_1 be a subpacking of Q where there is still some contact among the member disks or the walls of K. P_1 has at least one fewer disk than P. If P_1 is stable we stop. If P_1 is unstable, we continue as before to find a subpacking, etc. If this process eventually uses all of the disks of P, then clearly P was not locally maximally dense in K. Thus we can always be assured of finding a P_1 as above, that is stable in K, and is close to a subpacking of P.

Next in the complement (in K) of the open disks of P_1 we can find another maximally dense packing with the same number of disks as $P - P_1$ but with a radius $r_2 > r_1$. The above argument shows that when r_2 is maximal, then there is a corresponding packing that is stable (in the complement). We can continue to find a sequence of "nested" packings each stable in the complement of the (open disks of the) previous packings and each having larger radius than the previous packings. We call this a (sequence of) nested stable packing(s).

Proposition 3.1: Any locally maximally dense packing has a subpacking approximated by the first element of a sequence of nested stable packings.

On the other hand if we have a nested stable packing, the first packing in this sequence will insure that the whole packing is maximally locally dense.

The following is a picture of a (maximal) packing of a square by 7 circles. See M. Goldberg [18] Schaer [22]. The second element of the sequence is indicated by a dotted circle.

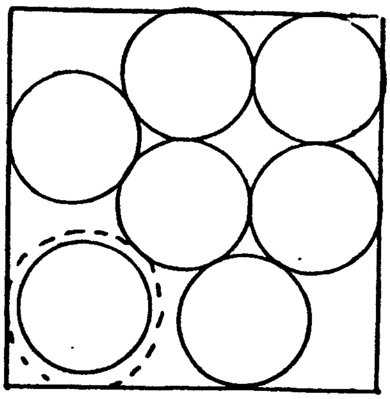


Figure 3.1

It is remotely conceiveable that we could have a locally maximally dense packing where there is no subpacking that is stable as it sits. However, if we have a situation where the only stable packings are rigid (which we shall see later happens often), then there must be such a rigid stable subpacking, given that the whole packing is locally maximally dense. It is concievable that a stable packing could continuously move to an unstable position or even one that is not locally maximally dense.

As an aside, the above ideas suggest that one might extend the notion of density for a packing. Namely for a given nested sequence of packings (not necessarity stable) in K with increasing radii, we associate a sequence $(D_1,\dots,D_m)$ of densities, where D_1 is the usual density of the union of all the packings but with all disks taken to be the radius of the first packing r_1. Next shrink all the packings in the sequence, except the first, to the radius r_2 of the second packing and D_2 is the density of that union, etc. We then consider the lexographic ordering on such sequences. Namely,

$$(D_1,\dots,D_{m_1}) < (\bar{D}_1,\dots,\bar{D}_{m_2})$$

if for some s, $D_1 = \bar{D}_1,\dots,D_s = \bar{D}_s$, $D_{s+1} < \bar{D}_{s+1}$, where the shorter sequence is extended by the same last element D_{m_1} or D_{m_2}. Then locally maximally dense nested packings in this sense will have each packing stable in the complement of the previous ones.

§ 4, Rigidity

We review and extend some of the basic results concerning the rigidity of frameworks.

4.1 Infinitessimal rigidity

Let $G(p)$ be a realization of a graph in X as in section 2. For each p_i, $(p_1,\dots,p_k) = p$, suppose we have a p_i' in the tangent space of X at p_i. We assume $p_i' = 0$ for the fixed vertices of G. Consider p_i and let $\bar{p}_j'$ be (the parallel) transport of p_j back to p_i along the (geodesic) arc of $G(p)$ from p_i to p_j, assuming i,j is a member of G. We say $p' = (p_1',\dots,p_R')$ is an <u>infinitessimal</u> <u>flex</u> of $G(p)$ if for each variable p_i

$$(4.1) \qquad (\bar{p}_j' - p_i')\cdot\bar{p}_j \begin{Bmatrix} \leq \\ = \\ \geq \end{Bmatrix} 0 \text{ for } i,j \text{ a } \begin{Bmatrix} \text{cable} \\ \text{rod} \\ \text{strut} \end{Bmatrix} ;$$

where $\bar{p}_j$ is a vector in the tangent space of p_i in the direction of the arc from p_i to p_j.

We say p' is a <u>trivial</u> <u>infinitessimal</u> <u>flex</u> if p' is the derivative (at $t=0$) of a rigid motion $R_t : K \to K$

$$\frac{d}{dt} R_t \big|_{t=0} p = p' .$$

This is an extension of the usual definition of infinitessimal rigidity for the case when $X=\mathbb{R}^n$. Then formula (4.1) becomes

$$(4.2) \qquad (p_i' - p_j')\cdot(p_i - p_j) \begin{Bmatrix} \leq \\ = \\ \geq \end{Bmatrix} 0, \text{ for } i,j \text{ a } \begin{Bmatrix} \text{cable} \\ \text{rod} \\ \text{strut} \end{Bmatrix} .$$

(4.2) can also be used for the case when $X=K=T^n$, where it is understood that p_i and p_j are taken so that the given arc in T^n lifts to the line segment from p_i to p_j in $\mathbb{R}^n$.

A similar remark holds for $X=\mathbb{H}^n$. See Asimow and Roth [1] or Connelly [9].

(4.1) and (4.2) are to be interpreted in the natural way when one of the vertices are at infinity. Namely, if p_i is a variable vertex of $G(p)$, and p_j is a fixed vertex at infinity, we replace $p_j - p_i$ by a unit vector in the direction of the geodesic from p_i determined by p_j.

We say $G(p)$ is <u>infinitessimally rigid</u> if $G(p)$ has only trivial infinitessimal flexs. An easy generalization of Connelly [9], Asimow and Roth [1], or Roth and Whiteley [21] now follows.

<u>Theorem 4.1</u>: <u>If</u> $G(p)$ <u>is infinitessimally rigid then it is rigid</u>.

In general the converse of this theorem is false. See [9], [1], or [21] again.

4.2 Recognizing Infinitessimal Rigidity

Since our graphs usually do not have all rods, the next result due to Roth and Whiteley [21] is very helpful in distinguishing when $G(p)$ is infinitessimally rigid.

Let $\omega = (\ldots, \omega_{ij}, \ldots)$ be real numbers, one corresponding to each member of G. We say ω is <u>proper</u> if

$$\omega_{ij} \begin{Bmatrix} \geq \\ \leq \end{Bmatrix} 0 \quad \text{for } i,j \text{ a } \begin{Bmatrix} \text{strut} \\ \text{cable} \end{Bmatrix} \text{ of } G.$$

There is no condition for a rod. We say ω is a <u>stress</u> for $G(p)$ if for each p_i the following holds in the tangent space at p_i

$$(4.3) \qquad \sum_j \omega_{ij} (p_j - p_i) = 0 ,$$

where the sum is taken over all j such that i,j is a member of G, and $p_j - p_i$ has the meaning as with equation (4.2). (4.3) is called the equilibrium equation at i.

For a graph G, $\bar{G}$ denotes the graph obtained by making all the members rods.

Theorem 4.2 (Roth and Whiteley): The following are equivalent:

(a.) $G(p)$ is infinitessimally rigid.

(b.) $G(p)$ is infinitessimally rigid and there is a proper stress ω for $G(p)$ such that $\omega_{ij} \neq 0$ for each cable and strut of G.

(c.) $\bar{G}(p)$ is infinitessimally rigid and there is a proper stress ω for $G(p)$ such that $\omega_{ij} \neq 0$ for each cable and strut of G.

The point of this theorem is that we can easily apply arguments counting the ranks of various matrices and generally work with $\bar{G}(p)$ using ordinary linear algebra. ω can be treated separately and often more easily by itself. We will apply this theorem later.

4.3 Infinitessimal Determinancy

It turns out that in some cases the converse of Theorem 4.1 is true. That is, a framework $G(p)$ is rigid if and only if it is infinitessimally rigid. Rigidity for $G(p)$ is determined by the infinitessimal case. This is essentially what happens for the following.

Theorem 4.3: Suppose the ambient space X has non-positive curvature. Let $G(p)$ be a framework in X with all members struts. Then $G(p)$ is rigid if and only if it is infinitessimally rigid. Furthermore if p' is an infinitessimal flex of $G(p)$ the continuous flex will increase all (strut) distances i,j where $p'_i \neq p'_j$ unless i or j is at infinity. In the case when the curvature of X is negative and (4.2) is equality, the strut distances will still increase in distance as long as $p'_i \neq 0$ or $p'_j \neq 0$.

Proof: First consider the case when the curvature of X is 0. We shall assume $X = \mathbb{R}^n$, but the motions and flexes may have certain periods which we wish to preserve. Let p' be an infinitessimal flex of $G(p)$, p' not trivial. Then define

$$p(t) = p + tp'$$

for $0 \leq t \leq 1$. Clearly $p(t)$ is a non-trivial flex of $G(p)$ and we compute for i,j a strut of G, neither i nor j at infinity.

$$\begin{aligned}|p_i(t) - p_j(t)|^2 &= |p_i - p_j + t(p'_i - p'_j)|^2 \\ &= |p_i - p_j|^2 + 2t(p_i - p_j)\cdot(p'_i - p'_j) + t^2|p'_i - p'_j|^2 \\ &\leq |p_i - p_j|^2 .\end{aligned}$$

Note that the inequality is strict unless $p'_i = p'_j$ and $(p_i - p_j)\cdot(p'_i - p'_j) = 0$.

In case p_j, say, is at infinity, then if we let e_j represent the unit vector in the direction determined at infinity we calculate

$$p_i(t)\cdot e_j = (p_i + tp'_i)\cdot e_j = p_i\cdot e_j + t\,p'_i\cdot e_j$$
$$\geq p_i\cdot e_j ,$$

which is what is desired.

Next we consider the case when the curvature of X is -1. We shall assume $X = \mathbb{H}^n$, and regard $\mathbb{H}^n$ as a subset of Minkowski space M^{n+1}, where M^{n+1} is the same as $\mathbb{R}^{n+1}$ as a set, but has the Lorentz indefinite inner product $< , >$ defined by

$$< x, y > = \sum_{i=1}^{n} x_i y_i - x_{n+1}\, y_{n+1} ,$$

where $x = (x_1,\ldots,x_{n+1})$, $y = (y_1,\ldots,y_{n+1})$. Then

$$\mathbb{H}^n = \left\{ x \in M^{n+1} \mid < x,y > = -1,\ x_{n+1} > 0 \right\}.$$

So for $x,y \in \mathbb{H}^n$, $<x-y, x-y> \geq 0$, $<x,y> \leq 1$. Also if x',y' are in the tangent space of some points in $\mathbb{H}^n$, then $<x',y'> \geq 0$.

Let $p_i(t)$ be the point in $\mathbb{H}^n$ at distance $t|p_i|$ on the geodesic from p_i in the direction p'_i. Then

$$<\frac{d}{dt}\, p_i(t) , \frac{d}{dt}\, p_i(t) > = < p'_i , p'_i > ,$$

a constant, and

$$< \frac{d^2}{dt^2} p_i(t) \ , \ p_i(t) > \ = \ - < p_i' \ , \ p_i' > \ .$$

So

$$\frac{d^2}{dt^2} p_i(t) = < p_i', p_i' > \ p_i(t) \quad .$$

Then

$$\frac{d^2}{dt^2} < p_i(t) - p_j(t), \ p_i(t) - p_j(t) >$$

$$= \ < \frac{d}{dt} p_i(t) - \frac{d}{dt} p_j(t), \frac{d}{dt} p_i(t) - \frac{d}{dt} p_j(t) > +$$

$$< \frac{d^2}{dt^2} p_i(t) - \frac{d^2}{dt^2} p_j(t) \ , \ p_i - p_j >$$

$$= \ < \frac{d}{dt} p_i(t) - \frac{d}{dt} p_j(t), \frac{d}{dt} p_i(t) - \frac{d}{dt} p_j(t) > +$$

$$(< p_i', p_i' > + < p_j', p_j' >) \ (-1 - < p_i, p_j >)$$

$$\geq \ 0.$$

Note that we have equality if and only if $p_i' = p_j' = 0$. Thus if $< p_i - p_j \ , \ p_i' - p_j' > \ > 0$, then clearly p_i and p_j will increase in distance. In the case of equality, the above computation implies p_i and p_j increase in distance near $t=0$. Note that the Minkowski inner product is positive if and only if the inner product after transport is positive.

4.4 Global Uniqueness

From the results of 4.2 and from more general results of Connelly [10], we can expect that a rigid framework $G(p)$ will have a proper stress $\omega \neq 0$. Here we present some results concerning the global uniqueness of such sort of realizations.

Let $G(p)$ be a framework in X. Here we allow $G(p)$ posibly to have some fixed points posibly at infinity.

Let $G(q)$ be a realization of G with posible different vertices. We say $G(p)$ and $G(q)$ are <u>homotopic</u> if they are homotopic as continuous maps of one-dimensional graphs into X. Of course fixed points are to remain fixed during the homotopy. (See any text with an introduction to algebraic topology for a definition of homotopy.)

We say $G(q)$ is <u>dominated by</u> $G(p)$ or $G(p)$ <u>dominates</u> $G(q)$, and we write $G(q) \leq G(p)$ if

$$(4.4) \qquad |p_i - p_j| \left\{ \begin{matrix} \leq \\ = \\ \geq \end{matrix} \right\} |q_i - q_j| \quad \text{for } i,j \text{ a } \left\{ \begin{matrix} \text{strut} \\ \text{rod} \\ \text{cable} \end{matrix} \right\} \text{ of } G ,$$

If for all $G(q)$ (homotopic to $G(p)$); $G(q) \leq G(p)$ implies p is congruent to q, then we say $G(p)$ is <u>uniquely</u> positioned <u>(up to homotopy) in</u> X.

Of course the distances implied above are interpreted to mean the geodesic lengths in X and is part of the information of a realization of G. One should also be aware that vertices may coincide and edges may cross in a general realization $G(p)$.

By wrapping geodesics around handles we can obtain many more realizations satisfying (4.4). So we narrow our attention to the case when $G(p)$ and $G(q)$ are homotopic. Of course this always happens when X is simply connected.

We now have an easy generalization of a result in Connelly [10] where the following frameworks were called spiderwebs.

Theorem 4.4: Let $G(p)$ be a framework with a proper stress ω such that:

(i) All the members of G are cables.

(ii) For each vertex i of G there is some $\omega_{ij} \neq 0$.

(iii) The curvature of X is non-negative (and constant).

Then $G(p)$ is uniquely positioned up to homotopy in X.

Proof: In the case when the curvature of X is 0, we lift the problem to $\mathbb{R}^n$ and then this follows from Connelly [10].

Similarly if the curvature of X is -1 we can lift the problem to $\mathbb{H}^n$. But then we can replace the stress ω by a stess $\bar{\omega}$ in M^{n+1}. In order to obtain the equilibrium equations in M^{n+1} we introduce a new vertex $0 = p_o$ in the framework and join all the variable points of G to 0, to get a graph $\hat{G}$, by a "Minkowskii" strut or "Euclidien" cable. More precisely we define $\omega_{0i} < 0$ such that the equilibrium equations (4.3) hold in M^{n+1}. Then we define an energy by

$$E(q) = \sum_{ij \in \hat{G}} \omega_{ij} < p_i - p_j, p_i - p_j > .$$

Since $< p_i, p_i > = -1 < 0$ and $\omega_{0i} < 0$, $E(q)$ is a positive quadratic function and global uniqueness follows easily as in the flat case.

Note that in both the 0 curvature case and -1 curvature case, if G has no fixed points it is implicitly assumed that the lift is periodic. To calculate the energy then it is only necessary to take one, but only one, representative for the lift of each member of $G(p)$. All other lifts will have the same energy, since they differ by an affine motion that preserves the $\langle\, , \rangle$ form. This finishes the proof of the Theorem.

The idea here is that with Theorem 4.2, when a graph with all struts is infinitessimally rigid, then it has a proper stress with all non-zero members. Then Theorem 4.3 says if we change all the struts to cables, there is no other realization in its homotopy class, unless some cable condition is violated. In our application all the members of $G(p)$ will have the same length and the above says that there can be at most one such infinitessimally rigid realization.

Theorem 4.3 and Theorem 4.4 should also be true in any Riemannian space X of non-positive sectional curvarture, constant or not. We do not need such a result here so we do not include that computation.

5. Applications to packings

5.1 Rigid and Stable Packings

There is a very simple situation when a stable packing is not rigid in $\mathbb{R}^n$. Namely, we can take a K that has a continuous group of symmetries and then remove a ball or some portion that is not left invariant by the symmetries. For instance we can take a rigid packing in a torus or a cylindrical strip and then remove a smaller ball that does not touch the packing as in the figure below.

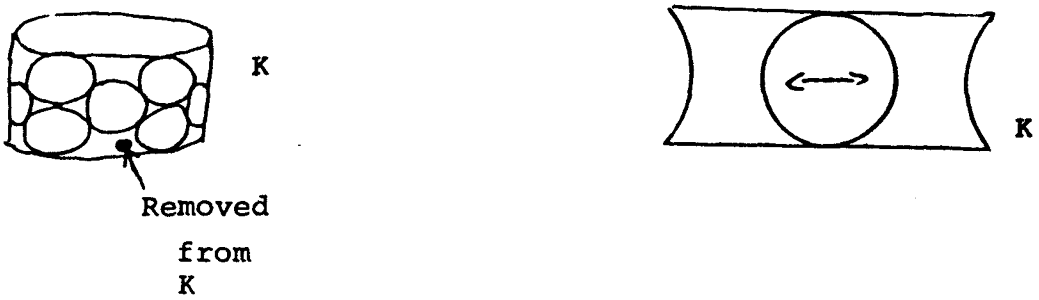


Figure 5.1

Theorem 5.1: *Let the ambient space X have non-positive (constant) curvature, where the container* K *is concave. Let* p *be a packing of* K*. Then* $K \subset \hat{K}$*, another concave container, such that the following are equivalent:*

(i) p *is stable in* $\hat{K}$

(ii) $G_p(p)$ *is rigid in* $\hat{K}$

(iii) $G_p(p)$ *is infinitessimally rigid in* $\hat{K}$,

where if p *is stable in* K, p *is stable in* $\hat{K}$.

Proof: (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) is always true.

Suppose (i) and let p' be an infinitessimal flex of $G_p(p)$. We shall show that $G_p(p)$ has only trivial infinitessimal flexes and thus is infinitessimally rigid.

Choose a disk i and consider the subpacking Q determined by those j such that $p'_j = p'_i$.

In case the curvature of X is -1, if $p'_i \neq 0$, then the motion $p(t)$ determined by Theorem 4.1 will have the disk i disjoint from all the others for all $t > 0$. Thus $p'_i=0$ for all i and we are done.

In case the curvature of X is 0, then $Q = p$ since otherwise p would be unstable again by Theorem 4.1. Also we can assume equation (4.1) is equality for the walls (i.e.boundary) of K, since if we have strict inequality we will remove that wall from K. If no member of p intersects some wall of K we will remove that wall as well to get $\hat{K}$.

If some wall that remains is not flat, then for $t > 0$, $p(t)$ will be such that no member of p will intersect that wall. Then p will be 1-unstable for the disk that intersected that wall (recall all the p'_i's are the same). This composite motion shows that p itself was unstable.

If all the walls of $\hat{K}$ are flat, then $p(t)$ leaves these walls invariant and $p(t)$ extends to a symmetry of $\hat{K}$, thus $p(t)$ is trivial and we are done.

Note that $\hat{K}$ may be infinite as for the second example of Figure 5.1, but this only happens when p consists of a single disk.

Note that the example of 5 circles in S^2 shows that some sort of condition about the curvature being nonpositive is necessary.

Let $\tilde{G}$ denote the graph obtained by reversing cables and struts.

Corollary 5.1: With the hypothesis of Theorem 5.1, if P is stable, then $\tilde{G}_p(p)$ is uniquely realized in X.

Proof: By Theorem 5.1, $G_p(p)$ is infinitessimally rigid in X. By Theorem 4.2 $G_p(p)$ has a proper stress ω such that $\omega_{ij} < 0$ for all $i\,j \in G_p$ (which are struts of G). Then $-\omega$ is a proper stress for $\tilde{G}_p(p)$.
Theorem 4.4 implies that $\tilde{G}_p(p)$ is uniquely realized in X.

The idea here is that in order to find maximally dense packings it is only necessary to consider stable packings in view of Proposition 3.1. In view of Theorem 5.1 we need only consider packings with infinitessimally rigid graphs. Corollary 5.1 says there is at most one such framework since one has to be smaller than the other. In principle one could perhaps search for such frameworks directly.

Suppose one has an algorithm such as discriped by Tarnai and Gáspár [25] where one starts with the graph of a packing G_p that is not locally maximally dense. After the application of the "heating" algorithm that increases edge lengths one may obtain another locally maximal graph G_Q which is obtained from G_p by adding new edges only. The final locally stable graph G_Q cannot be G_p by the Corollary 5.1. However, ironically the Corollary 5.1 and Theorem 5.1 do not apply to S^2

where Tarnai and Gåspår did their work.

It is also posible to obtain a bit more information from Corollary 5.1. Suppose $X = K = T^2$, the 2-torus, and each face determined by $G_p(p)$ has longest diagonal $\leq L$. If $\mathcal{Q}$ is as above, locally maximal, and if $L \leq 2\, r(\mathcal{Q})$ we have a contradiction to the Corollary, going from $\mathcal{Q}$ to p. Thus we know

$$L > 2\, r(\mathcal{Q}),$$

a bound on the maximum density for that kind of graph.

5.2. Calculations.

We now apply Theorem 4.2 to present some sufficient condition for being stable. In particular when Theorem 4.2 applies we will give conditions when $G_p(p)$ is infinitessily rigid.

Consider the function (the <u>rigidity map</u>)

$$f : X^k \longrightarrow \mathbb{R}^e,$$

where $f(p) = (\ldots,(p_i - p_j)^2,\ldots)$, where k is the number of variable vertices of G, i.e. the number of disks of $\mathcal{P}$, and e is the number of members of G_p. Let t_n be the dimension of the space of trivial infinitessimal flexes of X. Then following Asimow and Roth [1], or Connelly [9], we see that $\overline{G}_p(p)$ (the graph obtained by replacing all members by rods) is infinitessimally rigid if and only if

(5.2) $$\operatorname{rank}\, df_p = kn - t_n,$$

where n is the dimension of the ambient space X, and df_p is the differential of f at p. Thus $kn - t_n \leq e$ is a necessary condition for (5.1) to be true.

However, Theorem 4.2 implies that if $kn-t_n = e$ there is no dependency among the columns of df_p which is the same as a stress for $G_p(p)$. Thus we need

$$kn-t_n + 1 \le e \quad . \tag{5.2}$$

In our case the calculation of t_n is easy, which we give in the following table:

X	K	t_n
$\mathbb{R}^n$	compact	0
S^n	S^n	$n(n+1)/2$
$\mathbb{H}^n$	compact	0
T^n	T^n	n
$\mathbb{H}^n/\Gamma$	$\mathbb{H}^n/\Gamma$	0

Table 5.1

It is sometimes more convenient to calculate (5.2) instead in terms of the <u>average degree</u> of a vertex of G_p.

$$d_v = \sum_i d_i/k \quad , \tag{5.3}$$

where d_i is the <u>degree</u> or the number of members of G_p having i as a vertex, and i is a variable vertex. Also we have

$$2e = \sum_i d_i + e_b \quad , \tag{5.4}$$

where e_b is the number of edges of G having a fixed vertex at one end. Putting (5.2), (5.3), (5.4) together we get

$$(5.5) \qquad 2n - (2t_n - e_b - 2)/k \leq d_v .$$

In the case of finite stability and 0 curvature, we would expect that e_b would be at most of the order of $k^{\frac{n-1}{n}}$. Thus in the limit (if the limit exists) of larger and larger finite numbers of disks we find

$$(5.6) \qquad 2n \leq d_v .$$

In the case $n = 2$, we can also reformulate equation (5.2) in terms of the average degree of a face (assuming all faces are simply connected

$$(5.7) \qquad d_f = (\sum_i f_i)/f_t ,$$

where f_i is the number of edges of face i, and f_t is the total number of faces of $X - G$.

Suppose further that $K = X$ and has no boundary. Then we recall that the euler characteristic of K is

$$(5.8) \qquad \chi = k - e + f_t,$$

and similar to (5.3) we get

$$(5.9) \qquad 2e = \sum_i f_i .$$

Combining (5.9), (5.8), (5,7) and (5.2) we get

(5.10) $$d_f \leq 4 + 2\left(\frac{t_n - 1 - 2\chi}{f_t}\right) .$$

In the case $X = K = T^n$ we can be more precise. Let P be a packing of T^n. Then if we take an m-fold covering of T^n, the lifted packing has mk members and me edges in its graph. (5.2) becomes

$$mkn - t_n < me \quad , \quad \text{or}$$

$$kn - t_n/m < e.$$

Since $t_n = n$, when $m \geq n$, then

(5.11) $kn \leq e$,

and a similar analysis to the above shows that (5.6) holds for any packing of a torus that is an n-fold covering of another packing.

For $n = 2$ this also shows that

$$d_f \leq 4.$$

Thus G_p will have a triangle (multiple edge, or loop) unless all the degrees of all the vertices are 4. However, when P is a packing of equal circles it is possible to show that $G_p(p)$ is infinitessimally flexible. Thus G_p must always have a triangle when it is a double cover, assuming no loops or multiple edges in G_p.

For S^2, $\chi = 2$, $t_2 = 3$, but here the average degree cannot even equal 4, and so in order for $G(p)$ to be infinitessimally rigid on S^2 there must be at least one triangle also. This was observed in Tarnai and Gáspár [25].

5.3 Examples

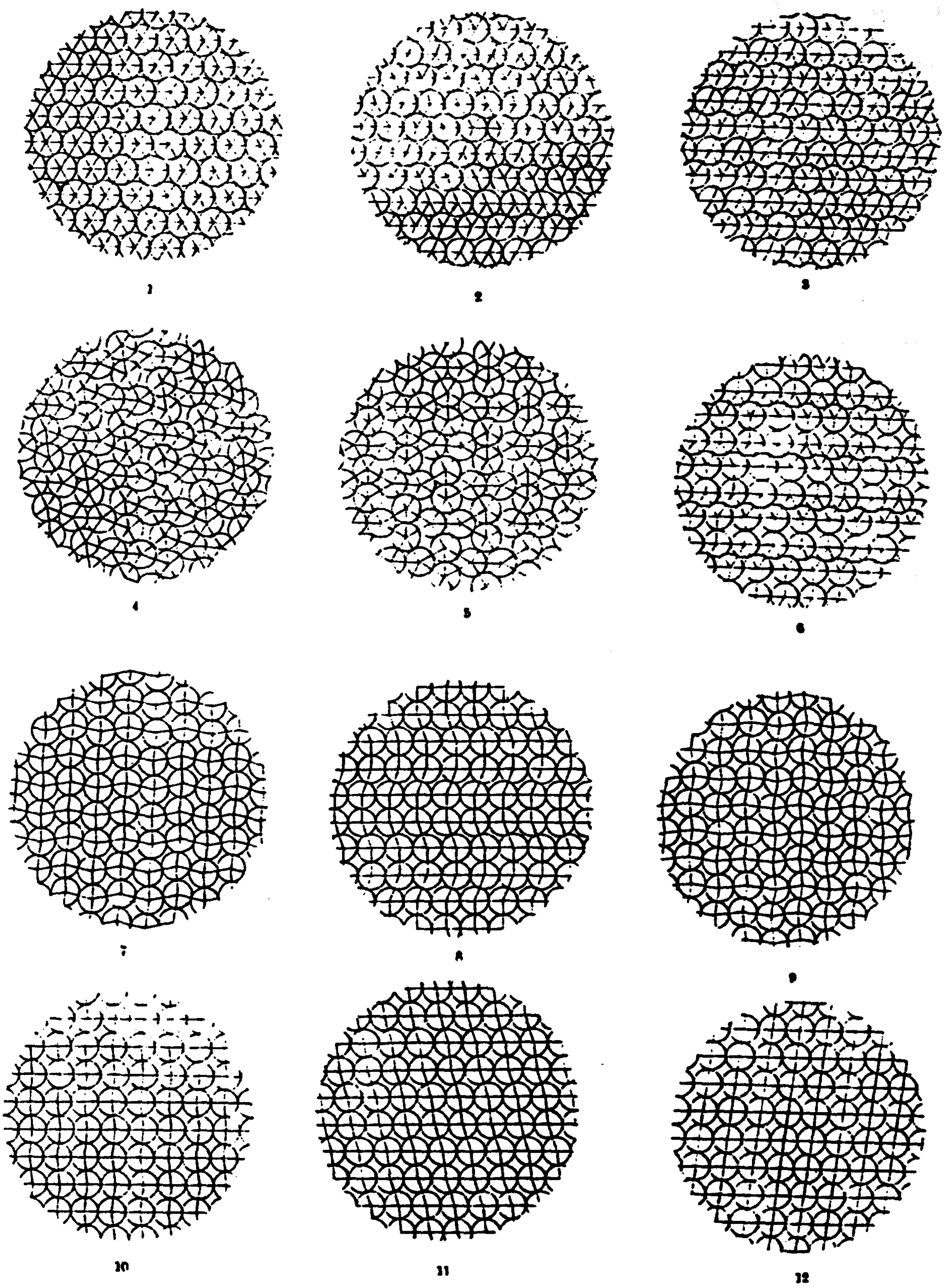
Using the above information we can calculate many examples when P is stable.The following is a list of "regular" circle packings, all periodic of course, in the plane, given by Niggli [19] and Sinogowitz [24],from the book of L. Fejes Toth [12]

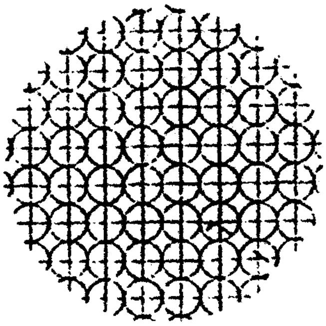
The following table shows which packings are stable and in which sense.

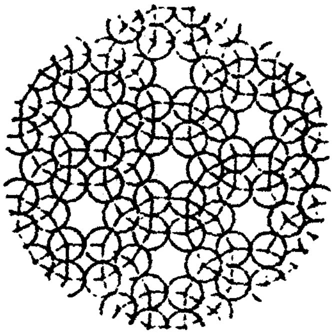
1-unstable or not Fejes Toth stable	finitly unstable	periodicly stable
22,20,31	16,18,19	1,4,5,24
	20,21,22	
	23,25,26	
	27,28,29	
	30,31	

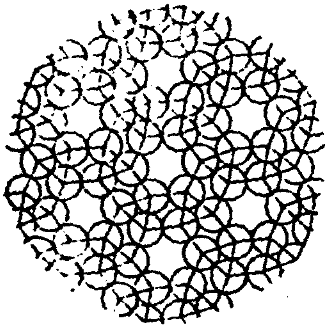
Table 5.2

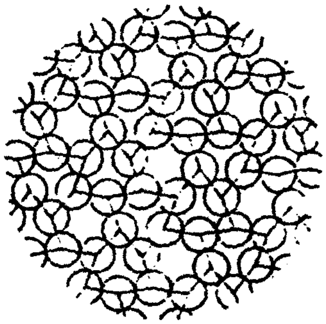
Figure 5.2

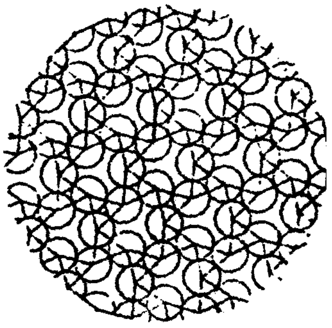


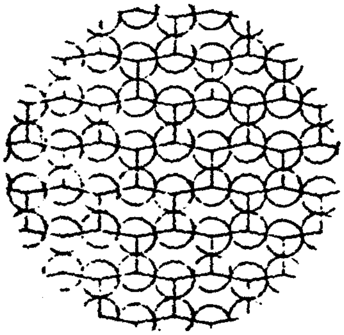

13

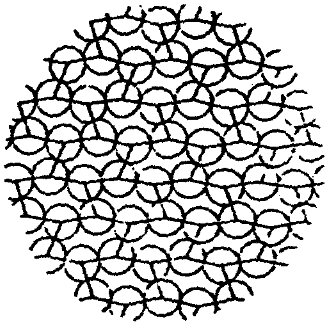

14

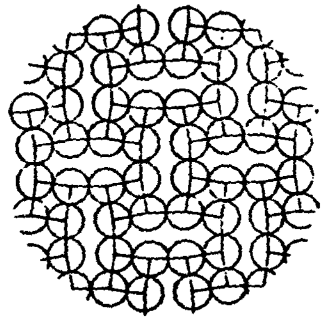

15

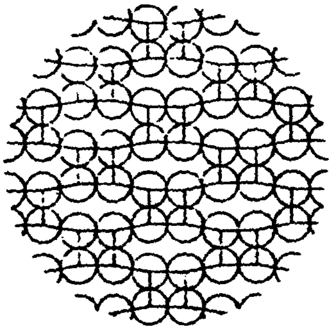

16

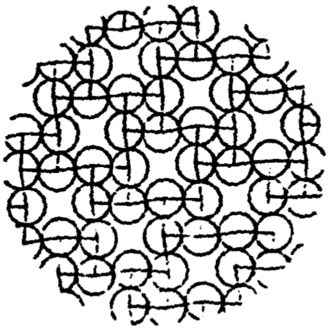

17

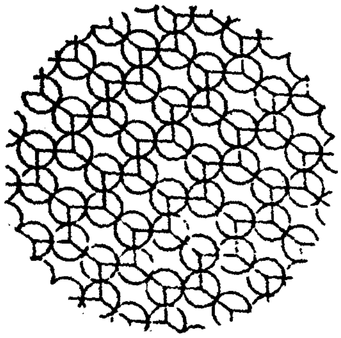

18

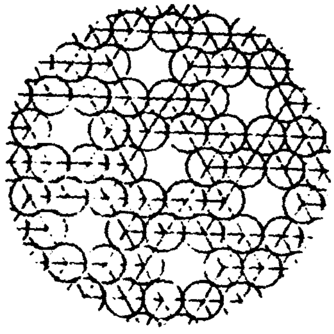

19

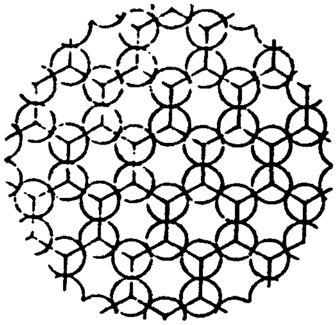

20


21

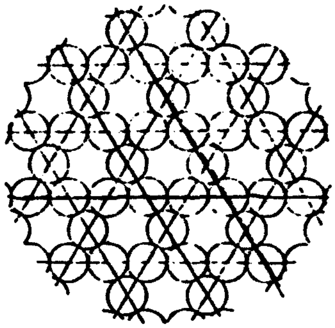

22


23

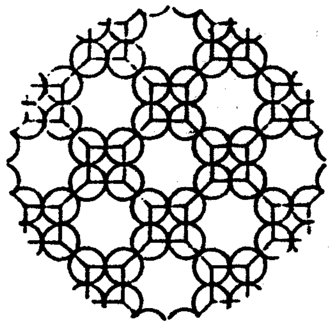

24



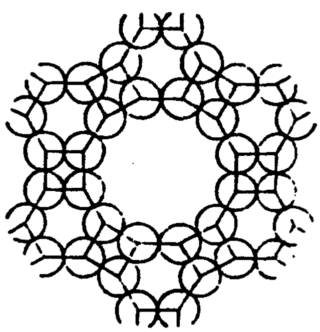
25



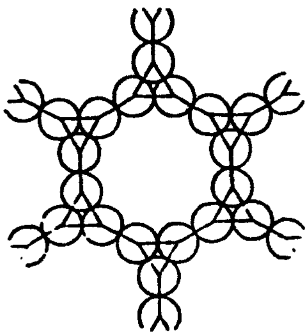
26



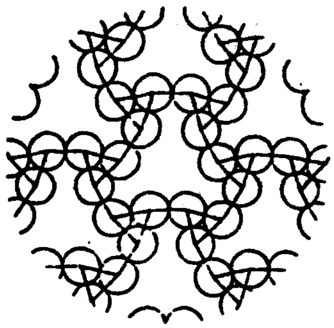
27



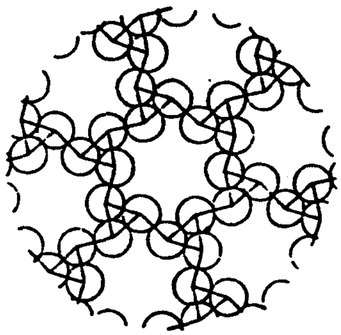
28



29



30



31

It seems that these packings are finitely stable if and only if $d_v \geq 4$. However, most of these packing are periodicaly unstable on a torus. These examples of (finitely) stable packings can be computed using Theorem 4.2. To tell when a rod framework $\bar{G}_p(p)$ is infinitessimally rigid, one can use the following. See Roth and Whiteley[21] for instance.

Lemma: Let $G(p)$ be a infinitessimally rigid rod framework in X. Add one more vertex $k+1$ to G to get H with at least two rods from $k+1$ to the old vertices of G. Let p_{k+1} be a realization such that the two new rods do not lie on the same geodesic. Then $H(p_1 \cdots p_{k+1})$ is infinitessimally rigid.

This can be used to "build up" the inside from the rigid outside.

The following is a packing of a square with 15 circles by circles due to M. Goldberg [18]

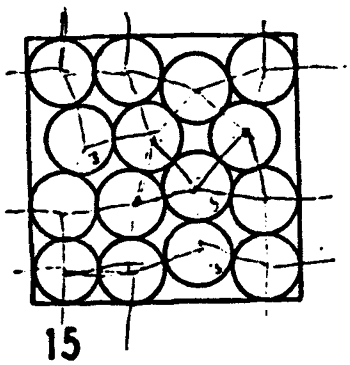


Figure 5.3

This has the property that it extends to a periodic stable packing of a "square" torus. For the smallest possible torus, the packing is stable, but for any lift it is not, since we can move along "shear" lines created by the square.

Also with these examples one should be aware that if a packing is periodic and has no finitely stable subpacking, then neither does a large enough lift for a torus packing. This is easy to see by moving the disks in a large circle inside an even larger fundamental domain. Then the rest of the disks all can be moved from the others since the remaining packing cannot even be 1-stable. Thus we can always improve the density of a periodic packing that has no 1-stable subpacking.

5.4 Physical Experiments:

L. Fejes Tóth [12] page 306 describes an experiment of Bernal where grains of shot are dropped into a vessel in a "loose packing" where on the average each sphere touches 6 others. He describes an explanation due to Heppes where each sphere must touch at least 3 of the spheres below it, and the probability of touching more than 3 spheres is 0.

Our above analysis can be applied here as well. Surely the rest position of the spheres must be stable. If not then a small perturbation would change this position. Thus (5.5) or essentially (5.6) would imply that $6 \leq d_v$, and this would be true no matter how the shot was placed into the vessel. An average degree much greater than 6 would imply a redundancy or indetermancy in the way the framework $G_p(p)$ "resolves" loads.

In another direction the instability of many of the torus packings of section 5.3 suggest that possibly a change in the container of a packing may break the symmetry of a previously stable packing. Also a break in the symmetry may "add" to the rigidity.

5.5 Nested table packings and packings with small densities

We can use the square packing of Figure 3.1 to create larger packings with even bigger holes that could hold circles that are free to move inside, the outside packing being stable (and thus rigid).

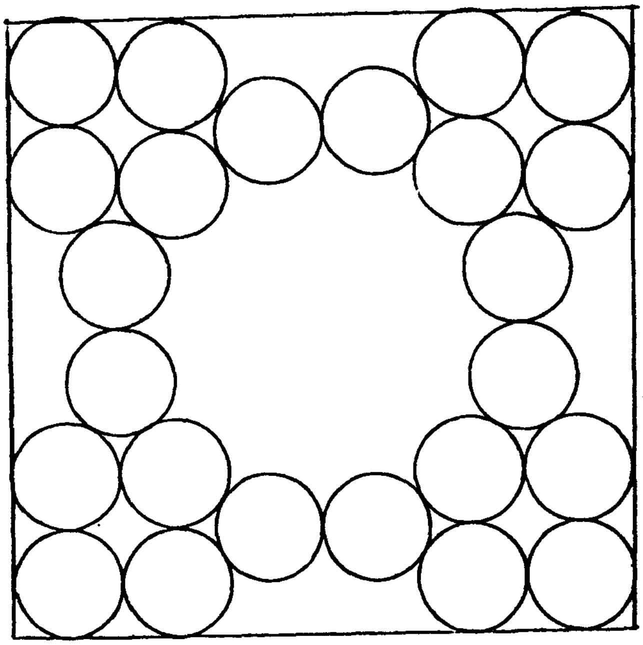


Figure 5.4

It is interesting to compare the packing of Figure 5.4 above with the one obtaining by filling in the "hexagonal holes" of the packing 28 of Figure 5.2. Note the packing of Figure 5.4 is not finitely stable when extended to a periodic packing since its average degree is less that 4.

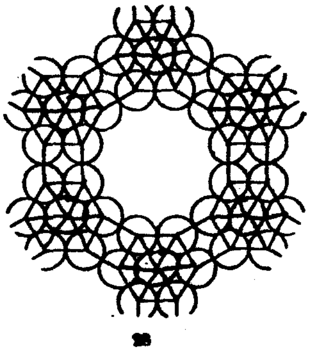


Figure 5.5

The packing of Figure 5.5 is now finitely stable but still not periodicly stable. This is the least dense that I have found so far. It is possible to find finitely stable or even periodicly stable packings of arbitrarily small densities? (Note that very large densities might be obtained by then filling these holes with a secondary packing.) Packing 24 of Figure 5.2 is the least dense periodicly stable packing I have found so far. Böröczky [5] has examples of 1-stable packings in the plane with arbitrarily small densities as in Figure 5.6 below (showing the graph of the packing), but the average degree is 3, and thus these packings are not even finitely stable.

What is the least dense packing one can obtain by removing disks from the standard triangular packing?

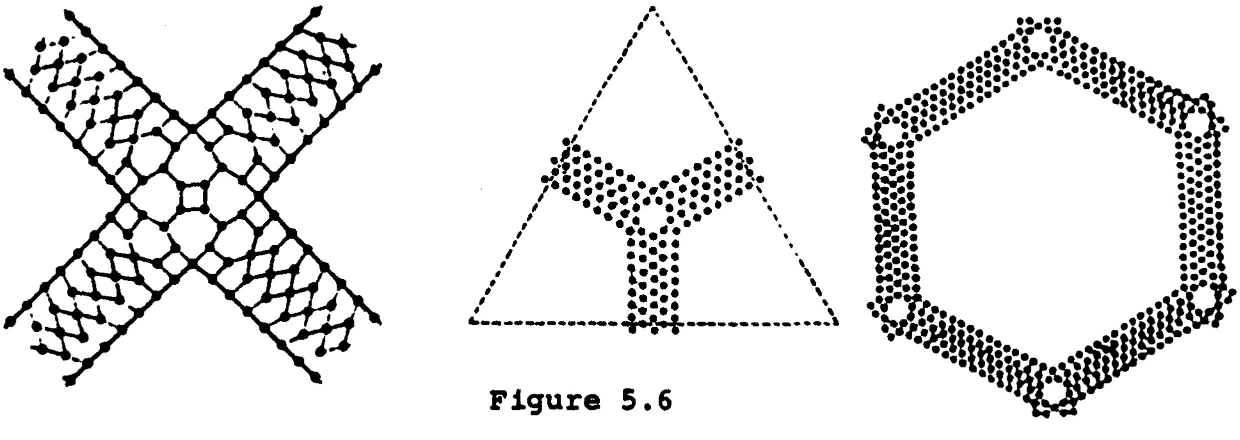


Figure 5.6

6. CONJECTURES AND QUESTIONS

6.1 S^n

From the point of view of guaranteeimg that the worst case does not happen, it would be good to know for S^n, and in particular for S^2, when a stable packing in S^n is rigid. Perhaps if the packing has a small enough radius, it is infinitessimally rigid. Does a stable packing for S^2 have a non-zero stress ω ?

Is $\omega_{ij} \neq 0$ for all members ij of G ?

Is such a $G_p(p)$ uniquely positioned in some reasonable class of realizations?

6.2 Random Packings

There have been several experiments which have involved various physical measurements of the densities of certain "packings". See for example Westman and Hugill [26], and Bernal [3]. On the other hand mathematical predictions description or simulations seem to be lacking. For instance Gilbert [17] models such phenomena as random packings by succesively choosing spheres of a smaller radius randomly and discarding any that overlap the preceedings ones. The problem with this approach is that there is no account made for stability. The packing density could increase considerably if one took a finite sequence of such points (randomly choosen) and then applied the "heating algorithm" of Tarnai and Gâspâr for instance. It would be interesting to confirm Bernal's calculation of 60 % for the packing density of such a "random" packing. See Bernal [3] also.

It also would be interesting to compute such densities for packing of a torus, where physical estimates are harder to obtain. Does a random packing of a torus always yield the same packing density regardless of the shape of the torus?

It should also be realized that most of what has been done for packings with disks of the same radius will work for packings with disks of different radii. The constant radii case is just simpler.

6.3 Improving known Packings

One main purpose of Tarnai and Gåspår was to take a known candidate for a good packing of S^2 by circles and make it better by their algorithm . Although in S^2 if they start with a packing with an infinitessimally flexible graph , that cannot be sure that they can improve the packing density, in practice they always are able to improve the density. This is not surprising.

One would apply this same idea ,for instance,to torus packings in diemension 3, where the best packing density is not known, even for the limiting case with a large number of disks. Presumably , one could start with interesting starting arrangements and see what happens after the algorithim is finished.

Lastly as a wild hope , it might be possible to take a clue from the global uniqueness for the position of a stable packing given the graph G_P. Is it possible to find a "formula " or "fast" algorithm that would compute the density of the packing P given only G_P but not its unique realization? It might be possible to calculate the overall maximal density by eliminating many graphs and only considering those that are left.

REFERENCES

[1] L. Asimow and B. Roth, The rigidity of graphs, Trans. Amer. Math. Soc. 245 (1978), p. 279-289.

[2] L. Asimow and B. Roth, The rigidity of graphs II, J. Math. Appl. 68, (1979), p. 171-190.

[3] J. D. Bernal, A geometric approach to the structure of liquids, Nature, vol. 183, (1959), p. 141-147 .

[4] A. Bezdek, Solid packing of circles in the hyperbolic plane, Studia Sci. Math. Hung. 14, (1979), p. 203-207.

[5] K. Böröczky, Über stabile Kreis und Kugel systeme, Ann. Univ. Sci. Budapest, Eötvös Sect. Math. 7, (1964), p. 79-82.

[6] K. Böröczky, Sphere packing in spaces of constant curvature I, (Hungarian), Math. Lapok 25, (1974), p. 265-306.

[7] M. Brown and R. Connelly, On graphs with a constant link, "New directions in the theory of graphs (Proc. Third Ann Arbor, Mich.), Academic Press, N.Y., (1973), p. 19 - 51.

[8] M. Brown and R. Connelly, On graphs with a constant link II, Discrete Math. 11, (1975), p. 199 - 232.

[9] R. Connelly, The rigidity of certain cabled frameworks and the second order rigidity of arbitrary triangulated convex surfaces, Advances in Math. 37, No. 3, (1980), p. 272-299.

[10] R. Connelly, Rigidity and Energy, Invent. Math. 6, (1982), p. 11-33.

[11] L. Danzer, Endliche Punktmengen auf der 2-Sphere mit möglichst großem Minimalabstand. Habilitationsschrift (Göttingen 1963).

[12] L. Fejes Tóth, Regular Figures, Pergaman Press, 1964.

[13] L. Fejes Tóth, Solid circle-packings and circle-coverings, Studia Sci. Math. Hungar. 3, (1968), p. 401-409.

[14] L. Fejes Tóth, Solid packing of circles in the hyperbolic plane, Studia Sci. Math. Hungar. 15, (1980), p. 299-302.

[15] L. Fejes Tóth and A. Heppes, Multisaturated packings of circles, Studia Sci. Math. Hungar., 15, (1980), p. 303-307.

[16] G. Fejes Tóth, New results in the theory of packing and covering, in Convexity and its applications, ed. by Gruber and Wills, Birkhäuser, Basel (1983), p. 318-359.

[17] E.N. Gilbert, Randomly packed and solidly packed sphere. Canad. J. Math. 16, (1964), p. 286-298.

[18] M. Goldberg, The packing of equal circles in a square., Math. Mag. 43, (1970), p. 24-30.

[19] P. Niggli, Geometrische Kristallgraphie des Diskontinuums, Leipzig (1919).

[20] C.A. Rogers, Packing and Covering, Cambridge University Press, 1964.

[21] B. Roth and W. Whiteley, Tensegrity Frameworks, Trans. of A.M.S., (1981), p. 410-446.

[22] J. Schaer, The densest packing of 9 circles in a square, Canad. Math. Bull. vol. 8, no. 3, April 1965, p. 273-277.

[23] K. Schütte and B.H. van der Waerden, Auf welcher Kugèl haben 5,6,7,8 oder 9 Punkte mit Mindestabstand Eins Platz?. Math. Ann., 123, (1951), p. 96-124.

[24] U. Sinogowitz, Die Kreislagen und Packungen Kongruenter Kreise in der Ebene, Zeit. f. Kristallographie, 100, p.461-508.

[25] T. Tarnai and Z.S. Gáspár, Improved packing of equal circles on a sphere and rigidity of its graph, Math. Proc. Camb. Phil. Soc. (1983), 93,p.191-218.

[26] A.E.R. Westman and H.R. Hugill, The packing of particles, J. Am. Ceram. Soc., 13, (1930), p. 767-779.

Department of Mathematics
Cornell University
Ithaca, N.Y., 14853
U.S.A.