Shapes of geodesic nets.

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August 14, 2006

Abstract

Let M^n be a closed Riemannian manifold of dimension n. In this paper we will show that either the length of a shortest periodic geodesic on M^n is bounded by $c(n)d$, where d is the diameter of M^n or there exists infinitely many geometrically distinct geodesic nets on this manifold. We will also show that either the length of a shortest periodic geodesic is bounded in terms of the volume of a manifold M^n, or there exists infinitely many geometrically distinct geodesic nets on M^n.

Introduction

It has been a long-standing question in Riemannian geometry that originated with H. Poincare whether every closed manifold has infinitely many periodic geodesics. The contributions to this questions were made by W. Klingenberg, ([Kl]), D. Gromoll and W. Meyer, ([GM]), who discovered that on a closed Riemannian manifold has infinitely many periodic geodesics, if the space of parametrized curves on this manifold has an unbounded sequence of betti numbers, and, later by M. Vigue-Poirrier and D. Sullivan, who in [VS] showed that this condition is satisfied if and only if the rational cohomology algebra of M requires at least two generators. In 1989 H-B. Rademacher has shown that generically, there are, indeed, infinitely many periodic geodesics on any closed Riemannian manifold, ([R]). In 1992 it was shown by V. Bangert [B]) using the work [F] of J. Franks that there exists infinitely many periodic geodesics on a manifold diffeomorphic to S^2. (Later N. Hingston found a lower bound for the number of geodesics of length $\leq x$ on a Riemannian 2-sphere, (see [H])).

On the other hand, already for a manifold diffeomorphic to S^3 it is not known whether there always exists more than one periodic geodesic.

The question is complicated by the fact that for some nonsymmetric Finsler metrics on S^n, $\mathbf{C}P^n$, $\mathbf{H}P^n$ and CaP^2 only finitely many periodic geodesics do exist, as it was shown by A. Katok, (see [K], [Z]). From this example one can conclude that, in general, topology of a manifold does not by itself guarantee an infinite number of periodic geodesics.

Geodesic nets can be viewed as a generalization of periodic geodesics. They, for example, are critical points of the length/energy functional on the space of graphs immersed in a manifold, similarly to geodesics that are critical points of the above functionals on the space of parametrized curves. That is, if Γ is a stationary geodesic net on M^n, then $\frac{dl_{X,\Gamma}}{dt}(0) = 0$, where. X is any smooth vector field on M^n, $\Phi_X(t)$ is the corresponding 1-parameter family of diffeomorphisms of M^n, and $l_{X,\Gamma}(t) = \text{length}(\Phi_X(t)(\Gamma))$.

Every geodesic net can be "made" into a stationary 1-cycle, if it is not already one, by doubling the multiplicity of each edge. Stationary 1-cycles can be considered as homological analogs of periodic geodesics.

Like periodic geodesics, two geodesic nets are geometrically distinct if one is not a multiple of another.

Conjecture A. *On any closed Riemannian manifold there exists infinitely many geometrically distinct geodesic nets.*

Note that for a generic analytic Riemannian manifold the set of geodesic nets is countable (by the same reasons as the set of closed geodesics is countable). Also, it seems that Morse-theoretic arguments do not help to conclude the existence of infinitely many geometrically distinct geodesic nets in the situations, when they do not help to conclude the existence of infinitely many closed geodesics. For example, if M^n is homeomorphic to S^3, then a well-known theorem by F. Almgren implies that the space of 1-cycles on M^n is homotopy equivalent to $K(\mathbf{Z}, 2) = \mathbf{C}P^\infty$ and, so does not have enough homology classes to conclude the existence of infinitely many stationary 1-cycles. Moreover, there are virtually no results establishing the existence of geodesic nets that are not closed geodesics (or are formed by several intersecting closed geodesics) (cf. [HM]). All this makes Conjecture A look as difficult for all practical purposes as the conjecture asserting the existence of infinitely many closed geodesics (although the latter conjecture is formally stronger).

Another open question is that of the connection between the length of a shortest periodic geodesic and other geometric parameters of the manifold. For example, one can ask whether the length of a shortest closed geodesic can be uniformly bounded in terms of the volume of a manifold, (this question is due to M. Gromov, see [Gr]), and whether the length of a shortest closed geodesic can be bounded in terms of the diameter of a manifold. Here is our conjecture

Conjecture B. *On any closed Riemannian manifold there exists a periodic geodesic of length at most $c(n)d$, where d is the diameter of a manifold.*

In fact, there is no counter-example indicating that the length of a shortest periodic geodesic is not, in fact, bounded by two times the diameter of a manifold.

Conjecture C ([Gr]). *On any closed Riemannian manifold there exists a periodic geodesic of length at most $\tilde{c}(n)vol(M^n)^{\frac{1}{n}}$, where $vol(M^n)$ is the volume of a manifold M^n.*

0.1 Minimal geodesic nets. In this paper we will prove two theorems: one of them stating that either Conjecture A or Conjecture B is always satisfied on a closed Riemannian manifold M^n, and the second one stating that on a closed Riemannian manifold either Conjecture A or Conjecture C always holds. Moreover, our techniques are purely topological and apply also in the Finsler situation, even in situations when the set of distinct closed geodesics is known to be finite.

Minimal geodesic nets that result from the proofs of the theorems will be of a particular type. Namely, they will be minimal (stationary) geodesic cages and geodesic flowers formally defined below.

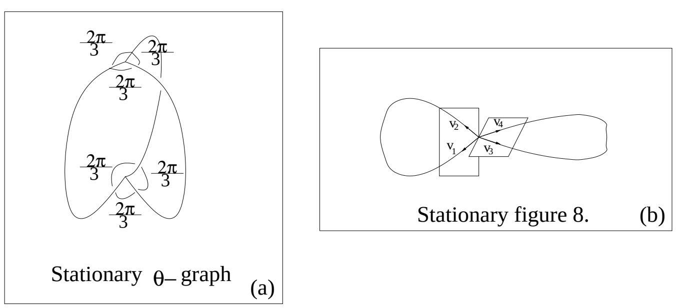

Figure 1: Geodesic Nets.

Definition 0.1 *(a) We define a minimal (or stationary) geodesic net Γ to be a graph immersed into a Riemannian manifold M^n that satisfies the following two conditions:*

(1) each edge of Γ *is a geodesic segment;*
(2) the sum of unit vectors at each vertex tangent to the edges and directed from this vertex equals to zero.

(b) If, in addition, all the vertices of Γ *have even degrees then* Γ *is called a stationary* 1*-cycle.*

(c) If a minimal geodesic net Γ *has two distinct vertices joined by at most* m *segments, (counted with multiplicities), or if* Γ *is a minimal geodesic flower, that is, a net that has one vertex and* m *or less geodesic loops based at that point, it is called a minimal geodesic* m*-cage, (or just a minimal geodesic cage), (see fig. 1).*

(d) A (not necessarily minimal or geodesic) immersion of a graph will be called a net. Also, nets that consist of a vertex together with at most m *(not necessarily geodesic) loops based at that point will be referred to simply as flowers and nets that are made of two vertices connected by at most* m *(not necessarily geodesic) segments or nets that are* m*-flowers will be referred to as* m*-cages, (or cages).*

Here, we allow a graph to have multiple edges between its vertices and to have loops. This object is sometimes referred to as a multigraph.

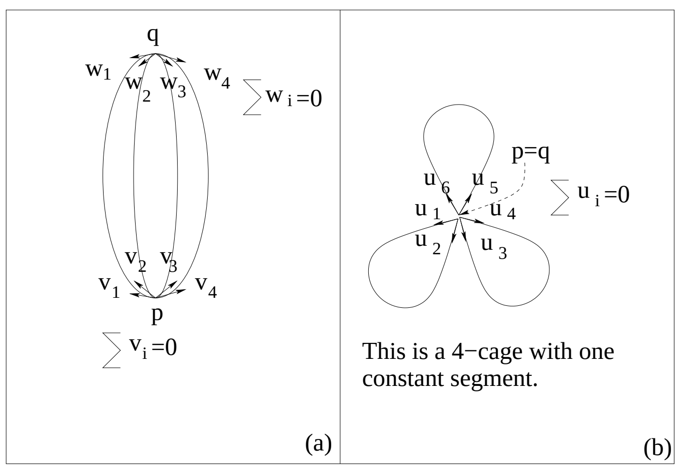

Figure 2: A non-degenerate stationary 4-cage and minimal geodesic 3-flower.

Example 1. *Obviously, a closed geodesic is a stationary* 2*-cage. It can also be considered as a geodesic flower with one petal. Geodesic loops are minimal geodesic cages if and only if they are periodic geodesics. A minimal*

θ-graph will be an example of a stationary 3*-cage, and so will be a stationary figure* 8*. In the latter case two points p and q coincide and the length of the third segment equals zero. Some examples of stationary* 4*-cages can be found on figure 2.*

In [NRo1] we have proved analogs of Conjecture 2 and Conjecture 3 for integral 1-cycles. That is we obtained diameter (and volume) estimates for the smallest length of a stationary 1-cycle.

In [NRo2] and [Ro1] we have obtained similar estimates for m-cages. Finally in [Ro2], we have obtained similar estimates for geodesic flowers.

The main idea of the papers [NRo2] and [Ro1] is that *in the absence of the stationary cages, contracting* 1*-skeleton of a sphere to a point leads to a homotopy contracting k-spheres to a point.* The 1-skeleton is contracted using "the cage shortening process" similar to the Birkhoff length shortening process for closed curves, sometimes denoted as BCSP, (cf. [C] for the detailed description). Note, that if one applies a length shortening process to a (non-degenerate) m-cage, it is possible for it to degenerate into a flower. That is, the length of one of its edges can become zero, and the two vertices will then coincide.

The idea we have used in [Ro2] is that we can define a weighted length functional on the space of cages such that its gradient flow will "force" critical cages to degenerate into geodesic flowers. In other words, it can be arranged so that a (non-degenerate) stationary m-cage is not a critical point of the new functional.

Definition 0.2 ([Ro2]) *(1) Let Γ be a (not necessarily geodesic) net with edges $e_1, ..., e_i, ..., e_k$. Then $L(\Gamma) = \Sigma_{i=1}^k m_i length(e_i)$, where $m_i \in \mathbf{Z}_+$ and $length(e_i)$ is the length of the edge e_i will be called a weighted length functional with weights $m_1, ..., m_k$. (Note that it corresponds to the regular length functional on the net, where each edge e_i is taken with a multiplicity m_i.)*
(2) A net N is critical with respect to a weighted length functional L with weights $m_i, i = 1, ..., k$ if for any one-parametric smooth flow of diffeomorphisms $\Phi_t, t = 0$ is a critical point of $\mu(t) = L(\Phi_t(N))$. It is equivalent to all edges being geodesic segments combined with the following stationarity condition satisfied at every vertex of N: the weighted sum of unit vectors tangent to edges of N at that vertex and directed from it, equals to zero.

The new idea of this paper is that the weighted length functional can be applied repeatedly with different weights, not with the goal of obtaining flowers, but with a goal of obtaining distinct critical points.

Example 2. *Let us consider the space of* 3*-cages and let* Γ *be an element of this space. That is* Γ *is a graph with two vertices* p *and* q *and three edges* e_1, e_2, e_3. *Define* $L_1(\Gamma) = length(e_1) + 2length(e_2) + 3length(e_3)$. *and* $L_2(\Gamma) = 3length(e_1) + 4length(e_2) + 5length(e_3)$. *We claim that critical points of* L_1 *and* L_2 *are stationary cages that are geometrically distinct, unless it is a periodic geodesic.*

To prove that let us examine their critical points. First, let us look at the possible critical points of L_1.

Note that a non-degenerate 3-cage can not be a critical point of L_1. Indeed, one of the conditions for it to be critical is that $v_1 + 2v_2 + 3v_3 = 0$, where v_1, v_2, v_3 are unit vectors tangent to e_1, e_2, e_3 respectively at p. Obviously, this condition can be satisfied if and only if the cage is a periodic geodesic. Therefore, if Γ is critical then either one of the e_i's degenerates into a point and the cage degenerates either into a "figure 8" or a periodic geodesic. In the case of a "figure 8", the two loops can have the following multiplicities: (a) 1 and 2; (b) 1 and 3; (c) 2 and 3.

Next, let us look at the critical points of L_2. Its critical points can be
1. a cage that consists of two vertices and three edges with multiplicities $3, 4, 5$, like in the original cage;
2.a "figure 8" with the following multiplicities (a) 3 and 4; (b) 3 and 5; (c) 4 and 5.
3. a periodic geodesic;

Even in the case of "figure 8", the stationary cages will be different, because the pair of multiplicities are not multiples of each other. Thus, if the appearance of periodic geodesics can be excluded, we will obtain different geodesic nets.

We then combine this idea with the techniques of [Ro1] that will be explained in the next section.

Note that although their critical points are not necessarily critical points of the length functional, one can make them into such by taking some of the edges with appropriate integer weights. Observe that if a stationary cage Cg that consists of k distinct edges $e_1, ..., e_k$ is a critical point for the weighted length functional with weights $m_1, ..., m_k$ then a stationary cage $\tilde{Cg}$ that consists of the geodesic edges e_i taken with multiplicities m_i, $i = 1, ..., k$ will be a stationary cage, that is a critical point for the regular length functional. This observation will be used throughout the paper.

0.2 Main results. In the next section we will prove the following two theorems.

Theorem 0.3 *Let M^n be a closed Riemannian manifold of dimension n and of diameter d. Let $q = \min\{\pi_i(M^n) \neq \{0\}\}$. Then either the length of a shortest closed geodesic is $\leq (q+1)d$ or there exists infinitely many stationary cages on M^n. In particular, there exists a constant $C(k,n)$, which can be explicitely calculated, such that there exists at least k distinct stationary geodesic nets of length $\leq C(k,n)d$.*

Theorem 0.4 *Let M^n be a closed Riemannian manifold of dimension n and of volume $vol(M^n)$. Then either there exists a periodic geodesic of length $\leq \tilde{c}(n)vol(M^n)^{\frac{1}{n}}$, or there exist infinitely many geometrically distinct minimal geodesic nets. In particular, there exists a constant $\tilde{C}(k,n)$, such that there exists at least k distinct stationary geodesic nets of length $\leq \tilde{C}(k,n)vol(M^n)^{\frac{1}{n}}$. These constants can be explicitely estimated.*

Remark 1. It can be arranged that nets that appear in Theorems 0.3 and 0.4 would be stationary geodesic flowers. However, to achieve that, the bounds on the length of a shortest periodic geodesic should be made significantly worse, (although, they still be of the form $c(n)d$ and $\tilde{c}(n)vol(M^n)^{\frac{1}{n}}$).

In this section we will prove Theorem 0.3 in the case when $q = 2$. Note that when $q = 1$, the length of a shortest closed geodesic is bounded above by $2d$, thus Theorem 0.3 is trivially satisfied.

Proof of Theorem 0.3 for q=2. Let $f : S^2 \longrightarrow M^n$ be a non-contractible map from a sphere endowed with a fine triangulation, (the diameter of each simplex $\leq \delta$) into M^n. The proof will be done by contradiction. That is, assuming there is no periodic geodesic of length $\leq 3d$ and only finitely many geometrically distinct cages, we will extend the map f to D^3 triangulated as a cone over the triangulation on S^2, thus reaching a contradiction.

The extension will be done as follows: it is trivial to extend f to the 0**-, **1**-, and** 2**-skeleta** of D^3. We map the center of the disc, $\tilde{p}$ to an arbitrary point $p \in M^n$, edges of the form $[\tilde{p}, \tilde{v}_i]$ to minimal geodesic segments denoted as $[p, v_i]$ connecting the point p with the vertex in the induced triangulation $v_i = f(\tilde{v}_i)$ and, finally, we map the 2-simplices of the form $\tilde{\sigma}_i^2 = [\tilde{p}, \tilde{v}_{i_1}, \tilde{v}_{i_2}]$ to the surface generated by a length-decreasing homotopy connecting $f(\partial\tilde{\sigma}_i^2)$ and a point. Here we are using the assumption that the length of a shortest periodic geodesic is bigger than $3d$.

Extending to the 3**-skeleton, (see fig. 3).** To extend to the 3-skeleton we will use a trick used in [NRo2], [Ro1], [Ro2], except we will use it

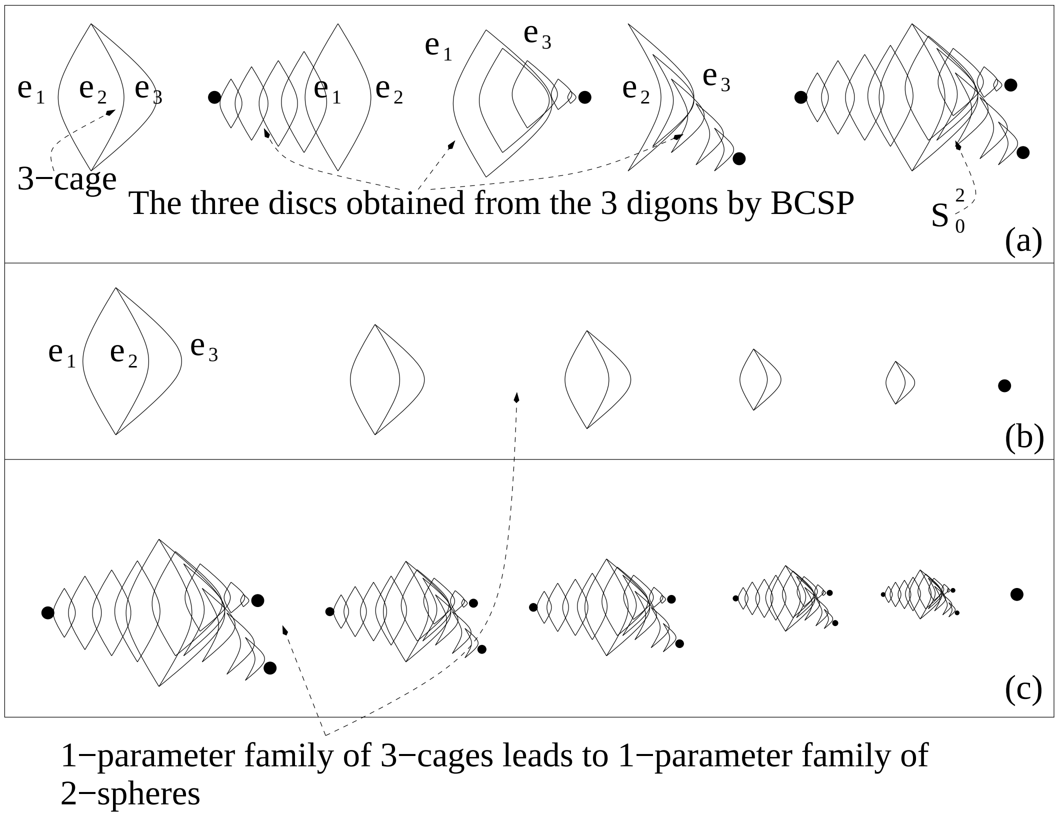

Figure 3: Extending to 3-skeleton.

infinitely many times. The trick is the following. Let $\tilde{\sigma}_i^3 = [\tilde{p}, \tilde{v}_{i_1}, \tilde{v}_{i_2}, \tilde{v}_{i_3}]$ be an arbitrary 3-simplex. Its boundary consists of four 2-simplices glued in an obvious way. One of the simplices $[\tilde{v}_{i_1}, \tilde{v}_{i_2}, \tilde{v}_{i_3}]$ and consequently $[v_{i_1}, v_{i_2}, v_{i_3}]$ is so small that it can be treated as a point q. This assertion can be made more rigourous, (see Remark 2.). Let us consider the image of 1-skeleton of $\tilde{\sigma}_i^3$. Assuming $[v_{i_1}, v_{i_2}, v_{i_3}]$ is a point, the image of the 1-skeleton consists of three edges that we will denote as e_1, e_2, e_3 respectively, connecting p and q. This is a 3-cage. This 3-cage corresponds, of course, to the 2-sphere $f(\partial\tilde{\sigma}_i^3)$ constructed from it by considering 3 of its digons and contracting each pair to a point by BCSP, (see fig. 3 (a)). The idea running through papers $[NRo2], [Ro1], [Ro2]$ is that if we can contract this cage to a point, then we can contract $f(\partial\tilde{\sigma}_i^3)$ to a point as well.

The new idea is that there are infinitely many ways we can try to contract this cage to a point. If one of them works then we have suceeded in extending the map to any 3-simplex in the triangulation of D^3. If none of them works, then for each try we get a new obstraction, which happens to be a stationary 3-cage. As with geodesics, we should be careful that we, indeed, get geometrically distinct nets and not a multiple of the same net.

First, let us explain why it suffices to contract the 3-cage to a point. As we would like to extend our map to the 3-simplex, we want to construct a 3-disc that has $f(\partial\tilde{\sigma}_i^3)$ as its boundary. This disc can be constructed as a 1-parameter family of spheres $S_\tau^2, 0 \leq \tau \leq 1$ that starts with the original sphere $f(\partial\tilde{\sigma}_i^3)$ and ends with a point, (see fig. 3 (c)). This one parameter family of spheres is constructed as follows: Let $N_\tau, 0 \leq \tau \leq 1$ be a 3-cage during some fixed deformation of the initial 3-cage to a point, (see fig. 3 (b). We can consider three digons formed by $(e_1)_\tau, (e_2)_\tau$, $(e_2)_\tau, (e_3)_\tau$ and $(e_3)_\tau, (e_1)_\tau$. Note that, it is possible that one or two of the segments have length zero. Each of these digons varies continuously with τ. Moreover, each of these digons can be contracted to a point without the length increase, assuming there is no short geodesics, thus generating 2-discs. It is essential that those discs depend continuously on the initial digon. Next, at each time τ we identify the three discs that correspond to the three digons as in the boundary of the 3-simplex, obtaining S_τ^2. Note also, that if we succeed at contracting the net to a point then S_1^2 is also a point and, thus, we have obtained a 3-disc.

Next, we would like to describe the many ways to contract a net. They correspond to many ways of assigning the multiplicity coefficients to the edges of the 3-net that result in geometrically distinct stationary geodesic nets. We, in fact, suggest a canonical way of assigning the coefficients, namely, $m, m+1, m+2, m \in \{1, 3, 5, ...\}$. We now would like to note that the infinite number of ways of assigning coefficients does, indeed, lead to infinite number of critical poins and not just multiples of finitely many geometrically distinct critical points.

First, note that during the deformation of the net one of the following three things can happen:

(a) An edge can disappear. Then the possible critical point will be a geodesic flower. In fact, it will be a figure 8, of one of the following three types, depending on which edge disappeared: (1) one petal has multiplicity m and the second petal has a multiplicity $(m+1)$; (2) one petal has multiplicity m and the second one has a multiplicity $(m+2)$; (3) one petal has multiplicity $(m+1)$ and the second petal has multiplicity $(m+2)$. The question is for which different positive integers m and k the corresponding weighted length functionals result in geometrically the same nets. Assume $m > k$. Then getting the same net would mean that the following two equations have to be simultaneoulsy satisfied:

$m + x_1 = \alpha(k + y_1)$ **and** $m + x_2 = \alpha(k + y_2)$**, where** $x_1, x_2, y_1, y_2 \in \{0, 1, 2\}$ **and** $x_1 \neq x_2$**, whereas** $y_1 \neq y_2$**.**

Analysing this equations we come to the conclusion that the following are the only non-trivial ways in which they can be satisfied by positive integers:

(1) $\alpha = 1, m = k+1$. This possibility can be excluded, once we assume that m, k are odd integers.

(2) $\alpha = 2, m = 2k$ or $\alpha = 2, m = 2k+2$. This possibility is also excluded when m is odd.

Thus, in the case of disappearence of an edge, our method does, indeed produce distinct cages.

Note, it is possible that after one edges disappears, the remaining two petals merge, but the only stationary geodesic net of this type would be a short periodic geodesic.

(b) Two edges disappear. In that case the resulting stationary cage can only be a periodic geodesic. Its length is at most $\frac{md+(m+1)d+(m+2)d}{m} = 3d + \frac{3d}{m}$. As m becomes large, this bound approaches $3d$.

(c) None of the edges disappear. In that case one can easily see that the stationary points must be different for different integers. $\square$

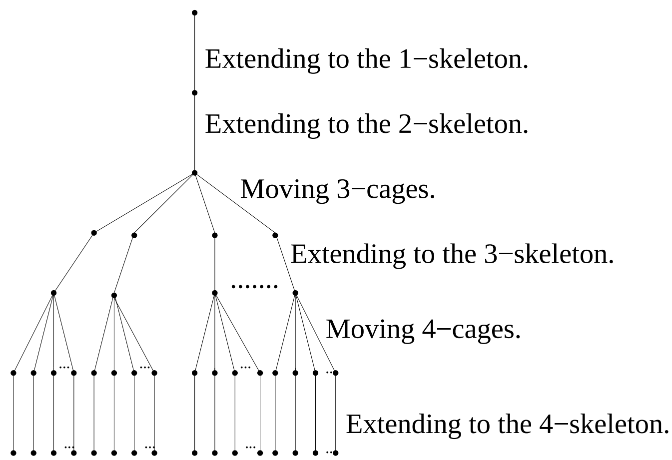

Figure 4: The scheme of the proof.

1 The scheme of the proofs of Theorems 0.3 and 0.4.

The rough scheme of the proof of Theorem 0.3 goes as follows. We begin with a non-contractible map of a sphere $S^q \longrightarrow M^n$ of a smallest possible dimension to a manifold M^n. Assuming the conclusion of Theorem 0.3 is not satisfied we will extend the map to a disc, thus reaching a contradiction. The extension process has a structure of **R**-tree. Let us describe it in more detail, (see fig. 4).

Step 1. First, let us note that it is trivial to extend the map to 0-, 1- and 2-skeleta of D^{q+1}. It will be done exactly as in the case of $q = 2$ with which we dealt in the previous section. The only assumption that we will need to use is that there is no "short" periodic geodesics on M^n.

Step 2. In the previous section we also dealt with extending to the 3-skeleton. Recall that we found infinitely many ways of extending to the 3-skeleton. They correspond to infinitely many weighted length functionals, $L^3(m^3_j)$, on the space of 3-cages, where $\{m^3_j\}$ is a sequence of natural numbers, . Note that either $L^3(m^3_j)$ indeed leads to extension to the 3-skeleton, or there exists a critical point corresponding to it. We must assume that there are only finitely many such critical points, or we are done. Let us represent each such extension attempt to the 3-skeleton by a branch of the tree. There are infinitely many branches. If the attempt leads to a critical point, we will call such branch a dead branch. If not there will be other branches growing out of it.

Step 3. Let us consider a particular extension to the 3-skeleton, let's say, $L^3(m^3_*)$. We will now extend the previous map to the 4-skeleton. Likewise, there will be infinitely many attempts (branches), corresponding to different ways of contracting 4-cages in accordance with different weighted length functionals $L^4(m^4_j)$, where $\{m^4_j\}$ is a sequence of natural numbers. Those attempts will be successful, unless $L^4(m^4_j)$ has a critical point. We are yet to show that different weighted length functionals correspond to different critical points. Nevertheless, each successful attempt corresponds to a 4-disc as follows. Consider a 1-parameter family of 4-cages $Cg^4_\tau, \tau \in [0,1]$ that starts with our cage and ends with a point. At time $\tau = 0$ it is a 1-skeleton of a 3-sphere S^3_0. We can construct a 1-parameter family of 3-spheres that begins with S^3_0 and ends with a point. This family will generate a disc. Now each sphere S^3_τ is constructed from four 3-discs by gluing them as in the boundary of the four-simplex, but keeping in mind that the fifth disc is simply a point. 3-discs are constructed exactly as in Step 2. That is, we consider four 3-cages obtained from Cg^4_τ, by forgetting one of the edges, to each of the triplets we apply the weighted length shorteing process associated with $L^3(m^*_3)$, etc. Step 2 is, thus, repeated at each τ giving the process the structure of an **R**-tree.

Remark 2. Here are some principles on which the proof is based:
1. Consider a non-contractible map $f : S^q \longrightarrow M^n$. The infinitely many distinct geodesic nets correspond to infinitely many attempts of extending this map to D^{q+1}. Of course, all of these attempts should fail. These nets are obstructions to extensions. They are the same only if they all happen to be the same "short" periodic geodesic.
2. The infinitely many extension attempts correspond to infinitely many different weighted length functionals, which in its turn correspond to infinitely many ways of contracting k-cages to a point with a controlled total length, which, in its turn leads to a 1-parameter family of cages, that begins with the original cage and ends with a point.
3. To each 1-parameter family of k-cages described in 2 there is a k-disc that "fills" it. This allows us to extend to the k-skeleton, $k \leq q+1$.
4. Discs are constructed from spheres, (of one dimension smaller), spheres are constructed from discs, (of the same dimension). A k-disc is constructed by producing a 1-parameter family of $(k-1)$-spheres that start with the original sphere and end with a point. This family of spheres is created by contracting the original k-cage to a point, (using an assumption

that there is no "small" geodesic cages) and at each time constructing a $(k-1)$-sphere, as it was discussed before. A sphere is obtained by gluing discs, just as we glue $(k-1)$-dimensional simplices in the boundary of a k-simplex to obtain a sphere, (only, since one of those simplices is small, we treat it as a point). These discs are obtained from $(k-1)$-cages. For example, when we extend to 3-skeleton, (see fig. 5), three discs are obtained from 3-cages as one parameter family of 2-spheres. 2-spheres are obtained by gluing three 2-discs, that are constructed by applying the BCSP to the three digons obtained at each $\tau, \tau \in [0,1]$ from a 3-cage as its 2-subcages.

5. Continuity is important. So, after we have extended to k-skeleton, using, let's say, the cage shortening process associated with $L^k(m_j^k)$, every time we have to "move" a k-cage, we have to use the same cage shortening process. Thus, from each "live" m_j^3-branch there are growing infinitley many m_j^4-branches. Also from each "live" m_j^4-branch there are growing infinitely many m_j^5-branches, etc. Here we also use a fact, that in the absence of minimal objects those spheres will change continuously. This proof uses a length shortening process for k-cages, which is an adaptation of a general length shortening process introduced in [NRo1]. A simplified version is also used in [NRo2], [Ro1], [Ro2]. For the sake of completeness we discuss how to adapt the process of [NRo1] to the case of k-cages in section 2.

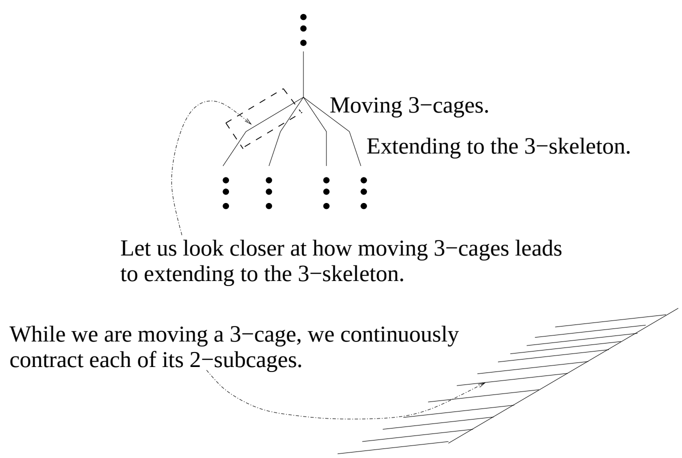

Figure 5: The scheme of the proof.

Theorem 0.3 will be proved in Section 2. In Section 3 we will prove Theorem 0.4. The proof is based on the combination of the ideas from the proof of Theorem 0.3 and the idea by M. Gromov from [Gr] involving filling

M^n by a polyhedron W^{n+1} in $L^\infty(M^n)$, attempting to extend the identity map on M^n to W^{n+1} and obtaining a geodesic net as an obstruction to this extension.

2 The proof of Theorem 0.3.

Lemma 2.1 *Let M^n be a closed Riemannian manifold of dimension n. There exists infinitely many different weighted length functionals $L^k(m_j^k)$, where m_j^k is a natural number and $j = 0, 1, 2, ...$ on the space of k-cages, such that their critical points are geometrically distinct, unless they are periodic geodesics.*

Proof. Consider the weighted length functional $L^k(m_j^k)$ with weights $m_j^k + c_1, ..., m_j^k + c_k$, where c_i is a natural number. We would like to find a sequence $\{m_j^k\}_{j=0}^{\infty}$, such that the corresponding functionals have different critical points unless they are closed geodesics.

Let us note that during the length shortening process one or both of the following two things can happen: (1) two or more edges can become one; (2) one or more edges can shrink to a point. In the later case, the cage will degenerate into a flower.

Now let us consider two functionals $L^k(\eta)$ and $L^k(m)$. Suppose there is a cage that is a critical point of both functionals. Let us denote its edges by $e_1, ..., e_s$, where $s \leq k$. The edges have multiplicities, which might be different, when we consider it as a critical point of $L^k(m)$ from those when we consider it as a critical point of $L^k(\eta)$. Let $a_i\eta + d_i$ denote the multiplicity of e_i with respect to $L^k(\eta)$ and $b_ik + f_i$ the multiplicity of e_i with respect to $L^k(m)$, $i = 1, ..., s$. Note that if the net is not a closed geodesic then $s \geq 2$. The only way, it can be a critical point of both of these functionals if the multiplicity coefficients satisfy the following s equations:

$$a_i\eta + d_i = \alpha(b_im + f_i), i = 1, ..., s$$

where α is a positive integer proportionality constant. Let us take a look at the first two such equations. The first one implies that $\eta = \frac{\alpha(b_1m+f_1)-d_1}{a_1}$. Combined with the second equation we will have $\alpha(a_2(b_1m + f_1) - a_1(b_2m + f_2)) = a_2d_1 - a_1d_2$, where $a_i, b_i, f_i, d_i, \alpha, m, \eta$ are all integers. Unless $a_2(b_1m + f_1) - a_1(b_2m + f_2) = 0$, $\alpha = \frac{a_2d_1-a_1d_2}{a_2(b_1m+f_1)-a_1(b_2m+f_2)} \leq \max\{a_2d_1, a_1d_2\} \leq k^2 \max_{i=1}^k c_i$. Note also, that in this case $\eta =$

$\frac{\alpha(b_1m+f_1)-d_1}{a_1}$. To make this case impossible define $m_j^k=(1+k^2\max_{i=1}^k c_i)^j$, Now let us examine the case when $a_2(b_1m+f_1)-a_1(b_2m+f_2)=0$. This equation has more than one solution $m(a_1,a_2,b_1,b_2,f_1,f_2)$ if and only if $f_1=\frac{a_1f_2}{a_2}$ and $b_1=\frac{a_1b_2}{a_2}$. We claim that we can easily find coefficients $c_1,..,c_k$, so that it never happens. Recall that a_i and b_i are just number of different edges merging into the considered edge. The numbers f_1 and f_2 are sums of c_j components of the wieghts of the mering edges. Therefore, let $c_1=k^2,...,c_i=k^{2i},...,c_k=k^{2k}$. WLOG, assume that $b_2>b_1$ and that $f_2>f_1$. Note that $\frac{b_2}{b_1}\leq k$. On the other hand, let us examine $\frac{f_2}{f_1}$. For some $r\leq k$ $\frac{f_2}{f_1}>\frac{k^{2r}}{(k-1)k^{2(r-1)}}>k$. Therefore, in such a case it cannot happen that $\frac{f_2}{f_1}=\frac{b_2}{b_1}$.

□

Proof of Theorem 0.3. The theorem will be proved by contradiction. Let M^n be a closed Riemannian manifold, such that $\pi_1(M^n)=...=\pi_{q-1}(M^n)=\{0\}$ and $\pi_q(M^n)\neq\{0\}$. Let $f:S^q\longrightarrow M^n$ be a non-contractible map of a finely triangulated sphere to M^n. Assuming there are no "small" periodic geodesics and only finetely many stationary geodesic nets, we will extend this map to the disc D^{q+1} of dimension $q+1$, thus reaching a contradiction. To construct this extension, we will triangulate the disc as the cone over the chosen triangulation of the sphere. The procedure will then be inductive on skeleta of D^{q+1}. To begin with, the center of the disc will be mapped to an arbitrary point in M^n and the edges will be mapped to minimal geodesic segments that connect this point with corresponding vertices of the triangulation of the image sphere. Finally, to extend to the 2-skeleton, we will consider simplices of the form $\tilde{\sigma}_i^2$. The boundary is mapped to a closed curve of length $\leq 2d+\delta$. As there are no short periodic geodesics, it can be contracted to a point using the regular Birkhoff curve shortening process. We will use the disc generated by this homotopy to extend to $\tilde{\sigma}_i^2$.

The rest of the extension procedure uses two ideas: 1. "filling" k-cages by k-discs for all values of $k\leq q+1$, which is an inductive bootstrap procedure similar to the one used in [Ro1] and [Ro2]: Assuming that we have extended our map to the k-skeleton, we will explain how to extend it to the $(k+1)$-skeleton of D^{q+1}. In order to do that we will extend f to each $(k+1)$-dimensional simplex of D^{q+1} and in order to do that it will be necessary to "fill" $(k+1)$-cages by $(k+1)$-dimensional discs. Note that from a previous step of the induction we already know how to "fill" $(k+1)$-cages by k-spheres.

That is, consider the image of the boundary of the above simplex. It consists of $k+2$ k-dimensional simplices, one of which is so small that it can be treated as a point, (see Remark 3). Here the idea is that we can contract this simplex to a point over itself, reducing our situation to the situation, where the simplex is treated as a point. The remaining $k+1$ k-simplices were already obtained during the previous step of the extension process. Thus, the inductive step of the extension reduces to contracting k-spheres "filling" k-cages.

2. In the absence of infinitely many stationary geodesic nets and short periodic geodesics, there is a homotopy connecting the cage with a point.

Now suppose we were able to extend the map $f : S^q \longrightarrow M^n$ to the k-skeleton of D^{q+1}. In fact, there are infinitely many such extensions, corresponding to infinitely many weighted length functionals $L^{k+1}(m_j^{k+1})$. We will extend each of these extensions to the $(k+1)$-skeleton, also in infinitely many ways. That will be done as follows: let $\tilde{\sigma}_i^{k+1}$ be a $k+1$-simplex in the triangulation of D^{q+1}. Consider a $(k+1)$-cage Cg that corresponds to the image of 1-skeleton of this simplex and apply a weighted length shortening process $L^{k+1}(m_j^{k+1})$ for m_j^{k+1} constructed as in Lemma 2.1. Our assumption imply that for infinitely many of them the cage contracts to a point unless it degenerates and gets stuck at a periodic geodesic. Observe that if there exist only finitely many stationary geodesic nets, then infinitely many such length-shortening process will lead to a homotopy that connects the net with a point.

Then the cage can be contracted to a point along a 1-parameter family of cages Cg_τ, $\tau \in [0,1]$ of smaller weighted length. We can now construct a 1-parameter family of spheres S_τ^k of dimension k that starts with the image of the boundary of the given simplex and ends with a point, and thus generates a $(k+1)$-dimensional disc. Spheres are constructed by the procedure of "filling" cages Cg_τ at each τ described in [Ro1]. That is given a $(k+1)$-cage Cg_τ, consider its $k+1$ "k-subcages", i.e. k-tuples $(e_1)_\tau, ..., (\hat{e}_j)_\tau, ..., (e_{k+1})_\tau$ obtained by ignoring one of the curves. By induction assumption, each of these subcages can be "filled" by discs of dimension k. (The base of induction is proved by contracting 2-cages, i.e. closed curves, by the usual Birkhoff curve shortening process. At this point we are using the assumption that there are no short periodic geodesics.)

We then glue these $(k+1)$ discs as in the boundary of $(k+1)$-simplex to obtain S_τ^k. It is important to note that this process is continuous with respect to Cg_τ. This one-parameter family of spheres generates the desired

$(k+1)$-dimensional disc that can be used to extend f.

The only thing it remains is to evaluate the constant in the theorem. For that, let us consider the $k+1$-cage and the length-shortening process associated with $L^{(k+1)}(m_j^{k+1})$ for some fixed j, with weights $m_j^{k+1}+c_1^{k+1},...,m_j^{k+1}+c_{k+1}^{k+1}$. For simplicity, let us $A^{k+1}=m_j^{k+1}$. Note that as we deform the cage, the total (not weighted) length of any pair of curves never exceeds $(k+1)d+\frac{const.(d)}{A^{k+1}}$, which approaches $(k+1)d$ as A^{k+1} approaches infinity. Thus, we can assume that if the length of a shortest periodic geodesic does not exceed this quantity, any digon formed by curves of this cage is contractible to a point by the Birkhoff length shortening process. Now let us consider a k-subcage of the $(k+1)$-cage at any stage during its deformation together with the weighted length shortening functional $L^k(m_J^k)$ with the weights $m_J^k+c_1^k,...,m_J^k+c_k^k$. Let $A^k=m_J^k$. We know that the original total weighted length should be not greater than the length of the total cage, which is $((k+1)A^{k+1}+\Sigma_{i=1}^{k+1}c_i^{k+1})d$. From this we can deduce that the total length of any pair of curves of the k-subcage during the deformation does not exceed $(k+1)d$ plus the terms that will approach zero, as A^k, A^{k+1} approach infinity. Similarly, we can consider $(k-1)$-subcages of k-subcages, etc. $\square$

Remark 3. Let us consider a sphere in the manifold M^n obtained by taking a small 2-simplex $[v_{i_1},v_{i_2},v_{i_3}]$ and a point p, connecting p with each v_{i_j} by a minimal geodesic segment $e_j, j=1,2,3$, and finally, by contracting each of the closed curves $e_j+[v_{i_j},v_{i_{j \bmod 3+1}}]-e_{j \bmod 3+1}$, where $j=1,2,3$ to a point, (see fig. 6 (a)). We claim that for all practical purposes $[v_{i_1},v_{i_2},v_{i_3}]$ can be treated as a point q. Simply take $q\in[v_{i_1},v_{i_2},v_{i_3}]$. Consider the boundary $\partial[v_{i_1},v_{i_2},v_{i_3}]=[v_{i_2},v_{i_3}]-[v_{i_1},v_{i_3}]+[v_{i_1},v_{i_2}]$. Let us denote each of the segments $[v_{i_j},v_{i_{j \bmod 3+1}}]$ as $s_j, j=1,2,3$. Without loss of generality, we can assume that $s_1+s_2+s_3$ can be contracted to q in $[v_{i_1},v_{i_2},v_{i_3}]$ without the length increase. Moreover, each of the vertices, v_{i_j} will trace a trajectory σ_j of length $\leq\varepsilon(\delta)$, such that ε approaches 0 as δ approaches 0. Let us denote the images of s_j under the homotopy as s_{j_t}, and the trajectories traced at the time t as σ_{j_t} Then instead of curves $e_j+[v_{i_j},v_{i_{j \bmod 3+1}}]-e_{j \bmod 3+1}$ consider new curves $e_j+\sigma_{j_t}+s_{j_t}-\sigma_{j \bmod 3+1_t}-e_{j \bmod 3+1}$ of length $\leq 2d+2\varepsilon(\delta)+3\delta$, (see fig. 6 (b), (c)). Each of those curves is contractible to a point without the length increase, assuming there is no short geodesics. Moreover, at $t=1$ we will obtain the sphere that is constructed as follows: take two points p and q and connect them with three segments $e_j^*=e_j+\sigma_j$, $j=1,2,3$. Then take three digons $e_j^* - e^*_{j \bmod 3+1}$ and contract them to the point, (see fig. 6 (d). Thus, the initial and the final 2-spheres are homotopic. We can eventually let δ go to 0. The same idea works in higher dimension as well.

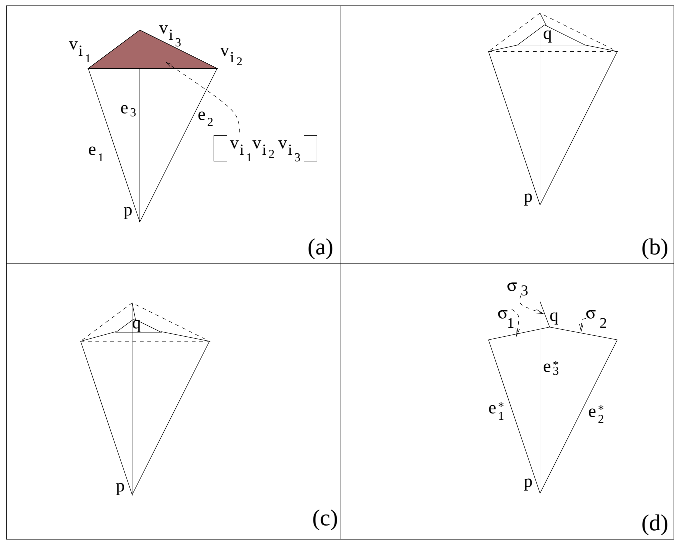

Figure 6: A small 2-simplex can be ignored.

3 The proof of Theorem 0.4.

Theorem 0.4 is proved by similar methods. However, we will need the definition of the Filling Radius below originally defined by M. Gromov in [Gr].

Definition 3.1 **[Gr]** *Let M^n be an abstract manifold and let $X = L^\infty(M^n)$ be the Banach space of bounded Borel functions f on M^n. Let M^n be isometrically imbedded in X, where the imbedding of M^n into X is the map that assigns to each point p of M^n the distance function $p \longrightarrow f_p = d(p,q)$. Then the filling radius $FillRadM^n$ is the infimum of $\varepsilon > 0$, such that M^n bounds in the ε-neighborhood $N_\varepsilon(M^n)$, i.e. homomorphism $H_n(M^n) \longrightarrow H_n(N_\varepsilon(M^n))$ vanishes, where $H_n(M^n)$ denotes the singular homology group of dimension n with coefficients in $\mathbf{Z}$, when M is orientable, and with coefficients in $\mathbf{Z}_2$, when M is not orientable.*

Alternatively, one can give a different definition of the filling radius of M^n by defining first $FillRad(M^n \subset X)$, the filling radius of M^n isometrically imbedded into some metric space X, as the smallest ε, such that M^n bounds in the ε-neighborhood of M^n and then taking the infimum over all of the isometric imbeddings. It was shown by M. Katz that $FillRadM^n \leq \frac{d}{3}$, where d is the diameter of M^n, (see [K]).

In [Gr] M. Gromov had found an estimate for the filling radius of a closed Riemannian manifold in terms of the volume of this manifold.

Theorem 3.2 [Gr] *Let M^n be a closed connected Riemannian manifold. Then $FillRadM^n \leq gc(n)(vol(M^n))^{\frac{1}{n}}$, where $gc(n) = (n+1)n^n(n+1)!^{\frac{1}{2}}$ and $vol(M^n)$ denotes the volume of M^n.*

In this section we will prove the following

Theorem 3.3 *Let M^n be a closed Riemannian manifold. Then either there exists a periodic geodesic on M^n of length $\leq a(n)FillRadM^n$ or there exists infinitely many geometrically distinct geodesic nets. In particular, there exists a constant $A(n,k), k \in \{1,2,3,...\}$, such that for any k there exist k geometrically distinct stationary geodesic nets of length $\leq A(n,k)FillRadM^n$. Moreover, $a(n)$ and $A(n,k)$ can be explicitely calculated.*

Theorem 3.3 combined with Theorem 3.2 leads to the volume bound in Theorem 0.4.

The proof of Theorem 3.3 is based on the combination of the ideas from the proof of Theorem 0.3 and an adaptation of the trick by M. Gromov from [Gr] involving filling M^n by a polyhedron W^{n+1} in $L^\infty(M^n)$, attempting to extend the identity map on M^n to W^{n+1} and otaining a short periodic geodesic or infinitely many geometrically distinct stationary geodesic nets as an obstruction to this extension.

The details of the proof of Theorem 3.3 are very similar to that of Theorem 0.3, except that instead of contracting k-cages, we will be contracting 1-skeletons of simplices. The spheres and discs are then built out of those 1-skeletons in a similar fashion. Also, for each k the weighted length functionals applied to 1-skeletal net will be $L^{\sigma^k}(m_j^k)$ with weights $m_j^k + c_1^k, ..., m_j^k + c^k_{\frac{k(k+1)}{2}}$, where the weight $m_j^k + c_i^k$ corresponds to edge e_i of the 1-skeleton of k-dimensional simplex σ^k. One can then prove lemma similar to Lemma 2.1 that one can find constants c_i^k and a sequence m_j^k, such that each such length functional has geometrically distinct critical points, unless it is a periodic geodesic.

Proof of Theorem 0.4. Suppose the conclusion of Theorem 0.4 is not satisfied, that is there is only finitely many geometrically distinct minimal geodesic nets and no "short" periodic geodesics.

By Definition 3.1 M^n bounds in the $(FillRadM^n + \delta)$-neighborhood of M^n in $L^\infty(M^n)$. Let W be a polyhedron, (see [Gr] for the proof of the existence), that "fills" M^n in the $(FillRadM^n + \delta)$-neighborhood of M^n. That is, $M^n = \partial W$, when M^n is orientable and $M^n = \partial W$ mod 2, when M^n is not orientable. We will begin by triangulating W and M^n be triangulated so that the diameter of any simplex in this triangulation is smaller than some small $\delta > 0$. We will extend the identity map $id : M^n \longrightarrow M^n$ to W, thus reaching a contradiction.

The extension procedure will be inductive to skeleta of W. It is different from the extension procedure employed in the proof of Theorem 0.3 only in the initial stages of extending to the $0,1$-skeleta of W.

Let us begin with the 0-**skeleton** of W. To each vertex $\tilde{w}_i \in W$ we will assign a vertex $w_i \in M^n$, that is closest to $\tilde{w}_i$. Thus, $d(\tilde{w}_i, w_i) \leq FillRadM^n + \delta$. Next, to extend to the 1-**skeleton**, we will assign to each edge of the form $[\tilde{w}_i, \tilde{w}_j] \subset W \backslash M^n$ a minimal geodesic segment $[w_i, w_j]$ connecting w_i and w_j of length $\leq 2FillRadM^n + 3\delta$. WLOG, we can assume that all edges in M^n are already short. Now, let us go to the 2-**skeleton**. Let $\tilde{\sigma}^2_{i_0,i_1,i_2} = [\tilde{w}_{i_0}, \tilde{w}_{i_1}, \tilde{w}_{i_2}]$ be an arbitrary 2-simplex. Its boundary is mapped to a closed curve of length $\leq 6FillRadM^n + 9\delta$. This curve can be contracted to a point without the length increase, using the Birkhoff curve shortening process, assuming, of course, that there is no periodic geodesics of smaller length. Moreover, assuming there is no "short" periodic geodesics, this homotopy depends continuously on the original curve. We will map $\tilde{\sigma}^2_{i_0,i_1,i_2}$ to a surface that is generated by above mentioned homotopy, that we will denote as $\sigma^2_{i_0,i_1,i_2}$.

Next let us go to the 3-**skeleton**. Consider an arbitrary 3-simplex $\tilde{\sigma}^3_{i_0,i_1,i_2,i_3} = [\tilde{w}_{i_0}, ..., \tilde{w}_{i_3}]$. By the previous step of the induction, its boundary is mapped to the following chain: $\Sigma^3_{j=0}(-1)^j \sigma^2_{i_i,...,\hat{i}_j,...,i_3}$. Consider its 1-skeleton. It will be a (not geodesic) net, that we will denote by K_i. Let us apply a weighted length shortening process $L^{\sigma^3}(m^3_j)$ to continuously deform it to a point. In fact, there exists infinitley many such weighted length functionals, as there is no "short" periodic geodesics and only finitely many distinct nets.

At each time $\tau, \tau \in [0,1]$ during this deformation, we can use the net $(K_i)_t$ to construct a 2-dimensional sphere S^2_τ in a way that is analogous to the similar construction in the proof of Theorem 0.3. This 1-parameter family of 2-spheres can be regarded as a 3-disc that we will denote as $\sigma^3_{i_0,\ldots,i_3}$. We will assign it to $\tilde{\sigma}^3_{i_0,\ldots,i_3}$. We can continue in a similar fashion until we reach the $(n+1)$-skeleton of W, thus constructing a singular chain on M^n, that has the fundamental class $[M^n]$ as its boundary, and therefore, arriving at a contradiction.

□

4 Length shortening proces for m-cages.

In this section we will describe a length shortening process for m-cages. A similar length shortening process for curves was introduced by G. Birkhoff and is described in detail in section 2 of [C]. Consider the length functional on the space C^m_L of the immersed m-cages of length $\leq L$. One can construct a flow on C^m_L that decreases the length functional, assuming there is no stationary m-cages of length $\leq L$. Note that closed curves and points can also be regarded as m-cages. We claim that in such a case there exists a deformation retraction of C^m_L to M^n, such that the length functional decreases along the trajectory of the deformation. Consider an m-cage consisting of two vertices a and b and m curves $\alpha_i, i = 1, ..., m$ that join those vertices.

The length shortening process we will describe is very similar to the Birkhoff Curve Shortening Process.

We will begin by replacing the curves α_i's by piecewise geodesics. This is done by subdividing each of the curves into many equal "small" segments, each of length $\leq injrad(M^n)/4$, where $injrad(M^n)$ denotes the injectivity radius of M^n, and then replacing each small segment by the minimal geodesic segment. Clearly, the original m-cage and the new piecewise geodesic m-cage will be homotopic by a length-decreasing homotopy. Moreover, this homotopy will continuously depend on the initial cage. (This observation is analogous to the starting point of Birkhoff Curve Shortening Process, (see [C])).

Thus, we find a deformation retraction of C^m_L to a finite dimensional space that we will denote FC^m_L, such that the length of an arbitrary edge does not increase during this deformation.

FC^m_L can be regarded as a subset of $(M^n)^N$ for a sufficiently large N.

Let $Cg^m \in FC^m_L$. We can define a vector of steepest descent tangent to FC^m_L at Cg^m. It will be defined as follows: consider all the vertices, (i.e. non-smooth points of m-cage). There will be many vertices, where two geodesic segments come together and two points a and b, where m geodesic segments come together. If $a = b$, there will be one point where $\leq 2m$ tangent vectors come together.

At each vertex consider the sum of the diverging unit vectors tangent to geodesic segments meeting at this vertex, (see fig. 7). This collection of vectors tangent to M^n constitutes the vector of the steepest descent for Cg^m. Note also, that it will also "work" for m-cages that are sufficiently close to Cg^m. That is, for any m-cage, sufficiently close to Cg^m, if we parallel transport our vector to that m-cage, we will obtain a vector such that the first variation of the length functional in the direction of this vector will be negative. Now choosing an appropriate locally finite partition of unity we can construct a vector field on FC_L^m such that the first variation of the length functional in the direction of this field is negative and FC_L^m deforms to FC_0^m in a finite time.

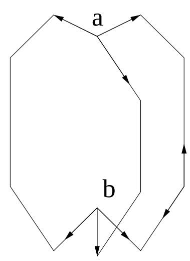

Figure 7: Length Shortening Process for θ-graph.

This process is a very much simplified version of the process described in paper [NR1], (see the proof of a Morse-theoretic type lemma for the space of 1-cycles made of at most k segments, (Lemma 3) in [NR1], in which we show that, assuming there are no non-trivial stationary 1-cycles in the space of 1-cycles Γ_k^x made of at most k segments of length $\leq x$, then the space Γ_k^0 of 1-cycles of 0 length is a deformation retract of Γ_k^x). All the technical difficulties that arise during this deformation were dealt with in [NR1]. One can find it summarized for θ-graphs in [NR2], (see section 3: Length-decreasing process for θ-graphs). During this length shortening process, it can happen that the length of one of the edges becomes 0 and the two points a and b coincide.

We will then have to move this unique vertex in the direction of the sum of all unit vectors tangent to geodesic segments and diverging from this vertex. Another difficulty is that despite the fact that the total length of each cage decreases, the distance between two neighboring vertices can increase. We want this distance to remain smaller than $injradM^n$. Otherwise we will not be able to connect the endpoints by a unique geodesic segment. Therefore, to resolve this difficulty, we apply the flow only for the time $t = \frac{injradM^n}{4}$. Then we stop, divide each segment into equal segments of length $\leq \frac{injradM^n}{4}$ and replace it by a piecewise geodesic curve, as it was done in the beginning. Then we apply the flow again for $t = \frac{injradM^n}{4}$ etc.

Under this curve shortening process the m-cage converges either to a stationary m-cage, (possibly degenerate, where two vertices coincide and lenghts of one or more segments equal zero), or to a point.

Acknowledgments. This paper was partially written during the authors' visit of the Max-Planck Institute, Bonn. The authors would like to thank the Max-Planck Institute for its kind hospitality. Both authors gratefully acknowledge the partial support from Natural Sciences and Engineering Research Council (NSERC) Research Grants. Regina Rotman also acknowledges the partial supposrt from NSERC University Faculty award. Both authors also gratefully acknowledge the partial support from their National Science Foundation research grants.

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