# Torus $n$-Point Functions for $\mathbb{R}$-graded Vertex Operator Superalgebras and Continuous Fermion Orbifolds

Geoffrey Mason*
Department of Mathematics,
University of California Santa Cruz,
CA 95064, U.S.A.

Michael P. Tuite, Alexander Zuevsky†
Department of Mathematical Physics,
National University of Ireland,
Galway, Ireland.

August 31, 2007

**Abstract**

We consider genus one $n$-point functions for a vertex operator superalgebra with a real grading. We compute all $n$-point functions for rank one and rank two fermion vertex operator superalgebras. In the rank two fermion case, we obtain all orbifold $n$-point functions for a twisted module associated with a continuous automorphism generated by a Heisenberg bosonic state. The modular properties of these orbifold $n$-point functions are given and we describe a generalization of Fay's trisecant identity for elliptic functions.

*Partial support provided by NSF, NSA and the Committee on Research, University of California, Santa Cruz

†Supported by a Science Foundation Ireland Frontiers of Research Grant, and by Max-Planck Institut für Mathematik, Bonn

# 1 Introduction

This paper is one of a series devoted to the study of $n$-point functions for vertex operator algebras on Riemann surfaces of genus one, two and higher [T], [MT1], [MT2], [MT3]. One may define $n$-point functions at genus one following Zhu [Z], and use these functions together with various sewing procedures to define $n$-point functions at successively higher genera [T], [MT2], [MT3]. In this paper we consider the genus one $n$-point functions for a Vertex Operator Superalgebra (VOSA) $V$ with a real grading (i.e. a chiral fermionic conformal field theory). In particular, we compute all $n$-point functions for rank one and rank two fermion VOSAs. In the latter case, we consider $n$-point functions defined over an orbifold $g$-twisted module for a continuous $V$ automorphism $g$ generated by a Heisenberg bosonic state. We also consider the Heisenberg decomposition (or bosonization) of $V$ and recover elliptic versions of Fay's generalized trisecant identity together with a new further generalization. The modular properties of the continuous orbifold $n$-point functions are also described.

In his seminal paper, Zhu defined and developed a constructive theory of torus $n$-point functions for a $\mathbb{Z}$-graded Vertex Operator Algebra (VOA) and its modules [Z]. In particular, he described various recursion formulae where, for example, an $n$-point function is expanded in terms of $n-1$-point functions and naturally occurring Weierstrass elliptic (and quasi-elliptic) functions. Indeed, one can prove the analytic, elliptic and modular properties of $n$-point functions for many VOAs from these recursion formulae (op.cit.). This technique has since been extended to include orbifold VOAs with a $g$-twisted module for a finite order automorphism $g$ [DLM1], $\frac{1}{2}\mathbb{Z}$-graded VOSAs [DZ1] and $\mathbb{Z}$-graded VOSAs [DZ2]. Here we consider a further generalization to obtain recursion formulae for torus $n$-point functions for an $\mathbb{R}$-graded VOSA. We consider $n$-point functions defined as the supertrace over the product of various vertex operators together with a general element of the automorphism group of the VOSA. The resulting recursion formula is expressed in terms of natural "twisted" Weierstrass elliptic functions periodic up to arbitrary multipliers in $U(1)$. Such elliptic functions already appear in ref. [DLM1] for multipliers of finite order. Here, we give a detailed description of twisted Weierstrass elliptic functions (and associated twisted Eisenstein series) for general $U(1)$ multipliers generalizing many results of the classical theory of elliptic functions.

We consider two applications of the Zhu recursion formula. The first example is that of the rank one $\frac{1}{2}\mathbb{Z}$-graded fermion VOSA. In this case, all $n$-point functions can be computed in terms of a single generating function. In particular, we obtain expressions for these $n$-point functions in a natural Fock basis in terms of the Pfaffian of an appropriate block matrix. The second example is that of the rank two fermion VOSA. As is well known, this VOSA contains a Heisenberg vector which generates a continuous automorphism $g$ and for which a $g$-twisted module can be constructed [Li]. The Heisenberg vector can also be employed to define a "shifted" Virasoro with real grading [MN2], [DM]. We demonstrate a general relationship between the $n$-point functions for orbifold $g$-twisted modules and the shifted VOSA. We next apply the recursion formula for $\mathbb{R}$-graded VOSAs in order to obtain all continuous orbifold $n$-point functions. These are expressible in terms of determinants of appropriate block matrices in a natural Fock basis and can again be obtained from a single generating function. Decomposing the rank two fermion VOSA into Heisenberg irreducible modules as a bosonic $\mathbb{Z}$-lattice VOSA (i.e. bosonization) we may employ results of ref. [MT1] to find alternative expressions for the $n$-point functions. In particular the generating function is expressible in terms of theta functions and the genus one prime form and we thus recover Fay's generalized trisecant identity for elliptic functions. We also prove a further generalization of Fay's trisecant identity based on the $n$-point function for $n$ lattice vectors. The paper concludes with a determination of the modular transformation properties for all rank two continuous orbifold $n$-point functions generalizing Zhu's results for $C_2$-cofinite VOAs [Z].

The study of $n$-point functions has a long history in the theoretical physics literature and we recover a number of well known physics results here. Thus the Pfaffian and determinant formulas for the rank one and two fermion generating functions and the relationship between Fay's generalized trisecant identity have previously appeared in physics [R1, R2, EO, RS, FMS, P]. However, it is important to emphasize that our approach is constructively based on the properties of a VOSA and that a rigorous and complete description of these $n$-point functions has been lacking until now. Thus, for example, no assumption is made about the local analytic properties of $n$-point functions as would normally be the case in physics. Similarly, other pure mathematical algebraic geometric approaches to $n$-point functions are based on an assumed local analytic structure [TUY]. Finally, apart from the intrinsic benefits of this rigorous approach, it is important to obtain a complete description of these $n$-point functions as the building blocks used in the construction of higher genus partition and $n$-point functions [T], [MT2], [MT3].

The paper is organized as follows. We begin in Section 2 with a review of classical Weierstrass elliptic functions and Eisenstein series. We introduce twisted Weierstrass functions which are periodic up to arbitrary elements of $U(1)$. We describe various expansions of these twisted functions, introduce twisted Eisenstein series and determine their modular properties. Section 3 contains one of the central results of this paper. We begin with the defining properties of an $\mathbb{R}$-graded VOSA $V$. We define $n$-point functions as a supertrace over a $V$-module and describe some general properties. We then formulate a generalization of Zhu's recursion formula [Z] to an $\mathbb{R}$-graded VOSA module making use of the twisted Weierstrass and Eisenstein series. Section 4 contains a discussion of a VOSA containing a Heisenberg vector. We prove the general relationship between the $n$-point functions for a VOSA with a Heisenberg shifted Virasoro vector and $g$-twisted $n$-point functions where $g$ is generated by the Heisenberg vector. In section 5 we apply the results of Section 3 to a rank one fermion VOSA. In particular, we compute all $n$-point functions in terms of a generating function given by a particular $n$-point function. We also discuss $n$-point functions for a fermion number-twisted module. Section 6 contains a description of a rank two fermion VOSA. We make use of the results of Section 3 and Section 4 to compute all $n$-point functions for a $g$-twisted module where $g$ is generated by a Heisenberg vector by means of a generating function. We next discuss the Heisenberg decomposition of this rank two theory - the bosonized theory. In particular, we derive an expression for the rank two generating function in terms of $\theta$-functions and prime forms related to Fay's generalized trisecant identity for elliptic functions. A further generalization for Fay's trisecant identity for elliptic functions is also discussed. Finally, we discuss the modular properties of all $n$-point functions for the rank two fermion VOSA. Properties of supertraces are recalled in the Appendix.

We collect here notation for some of the more frequently occurring functions and symbols that will play a role in our work. $\mathbb{Z}$ is the set of integers, $\mathbb{R}$ the real numbers, $\mathbb{C}$ the complex numbers, $\mathbb{H}$ the complex upper-half plane. We will always take $\tau$ to lie in $\mathbb{H}$, and $z$ will lie in $\mathbb{C}$ unless otherwise noted. For a symbol $z$ we set $q_z = \exp(z)$, in particular, $q = q_{2\pi i\tau} = \exp(2\pi i\tau)$.

## 2 Some Elliptic Function Theory

### 2.1 Classical Elliptic Functions

We discuss a number of modular and elliptic-type functions that we will need. We begin with some standard elliptic functions [La]. The Weierstrass $\wp$-function periodic in $z$ with periods $2\pi i$ and $2\pi i\tau$ is

$$\begin{aligned} \wp(z,\tau) &= \frac{1}{z^2} + \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} [\frac{1}{(z-\omega_{m,n})^2} - \frac{1}{\omega_{m,n}^2}]. &\quad (1)\\ &= \frac{1}{z^2} + \sum_{n\geq 4, n \text{ even}} (n-1)E_n(\tau)z^{n-2}, &\quad (2)\end{aligned}$$

for $(z,\tau)\in\mathbb{C}\times\mathbb{H}$ with $\omega_{m,n} = 2\pi i(m\tau+n)$. Here, $E_n(\tau)$ is equal to 0 for $n$ odd, and for $n$ even is the Eisenstein series [Se]

$$E_n(\tau) = -\frac{B_n(0)}{n!} + \frac{2}{(n-1)!}\sum_{r\geq 1}\frac{r^{n-1}q^r}{1-q^r}, \tag{3}$$

where $B_n(0)$ is the $n$th Bernoulli number (see (38) below). If $n\geq 4$ then $E_n(\tau)$ is a holomorphic modular form of weight $n$ on $SL(2,\mathbb{Z})$. That is, it satisfies

$$E_n(\gamma.\tau) = (c\tau+d)^n E_n(\tau), \tag{4}$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL(2,\mathbb{Z})$, where we use the standard notation

$$\gamma.\tau = \frac{a\tau+b}{c\tau+d}. \tag{5}$$

On the other hand, $E_2(\tau)$ is a quasimodular form [KZ] having the exceptional transformation law

$$E_2(\gamma.\tau) = (c\tau+d)^2E_2(\tau) - \frac{c(c\tau+d)}{2\pi i}. \tag{6}$$

We define $P_k(z,\tau)$ for $k\geq 1$ by

$$P_k(z,\tau) = \frac{(-1)^{k-1}}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}P_1(z,\tau) = \frac{1}{z^k} + (-1)^k\sum_{n\geq k}\binom{n-1}{k-1}E_n(\tau)z^{n-k}. \tag{7}$$

Then $P_2(z,\tau) = \wp(z,\tau) + E_2(\tau)$ whereas $P_1 - zE_2$ is the classical Weierstrass zeta function. $P_k$ has periodicities

$$\begin{aligned} P_k(z+2\pi i,\tau) &= P_k(z,\tau), \\ P_k(z+2\pi i\tau,\tau) &= P_k(z,\tau) - \delta_{k1}. \end{aligned} \tag{8}$$

We define the elliptic prime form $K(z,\tau)$ by [Mu]

$$K(z,\tau) = \exp(-P_0(z,\tau)), \tag{9}$$

where

$$P_0(z,\tau) = -\log(z) + \sum_{k\geq 2} \frac{1}{k} E_k(\tau) z^k, \tag{10}$$

so that

$$P_1(z,\tau) = -\frac{d}{dz} P_0(z,\tau) = \frac{1}{z} - \sum_{k\geq 2} E_k(\tau) z^{k-1}. \tag{11}$$

$K(z,\tau)$ has periodicities

$$\begin{aligned} K(z+2\pi i,\tau) &= -K(z,\tau), \\ K(z+2\pi i\tau,\tau) &= -q_z^{-1} q^{-1/2} K(z,\tau). \end{aligned} \tag{12}$$

We define the standard Jacobi theta function by[1] e.g. [FK]

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z,\tau) = \sum_{n\in\mathbb{Z}} \exp[i\pi(n+a)^2\tau + (n+a)(z+2\pi i b)], \tag{13}$$

with periodicities

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z+2\pi i,\tau) = e^{2\pi i a} \vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z,\tau), \tag{14}$$

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z+2\pi i\tau,\tau) = e^{-2\pi i b} q_z^{-1} q^{-1/2} \vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z,\tau). \tag{15}$$

We also note the modular transformation properties under the action of the standard generators $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ of $SL(2,\mathbb{Z})$ (with

[1] Note that the $z$ dependence of the theta function is chosen so that the periods are $2\pi i$ and $2\pi i\tau$ rather than the standard periods of 1 and $\tau$.

relations $(ST)^3 = -S^2 = 1$)

$$\vartheta\left[\begin{array}{c} a \\ b \end{array}\right](z,\tau+1) = e^{-i\pi a(a+1)}\vartheta\left[\begin{array}{c} a \\ b+a+\frac{1}{2} \end{array}\right](z,\tau), \tag{16}$$

$$\vartheta\left[\begin{array}{c} a \\ b \end{array}\right](-\frac{z}{\tau},-\frac{1}{\tau}) = (-i\tau)^{1/2}e^{2\pi iab}e^{-iz^2/4\pi\tau}\vartheta\left[\begin{array}{c} -b \\ a \end{array}\right](z,\tau). \tag{17}$$

$K(z,\tau)$ can be expressed in terms of half integral theta functions as

$$K(z,\tau) = \frac{\vartheta\left[\begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \end{array}\right](z,\tau)}{\frac{d}{dz}\vartheta\left[\begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \end{array}\right](0,\tau)} = \frac{-i}{\eta(\tau)^3}\vartheta\left[\begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \end{array}\right](z,\tau). \tag{18}$$

where the Dedekind eta-function is defined by

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n). \tag{19}$$

## 2.2 Twisted Elliptic Functions

Let $(\theta,\phi) \in U(1)\times U(1)$ denote a pair of modulus one complex parameters with $\phi = \exp(2\pi i\lambda)$ for $0 \leq \lambda < 1$. For $z \in \mathbb{C}$ and $\tau \in \mathbb{H}$ we define "twisted" Weierstrass functions for $k \geq 1$ as follows:

$$P_k\left[\begin{array}{c} \theta \\ \phi \end{array}\right](z,\tau) = \frac{(-1)^k}{(k-1)!}\sum_{n\in\mathbb{Z}+\lambda}{}' \frac{n^{k-1}q_z^n}{1-\theta^{-1}q^n}, \tag{20}$$

for $q = q_{2\pi i\tau}$ where $\sum'$ means we omit $n=0$ if $(\theta,\phi) = (1,1)$.

**Remark 2.1.** *(i) (20) was introduced in [DLM1] for rational $\lambda$, where it was denoted by $P_k(\phi,\theta^{-1},z,\tau)$. The alternative definition and notation used here is motivated by the modular and periodicity properties shown below and by the column vector notation for theta series.*

*(ii) (20) converges absolutely and uniformly on compact subsets of the domain $|q| < |q_z| < 1$ [DLM1].*

*(iii) For $k \geq 1$,*

$$P_k\left[\begin{array}{c} \theta \\ \phi \end{array}\right](z,\tau) = \frac{(-1)^{k-1}}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}P_1\left[\begin{array}{c} \theta \\ \phi \end{array}\right](z,\tau). \tag{21}$$

We now develop twisted versions of the standard results for the classical Weierstrass $\wp$-function reviewed above. A number of similar results appear in [DLM1]. However, the cases $k = 1, 2$ are treated separately there and only for rational $\lambda$ i.e. $\phi^N = 1$ for some positive integer $N$. The most canonical derivation of the periodic and modular properties of (20) for general $\lambda$ follow from the following theorem:

**Theorem 2.2.** *For $|q| < |q_z| < 1$ and $\phi \neq 1$,*

$$P_k \begin{bmatrix} \theta \\ \phi \end{bmatrix} (z, \tau) = \sum_{m \in \mathbb{Z}} \theta^m \left[ \sum_{n \in \mathbb{Z}} \frac{\phi^n}{(z - \omega_{m,n})^k} \right], \tag{22}$$

*whereas for $\theta \neq 1$,*

$$P_k \begin{bmatrix} \theta \\ \phi \end{bmatrix} (z, \tau) = \sum_{n \in \mathbb{Z}} \phi^n \left[ \sum_{m \in \mathbb{Z}} \frac{\theta^m}{(z - \omega_{m,n})^k} \right]. \tag{23}$$

**Remark 2.3.** *When* both *$\theta \neq 1$ and $\phi \neq 1$ then the double sums (22) and (23) are equal. For $k \geq 3$, they are absolutely convergent and equal for all $(\theta, \phi)$.*

In order to prove Theorem 2.2 it is useful to define the following convergent sum

$$S(x, \phi) = \sum_{n \in \mathbb{Z}} \frac{\phi^n}{x - 2\pi i n}. \tag{24}$$

Clearly

$$\begin{aligned} S(x + 2\pi i, \phi) &= \phi S(x, \phi), &(25)\\ S(x, \phi) &= -S(-x, \phi^{-1}). &(26) \end{aligned}$$

We then have:

**Lemma 2.4.** *For $\phi = \exp(2\pi i \lambda)$ with $0 \leq \lambda < 1$ we have*

$$S(x, \phi) = \frac{1}{2}\delta_{\lambda,0} + \frac{q_x^\lambda}{q_x - 1}. \tag{27}$$

**Proof.** Both $S(x,\phi)$ and $q_x^\lambda(q_x-1)^{-1}$ have simple poles at $x=2\pi in$ with residue $\phi^n$ for all $n\in\mathbb{Z}$. Furthermore, $q_x^\lambda(q_x-1)^{-1}$ is regular at the point at infinity for $0\le\lambda<1$. Thus $S(x,\phi)-q_x^\lambda(q_x-1)^{-1}$ is constant which from (25) and (26) must be given by $\frac{1}{2}\delta_{\lambda,0}$. $\square$

We first prove Theorem 2.2 for the case $k=1$ and $\phi\neq 1$ (i.e. $0<\lambda<1$). The double sum (22) is

$$\sum_{m\in\mathbb{Z}}\theta^m\left[\sum_{n\in\mathbb{Z}}\frac{\phi^n}{z-\omega_{m,n}}\right]=\sum_{m\in\mathbb{Z}}\theta^m S(x_m,\phi)=\sum_{m\in\mathbb{Z}}\theta^m\frac{q_{x_m}^\lambda}{q_{x_m}-1},$$

using Lemma 2.4 for $x_m=z-2\pi im\tau$ with $q_{x_m}=q_zq^{-m}$. Since $|q|<|q_z|<1$ we find for $m>0$ that $|q_{x_m}|>1$ and hence

$$\frac{q_{x_m}^\lambda}{q_{x_m}-1}=\sum_{r\le -1}q_z^{r+\lambda}(q^{-r-\lambda})^m.$$

Since $\left|\theta q^{-r-\lambda}\right|<1$ for $r\le -1$ we obtain

$$\begin{aligned}\sum_{m>0}\theta^m\left[\sum_{n\in\mathbb{Z}}\frac{\phi^n}{z-\omega_{m,n}}\right] &= \sum_{r\le -1}q_z^{r+\lambda}\sum_{m>0}(\theta q^{-r-\lambda})^m\\ &= -\sum_{r\le -1}\frac{q_z^{r+\lambda}}{1-\theta^{-1}q^{r+\lambda}}.\end{aligned}$$

Similarly for $m\le 0$ we have $|q_{x_m}|<1$, so that

$$\frac{q_{x_m}^\lambda}{q_{x_m}-1}=-\sum_{r\ge 0}q_z^{r+\lambda}(q^{-r-\lambda})^m.$$

Hence since $\left|\theta q^{r+\lambda}\right|<1$ for $r\ge 0$ we find

$$\sum_{m\le 0}\theta^m\left[\sum_{n\in\mathbb{Z}}\frac{\phi^n}{z-\omega_{m,n}}\right]=-\sum_{r\ge 0}\frac{q_z^{r+\lambda}}{1-\theta^{-1}q^{r+\lambda}}.$$

Altogether we obtain

$$\sum_{m\in\mathbb{Z}}\theta^m\left[\sum_{n\in\mathbb{Z}}\frac{\phi^n}{z-\omega_{m,n}}\right]=-\sum_{r\in\mathbb{Z}}\frac{q_z^{r+\lambda}}{1-\theta^{-1}q^{r+\lambda}}=P_1\left[\begin{array}{c}\theta\\ \phi\end{array}\right](z,\tau),$$

proving (22) for $k = 1$. The result for $k \geq 2$ follows after applying (21).

In order to prove (23) it is useful to first consider the following double sum for $\phi \neq 1$

$$A\left[\begin{array}{c}\theta\\ \phi\end{array}\right](z,\tau) = \sum_{m\in\mathbb{Z}} \theta^m \left[\sum_{n\in\mathbb{Z}} \phi^n \left(\frac{1}{z-\omega_{m,n}} - \frac{2}{z-\omega_{m,n-1}} + \frac{1}{z-\omega_{m,n-2}}\right)\right].$$

By (25) we find

$$\begin{aligned} A\left[\begin{array}{c}\theta\\ \phi\end{array}\right](z,\tau) &= \sum_{m\in\mathbb{Z}} \theta^m \left[S(x_m,\phi) - 2S(x_m+2\pi i,\phi) + S(x_m+4\pi i,\phi)\right] \\ &= (1-\phi)^2 P_1\left[\begin{array}{c}\theta\\ \phi\end{array}\right](z,\tau). \end{aligned} \tag{28}$$

On the other hand, we have

$$A\left[\begin{array}{c}\theta\\ \phi\end{array}\right](z,\tau) = \sum_{m\in\mathbb{Z}} \theta^m \left[\sum_{n\in\mathbb{Z}} \phi^n \frac{-8\pi^2}{(z-\omega_{m,n})(z-\omega_{m,n-1})(z-\omega_{m,n-2})}\right].$$

This sum is absolutely convergent since the summand is $O(|\omega_{m,n}|^{-3})$ for $|m|$, $|n|$ large. We may thus interchange the order of summation to find that, on relabelling, $A\left[\begin{array}{c}\theta\\ \phi\end{array}\right](z,\tau)$ becomes

$$\begin{aligned} &\sum_{m\in\mathbb{Z}} \phi^m \left[\sum_{n\in\mathbb{Z}} \theta^{-n} \left(\frac{1}{z-\omega_{-n,m}} - \frac{2}{z-\omega_{-n,m-1}} + \frac{1}{z-\omega_{-n,m-2}}\right)\right] \\ = \; &(-\frac{1}{\tau}) \sum_{m\in\mathbb{Z}} \phi^m \left[\sum_{n\in\mathbb{Z}} \theta^{-n} \left(\frac{1}{z'-\omega'_{m,n}} - \frac{2}{z'-\omega'_{m-1,n}} + \frac{1}{z'-\omega'_{m-2,n}}\right)\right] \\ = \; &(-\frac{1}{\tau}) \sum_{m\in\mathbb{Z}} \phi^m \left[S(x'_m,\theta^{-1}) - 2S(x'_{m-1},\theta^{-1}) + S(x'_{m-2},\theta^{-1})\right], \end{aligned} \tag{29}$$

where

$$z' = -\frac{z}{\tau}, \quad \tau' = -\frac{1}{\tau}, \quad \omega'_{m,n} = 2\pi i(m\tau' + n), \quad x'_m = z' - 2\pi i m\tau'. \tag{30}$$

Applying Lemma 2.4 with $\theta = \exp(-2\pi i\mu)$ for $0 \leq \mu < 1$, it follows that

$$\begin{aligned} &S(x'_m,\theta^{-1}) - 2S(x'_{m-1},\theta^{-1}) + S(x'_{m-2},\theta^{-1}) \\ = \; &(1-2+1).\frac{1}{2}\delta_{\mu,0} + \frac{q^{\mu}_{x'_m}}{q_{x'_m}-1} - 2\frac{q^{\mu}_{x'_{m-1}}}{q_{x'_{m-1}}-1} + \frac{q^{\mu}_{x'_{m-2}}}{q_{x'_{m-2}}-1}. \end{aligned}$$

We may next repeat the arguments above leading to (28) to find that (29) becomes

$$A\left[\begin{array}{c}\theta\\ \phi\end{array}\right](z,\tau)=(-\frac{1}{\tau})(1-\phi)^2P_1\left[\begin{array}{c}\phi\\ \theta^{-1}\end{array}\right](-\frac{z}{\tau},-\frac{1}{\tau}).$$

Comparing to (28), we find that for $\phi\neq 1$

$$P_1\left[\begin{array}{c}\theta\\ \phi\end{array}\right](z,\tau)=(-\frac{1}{\tau})P_1\left[\begin{array}{c}\phi\\ \theta^{-1}\end{array}\right](-\frac{z}{\tau},-\frac{1}{\tau}). \tag{31}$$

Considering this identity for $(z',\tau')$ of (30) and using

$$P_1\left[\begin{array}{c}\theta\\ \phi\end{array}\right](z,\tau)=-P_1\left[\begin{array}{c}\theta^{-1}\\ \phi^{-1}\end{array}\right](-z,\tau), \tag{32}$$

(which follows from (22)) it is clear that (31) holds for all $(\theta,\phi)\neq(1,1)$.

We may use (31) to prove (23) of Theorem 2.2 in the case $k=1$. The double sum of (23) becomes on relabelling

$$\begin{aligned}\sum_{m\in\mathbb{Z}}\phi^m\left[\sum_{n\in\mathbb{Z}}\frac{\theta^{-n}}{z-\omega_{-n,m}}\right] &= (-\frac{1}{\tau})P_1\left[\begin{array}{c}\phi\\ \theta^{-1}\end{array}\right](-\frac{z}{\tau},-\frac{1}{\tau})\\ &= P_1\left[\begin{array}{c}\theta\\ \phi\end{array}\right](z,\tau).\end{aligned}$$

The general result for $k\geq 2$ follows from (21). $\square$

Periodicity and modular properties now follow from Theorem 2.2. Thus we have

**Lemma 2.5.** *For $(\theta,\phi)\neq(1,1)$, $P_k\left[\begin{array}{c}\theta\\ \phi\end{array}\right](z,\tau)$ is periodic in $z$ with periods $2\pi i\tau$ and $2\pi i$ with multipliers $\theta$ and $\phi$ respectively.* $\square$

**Remark 2.6.** *Note that the periodicity in $2\pi i$ is determined by the second argument $\phi$ in contradistinction to the periodicity of the standard theta series (14). Periodicity for $(\theta,\phi)=(1,1)$ is given by (8).*

We now consider the modular properties. Define the standard left action of the modular group for $\gamma=\left(\begin{array}{cc}a&b\\ c&d\end{array}\right)\in\Gamma=SL(2,\mathbb{Z})$ on $(z,\tau)\in\mathbb{C}\times\mathbb{H}$ with

$$\gamma.(z,\tau)=(\gamma.z,\gamma.\tau)=(\frac{z}{c\tau+d},\frac{a\tau+b}{c\tau+d}). \tag{33}$$

We also define a *left* action of $\Gamma$ on $(\theta, \phi)$

$$\gamma. \begin{bmatrix} \theta \\ \phi \end{bmatrix} = \begin{bmatrix} \theta^a \phi^b \\ \theta^c \phi^d \end{bmatrix}. \tag{34}$$

Then we obtain:

**Proposition 2.7.** *For $(\theta, \phi) \neq (1,1)$ we have*

$$P_k\left(\gamma. \begin{bmatrix} \theta \\ \phi \end{bmatrix}\right)(\gamma.z, \gamma.\tau) = (c\tau + d)^k P_k \begin{bmatrix} \theta \\ \phi \end{bmatrix}(z, \tau). \tag{35}$$

**Proof.** Consider the case $k = 1$. It is sufficient to consider the action of the generators $S, T$ of $\Gamma$ where $S.(z,\tau) = (-\frac{z}{\tau}, -\frac{1}{\tau})$ and $T.(z,\tau) = (z, \tau+1)$. Then for $\gamma = S$, (35) is given by (31) whereas for $\gamma = T$, the result follows directly from definition (7). It is straightforward to check the relations $(ST)^3 = -S^2 = 1$ (using (32)) so that the result follows for $k = 1$. The general case follows from (21). $\square$

**Remark 2.8. (i)** *(35) is equivalent to Theorem 4.2 of [DLM1] for rational $\lambda$ after noting Remark 2.1 (i) and (34).*

**(ii)** *For $\gamma = -I$ one finds*

$$P_k \begin{bmatrix} \theta \\ \phi \end{bmatrix}(z, \tau) = (-1)^k P_k \begin{bmatrix} \theta^{-1} \\ \phi^{-1} \end{bmatrix}(-z, \tau). \tag{36}$$

We next introduce twisted Eisenstein series for $n \geq 1$, defined by

$$\begin{aligned} E_n \begin{bmatrix} \theta \\ \phi \end{bmatrix}(\tau) &= -\frac{B_n(\lambda)}{n!} + \frac{1}{(n-1)!} \sideset{}{'}\sum_{r \geq 0} \frac{(r+\lambda)^{n-1} \theta^{-1} q^{r+\lambda}}{1 - \theta^{-1} q^{r+\lambda}} \\ &\quad + \frac{(-1)^n}{(n-1)!} \sum_{r \geq 1} \frac{(r-\lambda)^{n-1} \theta q^{r-\lambda}}{1 - \theta q^{r-\lambda}}, \end{aligned} \tag{37}$$

where $\sum'$ means we omit $r = 0$ if $(\theta, \phi) = (1,1)$ and where $B_n(\lambda)$ is the Bernoulli polynomial defined by

$$\frac{q_z^\lambda}{q_z - 1} = \frac{1}{z} + \sum_{n \geq 1} \frac{B_n(\lambda)}{n!} z^{n-1}. \tag{38}$$

In particular, we note that $B_1(\lambda) = \lambda - \frac{1}{2}$.

**Remark 2.9.** *(i) (37) was introduced in [DLM1] for rational $\lambda$ where it was denoted by $Q_n(\phi, \theta^{-1}, \tau)$.*

*(ii)* $E_n \left[ \begin{array}{c} 1 \\ 1 \end{array} \right](\tau) = E_n(\tau)$, *the standard Eisenstein series for even $n \geq 2$, whereas* $E_n \left[ \begin{array}{c} 1 \\ 1 \end{array} \right](\tau) = -B_1(0)\delta_{n,1} = \frac{1}{2}\delta_{n,1}$ *for $n$ odd.*

We may obtain a Laurant expansion analogous to (7).

**Proposition 2.10.** *We have*

$$P_k \left[ \begin{array}{c} \theta \\ \phi \end{array} \right](z, \tau) = \frac{1}{z^k} + (-1)^k \sum_{n \geq k} \binom{n-1}{k-1} E_n \left[ \begin{array}{c} \theta \\ \phi \end{array} \right](\tau) z^{n-k}. \qquad (39)$$

**Proof.** Consider (20) for $k = 1$:

$$\begin{aligned} P_1 \left[ \begin{array}{c} \theta \\ \phi \end{array} \right](z, \tau) &= -\sum_{r \geq 0}{}' \frac{q_z^{r+\lambda}}{1 - \theta^{-1} q^{r+\lambda}} - \sum_{r \geq 1} \frac{q_z^{-r+\lambda}}{1 - \theta^{-1} q^{-r+\lambda}} \\ &= \frac{q_z^{\lambda}}{q_z - 1} - \sum_{r \geq 0}{}' q_z^{r+\lambda} \frac{\theta^{-1} q^{r+\lambda}}{1 - \theta^{-1} q^{r+\lambda}} \\ &\quad + \sum_{r \geq 1} q_z^{-r+\lambda} \frac{\theta q^{r-\lambda}}{1 - \theta q^{r-\lambda}} \\ &= \frac{1}{z} - \sum_{n \geq 1} E_n \left[ \begin{array}{c} \theta \\ \phi \end{array} \right](\tau) z^{n-1}, \end{aligned}$$

from (37) and (38). The general result then follows from (21). $\square$

**Remark 2.11.** *For $(\theta, \phi) = (1, 1)$ we have* $P_k \left[ \begin{array}{c} 1 \\ 1 \end{array} \right](z, \tau) = \frac{1}{2}\delta_{k,1} + P_k(z, \tau)$ *for $k \geq 1$.*

We also find

**Proposition 2.12.** *For $\phi \neq 1$ then*

$$E_k \left[ \begin{array}{c} \theta \\ \phi \end{array} \right](\tau) = \frac{1}{(2\pi i)^k} \sum_{m \in \mathbb{Z}} \theta^m \left[ \sum_{\substack{n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} \frac{\phi^n}{(m\tau + n)^k} \right], \qquad (40)$$

*whereas for* $\theta \neq 1$

$$E_k \left[ \begin{array}{c} \theta \\ \phi \end{array} \right] (\tau) = \frac{1}{(2\pi i)^k} \sum_{n \in \mathbb{Z}} \phi^n \left[ \sum_{\substack{m \in \mathbb{Z} \\ (m,n) \neq (0,0)}} \frac{\theta^m}{(m\tau + n)^k} \right]. \tag{41}$$

**Proof.** Expand the sum of (22) for $\phi \neq 1$ for $k = 1$ to find

$$P_1 \left[ \begin{array}{c} \theta \\ \phi \end{array} \right] (z, \tau) = \frac{1}{z} - \frac{1}{2\pi i} \sum_{m \in \mathbb{Z}} \theta^m \left[ \sum_{\substack{n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} \sum_{r \geq 1} (\frac{z}{2\pi i})^{r-1} \frac{\phi^n}{(m\tau + n)^r} \right].$$

Comparing with (39) then (40) follows. (41) similarly holds. $\square$

**Remark 2.13.** *When both* $\theta \neq 1$ *and* $\phi \neq 1$ *then (40) and (41) are equal. For* $k \geq 3$*, they are absolutely convergent and equal for all* $(\theta, \phi)$*. For* $k \geq 3$*, and* $(\theta, \phi) = (1, 1)$ *we obtain the standard Eisenstein series (3).*

From Proposition 2.7 it immediately follows that

**Proposition 2.14.** *For* $(\theta, \phi) \neq (1, 1)$*,* $E_k \left[ \begin{array}{c} \theta \\ \phi \end{array} \right]$ *is a modular form of weight* $k$ *where*

$$E_k \left( \gamma . \left[ \begin{array}{c} \theta \\ \phi \end{array} \right] \right) (\gamma . \tau) = (c\tau + d)^k E_k \left[ \begin{array}{c} \theta \\ \phi \end{array} \right] (\tau). \quad \square \tag{42}$$

**Remark 2.15.** *This is equivalent to Theorem 4.6 of [DLM1] for rational* $\lambda$*. (42) also holds for* $(\theta, \phi) = (1, 1)$ *for* $k \geq 3$*, whereas* $E_2$ *is quasi-modular.*

It is useful to note the analytic expansions:

$$P_1 \left[ \begin{array}{c} \theta \\ \phi \end{array} \right] (z_1 - z_2, \tau) = \frac{1}{z_1 - z_2} + \sum_{k,l \geq 1} C \left[ \begin{array}{c} \theta \\ \phi \end{array} \right] (k, l) z_1^{k-1} z_2^{l-1}, \tag{43}$$

$$P_1 \left[ \begin{array}{c} \theta \\ \phi \end{array} \right] (z + z_1 - z_2, \tau) = \sum_{k,l \geq 1} D \left[ \begin{array}{c} \theta \\ \phi \end{array} \right] (k, l, z) z_1^{k-1} z_2^{l-1}, \tag{44}$$

where for $k, l \geq 1$ we define

$$C \left[ \begin{array}{c} \theta \\ \phi \end{array} \right] (k, l, \tau) = (-1)^l \binom{k + l - 2}{k - 1} E_{k+l-1} \left[ \begin{array}{c} \theta \\ \phi \end{array} \right] (\tau), \tag{45}$$

$$D \left[ \begin{array}{c} \theta \\ \phi \end{array} \right] (k, l, \tau, z) = (-1)^{k+1} \binom{k + l - 2}{k - 1} P_{k+l-1} \left[ \begin{array}{c} \theta \\ \phi \end{array} \right] (\tau, z). \tag{46}$$

We also note that (36) implies

$$C\left[\begin{array}{c}\theta\\ \phi\end{array}\right](k,l,\tau) = -C\left[\begin{array}{c}\theta^{-1}\\ \phi^{-1}\end{array}\right](l,k,\tau), \tag{47}$$

$$D\left[\begin{array}{c}\theta\\ \phi\end{array}\right](k,l,\tau,z) = -D\left[\begin{array}{c}\theta^{-1}\\ \phi^{-1}\end{array}\right](l,k,\tau,-z). \tag{48}$$

Finally, we may also express the twisted Weierstrass functions in terms of theta series and the prime form as follows:

**Proposition 2.16.** *For $(\theta,\phi) \neq (1,1)$ with $\theta = \exp(-2\pi i\mu)$ and $\phi = \exp(2\pi i\lambda)$ then*

$$P_1\left[\begin{array}{c}\theta\\ \phi\end{array}\right](z,\tau) = \frac{\vartheta\left[\begin{array}{c}\lambda+\frac{1}{2}\\ \mu+\frac{1}{2}\end{array}\right](z,\tau)}{\vartheta\left[\begin{array}{c}\lambda+\frac{1}{2}\\ \mu+\frac{1}{2}\end{array}\right](0,\tau)}\frac{1}{K(z,\tau)}, \tag{49}$$

*whereas*

$$P_1\left[\begin{array}{c}1\\ 1\end{array}\right](z,\tau) = \frac{\frac{d}{dz}\vartheta\left[\begin{array}{c}\frac{1}{2}\\ \frac{1}{2}\end{array}\right](z,\tau)}{\frac{d}{dz}\vartheta\left[\begin{array}{c}\frac{1}{2}\\ \frac{1}{2}\end{array}\right](0,\tau)}\frac{1}{K(z,\tau)}. \tag{50}$$

**Proof.** For $(\theta,\phi) \neq (1,1)$ the result follows by comparing the periodicity and pole structure of each expression using (14) and (15). For $(\theta,\phi) = (1,1)$ the result follows from (11) and (18). □

# 3 $n$-Point Functions for $\mathbb{R}$-Graded Vertex Operator Superalgebras

## 3.1 Introduction to Vertex Operator Superalgebras

We discuss some aspects of Vertex Operator Superalgebra (VOSA) theory to establish context and notation. For more details see [B], [FHL], [FLM], [Ka], [MN1]. Let $V$ be a superspace i.e. a complex vector space $V = V_{\bar{0}} \oplus V_{\bar{1}} = \oplus_\alpha V_\alpha$ with index label $\alpha$ in $\mathbb{Z}/2\mathbb{Z}$ so that each $a \in V$ has a parity (fermion number) $p(a) \in \mathbb{Z}/2\mathbb{Z}$.

An $\mathbb{R}$-graded Vertex Operator Superalgebra (VOSA) is a quadruple $(V, Y, \mathbf{1}, \omega)$ as follows: $V$ is a superspace with a (countable) $\mathbb{R}$-grading where

$$V = \oplus_{r \geq r_0} V_r$$

for some $r_0$ and with parity decomposition $V_r = V_{\bar{0},r} \oplus V_{\bar{1},r}$. $\mathbf{1} \in V_{\bar{0},0}$ is the vacuum vector and $\omega \in V_{\bar{0},2}$ the conformal vector with properties described below. $Y$ is a linear map $Y : V \to (\mathrm{End}V)[[z, z^{-1}]]$, for formal variable $z$, so that for any vector (state) $a \in V$

$$Y(a, z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1}. \tag{51}$$

The component operators (modes) $a(n) \in \mathrm{End}V$ are such that $a(n)\mathbf{1} = \delta_{n,-1}a$ for $n \geq -1$ and

$$a(n)V_\alpha \subset V_{\alpha+p(a)}, \tag{52}$$

for $a$ of parity $p(a)$.

The vertex operators satisfy the locality property for all $a, b \in V$

$$(x - y)^N [Y(a, x), Y(b, y)] = 0, \tag{53}$$

for $N \gg 0$, where the commutator is defined in the graded sense, i.e.

$$[Y(a, x), Y(b, y)] = Y(a, x)Y(b, y) - (1)^{p(a)p(b)} Y(b, y)Y(a, x).$$

The vertex operator for the vacuum is $Y(\mathbf{1}, z) = Id_V$, whereas that for $\omega$ is

$$Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}, \tag{54}$$

where $L(n)$ forms a Virasoro algebra for central charge $c$

$$[L(m), L(n)] = (m - n)L(m + n) + \frac{c}{12}(m^3 - m)\delta_{m,-n}. \tag{55}$$

$L(-1)$ satisfies the translation property

$$Y(L(-1)a, z) = \frac{d}{dz} Y(a, z). \tag{56}$$

$L(0)$ describes the $\mathbb{R}$-grading with $L(0)a = wt(a)a$ for weight $wt(a) \in \mathbb{R}$ and

$$V_r = \{a \in V | wt(a) = r\}. \tag{57}$$

We quote the standard commutator property of VOSAs e.g. [Ka], [FHL], [MN1]

$$[a(m), Y(b, z)] = \sum\nolimits_{j\geq 0} \binom{m}{j} Y(a(j)b, z) z^{m-j}. \tag{58}$$

Taking $a = \omega$ this implies for $b$ of weight $wt(b)$ that

$$[L(0), b(n)] = (wt(b) - n - 1)b(n), \tag{59}$$

so that

$$b(n)V_r \subset V_{r+wt(b)-n-1}. \tag{60}$$

In particular, we define for $a$ of weight $wt(a)$ the zero mode

$$o(a) = \begin{cases} a(wt(a) - 1), & \text{for } wt(a) \in \mathbb{Z} \\ 0, & \text{otherwise,} \end{cases} \tag{61}$$

which is then extended by linearity to all $a \in V$.

## 3.2 Torus $n$-point Functions

In this section we will develop explicit formulas for the $n$-point functions for $\mathbb{R}$-graded VOSA modules at genus one [Z, DLM1, MT1, DZ1]. Let $(V, Y, \mathbf{1}, \omega)$ be an $\mathbb{R}$-graded VOSA. In order to consider modular-invariance of $n$-point functions at genus 1, Zhu introduced in ref. [Z] a second "square-bracket" VOA $(V, Y[,], \mathbf{1}, \tilde{\omega})$ associated to a given VOA $(V, Y(,), \mathbf{1}, \omega)$. We review some aspects of that construction here. The new square bracket vertex operators are defined by a change of co-ordinates, namely

$$Y[v, z] = \sum_{n\in\mathbb{Z}} v[n] z^{-n-1} = Y(q_z^{L(0)} v, q_z - 1), \tag{62}$$

with $q_z = \exp(z)$, while the new conformal vector is $\tilde{\omega} = \omega - \frac{c}{24}\mathbf{1}$. For $v$ of $L(0)$ weight $wt(v) \in \mathbb{R}$ and $m \geq 0$,

$$\begin{aligned} v[m] &= m! \sum_{i\geq m} c(wt(v), i, m) v(i), && (63) \\ \sum_{m=0}^{i} c(wt(v), i, m) x^m &= \binom{wt(v) - 1 + x}{i}. && (64) \end{aligned}$$

In particular we note that $v[0] = \sum\limits_{i\geq 0} \binom{wt(v)-1}{i} v(i)$.

We now define the torus $n$-point functions. Following (52) we let $\sigma \in \mathrm{Aut}(V)$ denote the parity (fermion number) automorphism

$$\sigma a = (-1)^{p(a)} a. \tag{65}$$

Let $g \in \mathrm{Aut}(V)$ denote any other automorphism which commutes with $\sigma$. Let $M$ be a $V$-module with vertex operators $Y_M$. We assume that $M$ is stable under both $\sigma$ and $g$ i.e. $\sigma$ and $g$ act on $M$ [DZ1]. The *n-point function* on $M$ for states $v_1, \ldots, v_n \in V$ and $g \in \mathrm{Aut}(V)$ is defined by[2]

$$\begin{aligned} F_M(g; v_1, \ldots v_n; \tau) &= F_M(g; (v_1, z_1), \ldots, (v_n, z_n); \tau) \\ &= \mathrm{STr}_M \left( g\, Y_M(q_1^{L(0)} v_1, q_1) \ldots Y_M(q_n^{L(0)} v_n, q_n) q^{L(0)-c/24} \right), \end{aligned} \tag{66}$$

$q = \exp(2\pi i \tau)$, $q_i = \exp(z_i)$, $1 \leq i \leq n$, for auxiliary variables $z_1, ..., z_n$ and where $\mathrm{STr}_M$ denotes the *supertrace* defined by

$$\mathrm{STr}_M(X) = Tr_M(\sigma X) = Tr_{M_{\bar{0}}}(X) - Tr_{M_{\bar{1}}}(X). \tag{67}$$

In Appendix A we describe some basic properties of the supertrace. Taking $g = 1$ and all $v_i = \mathbf{1}$ in (66) yields the *partition function* which we denote by

$$Z_M(\tau) = F_M(1; \tau) = \mathrm{STr}_M \left( q^{L(0)-c/24} \right). \tag{68}$$

We also denote the orbifold partition function for general $g$ by

$$Z_M(g, \tau) = F_M(g; \tau) = \mathrm{STr}_M \left( g q^{L(0)-c/24} \right). \tag{69}$$

For $g = 1$ (66) is defined by Zhu for a $\mathbb{Z}$-graded VOA [Z]. For $g$ of finite order, it is considered for $\mathbb{Z}$-graded VOAs in ref. [DLM1], $\frac{1}{2}\mathbb{Z}$-graded VOSAs in ref. [DZ1] and $\mathbb{Z}$-graded VOSAs in ref. [DZ2]. Here we generalize these results to an $\mathbb{R}$-graded VOSA for arbitrary $g$ commuting with $\sigma$.

For $n = 1$ in (66) we obtain the *1-point function* denoted by

$$Z_M(g, v_1, \tau) = F_M(g; (v_1, z_1); \tau) = \mathrm{STr}_M(g o(v_1) q^{L(0)-c/24}) \tag{70}$$

where $o(v_1)$ is the zero mode (61) and is independent of $z_1$. We note the following useful result relating any $n$-point function to a 1-point function:

[2] This $n$-point function would be denoted by $T((v_1, q_1), \ldots, (v_n, q_n), (1, g), q)$ in the notation of [DLM1] and [DZ1].

**Lemma 3.1.** *For states $v_1, v_2, \ldots, v_n$ as above we have*

$$\begin{aligned} & F_M(g;(v_1,z_1),\ldots,(v_n,z_n);\tau) \\ = \; & Z_M(g, Y[v_1,z_{1n}].Y[v_2,z_{2n}]\ldots Y[v_{n-1},z_{n-1n}].v_n,\tau) \qquad (71) \\ = \; & Z_M(g, Y[v_1,z_1].Y[v_2,z_2]\ldots Y[v_n,z_n].\mathbf{1},\tau), \qquad (72) \end{aligned}$$

*where $z_{ij} = z_i - z_j$.*

**Proof.** The proof follows Lemma 1 of ref. [MT1]. □

Every $n$-point function enjoys the following permutation and periodicity properties [Z], [MT1]:

**Lemma 3.2.** *Consider the $n$-point function $F_M$ for states $v_1, v_2, \ldots, v_n$, as above, where each $v_i$ is of weight $wt(v_i)$, parity $p(v_i)$ and is a $g$-eigenvector for eigenvalue $\theta_i^{-1}$.*

**(i)** *If $p(v_1) + \ldots + p(v_n)$ is odd then $F_M = 0$.*

**(ii)** *Permuting adjacent vectors,*

$$\begin{aligned} & F_M(g;(v_1,z_1),\ldots,(v_k,z_k),(v_{k+1},z_{k+1}),\ldots,(v_n,z_n);\tau) \\ = \; & (-1)^{p(v_k)p(v_{k+1})} F_M(g;(v_1,z_1),\ldots,(v_{k+1},z_{k+1}),(v_k,z_k),\ldots,(v_n,z_n);\tau). \end{aligned}$$

**(iii)** *$F_M$ is a function of $z_{ij} = z_i - z_j$ and is non-singular at $z_{ij} \neq 0$ for all $i \neq j$.*

**(iv)** *$F_M$ is periodic in $z_i$ with period $2\pi i$ and multiplier $\phi_i = \exp(2\pi i wt(v_i))$.*

**(v)** *$F_M$ is periodic in $z_i$ with period $2\pi i\tau$ and multiplier $\theta_i$.*

**Proof**. (i) This follows from definition (67).

(ii) Apply locality (53).

(iii) $F_M$ is a function of $z_{ij}$ from (71). Suppose $F_M$ is singular at $z_n = y$ for some $y \neq z_j$ for all $j = 1, \ldots, n-1$. We may assume that $z_0 = 0$ by redefining $z_i$ to be $z_i - z_0$ for all $i$. But from (72), $Y[v_n, z_n].\mathbf{1}|_{z_n=0} = v_n$ is non-singular at $z_{nj} \neq 0$. Applying (ii) the result follows for all $z_{ij}$.

(iv) This follows directly from the definition (66).

(v) Using (iii) we consider periodicity of $z_n$ wlog. Under $z_n \to z_n + 2\pi i\tau$ we have $F_M \to \hat{F}_M$ where

$$\begin{aligned}
\hat{F}_M &= q^{-c/24}\mathrm{STr}_M(gY(q_1^{L(0)}v_1, q_1)\ldots Y(q^{L(0)}q_n^{L(0)}v_n, qq_n)q^{L(0)}) \\
&= q^{-c/24}\mathrm{STr}_M(gY(q_1^{L(0)}v_1, q_1)\ldots q^{L(0)}Y(q_n^{L(0)}v_n, q_n)),
\end{aligned}$$

using $q^{L(0)}Y(b,z)q^{-L(0)} = Y(q^{L(0)}b, qz)$ (which follows from (59)). But

$$\begin{aligned}
&\mathrm{STr}_M(gY(q_1^{L(0)}v_1, q_1)\ldots q^{L(0)}Y(q_n^{L(0)}v_n, q_n)) \\
=\ & (-1)^{p(v_n)}\mathrm{STr}_M(Y(q_n^{L(0)}v_n, q_n)gY(q_1^{L(0)}v_1, q_1)\ldots Y(q_{n-1}^{L(0)}v_{n-1}, q_{n-1})q^{L(0)}) \\
=\ & \theta_n(-1)^{p(v_n)}\mathrm{STr}_M(gY(q_n^{L(0)}v_n, q_n)Y(q_1^{L(0)}v_1, q_1)\ldots Y(q_{n-1}^{L(0)}v_{n-1}, q_{n-1})q^{L(0)}) \\
=\ & \theta_n\mathrm{STr}_M(gY(q_1^{L(0)}v_1, q_1)\ldots Y(q_{n-1}^{L(0)}v_{n-1}, q_{n-1})Y(q_n^{L(0)}v_n, q_n)q^{L(0)}),
\end{aligned}$$

using $g^{-1}Y(v_n, q_n)g = Y(g^{-1}v_n, q_n) = \theta_n Y(v_n, q_n)$ and applying (ii) repeatedly. Thus $\hat{F}_M = \theta_n F_M$. $\square$

## 3.3 Zhu Recursion Formulas for $n$-Point Functions

We now prove a generalization of Zhu's $n$-point function recursion formula [Z] for the $n$-point function (66) for an $\mathbb{R}$-graded VOSA. We begin with the following Lemma which follows directly from (58):

**Lemma 3.3.** *Suppose that $u \in V$ is homogeneous of weight $wt(u) \in \mathbb{R}$. Then for $k \in \mathbb{Z}$ and $v \in V$ we have*

$$\left[u(k), Y(q_z^{L(0)}v, q_z)\right] = q_z^{k-wt(u)+1}\sum_{i\geq 0}\binom{k}{i}Y(q_z^{L(0)}u(i)v, q_z).\square \qquad (73)$$

**Corollary 3.4.** *Suppose that $u \in V$ is homogeneous of integer weight $wt(u) \in \mathbb{Z}$. Then*

$$\left[o(u), Y(q_z^{L(0)}v, q_z)\right] = Y(q_z^{L(0)}u[0]v, q_z). \qquad (74)$$

Similarly to Zhu's Proposition 4.3.1 (op.cit.) we find

**Proposition 3.5.** *Suppose that $v \in V$ is homogeneous of integer weight $wt(v) \in \mathbb{Z}$. Then for $v_1, \ldots, v_n \in V$, we have*

$$\sum_{r=1}^{n} p(v, v_1v_2\ldots v_{r-1})F_M(g; v_1; \ldots; v[0]v_r; \ldots v_n; \tau) = 0, \qquad (75)$$

*with $p(v, v_1v_2\ldots v_{r-1})$ of (147) in Appendix A.* $\square$

Let $v$ be homogeneous of weight $wt(v) \in \mathbb{R}$ and define $\phi \in U(1)$ by

$$\phi = \exp(2\pi i wt(v)). \tag{76}$$

We also take $v$ to be an eigenfunction under $g$ with

$$gv = \theta^{-1} v \tag{77}$$

for some $\theta \in U(1)$ so that

$$g^{-1} v(k) g = \theta v(k). \tag{78}$$

Then we obtain the following generalization of Zhu's Proposition 4.3.2 [Z] for the $n$-point function:

**Theorem 3.6.** *Let $v, \theta$ and $\phi$ be as as above. Then for any $v_1, \ldots v_n \in V$ we have*

$$\begin{aligned} & F_M(g; v, v_1, \ldots v_n; \tau) \\ = \quad & \delta_{\theta,1}\delta_{\phi,1} \mathrm{STr}_M \left( g o(v) Y_M(q_1^{L(0)} v_1, q_1) \ldots Y_M(q_n^{L(0)} v_n, q_n) q^{L(0)-c/24} \right) \\ & + \sum_{r=1}^{n} \sum_{m \geq 0} p(v, v_1 v_2 \ldots v_{r-1}) P_{m+1} \left[ \begin{array}{c} \theta \\ \phi \end{array} \right] (z - z_r, \tau) \cdot \\ & F_M(g; v_1, \ldots, v[m] v_r, \ldots, v_n; \tau). \end{aligned} \tag{79}$$

*(The twisted Weierstrass function is defined in (20)).*

**Proof.** We have

$$\begin{aligned} & q^{c/24} F_M(g; v, v_1, \ldots v_n; \tau) \\ = \quad & \sum_{k \in \mathbb{Z}} q_z^{-k-1+wt(v)} \mathrm{STr}_M \left( g \; v(k) Y_M(q_1^{L(0)} v_1, q_1) \ldots Y_M(q_n^{L(0)} v_n, q_n) q^{L(0)} \right). \end{aligned}$$

Thus we consider

$$\begin{aligned}
& \mathrm{STr}_M\left(g\ v(k)Y_M(q_1^{L(0)}v_1,q_1)\ldots Y_M(q_n^{L(0)}v_n,q_n)q^{L(0)}\right) \\
= \ & \mathrm{STr}_M\left(g\ [v(k),Y_M(q_1^{L(0)}v_1,q_1)\ldots Y_M(q_n^{L(0)}v_n,q_n)]q^{L(0)}\right) \\
& +p(v,v_1\cdots v_n)\mathrm{STr}_M\left(g\ Y_M(q_1^{L(0)}v_1,q_1)\ldots Y_M(q_n^{L(0)}v_n,q_n)v(k)q^{L(0)}\right) \\
= \ & \sum_{r=1}^{n}\sum_{i\geq 0}p(v,v_1\ldots v_{r-1})\binom{k}{i}q_r^{k+1-wt(v)}. \\
& \mathrm{STr}_M\left(g\ Y_M(q_1^{L(0)}v_1,q_1)\ldots Y_M(q_r^{L(0)}v(i)v_r,q_r)\ldots Y_M(q_n^{L(0)}v_n,q_n)q^{L(0)}\right) \\
& +\theta q^{k+1-wt(v)}\mathrm{STr}_M\left(g\ v(k)Y_M(q_1^{L(0)}v_1,q_1)\ldots Y_M(q_n^{L(0)}v_n,q_n)q^{L(0)}\right),
\end{aligned}$$

applying (59), (73), (78), (146) and Lemma 7.1 of Appendix A. Thus

$$\begin{aligned}
& \mathrm{STr}_M\left(g\ v(k)Y_M(q_1^{L(0)}v_1,q_1)\ldots Y_M(q_n^{L(0)}v_n,q_n)q^{L(0)}\right) \\
= \ & \frac{1}{1-\theta q^{k+1-wt(v)}}\sum_{r=1}^{n}\sum_{i\geq 0}p(v,v_1\ldots v_{r-1})\binom{k}{i}q_r^{k-wt(v)+1}. \\
& \mathrm{STr}_M\left(gY(q_1^{L(0)}v_1,q_1)\ldots Y(q_r^{L(0)}v(i)v_r,q_r)\ldots Y(q_n^{L(0)}v_n,q_n)q^{L(0)}\right),
\end{aligned}$$

provided $(\theta,\phi,k)\neq(1,1,-1+wt(v))$. This implies $F_M(g;v,v_1,\ldots v_n)$ is given by

$$\begin{aligned}
& \delta_{\theta,1}\delta_{\phi,1}\mathrm{STr}_M\left(go(v)Y_M(q_1^{L(0)}v_1,q_1)\ldots Y_M(q_n^{L(0)}v_n,q_n)q^{L(0)-c/24}\right) \\
& +\sum_{r=1}^{n}p(v,v_1\ldots v_{r-1}){\sum_{k\in\mathbb{Z}}}' \frac{(\frac{q_r}{q_z})^{k+1-wt(v)}}{1-\theta q^{k+1-wt(v)}}.F_M(g;v_1,\ldots\sum_{i\geq 0}\binom{k}{i}v(i)v_r,\ldots,v_n),
\end{aligned}$$

where the prime denotes the omission of $k=-1+wt(v)$ if $(\theta,\phi)=(1,1)$ and recalling (61). Now from (63) and (64) we find

$$\sum_{i\geq 0}\binom{k}{i}v(i)=\sum_{m\geq 0}\frac{(k+1-wt(v))^m}{m!}v[m].$$

The sum over $k$ can then be computed in terms of a twisted Weierstrass function (20) for $\lambda = wt(v)(\mathrm{mod}\mathbb{Z})$ as follows:

$$\begin{aligned}
&\frac{1}{m!}\sum_{k\in\mathbb{Z}}{}' \frac{(k+1-wt(v))^m(\frac{q_r}{q_z})^{k+1-wt(v)}}{1-\theta q^{k+1-wt(v)}} \\
&= (-1)^{m+1}P_{m+1}\left[\begin{array}{c}\theta^{-1}\\ \phi^{-1}\end{array}\right](z_r-z,\tau) - \frac{1}{2}\delta_{\theta,1}\delta_{\phi,1}\delta_{m,0} \\
&= P_{m+1}\left[\begin{array}{c}\theta\\ \phi\end{array}\right](z-z_r,\tau) - \frac{1}{2}\delta_{\theta,1}\delta_{\phi,1}\delta_{m,0},
\end{aligned}$$

using (36). Thus we find $F_M(g; v, v_1, \ldots v_n, \tau)$ is given by

$$\begin{aligned}
&\delta_{\theta,1}\delta_{\phi,1}\mathrm{STr}_M\left(go(v)Y_M(q_1^{L(0)}v_1, q_1)\ldots Y_M(q_n^{L(0)}v_n, q_n)q^{L(0)-c/24}\right) \\
&+\sum_{r=1}^{n}\sum_{m\geq 0} p(v, v_1v_2\ldots v_{r-1})P_{m+1}\left[\begin{array}{c}\theta\\ \phi\end{array}\right](z-z_r,\tau)F_M(g; v_1, \ldots, v[m]v_r, \ldots, v_n; \tau) \\
&-\frac{1}{2}\delta_{\theta,1}\delta_{\phi,1}\sum_{r=1}^{n} p(v, v_1\ldots v_{r-1})F_M(g; v_1, \ldots, v[0]v_r, \ldots, v_n; \tau).
\end{aligned}$$

Finally, it follows from (75) that the last sum is zero and hence (79) obtains.
□

**Remark 3.7.** *(i) Note that it is necessary for the $V$-grading to be real in order in order for $P_k\left[\begin{array}{c}\theta\\ \phi\end{array}\right]$ to converge. Thus, VOSAs with $\mathbb{C}$-grading such as those discussed in [DM] have divergent torus $n$-point functions.*

*(ii) From Lemma 2.5 it follows that $F_M$ is periodic in $z$ with periods $2\pi i\tau$ and $2\pi i$ with multipliers $\theta$ and $\phi$ respectively in agreement with Lemma 3.2.*

Other standard recursion formulas can be similarly generalized. Thus

**Proposition 3.8.** *With notation as above, for any states $v_1, \ldots v_n \in V$, and* *for $p \geq 1$ we have:*

$$
\begin{aligned}
& F_M(g; v[-p].v_1, \ldots v_n; \tau) \\
= \; & \delta_{\theta,1}\delta_{\phi,1}\delta_{p,1}\ \mathrm{STr}_M(go(v)Y(q_1^{L(0)}v_1, q_1)\ldots Y(q_n^{L(0)}v_n, q_n)q^{L(0)-c/24}) \\
& + \sum_{m\geq 0}(-1)^{m+1}\binom{m+p-1}{m} E_{m+p}\left[\begin{array}{c}\theta \\ \phi\end{array}\right](\tau) F_M(g; v[m]v_1, \ldots v_n; \tau) \\
& + \sum_{r=2}^{n}\sum_{m\geq 0} p(v, v_1 v_2 \ldots v_{r-1})(-1)^{p+1}\binom{m+p-1}{m} P_{m+p}\left[\begin{array}{c}\theta \\ \phi\end{array}\right](z_{1r}, \tau). \\
& F_M(g; v_1, \ldots v[m]v_r, \ldots v_n; \tau). \qquad (80)
\end{aligned}
$$

**Proof.** Using (72) of Lemma 3.1 and associativity of VOSAs (e.g. [FHL]) we have:

$$
\begin{aligned}
& F_M(g; (Y[v, z]v_1, z_1), \ldots (v_n, z_n); \tau) \\
= \; & Z_M(g, Y[Y[v, z]v_1, z_1]Y[v_2, z_2]\ldots Y[v_n, z_n]\mathbf{1}, \tau) \\
= \; & Z_M(g, Y[v, z + z_1]Y[v_1, z_1]Y[v_2, z_2]\ldots Y[v_n, z_n]\mathbf{1}, \tau) \\
= \; & F_M(g; (v, z + z_1), (v_1, z_1), \ldots (v_n, z_n); \tau). \qquad (81)
\end{aligned}
$$

Expanding the LHS of (81) in $z$ we find that the coefficient of $z^{p-1}$ is $F_M(v[-p].v_1, z_1; \ldots v_n, z_n; g; \tau)$. We can compare this to the expansion of the RHS in $z$ from (79) of Theorem 3.6. From (39) we find that for $p \geq 1$, the coefficient of $z^{p-1}$ in $P_{m+1}\left[\begin{array}{c}\theta \\ \phi\end{array}\right](z, \tau)$ is $(-1)^{m+1}\binom{m+p-1}{m}E_{m+p}\left[\begin{array}{c}\theta \\ \phi\end{array}\right](\tau)$. Furthermore for $r \neq 1$ the coefficient of $z^{p-1}$ in $P_{m+1}\left[\begin{array}{c}\theta \\ \phi\end{array}\right](z + z_{1r}, \tau)$ is given by $(-1)^{p+1}\binom{m+p-1}{m}P_{m+p}\left[\begin{array}{c}\theta \\ \phi\end{array}\right](z_{1r}, \tau)$. Lastly, for $p = 1$ the first term of (79) also contributes. Thus the stated result follows. $\square$

# 4 Shifted VOSAs and Heisenberg Twisted Modules

In this section we discuss the $n$-point functions for an orbifold $g$-twisted module for a VOSA where $g$ is a continuous symmetry generated by a Heisenberg vector. (For definitions and properties of twisted modules we refer the reader to refs. [Li], [DLM1], [DZ1]). In particular, we show below (Proposition 4.3) that every such $g$-twisted $n$-point function is related to an $n$-point function for the original VOSA but with a "shifted" Virasoro vector [MN2], [DM]. This generalizes a similar result for partition functions found in [DM] and allows us to apply Theorem 3.6 in order to compute all such $g$-twisted $n$-point functions. The general relationship at the operator level between these shifted and twisted formalisms is discussed elsewhere [TZ].

A Heisenberg bosonic vector is an element $h \in V_{\bar{0},1}$ such that [DM]

1. $h(0)$ is semisimple with real eigenvalues.
2. $h$ is a primary vector so that $L(n)h = 0$ for all $n \geq 1$.
3. $h(n)h = 0$ for all $n \geq 0$ except $n = 1$ for which $h(1)h = \xi_h \mathbf{1}$ for some $\xi_h \in \mathbb{C}$.
4. $[h(m), h(n)] = \xi_h m \delta_{m,-n}$.

**Remark 4.1.** *If the VOSA grading is non-negative and $V_0 = \mathbb{C}\mathbf{1}$ then (2)-(4) follow automatically for all $h \in V_{\bar{0},1}$ from (58).*

Given a Heisenberg vector $h$ then $h(0)$ generates a VOSA automorphism

$$g = \exp(2\pi i h(0)). \tag{82}$$

The order of $g$ is finite iff the eigenvalues of $h(0)$ are rational and otherwise is infinite. We can define [DLM2] and construct a $g$-twisted module in all cases as follows. We define [Li]

$$\Delta(h, z) = z^{h(0)} \exp\left(-\sum_{n\geq 1} \frac{h(n)}{n}(-z)^{-n}\right). \tag{83}$$

Noting $\Delta(h, z)^{-1} = \Delta(-h, z)$ one finds

$$\Delta(h, z)Y(v, z_0)\Delta(-h, z) = Y(\Delta(h, z + z_0)v, z_0). \tag{84}$$

This leads to:

**Proposition 4.2 ([Li]).** *Let $(M, Y_M)$ be a $V$-module. Defining*

$$Y_g(v, z) = Y_M(\Delta(-h, z)v, z), \tag{85}$$

*for all $v \in V$ then $(M, Y_g)$ is a $g$-twisted $V$-module[3].*

[3]Note that we apply the definition of $g$-twisted module of ref. [Li] which corresponds to a $g^{-1}$-twisted module in refs. [DLM1] and [DM]

Note for $v = \omega$ we find $\Delta(-h, z)\omega = \omega - hz^{-1} + \xi_h z^{-2}/2$ so that the $(M, Y_g)$ grading is determined by

$$L_g(0) = L(0) - h(0) + \frac{\xi_h}{2}. \tag{86}$$

We define the orbifold $g$-twisted $n$-point function for any automorphism $f$ commuting with $g$ and $\sigma$ by

$$F_M((f,g); v_1, \ldots, v_n; \tau) = \mathrm{STr}_M \left( f\ Y_g(q_1^{L(0)} v_1, q_1) \ldots Y_g(q_n^{L(0)} v_n, q_n) q^{L_g(0) - c/24} \right). \tag{87}$$

We denote the orbifold $g$-twisted partition function by $Z_M((f,g),\tau)$.

For each Heisenberg element $h$ we may also construct a VOSA $(V, Y, \mathbf{1}, \omega_h)$ with the original vector space and vertex operators but using a "shifted" conformal vector ([MN2], [DM])

$$\omega_h = \omega + h(-2)\mathbf{1}. \tag{88}$$

With $Y(\omega_h, z) = \sum_{n\in\mathbb{Z}} L_h(n) z^{-n-2}$ we find

$$L_h(n) = L(n) - (n+1)h(n), \tag{89}$$

and central charge

$$c_h = c - 12\xi_h. \tag{90}$$

In particular, $L_h(-1) = L(-1)$ and the grading is determined by

$$L_h(0) = L(0) - h(0). \tag{91}$$

We denote the partition function for a $V$-module $M$ with a $h$-shifted $L_h(0)$ by $Z_{M,h}(\tau)$. Following (66) the shifted $n$-point function is denoted by

$$F_{M,h}(f; v_1, \ldots, v_n; \tau) = \mathrm{STr}_M \left( f Y(q_1^{L_h(0)} v_1, q_1) \ldots Y(q_n^{L_h(0)} v_n, q_n) q^{L_h(0) - c_h/24} \right), \tag{92}$$

where $f$ commutes with $g$ and $\sigma$. We denote the $h$-shifted partition function by $Z_{M,h}(f,\tau)$. Comparing (86) and (91) we see that

$$L_g(0) - \frac{c}{24} = L_h(0) - \frac{c_h}{24},$$

so that ([DM])

$$Z_M((1,g),\tau) = Z_{M,h}(1,\tau). \tag{93}$$

This relationship can be generalized to relate all orbifold $g$-twisted $n$-point functions to $h$-shifted $n$-point functions as follows:

**Proposition 4.3.** *Let $M$ be a module for $V$ and let $g = \exp(2\pi i h(0))$ be generated by a Heisenberg state $h$. Then the $n$-point function for the orbifold $g$-twisted and the untwisted $n$-point function for $M$ with shifted $L_h(0)$-vertex operators are related as follows:*

$$F_M((f,g); v_1, \ldots, v_n; \tau) = F_{M,h}(f; Uv_1, \ldots, Uv_n; \tau), \tag{94}$$

*where $U = \Delta(-h, 1) = \exp\left(\sum_{n\geq 1} \frac{h(n)}{n}(-1)^n\right)$ and $f$ commutes with $g$ and $\sigma$.*

**Proof.** First we prove

$$\Delta(-h, q_z)\, q_z^{L(0)} = q_z^{L_h(0)}\, U. \tag{95}$$

From (83) one finds using $[L(0), h(n)] = -nh(n)$ that

$$\begin{aligned}
\Delta(-h, q_z) q_z^{L(0)} &= q_z^{-h(0)} \exp\left(\sum_{n>0} \frac{h(n)}{n}(-q_z)^{-n}\right) q_z^{L(0)} \\
&= q_z^{-h(0)} q_z^{L(0)} \exp\left(\exp(\mathrm{ad}_{-zL(0)}) \sum_{n>0} \frac{h(n)}{n}(-q_z)^{-n}\right) \\
&= q_z^{L_h(0)} \exp\left(\sum_{n>0} \frac{h(n)}{n}(-q_z)^{-n} q_z^n\right) = q_z^{L_h(0)} U.
\end{aligned}$$

Therefore from (85)

$$Y_g(q_z^{L(0)}v, q_z) = Y_M(\Delta(-h, q_z) q_z^{L(0)}v, q_z) = Y_M(q_z^{L_h(0)}Uv, q_z),$$

Thus the LHS of (94) is

$$\begin{aligned}
& \mathrm{STr}_M\left(f Y_g(q_1^{L(0)}v_1, q_1) \ldots Y_g(q_n^{L(0)}v_n, q_n) q^{L_g(0)-c/24}\right) \\
= \;& \mathrm{STr}_M\left(f Y_M(q_1^{L_h(0)}Uv_1, q_1) \ldots Y_M(q_n^{L_h(0)}Uv_n, q_n) q^{L_h(0)-c_h/24}\right) \\
= \;& F_{M,h}(f; Uv_1, \ldots, Uv_n; \tau).
\end{aligned}$$

□

We conclude this section by showing that $U$ maps between the square bracket vertex operators (62) of the original and shifted VOSAs. We let

$$Y[v, z]_h = Y(q_z^{L_h(0)}v, q_z - 1), \tag{96}$$

denote a square bracket vertex operator in the $h$-shifted VOSA. We then have

**Lemma 4.4.** *For $v \in V$ we have*

$$UY[v, z]U^{-1} = Y[Uv, z]_h. \tag{97}$$

**Proof.** Using associativity, (85) and (95) we obtain

$$\begin{aligned}
& Y_g(q_1^{L(0)}v_1, q_1)Y_g(q_2^{L(0)}v_2, q_2) \\
= \; & Y_g(Y(q_1^{L(0)}v_1, q_1 - q_2)\, q_2^{L(0)}v_2, q_2) \\
= \; & Y_g(q_2^{L(0)}Y[v_1, z_{12}]v_2, q_2) \\
= \; & Y(q_2^{L_h(0)}UY[v_1, z_{12}]v_2, q_2).
\end{aligned}$$

On the other hand

$$\begin{aligned}
& Y_g(q_1^{L(0)}v_1, q_1)Y_g(q_2^{L(0)}v_2, q_2) \\
= \; & Y(q_1^{L_h(0)}Uv_1, q_1)Y(q_2^{L_h(0)}Uv_2, q_2) \\
= \; & Y(q_2^{L_h(0)}Y[Uv_1, z_{12}]_h\, Uv_2, q_2).
\end{aligned}$$

Hence the result follows. □

# 5 Rank One Fermion VOSA

We begin with the example of the rank one "Neveu-Schwarz sector" fermion VOSA $V = V(H, \mathbb{Z} + \frac{1}{2})$ generated by one fermion [FFR], [Li]. This is a $\frac{1}{2}\mathbb{Z}$ graded VOSA with $H = \mathbb{C}\psi$ for a fermion vector $\psi$ of parity 1 and modes obeying

$$[\psi(m), \psi(n)] = \psi(m)\psi(n) + \psi(n)\psi(m) = \delta_{m+n+1,0}. \tag{98}$$

The superspace $V$ is spanned by Fock vectors of the form

$$\psi(-k_1)\psi(-k_2)\ldots\psi(-k_m)\mathbf{1}, \tag{99}$$

for integers $1 \leq k_1 < k_2 < \ldots k_m$ with $\psi(k)\mathbf{1} = 0$ for all $k \geq 0$ so that $V$ is generated by $Y(\psi, z)$. The conformal vector is $\omega = \frac{1}{2}\psi(-2)\psi(-1)\mathbf{1}$ of central charge $c = \frac{1}{2}$ for which the Fock vector (99) has $L(0)$ weight $\sum_{1\leq i\leq m}(k_i - \frac{1}{2}) \in \frac{1}{2}\mathbb{Z}$. In particular, $wt(\psi) = \frac{1}{2}$. The partition function is

$$Z_V(\tau) = \mathrm{STr}_V(q^{L(0)-\frac{1}{48}}) = q^{-\frac{1}{48}}\prod_{n\geq 0}(1 - q^{n+\frac{1}{2}}) = \frac{\eta(\frac{1}{2}\tau)}{\eta(\tau)}, \tag{100}$$

whereas for $g = \sigma$ of (65) we find

$$Z_V(\sigma,\tau) = \mathrm{STr}_V(\sigma q^{L(0)-\frac{1}{48}}) = q^{-\frac{1}{48}}\prod_{n\geq 0}(1+q^{n+\frac{1}{2}}) = \frac{\eta(\tau)^2}{\eta(2\tau)\eta(\frac{1}{2}\tau)}. \quad (101)$$

Let us next introduce the $n$-point function (66) for $V$ where $v_i = \psi$ for all $i = 1,\ldots n$:

$$G_n(g; z_1,\ldots,z_n;\tau) = F_V(g; (\psi, z_1),\ldots,(\psi, z_n);\tau), \quad (102)$$

which we will refer to as the *generating function*. We use the recursion formula (79) of Theorem 3.6 to compute $G_n$. Since $wt(\psi) = \frac{1}{2}$ we have $\phi = -1$ from (76) and $\theta = 1$ for $g = 1$ and $\theta = -1$ for $g = \sigma$ from (77). For $n = 1$, $G_1(g; z_1;\tau) = Z_V(g,\psi,\tau) = 0$ since $o(\psi) = 0$. For $n = 2$, (79) implies

$$G_2(g; z_1, z_2;\tau) = 0 + \sum_{m\geq 0} P_{m+1}\left[\begin{array}{c}\theta\\ -1\end{array}\right](z_{12},\tau)F_V(g;\psi[m]\psi;\tau).$$

Passing to the square bracket formalism (62) we find the same fermion commutator algebra as (98) obtains, namely

$$[\psi[m],\psi[n]] = \delta_{m+n+1,0}. \quad (103)$$

Thus it follows that $\psi[m]\psi = \delta_{m,0}\mathbf{1}$ giving

$$G_2(g; z_1, z_2;\tau) = P_1\left[\begin{array}{c}\theta\\ -1\end{array}\right](z_{12},\tau)Z_V(g,\tau). \quad (104)$$

We may similarly compute $G_n$ for all $n$ by repeated application of (79). It is easy to see that $G_n = 0$ for $n$ odd. For $n$ even $G_n$ is expressed in terms of a Pfaffian which is totally antisymmetric in $z_i$ as expected from Lemma 3.2 (ii). Let us first recall the definition of the Pfaffian of an anti-symmetric matrix $\mathbf{M} = (M(i,j))$ of even dimension $2m$ given by

$$\mathrm{Pf}(\mathbf{M}) = \sum_{\Pi}\varepsilon_{i_1 j_1\ldots i_m j_m}M(i_1,j_1)M(i_2,j_2)\ldots M(i_m,j_m), \quad (105)$$

where the sum is taken over the set of all partitions $\Pi$ of $\{1,2,\ldots,2m\}$ into pairs with elements

$$\{(i_1,j_1),(i_2,j_2)\ldots(i_m,j_m)\},$$

for $i_k < j_k$ and $i_1 < i_2 < \ldots i_m$ and where $\varepsilon_{i_1 j_1 \ldots i_m j_m}$ is the Levi-Civita symbol. We also note that

$$\mathrm{Pf}(\mathbf{M}) = \sqrt{\det \mathbf{M}}.$$

We then obtain:

**Proposition 5.1.** *For $n$ even and $g = 1$ or $\sigma$ we have*

$$G_n(g; z_1, \ldots, z_n; \tau) = \mathrm{Pf}(\mathbf{P}) Z_V(g, \tau), \tag{106}$$

*where $\mathbf{P}$ denotes the anti-symmetric $n \times n$ matrix with components*

$$\mathbf{P}(i,j) = P_1 \left[ \begin{array}{c} \theta \\ -1 \end{array} \right] (z_{ij}, \tau), \quad (1 \leq i \neq j \leq n), \tag{107}$$

*for $z_{ij} = z_i - z_j$ with $\theta = 1$ for $g = 1$ and $\theta = -1$ for $g = \sigma$.*

**Proof.** We first note that $\mathbf{P}$ is anti-symmetric from (36) since $\theta = \pm 1$. We prove the result by induction. For $n = 2$ the result is given in (104). For general $n$ we apply (79) to obtain

$$\begin{aligned} G_n(g; z_1, \ldots z_n; \tau) &= \sum_{r=2}^{n} (-1)^r P_1 \left[ \begin{array}{c} \theta \\ -1 \end{array} \right] (z_{1r}, \tau) G_{n-2}(g; z_2, \ldots, \hat{z}_r, \ldots z_n; \tau) \\ &= \sum_{r=2}^{n} (-1)^r \mathbf{P}(1,r) \mathrm{Pf}(\hat{\mathbf{P}}) Z_V(g, \tau), \end{aligned}$$

where $\hat{z}_r$ is deleted and $\hat{\mathbf{P}}$ is the "cofactor" matrix obtained by deleting the $1^{st}$ and $r^{th}$ rows and columns of $\mathbf{P}$. The result (107) follows from the definition (105).□

$G_n$ enjoys the following analytic properties following Remark 2.1 (ii):

**Corollary 5.2.** *$G_n$ is an analytic function in $z_i$ and converges absolutely and uniformly on compact subsets of the domain $|q| < \left|q_{z_{ij}}\right| < 1$ for all $z_{ij} = z_i - z_j$ with $i \neq j$.* □

We now show that all $n$-point functions can be computed from $G_n$. Consider a $V$ basis of square bracket Fock vectors denoted by

$$\Psi[-\mathbf{k}] = \psi[-k_1]\psi[-k_2] \ldots \psi[-k_m]\mathbf{1}, \tag{108}$$

where $\mathbf{k} = k_1, k_2, \ldots, k_m$ for integers $1 \leq k_1 < k_2 < \ldots k_m$. We will determine an explicit formula for all $n$-point functions for such Fock vectors. Thus the 1-point function $Z_V(g, \Psi[-\mathbf{k}], \tau)$ is the coefficient of $\prod_{i=1}^{m} z_i^{k_i-1}$ in $G_n$ since

$$\begin{aligned} G_n(g; z_1, \ldots, z_m; \tau) &= Z_V(g, Y[\psi, z_1] \ldots Y[\psi, z_m]\mathbf{1}, \tau) \\ &= \sum_{k_1, \ldots k_m \in \mathbb{Z}} Z_V(g, \psi[-k_1] \ldots \psi[-k_m]\mathbf{1}, \tau) z_1^{k_1-1} \ldots z_n^{k_m-1}. \end{aligned}$$

Examining (106) we can explicitly find this coefficient from the expansion of $P_1 \left[ \begin{array}{c} \theta \\ -1 \end{array} \right] (z_{ij}, \tau)$ given in (43). It follows that $Z_V(g, \Psi[-\mathbf{k}], \tau) = 0$ for $m$ odd whereas for $m$ even

$$Z_V(g, \Psi[-\mathbf{k}], \tau) = \mathrm{Pf}(\mathbf{C}) Z_V(g, \tau), \tag{109}$$

where $\mathbf{C}$ denotes the antisymmetric $m \times m$ matrix with $(i, j)$-entry

$$\mathbf{C}(i, j) = C \left[ \begin{array}{c} \theta \\ -1 \end{array} \right] (k_i, k_j, \tau),$$

(cf. (45)). $\mathbf{C}$ is antisymmetric from (47) since $\theta = \pm 1$.

We may similarly derive an expression for an arbitrary two-point function $F_V((\Psi[-\mathbf{k}^{(1)}], z_1), (\Psi[-\mathbf{k}^{(2)}], z_2); g; \tau)$ for $\mathbf{k}^{(1)} = k_1^{(1)}, \ldots k_{m_1}^{(1)}$ and $\mathbf{k}^{(2)} = k_1^{(2)}, \ldots k_{m_2}^{(2)}$. First consider the one-point function

$$Z_V(g, Y[Y[\psi, x_1] \ldots Y[\psi, x_{m_1}]\mathbf{1}, z_1].Y[Y[\psi, y_1] \ldots Y[\psi, y_{m_2}]\mathbf{1}, z_2]\mathbf{1}, \tau). \tag{110}$$

$F_V(g; (\Psi[-\mathbf{k}^{(1)}], z_1), (\Psi[-\mathbf{k}^{(2)}], z_2); \tau)$ is the coefficient of $\prod_{i=1}^{m_1} \prod_{j=1}^{m_2} x_i^{k_i^{(1)}-1} y_j^{k_j^{(2)}-1}$ in (110). By associativity (e.g. [FHL]) and using $Y[\mathbf{1}, z] = \mathrm{Id}_V$ we find (110) can be expressed as

$$\begin{aligned} & Z_V(g, Y[\psi, x_1 + z_1] \ldots Y[\psi, x_{m_1} + z_1].Y[\psi, y_1 + z_2] \ldots Y[\psi, y_{m_2} + z_2]\mathbf{1}, \tau) \\ = \ & G_n(g; x_1 + z_1, \ldots, x_{m_1} + z_1, y_1 + z_2, \ldots, y_{m_2} + z_2; \tau). \end{aligned}$$

The coefficient of $\prod_{i=1}^{m_1} \prod_{j=1}^{m_2} x_i^{k_i^{(1)}-1} y_j^{k_j^{(2)}-1}$ can then be extracted from the expansions (43) and (44). Thus the two point function vanishes for $m_1 + m_2$ odd, whereas for $m_1 + m_2$ even

$$F_V(g; (\Psi[-\mathbf{k}^{(1)}], z_1), (\Psi[-\mathbf{k}^{(2)}], z_2); \tau) = \mathrm{Pf}(\mathbf{M}) Z(g, \tau), \tag{111}$$

where $\mathbf{M}$ is the antisymmetric $(m_1+m_2)\times(m_1+m_2)$ block matrix

$$\mathbf{M} = \begin{pmatrix} \mathbf{C}^{(11)} & \mathbf{D}^{(12)} \\ \mathbf{D}^{(21)} & \mathbf{C}^{(22)} \end{pmatrix},$$

where for $a, b \in \{1, 2\}$

$$\begin{aligned}
\mathbf{C}^{(aa)}(i,j) &= C\left[\begin{array}{c}\theta \\ -1\end{array}\right](k_i^{(a)}, k_j^{(a)}, \tau), \quad (1 \le i, j \le m_a), \\
\mathbf{D}^{(ab)}(i,j) &= D\left[\begin{array}{c}\theta \\ -1\end{array}\right](k_i^{(a)}, k_j^{(b)}, \tau, z_a - z_b), \quad (1 \le i \le m_a, 1 \le j \le m_b),
\end{aligned} \tag{112}$$

(using (46)). $\mathbf{M}$ is antisymmetric from (47) and (48).

In a similar fashion we are lead to the general result:

**Proposition 5.3.** *Let $\Psi[-\mathbf{k}^{(a)}]$ for $a = 1\ldots n$ be $n$ Fock vectors for $\mathbf{k}^{(a)} = k_1^{(a)}, \ldots k_{m_a}^{(a)}$. Then the $n$-point function vanishes for odd $\sum_a m_a$ and for $\sum_a m_a$ even is given by*

$$F_V(g; (\Psi[-\mathbf{k}^{(1)}], z_1), \ldots (\Psi[-\mathbf{k}^{(n)}], z_n); \tau) = \mathrm{Pf}(\mathbf{M}) Z(g, \tau), \tag{113}$$

*where $\mathbf{M}$ is the antisymmetric block matrix*

$$\mathbf{M} = \begin{pmatrix} \mathbf{C}^{(11)} & \mathbf{D}^{(12)} & \ldots & \mathbf{D}^{(1n)} \\ \mathbf{D}^{(21)} & \mathbf{C}^{(22)} & & \\ \vdots & & \ddots & \\ \mathbf{D}^{(n1)} & \ldots & & \mathbf{C}^{(nn)} \end{pmatrix},$$

*with $\mathbf{C}^{(aa)}$ and $\mathbf{D}^{(ab)}$ of (112). (113) is an analytic function in $z_i$ and converges absolutely and uniformly on compact subsets of the domain $|q| < |q_{z_{ij}}| < 1$ for all $z_{ij} = z_i - z_j$ with $i \neq j$.* $\square$

We have also established

**Proposition 5.4.** *$G_n(g; z_1, \ldots, z_n; \tau)$ is a generating function for all $n$-point functions.* $\square$

We conclude this section by noting that we may also consider the "Ramond sector" $\sigma$-twisted module $V(H,\mathbb{Z})$ for $V(H,\mathbb{Z}+\frac{1}{2})$. This is discussed in detail in [FFR], [Li], [DZ1], [DZ2]. $V(H,\mathbb{Z})$ decomposes into two irreducible $\sigma$-twisted modules which are interchanged under the induced action of $\sigma$. For either irreducible $\sigma$-twisted module $M_\sigma$ the partition function is

$$\begin{aligned}
\mathrm{STr}_{M_\sigma}(q^{L(0)-\frac{1}{48}}) &= 0, \\
\mathrm{STr}_{M_\sigma}(\sigma q^{L(0)-\frac{1}{48}}) &= q^{\frac{1}{48}}\prod_{n\geq 0}(1+q^n) = \frac{\eta(2\tau)}{\eta(\tau)}.
\end{aligned}$$

We may similarly consider the generator of all $\sigma$-twisted $n$-point functions defined by

$$G_{M_\sigma,n}(g;z_1,\ldots,z_n;\tau) = F_{M_\sigma}(g;(\psi,z_1),\ldots,(\psi,z_n);\tau),$$

for $g=1$ or $\sigma$. This vanishes for all $n$ for $g=1$ and for $n$ odd for $g=\sigma$. By applying a VOSA orbifold Zhu reduction formula of ref. [DZ1] we find as in Proposition 5.1 that

**Proposition 5.5.** *For $n$ even we have*

$$G_{M_\sigma,n}(\sigma;z_1,\ldots,z_n;\tau) = \mathrm{Pf}(P_1\begin{bmatrix} -1 \\ 1 \end{bmatrix}(z_{ij},\tau))\frac{\eta(2\tau)}{\eta(\tau)}, \tag{114}$$

*for $z_{ij}=z_i-z_j$.* $\square$

One can similarly describe analytic properties as in Corollary 5.2 and determine all $\sigma$-twisted $n$-point functions by expanding this generating function along the same lines as Proposition 5.3, though we do not carry this out here.

# 6 Rank Two Fermion VOSA

## 6.1 $h$-Shifted and orbifold $g$-Twisted $n$-Point Functions

In this section we consider the rank two fermion VOSA formed from the tensor product of two copies of the rank one fermion VOSA and hence is generated by two free fermions $\psi_1 = \psi\otimes\mathbf{1}$ and $\psi_2 = \mathbf{1}\otimes\psi$. We may therefore compute all the untwisted and $\sigma$-twisted $n$-point functions based on the last section. However, as is well known, this VOSA contains a bosonic Heisenberg state $h = \alpha\psi \otimes \psi$ (for $\alpha \in \mathbb{C}$) and we will compute all $h$-shifted and $g$-twisted $n$-point functions where $g$ is generated by $h$ as discussed in Section 4.

It is convenient to introduce the off-diagonal basis $\psi^{\pm} = \frac{1}{\sqrt{2}}(\psi_1 \pm i\psi_2)$, where $\psi^{\pm}$-modes obey the commutation relations

$$[\psi^{+}(m), \psi^{-}(n)] = \delta_{m,-n-1}, \quad [\psi^{\pm}(m), \psi^{\pm}(n)] = 0, \tag{115}$$

The VOSA $V$ is generated by $Y(\psi^{\pm}, z) = \sum\limits_{n\in\mathbb{Z}} \psi^{\pm}(n)z^{-n-1}$ where the vector space $V$ is a Fock space with basis vectors of the form

$$\psi^{+}(-k_1)...\psi^{+}(-k_s)\psi^{-}(-l_1)...\psi^{-}(-l_t)\mathbf{1}, \tag{116}$$

for $1 \leq k_1 < k_2 < \ldots k_s$ and $1 \leq l_1 < l_2 < \ldots l_t$ with $\psi^{\pm}(k)\mathbf{1} = 0$ for all $k \geq 0$. We define the conformal vector to be

$$\omega = \frac{1}{2}[\psi^{+}(-2)\psi^{-}(-1)+\psi^{-}(-2)\psi^{+}(-1)]\mathbf{1}, \tag{117}$$

whose modes generate a Virasoro algebra of central charge 1. Then $\psi^{\pm}$ has $L(0)$-weight $\frac{1}{2}$ and the Fock state (116) has weight $\sum_{1\leq i\leq s}(k_i - \frac{1}{2}) + \sum_{1\leq j\leq t}(l_j - \frac{1}{2})$.

The weight 1 parity zero space is $V_{\bar{0},1} = \mathbb{C}a$ for (normalized) Heisenberg bosonic vector

$$a = \psi^{+}(-1)\psi^{-}(-1)\mathbf{1}, \tag{118}$$

with modes obeying

$$[a(m), a(n)] = m\delta_{m,-n},$$

and $\omega$ of (117) is nothing but the standard Heisenberg VOA conformal vector

$$\omega = \frac{1}{2}a(-1)^2\mathbf{1}.$$

Following Section 4 we define a one parameter family of Heisenberg vectors

$$h = \kappa a, \quad \kappa \in \mathbb{R}, \tag{119}$$

for which $\xi_h = \kappa^2$. The shifted conformal vector (88) is then $\omega_h = \omega + \kappa a(-2)\mathbf{1}$ with central charge $c_h = 1 - 12\kappa^2$ from (90). Then $\psi^{\pm}$ has $L_h(0) =$

$L(0) - \kappa a(0)$ weight $wt_h(\psi^{\pm}) = \frac{1}{2} \mp \kappa$ and the Fock state (116) has $L_h(0)$ weight $\sum_{1\leq i\leq s}(k_i - \frac{1}{2} - \kappa) + \sum_{1\leq j\leq t}(l_j - \frac{1}{2} + \kappa)$.

Noting that $\sigma = e^{\pi i a(0)}$ and following Section 4, we can construct a $\sigma g$-twisted module for $\sigma g = e^{2\pi i h(0)}$ so that

$$g = e^{2\pi i \beta a(0)}, \tag{120}$$

for real $\beta$ where

$$\beta = \kappa - \frac{1}{2}. \tag{121}$$

We also define $\phi \in U(1)$ by

$$\phi = \exp(2\pi i wt_h(\psi^+)) = e^{-2\pi i \beta}. \tag{122}$$

Introduce the automorphism

$$f = e^{2\pi i \alpha a(0)}, \quad \alpha \in \mathbb{R}, \tag{123}$$

which commutes with $g, \sigma$. Then $f\psi^{\pm} = \theta^{\mp 1}\psi^{\pm}$ for

$$\theta = e^{-2\pi i \alpha} \in U(1). \tag{124}$$

Finally, we denote the orbifold $\sigma g$-twisted trace by

$$Z_V \left[ \begin{array}{c} f \\ g \end{array} \right] (\tau) = Z_V((f, \sigma g), \tau).$$

We find using Proposition 4.3 that

$$Z_{V,h}(f,\tau) = Z_V \left[ \begin{array}{c} f \\ g \end{array} \right] (\tau) = q^{\kappa^2/2 - 1/24} \prod_{l\geq 1} (1 - \theta^{-1} q^{l-\frac{1}{2}-\kappa})(1 - \theta q^{l-\frac{1}{2}+\kappa}). \tag{125}$$

Note that $Z_{V,h}(f,\tau) = 0$ for $(\theta, \phi) = (1, 1)$, i.e. $(\alpha, \beta) \equiv (0, 0) \pmod{\mathbb{Z}}$.

**Remark 6.1.** *The RHS of (125) is related to a theta series via the Jacobi triple product formula as briefly reviewed below in Section 6.4. Hence* $Z_{V,h}(f,\tau)$ *depends on* $\alpha (\mathrm{mod}\ \mathbb{Z})$ *and* $\beta (\mathrm{mod}\ \mathbb{Z})$ *up to an overall* $\alpha$*-dependent constant.*

We next consider general $\sigma g$-twisted and $h$-shifted $n$-point functions which are related via Proposition 4.3. As in the rank one case, it is sufficient to consider $n$-point functions for the generating states $\psi^{\pm}$ only. To this end we define the $h$-shifted VOSA $n$-point generating function

$$\begin{aligned} & G_{2n,h}(f;x_1,...,x_n;y_1,...,y_n;\tau) \\ = \quad & F_{V,h}(f;(\psi^+,x_1),(\psi^-,y_1),...,(\psi^+,x_n),(\psi^-,y_n);\tau), \end{aligned} \tag{126}$$

**Remark 6.2.** *Note the choice of an alternating ordering of the operators with respect to the $\pm$ superscript here.*

We can also define a $\sigma g$-twisted $n$-point function denoted by

$$F_V \left[ \begin{array}{c} f \\ g \end{array} \right] ((v_1,z_1)...,(v_n,z_n);\tau) = F_V((f,\sigma g);(v_1,z_1)...,(v_n,z_n);\tau),$$

with generating function

$$\begin{aligned} & G_{2n} \left[ \begin{array}{c} f \\ g \end{array} \right] (x_1,...,x_n;y_1,...,y_n;\tau) \\ = \quad & F_V((f,\sigma g);(\psi^+,x_1),(\psi^-,y_1),...,(\psi^+,x_n),(\psi^-,y_n);\tau). \end{aligned}$$

Then noting that $U\ \psi^{\pm} = \psi^{\pm}$ and applying Proposition 4.3 we find

**Lemma 6.3.**

$$G_{2n} \left[ \begin{array}{c} f \\ \sigma g \end{array} \right] (x_1,...,x_n;y_1,...,y_n;\tau) = G_{2n,h}(f;x_1,...,x_n;y_1,...,y_n;\tau). \quad \square$$

These generating functions are totally antisymmetric in $x_i, y_j$ as expected from Lemma 3.2 (ii) and can be expressed in terms of a determinant computed by means of our recursion formula (79). Due to the leading term on the RHS of (79), we consider the cases $(\theta,\phi) \neq (1,1)$ and $(\theta,\phi) = (1,1)$ separately.

## 6.2 $n$-Point Functions for $(\theta,\phi) \neq (1,1)$.

**Proposition 6.4.** *For $(\theta,\phi) \neq (1,1)$ we have*

$$G_{2n,h}(f;x_1,...,x_n;y_1,...,y_n;\tau) = \det\mathbf{P}.\ Z_{V,h}(f;\tau), \tag{127}$$

*where* $\mathbf{P}$ *is the* $n \times n$ *matrix:*

$$\mathbf{P} = \left( P_1 \left[ \begin{array}{c} \theta \\ \phi \end{array} \right] (x_i - y_j, \tau) \right), \quad (1 \le i, j \le n), \tag{128}$$

*with* $\theta, \phi$ *of (124) and (122). Furthermore,* $G_{2n,h}$ *is an analytic function in* $x_i, y_j$ *and converges absolutely and uniformly on compact subsets of the domain* $|q| < \left|q_{x_i - y_j}\right| < 1$.

**Proof.** We apply Theorem 3.6 directly with $e^{2\pi i wt(\psi^+)} = \phi$ and $f\psi^+ = e^{2\pi i\beta}\psi^+ = \theta^{-1}\psi^+$ of (122) and (124). The Zhu recursion formula (79) results in a determinant similarly to the proof of Proposition 5.1. The region of analyticity follows as before. □

In order to describe general $n$-point functions, first note that

$$[\psi^+[m], \psi^-[n]] = \delta_{m,-n-1}, \quad [\psi^\pm[m], \psi^\pm[n]] = 0.$$

Now introduce

$$\begin{aligned} \Psi &= \Psi[-\mathbf{k}; -\mathbf{l}] = \psi^+[-k_1]...\psi^+[-k_s]\psi^-[-l_1]...\psi^-[-l_t]\mathbf{1}, && (129) \\ \Psi_h &= \Psi[-\mathbf{k}; -\mathbf{l}]_h = \psi^+[-k_1]_h...\psi^+[-k_s]_h\psi^-[-l_1]_h...\psi^-[-l_t]_h\mathbf{1}, && (130) \end{aligned}$$

where $\mathbf{k} = k_1, ..., k_s$ and $\mathbf{l} = l_1, ..., l_t$; these denote Fock vectors (116) in the square bracket and $h$-shifted square bracket formalisms respectively. From Lemma 4.4 and using $U\ \psi^\pm = \psi^\pm$ we have

$$\Psi[-\mathbf{k}; -\mathbf{l}]_h = U\Psi[-\mathbf{k}; -\mathbf{l}].$$

By expanding $G_{2n,h}$ appropriately and following the same approach that lead to Proposition 5.3, we obtain a determinant formula for every $n$-point function as follows:

**Proposition 6.5.** *Consider* $n$ *Fock vectors* $\Psi^{(a)} = \Psi^{(a)}[-\mathbf{k}^{(a)}; -\mathbf{l}^{(a)}]$ *and* $\Psi_h^{(a)} = \Psi^{(a)}[-\mathbf{k}^{(a)}; -\mathbf{l}^{(a)}]_h$ *for* $\mathbf{k}^{(a)} = k_1^{(a)}, ...k_{s_a}^{(a)}$ *and* $\mathbf{l}^{(a)} = l_1^{(a)}, ...l_{t_a}^{(a)}$ *with* $a = 1 \ldots n$. *Then for* $(\theta, \phi) \neq (1, 1)$ *the corresponding* $n$*-point functions are non-vanishing provided*

$$\sum_{a=1}^{n} (s_a - t_a) = 0.$$

*In this case they are given by*

$$\begin{aligned}
& F_V \left[ \begin{array}{c} f \\ g \end{array} \right] ((\Psi^{(1)}, z_1), \ldots, (\Psi^{(n)}, z_n); \tau) \\
= \; & F_{V,h}(f; (\Psi_h^{(1)}, z_1), \ldots, (\Psi_h^{(n)}, z_n); \tau) = \epsilon \, \det \mathbf{M}. \; Z_{V,h}(f; \tau), \quad (131)
\end{aligned}$$

*where* $\mathbf{M}$ *is the block matrix*

$$\mathbf{M} = \left( \begin{array}{cccc} \mathbf{C}^{(11)} & \mathbf{D}^{(12)} \ldots & \mathbf{D}^{(1n)} \\ \mathbf{D}^{(21)} & \mathbf{C}^{(22)} \ldots & \mathbf{D}^{(2n)} \\ \vdots & \ddots & \vdots \\ \mathbf{D}^{(n1)} & \ldots & \mathbf{C}^{(nn)} \end{array} \right),$$

*with*

$$\mathbf{C}^{(aa)}(i,j) = C \left[ \begin{array}{c} \theta \\ \phi \end{array} \right] (k_i^{(a)}, l_j^{(a)}, \tau), \quad (1 \leq i \leq s_a, 1 \leq j \leq t_a),$$

*for* $s_a, t_a \geq 1$ *with* $1 \leq a \leq n$ *and*

$$\mathbf{D}^{(ab)}(i,j) = D \left[ \begin{array}{c} \theta \\ \phi \end{array} \right] (k_i^{(a)}, l_j^{(b)}, \tau, z_{ab}), \quad (1 \leq i \leq s_a, 1 \leq j \leq t_b),$$

*for* $s_a, t_b \geq 1$ *with* $1 \leq a, b \leq n$ *and* $a \neq b$. $\epsilon$ *is the sign of the permutation associated with the reordering of* $\psi^{\pm}$ *to the alternating ordering of (126) following Remark 6.2. Furthermore, the* $n$*-point function (131) is an analytic function in* $z_a$ *and converges absolutely and uniformly on compact subsets of the domain* $|q| < |q_{z_{ab}}| < 1$. $\square$

**Example.** Consider the $n$-point function for $n$ vectors $\Psi = a$ for $a = \psi^+[-1]\psi^-[-1]\mathbf{1}$ and $(\theta, \phi) \neq (1,1)$. Then

$$F_V \left[ \begin{array}{c} f \\ g \end{array} \right] ((a, z_1), \ldots, (a, z_n); \tau) = \det M . Z_V \left[ \begin{array}{c} f \\ g \end{array} \right] (\tau),$$

for

$$\mathbf{M} = \left( \begin{array}{ccc} -E_1 \left[ \begin{array}{c} \theta \\ \phi \end{array} \right] (\tau) & P_1 \left[ \begin{array}{c} \theta \\ \phi \end{array} \right] (z_{12}, \tau) \ldots & P_1 \left[ \begin{array}{c} \theta \\ \phi \end{array} \right] (z_{1n}, \tau) \\ P_1 \left[ \begin{array}{c} \theta \\ \phi \end{array} \right] (z_{21}, \tau) & -E_1 \left[ \begin{array}{c} \theta \\ \phi \end{array} \right] (\tau) \ldots & P_1 \left[ \begin{array}{c} \theta \\ \phi \end{array} \right] (z_{2n}, \tau) \\ \vdots & \ddots & \vdots \\ P_1 \left[ \begin{array}{c} \theta \\ \phi \end{array} \right] (z_{n1}, \tau) & \ldots & -E_1 \left[ \begin{array}{c} \theta \\ \phi \end{array} \right] (\tau) \end{array} \right).$$

For $\theta, \phi \in \{\pm 1\}$, it follows from (36) that the diagonal Eisenstein terms vanish and that $\det \mathbf{M} = 0$ for odd $n$. Taking $n$ even and recalling that $\mathrm{Pf}(\mathbf{M}) = \sqrt{\det \mathbf{M}}$, we recover the square of the rank one generating function (106) for $\phi = -1$ and the rank one $\sigma$-twisted generating function (114) for $\phi = 1$.

## 6.3 $n$-Point Functions for $(\theta, \phi) = (1, 1)$.

We consider $(\alpha, \beta) = (0, 0)$ so that $(f, g) = (1, 1)$ and $(\theta, \phi) = (1, 1)$ with $\kappa = \frac{1}{2}$ (cf. Remark 6.1). We then have $wt_h(\psi^+) = 0$, $wt_h(\psi^-) = 1$ and $c_h = -2$. For $n = 1$, eqn. (126) can be computed from (79) to give the ($x, y$ independent) result:

$$\begin{aligned} G_{2,h}(1; x, y; \tau) &= F_{V,h}(1; (\psi^+, x), (\psi^-, y); \tau) \\ &= \mathrm{STr}_V \left( o_h(\psi^+) o_h(\psi^-) q^{L_h(0) + 1/12} \right) + 0, \end{aligned}$$

where $o_h(v) = v(wt_h(v) - 1)$ from (61) and recalling $Z_{V,h}(1; \tau) = 0$. Furthermore, $o_h(\psi^+) o_h(\psi^-) = \psi^+(-1)\psi^-(0)$ acts as a projection operator on $V$ preserving those Fock vectors (116) containing an $\psi^+(-1)$ operator. Hence we find

$$G_{2,h}(1; x, y; \tau) = q^{1/12}(-q^0) \prod_{k \geq 2} (1 - q^{k-1}) \prod_{l \geq 1} (1 - q^l) = -\eta(\tau)^2. \qquad (132)$$

We may proceed much as before to compute the generator $G_{2n,h}$ to find:

**Proposition 6.6.** *For $(\theta, \phi) = (1, 1)$ we have*

$$G_{2n,h}(1; x_1, ..., x_n; y_1, ..., y_n; \tau) = \det \mathbf{Q}.\eta(\tau)^2, \qquad (133)$$

*where $\mathbf{Q}$ is the $(n+1) \times (n+1)$ matrix:*

$$\mathbf{Q} = \begin{pmatrix} P_1(x_1 - y_1, \tau) & \ldots & P_1(x_1 - y_n, \tau) & 1 \\ \vdots & \ddots & & \vdots \\ P_1(x_n - y_1, \tau) & & P_1(x_n - y_n, \tau) & 1 \\ 1 & \ldots & 1 & 0 \end{pmatrix}. \qquad (134)$$

*($P_1(z, \tau)$ as in (11)). Furthermore, $G_{2n,h}$ is an analytic function in $x_i, y_j$ and converges absolutely and uniformly on compact subsets of the domain $|q| < |q_{x_i - y_j}| < 1$.*

**Proof.** We prove the result by induction. For $n = 1$ we obtain the result from (132). Assuming the result for $n-1$, we apply the Zhu recursive formula (79) to find

$$\begin{aligned}
& G_{2n,h}(1; x_1, ..., x_n; y_1, ..., y_n; \tau) \\
= \ & \mathrm{STr}_V \left( o(\psi^+) Y(q_{y_1}^{L(0)}\psi^-, q_{y_1}) \ldots Y(q_{x_n}^{L(0)}\psi^+, q_{x_n}) Y(q_{y_n}^{L(0)}\psi^-, q_{y_n}) q^{L(0)+1/12} \right) \\
& + \sum_{r=1}^{n} (-1)^{r-1} \mathbf{Q}(1, r) \det \hat{\mathbf{Q}} . \eta(\tau)^2,
\end{aligned}$$

where $\hat{\mathbf{Q}}$ denotes the matrix found from $\mathbf{Q}$ by deleting row 1 and column $r$. Next note from Lemma 3.2 (ii) and (115) that $G_{2n,h}$ vanishes for $x_1 = x_2$ so that

$$\begin{aligned}
& \mathrm{STr}_V \left( o(\psi^+) Y(q_{y_1}^{L(0)}\psi^-, q_{y_1}) \ldots Y(q_{y_n}^{L(0)}\psi^-, q_{y_n}) q^{L(0)+1/12} \right) \\
= \ & -\sum_{r=2}^{n} (-1)^{r} \mathbf{Q}(2, r) \det \hat{\mathbf{Q}} . \eta(\tau)^2.
\end{aligned}$$

Hence we find $G_{2n,h}$ is given by

$$\sum_{r=1}^{n} (-1)^{r-1} (\mathbf{Q}(1, r) - \mathbf{Q}(2, r)) \det(\hat{\mathbf{Q}}) \eta(\tau)^2 = \det \mathbf{Q} . \eta(\tau)^2,$$

on evaluating $\det \mathbf{Q}$ after subtracting row 2 from row 1. □

We may similarly obtain a determinant formula for all $n$-point functions along the same lines as Propositions 5.3 and 6.5.

## 6.4 Bosonization

As is well known, the rank two fermion VOSA $V$ can be constructed as a rank one bosonic $\mathbb{Z}$-lattice VOSA. $V$ is decomposed in terms of the Heisenberg subVOA $M$ generated by the boson $a$ of (118) and its irreducible modules $M \otimes e^m$ for $a(0)$ eigenvalue $m \in \mathbb{Z}$ (cf. [Ka]). In particular, the partition function $Z_{V,h}(f; \tau)$ and the generating function $G_{n,h}$ can be computed in this bosonic decomposition using the results of ref. [MT1], leading to the Jacobi triple product formula and Fay's trisecant identity (for elliptic functions) respectively. We also describe a further new generalization of Fay's trisecant identity for elliptic functions.

The highest weight lattice vector for the irreducible module $M\otimes e^m$ is

$$\mathbf{1}\otimes e^m = \begin{cases} \psi^+(-m)\psi^+(1-m)...\psi^+(-1).\mathbf{1}, & m>0, \\ \psi^-(m)\psi^-(1+m)...\psi^-(-1).\mathbf{1}, & m<0. \end{cases}$$

Then the partition function is

$$\begin{aligned} Z_{V,h}(f;\tau) &= Z_V\left[\begin{array}{c} f \\ g \end{array}\right](\tau) = \sum_{m\in\mathbb{Z}}(-1)^m e^{2\pi i m\alpha}\mathrm{Tr}_{M\otimes e^m}(q^{L(0)+\kappa^2/2-\kappa m-1/24}) \\ &= \frac{e^{2\pi i(\alpha+1/2)(\beta+1/2)}}{\eta(\tau)}\vartheta\left[\begin{array}{c} -\beta+\frac{1}{2} \\ \alpha+\frac{1}{2} \end{array}\right](0,\tau), \end{aligned} \tag{135}$$

in terms of the theta series (13). Comparing to (125) we obtain the standard Jacobi triple product formula.

We can also compute the generating function $G_{n,h}$ (and hence all $n$-point functions) in the bosonic setting based on results of ref. [MT1]. We illustrate this with the 2-point function generator (126). Recall from (127) and (132) that

$$G_{2,h}(f;x;y;\tau) = \begin{cases} P_1\left[\begin{array}{c} \theta \\ \phi \end{array}\right](x-y,\tau)Z_{V,h}(f;\tau), & (\theta,\phi)\neq(1,1), \\ -\eta(\tau)^2, & (\theta,\phi)=(1,1). \end{cases} \tag{136}$$

In the bosonic language we obtain:

$$\begin{aligned} G_{2,h}(f;x;y;\tau) &= \sum_{m\in\mathbb{Z}}(-1)^m e^{2\pi i m\alpha}F_{M\otimes e^m,h}(1;(\mathbf{1}\otimes e^{+1},x),(\mathbf{1}\otimes e^{-1},y);\tau) \\ &= \sum_{m\in\mathbb{Z}}(-1)^m e^{2\pi i m\alpha}\exp(-\kappa(x-y))q^{\kappa^2/2-\kappa m}. \\ &\quad F_{M\otimes e^m}(1;(\mathbf{1}\otimes e^{+1},x),(\mathbf{1}\otimes e^{-1},y);\tau), \end{aligned}$$

noting that $Y(q_z^{L_h(0)}e^{\pm1},q_z)=\exp(\mp\kappa z)Y(q_z^{L(0)}e^{\pm1},q_z)$. Using Propositions 4 and 5 of ref. [MT1] we obtain

$$F_{M\otimes e^m}(1;(\mathbf{1}\otimes e^{+1},x),(\mathbf{1}\otimes e^{-1},y);\tau) = \frac{q^{m^2/2}}{\eta(\tau)}\frac{\exp(m(x-y))}{K(x-y,\tau)},$$

where $K$ is the prime form (9). Altogether, it follows that

$$G_{2,h}(f;x;y;\tau) = \frac{e^{2\pi i(\alpha+1/2)(\beta+1/2)}}{\eta(\tau)}\frac{\vartheta\left[\begin{array}{c} -\beta+\frac{1}{2} \\ \alpha+\frac{1}{2} \end{array}\right](x-y,\tau)}{K(x-y,\tau)}.$$

Comparing with (136) we confirm the identities (49) for $(\theta,\phi)\neq(1,1)$ and (18) for $(\alpha,\beta)=(0,0)$, i.e. $(\theta,\phi)=(1,1)$.

In a similar fashion we can compute the general generating function $G_{2n,h}$ in the bosonic setting to obtain:

**Proposition 6.7.**

$$G_{2n,h}(f;x_1,...,x_n;y_1,...,y_n;\tau)=\frac{e^{2\pi i(\alpha+1/2)(\beta+1/2)}}{\eta(\tau)}\vartheta\left[\begin{array}{c}-\beta+\frac{1}{2}\\ \alpha+\frac{1}{2}\end{array}\right](\sum_{i=1}^{n}(x_i-y_i),\tau).$$
$$\frac{\prod\limits_{1\leq i<j\leq n}K(x_i-x_j,\tau)K(y_i-y_j,\tau)}{\prod\limits_{1\leq i,j\leq n}K(x_i-y_j,\tau)}.\quad\square$$

Comparing this to Proposition 6.4 for $(\theta,\phi)\neq(1,1)$ and Proposition 6.6 for $(\theta,\phi)=(1,1)$ we obtain the elliptic function version of Fay's Generalized Trisecant Identity [Fa]:

**Corollary 6.8.** *For $(\theta,\phi)\neq(1,1)$ we have*

$$\det(\mathbf{P})=\frac{\vartheta\left[\begin{array}{c}-\beta+\frac{1}{2}\\ \alpha+\frac{1}{2}\end{array}\right](\sum_{i=1}^{n}(x_i-y_i),\tau)}{\vartheta\left[\begin{array}{c}-\beta+\frac{1}{2}\\ \alpha+\frac{1}{2}\end{array}\right](0,\tau)}\frac{\prod\limits_{1\leq i<j\leq n}K(x_i-x_j,\tau)K(y_i-y_j,\tau)}{\prod\limits_{1\leq i,j\leq n}K(x_i-y_j,\tau)},\tag{137}$$

*with* $\mathbf{P}$ *as in (128). For $(\theta,\phi)=(1,1)$,*

$$\det(\mathbf{Q})=-\frac{K(\sum_{i=1}^{n}(x_i-y_i),\tau)\prod\limits_{1\leq i<j\leq n}K(x_i-x_j,\tau)K(y_i-y_j,\tau)}{\prod\limits_{1\leq i,j\leq n}K(x_i-y_j,\tau)},\tag{138}$$

*where* $\mathbf{Q}$ *is as in (134).* $\square$

We may generalize these identities using Propositions 4 and 5 of [MT1] again to consider the general lattice $n$-point function:

**Proposition 6.9.** *For integers $m_i,n_j\geq 0$ satisfying*

$$\sum_{i=1}^{r}m_i=\sum_{j=1}^{s}n_j,$$

*we have*

$$
\begin{aligned}
& F_V(f;(\mathbf{1}\otimes e^{m_1},x_1),...(\mathbf{1}\otimes e^{m_r},x_r),(\mathbf{1}\otimes e^{-n_1},y_1),...(\mathbf{1}\otimes e^{-n_s},y_s);\tau) \\
= \quad & \frac{e^{2\pi i(\alpha+1/2)(\beta+1/2)}}{\eta(\tau)}\vartheta\left[\begin{array}{c} -\beta+\frac{1}{2} \\ \alpha+\frac{1}{2}\end{array}\right](\sum_{i=1}^{r} m_i x_i - \sum_{j=1}^{s} n_j y_j,\tau). \\
& \frac{\prod\limits_{1\leq i<k\leq r} K(x_i-x_k,\tau)^{m_i m_k} \prod\limits_{1\leq j<l\leq s} K(y_j-y_l,\tau)^{n_j n_l}}{\prod\limits_{1\leq i\leq r, 1\leq j\leq s} K(x_i-y_j,\tau)^{m_i n_j}}. \quad \square
\end{aligned}
$$

Comparing this to Proposition 6.5 we obtain a new elliptic generalization of Fay's Trisecant Identity:

**Corollary 6.10.** *For $(\theta,\phi)\neq(1,1)$ we have*

$$
\begin{aligned}
\det(\mathbf{M}) \quad = \quad & \frac{\vartheta\left[\begin{array}{c} -\beta+\frac{1}{2} \\ \alpha+\frac{1}{2}\end{array}\right](\sum_{i=1}^{r} m_i x_i - \sum_{j=1}^{s} n_j y_j,\tau)}{\vartheta\left[\begin{array}{c} -\beta+\frac{1}{2} \\ \alpha+\frac{1}{2}\end{array}\right](0,\tau)}. \\
& \frac{\prod\limits_{1\leq i<k\leq r} K(x_i-x_k,\tau)^{m_i m_k} \prod\limits_{1\leq j<l\leq s} K(y_j-y_l,\tau)^{n_j n_l}}{\prod\limits_{1\leq i\leq r, 1\leq j\leq s} K(x_i-y_j,\tau)^{m_i n_j}},
\end{aligned}
$$

*where* $\mathbf{M}$ *is the block matrix*

$$
\mathbf{M} = \left(\begin{array}{ccc} \mathbf{D}^{(11)} & \dots & \mathbf{D}^{(1s)} \\ \vdots & \ddots & \vdots \\ \mathbf{D}^{(r1)} & \dots & \mathbf{D}^{(rs)} \end{array}\right),
$$

*with* $\mathbf{D}^{(ab)}$ *the* $m_a \times n_b$ *matrix*

$$
\mathbf{D}^{(ab)}(i,j) = D\left[\begin{array}{c} \theta \\ \phi \end{array}\right](i,j,\tau,x_a-y_b), \quad (1\leq i\leq m_a, 1\leq j\leq n_b),
$$

*for* $1\leq a\leq r$ *and* $1\leq b\leq s$. $\square$

A similar identity for $(\theta,\phi)=(1,1)$ generalizing (138) can also be described.

## 6.5 Modular Properties of $n$-Point Functions

In this section we consider the modular properties of all $n$-point functions for the rank two fermion VOSA. Despite the fact the twisted sectors are neither rational or $C_2$-cofinite we obtain modular properties similar to those found in [Z], [DZ1], [DZ2]. It is convenient to employ the twisted $n$-point function formalism to describe these modular properties. We firstly consider the partition function $Z_V \left[ \begin{array}{c} f \\ g \end{array} \right](\tau)$ and define a group action for $\gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL(2, \mathbb{Z})$ as follows:

$$Z_V \left[ \begin{array}{c} f \\ g \end{array} \right] \Bigg| \gamma(\tau) = Z_V \left( \gamma . \left[ \begin{array}{c} f \\ g \end{array} \right] \right) (\gamma.\tau), \tag{139}$$

with

$$\gamma . \left[ \begin{array}{c} f \\ g \end{array} \right] = \left[ \begin{array}{c} f^a g^b \\ f^c g^d \end{array} \right], \tag{140}$$

and $\gamma.\tau$ as in (33).

**Remark 6.11.** *(i) (140) is equivalent to left matrix multiplication on $\alpha, \beta$*

$$\gamma \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) = \left( \begin{array}{c} a\alpha + b\beta \\ c\alpha + d\beta \end{array} \right).$$

*(ii) In terms of the shifted VOA formalism, (139) reads*

$$Z_{V,h}(f;\tau)|\,\gamma = Z_{V,\gamma.h}(f^a g^b; \gamma.\tau),$$

*with $\gamma.h = (\gamma.\beta + \frac{1}{2})a = ((c\alpha + d\beta) + \frac{1}{2})a$, recalling (119) and (121).*

For $SL(2,\mathbb{Z})$ generators $S = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)$ and $T = \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right)$ we can use the theta function modular transformation properties (16) and (17) and thereby find from (135) that

$$Z_V \left[ \begin{array}{c} f \\ g \end{array} \right] \Bigg| S(\tau) = \varepsilon_S \left[ \begin{array}{c} f \\ g \end{array} \right] Z_V \left[ \begin{array}{c} f \\ g \end{array} \right](\tau), \tag{141}$$

$$Z_V \left[ \begin{array}{c} f \\ g \end{array} \right] \Bigg| T(\tau) = \varepsilon_T \left[ \begin{array}{c} f \\ g \end{array} \right] Z_V \left[ \begin{array}{c} f \\ g \end{array} \right](\tau), \tag{142}$$

where

$$\varepsilon_S \left[ \begin{array}{c} f \\ g \end{array} \right] = \exp(2\pi i(\frac{1}{2} + \beta)(\frac{1}{2} - \alpha)), \tag{143}$$

$$\varepsilon_T \left[ \begin{array}{c} f \\ g \end{array} \right] = \exp(\pi i(\beta(\beta+1) + \frac{1}{6})). \tag{144}$$

One can check that the relations $(ST)^3 = -S^2 = 1$ are satisfied so that $Z_V \left[ \begin{array}{c} f \\ g \end{array} \right] (\tau)$ is modular invariant as follows:

**Proposition 6.12.** *The partition function transforms under $\gamma \in SL(2,\mathbb{Z})$ with multiplier $\varepsilon_\gamma \left[ \begin{array}{c} f \\ g \end{array} \right] \in U(1)$ where*

$$Z_V \left[ \begin{array}{c} f \\ g \end{array} \right] \Bigg| \gamma(\tau) = \varepsilon_\gamma \left[ \begin{array}{c} f \\ g \end{array} \right] Z_V \left[ \begin{array}{c} f \\ g \end{array} \right] (\tau),$$

*with $\varepsilon_\gamma \left[ \begin{array}{c} f \\ g \end{array} \right]$ generated from $\varepsilon_S \left[ \begin{array}{c} f \\ g \end{array} \right]$ and $\varepsilon_T \left[ \begin{array}{c} f \\ g \end{array} \right]$.* $\square$

In order to discuss the modular properties of $n$-point functions we first define the left $SL(2,\mathbb{Z})$ action

$$F_V \left[ \begin{array}{c} f \\ g \end{array} \right] ((v_1, z_1)..., (v_n, z_n); \tau) \Bigg| \gamma = F_V \left( \gamma. \left[ \begin{array}{c} f \\ g \end{array} \right] \right) ((v_1, \gamma.z_1)..., (v_n, \gamma.z_n); \gamma.\tau), \tag{145}$$

and $\gamma.z$ as in (33). It is sufficient to consider the generating function:

**Proposition 6.13.** *The generating function $G_{2n} \left[ \begin{array}{c} f \\ g \end{array} \right]$ transforms under $\gamma \in SL(2,\mathbb{Z})$ with weight $n$ and multiplier $\varepsilon_\gamma \left[ \begin{array}{c} f \\ g \end{array} \right]$, that is*

$$G_{2n} \left[ \begin{array}{c} f \\ g \end{array} \right] (x_1...x_n; y_1...y_n; \tau) \Bigg| \gamma = (c\tau+d)^n \varepsilon_\gamma \left[ \begin{array}{c} f \\ g \end{array} \right] G_{2n} \left[ \begin{array}{c} f \\ g \end{array} \right] (x_1...x_n; y_1...y_n; \tau).$$

**Proof.** For $(\theta, \phi) \neq (1,1)$ we have

$$G_{2n} \left[ \begin{array}{c} f \\ g \end{array} \right] = \det(\mathbf{P})\ Z_V \left[ \begin{array}{c} f \\ g \end{array} \right] (\tau),$$

from Proposition 6.4. From Proposition 2.7 we have $P_1 \left[ \begin{array}{c} \theta \\ \phi \end{array} \right] (\gamma.z, \gamma.\tau) = (c\tau + d) P_1 \left[ \begin{array}{c} \theta \\ \phi \end{array} \right] (z, \tau)$. Hence using Proposition 6.12 the result follows.

For $(\theta, \phi) = (1, 1)$ we have

$$G_{2n} \left[ \begin{array}{c} f \\ g \end{array} \right] = \det \mathbf{Q}.\eta(\tau)^2,$$

from Proposition 6.6 with $\mathbf{Q}$ as in (134). From (4) and (6) it follows that $P_1(z, \tau)$ is quasi-modular:

$$P_1(\gamma.z, \gamma.\tau) = (c\tau + d)P_1(z, \tau) + \frac{c}{2\pi i}z.$$

However, $\det \mathbf{Q}$ is modular of weight $n - 1$ as follows. Subtract row 1 from rows $2 \ldots n$ and then subtract col 1 from cols $2 \ldots n$ to find $\det \mathbf{Q} = \det \mathbf{R}$ where for $2 \leq i, j \leq n$

$$R(i, j) = P_1(x_i - y_j, \tau) + P_1(x_1 - y_1, \tau) - P_1(x_i - y_1, \tau) - P_1(x_1 - y_j, \tau),$$

which is modular of weight 1. Hence the result follows. $\square$

The modular transformation properties for an arbitrary $n$-point function follows by appropriately expanding the generating function as before to find $n$-point functions for the Fock basis described in Proposition 6.5. We thus find

**Proposition 6.14.** *For $n$ vectors $v_a$ of $wt[v_a]$, $a = 1, \ldots, n$, the $n$-point function transforms under $\gamma \in SL(2, \mathbb{Z})$ with weight $K = \sum_a wt[v_a]$ and multiplier $\varepsilon_\gamma \left[ \begin{array}{c} f \\ g \end{array} \right]$:*

$$F_V \left[ \begin{array}{c} f \\ g \end{array} \right] ((v_1, z_1), \ldots, (v_n, z_n); \tau) \Big| \gamma = \\ (c\tau + d)^K \varepsilon_\gamma \left[ \begin{array}{c} f \\ g \end{array} \right] F_V \left[ \begin{array}{c} f \\ g \end{array} \right] ((v_1, z_1), \ldots, (v_n, z_n); \tau).$$

This result is a natural generalization for continuous orbifolds of the rank two fermion VOSA of Zhu's Theorem 5.3.2 for $C_2$-cofinite VOAs [Z].

# 7 Appendix A: Parity and Supertraces

A vertex operator $Y(a, z)$ has parity $p(a) \in \{0, 1\}$ if all its modes $a(n)$ have parity $p(a)$. Two operators $A, B$ on $V$ of parity $p(A), p(B)$ have commutator defined by

$$
\begin{aligned}
[A,B] &= AB - p(A,B)BA, \\
p(A,B) &= (-1)^{p(A)p(B)}.
\end{aligned}
$$

The commutator clearly obeys:

$$[A,B] = -p(A,B)[B,A],$$

and for $B_1 \ldots B_n$ of parity $p(B_1), \ldots p(B_n)$ respectively we have

$$
\begin{aligned}
& [A, B_1 \ldots B_n] \\
= & \sum_{r=1}^{n} p(A, B_1 \ldots B_{r-1}) B_1 \ldots B_{r-1} [A, B_r] B_{r+1} \ldots B_n,
\end{aligned} \tag{146}
$$

where

$$p(A, B_1 \ldots B_{r-1}) = \left\{ \begin{array}{cc} 1 & \text{for } r = 1 \\ (-1)^{p(A)[p(B_1)+\ldots+p(B_{r-1})]} & \text{for } r > 1 \end{array} \right. . \tag{147}$$

Let $V_\alpha = \bigoplus_{r \geq r_0} V_{\alpha,r}$ denoted the decomposition of $V_\alpha$ into $L(0)$ homogeneous spaces where $r_0$ is the lowest $L(0)$ degree. We assume that $\dim V_{\alpha,r}$ is finite for each $r, \alpha$. We define the Supertrace of an operator $A$ by:

$$
\begin{aligned}
\mathrm{STr}(Aq^{L(0)}) &= Tr(\sigma A q^{L(0)}) \\
&= Tr_{V_{\bar{0}}}(Aq^{L(0)}) - Tr_{V_{\bar{1}}}(Aq^{L(0)}) \\
&= \sum_{r \geq r_0} q^r [Tr_{V_{\bar{0},r}}(A) - Tr_{V_{\bar{1},r}}(A)].
\end{aligned}
$$

Clearly the supertrace is zero if $A$ has odd parity. We then note the following:

**Lemma 7.1.** *Suppose that $A$ is an operator on $V$ of parity $p(A)$ such that $A : V_{\alpha,r} \to V_{\alpha+p(A),r+s}$ for some real $s$. Then for any operator $B$ we have:*

$$\mathrm{STr}(ABq^{L(0)}) = q^s p(A,B)\, \mathrm{STr}(BAq^{L(0)}). \quad \square$$

Using (60) we find

**Corollary 7.2.** *For $v$ homogeneous of weight $wt(v)$ then*

$$\mathrm{STr}(v(k)\ B\ q^{L(0)}) = p(v,B) q^{wt(v)-k-1}\, \mathrm{STr}(Bv(k)q^{L(0)}). \quad \square \tag{148}$$

We also have

**Corollary 7.3.** *If $A : V_{\alpha,r} \to V_{\alpha+p(A),r}$ then for any operator $B$ we have*

$$\mathrm{STr}([A,B]q^{L(0)}) = 0. \quad \square \tag{149}$$

# References


[B] Borcherds, R.:Vertex algebras, Kac-Moody algebras and the Monster, Proc.Natl.Acad.Sci.U.S.A. **83,** 3068-3071 (1986).

[DLM1] Dong, C., Li, H. and Mason, G.: Modular-invariance of trace functions in orbifold theory and generalized moonshine. Comm. Math. Phys. **214**, 1-56 (2000).

[DLM2] Dong, C., Lin, Z. and Mason, G.: On Vertex operator algebras as $sl_2$-modules.Arasu, K. T. (ed.) et al., *Groups, difference sets, and the Monster*. Proceedings of a special research quarter, Columbus, OH, USA, Spring 1993. Berlin: Walter de Gruyter. Ohio State Univ. Math. Res. Inst. Publ. **4**, 349-362 (1996).

[DM] Dong, C. and Mason, G.: Shifted Vertex Operator Algebras. Math. Proc. Cambridge Philos. Soc. **141**, 67–80 (2006), math.QA/0411526.

[DZ1] Dong, C. and Zhao, Z.: Modularity in orbifold theory for vertex operator superalgebras. Comm. Math. Phys. **260**, 227–256 (2005).

[DZ2] Dong, C. and Zhao, Z.: Modularity of trace functions in orbifold theory for $\mathbb{Z}$-graded vertex operator superalgebras. Preprint math.QA/0601571.

[EO] Eguchi, T.; Ooguri, H. Chiral bosonization on a Riemann surface. Phys. Lett. **B187**, 127–134 (1987).

[Fa] Fay, J.D.: *Theta functions on Riemann surfaces.* Lecture Notes on Mathematics 352, New York:Springer-Verlag, 1973.

[FFR] Feingold, A.J., Frenkel, I.B. and Reis, J.F.X: Spinor construction of vertex operator algebras and $E_8^{(1)}$, Contemp. Math. **121** (1991).

[FHL] Frenkel, I., Huang, Y-Z. and Lepowsky, J.: On axiomatic approaches to vertex operator algebras and modules, Mem. Amer. Math. Soc. **104** no. 494, (1993).

[FK] Farkas, H. and Kra, I.M.: *Theta constants, Riemann surfaces and the modular group.* Graduate Studies in Mathematics, 37. American Mathematical Society, Providence, 2001.

[FLM] Frenkel, I., Lepowsky, J. and Meurman, A.: *Vertex operator algebras and the Monster.* New York: Academic Press, 1988.

[FMS] Di Francesco, Ph., Mathieu, P. and Sénéchal, D.: *Conformal field theory. Graduate Texts in Contemporary Physics.* New York: Springer-Verlag, 1997.

[Ka] Kac, V.: *Vertex Operator Algebras for Beginners.* University Lecture Series, Vol. 10, Boston:AMS, 1998.

[KZ] Kaneko, M. and Zagier, D.: A generalized Jacobi theta function and quasimodular forms, The Moduli Space of Curves (Texel Island, 1994), Progr. in Math. **129**, Birkhauser, (Boston, 1995).

[La] Lang S.: *Elliptic functions.* With an appendix by J. Tate. Second edition. Graduate Texts in Mathematics, 112. Springer-Verlag, New York, 1987.

[Li] Li, H.: Local systems of twisted vertex operators, vertex operator superalgebras and twisted modules, Contemp. Math. AMS. **193** (1996), 203-236.

[MN1] Matsuo, A. and Nagatomo, K.: Axioms for a Vertex Algebra and the locality of quantum fields. Math. Soc. Japan Memoirs **4**, (1999).

[MN2] Matsuo, A. and Nagatomo, K.: A note on free bosonic vertex algebras and its conformal vector, J.Alg. **212** 395-418 (1999).

[MT1] Mason, G. and Tuite, M.P.: Torus chiral n-point functions for free boson and lattice vertex operator algebras. Comm. Math. Phys. **235**, 47–68 (2003).

[MT2] Mason, G. and Tuite, M.P.: On genus two Riemann surfaces formed from sewn tori. Commun.Math.Phys. **270** 587-648 (2007).

[MT3] Mason, G. and Tuite, M.P.: The genus two partition function for free bosonic and lattice vertex operator algebras, to appear.

[Mu] Mumford, D.: *Tata lectures on Theta I.*, Boston:Birkhäuser, 1983.

[P] Polchinski, J: *String Theory, Volumes I and II.* Cambridge:Cambridge University Press, 1998.

[R1] Raina, A. K.: Fay's trisecant identity and conformal field theory. Comm.Math.Phys. **122**, 625–641 (1989).

[R2] Raina, A. K. :An algebraic geometry study of the *b*-*c* system with arbitrary twist fields and arbitrary statistics. Comm.Math.Phys. **140**, 373–397 (1991).

[RS] Sen S. and Raina A.K.: Grassmannians, multiplicative Ward identities and theta-function identities. Phys. Lett. B 203 (1988), no. 3, 256–262.

[Se] Serre, J-P.: *A course in arithmetic*, Springer-Verlag (Berlin 1978).

[T] Tuite, M.P.: Genus two meromorphic conformal field theory. CRM Proceedings and Lecture Notes **30**, 231-251 (2001).

[TUY] Tsuchiya, A, Ueno, K. and Yamada, Y.: Conformal field theory on universal family of stable curves with gauge symmetries, Adv.Stud.Pure.Math. **19** (1989) 459-566.

[TZ] Tuite, M.P. and Zuevsky, A.: Shifting, twisting and intertwining Heisenberg modules, to appear.

[Z] Zhu,Y.: Modular invariance of characters of vertex operator algebras. J. Amer.Math.Soc. **9**, 237–302 (1996).