# SPHERICAL UNITARY DUAL OF GENERAL LINEAR GROUP OVER NON-ARCHIMEDEAN LOCAL FIELD

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MPI/SFB 66-84

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## 1. Introduction

Let $G$ be a connected reductive group over a local non-archimedean field $F$. Fix a special good maximal compact subgroup $K$ of $G$ (in the sence of Bruhat and Tits). Let $\tilde{G}$ be the set of all equivalence classes of irreducible smooth representations of $G$, and let $\hat{G}$ be the subset of all unitarizable classes in $\tilde{G}$. The subset of all classes $\pi$ in $\tilde{G}$ such that $\pi$ restricted to $K$ contains the trivial representation of $K$ as a composition factor, is denoted by $\tilde{G}^s$. Let

$$\hat{G}^s = \tilde{G}^s \cap \hat{G} .$$

The set $\tilde{G}$ (resp. $\tilde{G}^s$) is called a non-unitary dual of $G$ (resp. non-unitary spherical dual of $G$) and $\hat{G}$ (resp. $\hat{G}^s$) is called the unitary dual of $G$ (resp. unitary spherical dual of $G$).

A basic problem of the harmonic analysis of the Gelfand pair $(G,K)$ is to describe $\hat{G}^s$.

Note that $\hat{G}^s$ is not very big part of the whole unitary dual $\hat{G}$, but it plays a very important role in the description of the unitary duals of adelic reductive groups.

The set $\tilde{G}^s$ is in the bijection with the set of all zonal spherical functions on $G$ with the respect to $K$, and $\hat{G}^s$

is in the bijection with the subset of all positive definite zonal spherical functions.

The spherical functions on a p-adic group, and related problems, were studied, for the first time in 1958, by F.I. Mautner ([7]). The group was PGL(2). This was the beginning of representation theory of p-adic reductive groups.

The non-unitary spherical dual $\tilde{G}^s$ was classified in 1963 by I. Satake ([9]). The basic ideas of the theory of spherical representations in the general seting, interesting for our point of view, seems to belong to I.M. Gelfand (for general ideas one can consult [4] and also [6]).

I. Satake obtained integral formulas for zonal spherical functions and I.G. Macdonald in 1968 computed that integrals and obtained explicit formulas for spherical functions ([6]). The same formulas, in a slightly more general situation, were obtained by W. Casselman in 1980, as a consequence of the general facts of representation theory of reductive p-adic groups ([3]).

The solution of the problem of spherical unitary dual for SL(2,F) is contained in [5]. One can obtain, in the same way, the answer for closely related rank one groups. These are all descriptions of spherical unitary duals, known to this author. It can happen that a description was also known for some other particular groups of low ranks, but this author does not know a reference for them. For example, one can easily obtain the spherical unitary dual of GL(3,F) using the same ideas as in the case of GL(2,F).

In this paper we classify all unitary spherical representations of the groups GL(n,F). Note that the notion of spherical representation does not depend on the choice of K, since in GL(n,F) all maximal compact subgroups are conjugated. In this way, one obtains also classification of positive zonal spherical functions on GL(n,F).

The description of spherical unitary dual will be obtained, using Zelevinsky classification, as a direct consequence of the Bernstein result which states that the induced representation of $GL(n,F)$ by an irreducible unitarizable representation of a parabolic subgroup, is irreducible ([ 1 ]). In fact, the bigger part of this short paper is devoted to the identification of $(GL(n,F)^{\sim})^{s}$ in Zelevinsky classification. The identification of $(GL(n,F)^{\wedge})^{s}$ is very short.

We present now our result. Let $U(F^{\times})$ be the set of all unramified unitary characters of the multiplicative group of $F$. The normalized absolute value of $F$ is denoted by $| \, |_F$.

First of all, we have very simple unitary spherical representations

$$\det_n : g \to \chi(\det_n g), \; g \in GL(n,F),$$

when $\chi \in U(F^{\times})$. Let $\pi(\chi(\det_n),\alpha)$ be the representation of $GL(2n,F)$ induced by

$$\chi(\det_n)|\det_n|_F^{\alpha} \otimes \chi(\det_n)|\det_n|_F^{-\alpha}.$$

If $a < \alpha < 1/2$, then $\pi(\chi(\det_n),\alpha)$ is unitarizable. That was shown by G.I.Olshansky in [8] (see also [1]).

<u>Theorem:</u> Fix a positive integer $t$.

(i) Let $n_1,\ldots,n_p,m_1,\ldots,m_q$ be a positive integers such that

$$n_1+\ldots+n_p + 2(m_1+\ldots+m_q) = t.$$

Let $\chi_1,\ldots,\chi_p,\mu_1,\ldots,\mu_q \in U(F^{\times})$ and let $0 < \alpha_1,\ldots,\alpha_q < 1/2$. Then the representation induced by

$$\chi_1(\det_{n_1}) \otimes \ldots \otimes \chi_p(\det_{n_p}) \otimes \pi(\mu_1(\det_{m_1}),\alpha_1) \otimes \ldots \otimes \pi(\mu_q(\det_{m_q}),\alpha_q)$$

from a suitable parabolic subgroup of $GL(t,F)$, is in $(GL(t,F)^\wedge)^s$.

(ii) Each unitary spherical representation can be obtained as it is described in (i), and the parameters

$$(\chi_1,n_1),\ldots,(\chi_p,n_p),(\mu_1,m_1,\alpha_1),\ldots,(\mu_q,m_q,\alpha_q)$$

are uniqueley determined, up to a permutation.

This theorem was announced in [10]. We shall now introduce some basic notation. The field of complex numbers is denoted by $\mathbb{C}$, the subfield of reals is denoted by $\mathbb{R}$, the subring of integers is denoted by $\mathbb{Z}$. The subset of positive integers is denoted by $\mathbb{N}$, and the subset of non-negative integers is denoted by $\mathbb{Z}_+$.

At the end, I would like to thank the Max-Planck-Institut für Mathematik for hospitality and excellent working conditions in which this paper was written.

## 2. Zelevinsky classification and identification on non-unitary spherical dual in this classification

Set $G_n = GL(n,F)$. The category of all smooth representations of $G_n$ of finite length is denoted by Alg $G_n$. The induction functor defines the mapping

$$\text{Alg } G_n \times \text{Alg } G_m \to \text{Alg } G_{n+m},$$
$$(\tau,\sigma) \to \tau \times \sigma$$

(see § 1 of [11]). Let $R_n$ be the Grothendieck group of the category Alg $G_n$. Let

$$R = \bigoplus_{n\geq 0} R_n.$$

The mapping $(\tau,\sigma) \to \tau \times \sigma$ induces a structure of a commutative graded algebra on $R$. We identify $\tilde{G}_n$ with a subset of $R_n$. Let $C(G_n)$ be the set of all cuspidal representations in $\tilde{G}_n$. Set

$$\text{Irr} = \bigcup_{n=0}^{\infty} \tilde{G}_n.$$
$$\text{Irr}^u = \bigcup_{n=0}^{\infty} \hat{G}_n$$
$$C = \bigcup_{n=0}^{\infty} C(G_n) .$$

If $X$ is a set then a function $f$ from $X$ into the non-negative integers with finite support will be called a multiset in $X$. If $\{x_1,\ldots,x_n\}$ is a support of $f$ we shall write $f$ also as

$$f = (\underbrace{x_1,\ldots,x_1}_{f(x_1)\text{-times}},\ \underbrace{x_2,\ldots,x_2}_{f(x_2)\text{-times}},\ldots,\underbrace{x_n,\ldots,x_n}_{f(x_n)\text{-times}})$$

The set of all multisets in $X$ is denoted by $M(X)$. The number $|f| = \sum_{x \in X} f(x)$ is called the cardinal number of $f$.

The representation $g \to |\det g|_F$ of $G_n$ is denoted $\nu$. For $\rho \in C$ and a non-negative integer $n$ we set

$$[\rho, \nu^n\rho] = \{\rho, \nu\rho, \ldots, \nu^{n-1}\rho, \nu^n\rho\}.$$

Then $\Delta = [\rho, \nu^n\rho]$ is called a segment in $C$. The set of all segments in $C$ is denoted by $S(C)$. We shall identify $C$ with a subset of $S(C)$ in a natural way, and also $M(C)$ with a subset of $M(S(C))$.

If $\Delta = [\rho, \nu^n\rho] \in S(C)$, then the representation $\rho \times \nu\rho \times \ldots \times \nu^n\rho$ contains a unique irreducible subrepresentation which is denoted by $\langle\Delta\rangle$. For $a = (\Delta_1, \ldots, \Delta_n) \in M(S(C))$ set

$$\pi(a) = \langle\Delta_1\rangle \times \ldots \times \langle\Delta_n\rangle \in R$$

We construct $\langle a\rangle \in \mathrm{Irr}$ like in 6.5 of [11]. Then $\langle a\rangle$ is a composition factor of $\pi(a)$.

The mapping

$$a \mapsto \langle a\rangle$$

is a bijection of $M(S(C))$ onto Irr. This is Zelevinsky classification.

Let $\Delta_i \in S(C)$, $i = 1,2$. Suppose that $\Delta_1 \cup \Delta_2$ is a segment and

$$\Delta_1 \cup \Delta_2 \notin \{\Delta_1, \Delta_2\}.$$

Then we say that $\Delta_1$ and $\Delta_2$ are linked.
We introduce an ordering on the set $M(S(C))$ like in 7.1. of [11]. By Theorem 7.1. of [11], $\langle a\rangle$ is a composition factor of $\pi(b)$ if and only if $a \leq b$.

Let $\mathcal{O}_F$ be the maximal compact subring of F. Let $K_n = GL(n, \mathcal{O}_F)$. Then $K_n$ is a maximal compact subgroup of $G_n$. A representation $\pi \in \tilde{G}_n$ is called spherical if it contains a non-trivial vector invariant under the action of $K_n$. Denote by $\tilde{G}_n^s$ the subset of all spherical representations in $\tilde{G}_n$. Let

$$\hat{G}_n^s = \tilde{G}_n^s \cap \hat{G}_n$$

$$\mathrm{Irr}^s = \bigcup_{n=0}^{\infty} \tilde{G}_n^s$$

$$\mathrm{Irr}^{su} = \bigcup_{n=0}^{\infty} \hat{G}_n^s \quad .$$

Note that $G_1$ is isomorphic to the multiplicative group $F^\times$ of F. We shall identify $G_1$ with $F^\times$. Then $\tilde{G}_1^s$ is identified also with the group of all unramified quasicharacters of $F^\times$ and $\hat{G}_1^s$ is identified with the group of all unramified unitary characters of $F^\times$. Let $\omega$ be a generator of the maximal ideal in $\mathcal{O}_F$. Then the mapping

$$\chi \to \chi(\omega)$$

is a bijection of $\tilde{G}_1^s$ onto $\mathbb{C}^\times$, and a bijection of $\hat{G}_1^s$ onto $\{z \in \mathbb{C}; |z| = 1\}$ Note that $\tilde{G}_1 = C(G_1)$.

Let us remind of some well known facts about spherical representations ([2],4.4). For each $a \in M(\tilde{G}_1^s)$ the representation $\pi(a)$ contains as a composition factor exactly one spherical representation which we shall denote by $s(a)$. Now

$$a \to s(a),\ M(\tilde{G}_1^s) \to \mathrm{Irr}^s$$

is a bijection. The restriction

$$\{a \in M(\tilde{G}_1^s); |a| = n\} \to \tilde{G}_n^s$$

is also a bijection. We want to describe $s(a)$ in terms of $M(S(C))$.

2.1 PROPOSITION: For $a \in M(G_1^s)$ let $m(a)$ be a minimal element, with respect to the ordering of $M(S(C))$, in the set

(*) $\{b \in M(S(C));\ b \leq a\}$ .

Then $m(a)$ is unique and

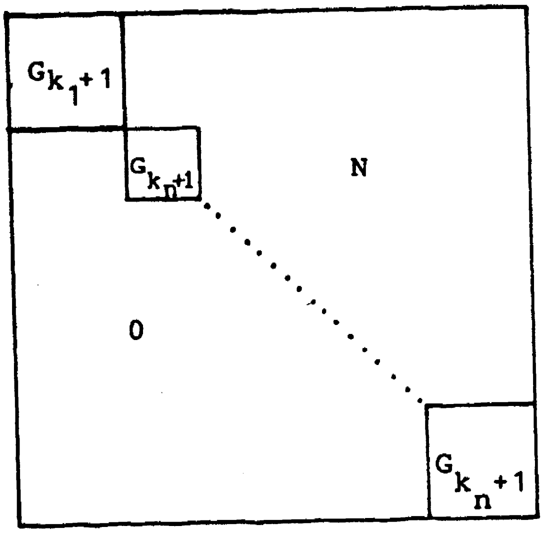
$$s(a) = \langle m(a)\rangle = \pi(m(a)).$$

Proof: Let $m(a) = (\Delta_1,\ldots,\Delta_n)$ be a minimal element of (*). Then $m(a)$ is minimal in (*) if and only if we have no linked segments in b. Now $\pi(m(a))$ is irreducible by Theorem 4.2 of [11]. Since $\langle m(a)\rangle$ is a composition factor of $\pi(m(a))$ we have $\langle m(a)\rangle = \pi(m(a)$ Now we shall prove that $\pi(m(a))$ is spherical.

Let $\Delta_i = [\chi_i, \nu^{k_i}\chi_i]$ , where $k_i$ are non-negative integers. Then $\langle\Delta_i\rangle \in \tilde{G}_{k_i+1}$ ,

$$\langle\Delta_i\rangle\ (g) = \nu^{k_i/2}\ \chi_i(\det g)$$

(see 3.2 of [11]). Now groups $G_{k_1+1},\ldots,G_{k_n+1}$ determine a parabolic subgroup P of $G_p$ where $p = \sum_{i=1}^{n}(k_i+1)$. Let $P = MN$ be a Levi decomposition of P such that $M = G_{k_1+1}\times\ldots\times G_{k_n+1}$ (see the following drawing illustration of P).




We consider $G_{k_i+1}$ as a subgroup of $G_p$ in a natural way. Let $\delta_p$ be a modular character of $P$. Clearly $\delta_p(M \cap K_p) = \{1\}$. We have the Iwasawa decomposition

$$G_p = PK_p .$$

Using this decomposition we can construct the following function defined by

$$f(u \cdot g_1 g_2 \dots g_n \cdot k) = \prod_{i=1}^{n} \delta_P^{\frac{1}{2}}(g_i)\,(\nu^{k_i/2}\chi_i)(\det g_i), u \in N, g_i \in G_{k_i+1}, k \in K_p$$

In a standard way we prove that $f$ is well defined. Now $f \neq 0$, $f \in \langle\Delta_1\rangle \times \dots \times \langle\Delta_n\rangle$ and $f$ is fixed for the action of $K_p$. Thus $\pi(m(a))$ is spherical.

Since $m(a) \leq a$ and $\langle m(a)\rangle$ is spherical, uniqueness of $s(a)$ in $\pi(a)$ implies $s(a) = \langle m(a)\rangle$. Now the fact that $b \to \langle b\rangle$ is a bijection implies uniqueness of $m(a)$.

## 3. Spherical unitary dual

We consider $R^* = R\setminus\{0\}$ as a commutative multiplicative semigroup with identity. Corollary 8.2 a) of [1] implies that $\mathrm{Irr}^u$ is a subsemigroup of $R^*$. From the proof of Proposition 2.1 one can obtain directly the following proposition. We give another proof.

3.1 PROPOSITION: $\mathrm{Irr}^{su}$ is a subsemigroup of $\mathrm{Irr}^u$.

Proof: Let $\pi_1, \pi_2 \in \mathrm{Irr}^{su}$. Choose $a_1, a_2 \in M(\tilde{G}_1^s)$ such that

$$\pi_i = s(a_i) = \langle m(a_i)\rangle = \pi(m(a_i)),\ i = 1,2.$$

Clearly $m(a_1 + a_2) \le m(a_1) + m(a_2)$. Now $\pi_1 \times \pi_2 = \pi(m(a_1)) \times \pi(m(a_2))$ is irreducible by Corollary 8.2 of [1]. Thus $\pi(m(a_1)) \times \pi(m(a_2)) = \pi(m(a_1) + m(a_2))$. The representation $\langle m(a_1+a_2)\rangle$ is a composition factor of $\pi(m(a_1) + m(a_2))$ since $m(a_1+a_1) \le m(a_1) + m(a_2)$. The irreducibility of $\pi(m(a_1) + m(a_2))$ implies

$$\pi_1 \times \pi_2 = \langle m(a_1+a_2)\rangle = s(a_1+a_2).$$

Thus $\pi_1 \times \pi_2 \in \mathrm{Irr}^{su}$.

Let $n$ be a positive integer and $\chi \in \tilde{G}_1^s$. Set

$$\Delta[n] = \{-(n-1)/2,\ 1-(n-1)/2, \dots, (n-1)/2\},$$
$$\Delta[n]^{(\chi)} = \{\nu^\alpha\chi\ ;\ \alpha \in \Delta[n]\}.$$

Note that the representation $\langle\Delta[n]^{(\chi)}\rangle$ is just

$$g \to \chi(\det_n g).$$

Therefore $\langle\Delta[n]^{(\chi)}\rangle \in Irr^{su}$ , $\chi \in \hat{G}_1^s$ .

Also

$$\pi(\langle\Delta[n]^{(\chi)}\rangle,\alpha) = (\nu^{\alpha} \langle \Delta[n]^{(\chi)}\rangle) \times (\nu^{-\alpha} \langle \Delta[n]^{(\chi)}\rangle), \ \alpha \in (0,1/2)$$

is irreducible by Theorem 4.2, of [11]. It is unitarizable by Theorem 2. of [8]. One can obtain this also from [1].

Let $S$ be a subsemigroup of $Irr^u$ generated by all $\langle\Delta[n]^{(\chi)}\rangle$, $\pi(\langle\Delta[n]^{(\chi)}\rangle,\alpha)$ where $n$ is a positive integer, $\chi \in \hat{G}_1^s$ and $0 < \alpha < 1/2$. Since $\langle\Delta[n]^{(\chi)}\rangle$, $\pi(\langle\Delta[n]^{(\chi)}\rangle,\alpha)$ are unitarizable spherical representations, we have $S \subseteq Irr^{su}$.

3.2. THEOREM: We have $S = Irr^{su}$.

Proof: We need to prove that $Irr^{su} \subseteq S$. Let $\pi \in Irr^{su}$. Then $\pi = \pi(m(a))$ for some $a \in M(\tilde{G}_1^s)$. Proposition 2.1 and the fact that $\pi$ is a Hermitian representation implies that

$$(*) \quad \pi = \pi(m(a)) = (\nu^{\alpha_1}\Delta[n_1]^{(\chi_1)} \times \nu^{-\alpha_1}\Delta[n_1]^{(\chi_1)}) \times \ldots$$

$$\ldots \times (\nu^{\alpha_k}\Delta[n_k]^{(\chi_k)} \times \nu^{-\alpha_k}\Delta[n_k]^{(\chi_k)}) \times$$

$$\times \Delta[m_1]^{(\mu_1)} \times \ldots \times \Delta[m_i]^{(\mu_i)}$$

where $\alpha_j > 0$, $n_j, m_j \in \mathbb{N}$, $\chi_j, \mu_j \in \hat{G}_1^s$ .

The theorem will be proved if we show that all $\alpha_j < 1/2$.

Let $\alpha_j \geq 1/2$ for some $j$. Now $\alpha_j \neq 1/2$ since (*) is irreducible. Let $\alpha_j = t + \beta$ for some $t \in (1/2)\mathbb{Z}$ and $0 \leq \beta < 1/2$. Now

$$(**) \quad \pi \times \nu^{\beta}\Delta[2t + n_j - 2]^{(\chi_j)} \times \nu^{-\beta}\Delta[2t + n_j - 2]^{(\chi_j)} \in Irr^{su}$$

by Proposition 3.1. But (**) reduces, since $\nu^{\beta}\Delta[2t + n_j - 2]^{(\chi_j)}$ and $\nu^{\alpha_j}\Delta[n_j]^{(\chi_j)}$ are linked segments. We have obtained contradiction and this proves the theorem.

3.3. <u>Remark:</u> One can also obtain Theorem 3.2. from Lemma 8.8 of [1], after one has Proposition 2.1.

One can also obtain Theorem 3.2. from the description of the whole unitary dual. A part of the solution of unitarizability problem for p-adic GL(n) is contained in [10] and the remaining part will appear soon.

Both of these proofs are using comphicated technics. The proof we presented in the section three is very short and used very simple idea.

## References


1. J.N. Bernstein, P-invariant distributions on GL(N) and the classification of unitary representations of GL(N) (non-archimedean case), in Lie Group Representations II , Proceedings, University of Maryland 1982-1983, Lecture Notes in Math. vol. 1041 Springer-Verlag, Berlin, 1983, 50-102.

2. P. Cartier, Representations of p-adic groups: a survey, in Proc. Sympos. Pure Math. vol. XXXIII, part 1, Amer. Math. Soc., Providence, R.I., 1979, 111-155.

3. W. Casselman, The unramified principal series of p-adic groups I, The spherical functions, Comp. Math. vol. 41 (1980), 387-406.

4. J. Dieudonne,Treatise on analysis, vol. VI, Academic Press, New York, 1978.

5. I.M. Gelfand, M.I. Graev, Representations of a group of the second order with elements from a locally compact field, Russian Math. Surveys 18 (1963), 29-100.

6. I.G. Macdonald, Spherical functions on a group of p-adic type, Rammanjan Institute, Univ. of Madras Publ., 1971.

7. F.I. Mautner, Spherical functions over p-adic fields I,II, Amer. J. Math. vol. 80 (1958), 441-457, vol.86 (1964) 171-200.

8. G.I. Olshansky, Intertwining operators and complementary series in the class of representations of the general group of matrices over a locally compact division algebra, induced from parabolic subgroups, Math. Sb. vol. 93, no.2 (1974), 218-253.

9. I. Satake, Theory of spherical functions on reductive algebraic groups over p-adic fields, Inst. Hautes Études Sci. Publ. Math. 18 (1963), 1-69.

10. M. Tadić, On the classification of irreducible unitary representations of GL(n) and the conjectures of Bernstein and Zelevinsky (non-archimedean case), to appear in Ann. Scient. Ecole Norm, Sup.

11. A.V. Zelevinsky, Induced representations of reductive p-adic groups II, Ann. Scient. Ecole Norm. Sup.13 (1980), 165-210.