TWISTED CONJUGACY CLASSES OF AUTOMORPHISMS OF BAUMSLAG-SOLITAR GROUPS

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Abstract. Let $\phi : G \to G$ be a group endomorphism where G is a finitely generated group of exponential growth, and denote by $R(\phi)$ the number of twisted ϕ-conjugacy classes. Fel'shtyn and Hill [7] conjectured that if ϕ is injective, then $R(\phi)$ is infinite. It was showed in [10] that this conjecture does not hold in general. Nevertheless in this paper, we show that the conjecture holds for the family of the Baumslag-Solitar groups. In particular it is true for automorphisms.

1. Introduction

The fixed point classes of a surface automorphism was introduced by J. Nielsen in [17]. Subsequently, K. Reidemeister [18] laid the algebraic foundation for what is now known as Nielsen fixed point theory. As result of Reidemeister's work, the twisted conjugacy classes of a group homomorphism was introduced. It turns out that the fixed point classes of a surface automorphism can easily be identified with the conjugacy classes of liftings of a surface automorphism, and this last one with the twisted conjugacy classes of the homomorphism induced in the fundamental group of the surface. Let G be a finitely generated group and $\phi : G \to G$ an endomorphism. Two elements $\alpha, \alpha' \in G$ are said to be $\phi - conjugate$ iff there exists $\gamma \in G$ with $\alpha' = \gamma\alpha\phi(\gamma)^{-1}$. The number of twisted ϕ-conjugacy classes is called

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the *Reidemeister number* of an endomorphism ϕ, denoted by $R(\phi)$. If ϕ is the identity map then the ϕ-conjugacy classes are the usual conjugacy classes in the group G. Let X to be a connected, compact polyhedron and $f : X \to X$ to be a continuous map. Recall that the Reidemeister number $R(f)$ is simply the cardinality of the set of twisted ϕ-conjugacy classes where $\phi = f_\#$ is the induced homomorphism on the fundamental group. $R(f)$ need not be finite, and determining of the finiteness of $R(f)$ plays an important role. For some connection with fixed point theory see for example [19], [11]. Also A. Fel'shtyn and R. Hill [7] established a connection between Nielsen fixed point theory and Reideimeister torsion for endomorphisms of certain groups via the Reidemeister zeta function $R_\phi(z)$, similar to the homological (Lefschetz) zeta function in dynamical systems. Such zeta functions are not defined unless the Reidemeister numbers of all iterates of ϕ are finite.

Let G be a non-elementary Gromov hyperbolic group and $\phi : G \to G$ be an automorphism. A. Fel'shtyn [6] showed that automorphisms of G have infinite number of twisted conjugacy classes. Furthermore, using co-Hofian property, A. Fel'shtyn showed that if in addition G is torsion-free and freely indecomposable then for every injective ϕ, $R(\phi)$ is infinite. This result gives supportive evidence to a conjecture of [7] which states that $R(\phi) = \infty$ if G is a finitely generated torsion-free group with exponential growth.

This conjecture was showed to be false in general. In [10] was constructed automorphisms $\phi : G \to G$ on certain finitely generated torsion-free exponential growth groups G that are not Gromov hyperbolic with $R(\phi) < \infty$.

In this paper we study the similar problem for a family of finitely generated groups which have exponential growth but are not Gromov hyperbolic. They are the Baumslag-Solitar groups. We recall its definition. They are indexed by pairs of natural numbers and have the following presentation:

$$B(m, n) = \langle a, b : a^{-1}b^m a^{-1} = b^n, m, n > 0\rangle.$$

It is easy to see that $B(m, n)$ is isomorphic to $B(n, m)$. This family has different features from the one given in [10], which is a family of groups which are metabelian having as kernel the group $\mathbb{Z}^n$. So they contain a subgroup isomorphic to $\mathbb{Z} + \mathbb{Z}$. In the case of Baumslag-Solitar groups this happens if and only if $m = n$. For $m = n = 1$ the group $B(1, 1) = \mathbb{Z}+\mathbb{Z}$ does not have exponential growth. So without loss of generality we will consider $m \leq n$, and for $m = n$ $n > 1$. For more about these groups see [1, 4].

Our results are also related with a question erased by Troitsky (criterion for amenability), namely: Is a discrete group amenable if and only if there is automorphism of this group with finite Reidemeister number? The Baumslag-Solitar groups $B(1, n)$ are amenable because they are solvable. Our calculation shows that the "only if" part of the Troitsky question has a negative answer. The other implication of the Troitsky's question is equivalent to a conjecture of Fel'shtyn and Troitsky [8] that any automorphism of nonamenable group has infinite Reidemeister number. Since for $m > 1$ the groups are nonamenable our calculations also give support to Fel'shtyn-Troitsky's conjecture.

The relevance of the Baumslag-Solitar groups also appears as result of the following Gersten's conjectures:
Conjecture 1: Let G be a finitely generated 1-relator group which contains no B-S(Baumslag-Solitar) subgroups. Then G is Gromov hyperbolic (exception: in the case q=-1 we have Klein bottle).
Conjecture 2: Let G be a group of "finite type" (e.g. has compact $K(G, 1)$) and G contains no B-S subgroups. Then G is Gromov hyperbolic.

It is worth to mention that very recently in [16], D. Kochloukova has described the Automorphism of Baumslag-Solitar groups. Likely several results in this work could be looked under the point of view of this classification.

Our main result is:

Theorem For any injective homomorphism of $B(m, n)$ we have that the Reidemeister number is infinite.

This result corresponds to the three results Theorem 3.4, Theorem 4.4 and Theorem 5.3 for the various values of m and n.

The paper is organized into four sections. In section 2, we develop some preliminaries about the Reidemeister classes of a pair of homomorphism between short exact sequences. In section 3 we study the case $B(1, n)$ for $n > 1$. The main result is Theorem 3.4. In section 4, we consider the cases $B(m, n)$ for $1 < m < n$. The main result is Theorem 4.4. In section 5, we consider the cases $B(n,n)$ for $n>1$. The main result is Theorem 5.3.

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2. Preliminaries

In this section we recall some facts about Reidemeister classes of a pair of homomorphism of a short exact sequences. The set of the Reidemeister classes will be denoted by $R[\ ,\ ]$ and the number of such classes by $R(\ ,\ )$. This will be used for the case where the two sequences are the same and one of the homomorphism is the identity. The main reference for this section is [9] and more details can be found there.

Let us consider a diagram of two short exact sequences of groups and maps between these two sequences:

$$
\begin{array}{ccccccccc}
1 & \to & H_1 & \stackrel{i_1}{\to} & G_1 & \stackrel{p_1}{\to} & Q_1 & \to & 1 \\
 & & f' \downarrow\downarrow g' & & f \downarrow\downarrow g & & \overline{f} \downarrow\downarrow \overline{g} & & \\
1 & \to & H_2 & \stackrel{i_2}{\to} & G_2 & \stackrel{p_2}{\to} & Q_2 & \to & 1
\end{array}
\tag{2.1}
$$

where $f' = f|_{H_1}$, $g' = g|_{H_1}$.

We recall that the Reidemeister classes $R[f_1, f_2]$ relative to homomorphisms $f_1, f_2 : K \to \pi$ are the equivalence classes of elements of π given by the following relation: $\alpha \sim f_2(\tau)\alpha f_1(\tau)^{-1}$ for $\alpha \in \pi$ and $\tau \in K$.

Proposition 1.2 in [9] says

Proposition 2.1 Given the diagram (2.1) we have a short sequence of sets

$$R[f', g'] \stackrel{\widehat{i}_2}{\to} R[f, g] \stackrel{\widehat{p}_2}{\to} R[\overline{f}, \overline{g}]$$

where $\widehat{p_2}$ is surjective and $\widehat{p_2}^{-1}[1] = \mathrm{im}\,(\widehat{i_2})$ where 1 is the identity element of Q_2.

An immediate consequence of this result is

Corollary 2.2 If $R(\overline{f}, \overline{g})$ is infinite then $R(f, g)$ is also infinite.

Proof. Since $\widehat{p_2}$ is surjective the result follows. □

In order to study the injectivity of the map $\widehat{i_2}$, for each element $\overline{\alpha} \in Q_2$ let $H_2(\overline{\alpha}) = p_2^{-1}(\overline{\alpha})$, $C_{\overline{\alpha}} = \{\overline{\tau} \in Q_1 | \overline{g}(\overline{\tau})\overline{\alpha}\overline{f}(\overline{\tau}^{-1}) = \overline{\alpha}\}$ and let $R_{\overline{\alpha}}[f', g']$ be the set of equivalence classes of elements of $H_2(\overline{\alpha})$ by the equivalence relation $\beta \sim g(\tau)\beta f(\tau^{-1})$ where $\beta \in H_2(\overline{\alpha})$ and $\tau \in p_1^{-1}(C_{\overline{\alpha}})$. Finally let $R[f_{\overline{\alpha}}, g_{\overline{\alpha}}]$ be the set of equivalence classes of elements of $H_2(\overline{\alpha})$ given by the relation $\beta \sim g(\tau)\beta f(\tau^{-1})$ where $\beta \in H_2(\overline{\alpha})$ and $\tau \in G_1$.

Proposition 1.2 in [9] says

Proposition 2.3 Two classes of $R(f_{\overline{\alpha}}, g_{\overline{\alpha}})$ represent the same class of $R(f, g)$ if and only if they belong to the same orbit by the action of $C_{\overline{\alpha}}$. Further the isotropy subgroup of this action at an element $[\beta]$ is $G_{[\beta]} = p_1(C_\beta) \subset C_{\overline{\alpha}}$ where $\beta \in [\beta]$.

An immediate consequence of this result is

Corollary 2.4 If $C_{\overline{\alpha}}$ is finite and $R(f_{\overline{\alpha}}, g_{\overline{\alpha}})$ is infinite, for some α, then $R(f, g)$ is also infinite.

Proof. The orbits of the action of $C_{\overline{\alpha}}$ on $R[f_{\overline{\alpha}}, g_{\overline{\alpha}}]$ are finite. So we have an infinite number of orbits and the result follows. □

3. THE CASES $B(1, n)$

As pointed out in the introduction, let $n > 1$ and $B(1, n) = \langle a, b : a^{-1}ba = b^n, n > 1\rangle$.

Recall from [4] that the Baumslag-Solitar groups $B(1, n)$ are finitely-generated solvable groups which are not virtually nilpotent.These groups have exponential growth[14] and they are not Gromov hyperbolic . Furthermore, those groups are metabelian and torsion free.

Consider the homomorphisms $|\ \ |_a : B(1,n) \longrightarrow Z$ which associates for each word $w \in B(1,n)$ the sum of the exponents of a in the word. It is easy to see that this is a well defined map into Z which is surjective.

Proposition 3.1 We have a short exact sequence

$$0 \longrightarrow K \longrightarrow B(1,n) \stackrel{|\ \ |_a}{\longrightarrow} Z \longrightarrow 1,$$

where K, the kernel of the map $|\ \ |_a$, is the set of the elements which have the sum of the powers of a equal to zero. Furthermore, $B(1,n) = K \rtimes Z$ - (semidirect product).

Proof. The first part is clear. The second part follows because Z is free, so the sequence splits. □

Proposition 3.2 The kernel K coincide with $N\langle b\rangle$ which is the normalizer of $\langle b\rangle$ in $B(1,n)$.

Proof. We have $N\langle b\rangle \subset K$. But the quotient $B/N\langle b\rangle$ has the following presentation: $\bar{a}^{-1}\bar{b}\bar{a} = \bar{b}^n, \bar{b} = 1$. Therefore this group is isomorphic to $\mathbb{Z}$ and the natural projection coincides with the map $|\ \ |_a$ under the obvious identification of $\mathbb{Z}$ with $B/N\langle b\rangle$. Consider the commutative diagram

$$\begin{array}{ccccc}
0\to N\langle b\rangle\to & B(1,n)\to & B/N\langle b\rangle\to & 1 \\
\downarrow & \downarrow & \downarrow & \\
0\to K\to & B(1,n)\to & \mathbb{Z}\to & 1
\end{array}$$

where the last vertical map is an isomorphism. From the five Lemma the result follows. □

We recall that the groups $B(1,n)$ are metabelian. Further $B(1,n)$ is isomorphic to $\mathbb{Z}[1/n] \rtimes_\theta \mathbb{Z}$ where the action of $\mathbb{Z}$ on $\mathbb{Z}[1/n]$ is given by $\theta(1)(x) = x/n$. To see this, first observe that the map defined by $\phi(a) = (0,1)$ and $\phi(b) = (1,0)$ extends to a unique homomorphism $\phi : B \to \mathbb{Z}[1/n] \rtimes \mathbb{Z}$ which is clearly surjective. It suffices to show that this homomorphism is injective. Given a word $w = a^{r_1}b^{s_1}...a^{r_t}b^{s_t}$ such that $r_1 + ... + r_t = 0$, so a word on K, using the relation of the group this word is equivalent to $b^{s_1/n^{r_1}}b^{s_2/n^{r_1+r_2}}....b^{s_{t-1}}/n^{r_1+...+r_{t-1}}.b^{s_t}$. If we apply ϕ to this element and assume that it is in the kernel of ϕ, we obtain that the sum of the powers $s_1/n^{r_1} + s_2/n^{r_1+r_2} + ... + s_{t-1}/n^{r_1+...+r_{t-1}} + s_t$ is zero. Bu this means that

w is the trivial element, hence ϕ restricted to K is injective. Therefore the result follows.

Propositon 3.3 Any homomorphism $\phi : B(1,n) \to B(1,n)$ is a homomorphism of the short exact sequence given in Proposition 3.2.

Proof. Let $\bar{\phi}$ be the homomorphism induced by ϕ on the abelianized of $B(1,n)$. The abelianized of $B(1,n)$, denoted by $B(1,n)_{ab}$, is isomorphic to $Z_{n-1} + Z$. The torsion elements of $B(1,n)_{ab}$ form a subgroup isomorphic to Z_{n-1} which is invariant under any homomorphism. The preimage of this subgroup under the projection to the abelianized $B(1,n) \to B(1,m)_{ab}$ is exactly the subgroup $N(b)$, i.e. the elements represented by words where the sum of the powers of a is zero. So it follows that $N(b)$ is mapped into $N(b)$ and the result follows. $\square$

Theorem 3.4 For any injective homomorphism of $B(1,n)$ we have that the Reidemeister number is infinite.

Proof. Let ϕ be a homomorphism. By Propositon 3.3 it is a homomorphism of short exact sequence. The induced map on the quotient is a nontrivial endomomorphism of Z. If the induced homomorphism $\bar{\phi}$ is the identity, by Corollary 2.2 the number of Reidemeister classes is infinite. Since ϕ is also injective, we can assume that $\bar{\phi}$ is multiplication by $k \neq 0,1$. Now we claim that there is no injective homomorphism of $B(1,n)$ such that the induced on the quotient is multiplication by k with $k \neq 0,1$. When we apply the homomorphism ϕ to the relation $a^{-1}ba = b^n$, using the description of $B(1,n) \to Z[1/n] \rtimes Z$ we get: $a^k\phi(b)a^{-k} = (\phi(b)/n^k, 0) = (n\phi(b), 0)$, which implies that either $n^{1-k} = 1$ or $\phi(b) = 0$. So the result follows. $\square$

4. THE CASE $B(m,n)$ $1 < m < n$

The groups in this section are more complicated than the ones in the previous section. Nevertheless in order to get the results we will use a similar procedure as the one in section 3. As pointed out in the introduction let

$1 < m < n$ and $B(m,n) = \langle a, b : a^{-1}b^m a = b^n \rangle$.
Recall that such groups are non-virtually solvable.

Consider the homomorphisms $|\ \ |_a : B(m,n) \longrightarrow \mathbb{Z}$ which associates to each word $w \in B(m,n)$ the sum of the powers of a in the word. It is easy to see that this is a well defined map into $\mathbb{Z}$ which is surjective.

Proposition 4.1 We have a short exact sequence

$$0 \longrightarrow K \longrightarrow B(m,n) \longrightarrow \mathbb{Z} \longrightarrow 1,$$

where K, the kernel of the map $|\ \ |_a$, is the set of the elements which have the sum of the powers of a equal to zero. Furthermore, $B(m,n) = K \rtimes \mathbb{Z}$ - semidirect product where the action is given with respect to some fixed section.

Proof. The first part is clear. The second part follows because $\mathbb{Z}$ is free so the sequence splits. Since the kernel K is not abelian the action is defined with respect to a specific section (see [2]) and the result follows. $\square$

Proposition 4.2 The kernel K coincides with $N\langle b \rangle$ which is the normalizer of $\langle b \rangle$ in $B(m,n)$.

Proof. Similar to Propositon 3.2. $\square$

Propositon 4.3 Any homomorphism $\phi : B(m,n) \to B(m,n)$ is a homomorphism of the short exact sequence given in Proposition 4.1.

Proof. Let $\bar{\phi}$ be the homomorphism induced by ϕ on the abelianized of $B(1,n)$. The abelianized of $B(m,n)$, denoted by $B(m,n)_{ab}$, is isomorphic to $\mathbb{Z}_{n-m}+\mathbb{Z}$. The torsion elements of $B(m,n)_{ab}$ form a subgroup isomorphic to $\mathbb{Z}_{n-m}$ which is invariant under any homomorphism. The preimage of this subgroup under the projection to the abelianized $B(1,n) \to B(1,m)_{ab}$ is exactly the subgroup $N(b)$, i.e. the elements represented by words where the sum of the powers of a is zero. So it follows that $N(b)$ is mapped into $N(b)$ and the result follows. $\square$

In order to have a homomorphism ϕ of $B(m,n)$ which has finite Reidemeister number, the induced map on the quotient $\mathbb{Z}$ must be different from the identity by the same argument used in the proof of Theorem 3.4. So we will assume this from now on that ϕ is not the identity.

Now we will give a presentation of the group K. The group K is generated by the elements $g_i = a^{-i}ba^i$ $i \in Z$ and they satisfy the following relations: $\{1 = a^{-j}(a^{-1}b^nab^{-m})a^j = g_{j+1}^n g_j^{-m}\}$ for all integers j. This presentation is a consequence of the Bass-Serre theory, see [3] Theorem 27 page 211.

Now we will prove the main result of this section. Denote by K_{ab} the abelianized of K.

Theorem 4.4 For any injective homomorphism of $B(m,n)$ we have that the Reidemeister number is infinite.

Proof. Let us consider the short exact sequence obtained from the short exact sequence given in Proposition 4.1 by taking the quotient with the commutators subgroup of K, i.e.

$$0 \longrightarrow K_{ab} \longrightarrow B(m,n)/[K,K] \longrightarrow \mathbb{Z} \longrightarrow 1.$$

So we obtain a short exact sequence where the kernel K_{ab} is abelian. From the presentation of K we obtain a presentation of K_{ab} given as follows: It is generated by the elements g_i $i \in Z$ and they satisfy the following relations: $\{1 = g_{j+1}^n g_j^{-m}, g_ig_j = g_jg_i\}$ for all integers i, j. This is the same as the quotient of the free abelian group generated by the elements g_i, $i \in Z$(so the direct summ of $Z's$ indexed by Z), module the subgroup generated by the relations $\{1 = g_{j+1}^n g_j^{-m}\}$. So an element can be regarded as an equivalence classe of a sequence of integers indexed by Z, where the elements of the sequence are zero but for finite number. By abuse of notation we denote the induced homomorphism on $B(m,n)/[K,K]$ also by ϕ. As in Theorem 3.5, we apply the injective homomorphism ϕ to the relation $a^{-1}b^ma = b^n$ regarded as elements of $B(m,n)/[K,K]$ and we obtain $\phi(a^{-1}(b)^ma^{-1}) = \phi(b^n) = \phi(b)^n$. Since the original homomorphism is injective it follows that the induced homomorphisms on the quotient and on K are also injective. But this implies that the homomorphism on K_{ab} is also injective and it follows that ϕ is injective on $B/[K,K]$. As in Theorem 3.5, we can assume that $\phi(a) = a^k\theta$ for $\theta \in K_{ab}$ and $k \neq 0, 1$, otherwise either the map would not be injective or it follows immediatly that the Reidemeister number is infinite, respectively. Since the kernel of the extension is abelian the equation becomes $\theta^{-1}a^{-k}\phi((b)^m)a^k\theta = a^{-k}\phi((b)^m)a^k = \phi(b^n) = n\phi(b)$. The element $\phi(b)$ is the equivalence class of a sequence of integers indexed in Z (this comes from the presentation of K_{ab}) where only a finite number of elements of the sequence is different from zero. So let $\phi(b) = (n_{i_1}, ..., n_{i_r})$ where $i_1 < i_2 < ... < i_r$ and $n_{i_s} \neq 0$ for all $s = 1, ..., r$, and let $t = i_r - i_1$. In the group K_{ab} we have identifications and $\xi_l^n = \xi_{l-1}^m$ (this follows from the presentation of the group). So if we take the power $b^{mn^{t+1}}$, then $\phi(b^{mn^{t+1}}) = \phi(b^{n^t})^{mn}$. ¿From the relations it follows that $\phi(b^{n^t}) = \phi(b)^{n^t} = (n^t n_{i_1}, ..., n^t n_{i_r}) = b_0$ for some $b_0 \in Z_{i_1}$. Now consider the equation

$$a^{-k}\phi((b^{n^t})^{m^k})a^k = \phi(b^{n^{t+k}}) = (\phi(b^{n^t}))^{n^k} = (b_0)^{n^k}.$$

But

$$\phi(a(b^{n^t})^{m^k})a^{-1}) = \phi(a(b^{m^k})^{n^t})a^{-1}) = \phi(a(b^{m^{k+1}})^{n^{t-1}})a^{-1}) = \phi((b^{n^{t-1}})^{m^{k+1}}) =$$

$$= (\phi(b^{nt}))^{n^{-1}m^{k+1}} = (b_0)^{n^{-1}m^{k+1}}.$$

So we have either $n^k = n^{-1}m^{k+1}$, which implies $m = n$, or b_0 is the trivial element. Since neither cases can happen, the result follows. □

5. THE CASE $B(n, n)$ $n > 1$

As it was point out before if $n = 1$ the group is $\mathbb{Z} + \mathbb{Z}$. Then we have an automorphism that has a finite number of Reidemeister classes. For $n > 1$ we describe the groups $B(n, n)$ as certain extensions and we study the injective homomorphisms. In the end we show that any injective homomorphism have infinite Reidemeister number.

These groups, in contrast with the other Baumslag-Solitar groups, have subgroups isomorphic to $\mathbb{Z}+\mathbb{Z}$. Let us point out that for $n = 2$ this is not the fundamental group of the Klein bottle. There is a surjection from $B(2, 2)$ into the fundamental group of the Klein bottle. We start by describing these groups. Let $| \ |_b : B(n, n) \to \mathbb{Z}$ be the homomorphism which associates to a word the sum of the powers of b which appears in the word. This is a well defined surjective homomorphism and we have:

Proposition 5.1 There is a splitting short exact sequence:

$$0 \to F \to B(n,n) \to \mathbb{Z} \to 1,$$

where F is the free group in n generators $x_1, ..., x_n$ and the last map is $|\ \ |_b$. Further, the action of the generator $1 \in Z$ is the automorphism of F which sends x_j to x_{j+1} for $j < n$, and x_n to x_1.

Proof. Let $F \rtimes \mathbb{Z}$ be the semi-direct product of F by $\mathbb{Z}$ where F is the free group on the set $x_1, ..., x_n$ and the action is given by the automorphism of F which maps x_j to x_{j+1} for $j < n$ and x_n to x_1. We will show that $B(n,n)$ is isomorphic to $F \rtimes \mathbb{Z}$. For, consider the map $\psi : \{a, x_1\} \to F \rtimes \mathbb{Z}$, which sends a to x_1 and b to $1 \in Z$. This map extends to a homomorphism $B(n,n) \to F \rtimes Z$, which we also denote by ψ, since the relation which defines the group $B(n,n)$ is preserved by the map. Also ψ is a homomorphism of short exact sequences. The map restricted to the kernel of $|\ \ |_b$ is surjective to the free group F. Also the kernel admits a set of generators with cardinality n. So the map restrict to the kernel is an isomorphism and the result follows. □

Proposition 5.2 We have a short exact sequence:

$$0 \to \Gamma \to B(n,n) \to \mathbb{Z} + \mathbb{Z} \to 1,$$

where the last map is $(|\ \ |_b, |\ \ |_a)$, and its kernel Γ is the commutator subgroup of $B(n,n)$, which is a free group. Furthermore, there is a subgroup of index n in $B(n,n)$ which is isomorphic to $\Gamma \rtimes (n\mathbb{Z} + \mathbb{Z})$ where the action of the generator $(n,0) \in (n\mathbb{Z} + \mathbb{Z})$ is trivial and the action of the generator $(0,1)) \in (n\mathbb{Z} + \mathbb{Z})$ is given as in Proposition 5.1.

Proof. The map $(|\ \ |_b, |\ \ |_a) : B(n,n) \to \mathbb{Z} + \mathbb{Z}$ is the abelianization and certainly factors through $|\ \ |_b : B(n,n) \to \mathbb{Z}$. So the first part follows. For the second part take the subgroup of $B(n,n)$ which is the preimage of the subgroup generated by $\{(n,0),(0,1)\}$. Then we obtain a short exact sequence where the map which sends $(n,0) \to b^n$ and $(0,1) \to a$ provides a splitting. So the result follows. □

Remark: The group $B(n,n)$ is also the middle term of another short exact sequence:

$$0 \to H \to B(n,n) \to \mathbb{Z} \to 1,$$

where the last map is $|\ \ |_a$, which of course splits. In this case the group H is a group which is the middle term of the following extension:

$$0 \to \mathbb{Z} \to H \to Q \to 1,$$

where Q is the free product of countable many copies of the cyclic group $\mathbb{Z}_n$, indexed by the integers. This normal subgroup $\mathbb{Z} \subset H$ corresponds to the subgroup generated by b^n and it is invariant by automorphism of either $B(n,n)$ or H. Furthermore, with respect to the former short exact sequence the action of the generator $1 \in Z$, with respect to the section which send $1 \to a$, is the automorphism of H which sends Z to itself by the identity and permutes cyclically the generators of the cyclic factors. Although we will not use these facts above, they might be interesting on its own right.

Theorem 5.3 Any injective homomorphism ϕ of $B(n,n)$ has infinite Reidemeister number.

Proof. Given any homomorphism ϕ let us consider the induced homomorphism of the short exact sequence given by Propositon 5.2. The induced homomorphism on the quotient $\bar{\phi}: \mathbb{Z} + \mathbb{Z} \to \mathbb{Z} + \mathbb{Z}$ either has Reidemeister number finite or infinite. In the case of infinite Reidemeister number the result follows from Corollary 2.2. If the Reidemeister number is finite this implies that the set of elements fixed by the homomorphism $\bar{\phi}$ has only one element. Now from Corollary 2.4 the number of Reidemeister classes is infinite if the set $R[f', g']$ is infinite and the set $C_{\bar{1}}$ is finite. But $C_{\bar{1}}$ is easy to see that is the trivial subgroup. So it sufices to show that any injective homomorphism of the group Γ has infinite Reidemeister number. Since from Propositon 5.2 Γ is free, any injective homomorphism of Γ has infinite Reidemeister number. So the result follows. $\square$

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