Spectra, associated points, and representations.

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Introduction

We start with explaining the words of the title.

Spectra in noncommutative algebraic geometry.

Topologizing subcategories and the spectrum $\mathbf{Spec}(-)$. Let C_X be an abelian category thought as the category of quasi-coherent or coherent sheaves on a 'space' X.

Recall that a full subcategory of C_X is called *topologizing* if it is closed under finite coproducts and taking subquotients. For any object M of C_X, let $[M]$ denote the smallest topologizing subcategory of C_X containing M. It is described explicitly as follows: objects of $[M]$ are all subquotients of a finite coproduct of copies of M. The spectrum $\mathbf{Spec}(X)$ consists of all nonzero subcategories of the form $[M]$ such that for any nonzero subobject N of M, the subcategories $[N]$ and $[M]$ coincide. It is regarded together with the preorder $\supseteq$ called, with a good reason, the *specialization* preorder. The specialization preorder determines a topology τ_X which is the finest among the reasonable topologies on $\mathbf{Spec}(X)$: the closure of a set consists of all specializations of its elements.

The closed points of $(\mathbf{Spec}(X), \tau_X)$ and simple objects of C_X. If M is a simple object of the category C_X, then objects of the subcategory $[M]$ are finite coproducts of copies of M. It follows that $[M]$ is a minimal element of $\mathbf{Spec}(X)$, hence it is a closed point of the topological space $(\mathbf{Spec}(X), \tau_X)$. This defines an injective map from the set of isomorphism classes of simple objects into the set $\mathbf{Spec}_0(X)$ of closed points of the space $(\mathbf{Spec}(X), \tau_X)$. If all nonzero objects of the category C_X have simple subquotients (say, C_X has *enough* objects of finite type), then this map is bijective: each closed point of $(\mathbf{Spec}(X), \tau_X)$ is of the form $[M]$ for a simple object M.

This relates the spectrum $\mathbf{Spec}(X)$ with classical representation theory.

The spectrum $\mathbf{Spec}(-)$ and the prime and completely prime spectra of rings. Recall that the *completely prime* spectrum, $Spec_1(R)$, of an associative unital ring R consists of all two-sided ideals $\mathfrak{p}$ of R such that $R - p$ is a multiplicative set. The *prime spectrum*, $Spec(R)$, of R is formed by all two-sided ideals $\mathbf{p}$ such that the set of all two-sided ideals of R which are not contained in $\mathbf{p}$ is closed under multiplication. These two notions coincide when the ring R is commutative. Notice that the completely prime spectrum is functorial with respect to (unital) ring morphisms – the preimage of a completely prime ideal is completely prime. The similar assertion for the prime spectrum is not true.

For an arbitrary associative unital ring A, the assignment $p \mapsto [A/p]$ is an injective map from $Spec_1(A)$ to $\mathbf{Spec}(X)$, where C_X is the category $A - mod$ of left A-modules. Much more subtle result [R, Ch.I] shows that the map $p \longmapsto [R/p]$ is an embedding of $Spec(R)$ into $\mathbf{Spec}(X)$, if R is a left noetherian ring. If the ring R is commutative (or, more generally, R is a PI ring), then the map $Spec(R) \longrightarrow \mathbf{Spec}(X)$ is bijective.

Local 'spaces' and categories. We call a 'space' X, and the representing it abelian category C_X, *local* if the category C_X has the smallest nonzero topologizing subcategory, or, equivalently, the intersection of all nonzero topologizing subcategories of C_X is a nonzero subcategory. The smallest nonzero topologizing subcategory of C_X is a unique closed point of $\mathbf{Spec}(X)$. Therefore, a local abelian category has at most one isomorphism class of simple objects. In particular, the category of modules over a commutative associative unital ring is local iff the ring is local.

Serre subcategories and the spectrum $\mathbf{Spec}^-(X)$. For a subcategory $\mathcal{T}$ of C_X, let $\mathcal{T}^-$ denote the full subcategory of C_X generated by all objects M such that any nonzero subquotient of M has a nonzero subobject which belongs to $\mathcal{T}$. One can show that the subcategory $\mathcal{T}^-$ is thick (that is topologizing and closed under extensions) and $(\mathcal{T}^-)^- = \mathcal{T}^-$. We call a subcategory $\mathcal{T}$ of C_X a *Serre* subcategory if $\mathcal{T} = \mathcal{T}^-$. In the case when C_X is a Grothendieck category, we recover the conventional notion: Serre subcategories are precisely thick subcategories closed under infinite coproducts.

The elements of the spectrum $\mathbf{Spec}^-(X)$ are all Serre subcategories $\mathcal{P}$ of the abelian category C_X such that the quotient category $C_X/\mathcal{P} = C_{X/\mathcal{P}}$ is *local.* Similarly to $\mathbf{Spec}(X)$, the spectrum $\mathbf{Spec}^-(X)$ is endowed with the *specialization* preorder $\supseteq$.

If C_X is a locally noetherian Grothendieck category, or, more generally, a Grothendieck category with a Gabriel-Krull dimension, then the map which assigns to every object E of C_X its left orthogonal, ${}^\perp E$ (– the full subcategory of C_X generated by all objects which have no nonzero morphisms to E) induces an isomorphism between the set of isomorphism classes of indecomposable injectives and $\mathbf{Spec}^-(X)$ [R, Ch.VI].

Thus, $\mathbf{Spec}^-(X)$ might be regarded as an extension of the Gabriel's injective spectrum to 'spaces' represented by arbitrary abelian categories.

A canonical embedding of $\mathbf{Spec}(X)$ into $\mathbf{Spec}^-(X)$. For any object M of an abelian category C_X, let $\langle M\rangle$ denote the full subcategory of C_X generated by all objects N such that M does not belong to the subcategory $[N]$. The following assertion [R4] explains the connection between the spectra $\mathbf{Spec}(X)$ and $\mathbf{Spec}^-(X)$.

The map $[M] \longmapsto \langle M\rangle$ induces an embedding of $\mathbf{Spec}(X)$ into $\mathbf{Spec}^-(X)$. Its image consists of all Serre subcategories $\mathcal{P}$ of C_X such that the intersection $\mathcal{P}^*$ of all topologizing subcategories properly containing $\mathcal{P}$ contains $\mathcal{P}$ properly too.

Localizations at points of $\mathbf{Spec}^-(X)$. Residue skew fields. Suppose an element of $\mathbf{Spec}^-(X)$ is such that $C_X/\mathcal{P}$ has a simple object. We denote its endomorphism ring by $k_{\mathcal{P}}$ and call it the *residue* skew field of the point $\mathcal{P}$. Since all simple objects of $C_X/\mathcal{P}$ are isomorphic to each other, the skew field $k_{\mathcal{P}}$ is determined uniquely up to isomorphism. The smallest non-trivial topologizing subcategory of $C_X/\mathcal{P}$ is isomorphic to the category of finite-dimensional vector spaces over the residue skew field $k_{\mathcal{P}}$.

In particular, to every point $\mathcal{Q} = [M]$ of the spectrum $\mathbf{Spec}(X)$ such that the quotient category has simple objects, we assign the residue field of the point $\widehat{\mathcal{Q}} = \langle M\rangle$ of $\mathbf{Spec}^-(X)$ and call it the residue field of the point $\mathcal{Q}$.

The spectra $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ and $\mathbf{Spec}^-_{\mathfrak{c}}(X)$. If C_X is an arbitrary abelian category, then the spectra $\mathbf{Spec}(X)$ and $\mathbf{Spec}^-(X)$ are the most reasonable choices. If C_X is an abelian category with exact filtered colimits (otherwise called here a category with the property (sup)), then there are two other spectra which might be more adequate – $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ and $\mathbf{Spec}^-_{\mathfrak{c}}(X)$. For instance, if C_X is the category of quasi-coherent modules or D-modules on a non-quasi-compact scheme (like the flag variety of a Kac-Moody Lie algebra), then $\mathbf{Spec}(X)$ and $\mathbf{Spec}^-(X)$ *must* be replaced by respectively $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ and $\mathbf{Spec}^-_{\mathfrak{c}}(X)$.

Recall that a subcategory $\mathcal{T}$ of a category C_X is called *coreflective* if the inclusion functor $\mathcal{T} \longrightarrow C_X$ has a right adjoint. The spectrum $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ is obtained by replacing in the definition of $\mathbf{Spec}(X)$ *topologizing* subcategories by *coreflective topologizing* subcategories. In particular, the smallest topologizing subcategory containing the object M is replaced by the smallest coreflective topologizing subcategory $[M]_{\mathfrak{c}}$ containing M.

The elements of $\mathbf{Spec}^-_{\mathfrak{c}}(X)$ are all Serre subcategories $\mathcal{P}$ of C_X such that the quotient category $C_X/\mathcal{P}$ has the smallest nonzero coreflective topologizing subcategory.

If C_X is the category of quasi-coherent sheaves on a scheme $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ such that the inclusion of all its points has a direct image functor, then the spectrum $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ endowed with Zariski topology (which can be defined in purely categorical terms) is isomorphic to the underlying topological space of the scheme. If the scheme in question is quasi-compact, then $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ coincides with $\mathbf{Spec}(X)$ and $\mathbf{Spec}^-_{\mathfrak{c}}(X)$ with $\mathbf{Spec}^-(X)$.

More generally, $\mathbf{Spec}^0_{\mathfrak{c}}(X) = \mathbf{Spec}(X)$ and $\mathbf{Spec}^-_{\mathfrak{c}}(X) = \mathbf{Spec}^-(X)$ if all nonzero objects of the category C_X have simple subquotients. In particular, the equalities hold if C_X is the category of modules over an associative ring, or, more generally, the category of quasi-coherent sheaves on a noncommutative quasi-compact scheme. By this reason, we shall mostly discuss the spectra $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ and $\mathbf{Spec}^-_{\mathfrak{c}}(X)$.

Associated points.

With each of the spectra, it is related the corresponding notion of *associated points*.

Let M be an object of the category C_X. An element $\mathcal{P}$ of $\mathbf{Spec}(X)$ is called an *associated point of* M *in* $\mathbf{Spec}(X)$ if M has a nonzero subobject L such that $\mathcal{P} = [L]$ and L is $\langle \mathcal{P} \rangle$-torsion free (equivalently, L is right orthogonal to $\langle \mathcal{P} \rangle$). We denote the set of associated points of M in $\mathbf{Spec}(X)$ by $\mathfrak{Ass}(M)$.

The set $\mathfrak{Ass}^-(M)$ of *associated points of the object* M *in* $\mathbf{Spec}^-(X)$ consists of all $\mathcal{P} \in \mathbf{Spec}^-(X)$ such that the localization $M_{\mathcal{P}}$ of M at $\mathcal{P}$ has a closed associated point; that is $M_{\mathcal{P}}$ has a nonzero subobject which belongs to the smallest topologizing subcategory of $C_X/\mathcal{P}$. If $C_X/\mathcal{P}$ has simple objects (which is the case when C_X is locally noetherian, or, more generally, has a Gabriel-Krull dimension), then the condition means precisely that $M_{\mathcal{P}}$ has a nonzero socle. If C_X is the category of coherent sheaves on a noetherian scheme, then this notion coincides with the Grothendieck's notion of associated points (prime cycles) of a coherent sheaf.

Associated points of M in $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ are defined similary to those in $\mathbf{Spec}(X)$, and their set is denoted by $\mathfrak{Ass}_{\mathfrak{c}}(M)$. The reader can now easily figure out what is the set $\mathfrak{Ass}^-_{\mathfrak{c}}(M)$ of associated points of M in $\mathbf{Spec}^-_{\mathfrak{c}}(X)$.

The natural embeddings

$$\begin{array}{ccc} \mathbf{Spec}(X) & \longrightarrow & \mathbf{Spec}^0_{\mathfrak{c}}(X) \\ \downarrow & & \downarrow \\ \mathbf{Spec}^-(X) & \longrightarrow & \mathbf{Spec}^-_{\mathfrak{c}}(X) \end{array} \tag{1}$$

induce the corresponding embeddings of the associated points

$$\begin{array}{ccc}
\mathfrak{Ass}(M) & \longrightarrow & \mathfrak{Ass}_{\mathfrak{c}}(M) \\
\big\downarrow & & \big\downarrow \\
\mathfrak{Ass}^-(M) & \longrightarrow & \mathfrak{Ass}^-_{\mathfrak{c}}(M)
\end{array}$$

All four types of associated points have properties analogous to the known properties of associated points of modules over commutative rings (see Appendix 3).

Representations.

Irreducible representations and the spectra. A classical problem of representation theory is the construction of (interesting classes of) irreducible representations. From the point of view of noncommutative algebraic geometry, this problem is a part of a more natural and more general problem of constructing objects representing elements of an appropriate spectrum. Like in commutative algebraic geometry, where the set of maximal ideals of a ring is replaced by its prime spectrum.

It is worth to mention that, under some mild general nonsense conditions (which hold in all cases we are interested in), all four spectra considered here have the same set of closed points; i.e. the maps (1) above induce isomorphisms between the sets of closed points.

On the experimental level, the work on the realizations of points of the spectrum started at the end of nineteen eighties with constructing realizations of the spectrum of several 'small' algebras which appear in representation theory and mathematical physics, like the first Weyl and Heisenberg algebras and their quantum analogs, (classsical and quantized) enveloping algebra of sl(2), quantum algebra of functions on SL(2). Some of the computations are gathered in Chapers II and IV of the monograph [R]. All these examples, however, are of a special nature – they belong to the class of so called 'hyperbolic' algebras or rank 1 [R, Ch.2] which is particularly convenient for spectral computations. Algebras of skew differential operators is the only other class of algebras whose spectrum was effectively computed "by hands"[R6].

Continuous, affine, and locally affine morphisms of 'spaces'. 'Spaces' of this work are represented by abelian categories and morphisms of 'spaces' $X \longrightarrow Y$ by isomorphism classes of additive, *inverse image*, functors $C_Y \longrightarrow C_X$ between the corresponding categories. A morphism of 'spaces' $X \xrightarrow{f} Y$ is called *continuous* if its inverse image functor, $C_Y \xrightarrow{f^*} C_X$, has a right adjoint, f_*, called the *direct image* functor of f.

A morphism of 'spaces' $\mathfrak{X} \xrightarrow{f} X$ is called *affine* if its inverse image functor $C_X \xrightarrow{f^*} C_{\mathfrak{X}}$ has a right adjoint, f_* (– the direct image functor of f) which is conservative (same as *faithful* in abelian case) and has a right adjoint $f^!$. It follows that f_* preserves all limits and colimits; in particular, it is exact.

A morphism of 'spaces' $\mathfrak{X} \xrightarrow{f} X$ is called *locally affine* if it admits an *affine cover*, which is a family of morphisms $\{U_i \xrightarrow{u_i} \mathfrak{X} \mid i \in J\}$ such that their inverse image functors, u_i^*, are exact localizations whose kernels are Serre subcategories of $C_{\mathfrak{X}}$, the intersection of all kernels is zero, and the compositions $f \circ u_i$ are affine morphisms for all $i \in J$.

Noncommutative schemes can be regarded as examples of locally affine morphisms.

Continuous morphisms and spectra. Induction problem. Let $\mathfrak{X}$ and X be 'spaces' represented by abelian categories, resp. $C_{\mathfrak{X}}$ and C_X, $\mathfrak{X} \xrightarrow{f} X$ a continuous morphism of 'spaces', $\mathcal{P}$ a point of the spectrum of X. The *induction problem* is to find representaves M of the spectrum of $\mathfrak{X}$ such that $\mathcal{P}$ is an associated point of $f_*(M)$.

Here by the *spectrum* of a 'space' X we understand usually $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ or $\mathbf{Spec}(X)$ and sometimes one the remaining two spectra, $\mathbf{Spec}^-_{\mathfrak{c}}(X)$ or $\mathbf{Spec}^-(X)$, more precisely, their 'dual' versions, $\mathbf{Spec}^{\mathfrak{c}}_-(X)$ and $\mathbf{Spec}_-(X)$ – natural extensions of respectively $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ and $\mathbf{Spec}(X)$ introduced in 2.9.4.

This work is concentrated around a (spectral version of the induction) construction which gives a solution of this problem in the case when f is a *locally affine* morphism and the pair $(f, \mathcal{P})$ satisfies certain additional conditions. We explain first its special case which can be formulated without preliminaries.

A special case of the construction. Let A and B be associative unital k-algebras, $C_{\mathfrak{X}} = B-mod$, $C_X = A-mod$, and the morhpism $\mathfrak{X} \xrightarrow{f} X$ is induced by a k-algebra morphism $A \xrightarrow{\varphi} B$. Fix a simple A-module P. Let $\widetilde{B}_P$ denote the class of all A-subbimodules N of B which are flat as right A-modules and such that $N \otimes_A P$ is isomorphic to a direct sum of copies of P. We call the supremum, B_P, of the family $\widetilde{B}_P$ the *stabilizer of P in* B. Pick a simple B_P-module M whose restriction to A is isomorphic to the direct sum of copies of P. The B-module $B \otimes_{B_P} M$ has the largest B-submodule, $\mathfrak{t}_P(B \otimes_{B_P} M)$, whose restriction to A does not have any subquotients isomorphic to P. We denote by $\mathfrak{L}_P(M)$ the quotient module $B \otimes_{B_P} M/\mathfrak{t}_P(B \otimes_{B_P} M)$. Under certain aditional conditions, the (multivalued in general) map $P \longmapsto \mathfrak{L}_P(M)$ produces simple B-modules.

An effect of noncommutativity. Notice that the above construction is useless if the algebras are commutative, because in this case, the stabilizer B_P coincides with the whole algebra B. In general, the size of B_P over A can be regarded as a measure of the noncommutativity of the data $(A \to B, P)$. In the best, noncommutative, case, the stabilizer B_P coincides with the image of A which makes the construction look particularly familiar: $\mathfrak{L}_P(M) \simeq B \otimes_A M/\mathfrak{t}_P(B \otimes_A M)$.

The insufficiency of the special case. With rare exceptions, most of isomorphism classes of simple B-modules cannot be reached this way. But, under certain finiteness conditions, all isomorphism classes of simple B-modules, more generally, all points of the spectrum of $\mathfrak{X}$ (where $C_{\mathfrak{X}} = B - mod$), can be realized if we allow P run through representatives of all, not necessarily closed points of the spectrum of the 'space' X, where $C_X = A - mod$. The construction in this case becomes more subtle.

Besides, it is important to consider a non-affine version of this construction in order to include into the picture D-modules on (quantized and classical) flag varieties and other (commutative and noncommutative) schemes. Therefore, algebras are replaced by 'spaces' represented by abelian categories and morphisms of algebras by *locally affine* morphisms of 'spaces'. The meaning of the last words is explained above.

Reduction to the affine case and glueing. It follows from the results of [R7] that if a locally affine morphism $\mathfrak{X} \xrightarrow{f} X$ admits a finite affine cover $\{U_i \xrightarrow{u_i} \mathfrak{X} \mid i \in J\}$, then the problem can be split into solving it for each affine morphism $U_i \xrightarrow{fu_i} X$ and checking certain *glueing conditions* (explained in Section 7). If the *spectrum* is $\mathbf{Spec}^0_{\mathfrak{c}}(X)$, then the same holds for arbitrary infinite covers as well.

The construction. A natural setting consists of an abelian category C_X endowed with an action of a svelte monoidal category $\widetilde{\mathcal{E}}$ on C_X given by a monoidal functor $\widetilde{\Phi}$ with values in exact continuous (i.e. having a right adjoint) endofunctors of C_X. If C_X has small limits and colimits (say, it is a Grothendieck category), then the forgetful functor φ_* from the category $C_{\mathfrak{A}}$ of $\widetilde{\Phi}$-modules to the category C_X is a direct image functor of an affine morphism, $\mathfrak{A} \xrightarrow{\varphi} X$, hence (by Beck's theorem) the category $C_{\mathfrak{A}}$ can be replaced by the category of modules over the monad $\mathcal{F}_\varphi$ associated with (the inverse and direct image functors of) φ and functor φ_* by the forgetful functor $\mathcal{F}_\varphi - mod \longrightarrow C_X$.

To each point $\mathcal{P}$ of the spectrum of X, there corresponds its *stabilizer* which is the full monoidal subcategory $\widetilde{\mathcal{E}}_{(\mathcal{P})}$ of $\widetilde{\mathcal{E}}$ (defined in 4.1.1). The category $C_{\mathfrak{A}_\mathcal{P}}$ of modules over the (induced by $\widetilde{\Phi}$) action $\widetilde{\Phi}_{(\mathcal{P})}$ of $\widetilde{\mathcal{E}}_{(\mathcal{P})}$ is equivalent to the category of modules over a monad $\mathcal{F}_{\varphi_\mathcal{P}}$, which is also called the *stabilizer* of $\mathcal{P}$. Thus, we have a commutative diagram

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{\mathfrak{f}_\mathcal{P}} & \mathfrak{A}_\mathcal{P} \\ \varphi \searrow & & \swarrow \varphi_\mathcal{P} \\ & X & \end{array}$$

of affine morphisms. Let $\mathfrak{L}_\mathcal{P}$ denote the composition of the functor $\mathfrak{f}^*_\mathcal{P}$ and the functor which assigns to every object of the category $C_{\mathfrak{A}}$ the quotient of this object by its $\varphi_*^{-1}(\widehat{\mathcal{P}})$-torsion, where $\widehat{\mathcal{P}}$ is the Serre subcategory of C_X corresponding to $\mathcal{P}$.

Let $Spec^0_{\mathfrak{c}}(\mathfrak{A})$ denote the family of all objects M such that $[M]_{\mathfrak{c}} = \mathcal{Q}$ is an element of the spectrum $\mathbf{Spec}^0_{\mathfrak{c}}(\mathfrak{A})$ and M is $\widehat{\mathcal{Q}}$-torsion free. In other words, objects of $Spec^0_{\mathfrak{c}}(\mathfrak{A})$ are *representatives* of elements of the spectrum. Let $Spec^{\mathcal{P}}_{\mathfrak{c}}(\mathfrak{A}_\mathcal{P})$ denote the family of all objects of $Spec^0_{\mathfrak{c}}(\mathfrak{A}_\mathcal{P})$ such that $\mathcal{P}$ is an associated point of their image in C_X. If the functor $\mathfrak{f}^*_\mathcal{P}$ is exact and faithful and the action $\widetilde{\Phi}$ satisfies certain 'ampleness' conditions, then the functor $\mathfrak{L}_\mathcal{P}$ transforms every object of $Spec^{\mathcal{P}}_{\mathfrak{c}}(\mathfrak{A}_\mathcal{P})$ into an object of the spectrum of the 'space' $\mathfrak{A}$. Moreover, every object of the spectrum of $\mathfrak{A}$ whose image in $C_{\mathfrak{A}_\mathcal{P}}$ has an associated point which belongs to $Spec^{\mathcal{P}}_{\mathfrak{c}}(\mathfrak{A}_\mathcal{P})$ is equivalent to the image of this associated point by the functor $\mathfrak{L}_\mathcal{P}$. The functor $\mathfrak{L}_\mathcal{P}$ maps simple objects from $Spec^{\mathcal{P}}_{\mathfrak{c}}(\mathfrak{A}_\mathcal{P})$ to simple objects of $C_{\mathfrak{A}}$ (see Theorem 4.2 for details).

Finiteness conditions. In the construction above, given a representative M of a point $\mathfrak{P}$ of the spectrum of $\mathfrak{A}$ such that $\varphi_*(M)$ has an associated point $\mathcal{P}$, one needs certain *finiteness conditions* which garantee that $\mathfrak{P}$ can be obtained via the construction; i.e. that it coincides with $[\mathfrak{L}_\mathcal{P}(V)]$ for some object V of $Spec^{\mathcal{P}}_{\mathfrak{c}}\mathfrak{A}_\mathcal{P}$. The most straightforward finiteness conditions say that $\mathcal{P}$ is an associated point of $\varphi_*(M)$ *of finite multiplicity.* The latter means that the local category $C_X/\mathcal{P}$ has simple objects and the localization of $\varphi_*(M)$ at $\mathcal{P}$ has a finite socle. The length of this socle is called the *multiplicity* of the associated point $\mathcal{P}$ in $u_*(M)$. This finiteness condition works for the spectra $\mathbf{Spec}^-(-)$ and $\mathbf{Spec}^-_{\mathfrak{c}}(-)$ and, in certain cases, for $\mathbf{Spec}(-)$ and $\mathbf{Spec}^0_{\mathfrak{c}}(-)$.

Holonomic objects. Given a continuous morphism $\mathfrak{A} \xrightarrow{\varphi} X$, we call an object M of the category $C_{\mathfrak{A}}$ *holonomic over* X if each nonzero subquotient of $\varphi_*(M)$ has associated points in $\mathbf{Spec}_{\mathfrak{c}}^-(X)$ and all these associated points are of finite multiplicity.

If C_X is the category of quasi-coherent sheaves on a smooth scheme $\mathcal{X}$ and $C_{\mathfrak{A}}$ is the category of D-modules on $\mathcal{X}$, then holonomic objects are precisely holonomic D-modules.

If C_X is the category of quasi-coherent sheaves on the quantum flag variety of a semisimple Lie algebra $\mathfrak{g}$ and $C_{\mathfrak{A}}$ is the category of quasi-coherent $U_q(\mathfrak{g})$-modules on X (cf. [LR2]), then holonomic objects are called *holonomic* quantum D-modules.

All simple holonomic objects can be obtained via the described above construction (i.e. by applying the functors $\mathfrak{L}_{\mathcal{P}}$). Thanks to their functorial properties, the desctiption of holonomic objects is directly reduced to their description on elements of any affine cover.

Organization of the text.

The first two sections contain preliminaries. Section 1 provides a short dictionary for 'spaces' and morphisms of 'spaces'. We remind the notions of *continuous, affine*, and *flat* morphisms of 'spaces' and basic facts about them needed in the main body of the text. Section 2 gives a short sketch of spectral theory of 'spaces' represented by abelian categories and related notions and facts.

Sections 3 and 4 are dedicated to the mentioned above construction of points of the spectrum $\mathbf{Spec}_{\mathfrak{c}}^0(\mathfrak{A})$. We conclude Section 4 with the reduction to the case when C_X is an element of $\mathbf{Spec}_{\mathfrak{c}}^0(X)$; i.e. C_X is the generic point of X. This reduction is useful for analyzing special cases. Two of them are considered in Section 5. The first one is when the functor $F_\varphi = \varphi_*\varphi^*$ is isomorphic to a direct sum of autoequivalences. The second case is when the functor F_φ *differential* and exact. The functor F_φ being differential implies that F_φ (as well as every its subquotient) preserves each Serre subcategory of C_X. In combination with the exactness, this implies that F_φ is compatible with localization at any Serre subcategory. In each of these two cases, we are able to obtain a much more detailed picture and in the first case a convenient variant of Theorem 4.2.

Curiously, both cases (which are, in a sense, perpendicular to each other) appear in the example of the Weyl algebra A_n. Recall that A_n is the k-algebra generated by $x_i,\ y_i$ subject to the relations $[x_i, y_j] = \delta_{ij},\ [x_i, x_j] = 0 = [y_i, y_j]$ for all $1 \le i, j \le n$.

Taking as C_X the category of modules over the polynomial algebra $k[\mathbf{y}] = k[y_1, \dots, y_n]$, and $C_{\mathfrak{A}} = A_n - mod$, we obtain a differential monad on X with $F_\varphi = A_n \otimes_{k[\mathbf{y}]} -$.

Taking as C_X the category of modules over the polynomial algebra $k[\xi] = k[\xi_1, \dots, \xi_n]$, where $\xi_i = x_i y_i,\ 1 \le i \le n$, we obtain the functor $F_\varphi = A_n \otimes_{k[\xi]} -$ which is a direct sum of autoequivalences of the category $k[\xi] - mod$.

This is discussed in more detail in Section C1 of "Complementary facts".

One of the main tools of studying spectra is the localization at appropriate Serre subcategories. The localization simplifies considerably the picture, so that in many cases it is not difficult to compute the spectrum of the quotient 'space'. But, unlike the commutative case, in general, not all points of the spectrum of the quotient 'space' corresponding to an 'open' subspace are localizations of points of the 'space' we started with. All we can say is that these points come from the contrepart $\mathbf{Spec}_-^{\mathfrak{c}}(X)$ of the spectrum $\mathbf{Spec}_{\mathfrak{c}}^-(X)$ (introduced in 2.9). Notice that $\mathbf{Spec}_{\mathfrak{c}}^-(-)$ and, therefore, $\mathbf{Spec}_-^{\mathfrak{c}}(-)$, are functorial with respect to localizations at Serre subcategories. These are some of the reasons why we need an analog of Theorem 4.2 for $\mathbf{Spec}^{\mathfrak{c}}_{-}(X)$ which is given in Section 6.

In Section 7, we remind local properties of the spectra which allow to construct elements of the spectra in the case of locally affine morphisms and simplify their construction in affine cases. We illustrate the general constructions of this work by a rough sketch of their applications to D-modules on classical and quantum flag varieties. In the classical case, the local properties of the spectra allow to reduce the study D-modules on the flag variety to the study of modules over the Weyl algebra A_n, where n is the dimension of the flag variety. Following the philosophy of this work, we study the spectrum of the affine scheme $\mathbf{Sp}(A_n)$ via hyperbolic coordinates, $k[\xi] \longrightarrow A_n$ mentioned above. Some details of this study are provided in "Complementary facts". It is worth to mention that Weyl algebras play also a crucial role in the representation theory of nilpotent Lie algebras: if $\mathfrak{g}$ is a finite-dimensional nilpotent Lie algebra over an algebraicly closed field of zero characteristic, then the set of primitive ideals of its universal enveloping algebra $U(\mathfrak{g})$ is parametrized by the orbits of adjoint action on the dual space $\mathfrak{g}^*$; and for any primitive ideal $\mathfrak{J}$, the quotient algebra $U(\mathfrak{g})/\mathfrak{J}$ is isomorphic to the Weyl algebra A_n.

In "Complementary facts", besides of a fragment of the spectral theory of Weyl algebras obtained via their hyperbolic structure, there are some remarks about application of our induction machinary to natural subalgebras of the enveloping algebras and their quantum analogs. Thus, we observe that highest weight modules are recovered by applying our induction functor together with Harish-Chandra homomorphism to Cartan subalgebras. Similarly in the case of quantized enveloping algebras. More curious possibilities appear if we use upper triangular part instead. There is no attempt to do 'real' applications here. Some of them will appear in consequent papers.

Appendix 1 contains facts on affine morphisms and differential monads, both play a big role in the main body of the text. Appendix 2 is dedicated to associated points and produces a noncommutative version of the classical facts of commutative algebra.

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1. Preliminaries on 'spaces' and morphisms of 'spaces'.

The purpose of this section is to fix notations and provide the reader with a required for this work part of the vocabulary of noncommutative algebraic geometry – an interpretation of geometric notions in which 'spaces' are represented by categories and morphisms of 'spaces' by (inverse image) functors. Although in this work we are mainly interested in the case when the categories representing 'spaces' are abelian and functors representing morphisms are additive, most of general notions (including the notion of a noncommutative scheme) and constructions do not depend on the abelian hypothesis and are more natural and useful without it. Besides, in the consequent papers, there will be non-abelian (derived and exact) versions of the results of this work.

Details (and proofs) can be found in [KR2].

1.1. 'Spaces' represented by categories. The *category of 'spaces'* is the category $|Cat|^o$ which has the same objects as the category Cat^{op} opposite to Cat; morphisms from X to Y are isomorphism classes of functors $C_Y \longrightarrow C_X$. For a morphism of 'spaces' $X \xrightarrow{f} Y$, we denote usually by f^* a representative of the class f and call it an *inverse image functor* of f. The composition of morphisms is defined naturally: $f \circ g$ is the isomorphism class of the the composition $g^* \circ f^*$ of inverse image functors.

1.1.1. The functors $\mathbf{Sp}_k$ and noncommutative affine k-schemes. Let k be a commutative unital ring. The category $\mathbf{Aff}_k$ of *noncommutative affine k-schemes* is the category opposite to the category Alg_k of associative unital k-algebras.

For an associative unital k-algebra R, the 'space' $\mathbf{Sp}_k(R)$ is defined by taking as $C_{\mathbf{Sp}_k(R)}$ the category $R-mod$ of left R-modules. To every k-algebra morphism $R \xrightarrow{\varphi} S$, there corresponds a 'space' morphism $\mathbf{Sp}_k(S) \longrightarrow \mathbf{Sp}_k(R)$ with the inverse image functor $\varphi^* = S \otimes_R - : R-mod \longrightarrow S-mod$. This defines a functor $\mathbf{Sp}_k$ from $\mathbf{Aff}_k = Alg_k^{op}$ to the category $|Cat|^o$ of 'spaces'.

1.2. Localizations and conservative morphisms. Let Y be a 'space' and Σ a class of arrows of the category C_Y. We denote by $\Sigma^{-1}Y$ the 'space' such that the corresponding category coincides with (the standard realization of) the quotient of the category C_Y by Σ (cf. [GZ, 1.1]): $C_{\Sigma^{-1}Y} = \Sigma^{-1}C_Y$. The canonical *localization functor* $C_Y \xrightarrow{p_\Sigma^*} \Sigma^{-1}C_Y$ is regarded as an inverse image functor of a morphism, $\Sigma^{-1}Y \xrightarrow{p_\Sigma} Y$.

For any morphism of 'spaces' $X \xrightarrow{f} Y$, we denote by Σ_f the family Σ_{f^*} of all arrows s of the category C_Y such that $f^*(s)$ is invertible. By the universal property of localizations, f^* is the composition of the localization functor $C_Y \xrightarrow{p_f^*} \Sigma_f^{-1}C_Y = C_{\Sigma_f^{-1}Y}$ and a uniquely determined functor $\Sigma_f^{-1}C_Y \xrightarrow{f_\mathfrak{c}^*} C_X$; hence $f = p_f \circ f_\mathfrak{c}$ for a unique morphism $X \xrightarrow{f_\mathfrak{c}} \Sigma_f^{-1}Y$.

A morphism $X \xrightarrow{f} Y$ is called a *localization* if $f_\mathfrak{c}$ is an isomorphism, i.e. the functor $\Sigma_f^{-1}C_Y \xrightarrow{f_\mathfrak{c}^*} C_X$ is an equivalence of categories.

A morphism $X \xrightarrow{f} Y$ is called *conservative* if its inverse image functor f^* is conservative, that is Σ_f consists of isomorphisms, or, equivalently, p_f is an isomorphism.

The morphism $f_\mathfrak{c}$ in the decomposition $f = p_f \circ f_\mathfrak{c}$ is conservative.

1.3. Left exact, right exact, and exact morphisms. A morphism $X \xrightarrow{f} Y$ is called *right exact* (resp. *left exact*, resp *exact*), if its inverse image functor preserves colimits (resp. limits, resp. both limits and colimits) of arbitrary finite diagrams.

The following assertion is a reformulation of Proposition 1.1.4 in [GZ].

1.3.1. Proposition. *Let $f = p_f \circ f_{\mathfrak{c}}$ be the canonical decomposition of a morphism $X \xrightarrow{f} Y$ into a conservative morphism $X \xrightarrow{f_{\mathfrak{c}}} \Sigma_f^{-1}Y$ and a localization $\Sigma_f^{-1}Y \xrightarrow{p_f} Y$. Suppose C_Y has finite limits (resp. finite colimits). Then f is left exact (resp. right exact) iff the class of arrows Σ_f is a left (resp. right) saturated multiplicative system. In this case both the localization p_f and the conservative morphism $f_{\mathfrak{c}}$ are left (resp. right) exact.*

In particular, if the category C_Y has limits and colimits of finite diagrams, then f is exact iff both the localization p_f and the conservative component $f_{\mathfrak{c}}$ are exact. The exactness of p_f is equivalent to that Σ_f is a (left and right) multiplicative system.

1.4. Dualization functor and dual notions. The *dualization functor*

$$^o : |Cat|^o \longrightarrow |Cat|^o$$

assigns to each 'space' the *dual 'space'* Y^o defined by $C_{Y^o} = C_Y^{op}$, and to each morphism f with an inverse image functor f^*, the morphism f^o having $(f^*)^{op}$ as an inverse image functor. It follows that the dialization functors is an automorphism of the category $|Cat|^o$ and its square is the identical functor.

The dualization functor maps left (resp. right) exact morphisms to right (resp. left) exact morphisms. Conservative morphisms and localizations are stable under the dualization. In particular, the dualization functor preserves the canonical decomposition: $f^o = p_f^o \circ f_{\mathfrak{c}}^o$ and $p_f^o = p_{f^o}$, $f_{\mathfrak{c}}^o = (f^o)_{\mathfrak{c}}$.

1.5. Continuous, flat, almost affine, and affine morphisms. A morphism $X \xrightarrow{f} Y$ is called *continuous* if its inverse image functor, $C_Y \xrightarrow{f^*} C_X$, has a right adjoint, f_*; the latter is called a *direct image functor* of f. One can show that a morphism f is continuous iff both the localization p_f and the conservative factor f_c are continuous.

A continuous morphism $X \xrightarrow{f} Y$ is called *flat* (resp. *faithfully flat*, or *fflat*) if its inverse image functor is exact (resp. exact and conservative).

It is called *almost affine* if its direct image functor is exact and conservative.

It is called *affine* if its direct image functor is conservative and has a right adjoint, $C_Y \xrightarrow{f^!} C_X$. It follows that every affine morphism is almost affine.

If $X \xrightarrow{f} Y$ is an affine morphism and C_Y is the category $R-mod$ of left modules over an associative ring R, then there exists a ring morphism $R \xrightarrow{\varphi} B$ and a natural commutative diagram

$$\begin{array}{ccc} C_X & \xrightarrow{\widetilde{f}_*} & B-mod \\ f_* \searrow & & \swarrow \varphi_* \\ & R-mod & \end{array}$$

in which the functor $\widetilde{f}_*$ is a category equivalence. The ring morphism $R \xrightarrow{\varphi} B$ is defined uniquely up to isomorphism. This diagram is interpreted as the daigram of inverse image functors of morphisms of 'spaces'

$$
\begin{array}{ccc}
X & \xrightarrow{\widetilde{f}} & \mathbf{Sp}(B) \\
f \searrow & & \swarrow \varphi \\
& \mathbf{Sp}(R) &
\end{array}
$$

1.6. 'Spaces' represented by abelian categories. 'Spaces' which appear in the main constructions of this work are represented by abelian categories and morphisms by additive functors. Therefore, we single out the subcategory $|\mathfrak{Ab}|^o$ of the category $|Cat|^o$ of 'spaces' whose objects are all 'spaces' X such that C_X is an abelian category and morphisms are morphisms of 'spaces' with additive inverse image functors.

2. Topologizing, thick, and Serre subcategories of an abelian category. Preliminaries on spectra.

This section contains a short overview a part of the spectral theory of abelian categories which is the starting point for this paper.

2.1. Topologizing subcategories. A full subcategory $\mathbb{T}$ of an abelian category C_X is called *topologizing* if it is closed under finite coproducts and subquotients.

A subcategory $\mathbb{S}$ of C_X is called *coreflective* if the inclusion functor $\mathbb{S} \hookrightarrow C_X$ has a right adjoint; that is every object of C_X has a biggest subobject which belongs to $\mathbb{S}$.

We denote by $\mathfrak{T}(X)$ the preorder (with respect to $\subseteq$) of topologizing subcategories and by $\mathfrak{T}_\mathfrak{c}(X)$ the preorder of coreflective topologizing subcategories of C_X.

2.1.1. The Gabriel product and infinitesimal neighborhoods of topologizing categories. The *Gabriel product*, $\mathbb{S} \bullet \mathbb{T}$, of the pair of subcategories $\mathbb{S}$, $\mathbb{T}$ of C_X is the full subcategory of C_X spanned by all objects M such that there exists an exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

with $L \in Ob\mathbb{T}$ and $N \in Ob\mathbb{S}$. It follows that $0 \bullet \mathbb{T} = \mathbb{T} = \mathbb{T} \bullet 0$ for any strictly full subcategory $\mathbb{T}$. The Gabriel product of two topologizing subcategories is a topologizing subcategory, and its restriction to topologizing categories is associative; i.e. $(\mathfrak{T}(X), \bullet)$ is a monoid. Similarly, the Gabriel product of coreflective topologizing subcategories is a coreflective topologizing subcategory, hence $\mathfrak{T}_\mathfrak{c}(X)$ is a submonoid of $(\mathfrak{T}(X), \bullet)$.

The n^{th} *infinitesimal neighborhood*, $\mathbb{T}^{(n+1)}$, of a subcategory $\mathbb{T}$ is defined by $\mathbb{T}^{(0)} = 0$ and $\mathbb{T}^{(n+1)} = \mathbb{T}^{(n)} \bullet \mathbb{T}$ for $n \geq 0$.

2.2. The preorder $\succ$ and topologizing subcategories. For any two objects, M and N, of an abelian category C_X, we write $M \succ N$ if N is a subquotient of a finite coproduct of copies of M. For any object M of the category C_X, we denote by $[M]$ the full subcategory of C_X whose objects are all $L \in ObC_X$ such that $M \succ L$. It follows that $M \succ N \Leftrightarrow [N] \subseteq [M]$. In particular, M and N are equivalent with respect to $\succ$ (i.e. $M \succ N \succ M$) iff $[M] = [N]$. Thus, the preorder $\big(\{[M] \mid M \in ObC_X\}, \supseteq\big)$ is a canonical realization of the quotient of $(ObC_X, \succ)$ by the equivalence relation associated with $\succ$.

2.2.1. Lemma. *(a) For any object M of C_X, the subcategory $[M]$ is the smallest topologizing subcategory containing M.*

(b) The smallest topologizing subcategory spanned by a family of objects $\mathcal{S}$ coincides with $\bigcup_{N\in\mathcal{S}_\Sigma}[N]$, where $\mathcal{S}_\Sigma$ denotes the family of all finite coproducts of objects of $\mathcal{S}$.

Proof. (a) Since $\succ$ is a transitive relation, the subcategory $[M]$ is closed with respect to taking subquotients. If $M \succ M_i$, $i = 1, 2$, then $M \succ M \oplus M \succ M_1 \oplus M_2$, which shows that $[M]$ is closed under finite coproducts, hence it is topologizing. Clearly, any topologizing subcategory containing M contains the subcategory $[M]$.

(b) The union $\bigcup_{N\in\mathcal{S}_\Sigma}[N]$ is contained in every topologizing subcategory containing the family $\mathcal{S}$. It is closed under taking subquotients, because each $[N]$ has this property. It is closed under finite coproducts, because if $N_1, N_2 \in \mathcal{S}_\Sigma$ and $N_i \succ M_i$, $i = 1, 2$, then $N_1 \oplus N_2 \succ M_1 \oplus M_2$. ■

For any subcategory (or a class of objects) $\mathcal{S}$, we denote by $[\mathcal{S}]$ (resp. by $[\mathcal{S}]_{\mathfrak{c}}$) the smallest topologizing resp. coreflective topologizing) subcategory containing $\mathcal{S}$.

2.2.2. Proposition. *Suppose that C_X is an abelian category with small coproducts. Then a topologizing subcategory of C_X is coreflective iff it is closed under small coproducts. The smallest coreflective topologizing subcategory spanned by a set of objects $\mathcal{S}$ coincides with $\bigcup_{N\in\widetilde{\mathcal{S}}}[N] = \bigcup_{N\in\widetilde{\mathcal{S}}}[N]_{\mathfrak{c}}$, where $\widetilde{\mathcal{S}}$ is the family of all small coproducts of objects of $\mathcal{S}$.*

Suppose that C_X satisfies (AB4), i.e. it has infinite coproducts and the coproduct of a set of monomorphisms is a monomorphism. Then, for any object M of C_X, the smallest coreflective topologizing subcategory $[M]_{\mathfrak{c}}$ spanned by M is generated by subquotients of coproducts of sets of copies of M.

Proof. The argument is similar to that of 2.2.1. ■

2.3. Thick subcategories. A topologizing subcategory $\mathbb{T}$ of the category C_X is called *thick* if $\mathbb{T} \bullet \mathbb{T} = \mathbb{T}$; in other words, $\mathbb{T}$ is thick iff it is closed under extensions.

We denote by $\mathfrak{Th}(X)$ the preorder of thick subcategories of C_X. For a thick subcategory $\mathcal{T}$ of C_X, we denote by $X/\mathcal{T}$ the *quotient 'space'* defined by $C_{X/\mathcal{T}} = C_X/\mathcal{T}$.

2.4. Serre subcategories. We recall the notion of a Serre subcategory of an abelian category as it is defined in [R, III.2.3.2]. For a subcategory $\mathbb{T}$ of C_X, let $\mathbb{T}^-$ denote the full subcategory of C_X generated by all objects L of C_X such that any nonzero subquotient of L has a nonzero subobject which belongs to $\mathbb{T}$.

2.4.1 Proposition. *Let $\mathbb{T}$ be a subcategory of C_X. Then*
(a) The subcategory $\mathbb{T}^-$ is thick.
(b) $(\mathbb{T}^-)^- = \mathbb{T}^-$.
(c) $\mathbb{T} \subseteq \mathbb{T}^-$ iff any subquotient of an object of $\mathbb{T}$ is isomorphic to an object of $\mathbb{T}$.

Proof. See [R, III.2.3.2.1]. ■

2.4.2. Definition. A subcategory $\mathbb{T}$ of C_X is called a *Serre subcategory* if $\mathbb{T}^- = \mathbb{T}$.

2.4.3. Remark. For any subcategory $\mathbb{T}$ of the category C_X, the associated Serre subcategory $\mathbb{T}^-$ is the largest topologizing subcategory of C_X such that every its nonzero object has a nonzero subobject from $\mathbb{T}$.

We denote by $\mathfrak{Se}(X)$ the preorder of all Serre subcategories of C_X.

2.4.4. The property (sup). Recall that X (or the corresponding category C_X) has the property (sup) if for any ascending chain, Ω, of subobjects of an object M, the supremum of Ω exists, and for any subobject L of M, the natural morphism

$$sup(N \cap L \mid N \in \Omega) \longrightarrow (sup\Omega) \cap L$$

is an isomorphism.

2.4.5. Lemma. *Any coreflective thick subcategory is a Serre subcategory. If C_X has the property (sup), then any Serre subcategory of C_X is coreflective.*

Proof. See [R, III.2.4.4]. ■

2.4.6. Proposition. *Let C_X have the property (sup). Then for any thick subcategory $\mathbb{T}$ of C_X, all objects of $\mathbb{T}^-$ are supremums of their subobjects contained in $\mathbb{T}$.*

Proof. Since C_X has the property (sup), the full subcategory $\mathbb{T}_{\mathfrak{s}}$ of C_X whose objects are supremums of objects from $\mathbb{T}$ is thick and coreflective, hence Serre, subcategory containing $\mathbb{T}$ and contained in $\mathbb{T}^-$. Therefore it coincides with $\mathbb{T}^-$. ■

2.5. The spectrum Spec(X). We denote by $Spec(X)$ the family of all nonzero objects M of the category C_X such that $L \succ M$ for any nonzero subobject L of M.

The spectrum $\mathbf{Spec}(X)$ of the 'space' X is the family of topologizing subcategories $\{[M] \mid M \in Spec(X)\}$ endowed with the *specialization* preorder $\supseteq$.

Let $\tau^{\succ}$ denote the topology on $\mathbf{Spec}(X)$ associated with the specialization preorder: the closure of $W \subseteq \mathbf{Spec}(X)$ consists of all $[M]$ such that $[M] \subseteq [M']$ for some $[M'] \in W$.

2.5.1. Proposition. *(a) Every simple object of the category C_X belongs to $Spec(X)$. The inclusion $Simple(X) \hookrightarrow Spec(X)$ induces an embedding of the set of the isomorphism classes of simple objects of C_X into the set of closed points of $(\mathbf{Spec}(X), \tau^{\succ})$.*

(b) If every nonzero object of C_X has a simple subquotient, then each closed point of $(Spec(X), \tau^{\succ})$ is of the form $[M]$ for some simple object M of the category C_X.

Proof. (a) If M is a simple object, then $Ob[M]$ consists of all objects isomorphic to coproducts of finite number of copies of M. In particular, if M and N are simple objects, then $[M] \subseteq [N]$ iff $M \simeq N$.

(b) If L is a subquotient of M, then $[L] \subseteq [M]$. If $[M]$ is a closed point of $\mathbf{Spec}(X)$, this implies the equality $[M] = [L]$. ■

2.6. Local 'spaces'. A 'space' X and the representing it abelian category C_X are called *local* if C_X has the smallest topologizing subcategory, $\mathbb{O}_X^*$.

It follows that $\mathbb{O}_X^*$ is the only closed point of $\mathbf{Spec}(X)$.

2.6.1. Proposition. *Let X be local, and let the category C_X have simple objects. Then all simple objects of C_X are isomorphic to each other, and every nonzero object of $\mathbb{O}_X^*$ is a finite coproduct of copies of a simple object.*

Proof. In fact, if M is a simple object in C_X, then $[M]$ is a closed point of $\mathbf{Spec}(X)$. If X is local, this closed point is unique. Therefore, objects of $[M]$ are finite coproducts of copies of M (see the argument of 2.5.1). ∎

2.7. $\mathbf{Spec}^-(X)$. By definition, $\mathbf{Spec}^-(X)$ is formed by all Serre subcategories $\mathcal{P}$ of C_X such that $X/\mathcal{P}$ is a local 'space'. It is endowed with the preorder $\supseteq$.

If C_X is a Grothendieck category with Gabriel-Krull dimension (say C_X is locally noetherian), then the elements of $\mathbf{Spec}^-(X)$ are in bijective correspondence with the set of isomorphism classes of indecomposable injectives of the category C_X. In other words, $\mathbf{Spec}^-(X)$ is isomorphic to the Gabriel spectrum of the category C_X.

2.7.1. Remark. If $C_X = R-mod$, where R is a commutative noetherian ring, then the Gabriel spectrum of C_X (hence $\mathbf{Spec}^-(X)$) is isomorphic to the prime spectrum of the ring R [Gab]. If R is a non-noetherian commutative ring, $\mathbf{Spec}^-(X)$ might be much bigger than the prime spectrum of R, while $\mathbf{Spec}(X)$ is naturally isomorphic to the prime spectrum of R (cf. [R], Ch.3).

More generally, if C_X is the category of quasi-coherent sheaves on a quasi-compact quasi-separated scheme, then $\mathbf{Spec}(X)$ endowed with the Zariski topology is homeomorphic to the underlying space to the scheme [R4]. In the case of a non-quasi-compact scheme, $\mathbf{Spec}(X)$ should be replaced by the spectrum $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ which is defined below.

2.8. The spectra $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ and $\mathbf{Spec}^-_{\mathfrak{c}}(X)$. We define $Spec^0_{\mathfrak{c}}(X)$ as the family of all nonzero objects M of C_X such that $[M]_{\mathfrak{c}} = [L]_{\mathfrak{c}}$ for any nonzero subobject L of M.

By definition, $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ is the preorder (with respect to $\supseteq$) formed by all coreflective subcategories of the form $[M]_{\mathfrak{c}}$, where $M \in Spec^0_{\mathfrak{c}}(X)$.

2.8.1. Note. It follows from 2.2.2 that if the category C_X satisfies (AB4) (i.e. it has small coproducts and a coproduct of a set of monomorphisms is a monomorphism), then $Spec^0_{\mathfrak{c}}(X)$ consists of all nonzero objects M which are subquotients of the coproduct of a set of copies of any of its nonzero subobjects.

2.8.2. $\mathbf{Spec}^-_{\mathfrak{c}}(X)$. The spectrum $\mathbf{Spec}^-_{\mathfrak{c}}(X)$ is a preorder with respect to $\supseteq$ of all coreflective thick subcategories $\mathcal{P}$ of C_X such that the quotient category $C_X/\mathcal{P}$ has the smallest nonzero coreflective topologizing subcategory.

2.8.3. Proposition. *Suppose that the category C_X has the property (sup). Then $\mathbf{Spec}^-(X)$ is contained in $\mathbf{Spec}^-_{\mathfrak{c}}(X)$ and the map which assigns to every topologizing subcategory $\mathcal{Q}$ of C_X the subcategory $[\mathcal{Q}]_{\mathfrak{c}}$ (which is the smallest coreflective topologizing subcagegory containing $\mathcal{Q}$) is an embedding $\mathbf{Spec}(X) \hookrightarrow \mathbf{Spec}^0_{\mathfrak{c}}(X)$.*

Proof. If C_X has the property (sup), then, by 2.4.6, for any topologizing subcategory $\mathcal{T}$ of C_X, the smallest coreflective subcategory containing $\mathcal{T}$ is generated by objects which are supremums of their subobjects from $\mathcal{T}$. The assertions follow from this fact. ∎

2.9. From Serre subcategories to spectra. Let $\mathcal{P}$ be a Serre subcategory of an abelian category C_X. Set

$$\mathcal{P}^* = \bigcap_{\mathcal{P}\subsetneq\mathcal{T}\in\mathfrak{T}(X)} \mathcal{T}, \quad \mathcal{P}^{\mathfrak{c}} = \bigcap_{\mathcal{P}\subsetneq\mathcal{T}\in\mathfrak{T}_{\mathfrak{c}}(X)} \mathcal{T}; \quad \mathcal{P}^{\circledast} = \bigcap_{\mathcal{P}\subsetneq\mathcal{T}\in\mathfrak{T}^{\mathcal{P}}(X)} \mathcal{T}, \quad \mathcal{P}^{\circledast}_{\mathfrak{c}} = \bigcap_{\mathcal{P}\subsetneq\mathcal{T}\in\mathfrak{T}^{\mathcal{P}}_{\mathfrak{c}}(X)} \mathcal{T}.$$

Here $\mathcal{T}(X)$ and $\mathcal{T}_{\mathfrak{c}}(X)$ are preorders of resp. topologizing and coreflective topologizing subcategories of C_X; and $\mathcal{T}^{\mathcal{P}}(X)$ (resp. $\mathcal{T}_{\mathfrak{c}}^{\mathcal{P}}(X)$) denotes the preorder of $\mathcal{P}$*-invariant* topologizing (resp. coreflective topologizing) subcategories.

Recall that a subcategory $\mathcal{S}$ is called $\mathcal{P}$-invariant if $\mathcal{S} = \mathcal{P} \bullet \mathcal{S} \bullet \mathcal{P}$.

Set

$$\mathcal{P}_* = \mathcal{P}^* \bigcap \mathcal{P}^{\perp}, \quad \mathcal{P}_*^{\mathfrak{c}} = \mathcal{P}^{\mathfrak{c}} \bigcap \mathcal{P}^{\perp}; \quad \mathcal{P}_{\circledast} = \mathcal{P}^{\circledast} \bigcap \mathcal{P}^{\perp}, \quad \mathcal{P}_{\circledast}^{\mathfrak{c}} = \mathcal{P}^{\mathfrak{c}} \bigcap \mathcal{P}^{\perp}. \tag{1}$$

2.9.1. Lemma. *(a) $\mathcal{P}^* \neq \mathcal{P}$ (resp. $\mathcal{P}^{\mathfrak{c}} \neq \mathcal{P}$, resp. $\mathcal{P}^{\circledast} \neq \mathcal{P}$, resp. $\mathcal{P}_{\mathfrak{c}}^{\circledast} \neq \mathcal{P}$) iff $\mathcal{P}_* \neq 0$ (resp. $\mathcal{P}_*^{\mathfrak{c}} \neq 0$, resp. $\mathcal{P}_{\circledast} \neq 0$, resp. $\mathcal{P}_{\circledast}^{\mathfrak{c}} \neq 0$).*

(b) Let $\mathcal{S}$ be one of the subcategories (1). Then the full subcategory of C_X generated by all quotients of objects of $\mathcal{S}$ is topologizing; i.e. it coincides with $[\mathcal{S}]$.

(c) If $\mathcal{S}$ is $\mathcal{P}_*^{\mathfrak{c}}$ or $\mathcal{P}_{\circledast}^{\mathfrak{c}}$ and C_X has small coproducts, then the $[\mathcal{S}] = [\mathcal{S}]_{\mathfrak{c}}$, i.e. the topologizing subcategory $[\mathcal{S}]$ is coreflective.

Proof. (a) Let $\mathbb{T}$ be a subcategory of C_X closed under taking subquotients and such that $\mathbb{T} \neq \mathcal{P}$, i.e. there is an object M of $\mathbb{T}$ which does not belong to $\mathcal{P}$. Since, by hypothesis, $\mathcal{P}$ is a Serre subcategory, i.e. $\mathcal{P} = \mathcal{P}^-$, the latter means that M has a nonzero subquotient, L, which is $\mathcal{P}$-torsion free. Since $\mathcal{P}$ is closed under taking quotients, it follows that L belongs to $\mathcal{P}^{\perp}$. On the other hand, $L \in Ob\mathbb{T}$, because $\mathbb{T}$ is closed under taking subquotients. This proves the implications $\Rightarrow$. The opposite implications are obvious.

(b) Each of the subcategories (1) is closed under taking subobjects and finite coproducts. This implies that the full subcategory of C_X generated by all quotients of objects of $\mathcal{S}$ is closed under taking subobjects, finite coproducts and quotients; i.e. it is topologizing.

(c) Suppose now that C_X is the category with small coproducts and $\mathcal{S}$ is $\mathcal{P}_*^{\mathfrak{c}}$ or $\mathcal{P}_{\circledast}^{\mathfrak{c}}$. Then, since any coproduct of epimorphisms is an epimorphism, it follows from (b) that $\mathcal{S}$ is closed under small coproducts. By 2.2.2, this implies that $\mathcal{S}$ is coreflective. ∎

For any subcategory (or a family of objects) $\mathcal{Q}$ of C_X, we denote by $\widehat{\mathcal{Q}}$ the full subcategory of C_X such that $Ob\widehat{\mathcal{Q}}$ is the union of classes of objects of all Serre subcategories of C_X which do not contain $\mathcal{Q}$.

Set $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X) = \{\mathcal{P} \in \mathfrak{Se}(X) \mid \mathcal{P}^* \neq \mathcal{P}\}$ and $\mathbf{Spec}_{\mathfrak{c}}^{1}(X) = \{\mathcal{P} \in \mathfrak{Se}(X) \mid \mathcal{P}^{\mathfrak{c}} \neq \mathcal{P}\}$, where $\mathfrak{Se}(X)$ is family of all Serre subcategories of C_X.

2.9.2. Proposition. *Let C_X be an abelian category with the property (sup).*

(a) The map $\mathcal{Q} \longmapsto \widehat{\mathcal{Q}}$ induces isomorphisms

$$\mathbf{Spec}(X) \xrightarrow{\sim} \mathbf{Spec}_{\mathfrak{t}}^{1,1}(X) \tag{2}$$

$$\mathbf{Spec}_{\mathfrak{c}}^{0}(X) \xrightarrow{\sim} \mathbf{Spec}_{\mathfrak{c}}^{1}(X), \tag{3}$$

(b) There are natural injective morphisms

$$\mathbf{Spec}(X) \longrightarrow \mathbf{Spec}_{\mathfrak{c}}^{0}(X) \quad \textit{and} \quad \mathbf{Spec}_{\mathfrak{t}}^{1,1}(X) \longrightarrow \mathbf{Spec}_{\mathfrak{c}}^{1}(X) \tag{4}$$

such that the diagram

$$\begin{array}{ccc}
\mathbf{Spec}(X) & \longrightarrow & \mathbf{Spec}^0_{\mathfrak{c}}(X) \\
\wr\downarrow & & \downarrow\wr \\
\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X) & \longrightarrow & \mathbf{Spec}^1_{\mathfrak{c}}(X)
\end{array} \tag{5}$$

commutes.

(c) If C_X has enough objects of finite type, then the morphisms (4) are isomorphisms.

Proof. (a1) The map inverse to (2) assigns to each element $\mathcal{P}$ of $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$ the topologizing subcategory $[\mathcal{P}_*]$ [R7, argument of 3.2].

(a2) The map inverse to (3) assigns to each element $\mathcal{P}$ of $\mathbf{Spec}^1_{\mathfrak{c}}(X)$ the coreflective topologizing subcategory $[\mathcal{P}^{\mathfrak{c}}_*]$ [R7, argument of 10.1.2].

(b) The map $\mathbf{Spec}(X) \longrightarrow \mathbf{Spec}^0_{\mathfrak{c}}(X)$ assigns to each element $\mathcal{Q}$ of the spectrum $\mathbf{Spec}(X)$ the smallest coreflective subcategory $[\mathcal{Q}]_{\mathfrak{c}}$ containing $\mathcal{Q}$ (see 2.8.3). The map $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X) \longrightarrow \mathbf{Spec}^1_{\mathfrak{c}}(X)$ is simply the inclusion.

(c) See the argument of [R7, 10.1.2(c)].

For omitted details and arguments, see proves of [R7, 3.2] and [R7, 10.1.2]. ■

2.9.3. Proposition. *(a)* $\mathbf{Spec}^-(X) = \{\mathcal{P} \in \mathfrak{Se}(X) \mid \mathcal{P}_{\circledast} \neq 0\}$.

(b) $\mathbf{Spec}^-_{\mathfrak{c}}(X) = \{\mathcal{P} \in \mathfrak{Se}(X) \mid \mathcal{P}^{\mathfrak{c}}_{\circledast} \neq 0\}$.

Proof. The first equality is one of the assertions of [R7, 5.3.1]. The equality (b) can be obtained via an argument similar to that of [R7, 5.3.1]. ■

2.9.4. The spectra $\mathbf{Spec}_-(X)$ and $\mathbf{Spec}^{\mathfrak{c}}_-(X)$. The elements of $\mathbf{Spec}_-(X)$ are all topologizing subcategories of the form $[\mathcal{P}_{\circledast}]$, where $\mathcal{P}$ runs through $\mathbf{Spec}^-(X)$.

Similarly, $\mathbf{Spec}^{\mathfrak{c}}_-(X)$ consists of all coreflective topologizing subcategories of the form $[\mathcal{P}_{\circledast}]_{\mathfrak{c}}$, where $\mathcal{P}$ runs through $\mathbf{Spec}^-_{\mathfrak{c}}(X)$.

Both $\mathbf{Spec}_-(X)$ and $\mathbf{Spec}^{\mathfrak{c}}_-(X)$ are endowed with the 'specialization' preorder $\supseteq$.

2.9.4.1. Proposition. *The map $\mathcal{Q} \longmapsto \widehat{\mathcal{Q}}$ induces isomorphisms*

$$\mathbf{Spec}_-(X) \xrightarrow{\sim} \mathbf{Spec}^-(X) \tag{6}$$

$$\mathbf{Spec}^{\mathfrak{c}}_-(X) \xrightarrow{\sim} \mathbf{Spec}^-_{\mathfrak{c}}(X), \tag{7}$$

Proof. The map (6) is inverse to the map $\mathcal{P} \longmapsto [\mathcal{P}_{\circledast}]$, and the map (7) is inverse to $\mathcal{P} \longmapsto [\mathcal{P}^{\mathfrak{c}}_{\circledast}]_{\mathfrak{c}}$. Details are left to the reader. ■

2.9.4.2. Diagrams. Let the category C_X have property (sup). The diagram (5) (see 2.9.2) is extended to the commutative diagram

$$\begin{array}{ccccccc}
\mathbf{Spec}_-(X) & \longleftarrow & \mathbf{Spec}(X) & \longrightarrow & \mathbf{Spec}^0_{\mathfrak{c}}(X) & \longrightarrow & \mathbf{Spec}^{\mathfrak{c}}_-(X) \\
\wr\downarrow & & \wr\downarrow & & \downarrow\wr & & \downarrow\wr \\
\mathbf{Spec}^-(X) & \longleftarrow & \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X) & \longrightarrow & \mathbf{Spec}^1_{\mathfrak{c}}(X) & \longrightarrow & \mathbf{Spec}^-_{\mathfrak{c}}(X)
\end{array} \tag{8}$$

in which all horizontal arrows are injective. Besides, there is a commutative diagram

$$\begin{array}{ccc}
\mathbf{Spec}_-(X) & \longrightarrow & \mathbf{Spec}^{\mathfrak{c}}_-(X) \\
\wr\downarrow & & \downarrow\wr \\
\mathbf{Spec}^-(X) & \longrightarrow & \mathbf{Spec}^-_{\mathfrak{c}}(X)
\end{array} \qquad (9)$$

One can show that if the category C_X has enough objects of finite type, then the horizontal maps of (9) are also bijections (like the central horizontal arrows in (8), according to 2.9.2(c)).

2.9.5. Remark. The diagram (8) exhibits the main characters of this work. All its vertical arrows are isomorphisms given by the same map, $\mathcal{Q} \longmapsto \widehat{\mathcal{Q}}$. In this work, the preorders in the upper row of (8) are called *spectra*. The *dual* Serre subcategory $\widehat{\mathcal{Q}}$ to a spectral point $\mathcal{Q}$ is regarded as a localization device at the point $\mathcal{Q}$. Our main construction uses pairs $(\mathcal{Q}, \widehat{\mathcal{Q}})$, where $\mathcal{Q}$ is a spectral point (for one of the spectra).

2.10. The spectrum $\mathbf{Spec}^1_{\mathfrak{s}}(X)$. For the sake of completeness, we recall one more spectrum, $\mathbf{Spec}^1_{\mathfrak{s}}(X)$ (introduced in [R5]), which is defined solely in terms of Serre subcategories. The elements of $\mathbf{Spec}^1_{\mathfrak{s}}(X)$ are Serre subcategories $\mathcal{P}$ of C_X such that the intersection $\mathcal{P}^{\mathfrak{s}}$ of all Serre subcategories properly containing $\mathcal{P}$ does not coincide with $\mathcal{P}$.

We denote by $\mathcal{P}_{\mathfrak{s}}$ the intersection $\mathcal{P}^{\mathfrak{s}} \bigcap \mathcal{P}^{\perp}$ and define $\mathbf{Spec}^0_{\mathfrak{s}}(X)$ as the preorder (with respect to $\supseteq$) of all coreflective topologizing subcategories of the form $[\mathcal{P}_{\mathfrak{s}}]_{\mathfrak{c}}$. The same map $\mathcal{Q} \longmapsto \widehat{\mathcal{Q}}$ induces an isomorphism

$$\mathbf{Spec}^0_{\mathfrak{s}}(X) \xrightarrow{\sim} \mathbf{Spec}^1_{\mathfrak{s}}(X)$$

which is inverse to the map $\mathcal{P} \longmapsto [\mathcal{P}_{\mathfrak{s}}]_{\mathfrak{c}}$.

We have a commutative diagram

$$\begin{array}{ccc}
\mathbf{Spec}^{\mathfrak{c}}_-(X) & \longrightarrow & \mathbf{Spec}^0_{\mathfrak{s}}(X) \\
\wr\downarrow & & \downarrow\wr \\
\mathbf{Spec}^-_{\mathfrak{c}}(X) & \longrightarrow & \mathbf{Spec}^1_{\mathfrak{s}}(X)
\end{array} \qquad (1)$$

where the lower horizontal arrow is the inclusion and the upper one is determined by the commutativity of the diagram.

If X has the Gabriel-Krull dimension (say, C_X is a locally noetherian category), then, by [R5, 8.7.2], the inclusion $\mathbf{Spec}^-_{\mathfrak{c}}(X) \longrightarrow \mathbf{Spec}^1_{\mathfrak{s}}(X)$ is the identical map. This implies that the map $\mathbf{Spec}^{\mathfrak{c}}_-(X) \longrightarrow \mathbf{Spec}^0_{\mathfrak{s}}(X)$ is a bijection, non-trivial in general.

2.11. Representatives of elements of the spectra. Fix an abelian category C_X. For any $\mathcal{Q} \in \mathbf{Spec}(X)$, the family $\{M \in Spec(X) \mid [M] = \mathcal{P}\}$ of representatives of the element $\mathcal{Q}$ coincides with the family of all nonzero elements of $\widehat{\mathcal{Q}}_* = \widehat{\mathcal{Q}}^* \bigcap \widehat{\mathcal{Q}}^{\perp} = \mathcal{Q} \bigcap \widehat{\mathcal{Q}}^{\perp}$.

If $\mathcal{Q}$ is an element of $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ (resp. $\mathbf{Spec}_-(X)$, resp. $\mathbf{Spec}^{\mathfrak{c}}_-(X)$, resp. $\mathbf{Spec}^0_{\mathfrak{s}}(X)$), then the nonzero objects of the subcategory $\mathcal{Q} \bigcap \widehat{\mathcal{Q}}^{\perp}$ are called *representatives* of $\mathcal{Q}$. The union of the representatives is denoted by $Spec^0_{\mathfrak{c}}(X)$ (resp. $Spec_-(X)$, resp. $Spec^{\mathfrak{c}}_-(X)$, resp. $Spec^0_{\mathfrak{s}}(X)$). Each of these families is endowed with the preorder induced by the specialization preorder (that is $\supseteq$) on the corresponding spectrum. It will be also called the *specialization preorder*. In the case of $\mathbf{Spec}(X)$, the specialization preorder coincides with the preorder $\succ$ recalled in 2.2: $M \succ L$ iff L is a subquotient of a finite coproduct of copies of the object M.

3. Actions of monoidal categories, stabilizers of points, induction functors.

3.1. Actions and continuous actions of monoidal categories. Let $\widetilde{\mathcal{E}} = (\mathcal{E}, \odot, \mathbb{I}, a; \ell, \mathfrak{r})$ be a svelte monoidal category with the product $\odot$, the unit object $\mathbb{I}$, the associativity constraint a, and natural isomorphisms $\mathbb{I} \odot Id_{\mathcal{E}} \xleftarrow{\ell} Id_{\mathcal{E}} \xrightarrow{\mathfrak{r}} Id_{\mathcal{E}} \odot \mathbb{I}$.

An *action* of the monoidal category $\widetilde{\mathcal{E}}$ on a svelte category C_X is a monoidal functor $\widetilde{\Phi} = (\Phi, \phi, \phi_0)$ from $\widetilde{\mathcal{E}}$ to the monoidal category $\widetilde{End}(C_X) = (End(C_X), \circ, Id_{C_X})$ of endofunctors of the category C_X. Recall that here Φ is a functor $\mathcal{E} \longrightarrow End(C_X)$, ϕ a functorial morphism $\Phi(V) \circ \Phi(W) \longrightarrow \Phi(V \odot W)$, and ϕ_0 a morphism from Id_{C_X} (– the unit object of $\widetilde{End}(C_X)$) to $\Phi(\mathbb{I})$ – the image of the unit object of $\widetilde{\mathcal{E}}$. These morphisms are related via the commutative diagrams

$$\begin{array}{ccccc}
\Phi(\mathcal{V}) \circ \Phi(\mathcal{W}) \circ \Phi(\mathcal{Z}) & \xrightarrow{\phi_{\mathcal{V},\mathcal{W}}\Phi(\mathcal{Z})} & \Phi(\mathcal{V} \odot \mathcal{W}) \circ \Phi(\mathcal{Z}) & \xrightarrow{\phi_{\mathcal{V}\odot\mathcal{W},\mathcal{Z}}} & \Phi((\mathcal{V} \odot \mathcal{W}) \odot \mathcal{Z}) \\
{\scriptstyle id}\downarrow & & & & \wr\downarrow{\scriptstyle \Phi(a_{\mathcal{V},\mathcal{W},\mathcal{Z}})} \\
\Phi(\mathcal{V}) \circ \Phi(\mathcal{W}) \circ \Phi(\mathcal{Z}) & \xrightarrow{\Phi(\mathcal{V})\phi_{\mathcal{W},\mathcal{Z}}} & \Phi(\mathcal{V}) \circ \Phi(\mathcal{W} \odot \mathcal{Z}) & \xrightarrow{\phi_{\mathcal{V},\mathcal{W}\odot\mathcal{Z}}} & \Phi((\mathcal{V} \odot \mathcal{W}) \odot \mathcal{Z})
\end{array} \qquad (1)$$

$$\begin{array}{ccccc}
\Phi(\mathcal{V}) \circ \Phi(\mathbb{I}) & \xleftarrow{\Phi(\mathcal{V})\phi_0} & \Phi(\mathcal{V}) & \xrightarrow{\phi_0\Phi(\mathcal{V})} & \Phi(\mathbb{I}) \circ \Phi(\mathcal{V}) \\
{\scriptstyle \phi_{\mathcal{V},\mathbb{I}}}\downarrow & & {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle \phi_{\mathbb{I},\mathcal{V}}} \\
\Phi(\mathcal{V} \odot \mathbb{I}) & \xleftarrow{\Phi(\ell_{\mathcal{V}})} & \Phi(\mathcal{V}) & \xrightarrow{\Phi(\mathfrak{r}_{\mathcal{V}})} & \Phi(\mathbb{I} \odot \mathcal{V})
\end{array} \qquad (2)$$

for all $\mathcal{V}, \mathcal{W}, \mathcal{Z} \in Ob\mathcal{E}$.

An action $\widetilde{\mathcal{E}} \xrightarrow{\widetilde{\Phi}} \widetilde{End}(C_X)$ will be called *continuous* if $\widetilde{\Phi}$ takes values in the full monoidal subcategory $\widetilde{End}_{\mathfrak{c}}(C_X) = (End_{\mathfrak{c}}(C_X), \circ, Id_{C_X})$ of $\widetilde{End}(C_X)$ generated by all continuous endofunctors of the category C_X.

3.1.1. Example: actions of the trivial monoidal category and monads. Let $\widetilde{\mathcal{E}}_\bullet$ be the trivial monoidal category; i.e. the category consisting of one object and one (hence identical) morphism. The category of actions of $\widetilde{\mathcal{E}}_\bullet$ on the category C_X is isomorphic to the category $\mathfrak{Mon}(C_X)$ of monads on the category C_X.

In fact, each action $\widetilde{\Phi} = (\Phi, \phi, \phi_0)$ is determined by the image, $F = \Phi(\mathbb{I})$, of the unique (unit) object of the category $\mathcal{E}_\bullet$ and the morphism $F \circ F \xrightarrow{\phi} F$. The fact that $\widetilde{\Phi}$ is a monoidal functor, means precisely that ϕ is associative, i.e. $\phi \circ F\phi = \phi \circ \phi F$, and $Id_{C_X} \xrightarrow{\phi_0} F$ is the unit of $\mathcal{F} = \Phi(\mathbb{I})$: $\phi \circ F\phi_0 = id_F = \phi \circ \phi_0 F$.

The map $\widetilde{\Phi} \longmapsto \mathcal{F}_{\widetilde{\Phi}} = (\Phi(\mathbb{I}), \phi)$ extends naturally to an isomorphism from the category of actions of $\widetilde{\mathcal{E}}_\bullet$ on C_X and the category of monads on C_X. This isomorphism induces an isomorphism between the category of continuous actions of $\widetilde{\mathcal{E}}_{\bullet}$ on C_X and the category of continuous monads on C_X.

3.2. Modules over an action and the associated monad. Fix a continuous action $\widetilde{\Phi} = (\Phi, \phi, \phi_0)$ of a svelte monoidal category $\widetilde{\mathcal{E}} = (\mathcal{E}, \odot, \mathbb{I}, a)$ on the category C_X. The forgetful functor

$$(\widetilde{\Phi}/X) - mod \xrightarrow{\varphi_*} C_X$$

preserves small limits. Suppose that the category C_X is *small-complete* (i.e. it has small limits). Since the categories C_X and $\mathcal{E}$ are svelte, this implies, by Freyd adjoint functor theorem, the existence of a left adjoint, φ^*, to φ_*. The functor φ_* is exact and conservative. Therefore, by Beck's theorem, the category $(\widetilde{\Phi}/X) - mod$ is equivalent to the category of modules over a monad, $\mathcal{F}_\varphi = (F_\varphi, \mu_\varphi)$, where $F_\varphi = \varphi_*\varphi^*$. More precisely, the forgetful functor φ_* is equivalent to the forgetful functor $\mathcal{F}_\varphi - mod \longrightarrow C_X$.

Notice that the latter implies that the category $(\widetilde{\Phi}/X) - mod$ is small-complete too.

Assume in addition that the category C_X has small colimits. It follows from the fact that the functor Φ takes values in the category of *continuous* endofunctors of C_X that the functor φ_* preserves small colimits, hence it has a right adjoint, $\varphi^!$. The latter is equivalent to the fact that the monad $\mathcal{F}_\varphi$ is continuous.

3.3. Colimits of actions. Identifying the category $(\widetilde{\Phi}/X) - mod$ of $\widetilde{\Phi}$-modules with the category $(\mathcal{F}_\varphi/X) - mod$, we can take as φ^* the functor which assigns to every object V of the category C_X the $\mathcal{F}_\varphi$-module $\mathcal{F}_\varphi(V) = (F_\varphi(V), \mu_\varphi(V))$. On the other hand, $\varphi^*(V)$ is an $(\widetilde{\Phi}/X)$-module; that is we have an action $\Phi(-) \circ F_\varphi(V) \xrightarrow{\xi_\varphi(V)} F_\varphi(V)$ of $\widetilde{\mathcal{E}}$ on $\mathcal{F}_\varphi(V)$ which is functorial in V. Taking the composition of this action with the morphism

$$\Phi(-) = \Phi(-) \circ Id_{C_X} \xrightarrow{\Phi(-)\eta_\varphi} \Phi(-) \circ F_\varphi(V)$$

(where η_φ is an adjunction arrow), we obtain a cone $\Phi(-) \xrightarrow{\gamma_\varphi} F_\varphi$. Note that monads on C_X can be identified with constant monoidal functors from $\widetilde{\mathcal{E}}$ to $\widetilde{End}(C_X)$. One can see that the cone $\Phi(-) \xrightarrow{\gamma_\varphi} F_\varphi$ is a morphism of monoidal functors $\widetilde{\Phi} \longrightarrow \mathcal{F}_\varphi$.

Let $\mathfrak{MF}(\widetilde{\mathcal{E}}, \widetilde{\mathcal{E}}')$ denote the category of monoidal functors from $\widetilde{\mathcal{E}}$ to a monoidal category $\widetilde{\mathcal{E}}'$ and $\mathfrak{Mon}(C_X)$ the category of monads on C_X. Let $\mathfrak{J}_X^*$ denote the embedding

$$\mathfrak{Mon}(C_X) \longrightarrow \mathfrak{MF}(\widetilde{\mathcal{E}}, \widetilde{End}(C_X))$$

which assigns to every monoid on C_X the corresponding constant monoidal functor; and let $\mathfrak{J}_{X*}$ be functor which assigns to each monoidal functor $\widetilde{\Phi}$ from $\widetilde{\mathcal{E}}$ to $\widetilde{End}(C_X)$ the monad $\mathcal{F}_\varphi$. The map which assigns to every monoidal functor $\widetilde{\Phi}$ from $\widetilde{\mathcal{E}}$ to $\widetilde{End}(C_X)$ the morphism $\widetilde{\Phi} \xrightarrow{\gamma_\varphi} \mathcal{F}_\varphi$ is an adjunction arrow $Id \xrightarrow{\gamma} \mathfrak{J}_{X*}\mathfrak{J}_X^*$. The other adjunction arrow is the identical morphism. This means that the monad $\mathcal{F}_\varphi$ corresponding to a monoidal functor $\widetilde{\Phi} = (\Phi, \phi)$ is the *colimit of this monoidal functor.*

3.4. Colimits of continuous actions. Suppose now that the category C_X has small limits and colimits. Let $\mathfrak{MF}_{\mathfrak{c}}(\widetilde{\mathcal{E}}, \widetilde{End}(C_X))$ denote the full subcategory of $\mathfrak{MF}(\widetilde{\mathcal{E}}, \widetilde{End}(C_X))$

whose objects are *continuous* actions of $\widetilde{\mathcal{E}}$ on the category C_X. And let $\mathfrak{Mon}_{\mathfrak{c}}(C_X)$ denote the category of continuous monads on C_X. The embedding

$$\mathfrak{Mon}(C_X) \xrightarrow{\mathfrak{J}_X^*} \mathfrak{MF}(\widetilde{\mathcal{E}}, \widetilde{End}(C_X))$$

induces an embedding

$$\mathfrak{Mon}_{\mathfrak{c}}(C_X) \xrightarrow{{}^{\mathfrak{c}}\mathfrak{J}_X^*} \mathfrak{MF}_{\mathfrak{c}}(\widetilde{\mathcal{E}}, \widetilde{End}(C_X)).$$

Since the monad $\mathcal{F}_\varphi$ corresponding to the continuous action $\widetilde{\Phi}$ is continuous, the right adjoint $\mathfrak{J}_{X*}$ to $\mathfrak{J}_X^*$ induces a right adjoint

$$\mathfrak{MF}(\widetilde{\mathcal{E}}, \widetilde{End}(C_X)) \xrightarrow{{}^{\mathfrak{c}}\mathfrak{J}_{X*}} \mathfrak{Mon}_{\mathfrak{c}}(C_X)$$

to the functor ${}^{\mathfrak{c}}\mathfrak{J}_X^*$ which assigns to every continous action $\widetilde{\Phi} = (\Phi, \phi)$ of the monoidal category $\widetilde{\mathcal{E}}$ on the category C_X its colimit – a continuous monad $\mathcal{F}_\varphi = (F_\varphi, \mu_\varphi)$.

It follows from the fact that functor Φ takes values in the category of continuous endofunctors, that the functor $F_\varphi = \varphi_*\varphi^*$ is the colimit of Φ.

3.5. Functorialities. These correspondences are functorial in the following sense: if $\widetilde{\mathcal{E}}'$ is another svelte monoidal category and

$$\begin{tikzcd}
\widetilde{\mathcal{E}}' \arrow[rr, "\widetilde{\Psi}"] \arrow[dr, "\widetilde{\Phi}'"'] & & \widetilde{\mathcal{E}} \arrow[dl, "\widetilde{\Phi}"] \\
& \widetilde{End}_{\mathfrak{c}}(C_X) &
\end{tikzcd}$$

is a quasi-commutative diagram of monoidal functors, then the monoidal functor $\widetilde{\Psi}$ induces a pull-back functor $(\widetilde{\Phi}/X) - mod \xrightarrow{\mathfrak{f}_*} (\widetilde{\Phi}'/X) - mod$ such that the diagram

$$\begin{tikzcd}
(\widetilde{\Phi}/X) - mod \arrow[rr, "\mathfrak{f}_*"] \arrow[dr, "\varphi_*"'] & & (\widetilde{\Phi}'/X) - mod \arrow[dl, "\varphi'_*"] \\
& C_X &
\end{tikzcd} \qquad (1)$$

commutes. If the category C_X is small-complete, then, by the argument above, the functors $(\widetilde{\Phi}/X) - mod \xrightarrow{\varphi_*} C_X$ and $(\widetilde{\Phi}'/X) - mod \xrightarrow{\varphi'_*} C_X$ are equivalent to the forgetful functors, respectively $(\mathcal{F}_\varphi/X) - mod \longrightarrow C_X$ and $(\mathcal{F}_{\varphi'}/X) - mod \longrightarrow C_X$.

The functor $\mathfrak{f}_*$ corresponds to the restriction functor $\mathcal{F}_\varphi - mod \xrightarrow{\psi_*} \mathcal{F}_{\varphi'} - mod$ along a monad morphism $\mathcal{F}_{\varphi'} \xrightarrow{\psi} \mathcal{F}_\varphi$. In particular, the functor $\mathfrak{f}_*$ has a left adjoint, $\mathfrak{f}^*$. Thus, the diagram (1) is equivalent to the diagram of canonical direct image functors of the commutative diagram

$$\begin{tikzcd}
\mathbf{Sp}(\mathcal{F}_\varphi/X) \arrow[rr, "\mathbf{Sp}(\psi)"] \arrow[dr, "\varphi"'] & & \mathbf{Sp}(\mathcal{F}_{\varphi'}/X) \arrow[dl, "\varphi'"] \\
& X &
\end{tikzcd} \qquad (2)$$

corresponding to a monad morphism $\mathcal{F}_{\varphi'} \xrightarrow{\psi} \mathcal{F}_{\varphi}$. Notice that the monads $\mathcal{F}_{\varphi}$ and $\mathcal{F}_{\varphi'}$, being colimits of monoidal functors, are defined uniquely up to isomorphism. By the universal property of colimits, the monad morphism ψ is determined uniquely, once the monads $\mathcal{F}_{\varphi}$ and $\mathcal{F}_{\varphi'}$ are fixed. Therefore, the map which assigns to the diagram (1) the monad morphism $\mathcal{F}_{\varphi'} \xrightarrow{\psi} \mathcal{F}_{\varphi}$ is a functor, Γ_{X*}, from the category $\mathfrak{Act}_{\mathfrak{c}}(C_X)$ of continuous actions of (svelte) monoidal categories on the category C_X to the category $\mathfrak{Mon}(C_X)$ of monads on C_X. The functor Γ_{X*} has a right adjoint, $\Gamma^!_X$, which assigns to each monad $\mathcal{F} = (F, \mu)$ on C_X the forgetful strict monoidal functor

$$\widetilde{End_{\mathfrak{c}}}(C_X)/\mathcal{F} \xrightarrow{\Gamma^!_X(\mathcal{F})} \widetilde{End_{\mathfrak{c}}}(C_X).$$

Suppose that, in addition, the category C_X is small-cocomplete. Then the monads $\mathcal{F}_{\varphi}$ and $\mathcal{F}_{\varphi'}$ are continuous, or, equivalently, all morphisms of the diagram (2) are affine. The category $End_{\mathfrak{c}}(C_X)/F$ has a canonical final object – the pair (F, id_F), which implies that the adjunction arrow $\Gamma_{X*} \circ \Gamma^!_X \longrightarrow Id$ is an isomorphism, or, what is the same, the functor $\Gamma^!_X$ is fully faithful; i.e. Γ_{X*} is equivalent to a localization functor.

The functor Γ_{X*} has a left adjoint (forcibly fully faithful), Γ^*_X, which assigns to every monad $\mathcal{F}$ on C_X the monoidal functor from the trivial monoidal category to $\widetilde{End_{\mathfrak{c}}}(C_X)$ sending the unique object to F (cf. 3.1.1).

3.5.1. Example: the stabilizer of a set of subcategories. Let $\mathcal{B}$ be a set of full subcategories of the category C_X; and let $\mathcal{E}_{\mathcal{B}}$ be the full subcategory of the category $\mathcal{E}$ generated by all objects L such that $\Phi(L)(\mathcal{A}) \subseteq \mathcal{A}$ for each $\mathcal{A} \in \mathcal{B}$. It follows that $\mathcal{E}_{\mathcal{B}}$ is a monoidal subcategory of $\widetilde{\mathcal{E}}$ and the restriction $\widetilde{\Phi}_{\mathcal{B}}$ of the monoidal functor $\widetilde{\Phi}$ to the subcategory $\mathcal{E}_{\mathcal{B}}$ is a continuous action of $\widetilde{\mathcal{E}}_{\mathcal{B}}$ on C_X. Thus, we have the category of $\widetilde{\Phi}_{\mathcal{B}}$-modules and the restriction functor $(\widetilde{\Phi}/X) - mod \xrightarrow{\mathfrak{f}_{\mathcal{B}*}} (\widetilde{\Phi}_{\mathcal{B}}/X) - mod$ corresponding to the embedding $\widetilde{\mathcal{E}}_{\mathcal{B}} \longrightarrow \widetilde{\mathcal{E}}$.

If the category C_X is small-complete, then the functor $(\widetilde{\Phi}_{\mathcal{B}}/X) - mod \xrightarrow{\varphi_{\mathcal{B}*}} C_X$ is equivalent to the forgetful functor $\mathcal{F}_{\varphi_{\mathcal{B}}} - mod \longrightarrow C_X$ for a monad $\mathcal{F}_{\varphi_{\mathcal{B}}}$ on C_X and we obtain the commutative diagram

$$\begin{tikzcd}[column sep=small] \mathbf{Sp}(\mathcal{F}_{\varphi}/X) \arrow[rr, "\psi_{\mathcal{B}}"] \arrow[dr, "\varphi"'] & & \mathbf{Sp}(\mathcal{F}_{\varphi_{\mathcal{B}}}/X) \arrow[dl, "\varphi_{\mathcal{B}}"] \\ & X & \end{tikzcd} \qquad (3)$$

corresponding to a monad morphism $\mathcal{F}_{\varphi_{\mathcal{B}}} \xrightarrow{\psi_{\mathcal{B}}} \mathcal{F}_{\varphi}$.

If, in addition, the category C_X is small-cocomplete, then the monads $\mathcal{F}_{\varphi}$ and $\mathcal{F}_{\varphi_{\mathcal{B}}}$ are continuous and, therefore, all morphisms in the commutative diagram (3) are affine.

3.6. Stabilizers of points and related functors. We fix a svelte abelian category C_X together with a continuous action of a svelte monoidal category $\widetilde{\mathcal{E}} = (\mathcal{E}, \odot, \mathbb{I}, a)$ on C_X given by a monoidal functor $\widetilde{\Phi} = (\Phi, \phi, \phi_0)$ from $\widetilde{\mathcal{E}}$ to the monoidal category $\widetilde{\mathfrak{Er}_{\mathfrak{c}}}(C_X)$ of continuous *exact* additive endofunctors of C_X. We shall assume that the category C_X has small limits and colimits.

3.6.1. The stabilizer of a point of the spectrum. Fix a point $\mathcal{P}$ of $\mathbf{Spec}^0_{\mathfrak{c}}(X)$. We shall write $(\mathcal{P})$ for pair $\{\mathcal{P}, \widehat{\mathcal{P}}\}$, where $\widehat{\mathcal{P}}$ is the corresponding to $\mathcal{P}$ Serre subcategory. We define the *stabilizer of the point* $\mathcal{P}$ as the stabilizer $\mathcal{E}_{(\mathcal{P})}$ of the pair $(\mathcal{P})$. We have a commutative diagram of affine morphisms

$$\begin{array}{ccc} \mathfrak{A} = \mathbf{Sp}(\mathcal{F}_{\varphi}/X) & \xrightarrow{\mathfrak{f}_{\mathcal{P}}} & \mathbf{Sp}(\mathcal{F}_{\varphi_{\mathcal{P}}}/X) = \mathfrak{A}_{\mathcal{P}} \\ \varphi \searrow & & \swarrow \varphi_{\mathcal{P}} \\ & X & \end{array} \qquad (1)$$

where $\mathfrak{f}_{\mathcal{P}} = \mathbf{Sp}(\psi_{\mathcal{P}})$ for a monad morphism $\mathcal{F}_{\varphi_{\mathcal{P}}} \xrightarrow{\psi_{\mathcal{P}}} \mathcal{F}_{\varphi}$.

3.6.2. The functor $\mathfrak{L}_{\mathcal{P}}$. Fix an element $\mathcal{P}$ of $\mathbf{Spec}^0_{\mathfrak{c}}(X)$. We denote by $\mathfrak{L}_{\mathcal{P}}$ the composition of the functors $C_{\mathfrak{A}_{\mathcal{P}}} \xrightarrow{\mathfrak{f}^*_{\mathcal{P}}} C_{\mathfrak{A}}$ and

$$C_{\mathfrak{A}} \xrightarrow{\Psi_{\mathcal{P}}} C_{\mathfrak{A}}, \quad M \longmapsto M/tors_{\varphi_*^{-1}(\widehat{\mathcal{P}})}(M).$$

Notice that since $\widehat{\mathcal{P}}$ is a Serre subcategory of C_X and φ_* has a right adjoint, the preimage $\varphi_*^{-1}(\widehat{\mathcal{P}})$ of $\widehat{\mathcal{P}}$ is a Serre subcategory of $C_{\mathfrak{A}}$. Thanks to the property (sup), every Serre subcategory, $\mathbb{S}$, of $C_{\mathfrak{A}}$ is coreflective, i.e. the inclusion functor $\mathbb{S} \hookrightarrow C_{\mathfrak{A}}$ has a right adjoint, $tors_{\mathbb{S}} : C_{\mathfrak{A}} \longrightarrow \mathbb{S}$ which assigns to every object M its $\mathbb{S}$-torsion. In particular, $tors_{\varphi_*^{-1}(\widehat{\mathcal{P}})}(M)$ is well defined for all $M \in ObC_{\mathfrak{A}}$.

3.6.2.1. Proposition. *Let $\mathcal{P} \in \mathbf{Spec}^0_{\mathfrak{c}}(X)$ be such that an inverse image functor $\mathfrak{f}^*_{\mathcal{P}}$ of the morphism $\mathfrak{A} \xrightarrow{\mathfrak{f}_{\mathcal{P}}} \mathfrak{A}_{\mathcal{P}}$ is exact. Then the functor $\mathfrak{L}_{\mathcal{P}}$ is exact.*

Proof. The functor $\mathfrak{L}_{\mathcal{P}}$ is the composition of two right exact functors, $\mathfrak{f}^*_{\mathcal{P}}$ and $\Psi_{\mathcal{P}}$, hence it is right exact. It remains to verify that $\mathfrak{L}_{\mathcal{P}}$ maps monomophisms to monomorphisms. Let $K \xrightarrow{\mathfrak{j}} M$ be a monomophism in $C_{\mathfrak{A}_{\mathcal{P}}}$. Consider the commutative diagram

$$\begin{array}{ccc} \mathfrak{f}^*_{\mathcal{P}}(K) & \xrightarrow{\mathfrak{f}^*_{\mathcal{P}}(\mathfrak{j})} & \mathfrak{f}^*_{\mathcal{P}}(M) \\ \mathfrak{e}_K \big\downarrow & & \big\downarrow \mathfrak{e}_M \\ \mathfrak{L}_{\mathcal{P}}(K) & \xrightarrow{\mathfrak{L}_{\mathcal{P}}(\mathfrak{j})} & \mathfrak{L}_{\mathcal{P}}(M) \end{array} \qquad (1)$$

and its image by the localization $C_{\mathfrak{A}} \xrightarrow{q^*} C_{\mathfrak{A}}/\varphi_*^{-1}(\widehat{\mathcal{P}})$. Since, by hypothesis, the functor $\mathfrak{f}^*_{\mathcal{P}}$ is exact and the localization functor q^* is exact, $q^*\mathfrak{f}^*_{\mathcal{P}}(K) \xrightarrow{q^*\mathfrak{f}^*_{\mathcal{P}}(\mathfrak{j})} q^*\mathfrak{f}^*_{\mathcal{P}}(M)$ is a monomorphism. The arrows $q^*\mathfrak{f}^*_{\mathcal{P}}(K) \xrightarrow{q^*(\mathfrak{e}_K)} q^*\mathfrak{L}_{\mathcal{P}}(K)$ and $q^*\mathfrak{f}^*_{\mathcal{P}}(M) \xrightarrow{q^*(\mathfrak{e}_M)} q^*\mathfrak{L}_{\mathcal{P}}(M)$ are isomorphisms. Therefore $q^*\mathfrak{L}_{\mathcal{P}}(K) \xrightarrow{q^*\mathfrak{L}_{\mathcal{P}}(\mathfrak{j})} q^*\mathfrak{L}_{\mathcal{P}}(M)$ is a monomorphism. Since the object $\mathfrak{L}_{\mathcal{P}}(K)$ is $Ker(q^*)$-torsion free, $\mathfrak{L}_{\mathcal{P}}(K) \xrightarrow{\mathfrak{L}_{\mathcal{P}}(\mathfrak{j})} \mathfrak{L}_{\mathcal{P}}(M)$ is a monomorphism. ∎

3.6.3. Remark. The notion of the stabilizer of a point, the definition of the functor $\mathfrak{L}_{\mathcal{P}}$, and Proposition 3.6.2.1 make sense if $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ is replaced by any of the remained spectra considered here: $\mathbf{Spec}(X)$, $\mathbf{Spec}_-(X)$, $\mathbf{Spec}^{\mathfrak{c}}_-(X)$, or $\mathbf{Spec}^0_{\mathfrak{s}}(X)$.

We need the following assertion which is of independent interest.

3.7. Proposition. *Let C_Y be an abelian category and $C_Y \xrightarrow{g^*} C_Z$ a functor having a right adjoint, g_*; and let $Id_{C_Y} \xrightarrow{\eta} g_*g^*$ be an adjunction arrow.*

(a) If the functor g^* is exact, then the adjunction morphism $M \xrightarrow{\eta(M)} g_*g^*(M)$ is a monomorphism for every $M \in Spec(Y)$ such that $g^*(M) \neq 0$.

(b) Suppose that the category C_Y satisfies (AB4), i.e. it has small coproducts and the coproduct of a set of monomorphisms is a monomorphism. If the functor g^* is exact and g_* has a right adjoint, then $M \xrightarrow{\eta(M)} g_*g^*(M)$ is a monomorphism for every $M \in Spec^0_{\mathfrak{c}}(Y)$ such that $g^*(M) \neq 0$.

Proof. (a1) Let M be an arbitrary object of C_Y, and let $K \xrightarrow{\mathfrak{j}} M$ be the kernel of the adjunction morphism $M \xrightarrow{\eta(M)} g_*g^*(M)$. Consider the commutative diagram

$$\begin{array}{ccc} K & \xrightarrow{\eta(K)} & g_*g^*(K) \\ {\scriptstyle \mathfrak{j}}\downarrow & & \downarrow {\scriptstyle g_*g^*(\mathfrak{j})} \\ M & \xrightarrow{\eta(M)} & g_*g^*(M) \end{array}$$

Since g^* is exact, the functor g_*g^* is left exact, in particular $g_*g^*(\mathfrak{j})$ is a monomorphism. Therefore, the equality $g_*g^*(\mathfrak{j}) \circ \eta(K) = \eta(M) \circ \mathfrak{j} = 0$ implies that $\eta(K) = 0$.

(a2) Suppose now that M belongs to $Spec(Y)$. If $K \neq 0$, then $K \succ M$, i.e. there exists a diagram $K^{\oplus n} \xleftarrow{\gamma} L \xrightarrow{\mathfrak{e}} M$ in which the left arrow is an epimorphism and the right arrow is a monomorphism. Consider the associated commutative diagram

$$\begin{array}{ccccc} K^{\oplus n} & \xleftarrow{\gamma} & L & \xrightarrow{\mathfrak{e}} & M \\ {\scriptstyle \eta(K^{\oplus n})}\downarrow & & {\scriptstyle \eta(L)}\downarrow & & \downarrow {\scriptstyle \eta(M)} \\ g_*g^*(K^{\oplus n}) & \xleftarrow{g_*g^*(\gamma)} & g_*g^*(L) & \xrightarrow{g_*g^*(\mathfrak{e})} & g_*g^*(M) \end{array} \qquad (2)$$

By (a1), the left vertical arrow in (2) is zero. Since $L \xrightarrow{\gamma} K^{\oplus n}$ is a monomorphism and the functor g_*g^* is, thanks to the exactness of g_*, left exact, $g_*g^*(\gamma)$ is a monomorphism. Therefore, the equality $g_*g^*(\gamma) \circ \eta(L) (= \eta(K^{\oplus n}) \circ \gamma) = 0$ implies that $\eta(L) = 0$. Then the commutativity of the right square of (2) yields the equality $\eta(M) \circ \mathfrak{e} = 0$. Since $\mathfrak{e}$ is an epimorphism, it follows that $\eta(M) = 0$. But, the equality $\eta(M) = 0$ means precisely that the object M belongs to the kernel of the functor g^*, i.e. $g^*(M) = 0$.

(b) Suppose that C_Y satisfies (AB4) and the functor g_* has a right adjoint.

By definition, an object M belongs to $Spec^0_{\mathfrak{c}}(Y)$ iff M is contained in the subcategory $[N]_{\mathfrak{c}}$ for any its nonzero subobject N. Since C_Y satisfies (AB4), each object of $[N]_{\mathfrak{c}}$

is a subquotients of the coproduct of a set of copies of the object N. In particular, if $K = Ker(\eta(M))$ is nonzero, there is a diagram $K^{\oplus J} \xleftarrow{\gamma} L \xrightarrow{\mathfrak{e}} M$, for some, infinite in general, set J, whose left (resp. right) arrow is a monomorphism (resp. an epimorphism). Thus, if $K \neq 0$, we have a commutative diagram

$$\begin{array}{ccccccc} K^{\oplus J} & \xrightarrow{id} & K^{\oplus J} & \xleftarrow{\gamma} & L & \xrightarrow{\mathfrak{e}} & M \\ {\scriptstyle \eta(K)^{\oplus J}}\big\downarrow & & {\scriptstyle \eta(K^{\oplus J})}\big\downarrow & & {\scriptstyle \eta(L)}\big\downarrow & & \big\downarrow{\scriptstyle \eta(M)} \\ g_*g^*(K)^{\oplus J} & \xrightarrow{\sim} & g_*g^*(K^{\oplus J}) & \xleftarrow{g_*g^*(\gamma)} & g_*g^*(L) & \xrightarrow{g_*g^*(\mathfrak{e})} & g_*g^*(M) \end{array} \qquad (3)$$

in which the lower left horizontal arrow is an isomorphism and the lower right horizontal arrow is an epimorphisms. Both observations follow from the fact that, since g_* has a right adjoint, the composition g_*g^* has a right adjoint, hence it preserves arbitrary colimits.

It follows from the commutativity of the diagram (3) and the equality $\eta(K) = 0$ established in (a1) above, that $\eta(K^{\oplus J}) = 0$. Repeating the argument (a2), we obtain the equality $g^*(M) = 0$. ■

3.7.1. Corollary. *Let C_Y be an abelian category, $C_Y \xrightarrow{g^*} C_Z$ a functor having a right adjoint, g_*, and $Id_{C_Y} \xrightarrow{\eta} g_*g^*$ an adjunction arrow.*

(a) If the functor g^* is exact and faithful, then $M \xrightarrow{\eta(M)} g_*g^*(M)$ is a monomorphism for every $M \in Spec(Y)$.

(b) If the category C_Y satisfies (AB4), the functor g_* has a right adjoint, and the functor g^* is exact and faithful, then the adjunction morphism $M \xrightarrow{\eta(M)} g_*g^*(M)$ is a monomorphism for every $M \in Spec^0_{\mathfrak{c}}(Y)$.

Proof. Since the functor g^* is faithful, $g^*(M) \neq 0$ for any nonzero object M, in particular, for any $M \in Spec^0_{\mathfrak{c}}(Y)$. The assertion follows from 3.7. ■

3.8. Proposition. *Let $\mathcal{P}$ be an element of $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ such that the inverse image functor $\mathfrak{f}^*_{\mathcal{P}}$ of the morphism $\mathfrak{A} \xrightarrow{\mathfrak{f}_{\mathcal{P}}} \mathfrak{A}_{\mathcal{P}}$ is exact and faithful. Let $Id_{\mathfrak{A}} \xrightarrow{\mathfrak{r}_{\mathcal{P}}} \mathfrak{f}_{\mathcal{P}*}\mathfrak{L}_{\mathcal{P}}$ be the composition of the adjunction arrow $Id_{\mathfrak{A}} \longrightarrow \mathfrak{f}_{\mathcal{P}*}\mathfrak{f}^*_{\mathcal{P}}$ and the epimorphism $\mathfrak{f}_{\mathcal{P}*}\mathfrak{f}^*_{\mathcal{P}} \longrightarrow \mathfrak{f}_{\mathcal{P}*}\mathfrak{L}_{\mathcal{P}}$. The morphism*

$$M \xrightarrow{\mathfrak{r}_{\mathcal{P}}(M)} \mathfrak{f}_{\mathcal{P}*}\mathfrak{L}_{\mathcal{P}}(M) \qquad (3)$$

is a monomorphism for every $M \in Spec(\mathfrak{A}_{\mathcal{P}})$ such that $\mathcal{P} \in Supp(\varphi^*_{\mathcal{P}}(M))$.

Proof. Let $K_M \xrightarrow{\mathfrak{j}_M} M$ denote the kernel of the morphism (3). The functor $\mathfrak{f}_{\mathcal{P}*}$ preserves colimits. By 3.7.1, the adjunction morphism $M \xrightarrow{\mathfrak{r}(M)} \mathfrak{f}_{\mathcal{P}*}\mathfrak{f}^*_{\mathcal{P}}(M)$ is a monomorphism for every $M \in Spec(\mathfrak{A}_{\mathcal{P}})$. Therefore $\varphi_{\mathcal{P}*}(K_M)$ is an object of $\widehat{\mathcal{P}}$. Since M belongs to the spectrum, if $K_M \neq 0$, then $M \in [K_M]_{\mathfrak{c}}$. The functor $\varphi_{\mathcal{P}*}$ is exact and preserves small colimits. Since for any object L, the subcategory $[L]_{\mathfrak{c}}$ is obtained from L by taking arbitrary small colimits and subobjects, $\varphi_{\mathcal{P}*}([L]_{\mathfrak{c}}) \subseteq [\varphi_{\mathcal{P}*}(L)]_{\mathfrak{c}}$. In particular, $\varphi_{\mathcal{P}*}(M)$ is an object of $[\varphi_{\mathcal{P}*}(K_M)]_{\mathfrak{c}}$. The latter implies that $\varphi_{\mathcal{P}*}(M)$ is also an object of the subcategory $\widehat{\mathcal{P}}$; that is $\mathcal{P} \notin Supp(\varphi_{\mathcal{P}*}(M))$. ■

4. Realization of points.

4.1. Assumptions and notations. We fix a Grothendieck category C_X together with a continuous action of a svelte monoidal category $\widetilde{\mathcal{E}} = (\mathcal{E}, \odot, \mathbb{I}, a)$ on C_X given by a monoidal functor $\widetilde{\Phi} = (\Phi, \phi, \phi_0)$ from $\widetilde{\mathcal{E}}$ to the full monoidal subcategory $\widetilde{\mathfrak{Ex}_{\mathfrak{c}}}(C_X)$ of $\widetilde{End}(C_X)$ generated by continuous *exact* endofunctors of C_X. Being a Grothendieck category, C_X has small limits and colimits, which garantees that continuous actions of svelte monoidal categories on C_X have colimits, and this colimits are continuous monads.

In particular, there is a (determined uniquely up to isomorphism) continuous monad $\mathcal{F}_\varphi = (F_\varphi, \mu_\varphi)$ and a universal morphism (or universal cone) $\widetilde{\Phi} \xrightarrow{\gamma_\varphi} \mathcal{F}_\varphi$ whose pull-back functor $(\mathcal{F}_\varphi/X) - mod \xrightarrow{\gamma_{\varphi *}} (\widetilde{\Phi}/X) - mod$ is an equivalence between the category of $\widetilde{\Phi}$-modules and the category of $\mathcal{F}_\varphi$-modules (see 3.3). The morphism γ_φ gives rise to a monoidal functor $\widetilde{\mathcal{E}} \xrightarrow{\widetilde{\Psi}_\varphi} \widetilde{\mathfrak{Ex}_{\mathfrak{c}}}(C_X)/\mathcal{F}_\varphi$ so that $\widetilde{\Phi}$ is the composition of $\widetilde{\Psi}_\varphi$ and the forgetful (strict) monoidal functor $\widetilde{\mathfrak{Ex}_{\mathfrak{c}}}(C_X)/\mathcal{F}_\varphi \xrightarrow{\widetilde{\mathfrak{F}}_X} \widetilde{\mathfrak{Ex}_{\mathfrak{c}}}(C_X)$.

In what follows, the monoidal category $\widetilde{\mathcal{E}}$ can be identified with its image in the strict monoidal category $\widetilde{\mathfrak{Ex}_{\mathfrak{c}}}(C_X)/\mathcal{F}_\varphi$. So, we assume, for convenience, that $\widetilde{\mathcal{E}}$ is a monoidal subcategory of $\widetilde{\mathfrak{Ex}_{\mathfrak{c}}}(C_X)/\mathcal{F}_\varphi$ and $\widetilde{\Phi}$ is the restriction to $\widetilde{\mathcal{E}}$ of the forgetful functor $\widetilde{\mathfrak{F}}_X$.

4.1.1. $\mathbf{Spec}^{\mathcal{P}}_{\mathfrak{c}}(\mathfrak{A}_{\mathcal{P}})$ and $\mathbf{Spec}^{\mathcal{P}}_{\mathfrak{c}}(\mathfrak{A})$. Fix a point $\mathcal{P}$ of $\mathbf{Spec}^0_{\mathfrak{c}}(X)$. Let $\widetilde{\mathcal{E}}_{(\mathcal{P})} = \widetilde{\mathcal{E}}_{\{\mathcal{P}, \widehat{\mathcal{P}}\}}$ be the stabilizer of the point $\mathcal{P}$, i.e. the full subcategory of $\widetilde{\mathcal{E}}$ generated by all $(U, U \xrightarrow{\mathfrak{v}} F_\varphi)$ such that $U(\mathcal{P}) \subseteq \mathcal{P}$ and $U(\widehat{\mathcal{P}}) \subseteq \widehat{\mathcal{P}}$. Let $\widetilde{\Phi}_{(\mathcal{P})}$ be the restriction of $\widetilde{\Phi}$ to $\widetilde{\mathcal{E}}_{(\mathcal{P})}$ and $\mathcal{F}_{\varphi_{\mathcal{P}}}$ the corresponding monad – the colimit of $\widetilde{\Phi}_{(\mathcal{P})}$ (cf. 3.5.1). By 3.6.1, we have a commutative diagram of affine morphisms

$$
\begin{array}{ccc}
\mathfrak{A} = \mathbf{Sp}(\mathcal{F}_\varphi/X) & \xrightarrow{\mathfrak{f}_{\mathcal{P}}} & \mathbf{Sp}(\mathcal{F}_{\varphi_{\mathcal{P}}}/X) = \mathfrak{A}_{\mathcal{P}} \\
\varphi \searrow & & \swarrow \varphi_{\mathcal{P}} \\
& X &
\end{array}
\qquad (1)
$$

corresponding to a monad morphism $\mathcal{F}_{\varphi_{\mathcal{P}}} \xrightarrow{\psi_{\mathcal{P}}} \mathcal{F}_\varphi$, where the 'space' $\mathfrak{A}_{\mathcal{P}}$ and the monad $\mathcal{F}_{\varphi_{\mathcal{P}}}$ (or, more precisely, the monad morphism $\psi_{\mathcal{P}}$) are called *stabilizers* of the point $\mathcal{P}$.

We denote by $Spec^{\mathcal{P}}_{\mathfrak{c}}(\mathfrak{A}_{\mathcal{P}})$ all objects $\widetilde{P}$ of $Spec^0_{\mathfrak{c}}(\mathfrak{A}_{\mathcal{P}})$ such that $\mathcal{P} \in Ass(\varphi^*_{\mathcal{P}}(\widetilde{P}))$, and we set $\mathbf{Spec}^{\mathcal{P}}_{\mathfrak{c}}(\mathfrak{A}_{\mathcal{P}}) = \{[\widetilde{P}]_{\mathfrak{c}} \mid \widetilde{P} \in Spec^{\mathcal{P}}_{\mathfrak{c}}(\mathfrak{A}_{\mathcal{P}})\}$.

Objects of $Spec^{\mathcal{P}}_{\mathfrak{c}}(\mathfrak{A})$ are all $M \in Spec_{\mathfrak{c}}(\mathfrak{A})$ such that the object $\mathfrak{f}^*_{\mathcal{P}}(M)$ has an associated point from $\mathbf{Spec}^{\mathcal{P}}_{\mathfrak{c}}(\mathfrak{A}_{\mathcal{P}})$. We set $\mathbf{Spec}^{\mathcal{P}}_{\mathfrak{c}}(\mathfrak{A}) = \{[M]_{\mathfrak{c}} \mid M \in Spec^{\mathcal{P}}_{\mathfrak{c}}(\mathfrak{A})\}$.

4.2. Theorem. *Let $\mathcal{P} \in \mathbf{Spec}^0_{\mathfrak{c}}(X)$ be such that the inverse image functor $\mathfrak{f}^*_{\mathcal{P}}$ of the morphism $\mathfrak{A} \xrightarrow{\mathfrak{f}_{\mathcal{P}}} \mathfrak{A}_{\mathcal{P}}$ is exact and faithful, and the following condition holds:*

(*) Let $P \in Spec^0_{\mathfrak{c}}(X)$ be representative of $\mathcal{P}$ and M a subobject of $\varphi^*(P)$ such that $\mathcal{P} \in Supp(\varphi_*(M))$. There exists $(U', \mathfrak{v}) \in Ob\mathcal{E}_{(\mathcal{P})}$ and a subobject P' of P such that the image of $U'(P')$ in $F_\varphi(P) = \varphi_*\varphi^*(P)$ is a subobject of $\varphi_*(M)$ whose support contains $\mathcal{P}$.

Then the functor $C_{\mathfrak{A}_{\mathcal{P}}} \xrightarrow{\mathfrak{L}_{\mathcal{P}}} C_{\mathfrak{A}}$ *induces a morphism*

$$\mathbf{Spec}^{\mathcal{P}}_{\mathfrak{c}}(\mathfrak{A}_{\mathcal{P}}) \xrightarrow{\mathfrak{L}_{\mathcal{P}}} \mathbf{Spec}^{\mathcal{P}}_{\mathfrak{c}}(\mathfrak{A}). \tag{1}$$

with the following properties:

(α) Every $[M] \in \mathbf{Spec}^0_{\mathfrak{c}}(\mathfrak{A})$ such that the image $\mathfrak{f}^*_{\mathcal{P}}(M)$ of M in $C_{\mathfrak{A}_{\mathcal{P}}}$ has an associated point from $\mathbf{Spec}^{\mathcal{P}}_{\mathfrak{c}}(\mathfrak{A}_{\mathcal{P}})$ belongs to the image of the map (1).

(β) The functor $\mathfrak{L}_{\mathcal{P}}$ maps simple objects from $Spec^{\mathcal{P}}_{\mathfrak{c}}(\mathfrak{A}_{\mathcal{P}})$ to simple objects of $C_{\mathfrak{A}}$.

Proof. (a) Let $\widetilde{P}$ be an object of $Spec^{\mathcal{P}}_{\mathfrak{c}}(\mathfrak{A}_{\mathcal{P}})$; i.e. $\widetilde{P}$ is an object of $Spec^0_{\mathfrak{c}}(\mathfrak{A}_{\mathcal{P}})$ and there exists a monomorphism $P \xrightarrow{\iota} \varphi^*_{\mathcal{P}}(\widetilde{P})$, where P is an object of $Spec^0_{\mathfrak{c}}(X)$ such that $\mathcal{P} = [P]_{\mathfrak{c}}$. The claim is that $\mathfrak{L}_{\mathcal{P}}(\widetilde{P})$ is an object of $Spec^{\mathcal{P}}_{\mathfrak{c}}(\mathfrak{A})$; i.e. $\mathfrak{L}_{\mathcal{P}}(\widetilde{P}) \in [M']_{\mathfrak{c}}$ for any nonzero subobject M' of $\mathfrak{L}_{\mathcal{P}}(\widetilde{P})$.

(i) Consider the composition $P \xrightarrow{\widehat{\mathfrak{v}}} \varphi_*\mathfrak{f}^*_{\mathcal{P}}(\widetilde{P})$ of the monomorphism $P \longrightarrow \varphi^*_{\mathcal{P}}(\widetilde{P})$ and the morphism $\varphi^*_{\mathcal{P}}(\widetilde{P}) \xrightarrow{\varphi^*_{\mathcal{P}}\eta(\widetilde{P})} \varphi^*_{\mathcal{P}}\mathfrak{f}_{\mathcal{P}*}\mathfrak{f}^*_{\mathcal{P}}(\widetilde{P}) = \varphi_*\mathfrak{f}^*_{\mathcal{P}}(\widetilde{P})$. By 3.7.1, the adjunction morphism $\widetilde{P} \xrightarrow{\eta(\widetilde{P})} \mathfrak{f}_{\mathcal{P}*}\mathfrak{f}^*_{\mathcal{P}}(\widetilde{P})$ is a monomorphism. Therefore its image by the exact functor $\varphi^*_{\mathcal{P}}$ is a monomorphism, which implies that $P \xrightarrow{\widehat{\mathfrak{v}}} \varphi_*\mathfrak{f}^*_{\mathcal{P}}(\widetilde{P})$ is a monomorphism.

In particular, the corresponding morphism $\varphi^*(P) \xrightarrow{\mathfrak{v}} \mathfrak{f}^*_{\mathcal{P}}(\widetilde{P})$ is nonzero.

(ii) Consider the cartesian square

$$\begin{tikzcd} P_1 \arrow[r, "h"] \arrow[d] & \widetilde{P} \arrow[d, "\eta(\widetilde{P})"] \\ \mathfrak{f}_{\mathcal{P}*}\varphi^*(P) \arrow[r, "\mathfrak{f}_{\mathcal{P}*}(\mathfrak{v})"] & \mathfrak{f}_{\mathcal{P}*}\mathfrak{f}^*_{\mathcal{P}}(\widetilde{P}) \end{tikzcd} \tag{2}$$

The functor $\varphi^*_{\mathcal{P}}$, being (left) exact, maps (2) to a cartesian square

$$\begin{tikzcd} \varphi^*_{\mathcal{P}}(P_1) \arrow[r, "h'"] \arrow[d] & \varphi^*_{\mathcal{P}}(\widetilde{P}) \arrow[d, "\varphi^*_{\mathcal{P}}\eta(\widetilde{P})"] \\ \varphi_*\varphi^*(P) \arrow[r, "\varphi_*(\mathfrak{v})"] & \varphi_*\mathfrak{f}^*_{\mathcal{P}}(\widetilde{P}) \end{tikzcd} \tag{3}$$

It follows from the commutative diagram

$$\begin{tikzcd} P \arrow[r, "\iota"] \arrow[d, "\eta_u(P)"'] & \varphi^*_{\mathcal{P}}(\widetilde{P}) \arrow[d, "\varphi^*_{\mathcal{P}}\eta(\widetilde{P})"] \\ \varphi_*\varphi^*(P) \arrow[r, "\varphi_*(\mathfrak{v})"] & \varphi_*\mathfrak{f}^*_{\mathcal{P}}(\widetilde{P}) \end{tikzcd} \tag{4}$$

and the universal property of cartesian squares (applied to the square (3)) that there exists a unique morphism $P \xrightarrow{\rho} \varphi^*_{\mathcal{P}}(P_1)$ such that the diagram

$$
\begin{array}{ccccc}
P & \xrightarrow{\rho} & \varphi^*_{\mathcal{P}}(P_1) & \xrightarrow{\varphi^*_{\mathcal{P}}(h)} & \varphi^*_{\mathcal{P}}(\widetilde{P}) \\
{\scriptstyle \eta_u(P)}\downarrow & & \downarrow & & \downarrow {\scriptstyle \varphi^*_{\mathcal{P}}\eta(\widetilde{P})} \\
\varphi_*\varphi^*(P) & \xrightarrow{id} & \varphi_*\varphi^*(P) & \xrightarrow{\varphi_*(\mathfrak{v})} & \varphi_*\mathfrak{f}^*_{\mathcal{P}}(\widetilde{P})
\end{array}
\qquad (5)
$$

commutes and the composition of $P \xrightarrow{\rho} \varphi^*_{\mathcal{P}}(P_1) \xrightarrow{\varphi^*_{\mathcal{P}}(h)} \varphi^*_{\mathcal{P}}(\widetilde{P})$ coincides with the monomorphism $P \xrightarrow{\iota} \varphi^*_{\mathcal{P}}(\widetilde{P})$ we started with. This shows, among other things, that the canonical morphism $P \xrightarrow{\rho} \varphi^*_{\mathcal{P}}(P_1)$ is a monomorphism and the morphism $P_1 \xrightarrow{h} \widetilde{P}$ is nonzero. Since $\widetilde{P}$ belongs to $Spec^0_{\mathfrak{c}}(\mathfrak{A}_{\mathcal{P}})$, the image, $\widetilde{P}_1$, of the morphism $P_1 \xrightarrow{h} \widetilde{P}$ is equivalent to $\widetilde{P}$, i.e. $[\widetilde{P}_1]_{\mathfrak{c}} = [\widetilde{P}]_{\mathfrak{c}}$. By 3.6.2.1, this implies that $[\mathfrak{L}_{\mathcal{P}}(\widetilde{P}_1)]_{\mathfrak{c}} = [\mathfrak{L}_{\mathcal{P}}(\widetilde{P})]_{\mathfrak{c}}$.

The decomposition of the morphism $P_1 \xrightarrow{h} \widetilde{P}$ into an epimorphism $P_1 \xrightarrow{h_1} \widetilde{P}_1$ and a monomorphism $\widetilde{P}_1 \longrightarrow \widetilde{P}$ induces the corresponding decomposition of (the right square of) the diagram (5):

$$
\begin{array}{ccccccc}
P & \xrightarrow{\rho} & \varphi^*_{\mathcal{P}}(P_1) & \xrightarrow{\varphi^*_{\mathcal{P}}(h_1)} & \varphi^*_{\mathcal{P}}(\widetilde{P}_1) & \longrightarrow & \varphi^*_{\mathcal{P}}(\widetilde{P}) \\
{\scriptstyle \eta_u(P)}\downarrow & & \downarrow & & \downarrow {\scriptstyle \varphi^*_{\mathcal{P}}\eta(\widetilde{P}_1)} & & \downarrow {\scriptstyle \varphi^*_{\mathcal{P}}\eta(\widetilde{P})} \\
\varphi_*\varphi^*(P) & \xrightarrow{id} & \varphi_*\varphi^*(P) & \xrightarrow{\varphi_*(\mathfrak{v}_1)} & \varphi_*\mathfrak{f}^*_{\mathcal{P}}(\widetilde{P}_1) & \longrightarrow & \varphi_*\mathfrak{f}^*_{\mathcal{P}}(\widetilde{P})
\end{array}
\qquad (5')
$$

Therefore, one can, replacing the object $\widetilde{P}$ by $\widetilde{P}_1$, assume that the morphism $P_1 \xrightarrow{h} \widetilde{P}$ is an epimorphism. We keep this assumption for the rest of the proof.

(iii) The epimorphness of $P_1 \xrightarrow{h} \widetilde{P}$ implies that the morphism $\varphi^*(P) \xrightarrow{\mathfrak{v}} \mathfrak{f}^*_{\mathcal{P}}(\widetilde{P})$ (defined in (i)) is an epimorphism.

In fact, the diagram (2) is equivalent (via adjunction) to the commutative diagram

$$
\begin{array}{ccc}
\mathfrak{f}^*_{\mathcal{P}}(P_1) & \xrightarrow{\mathfrak{f}^*_{\mathcal{P}}(h)} & \mathfrak{f}^*_{\mathcal{P}}(\widetilde{P}) \\
\downarrow & & \downarrow {\scriptstyle id} \\
\varphi^*(P) & \xrightarrow{\mathfrak{v}} & \mathfrak{f}^*_{\mathcal{P}}(\widetilde{P})
\end{array}
$$

The upper horizontal arrow is an epimorphism, because the functor $\mathfrak{f}^*_{\mathcal{P}}$ is right exact (as any functor having a right adjoint). Therefore $\varphi^*(P) \xrightarrow{\mathfrak{v}} \mathfrak{f}^*_{\mathcal{P}}(\widetilde{P})$ is an epimorphism.

(iii^{bis}) One can arrive to the above conclusions via a shorter argument taking into consideration that the morphism functor $\mathfrak{A}_{\mathcal{P}} \xrightarrow{\varphi_{\mathcal{P}}} X$ is continuous.

Indeed, let $\varphi^*_{\mathcal{P}}(P) \xrightarrow{\mathfrak{v}'} \widetilde{P}$ be the morphism of $C_{\mathfrak{A}_{\mathcal{P}}}$ corresponding to the monomorphism $P \longrightarrow \varphi_{\mathcal{P}*}(\widetilde{P})$. Since the morphism $\mathfrak{v}'$ is nonzero and $\widetilde{P}$ belongs to $Spec^0_{\mathfrak{c}}(\mathfrak{A}_{\mathcal{P}})$, the image of $\mathfrak{v}'$ is an object of $Spec(\mathfrak{A}_{\mathcal{P}})$ and is equivalent to $\widetilde{P}$. Thanks to 3.6.2.1, we can (and will) assume (replacing $\widetilde{P}$ by the image of $\mathfrak{v}'$) that $\mathfrak{v}'$ is an epimorphism. Since the functor $\mathfrak{f}^*_{\mathcal{P}}$ is right exact, $\mathfrak{f}^*_{\mathcal{P}}\varphi^*_{\mathcal{P}}(P) \xrightarrow{\mathfrak{f}^*_{\mathcal{P}}(\mathfrak{v}')} \mathfrak{f}^*_{\mathcal{P}}(\widetilde{P})$ is an epimorphism. Notice that $\varphi^* \simeq \mathfrak{f}^*_{\mathcal{P}}\varphi^*_{\mathcal{P}}$. Thus, we have an epimorphism $\varphi^*(P) \xrightarrow{\mathfrak{v}} \mathfrak{f}^*_{\mathcal{P}}(\widetilde{P})$.

(iv) We denote by $\varphi^*(P) \xrightarrow{\mathfrak{e}} \mathfrak{L}_{\mathcal{P}}(\widetilde{P})$ the composition of $\varphi^*(P) \xrightarrow{\mathfrak{v}} \mathfrak{f}^*_{\mathcal{P}}(\widetilde{P})$ and the epimorphism $\mathfrak{f}^*_{\mathcal{P}}(\widetilde{P}) \longrightarrow \mathfrak{L}_{\mathcal{P}}(\widetilde{P})$. Let $M' \xrightarrow{\mathfrak{j}} \mathfrak{L}_{\mathcal{P}}(\widetilde{P})$ be a nonzero monomorphism. Consider the cartesian square

$$\begin{array}{ccc} \varphi^*(P) & \xrightarrow{\mathfrak{e}} & \mathfrak{L}_{\mathcal{P}}(\widetilde{P}) \\ {\scriptstyle \widetilde{\mathfrak{j}}}\uparrow & & \uparrow{\scriptstyle \mathfrak{j}} \\ M & \xrightarrow{\widetilde{\mathfrak{e}}} & M' \end{array} \qquad (6)$$

and define the morphisms $\widetilde{P}_M \longrightarrow \mathfrak{f}^*_{\mathcal{P}}(M)$ and $\widetilde{P}_{M'} \longrightarrow \mathfrak{f}^*_{\mathcal{P}}(M')$ via the cartesian squares

$$\begin{array}{ccccc} \mathfrak{f}^*_{\mathcal{P}}(M) & \xrightarrow{\mathfrak{f}^*_{\mathcal{P}}(\widetilde{\mathfrak{e}})} & \mathfrak{f}^*_{\mathcal{P}}(M') & \xrightarrow{\mathfrak{f}^*_{\mathcal{P}}(\mathfrak{j})} & \mathfrak{f}^*_{\mathcal{P}}(\mathfrak{L}_{\mathcal{P}}(\widetilde{P})) \\ \uparrow & & \uparrow & & \uparrow \\ \widetilde{P}_M & \xrightarrow{\mathfrak{e}'} & \widetilde{P}_{M'} & \xrightarrow{\mathfrak{j}'} & \widetilde{P} \end{array} \qquad (7)$$

It follows from 3.8 that the right vertical arrow in the diagram (7) is a monomorphism. Therefore, by a well-known property of cartesian squares, the remaining vertical arrows are monomorphisms too.

Since $\mathfrak{f}^*_{\mathcal{P}}$ is an exact functor, $\mathfrak{f}^*_{\mathcal{P}}(\mathfrak{j})$ is a monomorphism and $\mathfrak{f}^*_{\mathcal{P}}(\mathfrak{e})$ is an epimorphism. Therefore, $\widetilde{P}_M \xrightarrow{\mathfrak{e}'} \widetilde{P}_{M'}$ is an epimorphism and $\widetilde{P}_{M'} \xrightarrow{\mathfrak{j}'} \widetilde{P}$ is a monomorphism.

(v) We claim that $\widetilde{P}_{M'} \neq 0$.

Notice that $\mathcal{P} \in Supp(\varphi_*(M'))$, because M' is a nonzero subobject of $\mathfrak{L}_{\mathcal{P}}(\widetilde{P})$, in particular it does not belong to the Serre subcategory $\varphi_*^{-1}(\widehat{\mathcal{P}})$. Since there is an epimorphism $M \longrightarrow M'$ (see (6) above) and φ_* is an exact functor, $\mathcal{P} \in Supp(\varphi_*(M))$.

By the condition (*), there exists $(U', \mathfrak{v}) \in \mathcal{E}_{(\mathcal{P})}$ and a subobject P' of P such that $U'(P') \longrightarrow F_{\varphi}(P)$ factors through $\varphi_*(M)$ and the support of its image contains $\mathcal{P}$. It follows from the construction that $U'(P')$ is a subobject of $\varphi^*_{\mathcal{P}}(\widetilde{P})$ and $\varphi_*(M)$. Therefore, $\mathcal{P} \in Supp(\varphi^*_{\mathcal{P}}(\widetilde{P}_M))$ which, in turn, implies that $\widetilde{P}_{M'} \neq 0$.

(vi) Consider the following commutative diagram

$$\begin{array}{ccc} \mathfrak{f}^*_{\mathcal{P}}(\widetilde{P}_{M'}) & \longrightarrow & M' \\ {\scriptstyle \mathfrak{f}^*_{\mathcal{P}}(\mathfrak{j}')}\downarrow & & \downarrow{\scriptstyle \mathfrak{j}} \\ \mathfrak{f}^*_{\mathcal{P}}(\widetilde{P}) & \longrightarrow & \mathfrak{L}_{\mathcal{P}}(\widetilde{P}) \end{array} \qquad (4)$$

corresponding to the right square of (7). Since $M' \xrightarrow{\mathfrak{j}} \mathfrak{L}_{\mathcal{P}}(\widetilde{P})$ is a monomorphism, the morphism $\mathfrak{f}^*_{\mathcal{P}}(\widetilde{P}_{M'}) \longrightarrow M'$ in (4) factors through $\mathfrak{L}_{\mathcal{P}}(\widetilde{P}_{M'}) \longrightarrow M'$. Thus, (4) induces a commutative diagram

$$\begin{array}{ccc} \mathfrak{L}_{\mathcal{P}}(\widetilde{P}_{M'}) & \xrightarrow{\iota} & M' \\ & {\scriptstyle \mathfrak{L}_{\mathcal{P}}(\mathfrak{j}')}\searrow \quad \swarrow{\scriptstyle \mathfrak{j}} & \\ & \mathfrak{L}_{\mathcal{P}}(\widetilde{P}) & \end{array}$$

By 3.6.2.1, the arrow $\mathfrak{L}_{\mathcal{P}}(\widetilde{P}_{M'}) \xrightarrow{\mathfrak{L}_{\mathcal{P}}(\mathfrak{j}')} \mathfrak{L}_{\mathcal{P}}(\widetilde{P})$ is a monomorphism, and $\mathfrak{L}_{\mathcal{P}}(\mathfrak{j}') = \mathfrak{j} \circ \iota$. Therefore $\mathfrak{L}_{\mathcal{P}}(\widetilde{P}_{M'}) \xrightarrow{\iota} M'$ is a monomorphism. In particular, $\mathfrak{L}_{\mathcal{P}}(\widetilde{P}_{M'}) \in [M']_{\mathfrak{c}}$. Since $\widetilde{P}_{M'}$ is a nonzero subobject of $\widetilde{P}$ and $\widetilde{P}$ belongs to $Spec^0_{\mathfrak{c}}(\mathfrak{A}_{\mathcal{P}})$, these objects are equivalent, that is $[\widetilde{P}_{M'}]_{\mathfrak{c}} = [\widetilde{P}]_{\mathfrak{c}}$. By 3.6.2.1, the functor $\mathfrak{L}_{\mathcal{P}}$ is exact, which implies that $\mathfrak{L}_{\mathcal{P}}(\widetilde{P}) \in \mathfrak{L}_{\mathcal{P}}(\widetilde{P}_{M'})$ (see the argument of 3.8). Therefore $\mathfrak{L}_{\mathcal{P}}(\widetilde{P}) \in [M']_{\mathfrak{c}}$.

(b) By Proposition 3.6.2.1, the functor $\mathfrak{L}_{\mathcal{P}}$ is exact. Therefore, by the argument of 3.8, the subcategory $[\mathfrak{L}_{\mathcal{P}}(M)]_{\mathfrak{c}}$ depends only on the subcategory $[M]_{\mathfrak{c}}$.

(c) *The inverse map.* Let $M \in Spec^{\mathcal{P}}_{\mathfrak{c}}(\mathfrak{A})$; i.e. M is an object of $Spec^0_{\mathfrak{c}}(\mathfrak{A})$, and there exists a monomorphism $\widetilde{P} \longrightarrow \mathfrak{f}^*_{\mathcal{P}}(M)$ such that $\mathcal{P} \in Ass(\varphi^*_{\mathcal{P}}(\widetilde{P}))$ and $\widetilde{P}$ belongs to $Spec^0_{\mathfrak{c}}(\mathfrak{A}_{\mathcal{P}})$. Note that the object M is $\varphi_*^{-1}(\mathcal{P})$-torsion free.

In fact, suppose that M has a nonzero subobject N which belongs to $\varphi_*^{-1}(\widehat{\mathcal{P}})$. Since $M \in Spec^0_{\mathfrak{c}}(\mathfrak{A})$, $M \in [N]_{\mathfrak{c}}$ which implies that $\varphi_*(M) \in Ob\widehat{\mathcal{P}}$. The latter contradicts to the fact that a representative of the subcategory $\mathcal{P}$ is a subobject of $\varphi_*(M)$.

Since the object M is $\varphi_*^{-1}(\widehat{\mathcal{P}})$-torsion free, the canonical morphism $\mathfrak{f}^*_{\mathcal{P}}(\widetilde{P}) \longrightarrow M$ factors through a morphism $\mathfrak{L}_{\mathcal{P}}(\widetilde{P}) \longrightarrow M$. Due to the fact that M belongs to the spectrum and the natural morphism $\mathfrak{L}_{\mathcal{P}}(\widetilde{P}) \longrightarrow M$ is nonzero, $M \in [\mathfrak{L}_{\mathcal{P}}(\widetilde{P})]_{\mathfrak{c}}$. To prove that $\mathfrak{L}_{\mathcal{P}}(\widetilde{P}) \in [M]_{\mathfrak{c}}$, it suffices to show that the morphism $\mathfrak{L}_{\mathcal{P}}(\widetilde{P}) \longrightarrow M$ is a monomorphism.

Consider the exact sequence $0 \longrightarrow K \xrightarrow{\kappa} \mathfrak{L}_{\mathcal{P}}(\widetilde{P}) \longrightarrow M$. It follows that the intersection of $\mathfrak{f}^*_{\mathcal{P}}(K) \xrightarrow{\mathfrak{f}^*_{\mathcal{P}}(\kappa)} \mathfrak{f}^*_{\mathcal{P}}(\mathfrak{L}_{\mathcal{P}}(\widetilde{P}))$ with the subobject $\widetilde{P} \longrightarrow \mathfrak{f}^*_{\mathcal{P}}(\mathfrak{L}_{\mathcal{P}}(\widetilde{P}))$ is zero. By the argument (v) above, this implies that $K = 0$. Therefore, the natural morphism $\mathfrak{L}_{\mathcal{P}}(\widetilde{P}) \longrightarrow M$ is a monomorphism.

(d) It remains to prove the last assertion of the theorem: if $\widetilde{P}$ is a simple object of the category $C_{\mathfrak{A}_{\mathcal{P}}}$ such that $\mathcal{P} \in Ass(\varphi^*_{\mathcal{P}}(\widetilde{P})$, then $\mathfrak{L}_{\mathcal{P}}(\widetilde{P})$ is a simple object of the category $\mathfrak{A}$.

In fact, let $K \xrightarrow{\mathfrak{j}} \mathfrak{L}_{\mathcal{P}}(\widetilde{P})$ be a nonzero monomorphism. By (v) above, the pull-back of monomorphisms $\mathfrak{f}^*_{\mathcal{P}}(K) \longrightarrow \mathfrak{f}^*_{\mathcal{P}}(\mathfrak{L}_{\mathcal{P}}(\widetilde{P})) \longleftarrow \widetilde{P}$ is nonzero. Since $\widetilde{P}$ is simple, it follows that the morphism $\widetilde{P} \longrightarrow \mathfrak{f}^*_{\mathcal{P}}(\mathfrak{L}_{\mathcal{P}}(\widetilde{P}))$ factors through $\mathfrak{f}^*_{\mathcal{P}}(K) \longrightarrow \mathfrak{f}^*_{\mathcal{P}}(\mathfrak{L}_{\mathcal{P}}(\widetilde{P}))$. Therefore the identical morphism $\mathfrak{L}_{\mathcal{P}}(\widetilde{P}) \longrightarrow \mathfrak{L}_{\mathcal{P}}(\widetilde{P})$ factors through $K \xrightarrow{\mathfrak{j}} \mathfrak{L}_{\mathcal{P}}(\widetilde{P})$ which shows that $K \xrightarrow{\mathfrak{j}} \mathfrak{L}_{\mathcal{P}}(\widetilde{P})$ is an isomorphism. ∎

4.2.1. The case of the trivial stabilizer. Suppose that the point $\mathcal{P}$ of the spectrum $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ has the trivial stabilizer. That is if $(U, U \to F_{\varphi})$ is an object of the stabilizer of $\mathcal{P}$, then U is a subfunctor of the identical functor. In this case, the condition (*) in 4.2 is equivalent to the condition

(†) If a subobject M of $\varphi^*(P)$ is such that $\mathcal{P} = [P]_{\mathfrak{c}} \in Supp(\varphi_*(M))$, then $\mathcal{P}$ is an associated point of $\varphi_*(M)$.

Evidently, the condition (†) holds if $Supp(\varphi_*\varphi^*(P)) = Ass(\varphi_*\varphi^*(P))$.

The latter equality holds if the functor $F_{\varphi} = \varphi_*\varphi^*$ is *differential* (cf. A1.6, A1.7) and $\mathcal{P} = [P]_{\mathfrak{c}}$ is a closed point. In this case, $Supp(\varphi_*\varphi^*(P)) = \{\mathcal{P}\} = Ass(\varphi_*\varphi^*(P))$.

The condition (†) also holds if $\mathcal{P}$ is a closed point and the functor F_{φ} is a coproduct of auto-equivalences.

4.3. A reduction. Once a point $\mathcal{P}$ of $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ is fixed, one can avoid dealing with the irrelevant parts of the categories C_X and $C_{\mathfrak{U}}$ proceeding as follows. We define the 'space' $X_{\mathcal{P}}$ by $C_{X_{\mathcal{P}}} = \mathcal{P}$. If C_X is the category of quasi-coherent sheaves on a scheme, then $\mathcal{P}$ corresponds to a point of the underlying space of this scheme and the category $C_{X_{\mathcal{P}}} = \mathcal{P}$ is naturally equivalent to the category of quasi-coherent sheaves on the closure of the point $\mathcal{P}$. Thus, the 'space' $X_{\mathcal{P}}$ can be regarded as the closure of the point $\mathcal{P}$ in X.

The inclusion functor $C_{X_{\mathcal{P}}} \xrightarrow{\mathfrak{j}^*_{\mathcal{P}}} C_X$ has a right adjoint $C_X \xrightarrow{\mathfrak{j}_{\mathcal{P}*}} C_{X_{\mathcal{P}}}$ which assigns to every object of C_X its $\mathcal{P}$-torsion. Let $\mathfrak{A} \xrightarrow{u_{\mathcal{P}}} X_{\mathcal{P}}$ denote the composition of the morphisms $\mathfrak{A} \xrightarrow{\varphi} X$ and $X \xrightarrow{\mathfrak{j}_{\mathcal{P}}} X_{\mathcal{P}}$. The morphism $u_{\mathcal{P}}$, being a composition of two continuous morphisms, is continuous. Its direct image functor is not, in general, right exact, because the functor $C_X \xrightarrow{\mathfrak{j}_{\mathcal{P}*}} C_{X_{\mathcal{P}}}$ is not necessarily right exact. Notice that the functor $\mathfrak{j}_{\mathcal{P}*}$ preserves supremums of objects; in particular, it preserves infinite coproducts. Since $u_{\mathcal{P}*} \simeq \mathfrak{j}_{\mathcal{P}*} \circ \varphi_*$ and φ_* preserves arbitrary colimits, the functor $u_{\mathcal{P}*}$ also preserves infinite coproducts.

We replace the category $C_{\mathfrak{A}_{\mathcal{P}}}$ by its full subcategory $C_{\mathfrak{A}'_{\mathcal{P}}}$ generated by all $\mathcal{F}_{\varphi_{\mathcal{P}}}$-modules (M, ξ) such that M is an object of $C_{X_{\mathcal{P}}} = \mathcal{P}$. The inclusion functor $C_{\mathfrak{A}'_{\mathcal{P}}} \xrightarrow{\widetilde{\mathfrak{j}}^*_{\mathcal{P}}} C_{\mathfrak{A}_{\mathcal{P}}}$ has a right adjoint, $\widetilde{\mathfrak{j}}_{\mathcal{P}*}$, induced by the functor $C_X \xrightarrow{\mathfrak{j}_{\mathcal{P}*}} C_{X_{\mathcal{P}}}$. We define the functor $C_{\mathfrak{A}} \xrightarrow{\widetilde{\mathfrak{f}}_{\mathcal{P}*}} C_{\mathfrak{A}'_{\mathcal{P}}}$ as the composition of the pull-back functor $C_{\mathfrak{A}} \xrightarrow{\mathfrak{f}_{\mathcal{P}*}} C_{\mathfrak{A}_{\mathcal{P}}}$ and the functor $C_{\mathfrak{A}_{\mathcal{P}}} \xrightarrow{\widetilde{\mathfrak{j}}_{\mathcal{P}*}} C_{\mathfrak{A}'_{\mathcal{P}}}$. Thus, we obtain a quasi-commutative diagram

$$\begin{array}{ccc} C_{\mathfrak{A}} & \xrightarrow{u_{\mathcal{P}*}} & C_{X_{\mathcal{P}}} \\ \widetilde{\mathfrak{f}}_{\mathcal{P}*} \searrow & & \nearrow \widetilde{\varphi}_{\mathcal{P}*} \\ & C_{\mathfrak{A}'_{\mathcal{P}}} & \end{array} \qquad (1)$$

interpreted as the diagram of direct image functors of the morphisms of the commutative diagram

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{u_{\mathcal{P}}} & X_{\mathcal{P}} \\ \widetilde{\mathfrak{f}}_{\mathcal{P}} \searrow & & \nearrow \widetilde{\varphi}_{\mathcal{P}} \\ & \mathfrak{A}'_{\mathcal{P}} & \end{array}$$

Let $\widetilde{\mathfrak{L}}_{\mathcal{P}}$ denote the restriction of the functor $\mathfrak{L}_{\mathcal{P}}$ to the subcategory $\mathfrak{A}'_{\mathcal{P}}$. The exactness of the functor $C_{\mathfrak{A}'_{\mathcal{P}}} \xrightarrow{\mathfrak{L}_{\mathcal{P}}} C_{\mathfrak{A}}$, depends now on the exactness of a left adjoint, $\widetilde{\mathfrak{f}}^*_{\mathcal{P}}$, to the functor $C_{\mathfrak{A}} \xrightarrow{\widetilde{\mathfrak{f}}_{\mathcal{P}*}} C_{\mathfrak{A}'_{\mathcal{P}}}$. The exactness of $\widetilde{\mathfrak{f}}^*_{\mathcal{P}}$ is a much weaker requirement than the exactness of a right adjoint $\mathfrak{f}^*_{\mathcal{P}}$ to the pull-back functor $C_{\mathfrak{A}} \xrightarrow{\mathfrak{f}_{\mathcal{P}*}} C_{\mathfrak{A}_{\mathcal{P}}}$ imposed in 4.2.

These considerations will be used in the following Sections.

5. Important special cases. Finiteness conditions.

5.1. In most of applications we have in mind (in particular, those mentioned in this work), the monad $\mathcal{F}_{\varphi} = (F_{\varphi}, \mu_{\varphi})$ belongs to one of the following two classes:

(a) The functor F_φ is a direct sum of a family of autoequivalences of the category C_X.

(b) The monad $\mathcal{F}_\varphi$ (i.e. the functor F_φ) is differential.

Below we consider each of these cases and give the corresponding specializations of Theorem 4.2.

5.2. The case of a direct sum of autoequivalences. Let C_X be an abelian category, and let $\mathcal{F}_\varphi = (F_\varphi, \mu_\varphi)$ a monad on C_X such that $F_\varphi = \bigoplus_{\alpha\in\mathfrak{J}} \theta_\alpha$, where θ_α are autoequivalences of the category C_X. We denote by $\mathfrak{A}$ the 'space' $\mathbf{Sp}(\mathcal{F}_\varphi/X)$ and by φ the canonical morphism $\mathfrak{A} \longrightarrow X$. We take as $\widetilde{\mathcal{E}}$ the full monoidal subcategory of monoidal category $\widetilde{\mathfrak{Er}_\mathfrak{c}}(C_X)/\mathcal{F}_\varphi$ generated by the coprojections $\theta_\alpha \xrightarrow{\pi_\alpha} F_\varphi,\ \alpha \in \mathfrak{J}$.

We are going to use the reduction described in 4.3; hence we assume for the rest of this section that the category C_X has the property (sup).

Fix an element $\mathcal{P}$ of $\mathbf{Spec}^0_\mathfrak{c}(X)$. Following the pattern of 4.3, we obtain a quasi-commutative diagram of functors

$$\begin{array}{ccc} C_{\mathfrak{A}} & \xrightarrow{u_{\mathcal{P}*}} & C_{X_\mathcal{P}} \\ {\scriptstyle \widetilde{\mathfrak{f}}_{\mathcal{P}*}} \searrow & & \nearrow {\scriptstyle \widetilde{\varphi}_{\mathcal{P}*}} \\ & C_{\mathfrak{A}'_\mathcal{P}} & \end{array} \qquad (1)$$

Here $C_{X_\mathcal{P}} = \mathcal{P}$, and $C_{\mathfrak{A}'_\mathcal{P}} = \mathbf{Sp}(\mathcal{F}_\mathcal{P}/X_\mathcal{P})$, where $\mathcal{F}_\mathcal{P}$ is a monad on $C_{X_\mathcal{P}}$ induced by the monad $\mathcal{F}_{\varphi_\mathcal{P}}$. In other words, $C_{\mathfrak{A}'_\mathcal{P}}$ is the full subcategory of the category $C_{\mathfrak{A}_\mathcal{P}}$ of $\mathcal{F}_{\varphi_\mathcal{P}}$-modules whose objects are modules (M,ξ) such that $M \in Ob\mathcal{P}$.

5.2.0. The Krull filtration of $\mathbf{Spec}^0_\mathfrak{c}(X)$ and the associated filtration of X. Fix an abelian category C_X. For every cardinal α, we define a subset $\mathfrak{S}_\alpha(X)$ of $\mathbf{Spec}^0_\mathfrak{c}(X)$ as follows.

$\mathfrak{S}_0(X) = \emptyset$;

if α is not a limit cardinal, then $\mathfrak{S}_\alpha(X)$ consists of all $\mathcal{P} \in \mathbf{Spec}^0_\mathfrak{c}(X)$ such that any $\mathcal{P}' \in \mathbf{Spec}^0_\mathfrak{c}(X)$ properly contained in $\mathcal{P}$ belongs to $\mathfrak{S}_{\alpha-1}(X)$;

if α is a limit cardinal, then $\mathfrak{S}_\alpha(X) = \bigcup\limits_{\beta<\alpha} \mathfrak{S}_\beta(X)$.

It follows from this definition (borrowed from [R, VI.6.3]) that $\mathfrak{S}_1(X)$ consists of all closed points of $\mathbf{Spec}^0_\mathfrak{c}(X)$.

We denote by $\mathfrak{S}_\omega(X)$ the union of all $\mathfrak{S}_\alpha(X)$. The filtration $\{\mathfrak{S}_\alpha(X)\}$ determines a filtration

$$C_{X_0} \hookrightarrow C_{X_1} \hookrightarrow \dots C_{X_\alpha} \hookrightarrow \dots \qquad (5)$$

of the category C_X (or the 'space' X) by taking as C_{X_α} the full subcategory of C_X generated by objects M such that $Supp^0_\mathfrak{c}(M) \subseteq \mathfrak{S}_\alpha(X)$. Recall that $Supp^0_\mathfrak{c}(M) = \{\mathcal{P} \in \mathbf{Spec}^0_\mathfrak{c}(X) \mid M \notin Ob\mathcal{P}\}$. In particular, C_{X_ω} is the full subcategory of C_X generated by all $M \in ObC_X$ such that $Supp^0_\mathfrak{c}(M) \subseteq \mathfrak{S}_\omega(X)$.

It follows from the general properties of supports that C_{X_α} is a Serre subcategory of C_X and $\mathbf{Spec}^0_\mathfrak{c}(X_\alpha)$ is naturally identified with $\mathfrak{S}_\alpha(X)$; in particular, $\mathbf{Spec}^0_\mathfrak{c}(X_\omega)$ is identified with $\mathfrak{S}_\omega(X)$.

5.2.0.1. Proposition. *For each cardinal α, the subset $\mathfrak{S}_\alpha(X)$ of the spectrum is stable under all auto-equivalences of the category C_X. Let $\mathcal{P} \in \mathfrak{S}_\omega(X)$. If θ is an auto-equivalence of the category C_X, such that $\theta(\mathcal{P}) \subseteq \mathcal{P}$, then $\theta(\mathcal{P}) = \mathcal{P}$.*

Proof. The assertion is true for $\mathcal{P} \in \mathfrak{S}_0(X)$, because any auto-equivalence maps spectral objects to spectral objects. So, if $\mathcal{P}$ is a closed point and $\theta(\mathcal{P}) \subseteq \mathcal{P}$, then $\mathcal{P} = \theta(\mathcal{P})$.

Suppose now that the fact is true if $\mathcal{P} \in \mathfrak{S}_\nu$ for any $\nu < \alpha$. The claim is that it holds for any $\mathcal{P} \in \mathfrak{S}_\alpha$. In fact, it holds by a trivial reason if α is a limit cardinal. Let α be a not a limit cardinal, $\mathcal{P} \in \mathfrak{S}_\alpha(X)$, and $\theta(\mathcal{P}) \subseteq \mathcal{P}$. If $\theta(\mathcal{P}) \neq \mathcal{P}$, then, by definition of $\mathfrak{S}_\alpha(X)$, the element $\theta(\mathcal{P})$ belongs to $\mathfrak{S}_{\alpha-1}(X)$. But then, by induction hypothesis, $\mathcal{P} \in \mathfrak{S}_{\alpha-1}(X)$, hence $\theta(\mathcal{P}) = \mathcal{P}$. ■

5.2.1. Proposition. *(a) Under the conditions above, the functor $C_{\mathfrak{A}} \xrightarrow{\widetilde{\mathfrak{f}}_{\mathcal{P}*}} C_{\mathfrak{A}'_{\mathcal{P}}}$ has a left adjoint; and the functor $C_{\mathfrak{A}'_{\mathcal{P}}} \xrightarrow{\widetilde{\varphi}_{\mathcal{P}*}} C_{X_{\mathcal{P}}}$ has a left adjoint which is faithfully flat.*

(b) Suppose that $\mathcal{P}$ belongs to $\mathfrak{S}_\omega(X)$. Then the functor $\mathfrak{f}^*_{\mathcal{P}}$ is faithful.

Proof. (a) Set $J_{\mathcal{P}} = \{\alpha \in J \mid \theta_\alpha \in \mathfrak{F}_{\mathcal{P}}\} = \{\alpha \in J \mid \theta_\alpha(\mathcal{P}) \subseteq \mathcal{P}\}$ and denote by $F_{\mathcal{P}}$ the endofunctor on $C_{X_{\mathcal{P}}} = \mathcal{P}$ (cf. 4.3) induced by $\bigoplus_{\alpha \in J_{\mathcal{P}}} \theta_\alpha$. The multiplication $F^2_\varphi \xrightarrow{\mu_\varphi} F_\varphi$ induces a multiplication $F^2_{\mathcal{P}} \xrightarrow{\mu_{\mathcal{P}}} F_{\mathcal{P}}$ on $F_{\mathcal{P}}$.

In fact, the monad structure on F_φ is determined by the compositions

$$\theta_\alpha \circ \theta_\beta \xrightarrow{\mu^\sigma_{\alpha,\beta}} \theta_\sigma, \quad \alpha, \beta, \sigma \in J, \tag{2}$$

of the embedding $\theta_\alpha \circ \theta_\beta \longrightarrow F_\varphi \circ F_\varphi$, the multiplication $F_\varphi \circ F_\varphi \xrightarrow{\mu_\varphi} F_\varphi$, and the projection $F_\varphi \longrightarrow \theta_\sigma$. Let $\alpha, \beta \in J_{\mathcal{P}} \not\ni \sigma$. Then the morphism $\theta_\alpha\theta_\beta(M) \xrightarrow{\mu^\sigma_{\alpha,\beta}(M)} \theta_\sigma(M)$ is zero for every object M of the subcategory $\mathcal{P}$.

(i) Suppose first that M is a representative of $\mathcal{P}$. Assume that $\mu^\sigma_{\alpha,\beta}(M) \neq 0$. Since θ_σ is an autoequivalence and $M \in Spec^0_{\mathfrak{c}}(X)$, the object $\theta_\sigma(M)$ belongs to $Spec^0_{\mathfrak{c}}(X)$ too. Therefore, the existence of a nonzero morphism $\theta_\alpha\theta_\beta(M) \longrightarrow \theta_\sigma(M)$ implies that the subcategory $[\theta_\alpha\theta_\beta(M)]_{\mathfrak{c}}$ contains $\theta_\sigma(M)$. Since $\theta_\alpha\theta_\beta$ stabilizes $\mathcal{P} = [M]_{\mathfrak{c}}$, it follows that $\theta_\sigma(M)$ belongs to $[M]_{\mathfrak{c}}$, which means precisely that θ_σ stabilizes $\mathcal{P}$. This, in turn, implies that θ_σ stabilizes $\widehat{\mathcal{P}}$. In fact, θ_σ not stabilizing $\widehat{\mathcal{P}}$ means that there exists $N \in ObC_X$ such that $M \notin [N]_{\mathfrak{c}}$, but, $M \in [\theta_\sigma(N)]_{\mathfrak{c}}$. Since θ_σ is an auto-equivalence, it preserves the relation $M \notin [N]_{\mathfrak{c}}$, that is $\theta_\sigma(M) \notin [\theta_\sigma(N)]_{\mathfrak{c}}$. But, this contradicts to the fact that $\theta_\sigma(M) \in [M]_{\mathfrak{c}}$ and $[M]_{\mathfrak{c}} \subseteq [\theta_\sigma(N)]_{\mathfrak{c}}$.

(ii) Suppose now that M is an arbitrary object of $\mathcal{P}$. Let L be a representative of $\mathcal{P}$. Then there exists a diagram $L^{\oplus J} \longleftarrow K \longrightarrow M$ whose the left arrow is a monomorphism and the right arrow is an epimorphism. Thus, we have a commutative diagram

$$\begin{array}{ccccc}
\theta_\alpha\theta_\beta(L^{\oplus J}) & \longleftarrow & \theta_\alpha\theta_\beta(K) & \longrightarrow & \theta_\alpha\theta_\beta(M) \\
\downarrow & & \downarrow & & \downarrow \\
\theta_\sigma(L^{\oplus J}) & \longleftarrow & \theta_\sigma(K) & \longrightarrow & \theta_\sigma(M)
\end{array}$$

whose left (resp. right) horizontal arrows are monomorphisms (resp. epimorphisms) and vertical arrows are values of the functor morphism $\mu^{\sigma}_{\alpha,\beta}$ on the objects respectively $L^{\oplus J}$, K and M. Suppose that $\alpha, \beta \in J_{\mathcal{P}}$, but $\sigma \notin J_{\mathcal{P}}$. Since $[L^{\oplus J}]_{\mathfrak{c}} = [L]_{\mathfrak{c}}$ and, by hypothesis, $[L]_{\mathfrak{c}} = \mathcal{P}$, it follows from (i) that the left vertical arrow in the diagram above is the zero morphism; in particular, the composition of the central vertical arrow,

$$\theta_\alpha\theta_\beta(K) \xrightarrow{\mu^{\sigma}_{\alpha,\beta}(K)} \theta_\sigma(K),$$

and the monomorphism $\theta_\sigma(K) \longrightarrow \theta_\sigma(L^{\oplus J})$ is zero, hence $\mu^{\sigma}_{\alpha,\beta}(K) = 0$. This, in turn, implies that the composition of the epimorphism $\theta_\alpha\theta_\beta(K) \longrightarrow \theta_\alpha\theta_\beta(M)$ and the left vertical arrow, $\theta_\alpha\theta_\beta(M) \xrightarrow{\mu^{\sigma}_{\alpha,\beta}(M)} \theta_\sigma(M)$, is zero which means that $\mu^{\sigma}_{\alpha,\beta}(M) = 0$.

(iii) Set $J^{\vee}_{\mathcal{P}} = J - J_{\mathcal{P}}$ and $F^{\vee}_{\mathcal{P}} = \bigoplus_{\beta\in J^{\vee}_{\mathcal{P}}} \theta_\beta$. Then $F_\varphi = F_{\mathcal{P}} \oplus F^{\vee}_{\mathcal{P}}$. It follows from the above argument that the compostition of $F^2_{\mathcal{P}} \longrightarrow F^2_\varphi \xrightarrow{\mu_\varphi} F_\varphi$ with the projection $F_\varphi \xrightarrow{\pi} F^{\vee}_{\mathcal{P}}$ is zero. Therefore the composition of $F^2_{\mathcal{P}} \longrightarrow F^2_\varphi \xrightarrow{\mu_\varphi} F_\varphi$ factors through the embedding $F_{\mathcal{P}} \hookrightarrow F_\varphi$, i.e. there exists a unique morphism $F^2_{\mathcal{P}} \xrightarrow{\mu_{\mathcal{P}}} F_{\mathcal{P}}$ such that the diagram

$$\begin{array}{ccc}
F^2_{\mathcal{P}} & \xrightarrow{\mu_{\mathcal{P}}} & F_{\mathcal{P}} \\
\big\downarrow & & \big\downarrow \\
F^2_\varphi & \xrightarrow{\mu_\varphi} & F_\varphi
\end{array}$$

commutes. Thus, the morphisms $\{\theta_\alpha\theta_\beta \xrightarrow{\mu^{\sigma}_{\alpha,\beta}} \theta_\sigma \mid \alpha, \beta, \sigma \in J_{\mathcal{P}}\}$ determine an associative multiplication $F^2_{\mathcal{P}} \xrightarrow{\mu_{\mathcal{P}}} F_{\mathcal{P}}$ on $F_{\mathcal{P}}$.

(a1) The forgetful functor $\mathfrak{F}_{\mathcal{P}}-mod \xrightarrow{\widetilde{\varphi}_{\mathcal{P}*}} C_{X_{\mathcal{P}}} = \mathcal{P}$ has a left adjoint, $\widetilde{\varphi}^*_{\mathcal{P}}$, which assigns to every object M of the category $C_{X_{\mathcal{P}}}$ the pair $(F_{\mathcal{P}}(M), \widetilde{\mu})$, where $\widetilde{\mu}$ is the obvious action of $\widetilde{\mathcal{E}}_{(\mathcal{P})}$ on $F_{\mathcal{P}}(M)$. It follows that $\mathcal{F}_{\mathcal{P}} = (F_{\mathcal{P}}, \mu_{\mathcal{P}})$ is the monad associated with the pair $\widetilde{\varphi}^*_{\mathcal{P}}$, $\widetilde{\varphi}^*_{\mathcal{P}}$ of adjoint functors. Since the functor $\widetilde{\varphi}^*_{\mathcal{P}}$ is exact and conservative, the category $(\widetilde{\Phi}_{\mathcal{P}}/X_{\mathcal{P}}) - mod$ is naturally equivalent to the category $\mathcal{F}_{\mathcal{P}} - mod$ of $\mathcal{F}_{\mathcal{P}}$-modules.

(a2) The latter implies the existence of a left adjoint, $C_{\mathfrak{A}'_{\mathcal{P}}} \xrightarrow{\widetilde{\mathfrak{f}}^*_{\mathcal{P}}} C_{\mathfrak{A}}$, to the functor $C_{\mathfrak{A}} \xrightarrow{\widetilde{\mathfrak{f}}_{\mathcal{P}*}} C_{\mathfrak{A}'_{\mathcal{P}}}$ (defined in 4.3).

In fact, identifying the category $C_{\mathfrak{A}'_{\mathcal{P}}}$ with $\mathcal{F}_{\mathcal{P}} - mod$, we take as $\widetilde{\mathfrak{f}}^*_{\mathcal{P}}$ the functor

$$\mathcal{F}_\varphi \otimes_{\mathcal{F}_{\mathcal{P}}} : (\mathcal{F}_{\mathcal{P}}/X_{\mathcal{P}}) - mod \longrightarrow (\mathcal{F}_\varphi/X) - mod. \qquad (3)$$

(b) If $\alpha, \beta \in J_{\mathcal{P}}$ and $\sigma \in J^{\vee}_{\mathcal{P}}$, then

$$\theta_\sigma\theta_\alpha(M) \xrightarrow{\mu^{\beta}_{\sigma,\alpha}(M)} \theta_\beta(M) \qquad (4)$$

is zero for every $M \in Ob[\mathcal{P}]_{\mathfrak{c}}$.

Suppose first that M is a representative of $\mathcal{P}$. Since $\mathcal{P} \in \mathfrak{S}_\omega(X)$, the inclusion $[\theta_\beta(M)]_{\mathfrak{c}} \subseteq [M]_{\mathfrak{c}}$ implies, by 5.2.0.1, the equality $[\theta_\beta(M)]_{\mathfrak{c}} = [M]_{\mathfrak{c}}$. If the morphism (4) is nonzero, then $[\theta_\sigma(M)]_{\mathfrak{c}} \supseteq [\theta_\sigma\theta_\alpha(M)]_{\mathfrak{c}} \supseteq [\theta_\beta(M)]_{\mathfrak{c}} \supseteq [M]_{\mathfrak{c}}$ But, $[\theta_\sigma(M)]_{\mathfrak{c}} \supseteq [M]_{\mathfrak{c}} \Leftrightarrow [M]_{\mathfrak{c}} = [\theta_\sigma(M)]_{\mathfrak{c}}$, which means that $\sigma \in J_{\mathcal{P}}$.

If M is an arbitrary object of $\mathcal{P}$, the argument is the same as the argument (ii) above.

(b1) The argument similar to that of (iii) shows that the multiplication $F_\varphi^2 \xrightarrow{\mu_\varphi} F_\varphi$ induces a morphism $F_{\mathcal{P}}^\vee F_{\mathcal{P}} \xrightarrow{\gamma_{\mathcal{P}}} F_{\mathcal{P}}^\vee$ which is a structure of a right $(F_{\mathcal{P}}, \mu_{\mathcal{P}})$-module on $F_{\mathcal{P}}^\vee$.

(b2) If follows that, as $(F_{\mathcal{P}}, \mu_{\mathcal{P}})$-module, F_φ is the direct sum of $F_{\mathcal{P}}^\vee$ and $F_{\mathcal{P}}$. Therefore, for every $\mathcal{F}_{\mathcal{P}}$-module (M, ξ),

$$\mathfrak{f}_{\mathcal{P}*}\mathfrak{f}_{\mathcal{P}}^*(M,\xi) = F_\varphi \otimes_{\mathcal{F}_{\mathcal{P}}} (M,\xi) \simeq \big(F_{\mathcal{P}}^\vee \otimes_{\mathcal{F}_{\mathcal{P}}} (M,\xi)\big) \oplus (M,\xi),$$

which immediately implies that $\mathfrak{f}_{\mathcal{P}}^*$ is a faithful functor. ■

The corresponding version of Theorem 4.2 is as follows.

5.2.2. Theorem. *Suppose that the category C_X has the property (sup). Let $F_\varphi = \bigoplus_{\alpha\in\mathfrak{J}} \theta_\alpha$, where θ_α are autoequivalences of the category C_X, and let $\mathfrak{F} = \{\theta_\alpha \mid \alpha \in \mathfrak{J}\}$.*

Suppose that an element $\mathcal{P}$ of $\mathfrak{S}_\omega(X)$ is such that the functor $C_{\mathfrak{A}'_{\mathcal{P}}} \xrightarrow{\widetilde{\mathfrak{f}}_{\mathcal{P}}^*} C_{\mathfrak{A}}$ is exact and the following condition holds:

(*) If P is a representative of $\mathcal{P}$ and M is a subobject of $\varphi^*(P)$ such that $\mathcal{P} \in Supp(\varphi_*(M))$, then there exists a subobject P' of P and $\alpha \in \mathfrak{J}$ such that $\theta_\alpha(P')$ is a subobject of $\varphi_*(M)$ and $[P'] \subseteq [\theta_\alpha(P')]_{\mathfrak{c}}$.

Then

(a) The composition $C_{\mathfrak{A}'_{\mathcal{P}}} \xrightarrow{\mathcal{L}_{\mathcal{P}}} C_{\mathfrak{A}}$, of the functors $C_{\mathfrak{A}'_{\mathcal{P}}} \xrightarrow{\widetilde{\mathfrak{f}}_{\mathcal{P}}^*} C_{\mathfrak{A}}$, and

$$C_{\mathfrak{A}} \xrightarrow{\Psi_{\mathcal{P}}} C_{\mathfrak{A}}, \quad M \longmapsto M/tors_{\varphi_*^{-1}(\widehat{\mathcal{P}})}(M),$$

induces a morphism

$$\mathbf{Spec}_{\mathfrak{c}}^{\mathcal{P}}(\mathfrak{A}'_{\mathcal{P}}) \xrightarrow{\mathfrak{L}_{\mathcal{P}}} \mathbf{Spec}_{\mathfrak{c}}^{\mathcal{P}}(\mathfrak{A}). \tag{5}$$

with the following property:

Every $[M]_{\mathfrak{c}} \in \mathbf{Spec}_{\mathfrak{c}}^0(\mathfrak{A})$ such that the image $\mathfrak{f}_{\mathcal{P}}^*(M)$ of M in $C_{\mathfrak{A}_{\mathcal{P}}}$ has an associated point from $\mathbf{Spec}_{\mathfrak{c}}^{\mathcal{P}}(\mathfrak{A}_{\mathcal{P}})$ belongs to the image of the map (1).

(b) The functor $\mathcal{L}_{\mathcal{P}}$ maps simple objects to simple objects.

Proof. The condition (*) is the specialization of the condition (*) in 4.2. Thus, the assertion is a consequence of 5.2.1 and Theorem 4.2. ■

5.2.3. Proposition. *Suppose that the category C_X has the property (sup). Each of the following conditions on a point $\mathcal{P}$ of $\mathfrak{S}_\omega(X)$ implies the condition (*) in 5.2.2:*

(a) the stabilizer of $\mathcal{P}$ is trivial;

(b) the local category $C_X/\widehat{\mathcal{P}}$ has simple objects.

Proof. (i) Set $J_{\mathcal{P}} = \{\alpha \mid [\theta_\alpha(P)] = \mathcal{P}\}$ and $J^{\mathcal{P}} = J - J_{\mathcal{P}}$. Let $M \hookrightarrow \varphi^*(P)$ be such that $\mathcal{P} \in Supp(\varphi_*(M))$. We denote by M' the kernel of the composition of the monomorphism $\varphi_*(M) \hookrightarrow F_\varphi(P)$ and the projection $F_\varphi(P) = \bigoplus_{\alpha\in J}\theta_\alpha(P)$ onto $\bigoplus_{\alpha\in J^{\mathcal{P}}}\theta_\alpha(P)$. It follows that M' is a subobject of $\bigoplus_{\alpha\in J_{\mathcal{P}}}\theta_\alpha(P)$. Since $Supp(\bigoplus_{\alpha\in J^{\mathcal{P}}}\theta_\alpha(P)) = \bigcup_{\alpha\in J^{\mathcal{P}}} Supp(\theta_\alpha(P))$ does not contain the point $\mathcal{P}$, and $Supp(\varphi_*(M)) = Supp(M')\bigcup Supp(M'')$, where M'' denotes the image of $\varphi_*(M)$ in $\bigoplus_{\alpha\in J^{\mathcal{P}}}\theta_\alpha(P)$, the condition $\mathcal{P} \in Supp(\varphi_*(M))$ is equivalent to that $\mathcal{P} \in Supp(M')$. In particular, $M' \neq 0$.

(ii) Suppose that $C_X/\widehat{\mathcal{P}}$ has simple objects. Replacing P with an appropriate subobject, we can and will assume that the image $q_{\widehat{\mathcal{P}}}(P)$ of P in $C_X/\widehat{\mathcal{P}}$ is a simple object. This implies that the image of $F_\varphi(P)$ (which coincides with the image of $\bigoplus_{\alpha\in J_{\mathcal{P}}}\theta_\alpha(P)$) in $C_X/\widehat{\mathcal{P}}$ is semisimple. Therefore, the image of M' in $C_X/\widehat{\mathcal{P}}$ is isomorphic to the image of $\bigoplus_{\alpha\in\mathcal{I}}\theta_\alpha(P)$ for some subset $\mathcal{I}$ of $J_{\mathcal{P}}$. This means that there exists a diagram

$$M' \xleftarrow{\ s\ } N \xrightarrow{\ t\ } \bigoplus_{\alpha\in\mathcal{I}}\theta_\alpha(P) \qquad (6)$$

in C_X such that $q_{\widehat{\mathcal{P}}}(s)$ and $q_{\widehat{\mathcal{P}}}(t)$ are monomorphisms. Since M' and $\bigoplus_{\alpha\in\mathcal{I}}\theta_\alpha(P)$ are $\widehat{\mathcal{P}}$-torsion free objects, the object N in the diagram (6) can and will be chosen $\widehat{\mathcal{P}}$-torsion free. The latter means that the morphisms s and t are monomorphisms. Since $q_{\widehat{\mathcal{P}}}(t)$ is an isomorphism and the localization functor is exact, the intersection $P'_\alpha = N\bigcap\theta_\alpha(P)$ (i.e. the pull-back of the monomorphism $N \xrightarrow{t} \bigoplus_{\alpha\in\mathcal{I}}\theta_\alpha(P)$ and the coprojection $\theta_\alpha(P) \longrightarrow \bigoplus_{\alpha\in\mathcal{I}}\theta_\alpha(P)$) is nonzero for every $\alpha\in\mathcal{I}$. Setting $P' = \theta_\alpha^{-1}(P'_\alpha)$, we obtain a subobject of the object P satisfying the condition (*) of 5.2.2. ■

5.2.4. Corollary. *Suppose that the category C_X has the property (sup). Let $F_\varphi = \bigoplus_{\alpha\in\mathfrak{J}}\theta_\alpha$, where θ_α are autoequivalences of the category C_X, and let $\mathfrak{F} = \{\theta_\alpha \mid \alpha\in\mathfrak{J}\}$.*

Suppose that an element $\mathcal{P}$ of $\mathfrak{S}_\omega(X)$ is such that the functor $C_{\mathfrak{A}'_{\mathcal{P}}} \xrightarrow{\widetilde{\mathfrak{f}}^*_{\mathcal{P}}} C_{\mathfrak{A}}$ is exact and the quotient category $C_X/\widehat{\mathcal{P}}$ has simple objects. Then

(a) The composition $C_{\mathfrak{A}'_{\mathcal{P}}} \xrightarrow{\ \mathcal{L}_{\mathcal{P}}\ } C_{\mathfrak{A}}$, of the functors $C_{\mathfrak{A}'_{\mathcal{P}}} \xrightarrow{\ \widetilde{\mathfrak{f}}^*_{\mathcal{P}}\ } C_{\mathfrak{A}}$, and

$$C_{\mathfrak{A}} \xrightarrow{\ \Psi_{\mathcal{P}}\ } C_{\mathfrak{A}}, \quad M \longmapsto M/tors_{\varphi_*^{-1}(\widehat{\mathcal{P}})}(M),$$

induces a morphism

$$\mathbf{Spec}^{\mathcal{P}}_{\mathfrak{c}}(\mathfrak{A}'_{\mathcal{P}}) \xrightarrow{\ \mathfrak{L}_{\mathcal{P}}\ } \mathbf{Spec}^{\mathcal{P}}_{\mathfrak{c}}(\mathfrak{A}). \qquad (5)$$

with the following property:

Every $[M] \in \mathbf{Spec}(\mathfrak{A})$ such that the image $\mathfrak{f}^*_{\mathcal{P}}(M)$ of M in $C_{\mathfrak{A}_{\mathcal{P}}}$ has an associated point from $\mathbf{Spec}^{\mathcal{P}}_{\mathfrak{c}}(\mathfrak{A}_{\mathcal{P}})$ belongs to the image of the map (1).

(b) The functor $\mathcal{L}_{\mathcal{P}}$ maps simple objects to simple objects.

Proof. The assertion follows from 5.2.2 and 5.2.3. ■

5.2.5. Proposition. *Suppose that the category C_X has the property (sup). Let $F_\varphi = \bigoplus_{\alpha\in\mathfrak{J}}\theta_\alpha$, where θ_α are autoequivalences of the category C_X.*

Suppose that an element $\mathcal{P}$ of $\mathfrak{S}_\omega(X)$ has a trivial stabilizer; i.e. $[\theta_\alpha(\mathcal{P})] = \mathcal{P}$ iff $\alpha = 0$ (here $\theta_0 = Id_{C_X}$). Then for every representative P of $\mathcal{P}$, the object $\mathcal{L}_{\mathcal{P}}(P) = \varphi^*(P)/tors_{\varphi_*^{-1}(\widehat{\mathcal{P}})}(P)$ belongs to $Spec^0_{\mathfrak{c}}(\mathfrak{A})$. If P is simple, then $\mathcal{L}_{\mathcal{P}}(P)$ is a simple object.

Proof. We adopt the notations of the part (i) of the argument of 5.2.3. Thanks to the property (sup), there exists a finite subset I of $J_{\mathcal{P}}$ such that the intersection $\widetilde{M} = M' \bigcap \big(\bigoplus_{\alpha \in I} \theta_\alpha(P) \big)$ is nonzero. Since $[\theta_\alpha(P)]_{\mathfrak{c}} = \mathcal{P}$ for every $\alpha \in I$, the object $\widetilde{M}$ belongs to $Spec^0_{\mathfrak{c}}(X)$ and $[\widetilde{M}]_{\mathfrak{c}} = \mathcal{P}$.

The assertion follows now from the observation 4.2.1 and Theorem 4.2. ■

5.3. Differential actions. For an abelian svelte category C_X, we denote by $\mathfrak{Der}_{\mathfrak{c}}(C_X)$ the full subcategory of the category $End(C_X)$ generated by all continuous exact differential endofunctors. Since the composition of differential endofunctors is a differential endofunctor, $\mathfrak{Der}_{\mathfrak{c}}(C_X)$ is a full monoidal subcategory of the monoidal category $\widetilde{End}(C_X)$.

We call an action $\widetilde{\Phi} = (\Phi, \phi, \phi_0)$ of a svelte monoidal category $\widetilde{\mathcal{E}} = (\mathcal{E}, \odot, \mathbb{I}, a; \ell, \mathfrak{r})$ on C_X *differential* if the functor Φ takes values in the subcategory $\mathfrak{Der}_{\mathfrak{c}}(C_X)$.

We assume until the end of the section that C_X is a Grothendieck category. This implies that C_X has small limits and colimits. Therefore, every continuous action $\widetilde{\Phi}$ of a svelte monoidal category $\widetilde{\mathcal{E}}$ has a colimit, $\mathcal{F}_\varphi = (F_\varphi, \mu_\varphi)$, which is a continuous monad. As in 4.1, we replace the monoidal category $\widetilde{\mathcal{E}}$ by its image in $\widetilde{\mathfrak{Ex}}(C_X)/\mathcal{F}_\varphi$ (determined by the universal cone $\widetilde{\Phi} \xrightarrow{\gamma_\varphi} \mathcal{F}_\varphi$) and identify the monoidal functor $\widetilde{\Phi}$ with the composition of the inclusion functor $\widetilde{\mathcal{E}} \longrightarrow \widetilde{\mathfrak{Ex}}(C_X)/\mathcal{F}_\varphi$ and the forgetful functor $\widetilde{\mathfrak{Ex}}(C_X)/\mathcal{F}_\varphi \longrightarrow \widetilde{\mathfrak{Ex}}(C_X)$.

If the action $\widetilde{\Phi}$ is differential, then $\widetilde{\mathcal{E}}$ is identified with a monoidal subcategory of $\widetilde{\mathfrak{Der}}(C_X)/\mathcal{F}_\varphi$ and the action $\widetilde{\Phi}$ with the restriction to $\widetilde{\mathcal{E}}$ of the forgetful monoidal functor $\widetilde{\mathfrak{Der}}(C_X)/\mathcal{F}_\varphi \longrightarrow \widetilde{\mathfrak{Der}}(C_X)$. In this case, the monad $\mathcal{F}_\varphi = (F_\varphi, \mu_\varphi)$ (that is the functor $F_\varphi = \varphi_*\varphi^*$) is differential (see A1.6, A1.7). Or, in other words, the affine morphism $\mathfrak{A} = \mathbf{Sp}(\mathcal{F}_\varphi/X) \xrightarrow{\varphi} X$ is differential.

For every $\mathcal{P} \in \mathbf{Spec}^0_{\mathfrak{c}}(X)$, we have a commutative diagram of affine morphisms

$$\begin{array}{ccc} \mathfrak{A} = \mathbf{Sp}(\mathcal{F}_\varphi/X) & \xrightarrow{\mathfrak{f}_{\mathcal{P}}} & \mathbf{Sp}(\mathcal{F}_{\varphi_{\mathcal{P}}}/X) = \mathfrak{A}_{\mathcal{P}} \\ {\scriptstyle\varphi} \searrow & & \swarrow {\scriptstyle\varphi_{\mathcal{P}}} \\ & X & \end{array}$$

corresponding to a monad morphism $\mathcal{F}_{\varphi_{\mathcal{P}}} \xrightarrow{\psi_{\mathcal{P}}} \mathcal{F}_\varphi$, where the 'space' $\mathfrak{A}_{\mathcal{P}}$ and the monad $\mathcal{F}_{\varphi_{\mathcal{P}}}$ (more precisely, the monad morphism $\psi_{\mathcal{P}}$) are *stabilizers* of the point $\mathcal{P}$ (see 4.1.1).

Therefore we have a well defined functor $C_{\mathfrak{A}_{\mathcal{P}}} \xrightarrow{\mathfrak{L}_{\mathcal{P}}} C_{\mathfrak{A}}$, which is the composition of $C_{\mathfrak{A}_{\mathcal{P}}} \xrightarrow{\mathfrak{f}^*_{\mathcal{P}}} C_{\mathfrak{A}}$, and the functor

$$C_{\mathfrak{A}} \xrightarrow{\Psi_{\mathcal{P}}} C_{\mathfrak{A}}, \quad M \longmapsto M/tors_{\varphi_*^{-1}(\widehat{\mathcal{P}})}(M).$$

Following the pattern of 4.3, consider the commutative diagram

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{u_{\mathcal{P}}} & X_{\mathcal{P}} \\ {\scriptstyle\tilde{\mathfrak{f}}_{\mathcal{P}}} \searrow & & \nearrow {\scriptstyle\tilde{\varphi}_{\mathcal{P}}} \\ & \mathfrak{A}'_{\mathcal{P}} & \end{array} \qquad (1)$$

associated with

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{\mathfrak{f}_{\mathcal{P}}} & \mathfrak{A}_{\mathcal{P}} \\ \varphi \searrow & & \swarrow \varphi_{\mathcal{P}} \\ & X & \end{array}$$

Notice that the composition $\widetilde{\mathfrak{f}}^*_{\mathcal{P}}$ of the inclusion functor $C_{\mathfrak{A}'_{\mathcal{P}}} \longrightarrow C_{\mathfrak{A}_{\mathcal{P}}}$ and the functor $C_{\mathfrak{A}_{\mathcal{P}}} \xrightarrow{\mathfrak{f}^*_{\mathcal{P}}} C_{\mathfrak{A}}$ is a left adjoint to the functor $C_{\mathfrak{A}} \xrightarrow{\widetilde{\mathfrak{f}}_{\mathcal{P}*}} C_{\mathfrak{A}'_{\mathcal{P}}}$.

5.3.1. Lemma. *The functors $u^*_{\mathcal{P}}$ and $\widetilde{\mathfrak{f}}^*_{\mathcal{P}}$ take values in the full subcategory $C_{\mathfrak{A}[\mathcal{P}^-]}$ of the category $C_{\mathfrak{A}}$ formed by all $\mathcal{F}_{\varphi}$-modules (M,ξ) such that $M \in Ob\mathcal{P}^-$.*

Proof. Recall that $\mathcal{P}^-$ is the smallest Serre subcategory containing $\mathcal{P}$.

The assertion is due to the fact that every differential endofunctor of the category C_X preserves every Serre subcategory of C_X ([LR1]). A more detailed argument is as follows.

(a) The subcategory $C_{\mathfrak{A}[\mathcal{P}^-]}$ coincides with the preimage, $\varphi_*^{-1}(\mathcal{P}^-)$ of a Serre subcategory. Therefore it is a Serre subcategory, because the functor φ_* preserves small colimits.

(b) The functor $u^*_{\mathcal{P}}$ is a restriction of the functor $C_X \xrightarrow{\varphi^*} C_{\mathfrak{A}},\ L \longmapsto (F_{\varphi}(L), \mu_{\varphi}(L))$, to the subcategory $C_{X_{\mathcal{P}}} = \mathcal{P}$. By hypothesis, the monad $\mathcal{F}_{\varphi} = (F_{\varphi}, \mu_{\varphi})$ (i.e. the functor $F_{\varphi} = \varphi_*\varphi^*$) is differential, hence $F_{\varphi}(L)$ is an object of $\mathcal{P}^-$ for every $L \in Ob\mathcal{P}^-$, in particular, for every $L \in Ob\mathcal{P}$.

(c) It follows from the construction of the functor $C_{\mathfrak{A}_{\mathcal{P}}} \xrightarrow{\mathfrak{f}^*_{\mathcal{P}}} C_{\mathfrak{A}}$ that for every $\widetilde{\widetilde{\mathfrak{F}}}$-module $\mathcal{M} = (M, \widetilde{\xi})$ (– an object of the category $C_{\mathfrak{A}_{\mathcal{P}}}$), there is an $\mathcal{F}_{\varphi}$-module epimorphism $\varphi^*(M) = (F_{\varphi}(M), \mu_{\varphi}(M)) \longrightarrow \mathfrak{f}^*_{\mathcal{P}}(\mathcal{M})$. Since, by (c), $\varphi^*(M)$ is an object of the Serre subcategory $C_{\mathfrak{A}[\mathcal{P}^-]}$, its quotient object $\mathfrak{f}^*_{\mathcal{P}}(\mathcal{M})$ belongs to the subcategory $C_{\mathfrak{A}[\mathcal{P}^-]}$ too. ■

The diagram (1) can be decomposed into a commutative diagram

$$\begin{array}{ccc} \mathfrak{A}[\mathcal{P}^-] & \xrightarrow{u_{\mathcal{P}}} & X_{\mathcal{P}} \\ \mathfrak{j}_{\mathcal{P}} \uparrow & & \uparrow \widetilde{\varphi}_{\mathcal{P}} \\ \mathfrak{A} & \xrightarrow{\widetilde{\mathfrak{f}}_{\mathcal{P}}} & \mathfrak{A}'_{\mathcal{P}} \end{array} \qquad (2)$$

Consider now the category $C_{\mathfrak{A}_{\mathcal{P}}}$. Its objects are pairs (M,ξ), where M is an object of the category $\mathcal{P}$ and ξ is an action of the differential monad $\mathcal{F}_{\varphi_{\mathcal{P}}}$ – the stabilizer of $\mathcal{P}$.

5.3.2. Proposition. *Let $C_{\mathcal{Y}_{\mathcal{P}}}$ be the full subcategory of the category $C_{\mathfrak{A}'_{\mathcal{P}}}$ formed by all (M,ξ) such that $M \in Ob\widehat{\mathcal{P}}$. Then $C_{\mathcal{Y}_{\mathcal{P}}}$ is a Serre subcategory of $C_{\mathfrak{A}'_{\mathcal{P}}}$ and $Spec(\mathfrak{A}'_{\mathcal{P}}) = Spec(\mathcal{Y}_{\mathcal{P}}) \coprod Spec^{\mathcal{P}}_{\mathfrak{c}}(\mathfrak{A}'_{\mathcal{P}})$. In particular, $\mathbf{Spec}(\mathfrak{A}'_{\mathcal{P}}) = \mathbf{Spec}(\mathcal{Y}_{\mathcal{P}}) \coprod \mathbf{Spec}^{\mathcal{P}}_{\mathfrak{c}}(\mathfrak{A}'_{\mathcal{P}})$.*

Proof (a) Let (M,ξ) be an object of $Spec(\mathfrak{A}'_{\mathcal{P}})$. Then either the object M is $\widehat{\mathcal{P}}$-torsion free, or $M \in Ob\widehat{\mathcal{P}}$, or, equivalently, $(M,\xi) \in ObC_{\mathcal{Y}_{\mathcal{P}}}$.

In fact, let $M_{\widehat{\mathcal{P}}}$ denote the $\widehat{\mathcal{P}}$-torsion of M. Any differential endofunctor of a category preserves all Serre subcategories of this category (see A1.7.3). Since objects of the monoidal category $\widetilde{\mathcal{E}}$, in particular objects of its subcategory $\widetilde{\mathcal{E}}_{(\mathcal{P})}$, are pairs $(U, U \to F_{\varphi})$, where U

is a differential endofunctor, the $\widehat{\mathcal{P}}$-torsion $M_{\widehat{\mathcal{P}}}$ of M is a submodule of the $\mathcal{F}_{\varphi_{\mathcal{P}}}$-module (M,ξ). Since (M,ξ) belongs to the spectrum, either $M_{\widehat{\mathcal{P}}} = 0$, or $[(M_{\widehat{\mathcal{P}}},\xi')]_{\mathfrak{c}} \supseteq [(M,\xi)]_{\mathfrak{c}}$. Here ξ' denotes the induced $\mathcal{F}_{\varphi_{\mathcal{P}}}$-module structure. Thanks to the exactness of the forgetful functor $\varphi_{\mathcal{P}*}$, the latter implies that $[M_{\widehat{\mathcal{P}}}]_{\mathfrak{c}} \supseteq [M]_{\mathfrak{c}}$, hence $M \in Ob\widehat{\mathcal{P}}$, i.e. $M = M_{\widehat{\mathcal{P}}}$.

(b) Let (M,ξ) belong to $Spec(\mathfrak{A}'_{\mathcal{P}}) - Spec(\mathcal{Y}_{\mathcal{P}})$. By (a), this implies that M is an object of the subcategory $\mathcal{P}\bigcap\widehat{\mathcal{P}}^{\perp}$ formed by $\widehat{\mathcal{P}}$-torsion free objects of the $\mathcal{P}$. It follows that M has a nonzero subobject, $L \hookrightarrow M$, with $L \in Ob\mathcal{P}\bigcap\widehat{\mathcal{P}}^{\perp}$. Pick a representative, P', of $\mathcal{P}$. The inclusion $L \in Ob\mathcal{P}$ means that $[P']_{\mathfrak{c}} \supseteq [L]_{\mathfrak{c}}$. The fact that $L \notin Ob\widehat{\mathcal{P}}$ means precisely that $[L]_{\mathfrak{c}} \supseteq [P']_{\mathfrak{c}}$. Every nonzero subobject L' of L has the same properties: $[P']_{\mathfrak{c}} \supseteq [L']_{\mathfrak{c}} \supseteq [P']_{\mathfrak{c}}$. Therefore $[L']_{\mathfrak{c}} \supseteq [L]_{\mathfrak{c}}$. This shows that L belongs to the spectrum $Spec^0_{\mathfrak{c}}(X)$ and $[L]_{\mathfrak{c}} = [P']_{\mathfrak{c}} = \mathcal{P}$.

The argument above shows that every nonzero object M of $\mathcal{P}\bigcap\widehat{\mathcal{P}}^{\perp}$ is a representative of the point $\mathcal{P}$. In particular, $Ass(M) = \{\mathcal{P}\}$. ∎

Now we shall make some observations related to the diagram (2) and the construction of the functor $\mathcal{L}_{\mathcal{P}}$.

Recall that an object M of the category C_X is called *$\mathcal{P}$-primary* if $Ass(M) = \{\mathcal{P}\}$.

5.3.3. Proposition. *Let $\mathbb{T}_{\mathcal{P}}$ denote the preimage $\varphi_*^{-1}(\widehat{\mathcal{P}})$ of the Serre subcategory $\widehat{\mathcal{P}}$ of C_X in $C_{\mathfrak{A}}$; and let $\mathcal{T}_{\mathcal{P}}$ denote the preimage in $C_{\mathfrak{A}[\mathcal{P}-]}$ of the subcategory $\widehat{\mathcal{P}} \cap C_{X_{\mathcal{P}}}$ (cf. the diagram (2)).*

(a) An object $\mathcal{M} = (M,\xi)$ of $C_{\mathfrak{A}} = (\mathcal{F}_{\varphi}/X) - mod$ is $\mathbb{T}_{\mathcal{P}}$-torsion free iff the object $\varphi_*(\mathcal{M}) = M$ is $\widehat{\mathcal{P}}$-torsion free.

(b) The image in $C_{X_{\mathcal{P}}}$ of every $\mathcal{T}_{\mathcal{P}}$-torsion free object of $C_{\mathfrak{A}[\mathcal{P}-]}$ is $\mathcal{P}$-primary.

Proof. (a) Let $\mathcal{M} = (M,\xi)$ be an $(\mathcal{F}_{\varphi}/X)$-module, and let $M_{\widehat{\mathcal{P}}}$ denote the $\widehat{\mathcal{P}}$-torsion of the object M. Since $\mathcal{F}_{\varphi} = (F_{\varphi},\mu_{\varphi})$, where F_{φ} is a differential functor, and all differential functors preserve Serre subcategories, the action $F_{\varphi}(M) \xrightarrow{\xi} M$ induces an action, ξ', of F_{φ} on the subobject $M_{\widehat{\mathcal{P}}}$. Clearly, $(M_{\widehat{\mathcal{P}}},\xi')$ belongs to the Serre subcategory $\mathbb{T}_{\mathcal{P}}$.

(b) By definition, $C_{\mathfrak{A}[\mathcal{P}-]}$ is a full subcategory of $C_{\mathfrak{A}}$ generated by $\mathcal{F}_{\varphi}$-modules (M,ξ) such that $M \in Ob\mathcal{P}^-$. Therefore, by (a), an object (M,ξ) of $C_{\mathfrak{A}[\mathcal{P}-]}$ is $\mathcal{T}_{\mathcal{P}}$-torsion free iff M is an object of $\mathcal{P}^-\bigcap\widehat{\mathcal{P}}^{\perp}$. If M is nonzero, it contains (by the definition of $\mathcal{P}^-$) a nonzero subobject L which belongs to $\mathcal{P}\bigcap\widehat{\mathcal{P}}^{\perp}$. But, nonzero objects of $\mathcal{P}\bigcap\widehat{\mathcal{P}}^{\perp}$ are precisely all the representatives of $\mathcal{P}$ (see the part (b) of the argument of 5.3.2). This shows that $\mathcal{P} \in Ass(M)$.

Suppose $N \hookrightarrow M$ is a subobject of M such that $N \in Spec(X)$. Then N has a nonzero subobject L which belongs to $\mathcal{P}\bigcap\widehat{\mathcal{P}}^{\perp}$. Therefore $[N]_{\mathfrak{c}} = [L]_{\mathfrak{c}} = \mathcal{P}$; i.e. $\mathcal{P}$ is the only element of $Ass(M)$. ∎

5.3.4. Proposition. *For every object $\mathcal{M}$ of $Spec^0_{\mathfrak{c}}(\mathfrak{A}[\mathcal{P}^-])$, its image in $C_{X_{\mathcal{P}}}$ either belongs to $\widehat{\mathcal{P}}$, or is $\mathcal{P}$-primary.*

Proof. Let $\mathcal{M} = (M,\xi)$ belong to $Spec(\mathfrak{A})$. By the argument of 5.3.3, the $\widehat{\mathcal{P}}$-torsion, $M_{\widehat{\mathcal{P}}}$, of the object M has a structure, ξ' of a submodule of $\mathcal{M}$. Therefore, if $M_{\widehat{\mathcal{P}}} \neq 0$, then

$[(M_{\widehat{\mathcal{P}}},\xi')]_{\mathfrak{c}} \supseteq [(M,\xi)]_{\mathfrak{c}}$ which implies that $M = M_{\widehat{\mathcal{P}}}$ (see the part (a) of the argument of 5.3.2). If $M_{\widehat{\mathcal{P}}} = 0$, then, by 5.3.3(b), the object M is $\mathcal{P}$-primary. ∎

5.3.5. Proposition. *The functor $C_{\mathfrak{A}'_{\mathcal{P}}} \xrightarrow{\mathcal{L}_{\mathcal{P}}} C_{\mathfrak{A}}$ takes values in the full subcategory of $C_{\mathfrak{A}}$ generated by $\mathcal{F}_\varphi$-modules (M,ξ) such that M is an object of the category $\mathcal{P}^- \bigcap \widehat{\mathcal{P}}$. In particular, M is either zero, or $\mathcal{P}$-primary.*

Proof. Recall that the functor $C_{\mathfrak{A}_{\mathcal{P}}} \xrightarrow{\mathfrak{L}_{\mathcal{P}}} C_{\mathfrak{A}}$ is the composition of a left adjoint, $C_{\mathfrak{A}_{\mathcal{P}}} \xrightarrow{\mathfrak{f}^*_{\mathcal{P}}} C_{\mathfrak{A}}$, the forgetful functor $C_{\mathfrak{A}} \xrightarrow{\mathfrak{f}_{\mathcal{P}*}} C_{\mathfrak{A}_{\mathcal{P}}}$ and the functor

$$C_{\mathfrak{A}} \xrightarrow{\Psi_{\mathcal{P}}} C_{\mathfrak{A}}, \quad M \longmapsto M/tors_{\varphi_*^{-1}(\widehat{\mathcal{P}})}(M). \tag{3}$$

By 5.3.3(a), the functor (3) takes values in the full subcategory of $C_{\mathfrak{A}}$ generated by all $\mathcal{F}_\varphi$-modules (M,ξ) such that M is a $\widehat{\mathcal{P}}$-torsion free object of C_X.

The functor $C_{\mathfrak{A}'_{\mathcal{P}}} \xrightarrow{\mathcal{L}_{\mathcal{P}}} C_{\mathfrak{A}}$ is the composition of the functor $C_{\mathfrak{A}_{\mathcal{P}}} \xrightarrow{\mathfrak{L}_{\mathcal{P}}} C_{\mathfrak{A}}$ and the inclusion functor $C_{\mathfrak{A}'_{\mathcal{P}}} \longrightarrow C_{\mathfrak{A}_{\mathcal{P}}}$; that is $\mathcal{L}_{\mathcal{P}}$ is the composition of the three functors

$$C_{\mathfrak{A}'_{\mathcal{P}}} \longrightarrow C_{\mathfrak{A}_{\mathcal{P}}} \xrightarrow{\mathfrak{f}^*_{\mathcal{P}}} C_{\mathfrak{A}} \xrightarrow{\Psi_{\mathcal{P}}} C_{\mathfrak{A}}.$$

The composition of the first two functors takes values (thanks to the fact that F_φ is differential) in the subcategory $C_{\mathfrak{A}[\mathcal{P}^-]} = \varphi_*^{-1}(\mathcal{P}^-)$. Therefore the functor $\mathcal{L}_{\mathcal{P}}$ takes values in the preimage in $C_{\mathfrak{A}}$ of the subcategory $\mathcal{P}^- \bigcap \widehat{\mathcal{P}} \subseteq C_X$, which is the full subcategory of $C_{\mathfrak{A}}$ formed by all $\mathcal{F}_\varphi$-modules (M,ξ) such that M is an object of $\mathcal{P}^- \bigcap \widehat{\mathcal{P}}$. In particular, M is either zero, or $\mathcal{P}$-primary. ∎

5.3.6. Localization. All exact differential endofunctors are compatible with localizations at Serre subcategories and induce exact differential endofunctors on the corresponding quotient categories (cf. A1.7.3). These endofunctors on quotient categories inherit *exactness* properties (like compatibility with limits or colimits of a certain class of diagrams, or having a right adjoint) of the initial endofunctors (see [KR2]). Thus, localization at any Serre subcategory $\mathbb{S}$ of the category C_X will transform our data (differential continuous monad (F_φ,μ_φ) and the family of exact continuous differential subfunctors of F_φ) to the same sort of data on $C_{X/\mathbb{S}}$. Taking an element $\mathcal{P}$ of the spectrum of $X/\mathbb{S}$, we obtain a relative version of the commutative diagram (1):

$$\begin{array}{ccc} C_{\mathfrak{A}/\mathbb{S}''} & \xrightarrow{u_{\mathcal{P}*}} & C_{(X/\mathbb{S})_{\mathcal{P}}} \\ {\scriptstyle \widetilde{\mathfrak{f}}^*_{\mathcal{P}}} \searrow & & \nearrow {\scriptstyle \widetilde{\varphi}^*_{\mathcal{P}}} \\ & C_{(\mathfrak{A}/\mathbb{S})'_{\mathcal{P}}} & \end{array} \tag{4}$$

where $\mathbb{S}'' = \varphi_*^{-1}(\mathbb{S})$ and $\mathbb{S}' = \varphi_{\mathcal{P}}^{*^{-1}}(\mathbb{S})$. The category $C_{\mathfrak{A}/\mathbb{S}''}$ here is naturally identified with the category $\mathcal{F}_{\mathbb{S}}$-modules, where $\mathcal{F}_{\mathbb{S}}$ is the monad on $C_{X/\mathbb{S}}$ uniquely determined by the monad $\mathcal{F}_\varphi = (F_\varphi,\mu_\varphi)$.

Applying this observation to the Serre subcategory $\widehat{\mathcal{P}}$ and the unique closed point of the quotient local category $C_{X/\widehat{\mathcal{P}}}$, we replace X by the local 'space' $X/\widehat{\mathcal{P}}$ and obtain (using the decomposition (2) in 5.3.1) the diagram

$$\begin{array}{ccc} C_{\mathfrak{A}_r[\mathcal{P}-]} & \xrightarrow{u_{\mathcal{P}^*}} & C_{X^r_{\mathcal{P}}} \\ \widetilde{\mathfrak{f}}^*_{\mathcal{P}} \searrow & & \nearrow \widetilde{\varphi}^*_{\mathcal{P}} \\ & C_{\mathfrak{A}^r_{\mathcal{P}}} & \end{array} \qquad (5)$$

in which $X^r_{\mathcal{P}}$ is the residue 'space' of X at the point $\mathcal{P}$, $C_{\mathfrak{A}^r_{\mathcal{P}}}$ is the category of $\widetilde{\mathfrak{F}}$-modules $(L, \widetilde{\xi})$, where L is an object of the residue category $C_{X^r_{\mathcal{P}}}$, $C_{\mathfrak{A}_r[\mathcal{P}-]}$ is the category of $\mathcal{F}_{\widehat{\mathcal{P}}}$-modules (M, ξ), where M is an object of the smallest nonzero Serre subcategory of $C_{X/\widehat{\mathcal{P}}}$.

If the local category $C_{X/\widehat{\mathcal{P}}}$ has simple objects (which is always the case if X has a Gabriel-Krull dimension) and C_X has infinite coproducts, then the residue category is equivalent to the category of vector spaces over the residue field $k_{\mathcal{P}}$ of the point $\mathcal{P}$.

6. Computing $\mathbf{Spec}_-(X)$.

6.1. The construction. We assume the setting of 4.1. That is we fix a Grothendieck category C_X endowed with an action $\widetilde{\Phi}$ of a svelte monoidal category $\widetilde{\mathcal{E}}$ taking values in the monoidal category $\widetilde{\mathfrak{Ex}}_{\mathfrak{c}}(C_X)$ of exact continuous endofunctors of C_X. As in 4.1, we identify the monoidal category $\widetilde{\mathcal{E}}$ with its image in $\widetilde{\mathfrak{Ex}}_{\mathfrak{c}}(C_X)/\mathcal{F}_\varphi$, where the continuous monad $\mathcal{F}_\varphi = (F_\varphi, \mu_\varphi)$ is the colimit of the monoidal functor $\widetilde{\mathcal{E}} \xrightarrow{\widetilde{\Phi}} \widetilde{\mathfrak{Ex}}_{\mathfrak{c}}(C_X)$. With this identification, $\widetilde{\Phi}$ becomes the restriction to $\widetilde{\mathcal{E}}$ of the forgetful monoidal functor $\widetilde{\mathfrak{Ex}}_{\mathfrak{c}}(C_X)/\mathcal{F}_\varphi \longrightarrow \widetilde{\mathfrak{Ex}}_{\mathfrak{c}}(C_X)$.

Fix an element $\mathcal{P}$ of $\mathbf{Spec}^{\mathfrak{c}}_-(X)$. Applying the pattern of 4.1.1 to $\mathcal{P}$, we obtain the *stabilizer* of $\mathcal{P}$ which is, by definition, the stabilizer $\widetilde{\mathcal{E}}_{(\mathcal{P})}$ of the pair $(\mathcal{P}) = \{\mathcal{P}, \widehat{\mathcal{P}}\}$, and the commutative diagram of affine morphisms

$$\begin{array}{ccc} \mathbf{Sp}(\mathcal{F}_\varphi/X) = \mathfrak{A} & \xrightarrow{\mathfrak{f}_{\mathcal{P}}} & \mathfrak{A}_{\mathcal{P}} = \mathbf{Sp}(\mathcal{F}_{\varphi_{\mathcal{P}}}/X) \\ \varphi \searrow & & \swarrow \varphi_{\mathcal{P}} \\ & X & \end{array} \qquad (1)$$

corresponding to a monad morphism $\mathcal{F}_{\varphi_{\mathcal{P}}} \xrightarrow{\psi_{\mathcal{P}}} \mathcal{F}_\varphi$, where the 'space' $\mathfrak{A}_{\mathcal{P}}$ and the monad $\mathcal{F}_{\varphi_{\mathcal{P}}}$ (or, more precisely, the monad morphism $\psi_{\mathcal{P}}$) are called *stabilizers* of the point $\mathcal{P}$.

6.2. $\mathbf{Spec}^{\mathfrak{c}}_-(\mathfrak{A}_{\mathcal{P}})_{\mathcal{P}}$ and $\mathbf{Spec}^{\mathfrak{c}}_-(\mathfrak{A})_{\mathcal{P}}$. For an element $\mathcal{P}$ of $\mathbf{Spec}^{\mathfrak{c}}_-(X)$, we denote by $Spec^{\mathfrak{c}}_-(\mathfrak{A}_{\mathcal{P}})_{\mathcal{P}}$ the family of all objects $\widetilde{P}$ of $Spec^{\mathfrak{c}}_-(\mathfrak{A}_{\mathcal{P}})$ such that $\mathcal{P} \in Ass(\varphi^*_{\mathcal{P}}(\widetilde{P}))$. We denote by $\mathbf{Spec}^{\mathfrak{c}}_-(\mathfrak{A}_{\mathcal{P}})_{\mathcal{P}}$ the correponding subset of $\mathbf{Spec}^{\mathfrak{c}}_-(\mathfrak{A}_{\mathcal{P}})$.

Similarly, $Spec^{\mathfrak{c}}_-(\mathfrak{A})_{\mathcal{P}}$ will denote the family of all objects M of $Spec^{\mathfrak{c}}_-(\mathfrak{A})$ such that $\mathcal{P}$ is an associated point of $\varphi_*(M)$, and denote by $\mathbf{Spec}^{\mathfrak{c}}_-(\mathfrak{A})_{\mathcal{P}}$ the corresponding subset of the spectrum $\mathbf{Spec}^{\mathfrak{c}}_-(\mathfrak{A})$.

6.3. Theorem. *Let $\mathcal{P} \in \mathbf{Spec}_-(X)$ be such that the inverse image functor $\mathfrak{f}^*_{\mathcal{P}}$ of the morphism $\mathfrak{A} \xrightarrow{\mathfrak{f}_{\mathcal{P}}} \mathfrak{A}_{\mathcal{P}}$ is exact and faithful, and the following conditions hold:*

(*) If P is a representative of $\mathcal{P}$ and M is a subobject of $\varphi^*(P)$ such that $\mathcal{P} \in Supp(\varphi_*(M))$, then there exists $(U', \mathfrak{v}) \in \mathfrak{F}_{\mathcal{P}}$ and a subobject P' of P such that the image of $U'(P')$ in $F_\varphi(P) = \varphi_*\varphi^*(P)$ is a subobject of $\varphi_*(M)$ whose support contains $\mathcal{P}$.

Then the functor $C_{\mathfrak{A}_{\mathcal{P}}} \xrightarrow{\mathfrak{L}_{\mathcal{P}}} C_{\mathfrak{A}}$ induces a surjective morphism

$$\mathbf{Spec}^{\mathfrak{c}}_{-}(\mathfrak{A}_{\mathcal{P}})_{\mathcal{P}} \xrightarrow{\mathfrak{L}_{\mathcal{P}}} \mathbf{Spec}^{\mathfrak{c}}_{-}(\mathfrak{A})_{\mathcal{P}}. \tag{1}$$

The functor $\mathfrak{L}_{\mathcal{P}}$ maps simple objects to simple objects.

Proof. The argument is similar to the proof of 4.2. Details are left to the reader. ■

6.4. Finiteness conditions.

6.4.1. Associated points of finite multiplicity. Let M be an object of C_X, and let $\mathcal{P} \in \mathbf{Spec}^{\mathfrak{c}}_{-}(X)$ be an associated point of M; i.e. M has a nonzero subobject which belongs to $\widehat{\mathcal{P}}^{\mathfrak{c}}_{\circledast} = \mathcal{P} \cap \widehat{\mathcal{P}}^{\perp}$. We say that the associated point $\mathcal{P}$ has a *finite multiplicity* if the $\widehat{\mathcal{P}}^{\circledast}_{\mathfrak{c}}/\widehat{\mathcal{P}}$-torsion of M belongs to $Spec(X/\widehat{\mathcal{P}})$.

If the quotient category $C_X/\widehat{\mathcal{P}}$ has simple objects, then the $\widehat{\mathcal{P}}^{\circledast}_{\mathfrak{c}}/\widehat{\mathcal{P}}$-torsion of the image of M in $C_X/\widehat{\mathcal{P}}$ coincides with its socle. The point $\mathcal{P}$ is of finite multiplicity in M iff this socle is of finite length. The latter is called *the multiplicity of $\mathcal{P}$ in M.*

6.4.2. Points of the spectrum finite over a point. Let $\mathfrak{A} \xrightarrow{\varphi} X$ be an affine morphism and $\mathcal{P}$ a point of $\mathbf{Spec}^{-}_{\mathfrak{c}}(X)$. It is not garanteed, in general, that $Spec^{-}_{\mathcal{P}}(\mathfrak{A}_{\mathcal{P}})$ is nonempty. We denote by $Spec^{0}_{\mathcal{P},\mathfrak{f}}(\mathfrak{A})$ the preorder of all $M \in Spec^{\mathfrak{c}}_{-}(\mathfrak{A})$ such that $\mathcal{P}$ is an associated point of $\varphi_*(M)$ of finite multiplicity.

6.4.2.1. Proposition. *Suppose that $\mathcal{P} \in \mathbf{Spec}^{-}_{\mathfrak{c}}(X)$ is such that the category $C_X/\mathcal{P}$ has simple objects. Then for every $M \in Spec^{0}_{\mathcal{P},\mathfrak{f}}(\mathfrak{A})$, the object $f_{\mathcal{P}*}(M)$ has a subobject $\widetilde{P}$ which belongs to $Spec^{\mathcal{P}}_{\mathfrak{c}}(\mathfrak{A}_{\mathcal{P}})$. For any such object $\widetilde{P}$, the corresponding element of $\mathbf{Spec}^{-}_{\mathfrak{c}}(\mathfrak{A}_{\mathcal{P}})$ is an associated point of $f_{\mathcal{P}*}(M)$ of finite multiplicity, and $\mathcal{P}$ is an associated point of $\widetilde{P}$ of finite multiplicity.*

Proof. Consider the set $\Omega_{\mathcal{P}}$ of all subobjects L of $f_{\mathcal{P}*}(M)$ such that $\varphi_{\mathcal{P}*}(L)$ is a representative of $\mathcal{P}$. Those of them with the smallest rank of $\varphi_{\mathcal{P}*}(L)$ belong to $Spec^{\mathcal{P}}_{\mathfrak{c}}(\mathfrak{A}_{\mathcal{P}})$. Details and the remaining observations are left to the reader. ■

6.5. Holonomic objects.

6.5.1. Definition. Let $\mathfrak{A} \xrightarrow{\varphi} X$ be a continuous morphism. We call an object M of the category $C_{\mathfrak{A}}$ *holonomic over X* (or, more precisely, *φ-holonomic*), if each nonzero subquotient of $\varphi_*(M)$ has associated points in $\mathbf{Spec}^{\mathfrak{c}}_{-}(X)$ and all these associated points are of finite multiplicity.

If C_X is the category of quasi-coherent sheaves on a smooth scheme $\mathcal{X}$ and $C_{\mathfrak{A}}$ is the category of D-modules on $\mathcal{X}$, then holonomic objects are precisely holonomic D-modules.

In the case C_X is the category of quasi-coherent sheaves on the quantum flag variety of a semisimple Lie algebra $\mathfrak{g}$ and $C_{\mathfrak{A}}$ is the category of quasi-coherent $U_q(\mathfrak{g})$-modules on X (cf. [LR2]), then holonomic objects are called *holonomic* quantum D-modules.

It follows from 6.4.2.1 that all holonomic objects over X which belong to $Spec^{\mathfrak{c}}_{-}(\mathfrak{A})$ are obtained via the construction of this work. Thanks to their functorial properties, the desctiption of holonomic objects is directly reduced to their description on an affine cover.

7. Local properties of spectra. Applications to D-modules on classical and quantum flag varieties.

7.1. Proposition. *Let $\{\mathcal{T}_i \mid i \in J\}$ be a set of coreflective thick subcategories of an abelian category C_X such that $\bigcap_{i\in J} \mathcal{T}_i = 0$; and let u_i^* denote the localization functor $C_X \longrightarrow C_X/\mathcal{T}_i$. The following conditions on a nonzero coreflective topologizing subcategory $\mathcal{Q}$ of C_X are equivalent:*

(a) $\mathcal{Q} \in \mathbf{Spec}^0_{\mathfrak{c}}(X)$,

(b) $[u_i^*(\mathcal{Q})]_{\mathfrak{c}} \in \mathbf{Spec}^0_{\mathfrak{c}}(X/\mathcal{T}_i)$ for every $i \in J$ such that $\mathcal{Q} \not\subseteq \mathcal{T}_i$.

Proof. See [R7, 10.4.3]. ∎

7.1.1. Note. The condition (b) of 7.1 can be reformulated as follows:
(b') For any $i \in J$, either $u_i^*(\mathcal{Q}) = 0$, or $[u_i^*(\mathcal{Q})]_{\mathfrak{c}} \in \mathbf{Spec}^0_{\mathfrak{c}}(X/\mathcal{T}_i)$.

7.2. Proposition. *Let C_X be an abelian category and $\mathfrak{U} = \{U_i \xrightarrow{u_i} X \mid i \in J\}$ a set of continous morphisms such that $\{C_X \xrightarrow{u_i^*} C_{U_i} \mid i \in J\}$ is a conservative family of exact localizations.*

(a) The morphisms $U_{ij} = U_i \cap U_j \xrightarrow{u_{ij}} U_i$ are continuous for all $i, j \in J$.

(b) Let L_i be an object of $Spec^0_{\mathfrak{c}}(U_i)$; i.e. $[L_i]_{\mathfrak{c}} \in \mathbf{Spec}^0_{\mathfrak{c}}(U_i)$ and L_i is $\langle L_i\rangle$-torsion free. The following conditions are equivalent:

(i) $L_i \simeq u_i^*(L)$ for some $L \in Spec^0_{\mathfrak{c}}(X)$;

(ii) for any $j \in J$ such that $u_{ij}^*(L_i) \neq 0$, the object $u_{ji*}u_{ij}^*(L_i)$ of C_{U_j} has an associated point; i.e. it has a subobject L_{ij} which belongs to $Spec^0_{\mathfrak{c}}(U_j)$.

Proof. The assertion follows from 7.1 (the argument is similar to that of [R7, 9.7.1]. ∎

7.2.1. Note. If the cover $\mathfrak{U} = \{U_i \xrightarrow{u_i} X \mid i \in J\}$ in 7.2 is finite, then $Spec^0_c(-)$ and $\mathbf{Spec}^0_c(-)$ can be replaced by resp. $Spec(-)$ and $\mathbf{Spec}(-)$.

7.2.2. Examples. (a) If C_X is the category of quasi-coherent sheaves on a quasi-separated scheme $\mathcal{X}$ and each U_i is the category of quasi-coherent sheaves on an open subscheme of $\mathcal{X}$, then the glueing conditions of 7.2 hold for any $L_i \in Spec^0_{\mathfrak{c}}(U_i)$; i.e. the spectrum $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ is naturally identified with $\bigcup_{i\in J} \mathbf{Spec}^0_{\mathfrak{c}}(U_i)$.

(b) Similarly, if C_X is the category of holonomic modules over a sheaf of twisted differential operators on a smooth scheme $\mathcal{X}$, and $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ is a cover of X corresponding to an open Zariski cover of $\mathcal{X}$, then $\bigcup_{i\in J} \mathbf{Spec}^0_{\mathfrak{c}}(U_i)$.

This is due to functoriality of sheaves of holonomic modules with respect to direct and inverse image functors of open immersions and the fact that holonomic modules are of finite length (hence they have associated closed points).

7.3. Proposition. *Let C_X be an abelian category and $\mathfrak{U} = \{U_i \xrightarrow{u_i} X \mid i \in J\}$ a finite set of morphisms of 'spaces' whose inverse image functors, $\{C_X \xrightarrow{u_i^*} C_{U_i} \mid i \in J\}$, form a conservative family of exact localizations, and $Ker(u_i^*)$ is a coreflective subcategory for every $i \in J$. Then $\mathbf{Spec}^-(X) = \bigcup_{i\in J} \mathbf{Spec}^-(U_i)$ and $\mathbf{Spec}^-_{\mathfrak{c}}(X) = \bigcup_{i\in J} \mathbf{Spec}^-_{\mathfrak{c}}(U_i)$.*

Proof. The first equality is proven in [R7, 9.5]. The argument for the second equality is similar to the proof of [R7, 9.5]. ∎

7.3.1. Proposition. *Let C_X be an abelian category and $\mathfrak{U} = \{U_i \xrightarrow{u_i} X \mid i \in J\}$ a set of continuous morphisms whose inverse image functors, $\{C_X \xrightarrow{u_i^*} C_{U_i} \mid i \in J\}$, form a conservative family of exact localizations. Suppose that $\mathbf{Spec}^-_{\mathfrak{c}}(X) = \bigcup_{i\in J} \mathbf{Spec}^-_{\mathfrak{c}}(U_i)$ (e.g. J is finite) and C_{U_i} is a Grothendieck category with a Gabriel-Krull dimension (for instance, U_i is locally noetherian; say $U_i \simeq \mathbf{Sp}(A_i)$ for a left noetherian ring) for each $i \in J$. Then $\mathbf{Spec}^-_{\mathfrak{c}}(X)$ is isomorphic to the set of isomorphism classes of indecomposable injectives of the category C_X.*

Proof. Each (isomorphism class of) indecomposable injective E of C_X corresponds to the element ${}^\perp E$ of $\mathbf{Spec}^-_{\mathfrak{c}}(X)$. Since direct image functors $C_{U_i} \xrightarrow{u_{i*}} C_X$ of morphisms u_i are right adjoints to exact functors, they map (indecomposable) injectives to (resp. indecomposable) injectives. For every 'space' Y such that C_Y is a Grothendieck category with a Gabriel-Krull dimension (in particular for each U_i), the isomorphism classes of indecomposable injectives are in bijective correspondence with elements of $\mathbf{Spec}^-_{\mathfrak{c}}(Y)$. ∎

7.4. Towards some applications. The assertions above allow to apply the results of the previous sections to *locally affine* morphisms; i.e. morphisms of 'spaces' $\mathfrak{A} \xrightarrow{\mathfrak{f}} X$ endowed with a set $\mathfrak{U} = \{U_i \xrightarrow{u_i} X \mid i \in J\}$ of morphisms such that $\{C_X \xrightarrow{u_i^*} C_{U_i} \mid i \in J\}$ is a conservative family of exact localizations whose kernels are coreflective subcategories of $C_{\mathfrak{A}}$, and for every $i \in J$, the compositions $\mathfrak{f} \circ u_i$ is an affine morphism.

A slightly more general setting we are interested in consists of a family of commutative diagrams

$$\begin{array}{ccc} \mathfrak{U}_i & \xrightarrow{\widetilde{\mathfrak{u}}_i} & \mathfrak{X} \\ {\scriptstyle \mathfrak{f}_i}\downarrow & & \downarrow{\scriptstyle \mathfrak{f}} \\ U_i & \xrightarrow{u_i} & X \end{array} \qquad i \in J, \tag{1}$$

where $\{C_X \xrightarrow{u_i^*} C_{U_i} \mid i \in J\}$ and $\{C_{\mathfrak{X}} \xrightarrow{\widetilde{\mathfrak{u}}_i^*} C_{\mathfrak{A}_i} \mid i \in J\}$ are conservative families of exact localizations with coreflective kernels and morphisms $\mathfrak{U}_i \xrightarrow{\mathfrak{f}_i} U_i$ are locally affine for all $i \in J$. Even when the morphism $\mathfrak{A} \xrightarrow{\mathfrak{f}} X$ is affine, the propositions 7.1 – 7.3.1 help to simplify the problem by using appropriate covers. In the examples below, the morphisms $\mathfrak{f}$ and $\mathfrak{f}_i$ are affine. We start with differential morphisms.

7.4.1. Affine differential morphisms. Let $\mathfrak{X} \xrightarrow{\mathfrak{f}} X$ be a differential affine morphism whose inverse image functor is exact. This means that the 'space' $\mathfrak{X}$ is naturally isomorphic to $\mathbf{Sp}(\mathcal{F}_{\mathfrak{f}}/X)$, where $\mathcal{F}_{\mathfrak{f}} = (F_{\mathfrak{f}}, \mu_{\mathfrak{f}})$ is the monad associated with $\mathfrak{f}$, and the functor $F_{\mathfrak{f}} = \mathfrak{f}_*\mathfrak{f}^*$ is exact, differential, and has a right adjoint.

Let $U \xrightarrow{u} X$ be a flat (i.e. continuous and exact) localization, and let $C_X \xrightarrow{F} C_X$ be an exact differential functor. Then there exists a unique exact differential functor $C_U \xrightarrow{F_U} C_U$ such that $u^* \circ F = F_U \circ u^*$. The functor F_U is naturally isomorphic to the composition u^*Fu_*. If the functor F is continuous, i.e. it has a right adjoint, $F^!$, then the functor F_U is continuous too: the composition $F_U^! = u^*F^!u_*$ is a right adjoint to F_U.

Let $\mathfrak{U} = \{U_i \xrightarrow{u_i} X \mid i \in J\}$ be a set of continuous morphisms whose inverse image functors $\{C_X \xrightarrow{u_i^*} C_{U_i} \mid i \in J\}$ form a conservative family of exact localizations. Then it follows from the discussion above and A1.5 that the differential affine morphism $\mathfrak{X} \xrightarrow{\mathfrak{f}} X$ gives rise to a uniquely determined commutative diagram (1) in which all morphisms $\mathfrak{f}_i$ are affine and differential.

7.4.2. Quasi-coherent sheaves of rings. Let $\mathcal{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a commutative scheme such that the embedding of each point of $\mathcal{X}$ into $\mathcal{X}$ has a direct image functor (e.g. $\mathcal{X}$ is quasi-separated). This condition implies that the scheme $\mathcal{X}$ can be canonically reconstructed (is naturally isomorphic to the *geometric center* of) the category $C_X = Qcoh_{\mathcal{X}}$ of quasi-coherent sheaves on $\mathcal{X}$. Let $\mathcal{A}_{\mathcal{X}}$ be a quasi-coherent sheaf of associative unital rings on $\mathcal{X}$ and $C_{\mathfrak{X}}$ the category of quasi-coherent sheaves of $\mathcal{A}_X$-modules. Let $\mathcal{O}_X \xrightarrow{\psi} \mathcal{A}_X$ be a morphism of sheaves of rings. The morphism ψ gives rise to an affine morphism $\mathfrak{X} \xrightarrow{\mathfrak{f}} X$ of 'spaces'. Fix an affine cover $\{\mathcal{U}_i \xrightarrow{\mathfrak{u}_i} \mathcal{X} \mid i \in J\}$ of $\mathcal{X}$. Then we have a commutative diagram

$$\begin{array}{ccc} \mathfrak{U}_i & \xrightarrow{\widetilde{\mathfrak{u}}_i} & \mathfrak{X} \\ {\scriptstyle \mathfrak{f}_i}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{f}} \\ U_i & \xrightarrow{u_i} & X \end{array} \qquad i \in J, \tag{1}$$

where $U_i = \mathbf{Sp}(\mathcal{O}_{\mathcal{X}}(\mathcal{U}_i))$, $\mathfrak{U}_i = \mathbf{Sp}(\mathcal{A}_{\mathcal{X}}(\mathcal{U}_i))$, $\mathfrak{f}_i$ is the affine morphism corresponding to the ring morphism $\mathcal{O}_{\mathcal{X}}(\mathcal{U}_i) \xrightarrow{\psi(\mathcal{U}_i)} \mathcal{A}_{\mathcal{X}}(\mathcal{U}_i)$, and the morphisms u_i and $\widetilde{\mathfrak{u}}_i$ have restriction functors to the open subset $\mathcal{U}_i$ as inverse image functors. Since u_i^* and $\widetilde{\mathfrak{u}}_i^*$ are localization functors, the commutative diagram (1) shows that $\mathfrak{X}$ is a (noncommutative in general) scheme, $\{\mathfrak{U}_i \xrightarrow{\widetilde{\mathfrak{u}}_i} \mathfrak{X} \mid i \in J\}$ its affine cover, and $\mathfrak{X} \xrightarrow{\mathfrak{f}} X$ is a scheme morphism.

Fix $i \in J$ and pick a point x of the open set $\mathcal{U}_i$. To the point x, there corresponds an element $\mathcal{P}_x^i$ of $\mathbf{Spec}(U_i) = \mathbf{Spec}_{\mathfrak{c}}^0(U_i)$. Since $\mathcal{U}_i$ is a Zariski open subset of the commutative scheme $\mathcal{X}$, the point $\mathcal{P}_x^i$ is the image of a uniquely determined point $\mathcal{P}_x$ of X.

We assume that the ring morphism $\mathcal{O}_X(\mathcal{U}_i) \xrightarrow{\psi(\mathcal{U}_i)} \mathcal{A}_{\mathcal{X}}(\mathcal{U}_i)$ is flat; i.e. the functor $\mathfrak{f}_i^* = \mathcal{A}_{\mathcal{X}}(\mathcal{U}_i) \otimes_{\mathcal{O}_{\mathcal{X}}(\mathcal{U}_i)} -$ from C_{U_i} to $C_{\mathfrak{U}_i}$ is exact. The stabilizer of the point $\mathcal{P}_x^i$ can be identified with the subring $\mathcal{A}_{\mathcal{P}_x^i}$ of the ring $\mathcal{A}_{\mathcal{X}}(\mathcal{U}_i)$ which contains the image of $\mathcal{O}_{\mathcal{X}}(\mathcal{U}_i)$ and such that the induced morphism $\mathcal{O}_{\mathcal{X}}(\mathcal{U}_i) \longrightarrow \mathcal{A}_{\mathcal{P}_x^i}$ (– the corestriction of $\psi(U_i)$) is flat.

7.4.2.1. Finiteness conditions. Let $C_{\mathfrak{X}_{\mathfrak{f}}^x}$ denote the full subcategory of the category $C_{\mathfrak{X}}$ generated by all objects M of $C_{\mathfrak{X}}$ such that x is an associated point of $\mathfrak{f}_*(M)$ of finite multiplicity (or, what is the same, $\mathcal{P}_x$ is an associated point of $\mathfrak{f}_*(M)$ of finite multiplicity). It follows from generalities on associated points (see A3.2) that the subcagtegory $C_{\mathfrak{X}_{\mathfrak{f}}^x}$ is closed under extensions. It follows from 6.4.2.1 and 6.3 that every object M of the subcategory $C_{\mathfrak{X}_{\mathfrak{f}}^x}$ has an associated point of the form $\mathfrak{L}_{\mathcal{P}_x}(V)$, where V is an element of the spectrum of the stabilizer $\mathbf{Sp}(\mathcal{A}_{\mathcal{P}_x^i})$ of the point $\mathcal{P}_x^i$ whose image in C_X is an element of $Spec(X)$ representing the point $\mathcal{P}_x^i$. Therefore, if M is the point of the $Spec_-(\mathfrak{U}_i)$, then M is equivalent to $\mathfrak{L}_{\mathcal{P}_x^i}(V)$.

7.4.2.2. Example. Let now $\mathcal{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a smooth scheme over $Spec(k)$; and let $\mathcal{A}_{\mathcal{X}}$ be the sheaf of algebras of twisted differential operators on $\mathcal{X}$. Then $Spec_{\mathfrak{c}}^-(\mathfrak{X}) \bigcap C_{\mathfrak{X}_{\mathfrak{f}}^x}$, consists of all semisimple holonomic $\mathcal{A}_X$-modules whose simple components are isomorphic to each other.

7.4.3. Remark. Given a cover $\mathfrak{U} = \{U_i \xrightarrow{u_i} \mathfrak{X} \mid i \in J\}$, Proposition 7.2 suggests a way of constructing points of $\mathbf{Spec}_{\mathfrak{c}}^0(\mathfrak{X})$ starting from a point $\mathcal{P}$ of $\mathbf{Spec}_{\mathfrak{c}}^0(X)$, taking its image in $\mathbf{Spec}_{\mathfrak{c}}^0(U_i)$ for some U_i containing $\mathcal{P}$ (i.e. $u_i^*(\mathcal{P}) \neq 0$) and an object M_i of $Spec_{\mathfrak{c}}^0(\mathfrak{U}_i)$ such that its image in C_{U_i} has $u_i^*(\mathcal{P})$ as an associated point. Notice that the object M_i can be obtained via our induction procedure applied to some other affine morphism, $\mathfrak{V}_i \xrightarrow{\varphi_i} \mathfrak{U}_i$, and a point $\mathcal{Q}_i$ of $\mathbf{Spec}_{\mathfrak{c}}^0(\mathfrak{V}_i)$. All we need to know is that the image of M_i in C_{U_i} has an associated point of the form $u_i^*(\mathcal{P})$ for some $\mathcal{P} \in \mathbf{Spec}_{\mathfrak{c}}^0(X)$. Thus, the glueing data related to this approach is described by the diagram

$$\begin{array}{ccccc} \mathfrak{V}_i & \xleftarrow{\varphi_i} & \mathfrak{U}_i & \xrightarrow{\widetilde{\mathfrak{u}}_i} & \mathfrak{X} \\ & & {\scriptstyle \mathfrak{f}_i}\downarrow & & \downarrow{\scriptstyle \mathfrak{f}} \\ & & U_i & \xrightarrow{u_i} & X \qquad i \in J, \end{array} \tag{3}$$

where $\{C_X \xrightarrow{u_i^*} C_{U_i} \mid i \in J\}$ and $\{C_{\mathfrak{X}} \xrightarrow{\widetilde{\mathfrak{u}}_i^*} C_{\mathfrak{A}_i} \mid i \in J\}$ are conservative families of continuous exact localizations and the morphisms $\mathfrak{V}_i \xleftarrow{\varphi_i} \mathfrak{U}_i \xrightarrow{\mathfrak{f}_i} U_i$ are affine for all $i \in J$.

7.4.4. Example: D-modules on flag varieties. Let $\mathfrak{g}$ be a semisimple Lie algebra over an algebraicly closed field of zero characteristic, G a connected simply connected algebraic group whose Lie algebra is isomorphic to $\mathfrak{g}$. Let $\mathcal{B}$ be a Borel subgroup of G, and $\mathcal{W}$ its Weyl group. The sheaf $\mathcal{D}_{G/\mathcal{B}}$ of algebras of differential operators on $G/\mathcal{B}$ defines a noncommutative scheme $\mathfrak{X}_{G/\mathcal{B}}$ represented by the category of D-modules on $G/\mathcal{B}$, together with the affine morphism $\mathfrak{X}_{G/\mathcal{B}} \xrightarrow{\mathfrak{f}} X_{G/\mathcal{B}}$ corresponding to the morphism $\mathcal{O}_{G/\mathcal{B}} \longrightarrow \mathcal{D}_{G/\mathcal{B}}$ of sheaves of rings. Here $X_{G/\mathcal{B}}$ denotes the 'space' corresponding to the scheme $G/\mathcal{B}$, i.e. $C_{X_{G/\mathcal{B}}}$ is the category of quasi-coherent sheaves on $G/\mathcal{B}$. By Beilinson-Bernstein theorem, the category $C_{\mathfrak{X}_{B/\mathcal{B}}} = \mathcal{D}_{G/\mathcal{B}} - mod$ of D-modules on the flag variety $G/\mathcal{B}$ is equivalent to the category $U_\rho(\mathfrak{g}) - mod$ of $U(\mathfrak{g})$-modules with the trivial central character.

Consider the canonical affine cover $\{\mathcal{U}_w \xrightarrow{u_i} G/\mathcal{B} \mid w \in \mathcal{W}\}$ of the flag variety by the translations of the big cell. Each open subscheme $\mathcal{U}_w$ is isomorphic to the affine space $\mathbb{A}^n$. Therefore, for all $w \in \mathcal{W}$, the algebra $\mathcal{D}_{G/\mathcal{B}}(\mathcal{U}_w)$ is isomorphic to the Weyl algebra A_n.

Thus we have commutative diagrams of 'spaces'

$$\begin{tikzcd}
\mathbf{Sp}(A_n) \arrow[r, "\widetilde{\psi}_w"] \arrow[d, "\varphi_n"'] & \mathfrak{U}_w \arrow[r, "\widetilde{\mathfrak{u}}_w"] \arrow[d, "\mathfrak{f}_w"'] & \mathfrak{X}_{G/\mathcal{B}} \arrow[d, "\mathfrak{f}"] \\
\mathbf{Sp}(\Gamma(\mathbb{A}_n)) \arrow[r, "\psi_w"] & U_w \arrow[r, "u_w"] & X_{G/\mathcal{B}} \qquad w \in \mathcal{W},
\end{tikzcd} \tag{1}$$

where left horizontal arrows are isomorphisms, φ_n is a morphism corresponding to the embedding of the algebra $k[\mathbf{y}] = \Gamma(\mathbb{A}_n)$ of polynomials in n variables to the Weyl algebra.

By 7.2, the construction of points of $\mathbf{Spec}^0_{\mathfrak{c}}(\mathfrak{X}_{G/\mathcal{B}})$ is reduced to

(i) the construction of points of $\mathbf{Spec}^0_{\mathfrak{c}}(\mathfrak{U}_w) = \mathbf{Spec}(\mathfrak{U}_w) \simeq \mathbf{Spec}(\mathbf{Sp}(A_n))$,

(ii) verifying the glueing conditions of 7.2(b).

As it is observed in 7.2.2(b), the glueing conditions hold automatically if we study holonomic D-modules. We look at the first, most important, problem.

7.4.4.1. The standard approach. The diagram (1) invites to apply the developed here induction machinary to the morphism φ_n corresponding to the standard embedding $k[\mathbf{y}] \hookrightarrow A_n$. It follows from 5.3.1 that for every closed irreducible subvariety $\mathcal{V}$ of $\mathbb{A}^n$ (– a point of the spectrum of $k[\mathbf{y}]$), the functor $\mathfrak{L}_{\mathcal{V}}$ produces A_n-modules supported in $\mathcal{V}$. If the subvariety $\mathcal{V}$ is smooth, then the stabilizer of V in A_n coincides with the ring of differential operators on $\mathcal{V}$. In this case, it follows from the Kashiwara's theorem, that the induction functor establishes an equivalence between the category $\mathcal{D}_{\mathcal{V}} - mod$ of D-modules on $\mathcal{V}$ and the full subcategory $A_n - mod_{\mathcal{V}}$ of the category $A_n - mod$ whose objects are A_n-modules supported on $\mathcal{V}$.

7.4.4.2. Hyperbolic coordinates. They are given by the k-algebra embedding $k[\bar{\xi}] = k[\xi_1, \dots, \xi_n] \xrightarrow{\psi} A_n$ which maps each indeterminate ξ_i to the product $x_i y_i$. The main advantage of this choice is that only a countable number of points of $Spec(k[\bar{\xi}])$ have a nontrivial stabilizer, and their stabilizer can be easily described and taken into account. Thus, we extend the diagram (1) to the diagram

$$\begin{tikzcd}
\mathbf{Sp}(k[\xi]) & \mathbf{Sp}(A_n) \arrow[l, "\widetilde{\psi}"'] \arrow[r, "\widetilde{\psi}_w"] \arrow[d, "\varphi_n"'] & \mathfrak{U}_w \arrow[r, "\widetilde{\mathfrak{u}}_w"] \arrow[d, "\mathfrak{f}_w"'] & \mathfrak{X}_{G/\mathcal{B}} \arrow[d, "\mathfrak{f}"] \\
 & \mathbf{Sp}(\Gamma(\mathbb{A}_n)) \arrow[r, "\psi_w"] & U_w \arrow[r, "u_w"] & X_{G/\mathcal{B}} \qquad w \in \mathcal{W},
\end{tikzcd} \tag{2}$$

and use the morphism $\widetilde{\psi} = \mathbf{Sp}(\psi)$ for constructing elements of the spectrum of $\mathbf{Sp}(A_n)$.

7.5. Quantized D-modules on quantum flag varieties.

7.5.1. The cone of a non-unital ring. Let R_0 be a unital associative ring and R_+ an associative (non-unital in general) ring in the category of R_0-bimodules; i.e. R_+ is endowed with an R_0-bimodule morphism $R_+ \otimes_{R_0} R_+ \xrightarrow{m} R_+$ satisfying the associativity condition. We denote by R the augmented unital ring $R_0 \oplus R_+$ and by $\mathcal{T}_{R_+}$ the full subcategory of $R - mod$ whose objects are all R-modules annihilated by R_+.

We define the 'space' *cone of* R_+ by taking as $C_{\mathbf{Cone}(R_+)}$ the quotient category $R-mod/\mathcal{T}^-_{R_+}$ of $R-mod$ by the Serre subcategory spanned by $\mathcal{T}_{R_+}$. The localization functor $R-mod \xrightarrow{u^*} R-mod/\mathcal{T}^-_{R_+}$ is an inverse image functor of a morphism of 'spaces' $\mathbf{Cone}(R_+) \xrightarrow{u} \mathbf{Sp}(R)$. The functor u^* has a (necessarily fully faithful) right adjoint, i.e. the morphism u is continuous. If R_+ is a unital ring, then u is an isomorphism (see [KR2, C3.2.1]). The composition of the morphism u with the canonical affine morphism $\mathbf{Sp}(R) \longrightarrow \mathbf{Sp}(R_0)$ is a continuous morphism $\mathbf{Cone}(R_+) \longrightarrow \mathbf{Sp}(R_0)$. Its direct image functor is (regarded as) the *global sections functor.*

7.5.2. The graded version: $\mathbf{Proj}_{\mathcal{G}}$. Let $\mathcal{G}$ be a monoid and $R = R_0 \oplus R_+$ a $\mathcal{G}$-graded ring with zero component R_0. Then we have the category $gr_{\mathcal{G}}R-mod$ of $\mathcal{G}$-graded R-modules and its full subcategory $gr_{\mathcal{G}}\mathcal{T}_{R_+} = \mathcal{T}_{R_+} \cap gr_{\mathcal{G}}R-mod$ whose objects are graded modules annihilated by the ideal R_+. We define the 'space' $\mathbf{Proj}_{\mathcal{G}}(R)$ by setting

$$C_{\mathbf{Proj}_{\mathcal{G}}(R)} = gr_{\mathcal{G}}R - mod/gr_{\mathcal{G}}\mathcal{T}^-_{R_+}.$$

Here $gr_{\mathcal{G}}\mathcal{T}^-_{R_+}$ is the Serre subcategory of the category $gr_{\mathcal{G}}R - mod$ spanned by $gr_{\mathcal{G}}\mathcal{T}_{R_+}$. One can show that $gr_{\mathcal{G}}\mathcal{T}^-_{R_+} = gr_{\mathcal{G}}R-mod \cap \mathcal{T}^-_{R_+}$. Therefore, we have a canonical projection

$$\mathbf{Cone}(R_+) \xrightarrow{\mathfrak{p}} \mathbf{Proj}_{\mathcal{G}}(R).$$

The localization functor $gr_{\mathcal{G}}R - mod \longrightarrow C_{\mathbf{Proj}_{\mathcal{G}}(R_+)}$ is an inverse image functor of a continuous morphism $\mathbf{Proj}_{\mathcal{G}}(R) \xrightarrow{\mathfrak{v}} \mathbf{Sp}_{\mathcal{G}}(R)$. The composition $\mathbf{Proj}_{\mathcal{G}}(R) \xrightarrow{\mathfrak{v}} \mathbf{Sp}(R_0)$ of the morphism $\mathfrak{v}$ with the canonical morphism $\mathbf{Sp}_{\mathcal{G}}(R) \xrightarrow{\phi} \mathbf{Sp}(R_0)$ defines $\mathbf{Proj}_{\mathcal{G}}(R)$ as a 'space' over $\mathbf{Sp}(R_0)$. Its direct image functor is called the *global sections functor.*

7.5.2.1. Standard example: cone and Proj of a $\mathbb{Z}_+$-graded ring. Let $R = \bigoplus_{n\geq 0} R_n$ be a $\mathbb{Z}_+$-graded ring, R_+ its 'irrelevant' ideal. Thus, we have $\mathbf{Cone}(R_+)$, $\mathbf{Proj}(R) = \mathbf{Proj}_{\mathbb{Z}}(R)$, and the canonical morphism $\mathbf{Cone}(R_+) \longrightarrow \mathbf{Proj}(R)$.

7.5.3. The category of D-modules on the flag variety of a reductive Lie algebra. Let $\mathfrak{g}$ be a reductive Lie algebra over $\mathbb{C}$ and $U(\mathfrak{g})$ the enveloping algebra of $\mathfrak{g}$. Let $\mathcal{G}$ be the group of integral weights of $\mathfrak{g}$ and $\mathcal{G}_+$ the semigroup of nonnegative integral weights. Let $R = \oplus_{\lambda \in \mathcal{G}_+} R_\lambda$, where R_λ is the vector space of the (canonical) irreducible finite dimensional representation with the highest weight λ. The module R is a $\mathcal{G}$-graded algebra with the multiplication determined by the projections $R_\lambda \otimes R_\nu \longrightarrow R_{\lambda+\nu}$, for all $\lambda, \nu \in \mathcal{G}_+$. It is well known that the algebra R is isomorphic to the algebra of regular functions on the *base affine space* of $\mathfrak{g}$. Recall that G/U, where G is a connected simply connected algebraic group with the Lie algebra $\mathfrak{g}$, and U is its maximal unipotent subgroup.

7.5.3.1. Base affine space and flag variety. The category $C_{\mathbf{Cone}(R)}$ is equivalent to the category of quasi-coherent sheaves on the base affine space Y of the Lie algebra $\mathfrak{g}$.

The category $Proj_{\mathcal{G}}(R)$ is equivalent to the category of quasi-coherent sheaves on the flag variety of $\mathfrak{g}$.

7.5.3.2. D-modules on the flag variety. Consider the cross-product $U(\mathfrak{g})\#R$ associated with the Hopf action of $U(\mathfrak{g})$ on R. This is a $\mathcal{G}$-graded algebra (with the grading induced by the grading of the algebra R). One can show that the category $C_{\mathbf{Proj}_{\mathcal{G}}(U(\mathfrak{g})\#R)}$ is equivalent to the category $\mathcal{D}-mod_{G/\mathcal{B}}$ of D-modules on the flag variety of the Lie algebra $\mathfrak{g}$. In other words, the 'space' represented by the category of D-modules on the flag variety is isomorphic to $\mathbf{Proj}(U(\mathfrak{g})\#R)$.

7.5.4. The quantum base affine 'space' and quantum flag variety of a semisimple Lie algebra. Let now $\mathfrak{g}$ be a semisimple Lie algebra over a field k of zero characteristic, and let $U_q(\mathfrak{g})$ be the quantized enveloping algebra of $\mathfrak{g}$. Define the $\mathcal{G}$-graded algebra $\mathfrak{R} = \bigoplus_{\lambda\in\mathcal{G}_+}\mathfrak{R}_\lambda$ the same way as above, i.e. $\mathfrak{R}_\lambda$ is a simple finite-dimensional module with the highest weight λ. This time, however, the algebra $\mathfrak{R}$ is not commutative. If $\mathfrak{g} = sl_2$, then $\mathfrak{R}$ is isomorphic to the algebra $k_{\mathfrak{v}}[x,y] = k\langle x,y\rangle/(xy - \mathfrak{v}yx)$ for an appropriate $\mathfrak{v}$. Following the classical example (and representing 'spaces' by the categories of quasi-coherent sheaves on them), we call $\mathbf{Cone}(\mathfrak{R})$ *the quantum base affine 'space'* and $\mathbf{Proj}_{\mathcal{G}}(\mathfrak{R})$ the *quantum flag variety* of the Lie algebra $\mathfrak{g}$. We call $\mathfrak{R}$ the *algebra of functions* on the quantum base affine 'space'.

7.5.4.1. Canonical affine covers of the quantum base affine 'space' and the quantum flag variety. Let W be the Weyl group of the Lie algebra $\mathfrak{g}$. Fix a $w\in W$. For any $\lambda\in\mathcal{G}_+$, choose a nonzero w-extremal vector $e^\lambda_{w\lambda}$ generating the one dimensional vector subspace of $\mathfrak{R}_\lambda$ formed by the vectors of the weight $w\lambda$. Set $S_w = \{k^*e^\lambda_{w\lambda}|\lambda\in\mathcal{G}_+\}$. It follows from the Weyl character formula that $e^\lambda_{w\lambda}e^\mu_{w\mu}\in k^*e^{\lambda+\mu}_{w(\lambda+\mu)}$. Hence S_w is a multiplicative set. It was proved by Joseph [Jo] that S_w is a left and right Ore subset in $\mathfrak{R}$. The Ore sets $\{S_w|w\in W\}$ determine a conservative family of affine localizations

$$\mathbf{Sp}(S_w^{-1}\mathfrak{R}) \longrightarrow \mathbf{Cone}(\mathfrak{R}),\quad w\in W, \tag{4}$$

of the quantum base affine 'space' and a conservative family of affine localizations

$$\mathbf{Sp}_{\mathcal{G}}(S_w^{-1}\mathfrak{R}) \longrightarrow \mathbf{Proj}_{\mathcal{G}}(\mathfrak{R}),\quad w\in W, \tag{5}$$

of the quantum flag variety. Here $\mathbf{Sp}_{\mathcal{G}}(S_w^{-1}\mathfrak{R})$ is the 'space' represented by the category $gr_{\mathcal{G}}S_w^{-1}\mathfrak{R}-mod$ of $\mathcal{G}$-graded $gr_{\mathcal{G}}S_w^{-1}\mathfrak{R}$-modules.

We claim that the category $gr_{\mathcal{G}}S_w^{-1}\mathfrak{R}-mod$ is naturally equivalent to $(S_w^{-1}\mathfrak{R})_0-mod$. By 1.5, it suffices to verify that the canonical functor $gr_{\mathcal{G}}S_w^{-1}\mathfrak{R}-mod \longrightarrow S_w^{-1}\mathfrak{R})_0-mod$ which assigns to every graded $S_w^{-1}\mathfrak{R}$-module its zero component is faithful; i.e. the zero component of every nonzero $\mathcal{G}$-graded $S_w^{-1}\mathfrak{R}$-module is nonzero. This is, really, the case, because if z is a nonzero element of λ-component of a $\mathcal{G}$-graded $S_w^{-1}\mathfrak{R}$-module, then $(e^\lambda_{w\lambda})^{-1}z$ is a nonzero element of the zero component of this module.

This shows that for every $w\in W$, the morphism $\mathbf{Sp}_{\mathcal{G}}(S_w^{-1}\mathfrak{R}) \longrightarrow \mathbf{Proj}_{\mathcal{G}}(\mathfrak{R})$ is isomorphic to the morphism $\mathbf{Sp}((S_w^{-1}\mathfrak{R})_0) \xrightarrow{\mathfrak{u}_w} \mathbf{Proj}_{\mathcal{G}}(\mathfrak{R})$. The morphism $\mathfrak{u}_w$ form an affine cover

$$\mathbf{Sp}((S_w^{-1}\mathfrak{R})_0) \xrightarrow{\mathfrak{u}_w} \mathbf{Proj}_{\mathcal{G}}(\mathfrak{R}),\quad w\in W \tag{6}$$

of the quantum flag variety $\mathbf{Proj}_{\mathcal{G}}(\mathfrak{R})$ turning it into a noncommutative scheme.

7.5.5. The quantum flag D-variety. Similar to 7.5.3.2, we consider the cross-product $U_q(\mathfrak{g})\#\mathfrak{R}$, where $\mathfrak{R}$ is the algebra of functions on the quantum base affine 'space' defined in 7.5.4, with $\mathcal{G}$-grading induced by the $\mathcal{G}$-grading of $\mathfrak{R}$. We call the 'space' $\mathbf{Proj}(U_q(\mathfrak{g})\#\mathfrak{R})$ the quantum flag D-variety. The objects of the category representing $\mathbf{Proj}_{\mathcal{G}}(U_q(\mathfrak{g})\#\mathfrak{R})$ are called *quantum D-modules* on the quantum flag variety $\mathbf{Proj}_{\mathcal{G}}(\mathfrak{R})$.

The natural algebra morphism $\mathfrak{R} \longrightarrow U_q(\mathfrak{g})\#\mathfrak{R}$ induces an affine morphism

$$\mathbf{Proj}(U_q(\mathfrak{g})\#\mathfrak{R}) \xrightarrow{\ \mathfrak{f}\ } \mathbf{Proj}(\mathfrak{R}).$$

As every affine morphism, the morphism $\mathfrak{f}$ is isomorphic to the natural morphism

$$\mathbf{Sp}(\mathcal{F}_{\mathfrak{f}}/\mathbf{Proj}_{\mathcal{G}}(\mathfrak{R})) \xrightarrow{\ \widetilde{\mathfrak{f}}\ } \mathbf{Proj}_{\mathcal{G}}(\mathfrak{R})$$

for a monad $\mathcal{F}_{\mathfrak{f}}$. The monad $\mathcal{F}_{\mathfrak{f}}$ can be chosen canonically: it is uniquely determined by the action of $U_q(\mathfrak{g})$ on the category $gr_{\mathcal{G}}\mathfrak{R} - mod$ of $\mathcal{G}$-graded $\mathfrak{R}$-modules, because this action is compatible with the localization $gr_{\mathcal{G}}\mathfrak{R} - mod \longrightarrow \mathbf{Proj}_{\mathcal{G}}(\mathfrak{R})$.

Moreover, the action of $U_q(\mathfrak{g})$ on $gr_{\mathcal{G}}\mathfrak{R} - mod$ becomes *differential* in an appropriate sense (explained in [LR1] and [LR2]). This implies, among other things, that the action of $U_q(\mathfrak{g})$ on $gr_{\mathcal{G}}\mathfrak{R} - mod$ is compatible with localizations at the Ore sets S_w for each $w \in W$. So that the cover of $Proj_{\mathcal{G}}(\mathfrak{R})$ described in 7.5.4.1(6) induces a cover

$$\mathbf{Sp}((S_w^{-1}(U_q(\mathfrak{g})\#\mathfrak{R})_0) \xrightarrow{\ \mathfrak{u}_w^{\rho}\ } \mathbf{Proj}_{\mathcal{G}}(U_q(\mathfrak{g})\#\mathfrak{R}), \quad w \in W \tag{7}$$

of the 'space' $\mathbf{Proj}(U_q(\mathfrak{g})\#\mathfrak{R})$ such that the diagram

$$\begin{array}{ccc} \mathbf{Sp}((S_w^{-1}(U_q(\mathfrak{g})\#\mathfrak{R}))_0) & \xrightarrow{\ \mathfrak{u}_w^{\rho}\ } & \mathbf{Proj}_{\mathcal{G}}(U_q(\mathfrak{g})\#\mathfrak{R}) \\ \downarrow & & \downarrow \\ \mathbf{Sp}((S_w^{-1}\mathfrak{R})_0) & \xrightarrow{\ \mathfrak{u}_w\ } & \mathbf{Proj}_{\mathcal{G}}(\mathfrak{R}) \qquad w \in W \end{array} \tag{8}$$

whose all four arrows are affine morphisms, commutes. In particular, the cover (7) turnes the 'space' $\mathbf{Proj}_{\mathcal{G}}(U_q(\mathfrak{g})\#\mathfrak{R})$ into a noncommutative separated scheme.

7.5.6. The global sections functor. For any $\mathcal{G}$-graded k-algebra $\mathcal{R}$, there is a canonical continuous morphism $\mathbf{Proj}_{\mathcal{G}}(\mathcal{R}) \xrightarrow{\ \gamma\ } \mathbf{Sp}(\mathcal{R}_0)$ whose direct image functor is the composition of the right adjoint $C_{\mathbf{Proj}(\mathcal{R})} \xrightarrow{\ \mathfrak{q}_*\ } gr_{\mathcal{G}}\mathcal{R} - mod$ and the functor

$$gr_{\mathcal{G}}\mathcal{R} - mod \xrightarrow{\ \mathfrak{p}_*\ } R_0 - mod$$

which assigns to every $\mathcal{G}$-graded $\mathcal{R}$-module M its zero component endowed with the action of the zero component $\mathcal{R}_0$ of the algebra $\mathcal{R}$. We call the direct image functor $\gamma_* = \mathfrak{p}_*\mathfrak{q}_*$ of the morphism γ the *global sections* functor.

Thus, if $\mathcal{R}$ is the algebra of functions on the quantum (or classical) flag variety of the Lie algebra $\mathfrak{g}$, then $\mathcal{R}_0 = k$. If $\mathcal{R} = U(\mathfrak{g})\#\mathfrak{R}$, then $\mathcal{R}_0 = U(\mathfrak{g})$; and the diagram

$$\begin{array}{ccc} \mathbf{Proj}_{\mathcal{G}}(U_q(\mathfrak{g})\#\mathfrak{R}) & \xrightarrow{\widetilde{\gamma}} & \mathbf{Sp}(U(\mathfrak{g})) \\ \mathfrak{f}\downarrow & & \downarrow \\ \mathbf{Proj}_{\mathcal{G}}(\mathfrak{R}) & \xrightarrow{\gamma} & \mathbf{Sp}(k) \end{array} \tag{9}$$

(where the right vertical arrow corresponds to the k-algebra structure on $U(\mathfrak{g})$) commutes.

By [LR2] (see also [T]), the morphism $\mathbf{Proj}_{\mathcal{G}}(U_q(\mathfrak{g})\#\mathfrak{R}) \xrightarrow{\widetilde{\gamma}} \mathbf{Sp}(U_q(\mathfrak{g}))$ is affine and its direct image function establishes an equivalence between the category $C_{\mathbf{Proj}_{\mathcal{G}}(U_q(\mathfrak{g})\#\mathfrak{R})}$ of quantum D-modules on the flag variety and the full subcategory $U_q(\mathfrak{g})_\rho - mod$ of $U_q(\mathfrak{g})$-modules with the trivial central character. Thus, we can replace the diagram (9) with the commutative diagram

$$\begin{array}{ccc} \mathbf{Proj}_{\mathcal{G}}(U_q(\mathfrak{g})\#\mathfrak{R}) & \xrightarrow{\gamma_\rho} & \mathbf{Sp}(U_q(\mathfrak{g})_\rho) \\ \mathfrak{f}\downarrow & & \downarrow \\ \mathbf{Proj}_{\mathcal{G}}(\mathfrak{R}) & \xrightarrow{\gamma} & \mathbf{Sp}(k) \end{array} \tag{10}$$

whose upper horizontal arrow is an isomorphism.Therefore, it induces isomorphisms between the corresponding spectra of these 'spaces'. In particular, the direct image functor $\gamma_{\rho*}$ of the morphism γ_ρ maps $Spec^0_{\mathfrak{c}}(\mathbf{Proj}_{\mathcal{G}}(U_q(\mathfrak{g})\#\mathfrak{R}))$ to $Spec^0_{\mathfrak{c}}(\mathbf{Sp}(U_q(\mathfrak{g})_\rho))$ and this map induces an isomorphism from $\mathbf{Spec}^0_{\mathfrak{c}}(\mathbf{Proj}_{\mathcal{G}}(U_q(\mathfrak{g})\#\mathfrak{R}))$ onto $\mathbf{Spec}^0_{\mathfrak{c}}(\mathbf{Sp}(U_q(\mathfrak{g})_\rho))$.

7.6. The twisted version. Fix a central character χ of the quantized enveloping algebra $U_q(\mathfrak{g})$ and consider the twisted cross-product $U_q(\mathfrak{g})\#_\chi\mathfrak{R}$. We call $\mathbf{Proj}_{\mathcal{G}}(U_q(\mathfrak{g})\#_\chi\mathfrak{R})$ the quantum D_χ-variety, or the quantum twisted D-variety. The constructions of 7.5 can be repeated literally for the twisted D-varieties and summarized in the commutative diagrams

$$\begin{array}{cccccc} \mathbf{Sp}((S_w^{-1}(U_q(\mathfrak{g})\#\mathfrak{R}))_0) & \xrightarrow{\mathfrak{u}_w^\chi} & \mathbf{Proj}_{\mathcal{G}}(U_q(\mathfrak{g})\#_\chi\mathfrak{R}) & \xrightarrow{\gamma_\chi} & \mathbf{Sp}(U_q(\mathfrak{g})_\chi) & \\ \downarrow & & \downarrow & & \downarrow & \\ \mathbf{Sp}((S_w^{-1}\mathfrak{R})_0) & \xrightarrow{\mathfrak{u}_w} & \mathbf{Proj}_{\mathcal{G}}(\mathfrak{R}) & \xrightarrow{\gamma} & \mathbf{Sp}(k) & w \in W \end{array} \tag{1}$$

It follows from [LR2] (and [T]) that if χ is regular, anti-dominant, then γ_χ is an isomorphism. In this case, computing the spectra of the twisted flag D-variety is the same as the computing the corresponding spectra of the affine scheme $\mathbf{Sp}(U_q(\mathfrak{g})_\chi)$.

As to the studying the spectra of the flag D_χ-variety, it is reduced to the study of the spectra of elements of the cover, $\mathbf{Sp}((S_w^{-1}(U_q(\mathfrak{g})\#\mathfrak{R}))_0)$, $w \in W$. The spectra of $\mathbf{Sp}((S_w^{-1}(U_q(\mathfrak{g})\#\mathfrak{R}))_0)$ can be studied via the affine morphism

$$\mathbf{Sp}((S_w^{-1}(U_q(\mathfrak{g})\#\mathfrak{R}))_0) \longrightarrow \mathbf{Sp}((S_w^{-1}\mathfrak{R})_0), \tag{2}$$

or, possibly, using a different affine morphism

$$\mathbf{Sp}((S_w^{-1}(U_q(\mathfrak{g})\#\mathfrak{R}))_0) \xrightarrow{\widetilde{\psi}_w} \mathbf{Sp}(\mathcal{A}_w). \tag{3}$$

7.7. Remarks.

7.7.1. These constructions for the usual enveloping algebras. If the quantized enveloping algebra $U_q(\mathfrak{g})$ is replaced by the enveloping algebra $U(\mathfrak{g})$ and the algebra $\mathfrak{R}$ of functions on the quantum base affine 'space' by the algebra R of functions on the base affine space, then the constructions of 7.5 and 7.6 become another, purely algebraic, description of D-modules on a flag variety, the related canonical covers of the flag variety, and the corresponding (twisted) D-scheme. In particular, the algebra $(S_w^{-1}R)_0$ is isomorphic to the polynomial algebra $k[\bar{y}] = k[y_1, \dots, y_n]$ – the coordinate algebra of the affine space $\mathbb{A}^n$, and $(S_w^{-1}(U(\mathfrak{g})\#R))_0$ is, therefore, isomorphic to the Weyl algebra A_n for all $w \in W$.

A sensible choice of the algebra $\mathcal{A}_w$ in (3) is the polynomial algebra $k[\bar{\xi}] = k[\xi_1, \dots, \xi_n]$ and the morphism (3) is induced by the algebra morphism $k[\bar{\xi}] \xrightarrow{\psi} A_n$ which maps each indeterminate ξ_i to the product $x_i y_i$ – *hyperbolic coordinates* (see 7.4.4.2). Why this choice is sensible is shown in Section C1 (see also [R, Chapters II and IV]).

7.7.2. Quantum hyperbolic coordinates. In the quantum case, the algebras $(S_w^{-1}\mathfrak{R})_0$ of functions on the *quantum* translations of the big cell are rather complicated noncommutative algebras, if $\mathfrak{g}$ is a simple Lie algebra of the rank higher than one. Finding their own spectra is already a problem, so that the *standard* method, i.e. using the morphism (2) for the construction (induction) of the points of the spectra of $(S_w^{-1}(U_q(\mathfrak{g})\#\mathfrak{R}))_0$ becomes unpractical. Amazingly, the second method, the induction along hyperbolic coordinates, survives. That is one can take as the algebra $\mathcal{A}_w$ in (3) the algebra of polynomials $k[\xi] = k[\xi_1, \dots, \xi_n]$ and a morphism

$$k[\xi] \xrightarrow{\psi_w} (S_w^{-1}(U_q(\mathfrak{g})\#\mathfrak{R}))_0 \tag{4}$$

which is a part of the hyperbolic structure. In the *classical limit* (i.e. after factorization by the ideal generated by $(q-1)$), the algebra $(S_w^{-1}(U_q(\mathfrak{g})\#\mathfrak{R}))_0$ becomes the Weyl algebra A_n and the morphism (4) turns into the canonical morphism $k[\bar{\xi}] \xrightarrow{\psi} A_n$ (see 7.4.4.2).

In the case when the Cartan matrix of the Lie algebra $\mathfrak{g}$ is of the type (A) or (C) and w is the longest element of the Weyl group, the construction of the hyperbolic structure on the algebra $(S_w^{-1}(U_q(\mathfrak{g})\#\mathfrak{R}))_0$, in particular the morphism (4), can be deduced from [Ha]. The construction is written explicitly (for a more general case) in [R, IV.C2.7].

The existence of the deformations (4) of the canonical map $k[\bar{\xi}] \xrightarrow{\psi} A_n$ (more precisely, of its composition with the isomorphism $A_n \xrightarrow{\sim} (S_w^{-1}(U(\mathfrak{g})\#\mathfrak{R}))_0$) implies that not only the highest weight simple $U_q(\mathfrak{g})$-modules are deformations of the highest weight simple $U(\mathfrak{g})$-modules (which is a well known result of G. Lusztig [L]), but also that 'almost all' representations of the quantized enveloping algebra $U_q(\mathfrak{g})$ parametrized by the points $\mathcal{P}$ of $Spec(k[\bar{\xi}])$ via the maps (4) and related functors $\mathfrak{L}_{\mathcal{P}}$ (hence these representations belong to the spectrum of the noncommutative 'space' $\mathbf{Sp}(U_q(\mathfrak{g})))$ are deformations of the representations of the enveloping algebra $U(\mathfrak{g})$ parametrized by the same points of $Spec(k[\bar{\xi}])$ via the maps $k[\bar{\xi}] \xrightarrow{\psi} A_n \xrightarrow{\sim} (S_w^{-1}(U(\mathfrak{g})\#\mathfrak{R}))_0$ and the functors $\mathfrak{L}_{\mathcal{P}}$ determined by the ring morphism ψ.

Note that the hyperbolic algebra structure works more or less the same way in all cases, so that the piece of spectral theory of $(S_w^{-1}(U_q(\mathfrak{g})\#\mathfrak{R}))_0$ (hence of $U_q(\mathfrak{g})$) related to the morphism (4) is produced approximately the same way as the piece of spectral theory of the Weyl algebra A_n related to hyperbolic coordinates $k[\bar{\xi}] \xrightarrow{\psi} A_n$. For the material supporting the latter assertion, we refer to the section C1 of this paper (see below) and Chapters II and IV of the monograph [R].

7.7.3. Hyperbolic coordinates and holonomic objects. One can show that all simple A_n-modules obtained via the functor $\mathfrak{L}_{\mathcal{P}}$ corresponding to the algebra morphism $k[\bar{\xi}] \xrightarrow{\psi} A_n$, where $\mathcal{P}$ runs through the closed points of $Spec(k[\bar{\xi}])$, are holonomic. This follows from the Roos criterium of the holonomicity, the formulas for the functors $\mathfrak{L}_{\mathcal{P}}$ in hyperbolic case, and the fact that the closed points of $Spec(k[\bar{\xi}])$ have the trivial stabilizer (see C1 below). Each simple holonomic module on an element of the cover (– translation of the big cell) determines a simple holonomic D-module on the flag variety.

Similar facts hold in the quantum case for the algebra morphisms (4).

A detailed exposition of the facts mentioned in the remarks 7.7.2 and 7.7.3 is fairly envolved and is a subject of a paper which is presently in preparation.

Complementary facts.

C1. Weyl and Heisenberg algebras.

The studying the spectra of universal enveloping algebra $U(\mathfrak{g})$ of a reductive Lie algebras over algebraicly closed fields of zero characteristic is reduced (via the passage to the categories of quasi-coherent modules over sheaves of twisted differential operators on flag variety and using the standard cover of the latter by translations of the big cell) to studying modules over Weyl algebras (see 7.4.4).

Weyl algebras play also a crucial role in representation theory of nilpotent Lie algebras: if $\mathfrak{g}$ is a finite-dimensional nilpotent Lie algebra over an algebraicly closed field of zero characteristic, then the set of primitive ideals of its universal enveloping algebra $U(\mathfrak{g})$ is parametrized by the orbits of adjoint action on the dual space $\mathfrak{g}^*$; and for any primitive ideal $\mathfrak{J}$, quotient algebra $U(\mathfrak{g})/\mathfrak{J}$ is isomorphic to the Weyl algebra A_n.

Recall that the Weyl algebra A_n is a k-algebra generated by $x_i,\ y_i$ subject to the relations

$$[x_i, y_j] = \delta_{ij},\ [x_i, x_j] = 0 = [y_i, y_j] \quad \text{for all } 1 \le i, j \le n. \tag{3}$$

We assume that k is a field of zero characteristic.

C1.1. The standard realization. Let now C_X be the category of modules over the polynomial algebra $k[\mathbf{y}] = k[y_1, \dots, y_n]$, and $C_{\mathfrak{A}} = A_n - mod \xrightarrow{\varphi_*} C_X$ the pull-back functor corresponding to the embedding $k[\mathbf{y}] \hookrightarrow A_n$. Then $C_{\mathfrak{A}} = \mathcal{F}_\varphi - mod$, where $\mathcal{F}_\varphi = (F_\varphi, \mu_\varphi)$ is a differential monad on X; i.e. $F_\varphi = A_n \otimes_{k[\mathbf{y}]} -$ is a differential functor.

Fix a point $\mathcal{P}$ of $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ and consider the related commutative diagram (see (2) in 5.3)

$$\begin{tikzcd} C_{\mathfrak{A}[\mathcal{P}-]} \arrow[rr, "u_{\mathcal{P}*}"] \arrow[dr, "\widetilde{\mathfrak{f}}^*_{\mathcal{P}}"'] & & C_{X_{\mathcal{P}}} \\ & C_{\mathfrak{A}'_{\mathcal{P}}} \arrow[ur, "\widetilde{\varphi}^*_{\mathcal{P}}"'] & \end{tikzcd} \tag{4}$$

Let $V_{\mathcal{P}}$ denote the Zariski closed irreducible subspace of $Spec(k[\mathbf{y}])$ corresponding to $\mathcal{P}$. The category $C_{\mathfrak{A}'_{\mathcal{P}}}$ is equivalent to the category $\mathcal{D}(V_{\mathcal{P}})-mod$ of modules over the ring $\mathcal{D}(V_{\mathcal{P}})$ of differential operators on the subvariety (corresponding to) $V_{\mathcal{P}}$. The category $C_{\mathfrak{A}[\mathcal{P}-]}$ is the category of A_n-modules whose support is contained in $V_{\mathcal{P}}$. If the subvariety $V_{\mathcal{P}}$ is smooth, then, by a Kashiwara's theorem, the functor $C_{\mathfrak{A}[\mathcal{P}-]} \xrightarrow{\widetilde{\mathfrak{f}}^*_{\mathcal{P}}} C_{\mathfrak{A}'_{\mathcal{P}}}$ in (4) is an equivalence of categories.

Thus, the problem of finding the part of the spectrum of $\mathfrak{A}$ corresponding to the point $\mathcal{P}$ such that $V_{\mathcal{P}}$ is a smooth subvariety, is reduced to the problem of classifying points of the spectrum of D-modules on the subvariety $V_{\mathcal{P}}$. If $\mathcal{P}$ is not a generic point, we reduce the dimension. The price to pay is studying D-modules on a possibly much more complicated variety.

Since we study only D-modules related to the point $\mathcal{P}$, we can localize at $\mathcal{P}$ and consider, together with the diagram (4), the diagram

$$\begin{tikzcd} C_{\mathfrak{A}_r[\mathcal{P}-]} \arrow[rr, "u_{\mathcal{P}*}"] \arrow[dr, "\widetilde{\mathfrak{f}}^*_{\mathcal{P}}"'] & & C_{X^r_{\mathcal{P}}} \\ & C_{\mathfrak{A}^r_{\mathcal{P}}} \arrow[ur, "\widetilde{\varphi}^*_{\mathcal{P}}"'] & \end{tikzcd} \tag{5}$$

Here $X^r_{\mathcal{P}}$ is the residue 'space' of X at the point $\mathcal{P}$; $C_{\mathfrak{A}^r_{\mathcal{P}}}$ is the category of $\widetilde{\mathfrak{F}}$-modules $(L, \widetilde{\xi})$, where L is an object of the residue category $C_{X^r_{\mathcal{P}}}$, and $C_{\mathfrak{A}_r[\mathcal{P}-]}$ is the category of $\mathcal{F}_{\widehat{\mathcal{P}}}$-modules (M, ξ), where M is an object of the *residue Serre subcategory* (which is by definition the smallest nonzero Serre subcategory) of $C_{X/\widehat{\mathcal{P}}}$ (cf. 5.3.6).

In the case of studying $\mathbf{Spec}_-(X)$, the diagram (4) can be replaced by (5).

The residue category $C_{X^r_{\mathcal{P}}}$ in (5) is equivalent to the category of vector spaces over the residue field $k_{\mathcal{P}}$ of the point $\mathcal{P}$. The category $C_{\mathfrak{A}^r_{\mathcal{P}}}$ is equivalent to the category of modules over the ring of differential operators on the subvariety $V_{\mathcal{P}}$ with rational coefficients. The category $C_{\mathfrak{A}_r[\mathcal{P}-]}$ is equivalent to the category of modules with support in the subvariety $V_{\mathcal{P}}$ over the algebra of differential operators with coefficients in the residue field $k_{\mathcal{P}}$.

If $\mathcal{P}$ is a generic point, then $V_{\mathcal{P}} = Spec(k[\mathbf{y}])$, $C_{\mathfrak{A}[\mathcal{P}-]} = C_{\mathfrak{A}'_{\mathcal{P}}} = C_{\mathfrak{A}}$, the residue field $k_{\mathcal{P}}$ is the field $k(\mathbf{y})$ of rational functions in $\mathbf{y} = (y_1, \dots, y_n)$.

Depending on the point $\mathcal{P}$ the algebras of differential operators with coefficients from the residue field $k_{\mathcal{P}}$, hence the categories of modules over them, might be quite complicated.

C1.2. The hyperbolic structure. Let C_X be the category of modules over the polynomial algebra $R = k[\xi_1, \dots, \xi_n]$, where $\xi_i = x_i y_i$, $1 \le i \le n$,. We take $C_{\mathfrak{A}} = A_n-mod$ and consider the morphism $\mathfrak{A} \xrightarrow{u} X$ corresponding to the embedding $k[\xi] \hookrightarrow A_n$. So the category $C_{\mathfrak{A}} = A_n - mod$ is realized as the category of modules over the monad $\mathcal{F}_\varphi = (F_\varphi, \mu_\varphi)$ on C_X, where $F_\varphi = A_n \otimes_R -$.

The algebra A_n is a free right R-module of rank $\mathbb{Z}^n$. Explicitly,

$$A_n = \bigoplus_{\mathbf{s},\mathbf{t}\in\mathbb{Z}_+^n,\ \mathbf{s}\cdot\mathbf{t}=0} \mathbf{x}^{\mathbf{s}}\mathbf{y}^{\mathbf{t}} R \tag{6}$$

Here $\mathbf{x}^{\mathbf{n}} = x_1^{s_1}\dots x_n^{s_n}$ and $\mathbf{s}\cdot\mathbf{t} = \sum_{1\le i\le n} s_i t_i$.

The left R-module structure and the rest of multiplication are given by

$$\begin{aligned} &r\mathbf{x}^{\mathbf{s}}\mathbf{y}^{\mathbf{t}} = \mathbf{x}^{\mathbf{s}}\mathbf{y}^{\mathbf{t}}\vartheta^{\mathbf{t}-\mathbf{s}}(r) \quad \text{for all } r\in R;\\ &x_i y_i = \xi_i, \quad y_i x_i = \vartheta_i^{-1}(\xi_i) = \xi_i - 1,\\ &[x_i, y_j] = [x_i, x_j] = [y_i, y_j] = 0 \quad \text{for all} \quad 1\le i,j\le n,\ i\ne j. \end{aligned} \tag{7}$$

Here $\vartheta^{\mathbf{s}} = \vartheta_1^{s_1}\circ\ldots\circ\vartheta_n^{s_n}$ and ϑ_i is an automorphism of the algebra R determined by $\vartheta_i(\xi_j) = \xi_j + \delta_{ij}$, where δ_{ij} is the Kronecker symbol.

It follows from this description that the functor $F_\varphi = A_n\otimes_R -$ is a direct sum of automorphisms of the category $C_X = R{-}mod$; namely, $F_\varphi = \bigoplus_{\mathbf{s}\in\mathbb{Z}^{\mathbf{n}}}\vartheta^{\mathbf{s}}$. The multiplication is defined by

$$\vartheta_i\circ\vartheta_j \xrightarrow{id} \vartheta_i\vartheta_j \quad \text{if } i\ne j, \quad \text{and} \quad \vartheta_i^n\circ\vartheta_i^m \xrightarrow{\xi_{n,m}^{(i)}} \vartheta_i^{n+m} \quad \text{for all } i,$$

where $\xi_{n,m}^{(i)} = id$ if n and m are both nonpositive or both nonnegative. For $n\ge m\ge 1$, the morphisms $\xi_{n,-m}^{(i)}$ and $\xi_{-n,m}^{(i)}$ are defined by

$$\xi_{n,-m}^{(i)} = \xi_{n-1,-m+1}^{(i)}\circ\vartheta_i^{n-1}\xi_i\vartheta_i^{-m+1} \quad \text{and} \quad \xi_{-n,m}^{(i)} = \xi_{-n+1,m-1}^{(i)}\circ\vartheta_i^{-n+1}\xi_i\vartheta_i^{m-1}. \tag{8}$$

Here ξ_i is the endomorphism of the identical functor which assigns to every object N of C_X (– an R-module) the action of the element ξ_i on N.

C1.3. The non-degenerate part of the spectrum. Points $\mathcal{P}$ of the spectrum of C_X are in bijective correspondence with irreducible Zariski closed subspaces $V_{\mathcal{P}}$ of $Spec(R)$. The point $\mathcal{P}$ has a non-trivial stabilizer iff the subvariety $V_{\mathcal{P}}$ is stable by the transformation $\theta_1^{m_1}\dots\theta_n^{m_n}$, where at least one of the integers m_i is nonzero. This shows that, generally, a point of $\mathbf{Spec}_{\mathfrak{c}}^0(X)$ has a trivial stabilizer.

C1.3.1. The description. If a point $\mathcal{P}$ of $\mathbf{Spec}_{\mathfrak{c}}^0(X)$ has a trivial stabilizer, then the functor $\mathfrak{f}_{\mathcal{P}}^*$ coincides with $\varphi^*: N\longmapsto (F_\varphi(N), \mu_\varphi(N))$. Let $M = R/p,\ p\in Spec(R)$, be a representative of $\mathcal{P}$. Then $M\xrightarrow{\lambda(M)} M$ is either zero or a monomorphism for any endomorphism λ of Id_{C_X}. In particular, either $\xi_i\vartheta_i^n(M)$ is a monomorphism for all n, or $\xi_i\vartheta_i^n(M) = 0$ for some unique n (see 8.1.3). The latter means that $\xi_i - n$ annihilates the R-module M; i.e. $\xi_i - n$ is an element of the prime ideal p.

If $\xi_i\vartheta_i^n(M)$ is a monomorphism for all n and all i, then one can show that the $\varphi_*(\langle M\rangle)$-torsion the $\mathcal{F}_\varphi$-module $\varphi^*(M) = (F_\varphi(M), \mu_\varphi(M))$ is zero. Therefore, by 4.2, $\varphi^*(M)$ is an object of $Spec(\mathfrak{A})$. The general case is as follows. We set

$$V_{i,n_i}(M) = \bigoplus_{m<n_i} \vartheta_i^m(M) \quad \text{if } n_i \geq 0, \quad \text{and} \quad V_{i,n_i}(M) = \bigoplus_{m\geq n_i} \vartheta_i^m(M) \quad \text{if } n_i < 0.$$

Let Ξ_M denote the set of all pairs (i, n_i) such that $\xi_i\vartheta_i^{n_i}(M) = 0$, or, equivalently, $\xi_i - n_i$ belongs to the prime ideal p. We set

$$V(M) = 0 \quad \text{if } \Xi_M = \emptyset, \quad \text{and} \quad V(M) = \bigoplus_{(i,n_i)\in\Xi_M} V_{i,n_i}(M) \quad \text{if } \Xi_M \neq \emptyset.$$

The $\mathcal{F}_\varphi$-submodule $\widetilde{V}(M)$ of $\varphi^*(M) = (F_\varphi(M), \mu_\varphi(M))$ generated by $V(M)$ coincides with the $\varphi_*^{-1}(\langle M\rangle)$-torsion of $\varphi^*(M)$. So, the quotient $\mathcal{F}_\varphi$-module $\varphi^*(M)/\widetilde{V}(M)$ is isomorphic to $\mathfrak{L}_\mathcal{P}(M)$. By 5.2.2, $\mathfrak{L}_\mathcal{P}(R/p)$ belongs to $Spec(\mathfrak{A})$.

C1.3.2. Note. We denote by $\mathbf{Spec}_{\varphi,0}(X)$ the subset of all points with trivial stabilizer and by $Spec_{\varphi,0}(R)$ the corresponding subset of $Spec(R)$. Let $\mathcal{P}_1,\ \mathcal{P}_2$ be points of $\mathbf{Spec}_{\varphi,0}(X)$, and let $p_1,\ p_2$ be the corresponding prime ideals – the elements of $Spec_{\varphi,0}(R)$. Set $M_i = R/p_i,\ i = 1,2$. It follows from the construction in C1.3.1 that if $\mathcal{P}_1 \subseteq \mathcal{P}_2$, or, equivalently, $p_2 \subseteq p_1$, then there is an epimorphism $L_{\mathcal{P}_1}(M_1) \longrightarrow L_{\mathcal{P}_2}(M_2)$. In particular, the point $[L_{\mathcal{P}_2}(M_2)]$ is a specialization of $[L_{\mathcal{P}_1}(M_1)]$.

C1.4. The degenerate part of the spectrum. For an element $\mathcal{P}$ of $\mathbf{Spec}_\mathfrak{c}^0(X)$, we set $\mathcal{G}_\mathcal{P} = \{\mathbf{t} \in \mathbb{Z}^n \mid \vartheta^\mathbf{t}(\mathcal{P}) = \mathcal{P}\}$. This is a subgroup of $\mathbb{Z}^n$ which we assume here to be nonzero, hence it is isomorphic to $\mathbb{Z}^m$ for some positive integer m. Let $\{\mathbf{t}_i \mid 1 \leq i \leq m\}$ be free generators of $\mathcal{G}_\mathcal{P}$. The category $C_{\mathfrak{A}_\mathcal{P}}$ is isomorphic to the category $R_\mathcal{P} - mod$ of left modules over the hyperbolic algebra $R_\mathcal{P}$ corresponding to the data $\{\widetilde{\vartheta}_i = \vartheta^{\mathbf{t}_i},\ \widetilde{\xi}_i = \xi(\mathbf{t}_i) \mid 1 \leq i \leq m\}$. Here $\xi(\mathbf{t}_i) = \prod_{1\leq j\leq n} \xi_j(t_{ij})$, where t_{ij} is the j-th component of $\mathbf{t}_i$, and

$$\begin{aligned}
&\xi_j(\nu) = 1 \quad \text{if } \nu = 0,\\
&\xi_j(\nu) = \prod_{0\leq s<\nu} \vartheta_j^s(\xi_j) = \prod_{0\leq s<\nu} (\xi_j + s) \quad \text{if } \nu > 0, \quad \text{and}\\
&\xi_j(\nu) = \vartheta_j^\nu(\xi_j(-\nu) \prod_{1\leq s\leq-\nu} (\xi_j - s) \quad \text{if } \nu < 0.
\end{aligned} \tag{9}$$

That is $R_\mathcal{P}$ is generated by the algebra R and by the indeterminates $\widetilde{x}_i,\ \widetilde{y}_i$ subject to the relations

$$\begin{aligned}
&\widetilde{x}_i r = \widetilde{\vartheta}_i(r)\widetilde{x}_i, \quad r\widetilde{y}_i = \widetilde{y}_i\widetilde{\vartheta}_i(r),\\
&\widetilde{x}_i\widetilde{y}_i = \widetilde{\xi}_i, \quad \widetilde{y}_i\widetilde{x}_i = \widetilde{\vartheta}_i^{-1}(\widetilde{\xi}_i);\\
&[\widetilde{x}_i, \widetilde{y}_j] = [\widetilde{x}_i, \widetilde{x}_j] = [\widetilde{y}_i, \widetilde{y}_j] = 0 \quad \text{for all} \quad r \in R, \text{ and } 1 \leq i,j \leq m \text{ such that } i \neq j.
\end{aligned} \tag{10}$$

The functor $C_{\mathfrak{A}} \xrightarrow{\mathfrak{f}_{\mathcal{P}*}} C_{\mathfrak{A}_{\mathcal{P}}}$ corresponds to the algebra morphism $R_{\mathcal{P}} \longrightarrow A_n$ which is identical on R and maps $\widetilde{x}_i$ to $x^{\mathbf{t}_i^+} y^{\mathbf{t}_i^-}$ and $\widetilde{y}_i$ to $x^{\mathbf{t}_i^-} y^{\mathbf{t}_i^+}$, $1 \le i \le m$. Here $\mathbf{t}_i^+$ and $\mathbf{t}_i^-$ are elements of $\mathbb{Z}_+^n$ uniquely defined by the conditions: $\mathbf{t}_i = \mathbf{t}_i^+ - \mathbf{t}_i^-$, $\mathbf{t}_i^+ \cdot \mathbf{t}_i^- = 0$.

The category $C_{\mathfrak{A}'_{\mathcal{P}}}$ is naturally equivalent to the category $R_{\mathcal{P}}/(p) - mod$. Here p is the prime ideal in R corresponding to the point $\mathcal{P}$ and (p) denote the two-sided ideal in $R_{\mathcal{P}}$ generated by p.

The points of $\mathbf{Spec}_{\mathfrak{c}}^{\mathcal{P}}(\mathfrak{A}_{\mathcal{P}})$ are identified with those points of $\mathbf{Spec}(\mathfrak{A}'_{\mathcal{P}})$ which survive the localization at $\mathcal{P}$. The latter is given by the localization of the algebra R at p. Thus, $\mathbf{Spec}_{\mathfrak{c}}^{\mathcal{P}}(\mathfrak{A}_{\mathcal{P}})$ is identified with a subset of the spectrum of $\mathfrak{A}_{\mathcal{P}}^{\mathfrak{r}}$ (cf. 5.3.6). The category $C_{\mathfrak{A}_{\mathcal{P}}^{\mathfrak{r}}}$ is naturally equivalent to the category modules over the algebra $k_{\mathcal{P}}[(\widetilde{x}_i, \widetilde{x}_i^{-1}; \widetilde{\vartheta}_i)]$ of skew Laurent polynomials in $(\widetilde{x}_i \mid 1 \le i \le m)$ with coefficients in the residue field $k_{\mathcal{P}}$ of the point $\mathcal{P}$ which can be identified with the residue field $K(R/p)$ of the prime ideal p.

Here we used the fact that the elements $\widetilde{\xi}_i$, $1 \le i \le m$, do not belong to the ideal p.

Indeed, it follows from the formulas (9) that if $\widetilde{\xi}_i \in p$, then there is s such that $\xi_s + t \in p$ and $t_{is} \neq 0$. Since $\vartheta^{\mathbf{t}_i}(p) = p$, the element $\vartheta^{\ell \mathbf{t}_i}(\xi_s) = \xi_s + \ell t_{is}$ belongs to the ideal p for any $\ell \in \mathbb{Z}$. But, since $char(k) = 0$, this is impossible.

C1.4.1. The points of the spectrum over the generic point. Since $char(k) = 0$, the only $\mathbb{Z}^n$-invariant point of $\mathbf{Spec}_{\mathfrak{c}}^0(X)$ is the generic point $\mathcal{P}_0$ corresponding to the zero ideal of the k-algebra $R = k[\xi_1, \dots, \xi_n]$.

The categories $C_{\mathfrak{A}}$, $C_{\mathfrak{A}_{\mathcal{P}_0}}$, and $C_{\mathfrak{A}'_{\mathcal{P}_0}}$ coincide, and the localization at $\mathcal{P}_0$ provides an embedding $\mathbf{Spec}_{\mathcal{P}_0}(\mathfrak{A}) \longrightarrow \mathbf{Spec}(\mathfrak{A}^{\mathfrak{r}}) = \mathbf{Spec}(\mathfrak{A}_{\mathcal{P}_0}^{\mathfrak{r}})$. The category $C_{\mathfrak{A}^{\mathfrak{r}}}$ here is equivalent to the category of modules over the algebra $k(\xi_1, \dots, \xi_n)[x_1^{\pm 1}, \dots, x_n^{\pm 1}; \theta_1, \dots, \theta_n]$ of skew Laurent polynomials in $x_1, \dots, x_n$ with coefficients in the field $k(\xi_1, \dots, \xi_n)$ of rational functions in $\xi_1, \dots, \xi_n$.

C1.5. Heisenberg algebras. Recall that the Heisenberg algebra $\mathcal{H}_n$ (– the enveloping algebra of the Heisenberg Lie algebra) is an associative k-algebra generated by x_i, y_i, and z subject to the relations

$$[x_i, y_j] = \delta_{ij} z, \quad [x_i, z] = [x_i, x_j] = 0 = [y_i, y_j] = [y_i, z] \quad \text{for all } 1 \le i, j \le n. \tag{1}$$

Let $R = k[z, \xi_1, \dots, \xi_n]$. The Heisenberg algebra $\mathcal{H}_n$ is a free right R-module with the basis formed by $\mathbf{x}^{\mathbf{s}} \mathbf{y}^{\mathbf{t}}$, where $\mathbf{s} \in \mathbb{Z}_+^n \ni \mathbf{t}$ are such that $\mathbf{s} \cdot \mathbf{t} = \sum_{1 \le i \le n} s_i t_i = 0$, $\mathbf{x}^{\mathbf{s}} = x_1^{s_1} \dots x_n^{s_n}$ (see C1.2):

$$\mathcal{H}_n = \bigoplus_{\mathbf{s}, \mathbf{t} \in \mathbb{Z}_+^n, \ \mathbf{s} \cdot \mathbf{t} = 0} \mathbf{x}^{\mathbf{s}} \mathbf{y}^{\mathbf{t}} R \tag{2}$$

The multiplication is given by

$$\begin{aligned} & r \mathbf{x}^{\mathbf{s}} \mathbf{y}^{\mathbf{t}} = \mathbf{x}^{\mathbf{s}} \mathbf{y}^{\mathbf{t}} \vartheta^{\mathbf{t} - \mathbf{s}}(r) \quad \text{for all } r \in R; \\ & x_i y_i = \xi_i, \quad y_i x_i = \vartheta_i^{-1}(\xi_i) = \xi_i - z, \\ & [x_i, y_j] = [x_i, x_j] = [y_i, y_j] = 0 \quad \text{for all} \quad 1 \le i, j \le n, \ i \neq j. \end{aligned} \tag{3}$$

Here $\vartheta^{\mathbf{s}} = \vartheta_1^{s_1} \circ \dots \circ \vartheta_n^{s_n}$ and ϑ_i, $1 \le i \le n$, are automorphisms of the algebra R defined by $\vartheta_i(\xi_j) = \xi_j + \delta_{ij} z$, $\vartheta_i(z) = z$.

The spectral picture corresponding to the embedding $R \hookrightarrow \mathcal{H}_n$ is recovered the same way (and in the same degree) as the spectrum of the Weyl algebra A_n regarded as a hyperbolic algebra over the ring of polynomials. We leave details to the reader.

C2. Remarks on enveloping algebras.

C2.1. The Harish-Chandra homomorphism and the highest weight simple modules. Let $\mathfrak{g}$ be a semisimple Lie algebra over a field k of zero characteristic. Fix its Cartan subalgebra $\mathfrak{h}$. We take $C_X = U(\mathfrak{h}) - mod$, $C_{\mathfrak{A}} = U(\mathfrak{g}) - mod$, and the functor $C_{\mathfrak{A}} \xrightarrow{\varphi_*} C_X$ corresponding to the embedding $U(\mathfrak{h}) \longrightarrow U(\mathfrak{g})$.

We consider the canonical grading $U(\mathfrak{g}) = \bigoplus_{\lambda \in \mathcal{Q}} U(\mathfrak{g})_\lambda$ defined by the adjoint action of $\mathfrak{g}$ on $U(\mathfrak{g})$ (cf. [D, 7.4]). The subalgebra $U(\mathfrak{g})_0$ is the centralizer of $U(\mathfrak{h})$ in $U(\mathfrak{g})$.

Let $\mathcal{P}$ be a point of $\mathbf{Spec}^0_{\mathfrak{c}}(X) = \mathbf{Spec}(X)$, and p the corresponding prime ideal of $U(\mathfrak{h})$. The category $C_{\mathfrak{A}_\mathcal{P}}$ is equivalent to the category of modules over the $\mathcal{Q}_\mathcal{P}$-graded subalgebra $U(\mathfrak{g})_\mathcal{P} = \bigoplus_{\lambda \in \mathcal{Q}_\mathcal{P}} U(\mathfrak{g})_\lambda$, where $\mathcal{Q}_\mathcal{P}$ is the subgroup of $\mathcal{Q}$ stabilizing $\mathcal{P}$ (i.e. the ideal p). In particular, the centralizer $U(\mathfrak{g})_0$ of $U(\mathfrak{h})$ stabilizes the subcategory $\mathcal{P} = U(\mathfrak{h})/p - mod$ for every point $\mathcal{P}$. For most of points $\mathcal{P}$, the subgroup $\mathcal{Q}_\mathcal{P}$ is trivial, hence the category $C_{\mathfrak{A}_\mathcal{P}}$ is naturally equivalent to the category $U(\mathfrak{g})_0 - mod$. In particular, $C_{\mathfrak{A}_\mathcal{P}}$ is equivalent to $U(\mathfrak{g})_0 - mod$ for all closed points $\mathcal{P}$ of $\mathbf{Spec}^0_{\mathfrak{c}}(X) = \mathbf{Spec}(X)$.

Set $C_{\mathfrak{A}_0} = U(\mathfrak{g})_0 - mod$. The Harish-Chandra homomorphism $U(\mathfrak{g})_0 \xrightarrow{\varphi_\mathcal{H}} U(\mathfrak{h})$ induces a full embedding $C_X \xrightarrow{\varphi_{\mathcal{H}*}} C_{\mathfrak{A}_0}$ which identifies the category C_X with a coreflective topologizing subcategory of $C_{\mathfrak{A}_0}$. Therefore, the embedding $\varphi_{\mathcal{H}*}$ determines an embedding $\mathbf{Spec}(X) \longrightarrow \mathbf{Spec}(\mathfrak{A}_0)$. So that every element $\mathcal{P}$ of $\mathbf{Spec}(X)$ is identified with the corresponding element of $\mathbf{Spec}^\mathcal{P}_{\mathfrak{c}}(\mathfrak{A}_\mathcal{P})$.

Let $M = U(\mathfrak{h})/p$. Then the composition of the embedding

$$C_X = U(\mathfrak{h}) - mod \longrightarrow U(\mathfrak{g})_0 - mod = C_{\mathfrak{A}_\mathcal{P}}$$

with the functor $\mathfrak{L}_\mathcal{P}$ assigns to M the highest weight module corresponding to the ideal p.

C2.1.1. Example. If $\mathfrak{g} = sl_2$, then $U(\mathfrak{g})$ is generated by indeterminates x, y, z subject to the relations

$$[x, y] = z, \quad [x, z] = \alpha x, \quad [y, z] = -\alpha y, \tag{1}$$

where α is a nonzero element of the base field k. Thus, $U(\mathfrak{h}) = k[z]$, $U(\mathfrak{g})_0 = k[z, \xi]$, and the Harish-Chandra homomorphism $k[z, \xi] \longrightarrow k[z]$ assigns to every polynomial $f(z, \xi)$ the element $f(z, 0)$ of $k[z]$. The corresponding map $Spec(U(\mathfrak{h})) \longrightarrow Spec(U(\mathfrak{g})_0)$ assigns to any prime ideal p in $k[z]$ the prime ideal (p, ξ). If $\mathcal{P}$ is a closed point (i.e. p is a maximal ideal), then $U(\mathfrak{g})_0$ is the stabilizer of $\mathcal{P}$ in the sense that $C_{\mathfrak{A}_\mathcal{P}}$ is equivalent to the category $U(\mathfrak{g})_0 - mod = k[z, \xi] - mod$. The functor $\mathfrak{f}^*_\mathcal{P}$ is isomorphic to $U(\mathfrak{g}) \otimes_{U(\mathfrak{g})_0} -$. The functor $\mathfrak{L}_\mathcal{P}$ assigns to the simple $U(\mathfrak{g})_0$-module $M = U(\mathfrak{g})_0/(p, \xi) \simeq U(\mathfrak{h})/p$ the corresponding Verma module $U(\mathfrak{g})/(p, y) = \bigoplus_{m \geq 0} x^m M$, if $p \neq (z - n/2)$ for any nonnegative integer n.

If $p = (z - n/2)$ for some nonnegative integer n (there is only one such integer n), then the module $M = k[z]/(z - n/2)$ is one-dimensional and $\mathfrak{L}_{\mathcal{P}}(M) = \bigoplus_{0 \le m \le n} x^m M$ has dimension $n + 1$ over the field k. In particular, if $n = 0$, then $\mathfrak{L}_{\mathcal{P}}(M)$ is the unique one-dimensional representation of $U(sl_2)$.

Set $R = k[z, \xi] = U(\mathfrak{g})_0$. The relations (1) are equivalent to the relations

$$xy = \xi, \quad yx = \theta^{-1}(\xi); \quad xr = \theta(r)x, \quad ry = y\theta(r) \tag{2}$$

for all $r \in R$. Here θ is the automorphism of the algebra R is defined by $\theta(f)(z, \xi) = f(z + \alpha, \xi + z + \alpha)$. In terminology of [R, Ch.II], (2) is the representation of $U(sl_2)$ as a *hyperbolic* ring over R. We take $C_X = R - mod$, $C_{\mathfrak{A}} = U(sl_2) - mod$ and the functor $C_{\mathfrak{A}} \xrightarrow{\varphi_*} C_X$ corresponding to the embedding $R \longrightarrow U(sl_2)$. Application the functors $\mathfrak{L}_{\mathcal{P}}$ gives a fairly complete description of the rest of the picture. Closed points of $\mathbf{Spec}(X) \simeq Spec(R)$ have trivial stabilizer, and the functor $\mathfrak{L}_{\mathcal{P}}$ for such point $\mathcal{P}$ coincides with the induction functor. By 4.2, $\mathfrak{L}_{\mathcal{P}}$ maps a simple module R/p representing $\mathcal{P}$ to a simple $U(sl_2)$-module. If $\mathcal{P}$ is a curve, then $[\mathfrak{L}_{\mathcal{P}}(R/p)]$ is a noncommutative curve in $\mathbf{Spec}(\mathfrak{A})$. If $\mathcal{P}$ is the generic point, then we localize at the multiplicative set of nonzero elements of R and reduce the problem to the description of simple modules over a skew polynomial ring $k(z, \xi)[x, \theta]$ which is a Eucledian domain. Therefore, its simple modules correspond to irreducible (skew) polynomials. See [R, II.4.3] for details.

If $\mathfrak{g}$ has a higher rank (starting from $\mathfrak{g} = sl_3$), then $U(\mathfrak{g})_0$ is a rather complicated noncommutative subalgebra of $U(\mathfrak{g})$. In particular, it is not clear how to approach to the description of $\mathbf{Spec}_{\mathfrak{c}}^{\mathcal{P}}(\mathfrak{A}_{\mathcal{P}})$.

C2.1.2. Remark. Similar facts on the connection of the Harish-Chandra homomorphism and highest weight simple modules hold for quantized enveloping algebras $U_q(\mathfrak{g})$ in the case when q is not a root of one [XT]. Also, $U_q(sl_2)$ has a hyperbolic structure over the ring $R = k[z, z^{-1}, \xi]$ which allows to get a description to the spectrum of $\mathfrak{A} = \mathbf{Sp}(U_q(sl_2)$ (see [R,II.4.2]).

Appendix 1: Monads and continuous morphisms.

A1.1. Monads and their categoric spectrum. Let Y be a 'space' represented by a category C_Y. A *monad on* Y is by definition a monad on C_Y, i.e. a pair (F, μ), where F is a functor $C_Y \longrightarrow C_Y$ and μ a morphism $F^2 \longrightarrow F$ (multiplication) such that $\mu \circ F\mu = \mu \circ \mu F$ and there exists a morphism $\eta : Id_{C_Y} \longrightarrow F$ uniquely determined by the equalities $\mu \circ F\eta = id_F = \mu \circ \eta F$ (a unit).

A morphism from a monad $\mathcal{F} = (F, \mu)$ to a monad $\mathcal{F}' = (F', \mu')$ is given by a functor morphism $F \longrightarrow F'$ such that $\varphi \circ \mu = \mu' \circ \varphi \odot \varphi$ and $\varphi \circ \eta = \eta'$. Here $\varphi \odot \varphi = F'\varphi \circ \varphi F$, and η, η' are units of the monads resp. $\mathcal{F}$ and $\mathcal{F}'$. The composition of morphisms is defined naturally, so that the map $\mathfrak{Mon}_Y \longrightarrow End(C_Y)$ forgetting monad structure, i.e. sending a monad morphism $(F, \mu) \xrightarrow{\varphi} (F', \mu')$ to the natural transformation $F \xrightarrow{\varphi} F'$, is a functor.

We denote by $\mathfrak{Mon}_Y$ the category of monads on Y.

Given a monad $\mathcal{F} = (F, \mu)$ on Y, we denote by $(\mathcal{F}/Y) - mod$, or simply by $\mathcal{F} - mod$, the category of $(\mathcal{F}/Y)$-modules. Its objects are pairs (M, ξ), where $M \in ObC_Y$ and ξ is a morphism $F(M) \longrightarrow M$ such that $\xi \circ F\xi = \xi \circ \mu(M)$ and $\xi \circ \eta(M) = id_M$. Morphisms from (M, ξ) to (M', ξ') are given by morphisms $g : M \longrightarrow M'$ such that $\xi' \circ Fg = g \circ \xi$.

We denote by $\mathbf{Sp}(\mathcal{F}/Y)$ the 'space' represented by the category $(\mathcal{F}/Y) - mod$. It is called sometimes the *categorical spectrum* of the monad $\mathcal{F}$.

The forgetful functor

$$(\mathcal{F}/Y) - mod \xrightarrow{f_*} C_Y, \quad (M, \xi) \longmapsto M,$$

is a right adjoint to the functor

$$C_Y \xrightarrow{f^*} (\mathcal{F}/Y) - mod, \quad L \longmapsto (F(L), \mu(L)), \ (L \xrightarrow{g} N) \longmapsto (f^*(L) \xrightarrow{F(g)} f^*(L')).$$

In other words, we have a canonical continuous morphism $\mathbf{Sp}(\mathcal{F}/Y) \xrightarrow{f} Y$.

A1.1.1. Example. Let R, S be unital associative rings. Any unital ring morphism $S \xrightarrow{\varphi} R$ defines a monad, $R_\varphi^\sim = (R_\varphi, \mu_\varphi)$, on $Y = \mathbf{Sp}(S)$. Here the functor R_φ is $M \longmapsto R \otimes_S M$, and the multiplication is induced by the multiplication on R. The canonical morphism $\mathbf{Sp}(R/\mathbf{Sp}(S)) \longrightarrow \mathbf{Sp}(S)$ has the pull-back functor $R - mod \xrightarrow{\varphi_*} S - mod$ as a direct image functor. Notice that the category $(R_\varphi^\sim/\mathbf{Sp}(S))$-modules is isomorphic to the category $R - mod$ of R-modules; in particular, $\mathbf{Sp}(R_\varphi^\sim/\mathbf{Sp}(S)) \simeq \mathbf{Sp}(R)$.

If $S = \mathbb{Z}$, i.e. $C_Y = \mathbb{Z} - mod$, the category $(R/\mathbf{Sp}\mathbb{Z}) - mod$ coincides with the category $R - mod$ of left R-modules. Consistent with our previous notations, we write $\mathbf{Sp}(R)$ instead of $\mathbf{Sp}(R/\mathbf{Sp}\mathbb{Z})$.

A1.2. Morphisms of monads and morphisms of their categoric spectra. Let Y be an object of $|Cat|^o$ and $\mathcal{F}$, $\mathcal{F}'$ monads on Y. Any monad morphism $\mathcal{F} \xrightarrow{\varphi} \mathcal{F}'$ induces the 'pull-back' functor

$$(\mathcal{F}'/Y) - mod \xrightarrow{\varphi_*} (\mathcal{F}/Y) - mod, \quad (M, \xi) \longmapsto (M, \xi \circ \varphi(M)).$$

This correspondence defines a functor $\mathfrak{Mon}_Y^{op} \longrightarrow Cat/C_Y$ which takes values in the full subcategory of Cat/C_Y objects of which are functors $C_Z \longrightarrow C_Y$ having a left adjoint.

A1.2.1. Reflexive pairs of arrows and weakly continuous functors and monads. Recall that a pair of arrows $M \underset{g_2}{\overset{g_1}{\rightrightarrows}} L$ in C_Y is called *reflexive*, if there exists a morphism $L \xrightarrow{h} M$ such that $g_1 \circ h = id_M = g_2 \circ h$.

We call a functor $C_Y \longrightarrow C_Z$ *weakly continous* if it preserves cokernels of reflexive pairs of arrows.

We call a monad $\mathcal{F} = (F, \mu)$ on Y *weakly continuous* if the functor $C_Y \xrightarrow{F} C_Y$ is weakly continuous. We denote by $\mathfrak{Mon}_Y^w$ the full subcategory of the category $\mathfrak{Mon}_Y$ whose objects are weakly continuous monads on Y.

A1.2.2. Lemma. *Let $\mathcal{F}$, $\mathcal{F}'$ be monads on Y and φ a monad morphism $\mathcal{F} \longrightarrow \mathcal{F}'$. Suppose the category C_Y has cokernels of reflexive pairs of morphisms and the monad $\mathcal{F}'$ is weakly continuous. Then the functor φ_* has a left adjoint.*

In particular, the map $(\mathcal{F}/Y) \longmapsto \mathbf{Sp}(\mathcal{F}/Y)$, $\varphi \longmapsto [u^*]$ is a functor,

$$\mathbf{Sp}_Y : \mathfrak{Mon}_Y^w \longrightarrow |Cat|^o, \tag{1}$$

which takes values in the subcategory $|Cat|^o_{cont}$ of $|Cat|^o$ formed by continuous morphisms.

Proof. The left adjoint, $(\mathcal{F}/Y) - mod \xrightarrow{u^*} (\mathcal{F}'/Y) - mod$ assigns to each $(\mathcal{F}/Y)$-module $(M, F(M) \xrightarrow{\xi} M)$ the cokernel of the pair of arrows

$$F'F(M) \underset{F'\xi}{\overset{\mu' \circ F'\varphi}{\rightrightarrows}} F'(M). \tag{2}$$

Since by hypothesis F' preserves cokernels of reflexive pairs and both arrows (1) are $\mathcal{F}'$-module morphisms, there exists a unique $\mathcal{F}'$-module structure on the cokernel of (1). Details are left to the reader. ∎

A1.2.3. Note. Suppose that the category C_X has colimits of certain type $\mathfrak{D}$, and let $\mathcal{F} = (F, \mu)$ be a monad on X such that the functor F preserves colimits of this type. Then the category $(\mathcal{F}/X) - mod$ has colimits of this type.

In fact, for a diagram $\mathfrak{D} \xrightarrow{\mathcal{D}} (\mathcal{F}/X) - mod$, the colimit of the composition $f_* \circ \mathcal{D}$ (where f_* is the forgetful functor $(\mathcal{F}/X) - mod \longrightarrow C_X$) has a unique $\mathcal{F}$-module structure, $\xi_{\mathcal{D}}$. The $\mathcal{F}$-module $(colim(f_* \circ \mathcal{D}), \xi_{\mathcal{D}})$ is a colimit of the diagram $\mathcal{D}$.

In particular, if $\mathcal{F} = (F, \mu)$ is a weakly continuous monad on X, and the category C_X has cokernels of reflexive pairs of arrows, then the category $(\mathcal{F}/X) - mod$ has cokernels of reflexive pairs of arrows.

The following assertion is one of the versions of Beck's theorem.

A1.3. The Beck's theorem. Let $X \xrightarrow{f} Y$ be a continuous morphism in with inverse image functor f^*, direct image functor f_*, and adjunction morphisms

$$Id_{C_Y} \xrightarrow{\eta_f} f_* f^* \quad \text{and} \quad f^* f_* \xrightarrow{\epsilon_f} Id_{C_X}.$$

Let $\mathcal{F}_f$ denote the monad (F_f, μ_f) on Y, where $F_f = f_*f^*$ and $\mu_f = f_*\epsilon_f f^*$. There is a commutative diagram

$$\begin{array}{ccc} C_X & \xrightarrow{\widetilde{f_*}} & (\mathcal{F}_f/Y) - mod \\ & {\scriptstyle f_*} \searrow \quad \swarrow {\scriptstyle \mathfrak{f}_*} & \\ & C_X & \end{array} \tag{3}$$

Here $\widetilde{f_*}$ is the canonical functor

$$C_X \longrightarrow (\mathcal{F}_f/Y) - mod, \quad M \longmapsto (f_*(M), f_*\epsilon_f(M)),$$

and $\mathfrak{f}^*$ is the forgetful functor $(\mathcal{F}_f/Y) - mod \longrightarrow C_Y$.

The following assertion is one of the versions of Beck's theorem.

A1.3.1. Theorem. *Let $X \xrightarrow{f} Y$ be a continuous morphism.*

(a) If the category C_Y has cokernels of reflexive pairs of arrows, then the functor $\bar{f}_*$ has a left adjoint, $\bar{f}^*$; hence $\bar{f}_*$ is a direct image functor of a continuous morphism $\bar{X} \xrightarrow{f} \mathbf{Sp}(\mathcal{F}_f/Y)$.

(b) If, in addition, the functor f_* preserves cokernels of reflexive pairs, then the adjunction arrow $\bar{f}^*\bar{f}_* \longrightarrow Id_{C_X}$ is an isomorphism, i.e. $\bar{f}_*$ is a localization.

(c) If, in addition to (a) and (b), the functor f_* is conservative, then $\bar{f}_*$ is a category equivalence.

Proof. See [MLM], IV.4.2, or [ML], VI.7. ■

A1.3.2. Corollary. *Let $X \xrightarrow{f} Y$ be an affine morphism (cf. 1.5). If the category C_Y has cokernels of reflexive pairs of arrows (e.g. C_Y is an abelian category), then the canonical morphism $X \xrightarrow{\mathfrak{f}} \mathbf{Sp}(\mathcal{F}_f/Y)$ is an isomorphism.*

A1.3.3. Monadic morphisms. A continuous morphism $X \xrightarrow{f} Y$ such that the functor

$$C_X \xrightarrow{\widetilde{f_*}} \mathcal{F}_f - mod, \quad M \longmapsto (f_*(M), f_*\epsilon_f(M)),$$

is an equivalence of categories.

A1.4. Continuous monads and affine morphisms. A functor F is called *continuous* if it has a right adjoint. A monad $\mathcal{F} = (F, \mu)$ on Y (i.e. on the category C_Y) is called *continuous* if the functor F is continuous.

A monad $\mathcal{F} = (F, \mu)$ is called *right exact* if the functor F is right exact.

A1.4.1. Proposition. *A monad $\mathcal{F} = (F, \mu)$ on Y is continuous (resp. right exact) iff the canonical morphism $\mathbf{Sp}(\mathcal{F}/Y) \xrightarrow{\hat{f}} Y$ is affine (resp. almost affine).*

Proof. A proof in the case of a continuous monad can be found in [KR2, 6.2], or in [R3, 4.4.1] (see also [R4, 2.2]).

The argument in the case of a right exact monad is straightforward and is left to the reader. ■

A1.4.2. Corollary. *Suppose that the category C_Y has cokernels of reflexive pairs of arrows. A continuous morphism $X \xrightarrow{f} Y$ is affine (resp. almost affine) iff its direct image functor $C_X \xrightarrow{f_*} C_Y$ is the composition of a category equivalence*

$$C_X \longrightarrow (\mathcal{F}_f/Y) - mod$$

for a continuous (resp. right exact) monad $\mathcal{F}_f$ on Y and the forgetful functor $(\mathcal{F}_f/Y) - mod \longrightarrow C_Y$. The monad $\mathcal{F}_f$ is determined by f uniquely up to isomorphism.

Proof. The conditions of the Beck's theorem are fullfiled if f is affine, hence f_* is the composition of an equivalence $C_X \longrightarrow (\mathcal{F}_f/Y) - mod$ for a monad $\mathcal{F}_f = (f_*f^*, \mu_f)$ in C_Y and the forgetful functor $(\mathcal{F}_f/Y) - mod \longrightarrow C_Y$ (see (1)). The functor $F_f = f_*f^*$ has a right adjoint $f_*f^!$, where $f^!$ is a right adjoint to f_*. The rest follows from A1.4.1. ■

A1.5. Localizations compatible with monadic morphisms. Fix a monadic morphism $X \xrightarrow{f} Z$ and a Serre localization $U \xrightarrow{u} Z$ (i.e. $C_Z \xrightarrow{u^*} C_U$ is the localization at a Serre subcategory) compatible with f. Here *compatible* means that the functor $F_f = f_*f^*$ maps $\Sigma_{u^*} \stackrel{\text{def}}{=} \{s \in HomC_Z \mid u^*(s) \in Iso(C_U)\}$ to Σ_{u^*}; or, equivalently, there exists a functor $C_U \xrightarrow{F_U} C_U$ such that $u^* \circ F_f = F_U \circ u^*$. Thanks to the universal property of localizations, the functor F_U is determined uniquely by the latter equality. The monad structure $F_f^2 \xrightarrow{\mu_f} F_f$ induces a monad structure $F_U \xrightarrow{\mu} F_U$, hence we obtain a monad $\mathcal{F}_U = (F_U, \mu)$. The localization functor u^* induces a functor $(\mathcal{F}_f/Z) - mod \xrightarrow{\widetilde{u}^*} (\mathcal{F}_U/U) - mod$ which maps an $\mathcal{F}_f$-module $(M, F_f(M) \xrightarrow{\xi} M)$ to the $\mathcal{F}_U$-module $(u^*(M), F_U u^*(M) \xrightarrow{u^*(\xi)} u^*(M))$.

It is easy to see that $\widetilde{u}^*$ is (isomorphic to) an exact localization and $Ker(\widetilde{u}^*)$ is generated by all $\mathcal{F}_f$-modules (M, ξ) with $M \in ObKer(u^*)$.

Suppose now that the localization φ is continuous, and let u_* is its direct image functor. The equality $F_U \circ u^* = u^* \circ F_f$ implies an isomorphism $u^*F_f u_* = F_U u^* u_* \xrightarrow{F_U \epsilon_u} F_U$, where ϵ_u is an adjunction isomorphism $u^*u_* \longrightarrow Id_{C_U}$. The compatibility of F_f with the localization functor u^* means precisely that the morphism $u^*F_f \xrightarrow{u^*F_f\eta_u} u^*F_f u_* u^*$, where η_u is an adjunction arrow $Id_{C_Z} \longrightarrow u_*u^*$, is an isomorphism. This isomorphism allows to write the multiplication $\widetilde{\mu}$ on $u^*F_f u_*$ as the composition of the isomorphism

$$(u^*F_f u_*)^2 = (u^*F_f u_* u^*) F_f u_* \xrightarrow{\sim} u^*F_f^2 u_* \quad \text{and} \quad u^*F_f^2 u_* \xrightarrow{u^*\mu_f u_*} u^*F_f u_*.$$

One can show that $\widetilde{\mu}$ is a monad structure on $u^*F_f u_*$ and the canonical isomorphism $u^*F_f u_* \xrightarrow{\sim} F_U$ described above is a monad isomorphism $(u^*F_f u_*, \widetilde{\mu}) \xrightarrow{\sim} (F_U, \mu)$.

One of the consequences of this isomorphism is a description of a canonical right adjoint $\widetilde{u}_*$ to the localization functor $\mathcal{F}_f - mod \xrightarrow{\widetilde{u}^*} \mathcal{F}_U - mod$.

In fact, let $\mathfrak{F}_U$ denote the monad $(u^*F_f u_*, \widetilde{\mu})$. Every morphism $u^*F_f u_*(M) \xrightarrow{\xi} M$ determines via adjunction (and is determined by) a morphism $F_f(u_*(M)) \xrightarrow{\widehat{\xi}} u_*(M)$.

If ξ is an $(u^*F_f u_*, \widetilde{\mu})$-module structure, then $\widehat{\xi}$ is an $\mathcal{F}_f$-module structure. This defines a functor $\mathfrak{F}_U - mod \xrightarrow{\widetilde{u}_*} \mathcal{F}_f - mod$. The functor $\widetilde{u}_*$ is a right adjoint to the functor $\mathcal{F}_f - mod \xrightarrow{\widetilde{u}_*} \mathfrak{F}_U - mod$ which maps an $\mathcal{F}_f$-module (M, ζ) to the $\mathfrak{F}_U$-module $(u^*(M), \zeta_u)$, where ζ_u is the composition of the isomorphism $u^*F_f u_* u^*(M) \xrightarrow{\sim} u^*F_f(M)$ and $u^*F_f(M) \xrightarrow{u^*(\zeta)} u^*(M)$. One can verify that the adjunction morphisms $u^*u_* \xrightarrow{\epsilon_u} Id_{C_U}$ and $Id_{C_Z} \xrightarrow{\eta_u} u_*u^*$ give rise to the adjunction morphisms $\widetilde{u}^*\widetilde{u}_* \longrightarrow Id_{\mathfrak{F}_U - mod}$ and $Id_{\mathcal{F}_f - mod} \longrightarrow \widetilde{u}_*\widetilde{u}^*$. In particular, $\widetilde{u}^*\widetilde{u}_* \longrightarrow Id_{\mathfrak{F}_U - mod}$ is an isomorphism, which shows that $\widetilde{u}^*$ is a localization. It follows from this description that the diagram

$$\begin{array}{ccc}
\mathfrak{F}_U - mod & \xrightarrow{\widetilde{u}_*} & \mathcal{F}_f - mod \\
\mathfrak{f}_{u*} \downarrow & & \downarrow \mathfrak{f}_* \\
C_U & \xrightarrow{u_*} & C_Z
\end{array}$$

quasi-commutes. Here the vertical arrows are forgetful functors.

A1.5.1. Lemma. *Let $U \xrightarrow{u} X$ be a continous morphism such that u^* is a localization, and $C_X \xrightarrow{F} C_X$ is a functor compatible with the localization u^*. If the functor F is continuous, then the induced endofunctor $C_U \xrightarrow{F_U} C_U$ is continuous.*

Proof. Let $F^!$ be a right adjoint to the functor F and $Id_{C_X} \xrightarrow{\eta} F^!F$, $FF^! \xrightarrow{\epsilon} Id_{C_X}$ adjunction arrows. By the argument above, the functor F_U uniquely determined by the equality $F_U \circ u^* = u^* \circ F$, is naturally isomorphic to u^*Fu_*, and the compatibility of F with the localization u^* (i.e. the existence of F_U is equivalent to that the natural morphism $u^*F \xrightarrow{u^*F\eta_u} u^*Fu_*u^*$ is an isomorphism. Here η_u is the adjunction arrow $Id_{C_X} \longrightarrow u_*u^*$. The claim is that the functor $u^*F^!u_*$ is a right adjoint to u^*Fu_* (hence to F_U).

In fact, there are natural morphisms

$$(u^*Fu_*)(u^*F^!u_*) = (u^*Fu_*u^*)(F^!u_*) \xrightarrow{\sim} u^*FF^!u_* \xrightarrow{u^*\epsilon u_*} u^*u_* \xrightarrow{\epsilon_u} Id_{C_U}$$

and

$$Id_{C_U} \xrightarrow{\epsilon_u^{-1}} u^*u_* \xrightarrow{u^*\eta u_*} u^*F^!Fu_* \xrightarrow{u^*F^!\eta_u Fu_*} u^*F^!u_*u^*F_*^u.$$

One can check that their respective compositions produce a pair of adjuction morphisms. Details are left to the reader. ∎

A1.6. Infinitesimal neighborhoods of the diagonal. Differential calculus. Fix a monoidal category $\mathcal{A}^\sim = (\mathcal{A}, \otimes, \mathbf{1}, a)$. Here $\mathbf{1}$ denotes the unit object and a the associativity constraint. In order to simplify the exposition, we assume that the category $\mathcal{A}$ is quasi-abelian (i.e. it is additive and every morphism has a kernel and cokernel) and that the functor $M \otimes - : L \longmapsto M \otimes L$ preserves small colimits.

Fix a full monoidal subcategory T of $\mathcal{A}$ closed with respect to colimits taken in $\mathcal{A}$. The pair $(\mathcal{A}^\sim, T)$ is the initial data for differential calculus.

Objects of the n^{th} neighborhood $T^{(n+1)}$ of the subcategory T are called T-*differential objects of order* $\le n$. In particular, zero objects are the only T-differential objects of the order -1, and T consists of T-differential objects of order ≤ 0. Sometimes we shall loosely call the subcategory T the 'diagonal'.

A1.6.1. Proposition. *The category* $T^{(\infty)} \overset{\mathrm{def}}{=} \bigcap_{n\ge 1} T^{(n)}$ *of* T*-differential objects is a monoidal subcategory of* $\mathcal{A}^\sim$.

Proof. See [RL1]. ∎

A1.6.2. Corollary. *The category* T^∞ *whose objects are colimits of objects of the category* $T^{(\infty)}$ *is a monoidal subcategory of* $\mathcal{A}^\sim$.

Proof. The assertion follows from A1.6.1 and the assumption that the functors $M\otimes : L \longmapsto M\otimes L,\ M \in Ob\mathcal{A}$, preserve colimits. ∎

A1.6.3. The smallest diagonal. We denote the *smallest 'diagonal'* (i.e. the monoidal subcategory of $\mathcal{A}^\sim$ closed with respect to colimits (taken in $\mathcal{A}$) and generated by the identity object $\mathbf{1}$) by $\Delta_{\mathcal{A}^\sim}$.

A1.7. Differential functors and differential monads.

A1.7.1. Differential functors and (co)monads. Let $\mathcal{A}^\sim$ be the monoidal category $\mathfrak{End}_{\mathfrak{r}}(C_X)$ of right exact endofunctors of an abelian category C_X and $\mathbb{T} = \Delta_{\mathcal{A}^\sim}$ the smallest diagonal of $\mathcal{A}^\sim$. Objects of the subcategory $\mathbb{T}^{(\infty)} = \Delta_{\mathcal{A}^\sim}^{(\infty)}$ are called *differential* functors.

A monad (F,μ) (resp. a comonad (G,δ) is called *differential* if the endofunctor F (resp. G) is differential.

A1.7.2. Differential bimodules. Let R be an associative unital ring and $\mathcal{A}^\sim$ the monoidal category of R-bimodules: $\mathcal{A}^\sim = R-bimod^\sim = (R-bimod, \otimes_R, R)$. In this case the smallest diagonal is the full subcategory of $R-bimod$ whose objects are all *central bimodules*, i.e. bimodules M generated by their center $C(M) \overset{\mathrm{def}}{=} \{z \in M|\ rz = zr \text{ for all } r\in R\}$. The corresponding differential objects are called differential bimodules.

Note that the monoidal category of differential bimodules is equivalent to the monoidal category of differential endofunctors $C_X \longrightarrow C_X$, where $C_X = R-mod$.

A1.7.3. Proposition. *(a) Let* C_X *be an abelian category and* $C_X \xrightarrow{F} C_X$ *a differential endofunctor. Then every thick subcategory* $\mathbb{T}$ *of the category* C_X *is* F*-stable, i.e.* $F(\mathbb{T}) \subseteq \mathbb{T}$.

(b) If, in addition, the functor F *is exact, then there exists a unique endofunctor* $F_{\mathbb{T}}$ *of the quotient category* $C_{X/\mathbb{T}}$ *such that* $F_{\mathbb{T}}\circ q_{\mathbb{T}}^* = q_{\mathbb{T}}^*\circ F$. *Here* $q_{\mathbb{T}}$ *is the localization functor* $C_X \longrightarrow C_{X/\mathbb{T}}$. *The functor* $F_{\mathbb{T}}$ *is exact and differential.*

(c) If the differential functor F *is exact and continuous (i.e. it has a right adjoint), then for every continuous exact localization* $C_X \xrightarrow{q_{\mathbb{T}}} C_{X/\mathbb{T}}$, *the induced endofunctor* $F_{\mathbb{T}}$ *of* $C_{X/\mathbb{T}}$ *is continuous.*

Proof. (a) If F belongs to the diagonal, then $F(\mathbb{S}) \subseteq \mathbb{S}$ for every full subcategory of C_X closed under coproducts and quotients (taken in C_X). In particular, every topologizing (hence every thick) subcategory of C_X is F-stable.

In general, an endofunctor F is differential iff it has an increasing filtration, $F_{-1} = 0 \hookrightarrow F_0 \hookrightarrow \ldots \hookrightarrow F_n = F$ such that all quotients $F_i/F_{i-1},\ 0 \le i \le n$, belong to the diagonal. In particular, for every object M of a thick subcategory $\mathbb{T}$, there is a filtration $0 \hookrightarrow F_0(M) \hookrightarrow \ldots \hookrightarrow F_n(M) = F(M)$ such that all quotients $F_i(M)/F_{i-1}(M),\ 0 \le i \le n$, belong to $\mathbb{T}$. Therefore, $F(M)$ is an object of $\mathbb{T}$.

(b) If a functor F stabilizes a thick subcategory $\mathbb{T}$ and is exact, then it determines a unique endofunctor $F_{\mathbb{T}}$ of the quotient category $C_{X/\mathbb{T}}$ such that $q_{\mathbb{T}}^* \circ F = F_{\mathbb{T}} \circ q_{\mathbb{T}}^*$. Since the functor $q_{\mathbb{T}}^* \circ F$ is exact, it follows from [GZ, 1.1.4] that the functor $F_{\mathbb{T}}$ is exact.

(c) If F_* is a right adjoint to the endofunctor F and $q_{\mathbb{T}*}$ is a right adjoint to the localization functor $C_X \xrightarrow{q_{\mathbb{T}}} C_{X/\mathbb{T}}$. The checking (or reading [KR2, C2.1]) is left to the reader. ∎

Appendix 2. Associated points and primary decomposition.

Fix an abelian category C_X. For every $M \in ObC_X$, the set $\mathfrak{Ass}(M)$ of associated points of M can be described as the set of all $\mathcal{Q} \in \mathbf{Spec}^0_{\mathfrak{c}}(X)$ such that there exists a nonzero monomorphism $L \hookrightarrow M$ with L from $\mathcal{Q} \cap \langle \mathcal{Q} \rangle^{\perp}$.

We define $\mathfrak{Ass}^{1,1}_{\mathfrak{t}}(M)$ as the set of all $\mathcal{P} \in \mathfrak{Th}(X)$ such that there exists a nonzero monomorphism $L \hookrightarrow M$ with L from $\mathcal{P}^{\mathfrak{t}} \cap \mathcal{P}^{\perp}$. It follows that $\mathfrak{Ass}^{1,1}_{\mathfrak{t}}(M) \subseteq \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$.

We define $\mathfrak{Ass}^{-}(M)$ as the set of all $\mathcal{P} \in \mathfrak{Th}(X)$ such that there exists a nonzero monomorphism $L \hookrightarrow M$ with L from $\mathcal{P}_{\circledast} = \mathcal{P}^{\circledast} \cap \mathcal{P}^{\perp}$.

It follows that $\mathfrak{Ass}^{-}(M) \subseteq \mathbf{Spec}^{-}(X)$.

We denote by $\mathfrak{Ass}^{0,1}(M)$ the set of elements $\mathcal{P}_{\circledast}$ of $\mathbf{Spec}_{-}(X)$ such that there is a nonzero subobject $L \hookrightarrow M$ with $L \in Ob\mathcal{P}_{\circledast}$.

Finally, $\mathfrak{Ass}_{\mathfrak{L}}(M)$ is the set of all $\mathcal{P} \in \mathfrak{Th}(X)$ such that there exists a nonzero monomorphism $L \hookrightarrow M$ with L from $\mathcal{P}_{\star} = \mathcal{P}^{\star} \cap \mathcal{P}^{\perp}$. In particular, $\mathfrak{Ass}_{\mathfrak{L}}(M) \subseteq \mathbf{Spec}^{1,1}_{\mathfrak{Th}}(X)$.

We denote by $\mathfrak{Ass}^{0,1}_{\mathfrak{Th}}(M)$ the set whose elements are $\mathcal{P}_{\star} = \mathcal{P}^{\star} \cap \mathcal{P}^{\perp}$ of $\mathbf{Spec}^{0,1}_{\mathfrak{Th}}(X)$ such that M has a nonzero subobject which belongs to $\mathcal{P}_{\star}$.

It follows from these definitions that the commutative diagram

$$\begin{array}{ccccc}
\mathbf{Spec}^0_{\mathfrak{c}}(X) & \xrightarrow{\alpha} & \mathbf{Spec}_{-}(X) & \xrightarrow{\beta} & \mathbf{Spec}^{0,1}_{\mathfrak{Th}}(X) \\
\wr\downarrow & & \wr\downarrow & & \downarrow\wr \\
\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X) & \longrightarrow & \mathbf{Spec}^{-}(X) & \longrightarrow & \mathbf{Spec}^{1,1}_{\mathfrak{Th}}(X)
\end{array} \qquad (1)$$

(see C2.6(5)) induces for any object M of the category C_X a commutative diagram

$$\begin{array}{ccccc}
\mathfrak{Ass}(M) & \longrightarrow & \mathfrak{Ass}^{0,1}(M) & \longrightarrow & \mathfrak{Ass}^{0,1}_{\mathfrak{Th}}(M) \\
\wr\downarrow & & \wr\downarrow & & \downarrow\wr \\
\mathfrak{Ass}^{1,1}_{\mathfrak{t}}(M) & \longrightarrow & \mathfrak{Ass}^{-}(M) & \longrightarrow & \mathfrak{Ass}_{\mathfrak{L}}(M)
\end{array} \qquad (2)$$

whose horizontal arrows are embeddings and the vertical arrows are isomorphisms.

It follows that

$$\begin{aligned} &\mathfrak{Ass}^{1,1}_{\mathfrak{t}}(M) = \mathfrak{Ass}^-(M) \bigcap \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X) = \mathfrak{Ass}_{\mathfrak{L}}(M) \bigcap \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X) \quad \text{and} \\ &\mathfrak{Ass}^-(M) = \mathfrak{Ass}^{1,1}_{\mathfrak{T}\mathfrak{h}}(M) \bigcap \mathbf{Spec}^-(X). \end{aligned} \tag{3}$$

A2.1. Remarks. (a) If X has a Gabriel-Krull dimension, then, by [R5, 8.7.1], the inclusion map $\mathbf{Spec}^-(X) \longrightarrow \mathbf{Spec}^{1,1}_{\mathfrak{T}\mathfrak{h}}$ in the diagram (1) is an isomorphism, hence the right horizontal arrows in the diagrams (1) and (2) are isomorphisms.

(b) The correspondence $M \longmapsto \mathfrak{Ass}_{\mathfrak{L}}(M)$ is studied in [R5, 10.8–10.10], where it is shown that $\mathfrak{Ass}_{\mathfrak{L}}(M)$ enjoys all general properties of associated points in the context of commutative algebra. Similar facts hold for the map $M \longmapsto \mathfrak{Ass}^{0,1}(M)$.

Here we sketch the facts about $M \longmapsto \mathfrak{Ass}(M)$ immitating [R5, 10.8–10.10] whenever it is possible to do.

A2.2. Proposition. *(a) For any exact sequence*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0,$$

$$\mathfrak{Ass}(M') \subseteq \mathfrak{Ass}(M) \subseteq \mathfrak{Ass}(M') \bigcup \mathfrak{Ass}(M'').$$

(b) Suppose X has the property (sup). Let an object M of C_X be a supremum of an ascending family, Ξ, of its subobjects. Then

$$\mathfrak{Ass}(M) = \bigcup_{M' \in \Xi} \mathfrak{Ass}(M').$$

(c) For every object M of C_X, any exact localization, $Y \xrightarrow{u} X$, induces an injective map $\mathfrak{Ass}(M) \cap \mathcal{U}_{\mathfrak{T}}(Ker(u^*)) \longrightarrow \mathfrak{Ass}(u^*(M))$. Here $\mathcal{U}_{\mathfrak{T}}(\mathbb{S}) = \{\mathbb{T} \in \mathfrak{T}(X) \mid \mathbb{T} \not\subseteq \mathbb{S}\}$.

(d) If M belongs to $Spec^0_{\mathfrak{c}}(X)$, then $\mathfrak{Ass}(M) = \{[M]\}$.

Proof. (a) The inclusion $\mathfrak{Ass}(M') \subseteq \mathfrak{Ass}(M)$ follows from definitions.

Let $\mathcal{P} \in \mathfrak{Ass}(M)$, i.e. there exists a nonzero subobject, L, of M such that $[L] = \mathcal{P}$. Suppose $L' = L \cap M' \neq 0$. Then L' is a nonzero subobject of M' and L. The latter implies that $[L'] = [L] = \mathcal{P}$, hence $\mathcal{P} \in \mathfrak{Ass}(M')$. If $L' = 0$, then the composition of $L \hookrightarrow M$ and the canonical epimorphism $M \longrightarrow M''$ is a monomorphism, hence $\mathcal{P} \in \mathfrak{Ass}(M'')$. This proves the inclusion $\mathfrak{Ass}(M) \subseteq \mathfrak{Ass}(M') \bigcup \mathfrak{Ass}(M'')$.

(b) It follows from (a) that the inclusion $\mathfrak{Ass}(M) \supseteq \bigcup_{M' \in \Xi} \mathfrak{Ass}(M')$ holds without any additional conditions on X.

Let $\mathcal{P} \in \mathfrak{Ass}(M)$, i.e. M has a nonzero subobject L such that $[L] = \mathcal{P}$. Since X has the property (sup), $L \cap M' \neq 0$ for some $M' \in \Xi$. Therefore $\mathcal{P} \in \mathfrak{Ass}(M')$ (see the argument in (a) above). This verifies the inverse inclusion, $\mathfrak{Ass}(M) \subseteq \bigcup\limits_{M' \in \Xi} \mathfrak{Ass}(M')$.

(c) Let u^* be an inverse image functor of $Y \xrightarrow{u} X$. Set $Ker(u^*) = \mathbb{S}$. The claim is that the injective map $\mathcal{U}_{\mathfrak{T}}(X) \longrightarrow \mathfrak{T}\mathfrak{h}(Y)$, $\mathcal{P} \longrightarrow \mathcal{P}/\mathbb{S}$, induces a (forcibly injective) map $\mathfrak{Ass}(M) \cap \mathcal{U}_{\mathfrak{T}\mathfrak{h}}(\mathbb{S}) \longrightarrow \mathfrak{Ass}(u^*(M))$.

Let $\mathcal{P} \in \mathfrak{Ass}(M) \cap \mathcal{U}_{\mathfrak{T}}(\mathbb{S})$, that is $\mathcal{P} \not\subseteq \mathbb{S}$, and there exists a nonzero subobject L of M such that $[L] = \mathcal{P}$. Since $\mathcal{P} \not\subseteq \mathbb{S}$, the object L is $\mathbb{S}$-torsion free. Therefore, $u^*(L)$ is a nonzero subobject of $u^*(M)$ which belongs to $Spec^0_{\mathfrak{c}}(X)$.

(d) The assertion follows from the definition of $Spec^0_{\mathfrak{c}}(X)$. ■

A2.3. Corollary. *(i) For any finite set, $\{M_i \mid i \in J\}$, of objects of C_X,*

$$\mathfrak{Ass}_{\mathfrak{L}}(\oplus_{i \in J} M_i) = \bigcup_{i \in J} \mathfrak{Ass}_{\mathfrak{L}}(M_i).$$

If X has the property (sup), then the finiteness condition can be dropped.

(ii) Let $\{L_i \mid i \in J\}$ be a finite set of subobjects of an object M such that $\bigcap_{i \in J} L_i = 0$. Then

$$\mathfrak{Ass}_{\mathfrak{L}}(M/(\cap_{i \in J} L_i)) \subseteq \bigcup_{i \in J} \mathfrak{Ass}_{\mathfrak{L}}(M/L_i).$$

Proof. (i) For a finite set $\{M_i \mid i \in J\}$, the assertion follows from A2.2(a). The infinite case is a consequence of A2.2(b).

(ii) The assertion follows from (i) and A2.2(a) applied to the canonical monomorphism $M/(\cap_{i \in J} L_i) \longrightarrow \oplus_{i \in J} M/L_i$. ■

A2.4. Corollary. *The full subcategory, $C_{X^{\emptyset}_{\mathfrak{Ass}}}$, of the category C_X whose objects, M, have no associated points, $\mathfrak{Ass}_{\mathfrak{L}}(M) = \emptyset$, is closed under extensions, taking subobjects, and colimits of filtered diagrams of monoarrows.*

Proof. The assertion is a consequence of A2.2(a) and (b). ■

A2.5. Proposition. *Let $Y \xrightarrow{u} X$ be an exact localization such that $\mathbb{S} = Ker(u^*)$ is a coreflective subcategory of the category C_X. Let $\mathcal{P} \in \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$ and $\mathbb{S} \subseteq \mathcal{P}$. Let M be an object of C_X such that $\mathfrak{Ass}(L) \neq \emptyset$ for any nonzero subobject, L, of M. Then the following conditions are equivalent:*

(a) $\mathfrak{Ass}^{1,1}_{\mathfrak{t}}(M) = \{\mathcal{P}\}$;

(b) $\mathfrak{Ass}^{1,1}_{\mathfrak{t}}(u^*(M)) = \{\mathcal{P}/\mathbb{S}\}$ and M is $\mathbb{S}$-torsion free.

Proof. (a)⇒(b). Let $\mathfrak{t}_{\mathbb{S}}M$ denote the $\mathbb{S}$-torsion of M. If $\mathfrak{t}_{\mathbb{S}}M \neq 0$, then, by hypothesis, $\mathfrak{Ass}(\mathfrak{t}_{\mathbb{S}}M) \neq \emptyset$, i.e. $\mathfrak{Ass}(\mathfrak{t}_{\mathbb{S}}M) = \{\mathcal{P}\}$. The latter means that $\mathfrak{t}_{\mathbb{S}}M$ has a nonzero subobject L such that to $\langle L \rangle = \mathcal{P}$; in particular, L is $\mathcal{P}$-, hence $\mathbb{S}$-torsion free, which contradicts to that L is a nonzero object of the subcategory $\mathbb{S}$.

Since M is $\mathbb{S}$-torsion free, it follows from A2.2(c) that $\mathfrak{Ass}^{1,1}_{\mathfrak{t}}(u^*(M)) = \{\mathcal{P}/\mathbb{S}\}$.

(b)⇒(a). There is a subobject N of M such that $\langle u^*(N) \rangle = \mathcal{P}/\mathbb{S}$. By hypothesis, since $N \neq 0$, $\mathfrak{Ass}(N) \neq \emptyset$; i.e. there exists a subobject $L \hookrightarrow N$ such that $[L] \in \mathbf{Spec}^0_{\mathfrak{c}}(X)$. Since L is $\mathcal{P}$-torsion free, it follows that $\mathcal{P} = \langle L \rangle$. ■

A2.6. Proposition. *Suppose X has the property (sup). Let $M \in ObC_X$, and let Φ be a subset of $\mathfrak{Ass}(M)$. Then there exists a subobject $L \longrightarrow M$ such that*

$$\mathfrak{Ass}(M/L) = \mathfrak{Ass}(M) - \Phi \quad \textit{and} \quad \mathfrak{Ass}(L) = \Phi. \tag{4}$$

Proof. (a) Let $\mathfrak{D}_\Phi$ be the set of subobjects, M', of M such that $\mathfrak{Ass}(M') \subseteq \Phi$. The set $\mathfrak{D}_\Phi$ is not empty, because it contains the zero subobject. It follows from A2.2(b) that $sup\Xi \in \mathfrak{D}_\Phi$ for every filtered subset Ξ of $\mathfrak{D}_\Phi$. Therefore, by Zorn's lemma, there exists a maximal element (subobject), L, in $\mathfrak{D}_\Phi$. We claim that the subobject L satisfies the conditions (4). Thanks to A2.2(a), it suffices to show that $\mathfrak{Ass}(M/L) \subseteq \mathfrak{Ass}(M) - \Phi$.

(b) Let $\mathcal{P} \in \mathfrak{Ass}(M/L)$, i.e. M/L has a subobject, $N \longrightarrow M/L$ such that $\mathcal{P} = [N]$. Consider the short exact sequence

$$0 \longrightarrow L \longrightarrow \widetilde{N} = M \times_{M/L} N \longrightarrow N \longrightarrow 0. \tag{5}$$

associated with $N \longrightarrow M/L$. By A2.2(a), $\mathfrak{Ass}(\widetilde{N}) \subseteq \mathfrak{Ass}(L) \bigcup \mathfrak{Ass}(N)$. By A2.2(d), $\mathfrak{Ass}(N) = \{\mathcal{P}\}$. Since L is a maximal element of $\mathfrak{D}_\Phi$ and a proper subobject of $\widetilde{N}$, the latter does not belong to $\mathfrak{D}_\Phi$. Therefore $\mathcal{P} \in \mathfrak{Ass}(\widetilde{N}) - \Phi$. ∎

A2.7. Primary decomposition.

A2.7.1. Definition. Let M be an object of an abelian category C_X. We call a subobject N of M *primary*, or $\mathcal{P}$*-primary*, if $\mathfrak{Ass}(M/N)$ consists of one element, $\mathcal{P}$.

A2.7.2. Proposition. *Let $\{N_i \mid i \in J\}$ be a finite set of $\mathcal{P}$-primary subobjects of an object M of an abelian category C_X. Then $\bigcap_{i\in J} N_i$ is a $\mathcal{P}$-primary subobject of M.*

Proof. The fact follows from A2.3(ii). ∎

A2.7.3. Definition. Let N be a subobject of an object M of the category C_X. A *primary decomposition of* $N \hookrightarrow M$ is a finite set, $\{N_i \mid i \in J\}$, of primary subobjects of M such that N is a subobject of $\bigcap_{i\in J} N_i$ and $\mathfrak{Ass}(\bigcap_{i\in J} N_i/N) = \emptyset$.

A2.7.3.1. Note. It follows from this definition and A2.3(ii) that if a subobject N of M has a primary decomposition, then $\mathfrak{Ass}(M/N)$ is a subset of $\{\mathcal{P}_i \mid i \in J\}$, in particular, $\mathfrak{Ass}(M/N$ is finite. Here $\mathfrak{Ass}(M/N_i) = \{\mathcal{P}_i\}$.

A2.7.4. Proposition. *Let N be a subobject of an object M such that $\mathfrak{Ass}(M/N)$ is finite. Then there exists a primary decomposition, $\{N_\mathcal{P} \mid \mathcal{P} \in \mathfrak{Ass}(M/N)\}$, such that $N_\mathcal{P}$ is $\mathcal{P}$-primary for every $\mathcal{P} \in \mathfrak{Ass}(M/N)$.*

Proof. Replacing M by M/N, we can and will assume that $N = 0$. By A2.6, for every $\mathcal{P} \in \mathfrak{Ass}(M)$, there exists a subobject $N_\mathcal{P}$ of M such that $\mathfrak{Ass}(M/N_\mathcal{P}) = \{\mathcal{P}\}$ and $\mathfrak{Ass}(N_\mathcal{P}) = \mathfrak{Ass}(M) - \{\mathcal{P}\}$. Set $M_0 = \bigcap\limits_{\mathcal{P}\in\mathfrak{Ass}(M)} N_\mathcal{P}$. For each $\mathcal{P} \in \mathfrak{Ass}(M)$, we have the inclusion $\mathfrak{Ass}(M_0) \subseteq \mathfrak{Ass}(N_\mathcal{P})$, hence $\mathfrak{Ass}(M_0) = \emptyset$. ∎

A2.7.5. Definition. Let N be a subobject of an object M such that $\mathfrak{Ass}(M/N)$ is finite. Let $\{N_i \mid i \in J\}$ be a primary decomposition of N in M with $\mathfrak{Ass}(M/N_i = \{\mathcal{P}_i\}$. The primary decompsition $\{N_i \mid i \in J\}$ is called *reduced* if

(a) for any $i \in J$, $\mathfrak{Ass}(\bigcap\limits_{J\ni j\neq i} N_j/\bigcap\limits_{j\in J} N_j) \neq \emptyset$; in particular, the intersection $\bigcap\limits_{J\ni j\neq i} N_j$ is not a subobject of N_i;

(b) if $i \neq j$, then $\mathcal{P}_i \neq \mathcal{P}_j$.

A2.7.5.1. Note. Starting with an arbitrary primary decomposition, one can obtain a reduced primary decomposition as follows. Let $\{N_i \mid i \in J\}$ be any primary decomposition of $N \hookrightarrow M$ with $\mathfrak{Ass}(M/N_i) = \{\mathcal{P}_i\},\ i \in J$. Set $\Phi = \{\mathcal{P}_i \mid i \in J\}$. Let J_0 is a minimal element of the set of subsets, I, of J such that $\{N_i \mid i \in I\}$ is a primary decomposition. Clearly, $\{N_i \mid i \in J_0\}$ satisfies the condition (a). For each $\mathcal{P} \in \Phi$, let $N_{\mathcal{P}} = \bigcap\limits_{\mathcal{P}_i = \mathcal{P}} N_i$. By A2.7.2, $N_{\mathcal{P}} \hookrightarrow M$ is $\mathcal{P}$-primary. Since $\bigcap\limits_{\mathcal{P} \in \Phi} N_{\mathcal{P}} = \bigcap\limits_{i \in J} N_i$, the set of subobjects $\{N_{\mathcal{P}} \mid \mathcal{P} \in \Phi\}$ is a reduced primary decomposition of $N \hookrightarrow M$.

A2.7.6. Proposition. *Let N be a subobject of an object M such that $\mathfrak{Ass}(M/N)$ is finite. Let $\{N_i \mid i \in J\}$ be a primary decomposition of N in M with $\mathfrak{Ass}(M/N_i) = \{\mathcal{P}_i\}$.*

(i) The following conditions are equivalent:

(a) The decomposition $\{N_i \mid i \in J\}$ is reduced.

(b) All $\mathcal{P}_i$ belong to $\mathfrak{Ass}(M/N)$ and $\mathcal{P}_i \neq \mathcal{P}_j$ if $i \neq j$.

(ii) If the equivalent conditions (a), (b) are fulfilled, then

$$\mathfrak{Ass}(M/N) = \{\mathcal{P}_i \mid i \in J\} \quad \textit{and}$$
$$\mathfrak{Ass}(N_i/N) = \{\mathcal{P}_j \mid j \in J,\ j \neq i\} \quad \textit{for all}\ \ i \in J.$$

Proof. (a)⇒(b). Let $\{N_i \mid i \in J\}$ be a reduced primary decomposition. By A2.7.3.1, $\mathfrak{Ass}(M/N)$ is a subset of $\{\mathcal{P}_i \mid i \in J\}$. Set $N_i^\vee = \bigcap\limits_{J \ni j \neq i} N_j$. We can and will assume that $N = \bigcap\limits_{j \in J} N_j = N_i^\vee \cap N_i$. Since the decomposition $\{N_i \mid i \in J\}$ is reduced, $\mathfrak{Ass}(N_i^\vee/N) \neq \emptyset$. Because $N_i^\vee/N$ is isomorphic to the subobject $sup(N_i^\vee, N_i)/N_i$ of M/N_i, this implies that $\mathfrak{Ass}(N_i^\vee/N) = \{\mathcal{P}_i\}$, whence the inverse inclusion: $\{\mathcal{P}_i \mid i \in J\} \subseteq \mathfrak{Ass}(M/N)$.

(b)⇒(a). If the condition (b) holds, $\{N_j \mid j \in J - \{i\}\}$ cannot be a primary decomposition, because this would imply that $\mathcal{P}_i \notin \mathfrak{Ass}(M/N)$. Therefore the primary decomposition $\{N_i \mid i \in J\}$ of $N \hookrightarrow M$ is reduced.

The equality $\mathfrak{Ass}(M/N) = \{\mathcal{P}_i \mid i \in J\}$ is already established. It remains to show that for any $i \in J$, $\mathfrak{Ass}(N_i/N) = \{\mathcal{P}_j \mid j \in J,\ j \neq i\}$. Applying A2.2(a) to the exact sequence

$$0 \longrightarrow N_i/N \longrightarrow M/N \longrightarrow M/N_i \longrightarrow 0,$$

we obtain inclusions

$$\mathfrak{Ass}(N_i/N) \subseteq \mathfrak{Ass}(M/N) \subseteq \mathfrak{Ass}(N_i/N) \bigcup \mathfrak{Ass}(M/N_i) = \mathfrak{Ass}(N_i/N) \bigcup \{\mathcal{P}_i\}.$$

This and the equality $\mathfrak{Ass}(M/N) = \{\mathcal{P}_j \mid j \in J\}$ imply that

$$\{\mathcal{P}_j \mid j \in J - \{i\}\} \subseteq \mathfrak{Ass}(N_i/N) \subseteq \{\mathcal{P}_j \mid j \in J\}.$$

On the other hand, since $N = \bigcap_{j \in J-\{i\}} (N_i \cap N_j)$, we have an inclusion

$$\mathfrak{Ass}(N_i/N) \subseteq \bigcup_{j \in J-\{i\}} \mathfrak{Ass}(N_i/(N_i \cap N_j)).$$

But, $N_i/(N_i \cap N_j)$ is isomorphic to the subobject $sup(N_i, N_j)/N_j$ of the object M/N_j, hence $\mathfrak{Ass}(N_i/(N_i \cap N_j)) \subseteq \mathfrak{Ass}(M/N_j) = \{\mathcal{P}_j\}$. This gives the inverse inclusion: $\mathfrak{Ass}(N_i/N) \subseteq \{\mathcal{P}_j \mid j \in J - \{i\}\}$. ■

A2.7.7. Corollary. *Let $\{N_i \mid i \in J\}$ be a primary decomposition of a subobject N of an object M. Then $Card(\mathfrak{Ass}(M/N)) \leq Card(J)$. The decomposition $\{N_i \mid i \in J\}$ is reduced iff $Card(\mathfrak{Ass}(M/N)) = Card(J)$.*

Proof. Following the procedure described in A2.7.5.1, one can obtain, starting from $\{N_i \mid i \in J\}$, a reduced primary decomposition, $\{\widetilde{N}_j \mid j \in I\}$ such that $Card(I) \leq Card(J)$. The rest follows from A2.7.6. ■

For any object M of the category C_X, let $\mathfrak{D}_{\wp}(M)$ denote the set of reduced primary decompositions of $0 \hookrightarrow M$. By A2.7.6, each element of $\mathfrak{D}_{\wp}(M)$ is a set, $\{N_{\mathcal{P}} \mid \mathcal{P} \in \mathfrak{Ass}(M)\}$ of subobjects of M such that $\mathfrak{Ass}(M/N_{\mathcal{P}}) = \{\mathcal{P}\}$ and $\mathfrak{Ass}\big(\bigcap_{\mathcal{P} \in \mathfrak{Ass}(M)} N_{\mathcal{P}}\big) = \emptyset$.

A2.7.8. Proposition. *Let $\{N_{\mathcal{P}} \mid \mathcal{P} \in \mathfrak{Ass}(M)\}$ and $\{\widetilde{N}_{\mathcal{P}} \mid \mathcal{P} \in \mathfrak{Ass}(M)\}$ be two elements of $\mathfrak{D}_{\wp}(M)$, and let Φ be a subset of $\mathfrak{Ass}(M)$. Then $\{N_{\mathcal{P}} \mid \mathcal{P} \in \Phi\} \bigcup \{\widetilde{N}_{\mathcal{P}} \mid \mathcal{P} \in \mathfrak{Ass}(M) - \Phi\}$ is an element of $\mathfrak{D}_{\wp}(M)$.*

Proof. Set $N_\Phi = \bigcap_{\mathcal{P} \in \Phi} N_{\mathcal{P}}$ and $\widetilde{N}_\Phi^\vee = \bigcap_{\mathcal{P} \in \mathfrak{Ass}(M) - \Phi} \widetilde{N}_{\mathcal{P}}$. Since $\mathfrak{Ass}(M/N_{\mathcal{P}}) = \{\mathcal{P}\}$ and $\mathfrak{Ass}(M/\widetilde{N}_{\mathcal{P}}) = \{\mathcal{P}\}$ for all $\mathcal{P} \in \mathfrak{Ass}(M)$, it suffices to verify (thanks to A2.7.6) that $\mathfrak{Ass}(N_\Phi \bigcap \widetilde{N}_\Phi^\vee) = \emptyset$.

By A2.7.6(ii), $\mathfrak{Ass}(N_{\mathcal{P}}) = \mathfrak{Ass}(M) - \{\mathcal{P}\}$, in particular, $\mathcal{P} \notin \mathfrak{Ass}(N_{\mathcal{P}})$. Therefore, every element of Φ does not belong to $\mathfrak{Ass}(N_\Phi)$, i.e. $\Phi \bigcap \mathfrak{Ass}(N_\Phi) = \emptyset$. Similarly $(\mathfrak{Ass}(M) - \Phi) \bigcap \mathfrak{Ass}(\widetilde{N}_\Phi^\vee) = \emptyset$. Thus, $\mathfrak{Ass}(N_\Phi \bigcap \widetilde{N}_\Phi^\vee) \subseteq \Phi \bigcap (\mathfrak{Ass}(M) - \Phi) = \emptyset$. ■

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